NRL REPORT 3557

FR-3557

A NOTE ON THE ANALOG COMPUTATION OF DETERMINANTS

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October 18, 1949

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Naval Res. Sec., Science Div., Library of Congress Attn: Mr. J. H. Heald	2

CONTENTS

ABSTRACT

A method for evaluating determinants, depending upon the solution of a sequence of linear algebraic equations, is given. The mechanization is arranged so that the McCool stabilizing procedure for these equations can be applied directly at the first appearance of instability. The size of the REAC installation limits the order of determinants which can be evaluated by this method. Considerations of accuracy in solving algebraic equations have not yet been investigated sufficiently to determine the accuracy, and, therefore, the practical utility, of this evaluation method.

PROBLEM STATUS

This is an interim report. Work is continuing on the problem.

AUTHORIZATION

NRL Problem P10-01R
NR490-010

A NOTE ON THE ANALOG COMPUTATION OF DETERMINANTS

INTRODUCTION

The evaluation of determinants of order higher than three or four is a tedious task by the ordinary expansion method, using pencil, paper, and desk calculator. The availability of a Reeves Electronic Analog Computer (REAC) at the Naval Research Laboratory made it interesting to seek a method of evaluation, using this computer, for problems in which high accuracy is less important than speed and ease of computation.

The method presented occurred independently to the authors, but was not practicable when first conceived because of the difficulties sometimes encountered in obtaining stable solutions to linear algebraic equations with the REAC. Recently, however, W. McCool[1] has given a relatively simple method by which stable solutions of such equations can be assured. In conjunction with the McCool stabilizing procedure, the method developed here for the evaluation of determinants has become applicable in practice.

THE EVALUATION METHOD

Let a_{ij} $(i,j = 1,2,\ldots,n)$ be a set of real numbers whose determinant,

$$A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}, \qquad (1)$$

it is desired to evaluate. The philosophy of the evaluation method can be given very simply as follows.

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[1] *W. McCool, "D-C Analog Solution of Simultaneous Linear Algebraic Equations; Circuit Stability Considerations," NRL Report 3533, September 9, 1949*

Suppose that the members of this set of numbers are regarded as the coefficients of a set of linear algebraic equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \quad \cdots \quad + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots \\ a_{n-1,1}x_1 + a_{n-1,2}x_2 + \cdots + a_{n-1,n}x_n = 0 \\ a_{n1}x_1 + a_{n2}x_2 + \quad \cdots \quad + a_{nn}x_n = 1 \end{array}\right\} \quad (2)$$

Using Cramer's Rule for an analytic solution of these equations for x_n, one has

$$x_n = \frac{1}{A}\begin{vmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1,n-1} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1,1} & & \cdots & & a_{n-1,n-1} & 0 \\ a_{n1} & \cdot & \cdot & \cdot & a_{n,n-1} & 1 \end{vmatrix} = \frac{A_{n-1}}{A} \quad (3)$$

where A_{n-1} is the determinant of the minor of a_{nn}. Thus, A is given in terms of a value x_n, which may be observed on the computer, and a new determinant A_{n-1} of rank one less than A. The same procedure may be used to relate A_{n-1} to another observed x-value, corresponding to a new set of n-1 equations, and another unknown determinant of rank two less than A. Indeed, the same process may be continued until the last remaining "unknown" determinant is the known constant a_{11}. At this point, the observed x-values may be combined (by multiplication on a desk calculator) to give A.

The evaluation process may now be considered in greater detail. It will be convenient to use the matrix notation listed below:

$\mathfrak{a}$ the matrix of the original coefficients a_{ij} ;

$\mathfrak{a}_r$ the matrix formed from $\mathfrak{a}$ by striking out all rows and columns whose indices exceed r;

A, A_r the determinants of $\mathfrak{a}, \mathfrak{a}_r$ respectively;

$\mathfrak{X}^{(i)}$ an (i x 1) column matrix of unspecified elements;

x_i the element in the ith (last) row of the matrix $\mathfrak{X}^{(i)}$

$\mathcal{E}^{(i)}$ an (i x 1) column matrix, all of those elements are zero except that in the ith row, which is unity.

Equation 2 can be written in these terms as

$$\mathfrak{a}\mathfrak{X}^{(n)} = \mathcal{E}^{(n)} \qquad (2')$$

and its solution for x_n as

$$x_n = \frac{A_{n-1}}{A} \qquad (3')$$

Note that the physical representation of the matrix $\mathcal{E}^{(i)}$ is a set of zero-inputs to all adders used in the computing circuit except the ith (last) adder, where the input is unity.

A schematic diagram (Figure 1) shows the computer arrangement for solving Equation 2. Having set the appropriate coefficients, a_{ij}, and inputs on the computer, and having observed the value of x_n (in general, after a stabilizing process), the nth equation may be removed from consideration. This means merely that the output of the nth adder is severed, and the x_n-buss grounded. If, then, the inputs are changed to correspond to $\mathcal{E}^{(n-1)}$ rather than $\mathcal{E}^{(n)}$, one is left with a mechanization of the reduced system of equations

$$\mathfrak{a}_{n-1}\mathfrak{X}^{(n-1)} = \mathcal{E}^{(n-1)} \qquad (4)$$

on the computer.

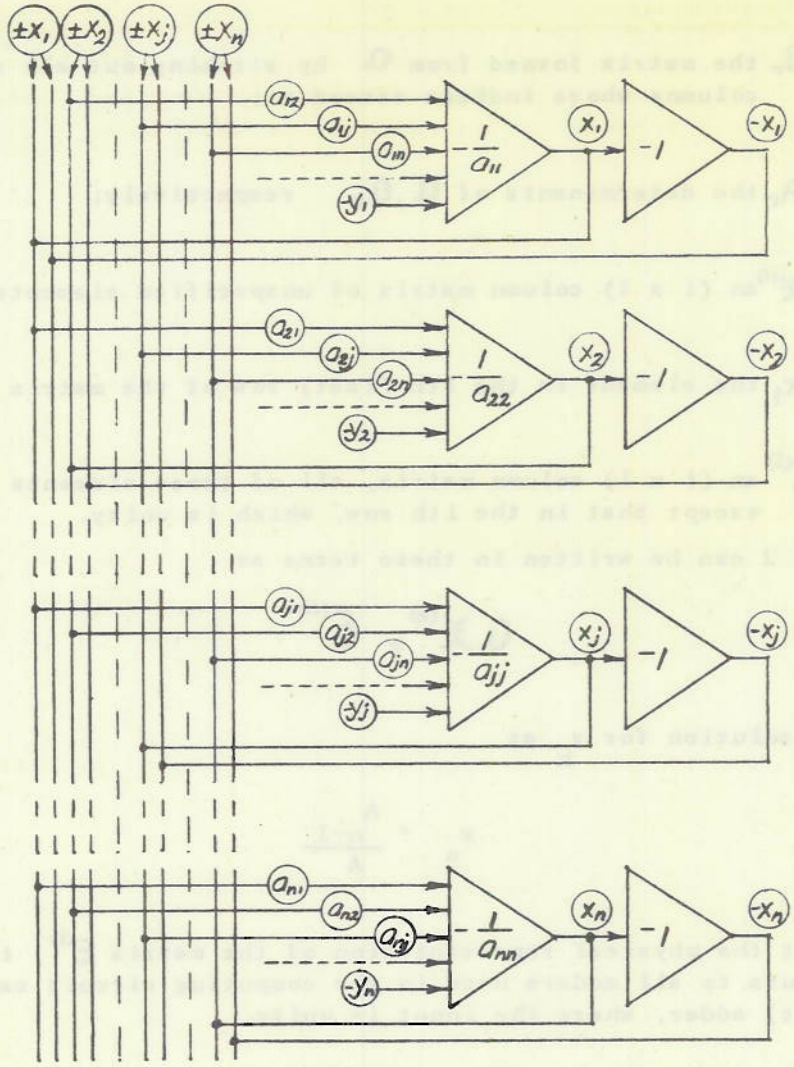

Figure 1 - Computing circuit diagram

The solution for x_{n-1} of Equation 4, analytically related to A_{n-1} by

$$x_{n-1} = \frac{A_{n-2}}{A_{n-1}}, \qquad (5)$$

can now be obtained from the computer. Thereafter, the output of the (n-1)st adder can be severed, and the new inputs $\mathcal{E}^{(n-2)}$ put on the computer. The process can be continued as described, obtaining at each step as a reading on the computer, a value

$$x_i = \frac{A_{i-1}}{A_i} \qquad (i=n,\ldots,2) . \tag{6}$$

As a final step, this set of observed values may be multiplied together to give

$$x_n x_{n-1} x_{n-2} \cdots x_2 = \frac{A_{n-1}}{A} \frac{A_{n-2}}{A_{n-1}} \frac{A_{n-3}}{A_{n-2}} \cdots \frac{A_1}{A_2} = \frac{a_{11}}{A} , \tag{7}$$

where $A_1 = a_{11}$ has been used. The solution to our problem, therefore, is

$$A = \frac{a_{11}}{x_n x_{n-1} \cdots x_2} \tag{8}$$

with a_{11} known and x_i $(i = 2,\ldots,n)$ observed numbers.

SOME PRACTICAL CONSIDERATIONS

Ease of application of the evaluation process is an important consideration in weighing the utility of the method. Mechanically, it consists of setting an array of coefficients on the REAC, and wiring the computer to solve a set of n linear algebraic equations with certain simple inputs. Thereafter, the components are disconnected one by one, a single input is changed, and a single reading is taken at each stage. After reducing the number of mechanized equations to two, the observed x-values are combined according to Equation 8 to give the value of the determinant A. This procedure is conceptually simple, and will be so in practice provided stability difficulties are not encountered during the reduction of the order of the equations.

Some remarks on stability are pertinent. If the original set of equations is found to be stable upon mechanization, all of the reduced equations will be stable. This is evident when one recalls that the stability of the original system depends[2] upon the positive-definiteness of the system matrix $\mathfrak{a}$, and that a necessary and sufficient condition that $\mathfrak{a}$ be positive-definite is that all the determinants A_r $(r = 1,\ldots,n)$ have the same sign. Thus, the stability of Equation 2 implies the positive-definiteness not only of $\mathfrak{a}$, but of all the reduced matrices $\mathfrak{a}_r$. This fact, in turn means that all of the reduced equations are stable.

The procedure outlined above is a logical one, and one which will work in practice. However, the construction of the procedure has not taken account of the details of the stabilizing process. Since the equations originally mechanized will generally be unstable, the possibility arises of modifying the determinant evaluation process in such a way that the stabilization will be included automatically, with but little increase in effort. A modification exists, substantially an inversion of the order of steps outlined in the explanation of the theory, which accomplishes the stated purpose.

Attempts to carry out an evaluation in the logical way described on an initially unstable system, reveal that the first step in stabilization consists of removing the problem from the machine, and then building up the set of equations one by one. Therefore, rather than first setting up the extended system represented by Equation 2, with a possibility of having an unstable system and being required to return again to the beginning, it is well to begin with the most reduced system, represented by

$$x_2 = \frac{a_{11}}{A_2} \qquad (9)$$

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[2]*Murray, F. J., "The Theory of Mathematical Machines," Revised Ed. pp. 11-14, New York: King's Crown Press, 1948*

and carry out the determinant evaluation by proceeding to $x_3, x_4, \ldots, x_n$ in succession. In this way, the question of stability may be settled at once, wherever it arises upon the addition of a new equation. It is clear that this reversal of the order of procedure in no way affects the answer obtained, but that it does avoid the necessity for de-mechanizing an unstable system in order to use a logical stabilizing process.

An important consideration now remaining is that of accuracy. Unfortunately, no general statement can yet be made as to the error encountered in solving linear algebraic equations with the REAC. The answer to such a question depends not only upon the inherent accuracy of the machine, but upon the algebraic signs and magnitudes of the coefficients, and to a considerable extent upon the arrangement of the equations. It is possible that a suitable rearrangement of the determinant before placing its elements on the machine would be very effective in reducing errors. Until a more general study of such errors is made, and a number of examples are tried by the method given here, the expected machine error can only be used as the basis for a guess as to the error of a single reading. Since the individual readings are compounded by multiplication to obtain A, the percentage errors of these readings will be compounded additively. It may be, therefore, that the REAC evaluation of a determinant would prove of most value to establish orders of magnitudes (with perhaps 10% or 20% error) for relatively high order determinants.[3]

CONCLUSION

A method has been outlined for the evaluation of determinants, using the REAC. The essence of this method is the solution of a set of linear algebraic equations of decreasing order, and a suitable combination of the observed solutions. In practice, a better procedure is to follow the steps of the method in reverse order, thereby making stabilization simple at each step.

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[3] *The limits to the size determinant which may be considered, depend upon the REAC installation. There is a limited number of inputs to any summing amplifier used in the mechanization, but this limitation can be circumvented if enough additional adders are available.*

The size determinants which can be evaluated by the method given is limited by the number of inputs available to the summing amplifiers of the REAC, or by the total number of summing amplifiers in the machine.

The accuracy of this method may be satisfactory for many purposes. However, the general question of the dependence of the errors upon the nature of the elements of the determinant is still to be investigated. A final decision as to the merit of the method presented here will depend upon the results of this investigation.

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