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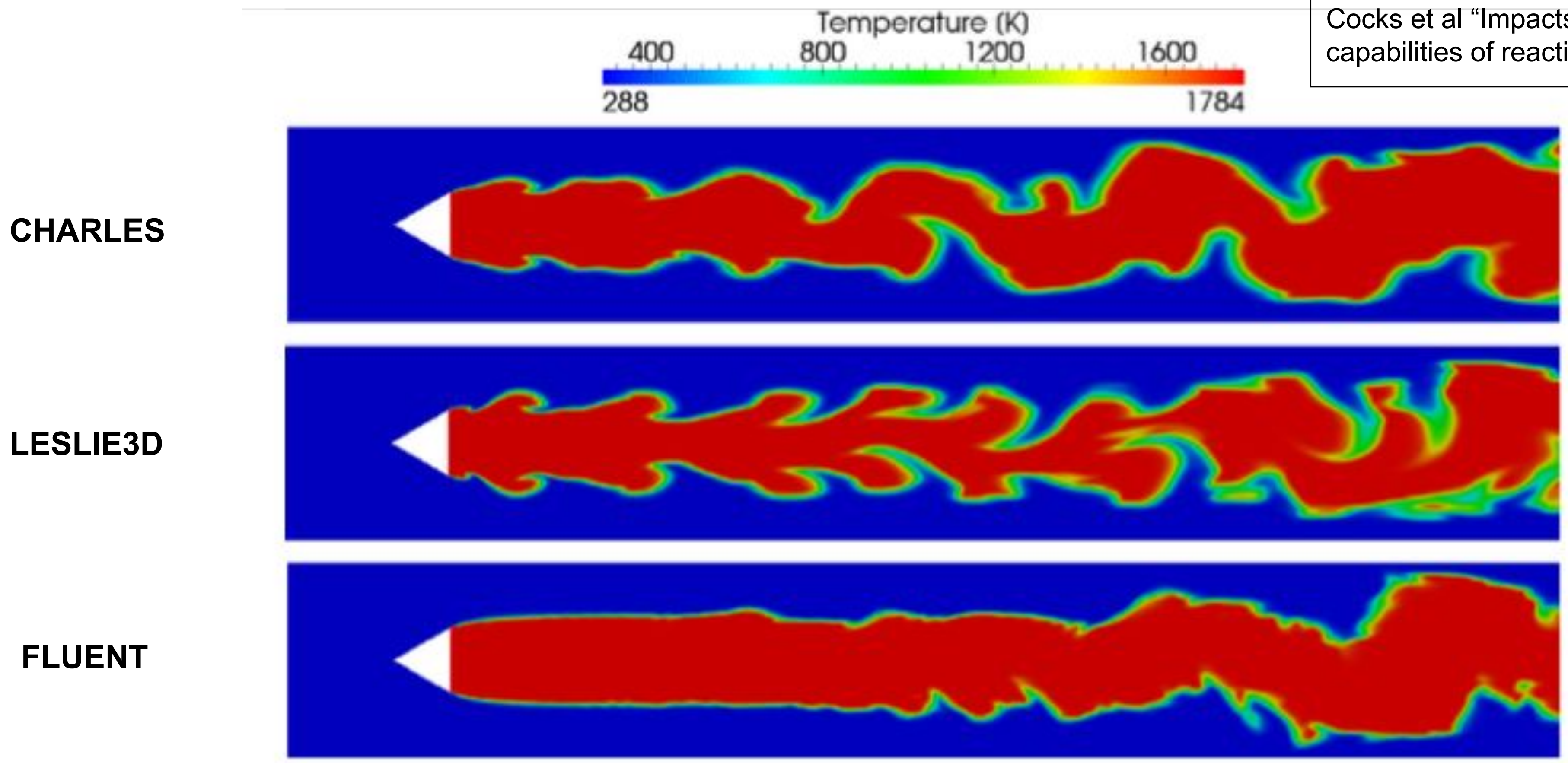
Split Forms for Secondary Preservation of the Navier-Stokes Equations

Ayaboe Edoh (Jacobs Engineering Inc.)

Motivation

Complex dynamics yield heightened sensitivity of the solution to numerics

Cocks et al "Impacts of numerics on predictive capabilities of reacting LES" CnF (2015)



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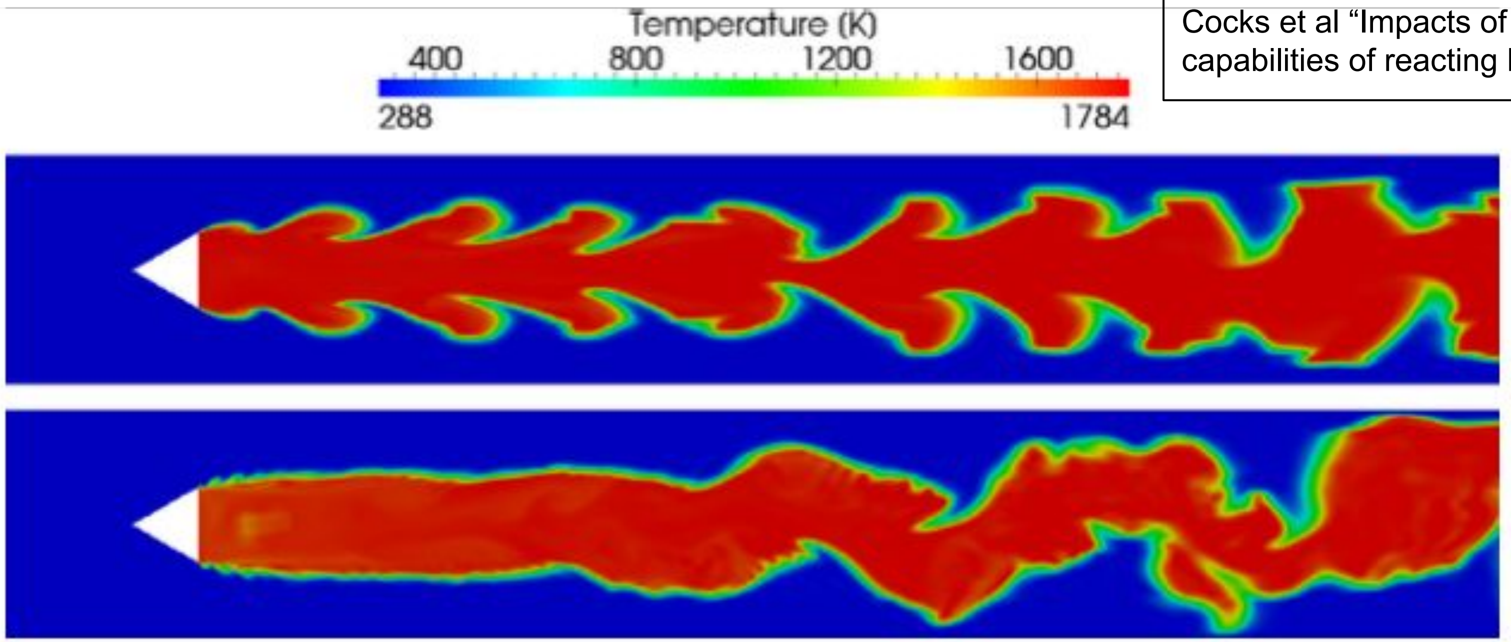

LESLIE3D
(MUSCL scheme)

LESLIE3D
(central scheme)

Enforce *additional* key physical properties at the discrete level: **secondary preservation**

Overview

1. Splitting schemes
    a. local conservation
    b. aliasing considerations

2. Incompressible Navier-Stokes convective splittings
    a. secondary quantities and stability
    b. kinetic-energy-preserving (KEP) methods

3. Compressible Navier-Stokes convective splittings
    a. kinetic energy preservation and non-linear robustness
    b. parameterized KEP methods
    c. solution filtering for enhancing stability and accuracy
    d. numerical demonstrations

4. Summary and Future Work

Overview

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4. Summary and Future Work

Finite Difference Splitting: Local Conservation

Congruence of **finite differencing** to a locally telescoping **finite volume** type representation

$$\partial_x \phi\big|_i \approx \delta_x \phi\big|_i = \frac{1}{\Delta x} \sum_{r \geq 1} \beta_r (\phi_{i+r} - \phi_{i-r})$$

$$= \frac{\bar{f}_{i+1/2} - \bar{f}_{i-1/2}}{\Delta x}$$

Finite Difference Splitting: Local Conservation

Congruence of **finite differencing** to a locally telescoping **finite volume** type representation

$$\partial_x \phi\big|_i \approx \delta_x \phi\big|_i = \frac{1}{\Delta x}\sum_{r\geq 1}\beta_r(\phi_{i+r}-\phi_{i-r}) = \frac{\bar{f}_{i+1/2}-\bar{f}_{i-1/2}}{\Delta x}$$

For non-linear terms, a quadratic splitting yields a **discrete product rule**

$$\overbrace{a_i(b_{i+r}-b_{i-r})+b_i(a_{i+r}-a_{i-r})}^{\partial(ab)\,\approx\, a\delta b+b\delta a} = \frac{a_i b_{i+r}+a_{i+r}b_i}{2} - \frac{a_i b_{i-r}+a_{i-r}b_i}{2}$$

$$\frac{a_i b_{i+r}+a_{i+r}b_i}{2} = 2\cdot\frac{a_i+a_{i+r}}{2}\,\frac{b_i+b_{i+r}}{2} - \frac{(ab)_i+(ab)_{i+r}}{2}$$

Finite Difference Splitting: Local Conservation

Congruence of **finite differencing** to a <u>locally telescoping</u> **finite volume** type representation

$$\partial_x \phi\Big|_i \approx \delta_x \phi\Big|_i = \frac{1}{\Delta x}\sum_{r\geq 1}\beta_r(\phi_{i+r}-\phi_{i-r})$$
$$= \frac{\bar{f}_{i+1/2}-\bar{f}_{i-1/2}}{\Delta x}$$

For non-linear terms, a <u>quadratic</u> splitting yields a **discrete product rule**

$$\partial_x(ab) \approx \begin{cases} \delta_x(ab) \rightarrow \bar{f}_{i+1/2} = 2\sum_r \beta'_r \cdot \dfrac{\overbrace{(ab)_i + (ab)_{i+r}}^{\overline{ab}}}{2} \\ a\delta_x b + b\delta_x a \rightarrow \bar{f}_{i+1/2} = 2\sum_r \beta'_r \cdot \dfrac{\overbrace{a_i b_{i+r} + a_{i+r} b_i}^{2\bar{a}\bar{b}-\overline{ab}}}{2} \end{cases}$$

Valid on bounded domains via SBP operators (i.e. flux differencing)

Finite Difference Splitting: Impacts on Aliasing

Representations are *analytically equivalent*, but are *discretely distinct*

$$\phi(x) = a \cdot b = \sum_{n=-N/2+1}^{N/2} \sum_{m=-N/2+1}^{N/2} \hat{a}_n \hat{b}_m \, e^{\imath(k_n+k_m)x}$$

$$= \sum_{p=-N/2+1}^{N/2} \hat{\phi}_p e^{\imath k_p x}$$

$$\text{with} \quad \hat{\phi}_p = \sum_{n+m=p} \hat{a}_n \hat{b}_m + \boxed{\sum_{n+m=p\pm N} \hat{a}_n \hat{b}_m}$$

(aliased modes)

Finite Difference Splitting: Impacts on Aliasing

Representations are *analytically equivalent*, but are *discretely distinct*

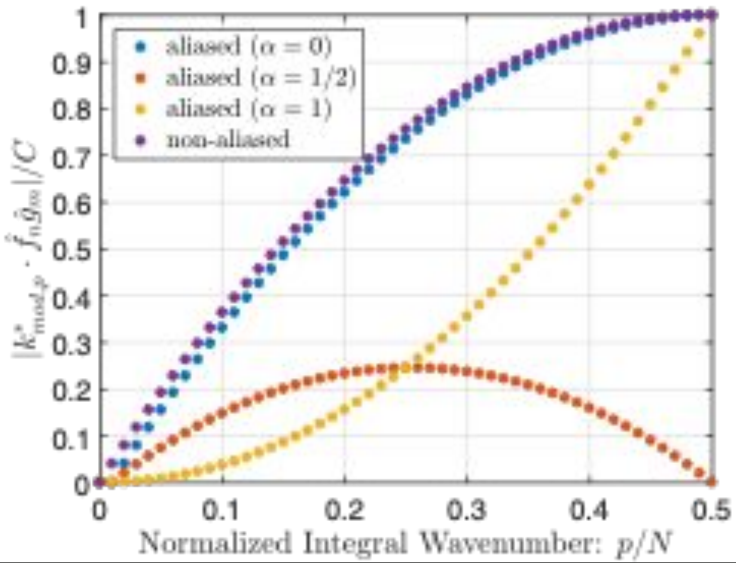
(aliased modes)

$$\partial_x(\phi) \approx \alpha \cdot \delta_x(ab) + (1-\alpha)\cdot[a\delta_x b + b\delta_x a]$$

$$\rightarrow \sum_{p=-N/2+1}^{N/2} \left\{ \left(\sum_{n+m=p} k^*_{mod,p} \cdot \hat{a}_n \hat{b}_m \right) + \left(\sum_{n+m=p\pm N} k^*_{mod,p} \cdot \hat{a}_n \hat{b}_m \right) \right\} e^{\imath k_p x}$$

$$\text{where} \quad k^*_{mod,p} = [\alpha\ k_{mod,p} + (1-\alpha)(k_{mod,n} + k_{mod,m})]$$

(Fourier basis)

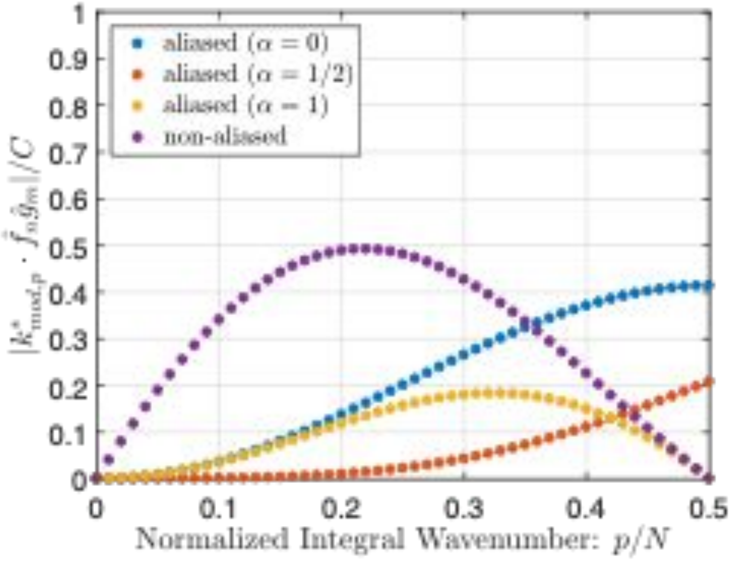


Edoh et al "Balancing aspects of numerical dissipation, dispersion, and aliasing in time-accurate simulations" IJNMF (2020)

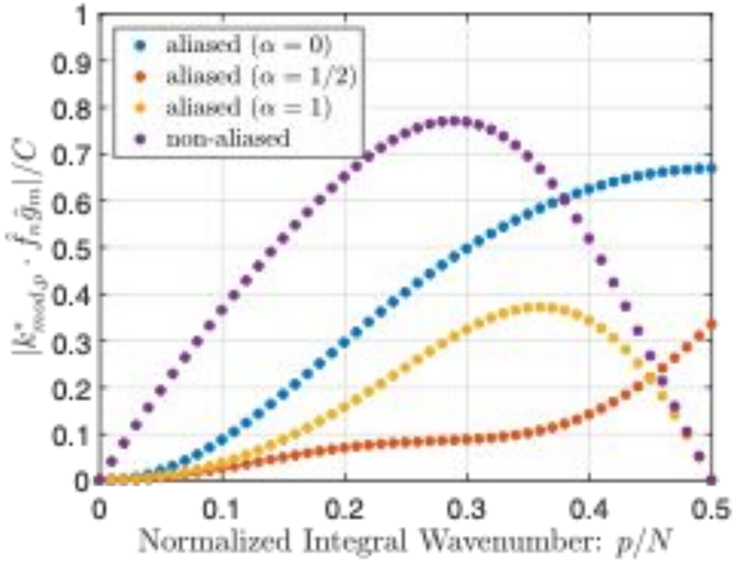
Finite Difference Splitting: Impacts on Aliasing

Averaging amongst different representations can improve aliasing performance, and therefore improve **robustness** and accuracy.

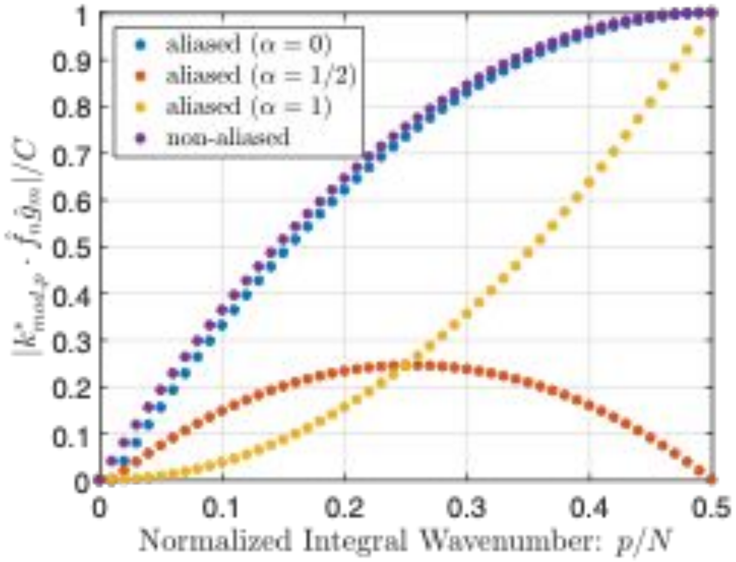
(2nd-order central)



(4th-order central DRP)



(Fourier basis)



May require numerical dissipation to attenuate high-wavenumber errors

Edoh et al "Balancing aspects of numerical dissipation, dispersion, and aliasing in time-accurate simulations" IJNMF (2020)

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4. Summary and Future Work

Incompressible Navier Stokes Equations (iNS)

$$\partial_t u_i + \overbrace{\partial_j u_j u_i}^{\mathcal{C}^{(j)}(\underline{u})} = -\partial_i p + \partial_j \tau_{ij}, \text{ with } \partial_i u_i = 0$$

Non-linear convective term provides opportunity to employ different representations

Divergence	solenoidal →	Advective
$\partial_j u_j u_i$		$u_j \partial_j u_i$
$u_j \partial_j u_i + u_i \partial_j u_j$		$u_j \overbrace{(\partial_j u_i + \partial_i u_j)}^{2S_{ij}} - \frac{1}{2}\partial_i u_j u_j$
		$\overbrace{u_j(\partial_j u_i - \partial_i u_j)}^{2u_j\Omega_{ij} = -\underline{u}\times\underline{\omega}} + \frac{1}{2}\partial_i u_j u_j$ (rotational form)

iNS: Supra-conservation

Primary	linear momentum	$\int_\Omega \partial_t \underline{u}$
Secondary	angular momentum	$\int_\Omega (\underline{x} \times \partial_t \underline{u})$
	helicity	$\int_\Omega \partial_t (\underline{\omega} \cdot \underline{u})$
	kinetic energy	$\int_\Omega \partial_t (\underline{u} \cdot \underline{u})$
		$\vdots$

"Supra"

Veldman "Supraconservative finite-volume methods for the Euler equations of subsonic compressible flow" SIAM Rev (2021)

iNS: Supra-conservation

Primary	linear momentum	$\int_\Omega \partial_t \underline{u}$
Secondary	angular momentum	$\int_\Omega (\underline{x} \times \partial_t \underline{u})$
	helicity	$\int_\Omega \partial_t (\underline{\omega} \cdot \underline{u})$
	kinetic energy	$\int_\Omega \partial_t (\underline{u} \cdot \underline{u})$

"Supra"

Veldman "Supraconservative finite-volume methods for the Euler equations of subsonic compressible flow" SIAM Rev (2021)

Kinetic-energy-preservation (**KEP**) as a main guidance wrt building non-linear numerical robustness:

$$\int_\Omega u_i \partial_t u_i = \int_\Omega u_i(-\partial_j u_j u_i - \partial_i p + \partial_j \tau_{ij})$$

$$\rightarrow \underbrace{\int_\Omega \frac{1}{2}\partial_t u_i u_i}_{\frac{1}{2}\partial_t \|u\|_2^2} = \int_\Omega \left[-\partial_j \left(\frac{1}{2}u_j u_i u_i + p u_j - \tau_{ij} u_j\right) - \overbrace{\tau_{ij}\partial_j u_i}^{\geq 0}\right]$$

$$\leq \left(-\frac{1}{2}u_j u_i u_i - p u_j + \tau_{ij} u_j\right)\Big|_{\partial\Omega}$$

iNS: KEP method of Morinishi *et al* (1998)

Morinishi et al "Fully consevative higher order finite difference schemes for incompressible flow" JCP (1998)

Primary conservation

$$\mathcal{C}^{(j)}_{\text{Mori.}} \equiv \frac{1}{2}\delta_j u_j u_i + \frac{1}{2}\left(u_j \delta_j u_i + u_i \delta_j u_j\right)$$

Secondary conservation

$$\begin{aligned} u_i \mathcal{C}^{(j)}_{\text{Mori.}} &= u_i \frac{1}{2}\delta_j u_j u_i + \frac{1}{2} u_i u_j \delta_j u_i + \frac{1}{2} u_i u_i \cancelto{0}{\delta_j u_j} \\ &= \frac{1}{2}\left(u_i \delta_j u_j u_i + u_i u_j \delta_j u_i\right) \end{aligned}$$

Valid on bounded domains via diagonal-norm SBP operators

- Recovers skew-symmetric representation for solenoidal velocity fields
- Flux forms in both momentum and induced KE equation
- Average of quadratic forms known to have enhanced aliasing performance

iNS: KEP methods of Edoh (2022)

Edoh "A new kinetic-energy-preserving method based on the convective rotational form" JCP (2022)

Primary conservation

$$\mathcal{C}^{(j)}_{\text{Div-rot}} \equiv u_i\delta_j u_j + u_j\delta_j u_i + \overbrace{\left(\frac{1}{2}\delta_i u_j u_j - u_j\delta_i u_j\right)}^{O(\Delta x^p)}$$

(correction term, group w/ pressure gradient)

Secondary conservation

(intrinsically multi-dimensional cancellations)

$$u_i\mathcal{C}^{(j)}_{\text{Div-rot}} = u_i u_i \cancelto{0}{\delta_j u_j} + \overbrace{u_i u_j\delta_j u_i - u_i u_j \delta_i u_j}^{=0} + u_i\frac{1}{2}\delta_i u_j u_j$$

$$= \frac{1}{2}u_j\delta_j u_i u_i + \overbrace{\frac{1}{2}u_i u_i \delta_j u_j}^{=0}$$

Valid on bounded domains via diagonal-norm SBP operators

- Recovers rotational representation for solenoidal velocity fields
- Flux forms in both momentum and induced KE equation
- Ties to advective form suggests poor aliasing performance

iNS: KEP methods of Edoh (2022)

Edoh "A new kinetic-energy-preserving method based on the convective rotational form" JCP (2022)

Primary conservation

$$\mathcal{C}^{(j)}_{\text{Div-str}} = 2\cdot\mathcal{C}^{(j)}_{\text{Mori.}} - \mathcal{C}^{(j)}_{\text{Div-rot}} \equiv \delta_j u_j u_i + \overbrace{\left(u_j\delta_i u_j - \frac{1}{2}\delta_i u_j u_j\right)}^{O(\Delta x^p)}$$

(correction term, group w/ pressure gradient)

Secondary conservation

$$\begin{aligned} u_i\mathcal{C}^{(j)}_{\text{Div-rot}} &= u_i\left(2\cdot\mathcal{C}^{(j)}_{\text{Mori.}} - \mathcal{C}^{(j)}_{\text{Div-rot}}\right) \\ &= \left(u_i\delta_j u_j u_i + u_i u_j \delta_j u_i\right) - \frac{1}{2}\left(u_j\delta_j u_i u_i + u_i u_i \delta_j u_j\right) \end{aligned}$$

Valid on bounded domains via diagonal-norm SBP operators

- Extrapolation of Morinishi and Div-rot splittings
  - Extrapolation of skew-symmetric and rotational forms for solenoidal velocity fields
- Flux forms in both momentum and induced KE equation
- Ties to product form suggests poor aliasing performance

iNS: A one-parameter family of KEP methods

Edoh “A new kinetic-energy-preserving method based on the convective rotational form” JCP (2022)

$$\partial_t u_i + \mathcal{C}^{(j)}(\underline{u}) = -\delta_i p + \delta_j \tau_{ij}, \ \text{with } \delta_i u_i = 0$$

$$\begin{aligned} \mathcal{C}^{(j)}_{\text{KEP},\alpha} &\equiv \alpha \cdot \mathcal{C}^{(j)}_{\text{Div-str}} + (1-\alpha) \cdot \mathcal{C}^{(j)}_{\text{Div-rot}} \\ &= \alpha \cdot \delta_j u_j u_i + (1-\alpha) \cdot (u_j \delta_j u_i + u_i \delta_j u_j) + \frac{(1-2\alpha)}{2} \cdot \overbrace{\left(\frac{1}{2}\delta_i u_j u_j - u_j \delta_i u_j\right)}^{O(\Delta x^p)} \end{aligned}$$

- Primary conservative and KEP family of schemes
  - characterized by a small-scale correction to the pressure gradient

iNS: A one-parameter family of KEP methods

Edoh "A new kinetic-energy-preserving method based on the convective rotational form" JCP (2022)

$$\partial_t u_i + \mathcal{C}^{(j)}(\underline{u}) = -\delta_i p + \delta_j \tau_{ij}, \text{ with } \delta_i u_i = 0$$

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	Momentum	Kinetic Energy	Angular Momentum	Vorticity	Enstrophy	Helicity
Div.	⊙		⊙	◯		
Prod.	⊙			◯		
Mori.	⊙	◯		◯		
Div-rot	⊙	◯		◯		◯
Div-str	⊙	◯	⊙	◯		

Table 1: Discrete global preservation properties associated with the different *a priori* momentum conserving splittings in the incompressible setting: ⊙ is preserving *a priori* and ◯ is preserving if $\delta_j u_j = 0$.

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Compressible Navier Stokes Equations (cNS)

$$\overbrace{\partial_t \begin{bmatrix} \rho \\ \rho u_i \\ \rho e_{\text{tot}} \end{bmatrix}}^{\partial_t q} = -\overbrace{\partial_j \begin{bmatrix} \rho u_i \\ \rho u_j u_i + \delta_{ij} \cdot P \\ \rho u_j e_{\text{tot}} + P u_j \end{bmatrix}}^{r^{\text{inv}} = \partial_j f^{\text{inv},(j)}} + \overbrace{\partial_j \begin{bmatrix} 0 \\ \tau_{ij} \\ u_i \tau_{ij} + q_j \end{bmatrix}}^{r^{\text{visc}} = \partial_j f^{\text{visc},(j)}}, \quad \text{for } i,j \in [1,d]$$

$$\begin{aligned} \tau_{ij} &= \mu \overbrace{(\partial_j u_i + \partial_i u_j)}^{2S_{ij}} - \tfrac{2}{3}\mu\delta_{ij} \cdot (\partial_k u_k) & e_{\text{int}} &= P[\rho(\gamma-1)]^{-1} = e_{\text{tot}} - \tfrac{1}{2}u_k u_k \\ q_j &= \kappa \partial_j T & \gamma &= 1 + \frac{R}{c_v} = c_p/c_v \end{aligned}$$

Kinetic energy no longer functions as bounded estimate on the solution variables

$$\begin{aligned} \frac{1}{2}\partial_t(\rho u_i u_i) &= u_i \cdot \partial_t(\rho u_i) - \frac{1}{2}u_i^2 \cdot \partial_t \rho \\ &= -\frac{1}{2}\partial_j \rho u_j u_i u_i - u_j \partial_j P + u_i \partial_j \tau_{ij} \\ &= -\partial_j \rho u_j (e_{\text{tot}} - e_{\text{int}}) - \partial_j u_j P + P \partial_j u_j + \partial_j \tau_{ij} - \overbrace{\tau_{ij}\partial_j u_i}^{\geq 0} \end{aligned}$$

cNS: KEP and non-linear stability

Primary conservation as **Secondary** conservation:

$$\begin{aligned}\int_\Omega [\partial_t \rho + \partial_t \rho \overbrace{(e_{\text{int}} + e_{\text{kin}})}^{e_{\text{tot}}}] &= 0 \\ &= \int_\Omega (2\sqrt{\rho} \cdot \partial_t \sqrt{\rho} + \sqrt{\rho} u_i \cdot \partial_t \sqrt{\rho} u_i + 2\sqrt{\rho e_{\text{int}}} \cdot \partial_t \sqrt{\rho e_{\text{int}}}) \\ &= \partial_t \|q_p\|_C^2 \\ &\quad \text{with } C = [diag\{1, 0.5, 1\}], q_p = [\sqrt{\rho}, \sqrt{\rho} u_i, \sqrt{\rho e_{\text{int}}}]^T\end{aligned}$$

Brouwer et al "Conservative time integrators of arbitrary order for skew-symmetric finite-difference discretizations of compressible flow" JCP (2014)
Rozema et al "A symmetry-preserving discretization and regularization model for compressible flow with application to turbulent channel flow" J. of Turb. (2014)
Nordstrom "A new energy stable formulation of the compressible Euler equations" arxiv (2022)

cNS: KEP and non-linear stability

Entropy consistency as a notion of non-linear robustness

$$\int_\Omega \partial_t \rho s = \int_\Omega w^T \partial_t q \geq 0$$

Kinetic energy preservation, as part of energy consistency, is essential

$$w = \frac{\partial \rho s}{\partial q} = \frac{1}{T} \cdot \begin{bmatrix} sT - h + \frac{1}{2}u_i u_i \\ -u_i \\ 1 \end{bmatrix}$$

$$\begin{aligned} w^T \partial_t q &= \frac{1}{T} \cdot \partial_t \rho e_{\text{tot}} - \frac{u_i}{T} \cdot \partial_t \rho u_i + \frac{u_i u_i}{2T} \cdot \partial_t \rho + \frac{sT - h}{T} \cdot \partial_t \rho \\ &= \frac{1}{T} \cdot \partial_t \rho e_{\text{int}} + \frac{sT - h}{T} \cdot \partial_t \rho \end{aligned}$$

cNS: parameterized convective compressible KEP schemes

$$\widetilde{(a,b)} \equiv 2\bar{a}\bar{b} - \overline{ab}$$

$$\overline{a}^{\text{geo}} = \sqrt{a_i a_{i+r}}$$

Parameterization of Coppola et al (2019):

$$\begin{aligned}
\text{mass}: &\quad \mathcal{M}^{(j)}_{\text{Copp.},\alpha} &\rightarrow&\quad \bar{f}^{(j)}_{\rho} &= \alpha \cdot \overline{\rho u_j} + (1-\alpha)\cdot \widetilde{(\rho, u_j)} \\
\text{momentum}: &\quad \mathcal{C}^{(j)}_{\text{Copp.},\alpha} &\rightarrow&\quad \bar{f}^{(j)}_{\rho u_i} &= \alpha \cdot \overline{\rho u_j}\ \overline{u_i} + (1-\alpha)\cdot \widetilde{(\rho, u_j)}\ \overline{u_i} \\
\text{kinetic energy}: &\quad u_i \cdot \mathcal{C}^{(j)}_{\text{Copp.},\alpha} - \frac{u_i^2}{2}\cdot \mathcal{M}^{(j)}_{\text{Copp.},\alpha} &\rightarrow&\quad \bar{f}^{(j)}_{\rho u_i^2/2} &= \alpha \cdot \frac{1}{2}\overline{\rho u_j}\ (\overline{u_i}^{\text{geo}})^2 + (1-\alpha)\cdot \frac{1}{2}\widetilde{(\rho, u_j)}\ (\overline{u_i}^{\text{geo}})^2
\end{aligned}$$

Coppola et al "Numerically stable formulations of convective terms for turbulent compressible flows" JCP (2019)

cNS: parameterized convective compressible KEP schemes

$$\widetilde{(a,b)} \equiv 2\bar{a}\bar{b} - \overline{ab}$$

$$\overline{a}^{\text{geo}} = \overline{\sqrt{a_i a_{i+r}}}$$

Parameterization of Coppola et al (2019):

$$\text{mass}: \quad \mathcal{M}^{(j)}_{\text{Copp.},\alpha} \rightarrow \bar{f}^{(j)}_{\rho} = \alpha \cdot \overline{\rho u_j} + (1-\alpha)\cdot \widetilde{(\rho, u_j)}$$

$$\text{momentum}: \quad \mathcal{C}^{(j)}_{\text{Copp.},\alpha} \rightarrow \bar{f}^{(j)}_{\rho u_i} = \alpha \cdot \overline{\rho u_j}\;\overline{u_i} + (1-\alpha)\cdot \widetilde{(\rho, u_j)}\;\overline{u_i}$$

$$\text{kinetic energy}: \quad u_i \cdot \mathcal{C}^{(j)}_{\text{Copp.},\alpha} - \frac{u_i^2}{2}\cdot \mathcal{M}^{(j)}_{\text{Copp.},\alpha} \rightarrow \bar{f}^{(j)}_{\rho u_i^2/2} = \alpha \cdot \tfrac{1}{2}\overline{\rho u_j}\;(\overline{u_i}^{\text{geo}})^2 + (1-\alpha)\cdot \tfrac{1}{2}\widetilde{(\rho, u_j)}\;(\overline{u_i}^{\text{geo}})^2$$

Parameterization of Edoh (2022):

$$\overline{\epsilon_{ke}} = \frac{1}{2}\overline{\rho}^{\text{geo}}\left[\overline{u_i u_i} - \widetilde{(u_i, u_i)}\right]$$

conservative correction term that groups w/ the pressure gradient

$$\text{mass}: \quad \mathcal{M}^{(j)}_{\text{Edoh}} \rightarrow \bar{f}^{(j)}_{\rho} = \overline{\rho}^{\text{geo}}\overline{u_j}$$

$$\text{momentum}: \quad \mathcal{C}^{(j)}_{\text{Edoh},\alpha} \rightarrow \bar{f}^{(j)}_{\rho u_i} = \alpha \cdot \overline{\rho}^{\text{geo}}\overline{u_j u_i} + (1-\alpha)\cdot \overline{\rho}^{\text{geo}}\widetilde{(u_j, u_i)} + \delta_{ij}\cdot(1-2\alpha)\overline{\epsilon_{ke}}$$

$$\text{kinetic energy}: \quad u_i \cdot \mathcal{C}^{(j)}_{\text{Edoh},\alpha} - \frac{u_i^2}{2}\cdot \mathcal{M}^{(j)}_{\text{Edoh}} \rightarrow \bar{f}^{(j)}_{\rho u_i^2/2} = 2\alpha \cdot \tfrac{1}{2}\overline{\rho}^{\text{geo}}\overline{u_j}\;(\overline{u_i}^{\text{geo}})^2 + (1-2\alpha)\cdot \tfrac{1}{2}\overline{\rho}^{\text{geo}}\widetilde{(u_j, u_i u_i)}$$

cNS: parameterized convective compressible KEP schemes

$\alpha = 1/2$

Parameterization of Coppola et al (2019):

$$\begin{aligned}
\text{mass}: &\quad \mathcal{M}^{(j)}_{\text{Copp.},\alpha=1/2} &\rightarrow\quad \bar{f}^{(j)}_{\rho} &= \overline{\rho}\ \overline{u_j} \\
\text{momentum}: &\quad \mathcal{C}^{(j)}_{\text{Copp.},\alpha=1/2} &\rightarrow\quad \bar{f}^{(j)}_{\rho u_i} &= \overline{\rho}\ \overline{u_j}\ \overline{u_i} \\
\text{kinetic energy}: &\quad \left(u_i \cdot \mathcal{C}^{(j)}_{\text{Copp.},} - \frac{u_i^2}{2} \cdot \mathcal{M}^{(j)}_{\text{Copp.}}\right)_{\alpha=1/2} &\rightarrow\quad \bar{f}^{(j)}_{\rho u_i^2/2} &= \frac{1}{2}\overline{\rho}\ \overline{u_j}\ (\overline{u_i}^{\text{geo}})^2
\end{aligned}$$

* Equivalent to cubic KG splitting

Kennedy and Gruber "Reduced aliasing formulations of the convective terms within the Navier-Stokes equations for a compressible fluid" JCP (2008)

Parameterization of Edoh (2022):

$$\begin{aligned}
\text{mass}: &\quad \mathcal{M}^{(j)}_{\text{Edoh}} &\rightarrow\quad \bar{f}^{(j)}_{\rho} &= \overline{\rho}^{\text{geo}}\ \overline{u_j} \\
\text{momentum}: &\quad \mathcal{C}^{(j)}_{\text{Edoh},\alpha=1/2} &\rightarrow\quad \bar{f}^{(j)}_{\rho u_i} &= \overline{\rho}^{\text{geo}}\ \overline{u_j}\ \overline{u_i} \\
\text{kinetic energy}: &\quad \left(u_i \cdot \mathcal{C}^{(j)}_{\text{Edoh}} - \frac{u_i^2}{2} \cdot \mathcal{M}^{(j)}_{\text{Edoh}}\right)_{\alpha=1/2} &\rightarrow\quad \bar{f}^{(j)}_{\rho u_i^2/2} &= \frac{1}{2}\overline{\rho}^{\text{geo}}\ \overline{u_j}\ (\overline{u_i}^{\text{geo}})^2
\end{aligned}$$

* Gives cubic splitting of Rozema et al

Rozema et al "A symmetry-preserving discretization and regularization model for compressible flow with application to turbulent channel flow" J. of Turb. (2014)

cNS: treatment of the total energy equation

Explicitly enforce consistency of energy transfers

Kuya et al "Kinetic energy and entropy preserving schemes for compressible flows by split convective terms" JCP (2018)

$$
\begin{aligned}
\partial_t \rho e_{\text{tot}} &= \partial_t \rho e_{\text{kin}} + \partial_t \rho e_{\text{int}} \\
&= u_i \cdot \partial_t \rho u_i - \frac{1}{2} u_i^2 \cdot \partial_t \rho + \partial_t \rho e_{\text{int}} \\
&= -\left(u_i \cdot \partial_j \rho u_j u_i - \frac{1}{2} u_i^2 \partial_j \rho u_j + u_j \partial_j P + u_i \right) - \left(\partial_j \rho u_j e_{\text{int}} + P \partial_j u_j \right) \\
&\rightarrow -\left(u_i \cdot \mathcal{C}^{(j)} - \frac{u_i^2}{2} \cdot \mathcal{M}^{(j)} \right)_{KEP} - \underline{\partial_j \rho u_j e_{\text{int}}} - \left(u_j \delta_j P + P \delta_j u_j \right)
\end{aligned}
$$

flexibility in its discretization

Overview

1. Splitting schemes
   a. local conservation
   b. aliasing considerations

2. Incompressible Navier-Stokes convective splittings
   a. secondary quantities and stability
   b. kinetic-energy-preserving (KEP) methods

3. Compressible Navier-Stokes convective splittings
   a. kinetic energy preservation and non-linear robustness
   b. parameterized KEP methods
   c. **solution filtering for enhancing stability and accuracy**
   d. numerical demonstrations

4. Summary and Future Work

Solution Filtering

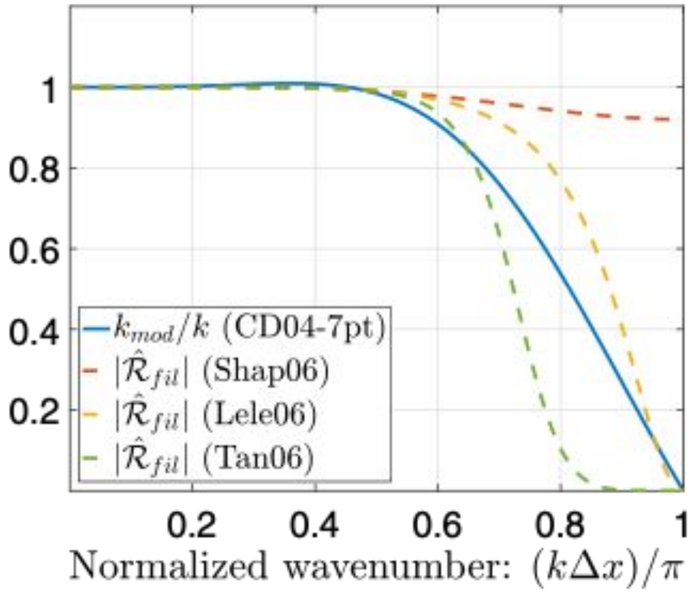
$$\bar{\phi}_i + \sum_{\ell}^{L} a_\ell(\bar{\phi}_{i-\ell} + \bar{\phi}_{i+\ell}) = b_0\phi_i + \sum_{r}^{R} b_r(\phi_{i-r} + \phi_{i+r})$$

$$\rightarrow \left[1 + \sum_{\ell\geq 1}^{L} \epsilon_{IF,2\ell}(\Delta x)^{2\ell}\delta_x^{2\ell}\right]\bar{\phi}_i = \left[1 + \sum_{r\geq 1}^{R} \epsilon_{EF,2r}(\Delta x)^{2r}\delta_x^{2r}\right]\phi_i$$

Edoh et al "Comparison of artificial-dissipation and solution-filtering stabilization schemes for time-accurate simulations" JCP (2018)
Edoh et al "Balancing aspects of numerical dissipation, dispersion, and aliasing in time-accurate simulations" IJNMF (2020)

Intermittently filter the solution variables to remove high wavenumber error content

- temporally consistent formulations
- tunable attenuation
  - choose scale-discriminant filter (e.g. Tangent stencils)
  - tune relative to resolvability of base scheme



Solution Filtering and Entropy Stability

Edoh "A new kinetic-energy-preserving method based on the convective rotational form" JCP (2022)

Interpreting solution filtering as artificial dissipation…

$$d_t q = r_o + r_{AD}(q), \text{ with } \begin{cases} r_{AD}(q) & \equiv & \bar{\mathcal{D}} q \\ \bar{\mathcal{D}} & \sim & \frac{|\lambda_{|u|+c}|}{\Delta x} \sum_n \epsilon_{2n} \delta^{2n} \leq 0 \end{cases}$$

Effect of filtering all solution variables is manifested as source terms in the entropy dynamics:

$$\int_\Omega \frac{1}{T} \overbrace{\left[\partial_t \rho e_{\text{int}} + \underbrace{\left(sT - e_{\text{int}} - \frac{P}{\rho} \right)}_{-G} \cdot \partial_t \rho \right]}^{\partial_t \rho s} = \cdots + \mathbf{1}^T T^{-1} \left[\overbrace{\bar{\mathcal{D}} R e_{\text{tot}}}^{r_{AD,\rho e_{\text{tot}}}} - \overbrace{\left(U_i \bar{\mathcal{D}} R \mathbf{u}_i - \frac{1}{2} U_i U_i \bar{\mathcal{D}} R 1 \right)}^{u_i \cdot r_{AD,\rho u_i} - \frac{1}{2} u_i^2 \cdot r_{AD,\rho}} - \overbrace{G \bar{\mathcal{D}} R 1}^{G \cdot r_{AD,\rho}} \right]$$

Only filer the momentum variable (quasi entropy stable)

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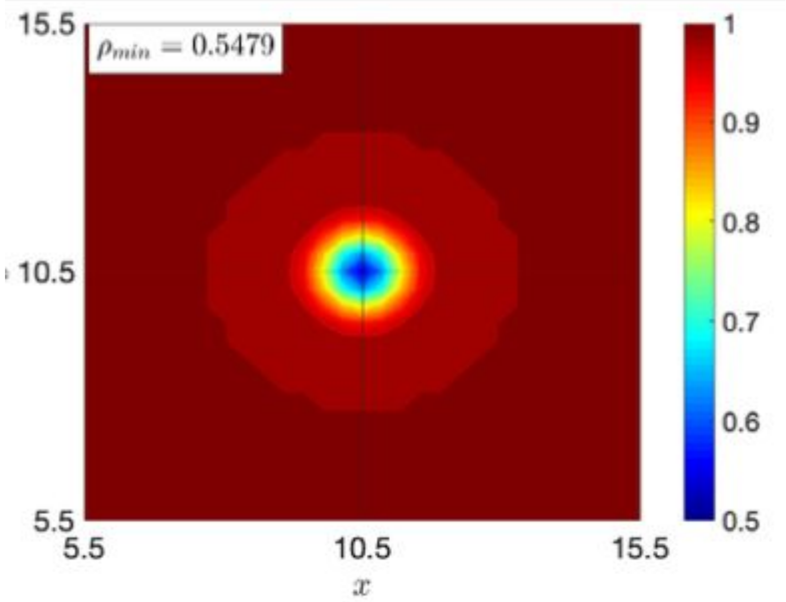
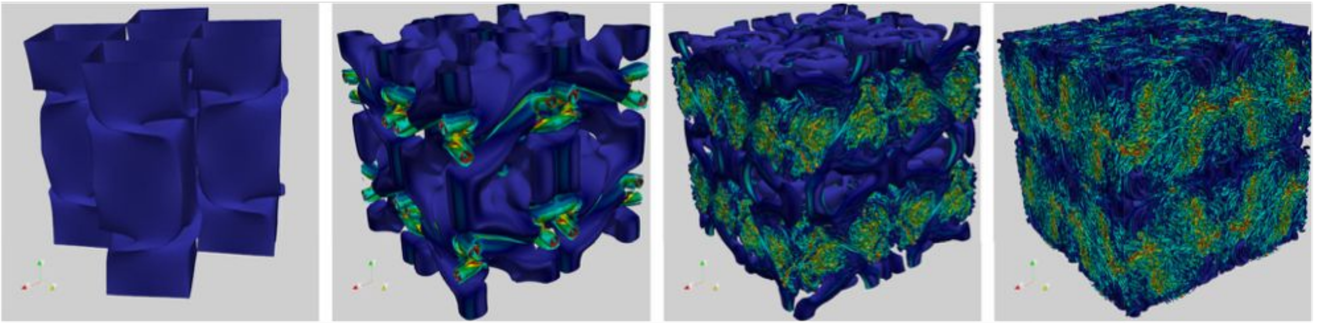
   c. solution filtering for enhancing stability and accuracy
   d. **numerical demonstrations**

4. Summary and Future Work

Numerical Demonstrations: Test Cases





(A) initial condition($t = 0$) (B) vortical roll-up ($t = 5t_c$) (C) coherent breakdown($t = 9t_c$) (D) turbulent decay($t = 14t_c$)

2D Isentropic vortex transport:

- Euler equations
- 6 points across vortex (60^2 DOF)
- 45% density perturbation
- 6th-order spatial discretizations
- 4th-order RK integration

3D Taylor Green vortex:

- Navier-Stokes equations
- Re = 1600
- 64^3 DOF
- 6th-order spatial discretizations
- 4th-order RK integration

cNS: reference schemes

$$\text{mass}: \quad \bar{f}_{\rho}^{(j)} = \overline{\rho u_j}$$

$$\text{momentum}: \quad \bar{f}_{\rho u_i}^{(j)} = \overline{\rho u_j u_i} + \delta_{ij} \cdot \overline{P}$$

$$\text{total energy}: \quad \bar{f}_{\rho e_{tot}}^{(j)} = \overline{\rho u_j e_{tot}} + \widetilde{(u_j, P)}$$

Standard discretization

$$\text{mass}: \quad \bar{f}_{\rho}^{(j)} = \overline{\rho}\ \overline{u_j}$$

$$\text{momentum}: \quad \bar{f}_{\rho u_i}^{(j)} = \overline{\rho}\ \overline{u_j}\ \overline{u_i} + \delta_{ij} \cdot \overline{P}$$

$$\text{total energy}: \quad \bar{f}_{\rho e_{tot}}^{(j)} = \frac{1}{2}\overline{\rho}\ \overline{u_j}\ (\overline{u_i}^{\text{geo}})^2 + \overline{\rho}\ \overline{u_j}\ \overline{e_{\text{int}}} + \widetilde{(u_j, P)}$$

Discretization of Kuya (2018) …

- primary conservative (locally telescoping)
- kinetic energy preserving
- quasi entropy preserving
- mitigated aliasing effects ("cubically split")

"Supra" conservative properties

cNS: schemes to be evaluated

$$\overline{\epsilon_{ke}} = \frac{1}{2}\overline{\rho}^{\text{geo}}\left[\overline{u_i u_i} - \widetilde{(u_i, u_i)}\right]$$

$$\text{mass}: \quad \bar{f}_{\rho}^{(j)} = \overline{\rho}^{\text{geo}}\ \overline{u_j}$$

$$\text{momentum}: \quad \bar{f}_{\rho u_i}^{(j)} = \alpha \cdot \overline{\rho}^{\text{geo}}\overline{u_j u_i} + (1-\alpha)\cdot \overline{\rho}^{\text{geo}}\widetilde{(u_j, u_i)} + \delta_{ij}\cdot\left[\overline{P} + (1-2\alpha)\overline{\epsilon_{ke}}\right]$$

$$\text{total energy}: \quad \bar{f}_{\rho e_{\text{tot}}}^{(j)} = \left[2\alpha \cdot \frac{1}{2}\overline{\rho}^{\text{geo}}\overline{u_j}\ (\overline{u_i}^{\text{geo}})^2 + (1-2\alpha)\cdot\frac{1}{2}\overline{\rho}^{\text{geo}}\widetilde{(u_j, u_i u_i)}\right] + \overline{\rho}^{\text{geo}}\ \overline{u_j}\ \overline{e_{\text{int}}}^{\text{geo}} + \widetilde{(u_j, P)}$$

Working off of the parameterization from Edoh (2022) …

- primary conservative (locally telescoping)
- kinetic energy preserving
- quasi entropy preserving

cNS: schemes to be evaluated

$$\overline{\epsilon_{ke}} = \frac{1}{2}\overline{\rho}^{\text{geo}}\left[\overline{u_i u_i} - \widetilde{(u_i, u_i)}\right]$$

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$$\text{momentum}: \quad \bar{f}_{\rho u_i}^{(j)} = \alpha \cdot \overline{\rho}^{\text{geo}}\overline{u_j u_i} + (1-\alpha)\cdot\overline{\rho}^{\text{geo}}\widetilde{(u_j, u_i)} + \delta_{ij}\cdot\left[\overline{P} + (1-2\alpha)\overline{\epsilon_{ke}}\right]$$

$$\text{total energy}: \quad \bar{f}_{\rho e_{tot}}^{(j)} = \left[2\alpha\cdot\tfrac{1}{2}\overline{\rho}^{\text{geo}}\overline{u_j}\ (\overline{u_i}^{\text{geo}})^2 + (1-2\alpha)\cdot\tfrac{1}{2}\overline{\rho}^{\text{geo}}\widetilde{(u_j, u_i u_i)}\right] + \overline{\rho}^{\text{geo}}\ \overline{u_j}\ \overline{e_{\text{int}}}^{\text{geo}} + \widetilde{(u_j, P)}$$

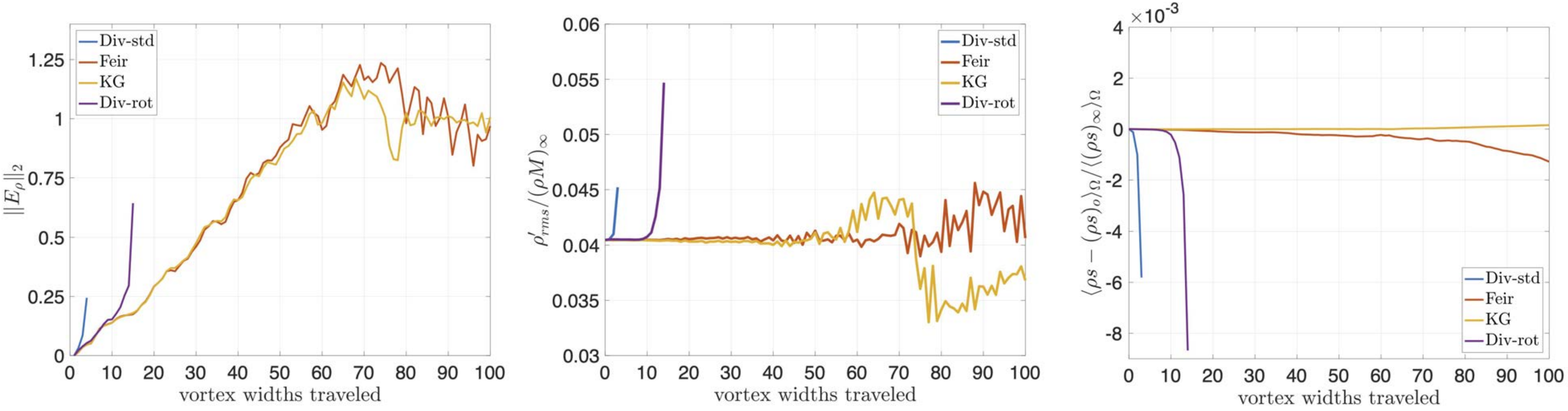
Working off of the parameterization from Edoh (2022) …

- primary conservative (locally telescoping)
- kinetic energy preserving
- quasi entropy preserving
- pressure equilibrium preserving
- alpha = 0
  - helicity preserving at incompressible limit
  - mitigated spurious vorticity and enstrophy effects at incompressible limit
- alpha = 1
  - angular momentum preserving at incompressible limit
- alpha = ½
  - mitigated aliasing effects (“cubically split”)

“Supra” conservative properties

cNS: Isentropic Vortex

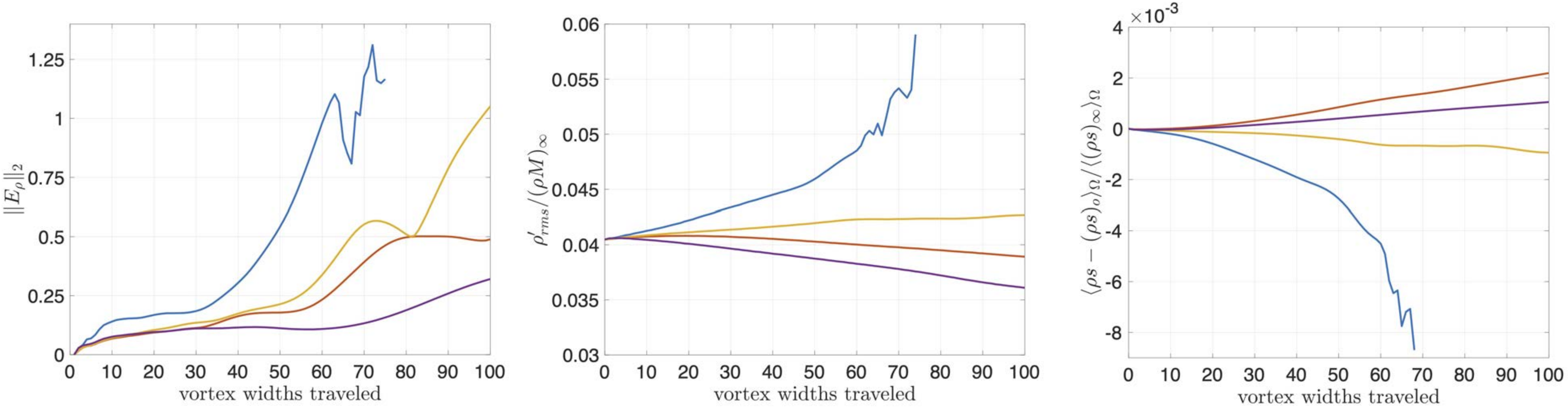
(no solution filtering)



- Stability enhanced by secondary preserving methods
  - still prone to accumulated small scale noise/errors

cNS: Isentropic Vortex

(w/ solution filtering)



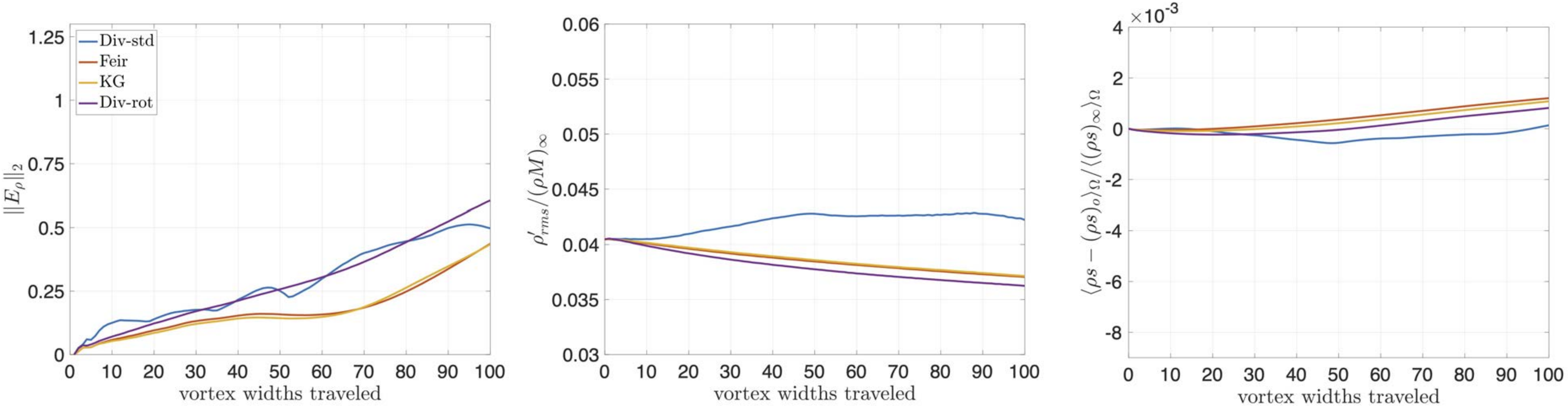
Solution filtering removes small scale errors

- tuned to preserve accuracy of base scheme

* Issues w/ entropy stability from solution filtering

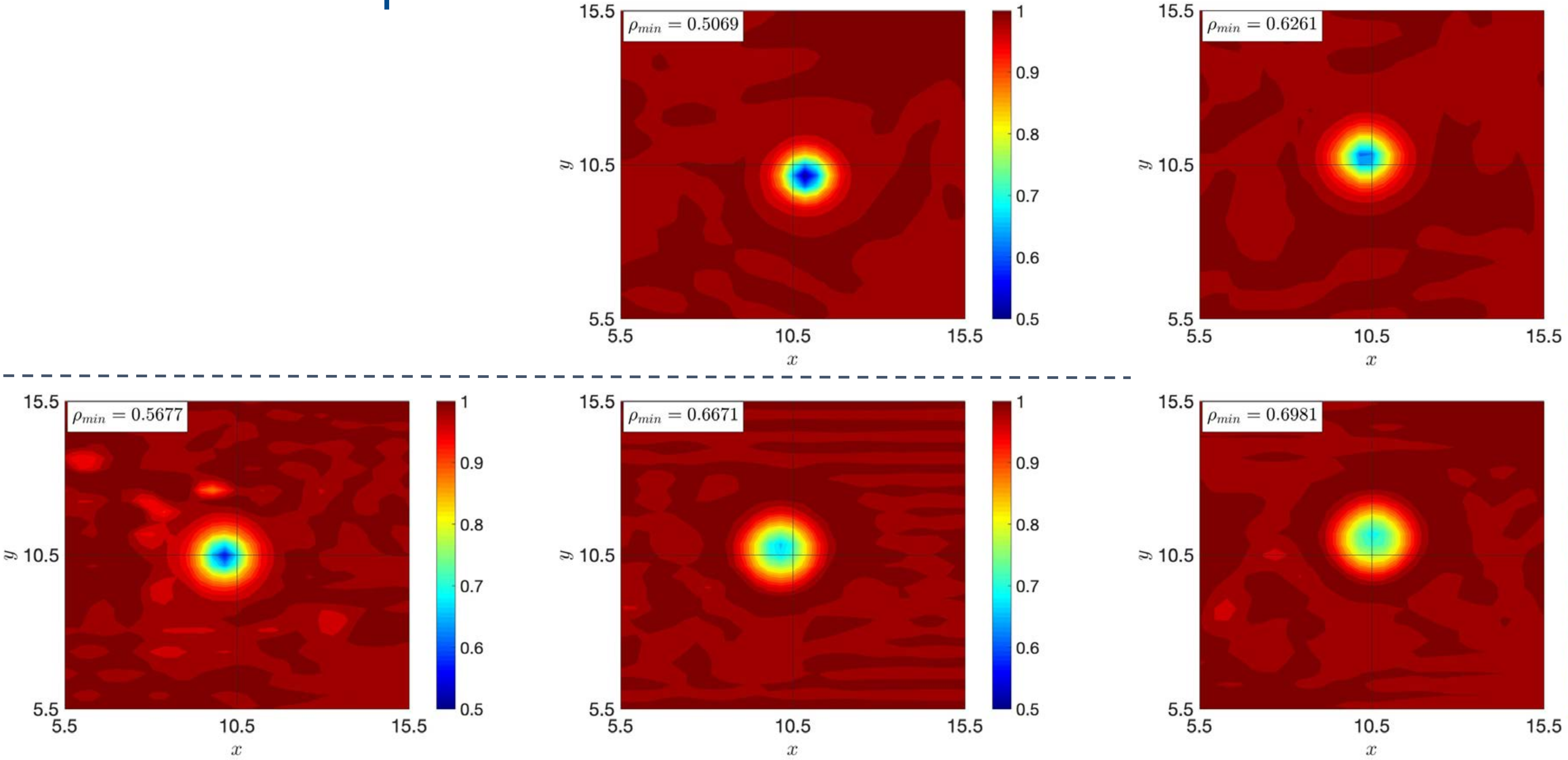
cNS: Isentropic Vortex

(w/ quasi entropy stable solution filtering)



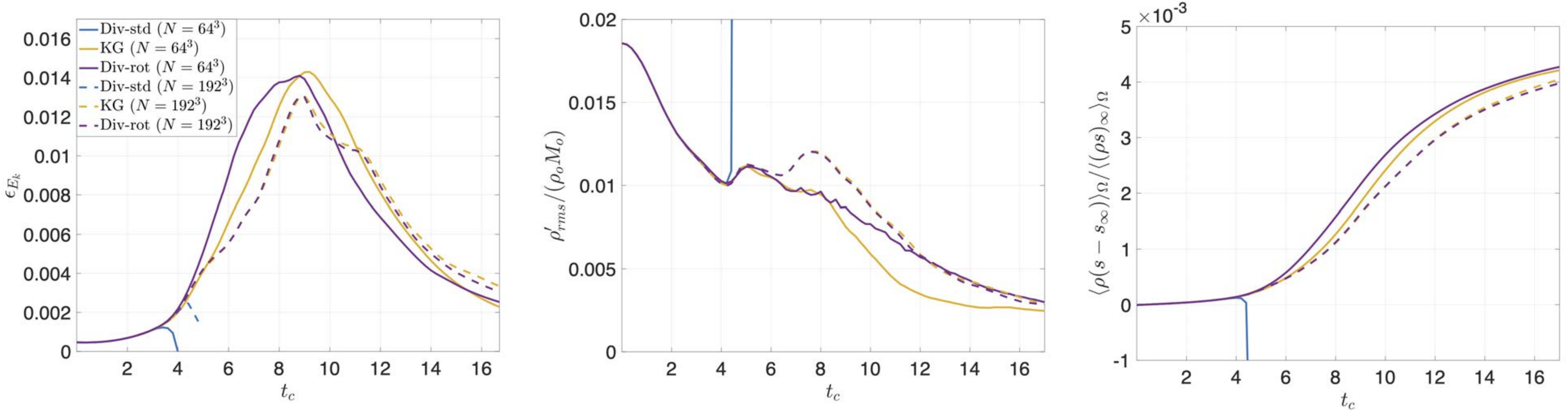
Filtering only the momentum variables admits more errors but improves entropy performance

cNS: Isentropic Vortex



cNS: Taylor Green vortex

Evolution of global quantities

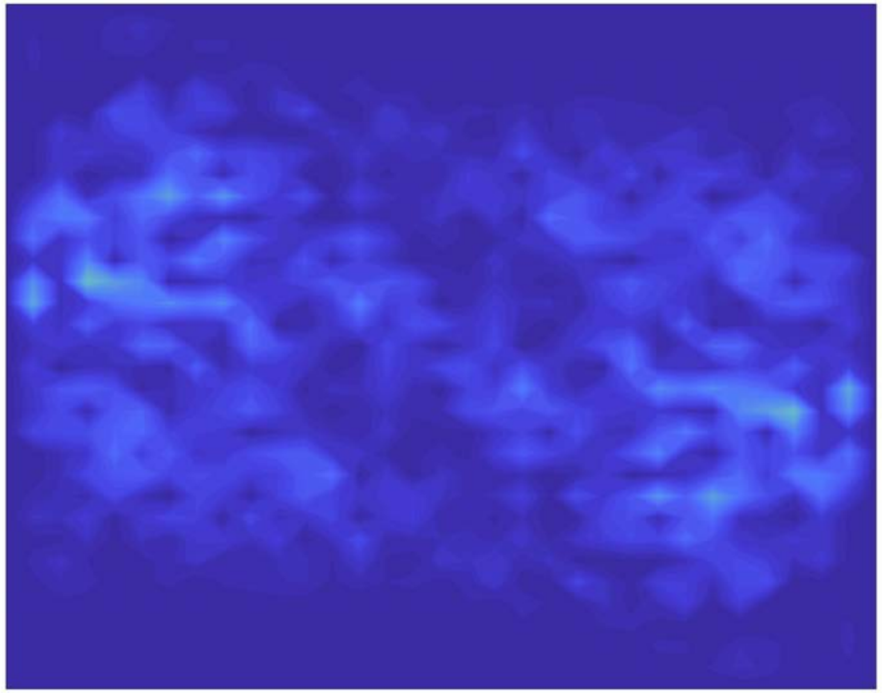


cNS: Taylor Green vortex

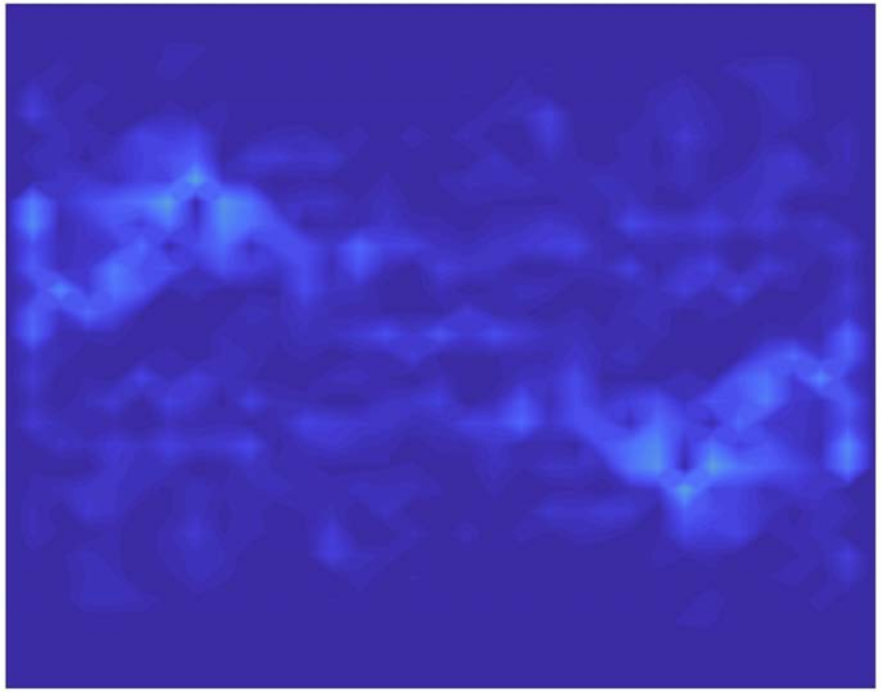
Vorticity magnitude contours:

- **64^3 grid, at time of peak dissipation rate**

<u>KG Splitting</u>



<u>Div-Rot Splitting</u>

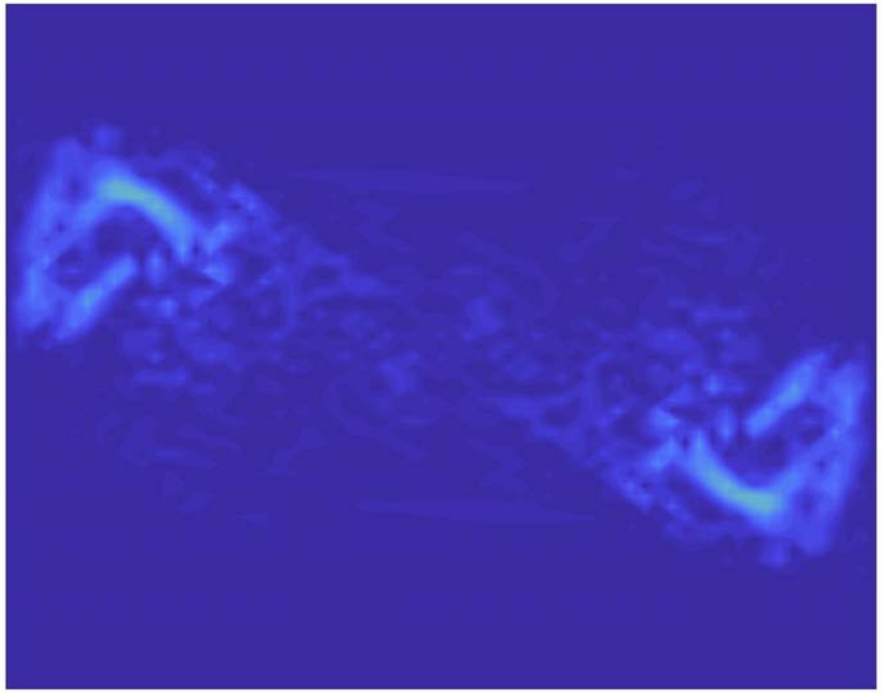


Rotationally-consistent formulation admitting less noise in voriticity field

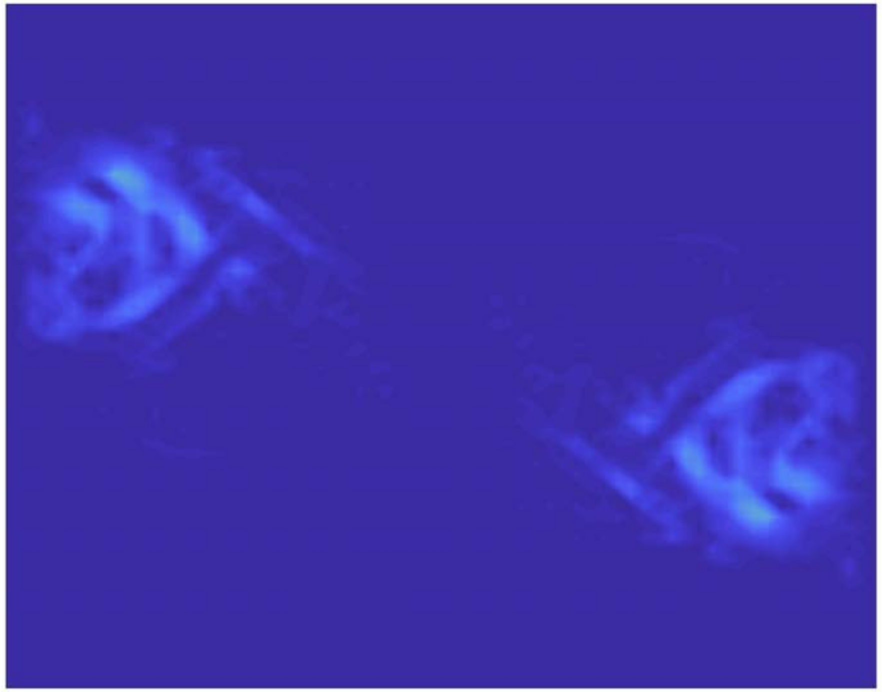
cNS: Taylor Green vortex

Vorticity magnitude contours:
- **128^3 grid, at time of peak dissipation rate**

KG Splitting



Div-Rot Splitting



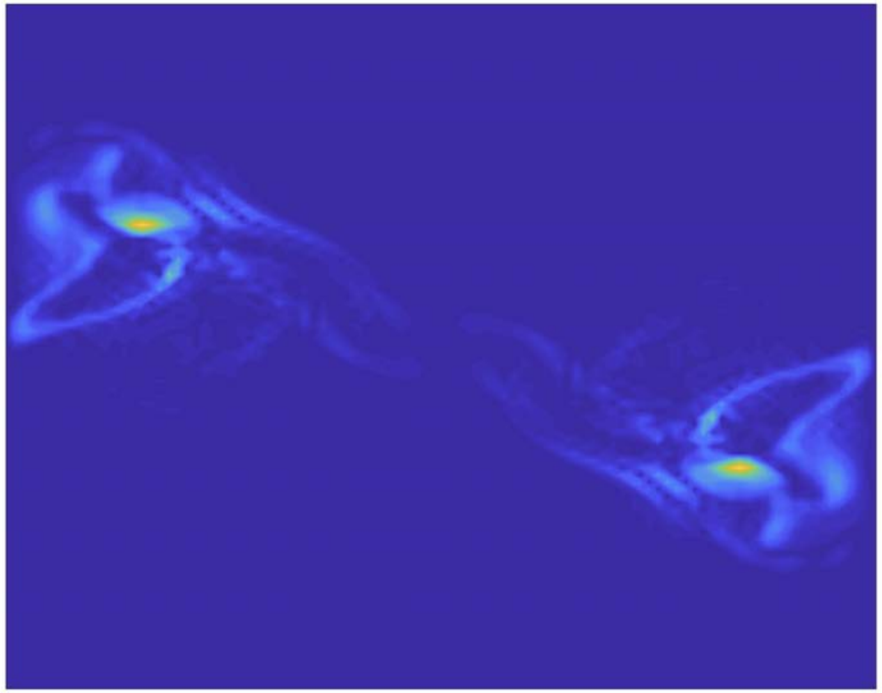
Rotationally-consistent formulation admitting less noise in voriticity field

cNS: Taylor Green vortex

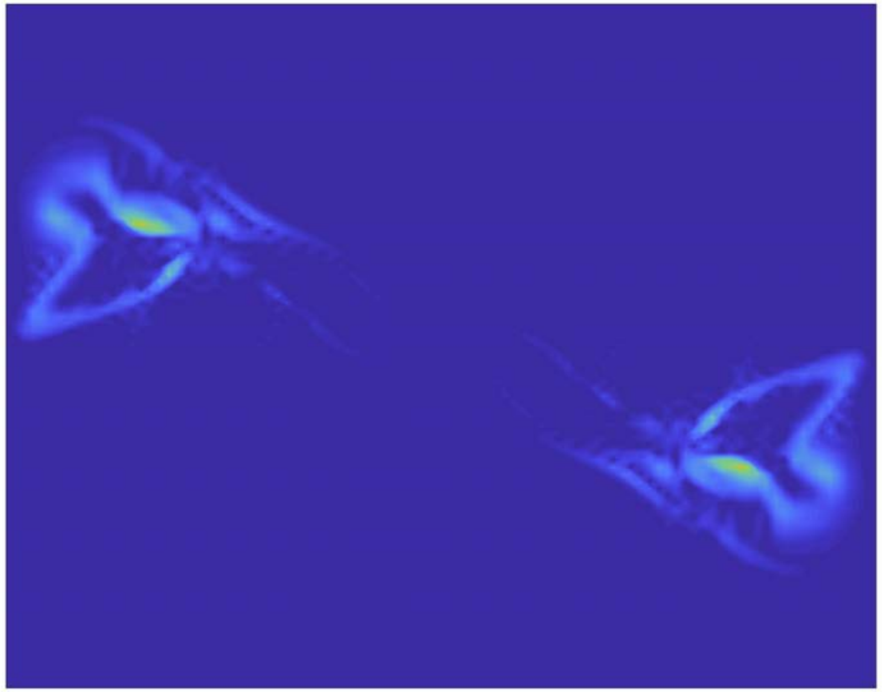
Vorticity magnitude contours:

- **192^3 grid, at time of peak dissipation rate**

KG Splitting



Div-Rot Splitting



Rotationally-consistent formulation admitting less noise in voriticity field

Summary

- Quadratic splittings producing distinct numerical properties

- Secondary preservation for enhancing quality of solution
  - accuracy and non-stability
  - kinetic energy preservation as a guiding principle
    - new supra conservative parameterized families of splittings

- Solution filtering for improving accuracy and stability
  - filter only momentum variable (quasi entropy stable)

Summary

- Quadratic splittings producing distinct numerical properties

- Secondary preservation for enhancing quality of solution
    - accuracy and non-stability
    - kinetic energy preservation as a guiding principle
        - new supra conservative parameterized families of splittings

- Solution filtering for improving accuracy and stability
    - filter only momentum variable (quasi entropy stable)

<u>Related on-going work</u>:

- Entropy consistent discretizations in split form
- Evaluating impact of secondary conservation on the reactive bluff body case (SciTech 2023)
- Employing secondary consistencies in the context of Large-Eddy-Simulations

Referenced and Relevant Text

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- Charnyi *et al* “On conservation laws of Navier-Stokes Galerkin discretizations” JCP (2017)
- Cocks *et al* “Impacts of numerics on predictive capabilities of reacting LES” CnF (2015)
- Coppola et al “Numerically stable formulations of convective terms for turbulent compressible flows” JCP (2019)

- **Edoh** *et al* “Comparison of artificial-dissipation and solution-filtering stabilization schemes for time-accurate simulations” JCP (2018)
- **Edoh** *et al* “Balancing aspects of numerical dissipation, dispersion, and aliasing in time-accurate simulations” IJNMF (2020)
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