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D-C ANALOG SOLUTION OF SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS: CIRCUIT STABILITY CONSIDERATIONS

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ABSTRACT

A simple and novel technique has been developed for positively stabilizing the electronic computing circuit usually employed to solve a system of simultaneous linear algebraic equations. The stabilized circuit solves a corresponding system of differential equations, the steady-state solution of which is identical to the desired solution of the algebraic equations. Heretofore, elaborate transformation of the given system of equations, in the general case, has been required to insure a stable "mechanizing" circuit. When this new technique is applied, however, such a transformation is not required and there is no sacrifice of computing accuracy. The theory, explaining how stabilization is effected and how the general stability criteria for such computing circuits are satisfied, has been developed in some detail. As a practical example, the solution of a system of five equations has been obtained by applying the stabilizing technique in several different ways.

PROBLEM STATUS

This is an interim report on one phase of the problem; work is continuing.

AUTHORIZATION

NRL Problem P10-01R

D-C ANALOG SOLUTION OF SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS: CIRCUIT STABILITY CONSIDERATIONS

INTRODUCTION

The solution of a system of n simultaneous linear algebraic equations--a frequently recurring problem in many fields of science and engineering—is theoretically a simple and straightforward process, and as long as n is small, the practical solution of a given set of equations is more or less readily arrived at. When n is large, however, the numerical solution presents a formidable problem and one which has been the subject of intensive study, at least since the days of Lord Kelvin. As early as 1890,[1] he developed a device for the mechanical solution of simultaneous equations. But even with the aid of modern-day computers, the task, when n is large, is still a difficult one. If, for example, the equations are to be solved exactly by Cramer's rule, n^2 determinants of order n-1 and one determinant of order n must be evaluated.

Ostensibly, it should be possible to solve the equations with a high-speed digital computer, for which iterative methods (for example, the Gauss-Seidel[2]) are most applicable. This is sometimes the case. The operating cost, however, and the complexity of coding, may not justify the use of this type of computer. Moreover, an iterative process may converge too slowly, thereby producing excessive accumulated error due to neglect of digits beyond computer capacity ("round-off"). Or it may not converge at all.

Instead of the complex digital computer, a much simpler computer of the analog type may be used, several of which satisfactorily solve systems of algebraic equations. In particular, the electronic analog computer employing d-c feedback amplifiers, although not the simplest type, operates almost automatically after the equations have been set in. However, since this type of computer involves many closed electrical circuits or loops containing amplifiers, the complete computer circuit is usually unstable, that is, it is driven out of control by transients which "grow" until limited by saturation of one or more of the amplifiers.

In the past, circuit stability has been insured by appropriate mathematical manipulation of the algebraic equations. The basic problem in mechanizing a given system is to ascertain a transformation (proper arrangement of the order of the equations, change of variables, and scaling of the coefficients) which will insure circuit stability, preferably prior to setting the equations into the computer. One approach to this problem consists

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[1] *"Treatise on Natural Philosophy," Part I, Appendix BII. London, 1890*

[2] *Murray, F. J., "The Theory of Mathematical Machines," King's Crown Press, New York, 1947, Part III, Chapter IV*

of applying a transformation which makes the system matrix "positive definite."[3] It turns out, however, that applying such a satisfactory transformation often is just as difficult as solving the equations themselves.[4]

Goldberg and Brown[5] have described a simple technique for evolving a transformed system of equations which yields a stable circuit, but unfortunately, to accomplish this end, the number of equations is doubled.

A NEW APPROACH

An interesting and fruitful alternative approach to this stability problem involves the transformation of the mechanizing circuit rather than of the equations, and a simple, novel, and widely applicable technique—requiring no major computing-circuit alterations and demanding no sacrifice in computing accuracy—has been worked out at NRL for positively stabilizing the electronic circuit commonly employed. It arises from recognition of the fact that the basic difficulty is associated with the computing machine itself and that, consequently, stability must be realized by proper selection and arrangement of computing components.

Although a high degree of precision is characteristic of the components (that is, the inverting and summing amplifiers) used in the d-c analog computer, none of these components is perfect, the inherent error introduced by each being of the order of several tenths of one percent. Precisely because of these residual errors, and because of the manner in which they enter the exact computing equations of the components, the machine inherently solves a system of differential equations even though the given system is algebraic. In general, the solution of a differential equation having a driving function involves both a transient and a steady-state solution. It is the steady-state (that is, the particular) solution that is wanted, and that only. Hence, it is necessary that the complementary (transient) solution die out in time in order that the desired answer may be obtained. If, instead, the complementary solution "grows" with time, computer instability results.

Fortunately, the transient portion of any solution may, without affecting the steady-state solution, be so modified that it must decay in a reasonable length of time. This result is obtained by adding a single derivative term (of the proper sign and magnitude) to each equation causing instability and then mechanizing the augmented equations. The introduction of these derivative terms in no way effects the particular or steady-state solution for the simple reason that the derivative of a constant is zero, but the added terms are sufficient always to insure a stable circuit arrangement. By means of this technique, circuit stability can be simply and positively assured. The stabilized circuit thus solves a corresponding system of differential equations, the steady-state solution of which is identical with the desired solution of the algebraic equations.

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[3] *Bode, H.W., "Network Analysis and Feedback Amplifier Design," Van Nostrand, New York, 1945*

[4] *As Korn points out (Proc. IRE, 37:1000-1002, Sept. 1949), the gain function of the basic high-gain amplifier must satisfy a stability criterion even when the matrix of the given system of equations is positive definite.*

[5] *"An Electronic Simultaneous Equation Solver," J. App. Phys. 10:330-345, April, 1948*

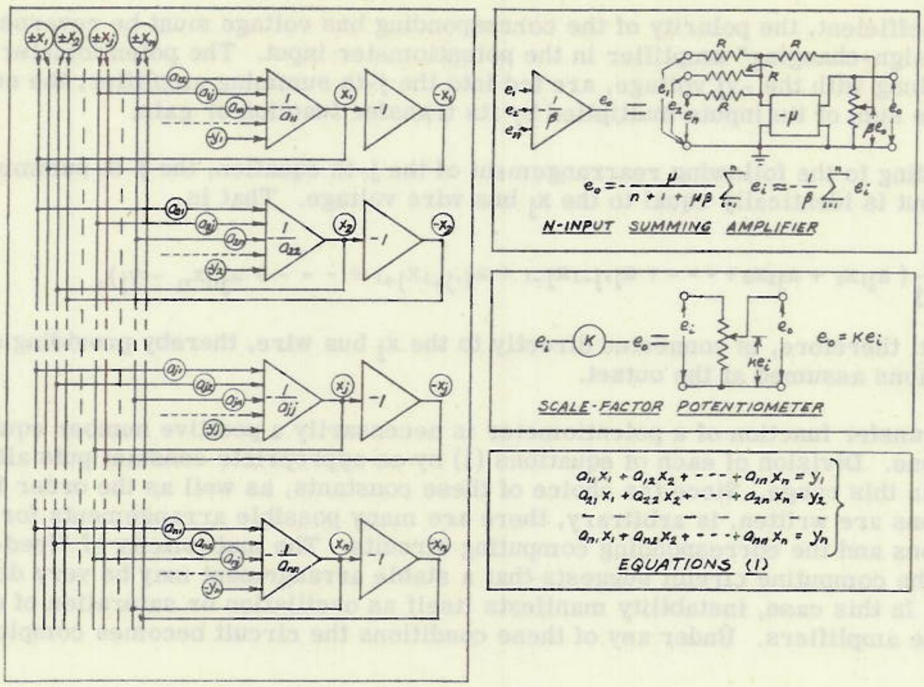

Figure 1 - Computing circuit

THE THEORETICAL ANALYSIS

The solution of the general system of linear algebraic equations,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + - - - + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + - - - + a_{2n}x_n = y_2 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \\ a_{n1}x_1 + a_{n2}x_2 + - - - + a_{nn}x_n = y_n \end{array}\right\}, \qquad (1)$$

can be determined to any desired degree of accuracy[6] by the d-c electronic analog computing circuit[7] of Figure 1, wherein voltages correspond with the variables, x_i (i = 1, 2, . . . n), and with the right-hand members, y_i, while the transfer functions (ratio of output to input) of the potentiometers and amplifiers correspond with the coefficients, a_{ij}.

The circuit is developed in the following way. Voltages proportional to the variables, x_i, are assumed to exist on a set of bus wires having a common return. The j-th equation is "mechanized" by connecting each bus wire except the j-th to the input of a potentiometer, each of which is adjusted according to the corresponding coefficient, a_{jk} $(k \neq j)$. For any

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[6] *Goldberg and Brown, op. cit.*

[7] *Although d-c computing voltages are indicated throughout this discussion, there appears to be no basic reason why the stabilizing technique can not be applied to analog computers employing a-c computing voltages (modulated carriers).*

negative coefficient, the polarity of the corresponding bus voltage must be reversed by inserting a "sign-changing" amplifier in the potentiometer input. The potentiometer output voltages, along with the -yj voltage, are fed into the j-th summing amplifier, the output of which is the sum of its inputs multiplied by its transfer function or gain.

According to the following rearrangement of the j-th equation, the j-th summing amplifier output is identically equal to the x_j bus wire voltage. That is,

$$x_j = -\frac{1}{a_{jj}}\left(a_{j1}x_1 + a_{j2}x_2 + - - - + a_{j,j-1}x_{j-1} + a_{j,j+1}x_{j+1} + - - - + a_{jn}x_n - y_j\right). \tag{2}$$

This output, therefore, is connected directly to the x_j bus wire, thereby providing one of the excitations assumed at the outset.

The transfer function of a potentiometer is necessarily a positive number equal to or less than one. Division of each of equations (1) by an appropriate constant puts all the coefficients in this range. Since the choice of these constants, as well as the order in which the equations are written, is arbitrary, there are many possible arrangements for both the equations and the corresponding computing circuits. The multiplicity of "feed-back loops" in the computing circuit suggests that a stable arrangement may be very difficult to devise. In this case, instability manifests itself as oscillation or saturation of one or more of the amplifiers. Under any of these conditions the circuit becomes completely useless.

Since considerable negative feed-back is applied to the summing amplifiers, which are of the multi-input d-c type, their normal characteristics are essentially independent of vacuum-tube variations. Without feed-back, the basic amplifier has very high gain. For an amplifier with only one input, the "gain" or "transfer function" is defined simply as the ratio of output response to input excitation. When there are several inputs, on the other hand, the amplifier gain for any one input is defined in the same way. The gain of each input differs from the others only by a constant factor which corresponds to the nominal gain of the particular input. The over-all gain of the amplifier can then be defined as the ratio of output response to the <u>sum</u> of the input excitations, each weighted according to their respective nominal gains. In this paper the discussion is simplified by assuming that each of these nominal gains is unity. Accordingly, the gain of a summing amplifier, which has the same form as the usual relation for a single-loop feed-back amplifier, is

$$\frac{e_0}{\Sigma e_i} = \frac{\mu'}{1 - \mu'\beta} = -\frac{\mu/k(n)}{1 + \mu\beta/k(n)} = \frac{\mu}{k(n) + \mu\beta} \tag{3}$$

where e_0 = $e_0(t)$, instantaneous output voltage;

e_i = $e_i(t)$, instantaneous voltage applied at the i-th input;

μ' = $-\mu/k$, effective gain of the amplifier without feed-back in which loading by the feed-back network is taken into account;

μ = $\mu(p)$, operational or generalized transfer function of the basic amplifier without feed-back;

β = transfer function of the feed-back network; and

$k(n)$ = a parameter which is a function of the number of inputs.

The negative sign associated with μ indicates that the amplifier has an odd number of stages. In order to study the stability problem, equation (3) must be applied to the amplifiers mechanizing the given system of equations. The gain of the j-th summing amplifier is

$$\frac{e_0}{\Sigma e_i} = -\frac{1}{a_{jj}} = -\frac{\mu}{k(n) + \mu\beta_j} \approx -\frac{1}{\beta_j}, \ (\mu >> 1), \tag{4}$$

so that the coefficient, a_{jj}, has the form,

$$a_{jj} = \beta_j + \frac{k(n)}{\mu} = a'_{jj} + \frac{k(n)}{\mu} \approx a'_{jj}. \tag{5}$$

The gain of the sign-changing amplifier is

$$\frac{e_0}{e_i} = -\frac{\mu}{2 + \mu} \approx -1, \tag{6}$$

so that the coefficient, a_{jk}, when it is negative, has the form,

$$a_{jk} = a'_{jk}\left(1 + \frac{2}{\mu}\right), \ a_{jk} < 0. \tag{7}$$

If equation (5), and equation (7) as necessary, are substituted in equations (1), it is then apparent that the system of equations actually mechanized is

$$\left.\begin{array}{l} \left(a'_{11} + \frac{k}{\mu}\right)x_1 \ a_{12}x_2 + - - - + a_{1n}x_n = y_1 \\ a_{21}x_1 + \left(a'_{22} + \frac{k}{\mu}\right)x_2 + - - - + a_{2n}x_n = y_2 \\ - \\ a_{n1}x_1 + a_{n2}x_2 + - - - + \left(a'_{nn} + \frac{k}{\mu}\right)x_n = y_n \end{array}\right\} . \tag{8}$$

Since μ is a function of the differential operator, $p = d/dt$, equations (8) are really the operational transforms of a system of differential equations, the particular integrals of which are the desired steady-state solutions, the x_i. Moreover, any physical system accurately described by equations (8) is stable if, and only if, the roots of the characteristic equation,

$$\Delta(p) = 0, \tag{9}$$

in which $\Delta(p)$ is the system determinant of equations (8), have negative real parts.[8] In other words, when the system is stable, the transient or complementary solution dies out in time leaving only the steady state, whereas if the system is unstable, the transient solution "grows" until limited by saturation of the amplifiers. Physically, the excitation for these transients is derived from the y_i voltages (generally applied as step functions because of the necessary switching operations) or from tube and circuit noise.

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[8] *Bode, op. cit.*

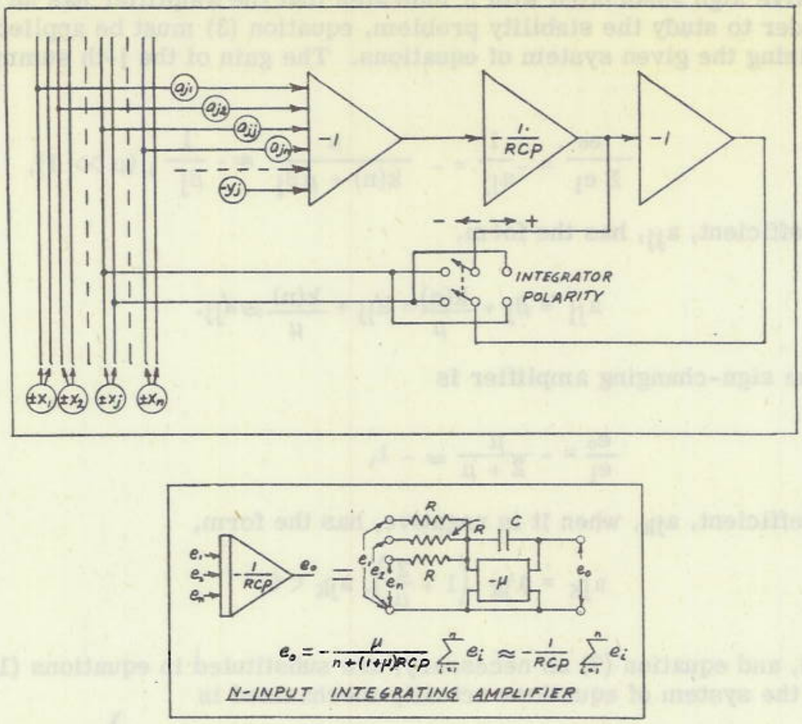

Figure 2 - Circuit for stabilizing the j-th amplifier

The procedure for positively stabilizing the mechanizing circuit by the "alternative" approach is as follows. The given equations are arranged in the order in which each of the main diagonal coefficients of the system matrix is, in its respective row and column, the largest, or almost the largest, in absolute value. Despite the fact that this arrangement is the most likely to produce a stable circuit, it must be assumed, in the general case, that the mechanizing circuit for the rearranged equations is still unstable. Suppose, for example, that the circuit of Figure 1 is unstable for the particular arrangement of a given system of equations. Suppose further that the circuit becomes stable when the j-th summing amplifier output is disconnected from the x_j bus wire (the equivalent of removing the amplifier from the circuit althogether). This reduced circuit represents a system of n-1 equations formed by setting $x_j = 0$ and deleting the j-th equation in the original system.

Stabilization of the circuit of Figure 1 can be effected by making the following modifications around the j-th summing amplifier as indicated in Figure 2:

(a) inserting an integrating amplifier in the output of the j-th summing amplifier,

(b) connecting the x_j bus wire to the input of the j-th summing amplifier through an additional potentiometer set at the coefficient, a_{jj}, and

(c) selecting the integrating amplifier polarity which yields stability.

The j-th equation is now approximately mechanized as

$$a_{j1}x_1 + a_{j2}x_2 + - - - + a_{jj}x_j + - - - + a_{jn}x_n - y_j = \pm RCpx_j , \tag{10}$$

which finally reduces to the given j-th equation because the right-hand member is zero for the steady-state (the derivative of a constant being zero).

It is necessary, in general, to disconnect the outputs of more than one summing amplifier from the x_j bus wires to find a stable circuit. In practice, when it has been determined that the initial circuit is unstable, the procedure is simplified by first disconnecting the outputs of all the summing amplifiers and then systematically reconnecting only those which do not make the circuit unstable. It is not necessary to find the optimum combination, that is, the stable circuit containing the greatest number of amplifiers. The remaining amplifiers are each reconnected, one at a time, according to the stabilizing circuit of Figure 2.

The function of the integrating amplifier in the stabilizing process is analogous to that of resistance-capacitance "anti-sing" networks employed in ordinary feed-back amplifiers. From another point of view, the right-hand member of equation (10) may be regarded as an error function which, when "amplified" by the integrating amplifier, is finally "forced" to be identically x_j in the steady state. Thus, the charge in the integrating capacitor (see the integrating amplifier circuit in Figure 2) is just sufficient to maintain this output voltage, x_j.

It is interesting to note that the gain of the integrating amplifier increases with time, t. At t = 0, therefore, this gain is momentarily zero, at which time the j-th summing amplifier is effectively disconnected and the remainder of the circuit (representing the n-1 equations) rapidly stabilizes. As t increases, the gain of the integrating amplifier is no longer zero and becomes theoretically infinite in the steady state. Evidently, the properties of the j-th summing amplifier and its associated circuitry, originally responsible for the instability of the entire circuit, are masked or isolated by the presence of the integrating amplifier. Furthermore, the output of the j-th summing amplifier becomes theoretically zero in the steady state. Hence, according to the given j-th equation, a voltage corresponding with the term, $a_{jj}x_j$, must be added to the input of the j-th summing amplifier for accurate mechanization.

In order to insure stabilization of any mechanizing circuit, the ranges of the available parameters in the integrating amplifier transfer function, for which the roots of the characteristic equation of the stabilized circuit will have negative real parts, must be established. This transfer function is (see Figure 2)

$$\frac{e_o}{e_i} = -\frac{\mu}{(1+\mu)RCp+1} \approx -\frac{1}{RCp}, \; (\mu >> 1), \tag{11}$$

in which the resistance, R, and the capacitance, C, are passive computing components; μ is the amplifier gain without feed-back (the same as the summing amplifiers); and p is the usual differential operator. The transfer function of the j-th summing amplifier (corresponding to equation (4)) in the modified circuit of Figure 2 is

$$\frac{e_o}{\Sigma e_i} = -\frac{\mu/k(n)}{1+\mu/k(n)} = -\frac{\mu}{k(n)+\mu} \approx -1, \tag{12}$$

and the transfer function for the sign-changing amplifier is

$$\frac{e_o}{e_i} = -\frac{\mu}{2+\mu} \approx -1. \tag{6}$$

When the integrating amplifier polarity is negative, therefore, the j-th equation is exactly mechanized as

$$a_{j1}x_1 + a_{j2}x_2 + - - - + a_{jj}x_j + - - - + a_{jn}x_n = \left(\frac{k(n) + \mu}{\mu}\right)\left\{\frac{(1 + \mu)\, RCp + 1}{\mu}\right\} x_j , \tag{13}$$

and, when the polarity is positive, it is mechanized as

$$a_{j1}x_1 + a_{j2}x_2 + - - - + a_{jj}x_j + - - - + a_{jn}x_n = - \frac{k(n) + \mu}{\mu}\bigg)\left(\frac{2 + \mu}{\mu}\right)\left\{\frac{(1 + \mu)\, RCp + 1}{\mu}\right\} x_j . \tag{14}$$

In equations (13) and (14) any negative coefficients, a_{jk} $(j \neq k)$, must be modified according to equation (7). Omitting the negligible terms on the right-hand side of equations (13) and (14), they can be written together as

$$a_{j1}x_1 + a_{j2}x_2 + - - - + \left(a_{jj} \mp \frac{1}{\mu} \mp RCp\right) x_j + - - - + a_{jn}x_n - y_j = 0. \tag{15}$$

By applying the rule for addition of determinants, the characteristic equation is written in the following form:

$$\Delta(p) = \mp \Delta^0(p) + \Delta_{jj}(p)\, RCp = 0, \tag{16}$$

in which $\Delta(p)$ is the stabilized system determinant,

$$\mp \Delta^0(p) = \begin{vmatrix} \left(a_{11} + \frac{k}{\mu}\right) & a_{12} & - & - & - & - & - & -a_{1n} \\ a_{21} & \left(a_{22} + \frac{k}{\mu}\right) & - & - & - & - & - & -a_{2n} \\ - & - & - & - & - & - & - & - \\ \mp a_{j1} & \mp a_{j2} & - & \left(\mp a_{jj} + \frac{1}{\mu}\right) & & & - & \mp a_{jn} \\ - & - & - & - & - & - & - & - \\ a_{1n} & a_{2n} & - & - & - & - & - & \left(a_{nn} + \frac{k}{\mu}\right) \end{vmatrix} , \tag{17}$$

and $\Delta_{jj}(p)$ is the j-th row, j-th column co-factor. In equation (17), the minus signs correspond to equation (13), and the plus signs correspond to equation (14). When $\Delta^0(p)$ is expanded as a polynomial in p, the magnitude of the constant term is equal to, within a negligible error (of the order of $1/\mu$), the value of the determinant of the given system of equations. The sign of this constant term can be reversed arbitrarily by the choice of the integrating amplifier polarity.

The amplifier gain without feed-back, $\mu(p)$, which is assumed to be the same for all the amplifiers (without loss of generality in this analysis) has the form,

$$\mu(p) = A \frac{f(p)}{g\,(p)} , \tag{18}$$

where A = a positive constant,

$f(p)$ = a polynomial in p of degree k, and

$g(p)$ = a polynomial in p of degree $k+1$ at least.

Equation (16) reduces to the sum of two polynomials in p of degree $(2nk + n - k)$ when equation (18) is substituted therein and after fractions are eliminated.

The second of these two polynomials contains roots corresponding to the roots of the cofactor, $\Delta_{jj}(p)$, as well as to the roots of the amplifier gain function, $\mu(p)$; all these roots are negative or have negative real parts because: (1) $\Delta_{jj}(p)$ is the determinant of the stable mechanizing circuit for the system of $(n-1)$ order, and (2) by hypothesis, $\mu(p)$ has no poles or zeros in the right half of the complex p-plane. Since this second polynomial is multiplied by RCp, as indicated in equation (16), it contains no constant term and the magnitudes of its coefficients can be linearly adjusted by the arbitrary R-C product. If this product is made sufficiently large, all the coefficients of the first polynomial, except the constant one, can be neglected. Since the relative magnitude of this constant term is so small, the $n-1$ largest roots of equation (16) are essentially the same as the roots of the second polynomial and, therefore, are negative or have negative real parts. This means that the magnitude of the nth root is much smaller than the other roots. According to the theory of equations, then, this root has negligible influence on all coefficients except the constant term. Furthermore, its sign is determined by the constant term which, as it can be seen in equation (16) again, can be arbitrarily selected by the integrator polarity yielding a stable mechanizing circuit. Evidently, there are no restrictions on the amplifier gain, $\mu(p)$, other than it must be stable without feedback.

The necessary choice of polarity sheds some light on the nature of the particular roots of equation (9) which are responsible for the instability. If there are an odd number of such roots (any two of which may be complex), the constant term of the characteristic equation of the unstable circuit is negative. In this case, the negative sign in equation (16) is chosen. If there are an even number of such roots (any two of which may be complex), this constant term is positive and the positive sign in equation (16) is chosen. It is interesting to note that this constant term, which is the value of the determinant of the given system of equations, indicates instability only when its sign is negative. Also, in the case where the j-th equation is not independent of one or more of the other equations, the value of the system determinant is zero, so that the mechanizing circuit can not be stabilized by either polarity of the integrating amplifier. This is precisely as it should be because no solution exists.

VERIFICATION

With a view toward experimental verification, the system of equations,

$$\left\{\begin{array}{rrrrrcr}
5x_1 & + \; x_2 & - \; 2x_3 & + \; 3x_4 & - \; x_5 & = & -48 \\
x_1 & + \; 4x_2 & & + \; x_4 & + \; 2x_5 & = & 0 \\
-2x_1 & & + \; 4x_3 & - \; x_4 & - \; 3x_5 & = & 0 \\
3x_1 & + \; x_2 & - \; x_3 & + \; 2x_4 & & = & 0 \\
-x_1 & + \; 2x_2 & - \; 3x_3 & & + \; 5x_5 & = & 0
\end{array}\right. \qquad (19)$$

was mechanized according to the circuit of Figure 1, employing a general purpose d-c electronic analog computer. Instability of the mechanizing circuit made it necessary to apply the stabilizing technique, and four possible stable combinations of amplifiers were found. These and the corresponding solutions of equations (6) are listed in Table 1. Although the computation errors in these experimental solutions appear to be large in the case of the magnitude of the smallest variable, they can be made as small as desired by the simple and rapidly converging iterative process described by Goldberg and Brown. The stable combination employing two integrating amplifiers is included to demonstrate successive applications of the technique.

TABLE 1

Amplifier Combination*	x_1	x_2	x_3	x_4	x_5
1-Int., 2, 3, 4, 5.	34.0	-34.0	56.3	-5.9	54.2
1, 2, 3-Int., 4, 5.	34.2	-34.2	56.4	-6.0	54.3
1, 2, 3, 4, 5-Int.	34.3	-34.1	55.8	-6.1	54.3
1, 2-Int., 3, 4+Int., 5.	34.8	-34.2	56.4	-6.4	54.3
Exact solution by Cramer's Rule	34.0	-34.0	56.0	-6.0	54.0

* Number of amplifier refers to the equation it mechanizes.

± Int. refers to the integrating amplifier with the indicated polarity and associated summing amplifier.

THE SIGNIFICANCE

In many phases of scientific and engineering work—in the determination of molecular weights; in the establishment of crystallographic axes from X-ray data; in work with the mass spectrograph; in the evaluation of flutter, stability, and vibration in aircraft; in the analysis of electrical and electronic circuits, to name a few—the solution of systems of linear algebraic equations of high order is a frequent requirement. In fact, the smoothing of almost any experimental data by the method of least squares, a widely applicable procedure, calls for the solution of such a system of equations; and the higher the order of the equations, the greater the precision. Other examples may be cited.

Except for the disadvantage of instability, the analog computing circuit with which we are here concerned probably provides the most practical means now available for solving systems of linear algebraic equations. The stabilizing procedure of Goldberg and Brown, while simplifying the transformation of equations, nevertheless halves the capacity of any given computer. The new NRL technique for circuit stabilization eliminates this final shortcoming.

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