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FREQUENCY ANALYSIS OF BEAM-RIDER TURNING RATES



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C. E. Corum and C. L. Key, Jr.

November 21, 1949

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CONTENTS

Abstract	iv
Problem Status	iv
Authorization	iv
INTRODUCTION	1
BEAM-RIDER TRAJECTORY	1
Derivation of the Differential Equation of the Trajectory	1
Solution of the Differential Equation of the Trajectory	3
MISSILE TURNING-RATE CURVE	7
Derivation of the Equation of the Turning-Rate Curve	8
Polynomial Approximation to the Turning-Rate Curve	9
FREQUENCY SPECTRUM OF THE TURNING-RATE CURVE	11
SAMPLE COMPUTATION	12
Bandwidth and Crossover Range	17
CONCLUSIONS	17
ACKNOWLEDGMENTS AND REMARKS	18
BIBLIOGRAPHY	19
APPENDIX: COMPUTATIONAL DATA	21

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ABSTRACT

The differential equation for the theoretical trajectory of a beam-rider missile is derived and solved. A constant-velocity target and a constant-speed missile are assumed, and no consideration is given to such matters as stability and the effects of gravity. From the differential equation and its solution the turning rate is derived as a function of time. In order to get the frequency spectrum of the turning rate, it is necessary first to approximate the turning-rate function by a polynomial before applying the Fourier transform. It is shown that the relative amplitudes of the frequency components decrease at six db per octave, beyond the point where the frequency equals the reciprocal of the time of flight. The report includes curves showing trajectories, turning rates, and frequency spectra for various conditions; and an appendix giving computational details.

PROBLEM STATUS

This is an interim report; work on the problem is continuing.

AUTHORIZATION

NRL Problem R05-16D (BuAer Problem A-156)
NL 480-042

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FREQUENCY ANALYSIS OF BEAM- RIDER TURNING RATES

INTRODUCTION

The direction of flight of a beam-rider missile is controlled by the deflection of flaps. By this means a rate of turn is introduced which is a function of the flap deflection, and is usually directly proportional to the control voltage at some point in the control system over a limited range of frequencies.

In this analysis, it is assumed that this direct proportionality holds exactly over all frequencies, so that there is a linear relationship between the instantaneous rate of turn of the missile and the control voltage during the beam-rider flight. The Fourier transform of the rate-of-turn function then represents the frequency spectrum which must be passed by the entire system to follow the theoretical beam-rider trajectory exactly.

The turning-rate function is derived from the solution of the differential equation of the trajectory of a constant-speed beam-rider missile against a constant-speed target flying a straight-line pass course.

BEAM-RIDER TRAJECTORY

A line-of-sight or beam-rider course is one in which the missile and the target both lie on a straight line joining the target and the point of control. In this analysis the missile and target speeds are considered constant, the target flying a straight line and the point of control stationary. For the determination of the trajectory, the coordinates are set up in a plane determined by the target velocity vector and the line joining the target velocity vector with the point of origin. This makes the solution general for any straight-line constant-speed target course.

Derivation of the Differential Equation of the Trajectory

The following symbols are employed: r and θ are the conventional polar coordinates of the instantaneous missile position measured from the point of control (radar dish at origin); ρ and θ are the conventional polar coordinates of the target position measured from the point of control; v_m = speed of missile and v_t = speed of target; $p = v_m/v_t$; and R = the minimum pass distance from the point of control to the target.

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It can be seen from Figure 1 that the distance the target travels in time dt is dX and

$$dX = v_t dt = \sqrt{d\rho^2 + \rho^2 d\theta^2}; \quad (1)$$

the distance the missile travels in time dt is ds and

$$ds = v_m dt = \sqrt{dr^2 + r^2 d\theta^2}; \quad (2)$$

and

$$R = \rho \sin \theta \text{ or } \rho = R \csc \theta. \quad (3)$$

Differentiating (3):

$$d\rho = -R \csc \theta \cot \theta d\theta. \quad (4)$$

Then, from equation (1):

$$\begin{aligned} dX^2 &= v_t^2 dt^2 = d\rho^2 + \rho^2 d\theta^2 \\ &= R^2 \csc^2 \theta \cot^2 \theta d\theta^2 + R^2 \csc^2 \theta d\theta^2 \\ &= R^2 \csc^2 \theta (1 + \cot^2 \theta) d\theta^2 \\ &= R^2 \csc^4 \theta d\theta^2, \end{aligned}$$

so that

$$dX = v_t dt = R \csc^2 \theta d\theta. \quad (5)$$

From (2):

$$dt = (1/v_m) \sqrt{dr^2 + r^2 d\theta^2},$$

and from (5):

$$dt = (1/v_t) R \csc^2 \theta d\theta.$$

Hence, eliminating dt ,

$$\begin{aligned} (R/v_t) \csc^2 \theta d\theta &= (1/v_m) \sqrt{dr^2 + r^2 d\theta^2} \\ dr^2 + r^2 d\theta^2 &= (v_m/v_t)^2 R^2 \csc^4 \theta d\theta^2 = p^2 R^2 \csc^4 \theta d\theta^2 \end{aligned}$$

* All figures appear at end of report.

or for $k = pR$

$$(dr/d\theta)^2 + r^2 = r'^2 + r^2 = k^2 \csc^4 \theta. \quad (6)$$

Under the assumption that the missile enters the beam at time $t = 0$ along the direction of the beam at that time, the beam-rider differential equation has the initial conditions

$$r = 0$$

$$\theta = \theta_c = \arctan m \quad (7)$$

where m is the initial slope of the trajectory and θ_c is the initial launching angle. Equation (6) and conditions (7) constitute the differential equation of the missile trajectory against a target flying a straight-line course, when both missile and target speeds are constant. Equation (6) was derived on the basis that the distance traveled by the missile in its curved path is v_m/v_t times the distance traveled by the target along a straight line pass course in the same length of time.

Solution of the Differential Equation of the Trajectory

This differential equation is of the first order, but is non-linear and not solvable in closed form by ordinary methods. Although it is possible to exhibit the solution in the form of a definite integral,¹ it is more convenient for the present purpose to employ an infinite-series solution.²

The solution may be written as a Taylor series:

$$r = r_0 + r_0' (\theta - \theta_0) + r_0'' (\theta - \theta_0)^2/2! + \dots + r_0^{(n)} (\theta - \theta_0)^n/n! + \dots \quad (8)$$

To obtain r explicitly it is necessary to evaluate the coefficients $r_0, r_0', r_0'',$ etc., in terms of the initial boundary conditions (7). The zero subscripts associated with these coefficients indicate that they are to be evaluated at $\theta = \theta_0$.

From (7), $r_0 = 0$ and, from (6),

$$(r_0')^2 + r_0^2 = (r_0')^2 = k^2 \csc^4 \theta_0;$$

¹ See Munro, G. C., "Line-of-Sight Paths," APL Bumblebee Report No. 7, 15 January 1945 (Confidential)

² This solution was derived by Dr. D. C. Harkin, Radio Division III, Naval Research Laboratory.

hence $r_o' = k \csc^2 \theta_o$. (9)

Now, since $\csc^2 \theta_o = 1 + \cot^2 \theta_o = 1 + (1/m^2) = m^{-2}(1 + m^2)$, then

$$r_o' = km^{-2}(1 + m^2).$$

To get r_o'' it is necessary to differentiate equation (6):

$$r'r'' + rr' = -2k^2 \csc^4 \theta \cot \theta.$$

At $\theta = \theta_o$, $\csc^4 \theta = (1 + \cot^2 \theta)^2 = [1 + 1/m^2]^2 = m^{-4}(1 + m^2)^2$,

hence

$$\begin{aligned} r_o'r_o'' + r_or_o' &= r_o'r_o'' = -2k^2 m^{-5}(1 + m^2)^2 \\ r_o'' &= [-2k^2 m^{-5}(1 + m^2)^2]/r_o' = [-2k^2 m^{-5}(1 + m^2)^2]/[km^{-2}(1 + m^2)] \\ r_o'' &= -2km^{-3}(1 + m^2). \end{aligned} \quad (10)$$

This procedure must be continued in order to obtain succeeding coefficients. The higher order derivatives become somewhat cumbersome; it is therefore convenient to differentiate the left- and right-hand sides of equation (6) separately and tabulate the successive derivatives.

The left-hand side of (6) is represented by $2f$, thus:

$$2f = r'^2 + r^2.$$

The successive derivatives are:

$$f' = r'r'' + rr'$$

$$f'' = r'r''' + r''^2 + r'^2 + rr''$$

$$f''' = r'r^{iv} + 3r''r''' + 3r'r'' + rr'''$$

$$f^{iv} = r'r^v + 4r''r^{iv} + 3r'''^2 + 3r''^2 + 4r'r''' + rr^{iv}$$

$$f^v = r'r^{vi} + 5r''r^v + 10r'''r^{iv} + 10r''r''' + 5r'r^{iv} + rr^v$$

$$f^{vi} = r'r^{vii} + 6r''r^{vi} + 15r'''r^v + 10r^{iv^2} + 10r'''^2 + 15r''r^{iv} + 6r'r^v + rr^{vi}$$

$$f^{vii} = r'r^{viii} + 7r''r^{vii} + 21r'''r^{vi} + 35r^{iv^2} + 35r'''r^{iv} + 21r''r^v$$

$$+ 7r'r^{vi} + rr^{vii}$$

$$f^{viii} = r' r^{ix} + 8r'' r^{viii} + 28r''' r^{vii} + 56r^{iv} r^{vi} + 35r^{v^2} + 35r^{iv^2} + 56r''' r^v \\ + 28r'' r^{vi} + 8r' r^{vii} + r r^{viii}$$

$$f^{ix} = r' r^x + 9r'' r^{ix} + 36r''' r^{viii} + 84r^{iv} r^{vii} + 126r^v r^{vi} + 126r^{iv} r^v \\ + 84r''' r^{vi} + 36r'' r^{vii} + 9r' r^{viii} + r r^{ix}.$$

Introduce the notation $s = \sin \theta$ and $c = \cos \theta$. Then let the right-hand side of equation (6) be represented by

$$2k^2 g = k^2 \csc^4 \theta = k^2 s^{-4}. \quad (12)$$

The successive derivatives are:

$$g' = -2s^{-5}c$$

$$g'' = 10s^{-6} - 8s^{-4}$$

$$g''' = -60s^{-7}c + 32s^{-5}c$$

$$g^{iv} = 420s^{-8} - 520s^{-6} + 128s^{-4}$$

$$g^v = -3360s^{-9}c + 3120s^{-7}c - 512s^{-5}c$$

$$g^{vi} = 30240s^{-10} - 48720s^{-8} + 21280s^{-6} - 2048s^{-4}$$

$$g^{vii} = -302400s^{-11}c + 389760s^{-9}c - 127680s^{-7}c + 8192s^{-5}c$$

$$g^{viii} = 3,326,400s^{-12} - 6,531,840s^{-10} + 4,011,840s^{-8} - 807,040s^{-6} + 32768s^{-4}$$

$$g^{ix} = -39,916,800s^{-13}c + 65,318,400s^{-11}c - 32,094,720s^{-9}c + 4,842,240s^{-7}c$$

$$- 131,072s^{-5}c.$$

Only the coefficients in these expressions demand special attention; since, aside from the coefficients, their form is quite obvious.

Each of the g -derivatives is written in ascending powers of $\sin \theta$. Let the coefficients of one of the derivatives be designated by a_1, a_2, a_3 , etc., and those of the next higher derivative by b_1, b_2, b_3 , etc. In order to obtain the n th derivative, given the $(n-1)$ st derivative, the following recursion formulas are employed:

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when n is odd,

$$\begin{aligned} b_1 &= -(n+3)\alpha_1 \\ b_2 &= -(n+1)\alpha_2 \\ b_3 &= -(n-1)\alpha_3 \\ b_4 &= -(n-3)\alpha_4 \\ b_5 &= -(n-5)\alpha_5, \text{ etc.}, \end{aligned}$$

and when n is even,

$$\begin{aligned} b_1 &= -(n+3)\alpha_1 \\ b_2 &= (n+2)\alpha_1 - (n+1)\alpha_2 \\ b_3 &= n\alpha_2 - (n-1)\alpha_3 \\ b_4 &= (n-2)\alpha_3 - (n-3)\alpha_4 \\ b_5 &= (n-4)\alpha_4 - (n-5)\alpha_5 \text{ etc.} \end{aligned}$$

For example, compute g^{iv} given $g''' = -60s^{-7}c + 32s^{-5}c = \alpha_1 s^{-7}c + \alpha_2 s^{-5}c$.

Here $n = 4$ (an even number), since it is desired to find the fourth derivative.

$$\begin{aligned} \text{Hence } b_1 &= -(n+3)\alpha_1 = -(4+3)(-60) = 420 \\ b_2 &= (n+2)\alpha_1 - (n+1)\alpha_2 = (4+2)(-60) - (4+1)(32) = -520 \\ b_3 &= n\alpha_2 - (n-1)\alpha_3 = 4(32) = 128, \\ \text{hence } g^{iv} &= 420s^{-8} - 520s^{-6} + 128s^{-4}. \end{aligned}$$

The cosine occurs only in the derivatives of odd order. To find g^v (odd order): $n = 5$, $\alpha_1 = 420$, $\alpha_2 = -520$, $\alpha_3 = 128$, hence

$$\begin{aligned} b_1 &= -(n+3)\alpha_1 = -(5+3)(420) = -3360 \\ b_2 &= -(5+1)(-520) = 3120 \\ b_3 &= -(5-1)(128) = -512. \end{aligned}$$

$$\text{Therefore } g^v = -3360s^{-9}c + 3120s^{-7}c - 512s^{-5}c.$$

Using the process illustrated before, by which the values of r_o' and r_o'' were obtained, the following set of values is found:

$$\begin{aligned} r_o &= 0 \\ r_o' &= C \end{aligned}$$

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$$r_o'' = -2Cm^{-1}$$

$$r_o''' = Cm^{-2}(6 + m^2)$$

$$r_o^{iv} = -Cm^{-3}(24 + 16m^2)$$

$$r_o^v = Cm^{-4}(120 + 120m^2 + 21m^4)$$

$$r_o^{vi} = -Cm^{-5}(720 + 960m^2 + 282m^4)$$

$$r_o^{vii} = Cm^{-6}(5040 + 8400m^2 + 3606m^4 + 301m^6)$$

$$r_o^{viii} = -Cm^{-7}(40320 + 80640m^2 + 48384m^4 + 6816m^6)$$

$$r_o^{ix} = Cm^{-8}(362,880 + 846,720m^2 + 645,120m^4 + 169,680m^6 + 4681m^8)$$

$$r_o^x = -Cm^{-9}(3,628,800 + 9,676,800m^2 + 8,951,040m^4 + 3,310,320m^6 + 352,562m^8)$$

where $C = km^{-2}(1 + m^2)$. The series solution can therefore be written:

$$r = km^{-2}(1 + m^2)[(\theta - \theta_o) - 2m^{-1}(\theta - \theta_o)^2/2! + m^{-2}(6 + m^2)(\theta - \theta_o)^3/3! - m^{-3}(24 + 16m^2)(\theta - \theta_o)^4/4! + m^{-4}(120 + 120m^2 + 21m^4)(\theta - \theta_o)^5/5! - \dots]. \quad (13)$$

In computing trajectories corresponding to different values of the speed ratio p , labor may be saved by plotting $r/k = r/pR$, instead of r , using a scale such that $pR = 1$. If a trajectory is now plotted for $p = 1$, the scale will be such that $R = 1$. To obtain the trajectory for $p = 2$, draw a horizontal line at the ordinate $1/2$, representing the target path. The trajectory for $p = 2$ is the portion of the original trajectory (for $p = 1$) included between the x-axis and the target path at ordinate $1/2$. For purposes of comparison, the vertical scale may be expanded (in this case doubled) until the minimum pass distance R is again represented by the ordinate 1. A similar procedure is used for other speed ratios. (See Figures 2 - 5)

MISSILE TURNING-RATE CURVE

The only control over the missile after it is launched is the power to move the flaps in the direction and amount desired. To a good approximation the flap angle δ is proportional to the rate of turn of the missile; and by means of a suitable autopilot, the relation can be made exact. It is therefore a matter of interest to determine the

control voltage necessary to produce the flap deflections required in following the trajectories just derived.

Derivation of the Equation of the Turning-Rate Curve

Assume the flap deflection δ proportional to the turning rate $d\phi/dt$ as well as to the control voltage E_{in} , and assume an ideal following system, that is, with infinite bandpass, no time delays, etc. The slope of the trajectory is then given by

$$\begin{aligned}\tan \phi &= dy/dx = d(r \sin \theta)/d(r \cos \theta) = [d(r \sin \theta)/d\theta]/[d(r \cos \theta)/d\theta] \\ &= (r' \sin \theta + r \cos \theta)/(r' \cos \theta - r \sin \theta) = (r's + rc)/(r'c - rs), \quad (14)\end{aligned}$$

where $c = \cos \theta$, $s = \sin \theta$ and the primes stand for differentiation with respect to θ . Differentiating $\tan \phi$ with respect to θ :

$$\begin{aligned}\sec^2 \phi \, d\phi/d\theta &= [(r'c - rs)(r''s + 2r'c - rs) \\ &\quad - (r's + rc)(r''c - 2r's - rc)]/(r'c - rs)^2\end{aligned}$$

After simplifying the numerator on the right-hand side, the result is

$$\sec^2 \phi \, d\phi/d\theta = (2r'^2 - rr'' + r^2)/(r'c - rs)^2$$

$$\begin{aligned}\text{Now} \quad \sec^2 \phi &= 1 + \tan^2 \phi = 1 + [(r's + rc)/(r'c - rs)]^2 \\ &= (r'^2 + r^2)/(r'c - rs)^2.\end{aligned}$$

$$\text{Therefore} \quad d\phi/d\theta = (2r'^2 - rr'' + r^2)/(r'^2 + r^2).$$

Comparing with equation (6), it follows that

$$d\phi/d\theta = (2r'^2 - rr'' + r^2)/k^2 s^{-4}.$$

$$\text{Also} \quad v_m dt = \sqrt{dr^2 + r^2 d\theta^2} = k \csc^2 \theta \, d\theta = (v_m/v_t) R \csc^2 \theta \, d\theta;$$

hence

$$d\theta/dt = (v_t/R) \sin^2 \theta.$$

Now

$$d\phi/dt = (d\phi/d\theta)(d\theta/dt) = [(2r'^2 - rr'' + r^2)/k^2 s^{-4}][v_t/R s^2]$$

$$\therefore d\phi/dt = k^{-2} s^2 (v_t/R) (2r'^2 - rr'' + r^2). \quad (15)$$

Differentiate equation (6) and solve for r'' :

$$r'r'' + rr' = -2k^2 s^{-5} c$$

$$r'' = -(2k^2 s^{-5} c/r') - r.$$

By using this and equation (6), it is seen that

$$k^{-2}(2r'^2 - rr'' + r^2) = 2s^{-4} + 2s^{-5} cr/r' = 2s^{-4}[1 + (s^{-1} cr)/r'].$$

Therefore

$$\begin{aligned} d\phi/dt &= 2s^{-4}[1 + (s^{-1} cr)/r'] (v_t/R) s^6 \\ &= 2(v_t/R) s^2 [1 + (r \cot \theta / r')]; \end{aligned}$$

Thus the final expression for the turning rate is

$$d\phi/dt = 2(v_t/R) \sin^2 \theta [1 + (r \cot \theta) / \sqrt{k^2 \csc^4 \theta - r^2}]. \quad (16)$$

By using the solution of the differential equation of the trajectory, $d\phi/dt$ may be plotted. Although the solution of the differential equation gives r in terms of θ , t may be computed from θ by using the simple relationship

$$v_t t = R(\cot \theta_0 - \cot \theta)$$

or

$$t = (R/v_t)(\cot \theta_0 - \cot \theta). \quad (17)$$

Hence $d\phi/dt$ may be plotted against t .

Let a function be defined by the following conditions: (a) it is equal to the function defined by equations (16) and (17) for values of time t between 0 and t_c , the time of contact, and (b) it is equal to zero for values of time less than zero or greater than t_c . The Fourier spectrum of this function represents the turning rate for time between 0 and t_c . Hence this function will be regarded as the turning rate for the present purposes. (See Figures 6 - 21).

Polynomial Approximation to the Turning-Rate Curve

It is clear from inspection that the Fourier transform would be difficult, if not impossible, to apply to such a complicated function as is given in (16). If the computed values are plotted against time and a smooth curve passed through them, a polynomial may then be

fitted to this curve. It is found that a cubic polynomial is adequate to fit any turning-rate curve which is discussed in this report, within 0.5% of the maximum value of the turning rate in each case. It is relatively easy to obtain the Fourier transform of such a polynomial.

Let the analytic expression for the turning rate be written

$$d\phi/dt = [(2v_t \sin^2 \theta)/R] \left\{ 1 + (r/k) \cot \theta / \sqrt{[\csc^4 \theta - (r^2/k^2)]} \right\}. \quad (18)$$

Since from equation (13) r/k is independent of p , it is plain that the turning-rate curve has the same shape for different values of p , for a given θ_0 . The constant factor $2v_t/R$ does not affect the shape of the curve or the relative turning rate. The only effect of a change in p is that θ_c , the polar angle at the time of contact of missile with target, is changed. That is to say, the curve is "cut off" at a different time for each different p . The time of contact is given by

$$v_t t_c = R(\cot \theta_0 - \cot \theta_c)$$

or, by suitable change of scale,

$$t_c = \cot \theta_0 - \cot \theta_c. \quad (19)$$

Hence, after the graph of $d\phi/dt$ is once plotted for a given θ_0 and p , say $p = 1.5$, a portion of the same graph may be used to represent $p = 2, 3, 5$, etc., provided the time of contact t_c is known. The values of t_c corresponding to $p = 2, 3, 5$, etc. may be computed from equation (19) after reading θ_c from the graph of trajectories in each case or by computation of θ_c by interpolation. The turning-rate curve may then be cut off at that value of t instead of at the time of contact of the original curve. The resulting graph may then be transformed by scale expansion so that $t_c = 10$ units and $(d\phi/dt)_c = 10$ units, for convenience in applying the Fourier transform. This may be achieved by taking the polynomial for $p = 1.5$, which may be written

$$d\phi/dt = y = a_0 + a_1 t + a_2 t^2 + a_3 t^3,$$

and computing new coefficients. For example, suppose that the time of contact for the new value of p is t_c instead of 10, and that from the turning-rate curve it is found that the corresponding ordinate is y_c . It is desired to change scale so that the new time of contact is 10 units and the corresponding ordinate is 10 units. The old variable t must therefore be multiplied by $t_c/10$ and the old y by $y_c/10$. Then

$$(y_c/10)y = a_0 + a_1(t_c/10)t + a_2(t_c/10)^2 t^2 + a_3(t_c/10)^3 t^3.$$

If the new polynomial expression is

$$y = a_0' + a_1' t + a_2' t^2 + a_3' t^3,$$

then the new coefficients are

$$a_0' = a_0(10/y_C); \quad a_1' = a_1 t_C / y_C;$$

$$a_2' = a_2(t_C^2 / 10 y_C); \quad a_3' = a_3(t_C^3 / 100 y_C).$$

The polynomial for the new speed ratio may also be obtained by the following alternative series of steps: (a) computing θ_C from the trajectory data or taking it from the trajectory, (b) computing t_C from θ_C , using equation (19) and (c) plotting the graph for $p = 1$ and using it as far out as the point (y_C, t_C) . This portion of the turning-rate curve may be expanded by multiplying the ordinates by $10/y_C$ and the abscissas by $10/t_C$, so that the final result has the point of contact at $(10, 10)$ on the new scale, and may now be compared with the other turning-rate curves.³

FREQUENCY SPECTRUM OF THE TURNING-RATE CURVE

The frequency spectrum of the turning rate may be obtained for each curve by applying the Fourier transform to each polynomial term by term.

To illustrate the method in general, consider a polynomial of the form

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3. \quad (20)$$

The frequency spectrum is given by the Fourier transform

$$G(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} dt = \int_0^{10} y(t) e^{-i\omega t} dt \quad (21)$$

if $y = 0$ for t less than 0 or greater than 10.

Each term must be integrated by parts and the results added. For example, the evaluation of

$$\int_0^{10} t^2 e^{-i\omega t} dt$$

³ If only one turning-rate curve is to be computed, the expression for $d\phi/dt$ may be used directly, using the appropriate value of k . This is done in the example given later.

gives

$$[(100i/\omega) + (20/\omega^2) - (2i/\omega^3)] e^{-10\omega i} + 2i/\omega^3.$$

Using Euler's relation

$$e^{-ix} = \cos x - i \sin x$$

and collecting real and imaginary terms, the final result is

$$\begin{aligned} \int_0^{10} t^2 e^{-i\omega t} dt &= (20/\omega^2) \cos 10\omega + [(100/\omega) - (2/\omega^3)] \sin 10\omega \\ &+ i \left\{ [(100/\omega) - (2/\omega^3)] \cos 10\omega \right. \\ &\left. - (20/\omega^2) \sin 10\omega + 2/\omega^2 \right\}. \end{aligned} \quad (22)$$

Each term of the polynomial will give a similar result. In order to obtain the amplitudes of the frequency components, without regard to phase angle, it is necessary to evaluate the resulting complex number and take its absolute value. This gives the amplitude of the component of the turning rate corresponding to a given ω . Of course only relative amplitudes are in question here.

The function representing the turning rate is set equal to zero outside the limits 0 to 10. The resulting function is defined from $-\infty$ to $+\infty$, and its frequency spectrum represents the turning rate between the limits 0 and 10. (See Figures 22-37.)

SAMPLE COMPUTATION

As an example, the complete computation for the case $\theta_0 = 15^\circ$, $p = 2$ will be given (Figure 23).

It is necessary first to compute the coefficients of the series for the solution of the differential equation of motion. The general expression for the sum of the first ten terms of the series is

$$\begin{aligned} r &= km^{-2}(1 + m^2) [\Delta - 2m^{-1}\Delta^2/2! + m^{-2}(6 + m^2)\Delta^3/3! \\ &- m^{-3}(24 + 16m^2)\Delta^4/4! + \dots]. \end{aligned}$$

Here, for brevity, Δ is written for $\theta - \theta_0$. For $\theta_0 = 15^\circ$, $m = \tan 15^\circ = 0.26795$. Here $k = 2$, since $p = 2$, $R = 1$ and $k = pR$. Substituting

into the above equation,

$$r = 29.85624 [\Delta - 3.73204\Delta^2 + 14.09482\Delta^3 - 54.46820\Delta^4 + 208.0955\Delta^5 - 794.3571\Delta^6 + 3035.295\Delta^7 - 11594.77\Delta^8 + 44289.04\Delta^9 - 169171.9\Delta^{10}]. \quad (23)$$

Computations for various angles (which must be converted to radians when substituting into the equation above) yield the following data:

θ	r	p = 2	
		x	y
16°	0.4892	0.4702	0.1348
17	0.9222	0.8819	0.2696
18	1.3080	1.2440	0.4042
19	1.6540	1.5639	0.5385
20	1.9648	1.8463	0.6720
22	2.5037	2.3214	0.9379
23	2.7340	2.5166	1.0683

Here x and y are the rectangular coordinates corresponding to the polar coordinates r and θ . These data are plotted (Figure 2) and the intersection point with the target path determined. This determines the angle θ_c at which contact is made; this angle is subsequently used to determine the time of flight.

It is now necessary to compute the turning rate for each value of θ , using the formula

$$d\phi/dt = \sin^2 \theta [1 + (r \cot \theta) / \sqrt{(4 \csc^4 \theta - r^2)}], \quad (24)$$

which is obtained from equation (18) by neglecting the scale factor $2v_t/R$ and using $k = 2$. The factor $2v_t/R$ only introduces a scale change, and the scale will be changed anyway for convenience in applying the Fourier integral. The values of θ are then converted to time t by use of the formula $t = \cot \theta_o - \cot \theta$. From the plot of the trajectory, the coordinates of the point of contact are (2.424, 1). Hence $\theta_c = \arctan (1/2.424) = \arctan 0.4125 = 22.42^\circ$. Also, $t_c = \cot \theta_o - \cot \theta_c = 3.7321 - 2.4240 = 1.3081$. The value of r at contact for $p = 2$ is equal to $\csc \theta_c = \csc 22.42^\circ = 2.6221$. Substituting into the turning-rate formula, $(d\theta/dt)_c = 0.21392$. Now it is desired to expand the scale so that the maximum abscissa $t_c = 10$ units, and so that the maximum ordinate $(d\theta/dt)_c = 10$ units. This means that each abscissa must be multiplied by $10/1.3081$, and each ordinate computed from the turning-rate formula must be multiplied by $10/0.21392$. The points are then plotted and a smooth curve passed through them (Figure 7). From this curve ten points are taken (the exact number is, of course, optional; four points are necessary to

DECLASSIFIED

14

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determine a cubic polynomial and the others may be used to check it after it is derived). Let the points be selected as follows:

t	y = dφ/dt
0	3.13
1	3.47
2	3.85
3	4.32
4	4.81
5	5.37
6	5.96
7	6.80
8	7.70
9	8.76
10	10.00

The polynomial will have the form

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3,$$

and it is necessary to evaluate the coefficients

$$y(0) = 3.13 = a_0$$

$$y(3) = 4.32 = 3.13 + 3a_1 + 9a_2 + 27a_3$$

$$y(6) = 5.96 = 3.13 + 6a_1 + 36a_2 + 216a_3$$

$$y(10) = 10 = 3.13 + 10a_1 + 100a_2 + 1000a_3.$$

Hence:

$$3a_1 + 9a_2 + 27a_3 = 1.19$$

$$6a_1 + 36a_2 + 216a_3 = 2.83$$

$$10a_1 + 100a_2 + 1000a_3 = 6.87,$$

or:

$$a_1 + 3a_2 + 9a_3 = .3967$$

$$a_1 + 6a_2 + 36a_3 = .4717$$

$$a_1 + 10a_2 + 100a_3 = .6870.$$

The solution of this system of simultaneous linear equations is:

$$a_1 = .3958; a_2 = -.01206; a_3 = .004118.$$

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Hence the polynomial approximation to the turning rate is

$$y = 3.13 + .3958t - .01206t^2 + .004118t^3. \quad (25)$$

It remains to obtain the frequency spectrum of the turning-rate function which this polynomial approximates. It is necessary to take

$$G(\omega) = \int_0^{10} (\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3) e^{-i\omega t} dt = \alpha_0 G_0 + \alpha_1 G_1 + \alpha_2 G_2 + \alpha_3 G_3$$

for various values of ω . On integrating by parts or consulting a table of integrals, it is found that

$$G_0(\omega) = \int_0^{10} e^{-i\omega t} dt = (1/\omega) [\sin 10\omega + i(\cos 10\omega - 1)], \text{ etc.}$$

The spectrum is now obtained by assuming various values of ω and computing G_0 , G_1 , G_2 and G_3 for each ω . The resulting complex numbers are then multiplied by α_0 , α_1 , α_2 , and α_3 respectively and the products added. A complex number $A + Bi$ is obtained and its modulus $\sqrt{(A^2 + B^2)}$ is calculated, since the important thing here is the magnitudes of the frequency components and not the phase relations.

At $\omega = 0$,

$$\begin{aligned} G(0) &= \int_0^{10} (\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3) dt \\ &= [3.13t + .3958t^2/2 - .01206t^3/3 + .004118t^4/4]_0^{10} \\ &= 31.3 + 19.79 - 4.02 + 10.295 \\ &= 57.365. \end{aligned}$$

This is the zero-frequency component.

As a typical example of the computation of the components of the frequency spectrum, $G(\omega)$ is computed for $\omega = \pi/10$: Here $\sin 10\omega = \sin \pi = 0$ and $\cos 10\omega = \cos \pi = -1$.

$$G_0(\omega) = 10/\pi [0 + i(-2)] = -20i/\pi = -6.36620i$$

$$G_1 = (100/\pi^2) [(-1-1) - i\pi] = -(1/\pi^2) [200 + 100\pi i] = -20.26425 - 31.83102i$$

$$G_2 = (1000/\pi^3) [-2\pi + i(-\pi^2 + 4)] = -202.64246 - 189.30412i$$

$$G_3 = (10000/\pi^4) [-3\pi^2 + 12 + i(-\pi^3 + 6\pi)] = -1807.71910 - 1248.01094i$$

$$\begin{aligned} G(\omega) &= \alpha_0 G_0 + \alpha_1 G_1 + \alpha_2 G_2 + \alpha_3 G_3 = 3.13(-6.36620i) + .3958(-20.26425 - 31.83102i) \\ &\quad - .01206(-202.64246 - 189.30412i) + .004118(-1807.71910 - 1248.01094i) \end{aligned}$$

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$$= -13.13299 - 35.45861i$$

$$|G| = \sqrt{(13.13299^2 + 35.45861^2)} = \sqrt{(172.47543 + 1257.31302)} \\ = \sqrt{(1429.78845)} = 37.8125.$$

Using the same procedure for other values of ω , the following set of data is obtained:

<u>(Radians per Time Unit)</u>	<u>G</u>
0	57.365
$\pi/25$	53.915
$3\pi/50$	49.716
$2\pi/25$	44.210
$\pi/10$	37.812
$2\pi/10$	10.329
$4\pi/10$	5.418
π	2.197
2π	1.098
4π	0.550
10π	0.220

When these values are plotted on logarithmic paper, a curve results which is shown in Figure 23. The zero frequency component has been plotted at the minimum abscissa, which actually represents $\pi/50$, but the difference is very small.

It may be desired to obtain the frequencies in cycles per second instead of in radians per arbitrary time unit, given actual values for R and v_t . The formula for time of flight in seconds is

$$t_c = (R/v_t) (\cot \theta_o - \cot \theta_c).$$

The time of flight in the time units used for the curves is

$$t_c = (10/t_c) (\cot \theta_o - \cot \theta_c).$$

Hence time in seconds = time from graph times $(r/v_t)(t_c/10)$:

$$\omega \text{ in radians/sec} = \omega \text{ from graph multiplied by time in} \\ \text{arbitrary time units divided by time} \\ \text{in seconds;}$$

$$\omega \text{ in cycles/sec} = 1/2\pi \text{ multiplied by } \omega \text{ in radians/sec.}$$

By the same general process, computations have been carried out for

$\theta = 15^\circ, 22.5^\circ, 30^\circ$ and 45° ; and for $p = 1.5, 2, 3$ and 5 with each θ_o . The results — shown graphically in Figures 2-37 — are tabulated in the appendix, together with certain computational details.

Bandwidth and Crossover Range

It is of interest to observe the interdependence of crossover range and bandwidth. If the bandwidth B in radians per second is taken as 2π divided by the time of flight t_c (this bandwidth appears as $2\pi/10$ on the generalized frequency spectrum plots, and coincides with an abrupt change in their slope) and if the crossover range R is defined as the minimum pass distance from target to origin, then the crossover range may be computed for given v_m, v_t and θ_o by using the formula

$$t_c = (R/v_t)(\cot \theta_o - \cot \theta_c)$$

or

$$R = v_t t_c / (\cot \theta_o - \cot \theta_c)$$

or

$$R = (2\pi/B) v_t / (\cot \theta_o - \cot \theta_c).$$

The value of θ_c may be taken from the appropriate trajectory plot. The resulting plots for several values of v_m, v_t and θ_o are shown in Figures 38, 39 and 40.

CONCLUSIONS

The differential equation for the beam-rider trajectory has been solved in Taylor series form, using recursion formulas to obtain the coefficients of the series.

Using this result, an analytical expression for the turning rate of the missile has been derived. Since this expression is of relatively complicated form, a simplification was introduced by fitting a polynomial to the turning-rate curve.

In order to get the frequency spectrum of the turning rate, the Fourier transform was then applied to this polynomial.

The relative amplitudes of the frequency components of the turning-rate function fall off at six decibels per octave, provided

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18

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the frequency of the components (in cycles per second) is greater than the reciprocal of the time of flight. The spectrum curves may be used as an indication of the pass band required or desired in specific applications.

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One of the authors, Mr. Carroll L. Key, Jr., is now at Pennsylvania State College.

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APPENDIX COMPUTATIONAL DATA

1. General Expressions for Series Solutions of Differential Equation of Motion:

Let the following definitions be made:

$$m = \tan \theta_0$$

$$C = m^{-2}(1 + m^2)$$

The general term is

$$[r_0^{(n)} / n!] \Delta^n, \text{ where } \Delta = \theta - \theta_0, \text{ expressed in radians.}$$

The values of the successive derivatives are:

$$r_0 = 0$$

$$r_0' = C$$

$$r_0'' = -2Cm^{-1}$$

$$r_0''' = Cm^{-2}(6 + m^2)$$

$$r_0^{iv} = -Cm^{-3}(24 + 16m^2)$$

$$r_0^v = Cm^{-4}(120 + 120m^2 + 21m^4)$$

$$r_0^{vi} = -Cm^{-5}(720 + 960m^2 + 282m^4)$$

$$r_0^{vii} = Cm^{-6}(5040 + 8400m^2 + 3606m^4 + 301m^6)$$

$$r_0^{viii} = -Cm^{-7}(40320 + 80640m^2 + 48384m^4 + 6816m^6)$$

$$r_0^{ix} = Cm^{-8}(362880 + 846720m^2 + 645120m^4 + 169680m^6 + 4681m^8)$$

$$r_0^x = -Cm^{-9}(3628800 + 9676800m^2 + 8951040m^4 + 3310320m^6$$

$$+ 352562m^8).$$

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22

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Then the following expressions hold for particular cases:

For $\theta_0 = 15^\circ$ and $p = 1$:

$$r = 14.92812(\Delta - 3.73204\Delta^2 + 14.09483\Delta^3 - 54.46820\Delta^4 + 208.0955\Delta^5 - 794.3571\Delta^6 + 3035.295\Delta^7 - 11594.77\Delta^8 + 44289.04\Delta^9 - 169171.9\Delta^{10}).$$

For $\theta_0 = 22.5^\circ$ and $p = 1$:

$$r = 6.828527(\Delta - 2.414234\Delta^2 + 5.995194\Delta^3 - 15.68092\Delta^4 + 39.97526\Delta^5 - 101.7232\Delta^6 + 258.8546\Delta^7 - 659.3561\Delta^8 + 1679.223\Delta^9 - 4276.341\Delta^{10}).$$

For $\theta_0 = 30^\circ$ and $p = 1$:

$$r = 4(\Delta - 1.732051\Delta^2 + 3.166667\Delta^3 - 6.35086\Delta^4 + 12.175\Delta^5 - 23.19506\Delta^6 + 44.20615\Delta^7 - 84.47046\Delta^8 + 161.4157\Delta^9 - 308.3638\Delta^{10}).$$

For $\theta_0 = 45^\circ$ and $p = 1$:

$$r = 2\Delta - 2\Delta^2 + 2.333333\Delta^3 - 3.333333\Delta^4 + 4.35\Delta^5 - 5.45\Delta^6 + 6.88373\Delta^7 - 8.73810\Delta^8 + 11.18321\Delta^9 - 14.28545\Delta^{10}.$$

Sometimes, for large Δ , ten terms do not converge closely enough. In such a case a good approximation may be obtained by adding to the first nine terms a geometric series whose ratio is the ratio of the tenth term to the ninth term. For example, using the 45° case above, the ratio of the geometric series would be $-14.28545\Delta^{10}/11.18321\Delta^9 = -1.277401\Delta$, and the sum to be added to the first nine terms would be

$$-14.28545\Delta^{10}/(1 + 1.277401\Delta)$$

since a geometric series whose first term is a and whose ratio is q has the sum $a/(1 - q)$.

2. Computed values of r .

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$$a. \theta_0 = 15^\circ.$$

θ	p = 1		p = 1.5		p = 2			p = 3			p = 5		
	r	r	x	y	r	x	y	r	x	y	r	x	y
16°	.2446	.3669	.3527	.1011	.4892	.4702	.1348	.7338	.7054	.2023	1.2231	1.1757	.3371
16.5	.3561							1.0683	1.0243	.3034	1.7805	1.7072	.5057
17	.4611	.6917	.6614	.2022	.9222	.8819	.2696	1.3833	1.3228	.4044	2.3055	2.2047	.6741
17.5	.5602							1.6806	1.6028	.5054	2.8010	2.6714	.8423
18	.6540	.9810	.9330	.3031	1.3080	1.2440	.4042	1.9620	1.8660	.6063	3.2700	3.1100	1.0105
19	.8270	1.2405	1.1729	.4039	1.6540	1.5639	.5385	2.4810	2.3458	.8077			
20	.9824	1.4736	1.3847	.5040	1.9648	1.8463	.6720	2.9472	2.7695	1.008			
20.73	1.0872	1.6308	1.5252	.5773	2.1744	2.0336	.7697						
22	1.2518	1.8778	1.7411	.7034	2.5037	2.3214	.9379						
23	1.3670	2.0504	1.8874	.8012	2.7340	2.5166	1.0683						
25	1.581	2.372	2.150	1.002									

$$b. \theta_0 = 22.5^\circ.$$

θ	p = 1			p = 1.5			p = 2			p = 3			p = 5		
	r	r	x	y	r	x	y	r	x	y	r	x	y		
23.5°	.1144	.1716	.1574	.068	.2288	.2098	.091	.3432	.315	.137	.5720	.5246	.228		
24.5	.2199	.3299	.3002	.137	.4398	.400	.182	.6597	.600	.274	1.0995	1.001	.456		
25.5	.3175	.4763	.430	.205	.6350	.573	.273	.9525	.860	.410	1.5875	1.433	.683		
26.5	.4081	.6122	.548	.273	.8162	.730	.364	1.224	1.095	.546	2.0405	1.826	.910		
27.5	.4925	.7388	.655	.341	.9850	.874	.455	1.478	1.311	.682	2.463	2.185	1.137		
28.5	.5711	.8567	.753	.409	1.142	1.004	.545	1.713	1.505	.817					
29.5	.6447	.9671	.842	.476	1.289	1.122	.635	1.934	1.683	.952					
30.5	.7135	1.070	.922	.543	1.427	1.230	.724	2.141	1.845	1.087					
31.5	.7780	1.167	.995	.610	1.556	1.327	.813								
32.5	.8387	1.258	1.061	.676	1.677	1.414	.901								
33.5	.8957	1.344	1.121	.742	1.791	1.494	.988								
34.5	.9496	1.424	1.174	.807	1.899	1.565	1.076								
35.5	1.0005	1.501	1.222	.872											
36.5	1.049	1.574	1.265	.936											
37.5	1.095	1.643	1.303	1.000											
38	1.118														

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c. $\theta_0 = 30^\circ$.

θ	p = 1	p = 1.5			p = 2			p = 3			p = 5		
	r	r	x	y	r	x	y	r	x	y	r	x	y
31°	.0678	.1017	.0872	.0524	.1356	.1162	.0698	.2034	.1743	.1048	.3390	.2906	.1746
32	.1317	.1976	.1675	.1047	.2634	.2234	.1396	.3951	.3351	.2094	.6585	.5584	.3490
33	.1921	.2882	.2417	.1570	.3842	.3222	.2093	.5763	.4833	.3139	.9605	.8055	.5231
34	.2493	.3740	.3101	.2091	.4986	.4134	.2788	.7479	.6200	.4182	1.2465	1.0334	.6970
35	.3035	.4552	.3729	.2611	.6070	.4972	.3482	.9105	.7458	.5222	1.5175	1.243	.8704
36	.3549	.53235	.4307	.3129	.7098	.5742	.4172	1.0647	.8614	.6258	1.7745	1.436	1.043
37	.4038	.6057	.4837	.3645	.8076	.6450	.4860	1.2114	.9675	.7290			
38	.4503	.6755	.5323	.4159	.9006	.7097	.5545	1.3509	1.0645	.8317			
39	.4946	.7419	.5766	.4669	.9892	.7688	.6225	1.4838	1.1531	.9338			
40	.5368	.8052	.6168	.5176	1.0736	.8224	.6901	1.6104	1.2336	1.035			
41	.5770	.8655	.6532	.5678	1.154	.8709	.7571						
42	.6154	.9231	.6860	.6177	1.2308	.9147	.8236						
43	.6521	.9781	.7153	.6671	1.3042	.9538	.8895						
44	.6871	1.0306	.7414	.7159	1.3742	.9885	.9546						
45.5	.7368	1.1052	.7746	.7883	1.4736	1.0329	1.051						
47	.7833	1.1750	.8014	.8593									
48	.8128	1.2192	.8158	.9060									
50	.8652	1.2978	.8342	.9942									
51	.8902	1.3353	.8403	1.038									

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d. $\theta_0 = 45^\circ$.

θ	p = 1	p = 1.5			p = 2			p = 3			p = 5		
	r	r	x	y	r	x	y	r	x	y	r	x	y
46°	.0343	.0515	.0358	.0370	.0686	.0477	.0493	.1029	.0715	.0740	.1715	.1191	.1234
47	.0675	.1013	.0691	.0740	.1350	.0921	.0987	.2025	.1381	.1481	.3375	.2302	.2468
48	.0995	.1493	.0999	.1110	.1990	.1332	.1479	.2985	.1997	.2218	.4975	.3329	.3697
49	.1306	.1959	.1285	.1478	.2612	.1714	.1971	.3918	.2570	.2957	.6530	.4284	.4928
50	.1607	.2411	.1550	.1847	.3214	.2066	.2462	.4821	.3099	.3693	.8035	.5165	.6155
51	.1898	.2847	.1792	.2213	.3796	.2389	.2950	.5694	.3583	.4425	.9490	.5972	.7375
52	.2181	.3272	.2014	.2578	.4362	.2686	.3437	.6543	.4028	.5156	1.0905	.6714	.8593
53	.2455	.3683	.2217	.2941	.4910	.2955	.3921	.7365	.4432	.5882	1.2275	.7387	.9803
54	.2722	.4083	.2400	.3303	.5444	.3200	.4404	.8166	.4800	.6606	1.361	.8000	1.101
55	.2980	.4470	.2564	.3662	.5960	.3419	.4882	.8940	.5128	.7323			
56	.3232	.4848	.2711	.4019	.6464	.3615	.5359	.9696	.5422	.8038			
57	.3476	.5214	.2840	.4373	.6952	.3786	.5830	1.0428	.5680	.8746			
58	.3713	.5570	.2952	.4724	.7426	.3935	.6298	1.1139	.5903	.9446			
59	.3943	.5915	.3046	.5070	.7886	.4062	.6760	1.1829	.6092	1.014			
60	.4168	.6252	.3126	.5414	.8336	.4168	.7219						
62	.4598	.6897	.3238	.6090	.9196	.4317	.8120						
63	.4804	.7206	.3271	.6421	.9608	.4362	.8561						
65	.5200	.7800	.3296	.7069	1.0400	.4395	.9426						
66	.5390	.8084	.3288	.7385	1.0779	.4384	.9847						
67	.5574	.8361	.3267	.7696	1.1148	.4356	1.026						
70	.6100	.9150	.3129	.8598									
71	.6259	.9388	.3056	.8877									
74	.6723	1.0085	.2780	.9694									
75	.6869	1.0304	.2667	.9952									

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3. Computation of Point of Contact.

The y-coordinate y_c of the point of contact is equal to 1. The x-coordinate x_c may be computed by interpolation or taken from the graph of the trajectory. The following results were obtained:

$$\theta_o = 15^\circ; \quad p = 1.5; \quad x_c = 2.149; \quad \theta_c = \operatorname{arccot} 2.149 = 24.96^\circ$$

$$p = 2; \quad x_c = 2.424; \quad \theta_c = \operatorname{arccot} 2.424 = 22.42^\circ$$

$$p = 3; \quad x_c = 2.767; \quad \theta_c = \operatorname{arccot} 2.767 = 19.87^\circ$$

$$p = 5; \quad x_c = 3.071; \quad \theta_c = \operatorname{arccot} 3.071 = 18.04^\circ$$

$$\theta_o = 22.5^\circ; \quad p = 1.5; \quad x_c = 1.303; \quad \theta_c = 37.5^\circ$$

$$p = 2; \quad x_c = 1.503; \quad \theta_c = 33.63^\circ$$

$$p = 3; \quad x_c = 1.742; \quad \theta_c = 29.86^\circ$$

$$p = 5; \quad x_c = 1.971; \quad \theta_c = 26.9^\circ$$

$$\theta_o = 30^\circ; \quad p = 1.5; \quad x_c = .8351; \quad \theta_c = 50.13^\circ$$

$$p = 2; \quad x_c = 1.010; \quad \theta_c = 44.71^\circ$$

$$p = 3; \quad x_c = 1.207; \quad \theta_c = 39.65^\circ$$

$$p = 5; \quad x_c = 1.389; \quad \theta_c = 35.75^\circ$$

$$\theta_o = 45^\circ; \quad p = 1.5; \quad x_c = .2646; \quad \theta_c = 75.18^\circ$$

$$p = 2; \quad x_c = .436; \quad \theta_c = 66.44^\circ$$

$$p = 3; \quad x_c = .605; \quad \theta_c = 58.82^\circ$$

$$p = 5; \quad x_c = .748; \quad \theta_c = 53.2^\circ$$

4. Turning Rates.

Here $y = d\phi/dt$ and $t = \cot \theta_o - \cot \theta$; y' and t' are obtained by multiplying y and t by $10/y_c$ and $10/t_c$ respectively.

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$$\theta_0 = 15^\circ, p = 1.5$$

θ	y	t	y'	t'
15°	.0670	0	2.299	0
16	.0809	.2447	2.777	1.541
17	.0965	.4612	3.312	2.905
18	.1139	.6544	3.908	4.122
19	.1331	.8279	4.567	5.215
20	.1542	.9846	5.291	6.202
22	.2023	1.2570	6.943	7.918
23	.2294	1.3762	7.874	8.668
25	.2914	1.5876	10.000	10.000

$$\theta_0 = 22.5^\circ, p = 1.5$$

θ	y	t	y'	t'
22.5°	.14644	0	2.5029	0
23.5	.16565	.1144	2.8312	1.0297
24.5	.18625	.2199	3.1833	1.9793
25.5	.20623	.3177	3.5248	2.8596
26.5	.23164	.4085	3.9590	3.6769
27.5	.25646	.4932	4.3833	4.4392
28.5	.28268	.5724	4.8314	5.1521
29.5	.31031	.6467	5.3036	5.8209
30.5	.33935	.7165	5.8000	6.4491
31.5	.37077	.7823	6.3370	7.0414
32.5	.40176	.8445	6.8666	7.6013
33.5	.43522	.9034	7.4385	8.1314
34.5	.47008	.9592	8.0343	8.6337
35.5	.50666	1.0123	8.6595	9.1116
36.5	.54498	1.0628	9.3145	9.5662
37.5	.58509	1.1110	10.0000	10.0000

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$$\theta_0 = 30^\circ, p = 1.5$$

θ	y	t	y'	t'
30°	.25000	0	2.8340	0
31	.27004	.0678	3.0612	.7558
32	.29745	.1318	3.3719	1.4693
33	.32270	.1922	3.6581	2.1426
34	.34894	.2495	3.9556	2.7814
36	.40425	.3557	4.5826	3.9653
38	.46308	.4522	5.2495	5.0411
40	.52519	.5403	5.9536	6.0233
43	.62387	.6597	7.0722	7.3543
45.5	.71086	.7494	8.0583	8.3543
48	.80206	.8317	9.0922	9.2718
50.8	.88211	.8970	10.0000	10.0000

$$\theta_0 = 45^\circ; p = 1.5$$

θ	y	t	y'	t'
45°	.50000	0	4.374	0
48	.57964	.09960	5.071	1.354
51	.66039	.19022	5.777	2.587
54	.74060	.27346	6.479	3.719
57	.81857	.35059	7.161	4.767
60	.89250	.42265	7.808	5.747
63	.96073	.49047	8.405	6.669
67	1.04002	.57553	9.099	7.826
71	1.10184	.65567	9.640	8.916
75.18	1.14304	.7354	10.000	10.000

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5. Polynomial Approximations to Turning Rates.

$$\theta_0 = 15^\circ$$

$$p = 1.5: y = 2.30 + .3486t - .01262t^2 + .005476t^3$$

$$p = 2: y = 3.13 + .3958t - .01206t^2 + .004118t^3$$

$$p = 3: y = 4.427 + .4079t - .008978t^2 + .002368t^3$$

$$p = 5: y = 5.823 + .3675t - .005540t^2 + .001001t^3$$

$$\theta_0 = 22.5^\circ$$

$$p = 1.5: y = 2.50 + .30595t + .0098810t^2 + .0034524t^3$$

$$p = 2: y = 3.3157 + .33280t + .0088151t^2 + .0025261t^3$$

$$p = 3: y = 4.5372 + .33594t + .0065647t^2 + .0013878t^3$$

$$p = 5: y = 6.0533 + .29552t + .0038074t^2 + .00053068t^3$$

$$\theta_0 = 30^\circ$$

$$p = 1.5: y = 2.834 + .33970t + .016691t^2 + .0021t^3$$

$$p = 2: y = 3.6567 + .35266t + .013942t^2 + .0014114t^3$$

$$p = 3: y = 4.8527 + .34074t + .0098073t^2 + .00072285t^3$$

$$p = 5: y = 6.3048 + .28899t + .0054297t^2 + .00026125t^3$$

$$\theta_0 = 45^\circ$$

$$p = 1.5: y = 4.37 + .42014t + .055715t^2 - .0041429t^3$$

$$p = 2: y = 4.8502 + .35766t + .036378t^2 - .0020747t^3$$

$$p = 3: y = 5.7652 + .29765t + .021195t^2 - .00084636t^3$$

$$p = 5: y = 6.9365 + .22874t + .010404t^2 - .00026537t^3$$

6. Frequency Spectra.

Absolute Magnitude of Frequency Components

$$\theta_0 = 15^\circ$$

ω	$p = 1.5$	$p = 2$	$p = 3$	$p = 5$
0	49.913	57.365	67.596	77.257
$\pi/25$	46.950	53.915	63.348	72.346
$3\pi/50$	43.427	49.716	58.278	66.480
$2\pi/25$	38.822	44.210	51.614	58.760
$\pi/10$	33.445	37.812	43.760	49.637
$2\pi/10$	11.484	10.329	8.368	6.336
$4\pi/10$	6.027	5.418	4.357	3.252
π	2.444	2.197	1.763	1.310
2π	1.222	1.098	.881	.655
4π	.613	.550	.442	.328
10π	.245	.220	.177	.131

$$\theta_0 = 22.5^\circ$$

0	52.222	59.050	67.827	77.904
$\pi/25$	49.120	55.426	63.556	72.930
$3\pi/50$	45.430	51.109	58.459	66.988
$2\pi/25$	40.602	45.446	51.760	59.167
$\pi/10$	34.957	38.800	43.864	49.926
$2\pi/10$	11.533	10.382	8.380	6.055
$4\pi/10$	5.916	5.317	4.272	3.065
π	2.384	2.141	1.718	1.230
2π	1.191	1.070	.859	.615
4π	.597	.536	.430	.308
10π	.239	.214	.172	.123

$$\theta_0 = 30^\circ$$

0	56.138	62.376	70.639	79.960
$\pi/25$	52.777	58.535	66.189	74.853
$3\pi/50$	48.774	53.955	60.877	68.753
$2\pi/25$	43.529	47.942	53.891	60.723
$\pi/10$	37.381	40.871	45.651	51.232
$2\pi/10$	11.167	9.898	8.028	5.835
$4\pi/10$	5.672	5.016	4.054	2.934
π	2.279	2.014	1.626	1.176
2π	1.139	1.007	.813	.588
4π	.570	.504	.407	.294
10π	.228	.202	.163	.118

$$\theta_0 = 45^\circ$$

0	72.921	73.323	77.483	83.607
$\pi/25$	68.516	68.782	72.580	78.248
$3\pi/50$	63.255	63.360	66.724	71.846
$2\pi/25$	56.336	56.227	59.017	63.415
$\pi/10$	48.171	47.811	49.914	53.447
$2\pi/10$	9.968	8.686	6.981	4.949
$4\pi/10$	4.606	4.153	3.409	2.447
π	1.800	1.640	1.354	.975
2π	.899	.820	.677	.488
4π	.448	.409	.338	.244
10π	.179	.164	.135	.097

* * *

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 To *Unclassified*
 By Authority Of *1505 Memo 505-407/54 of 24 May '50*

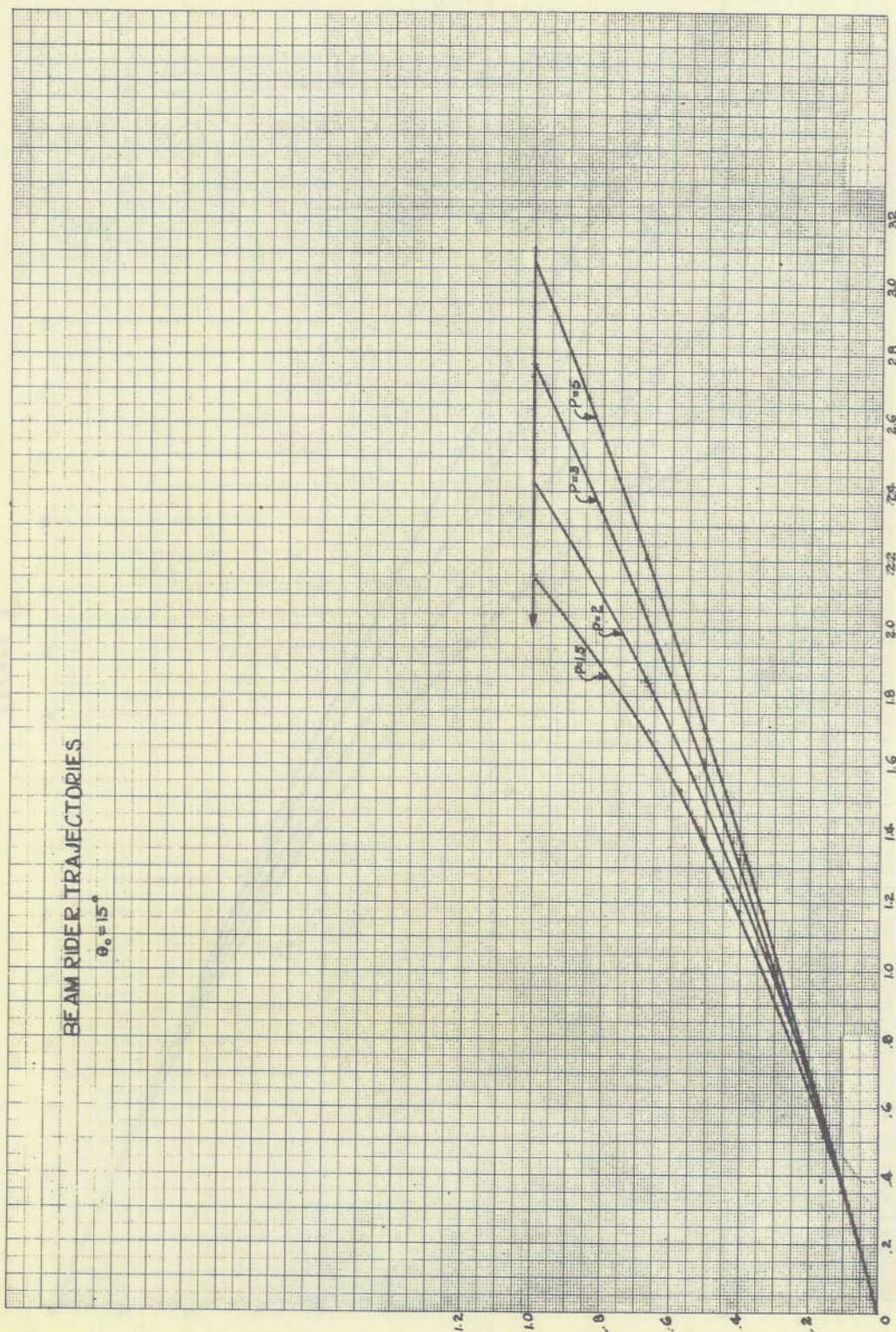


Figure 2

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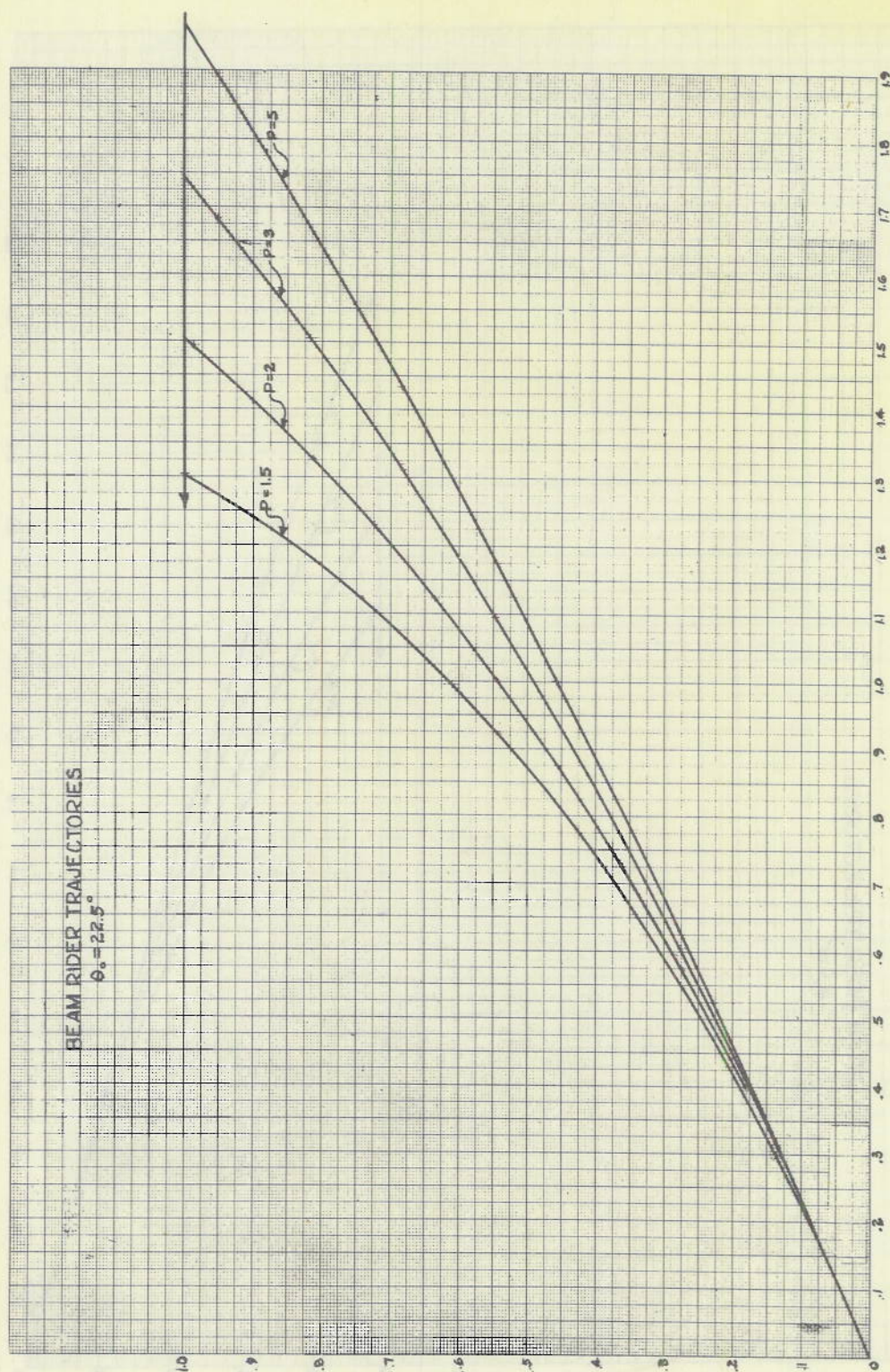


Figure 3

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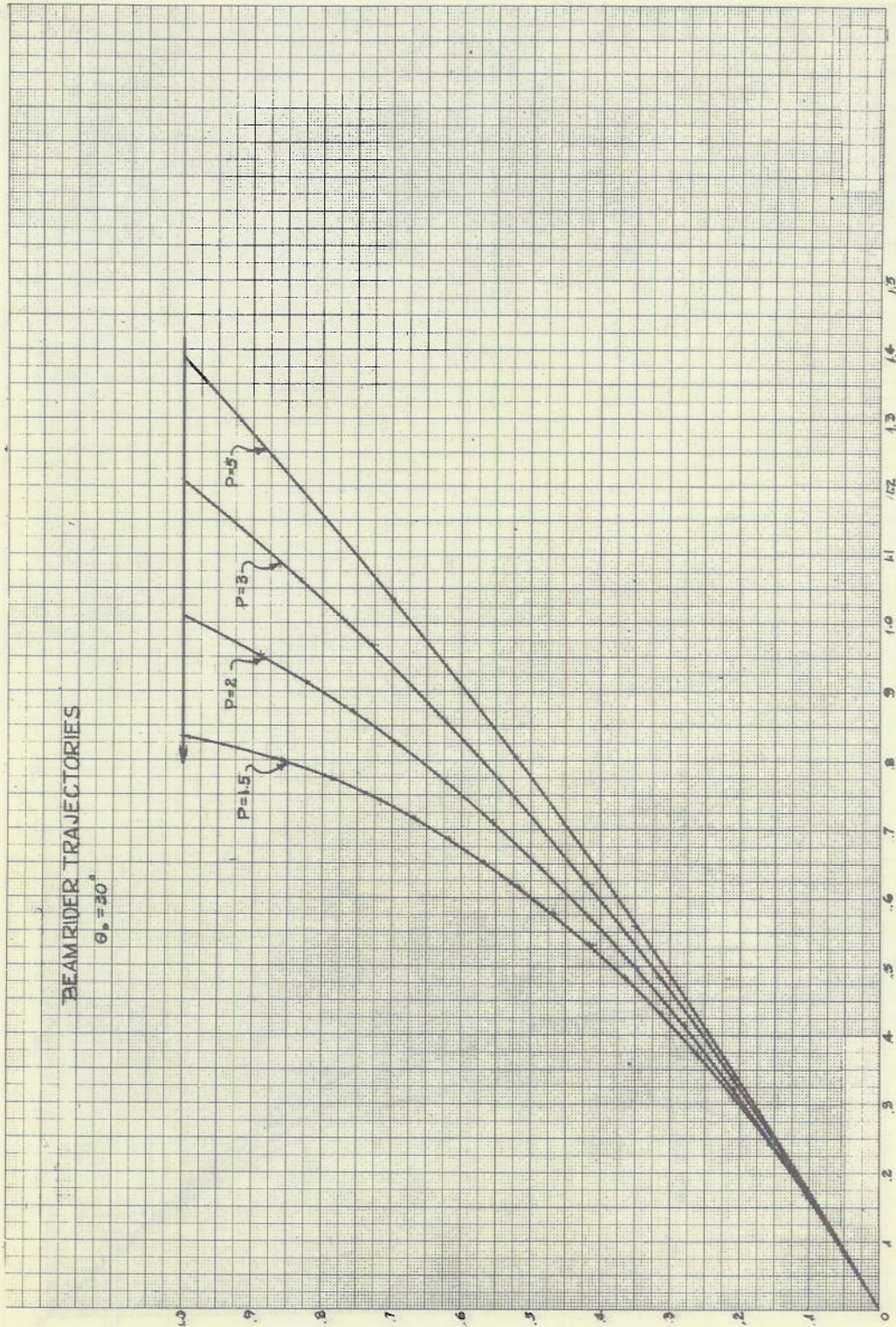


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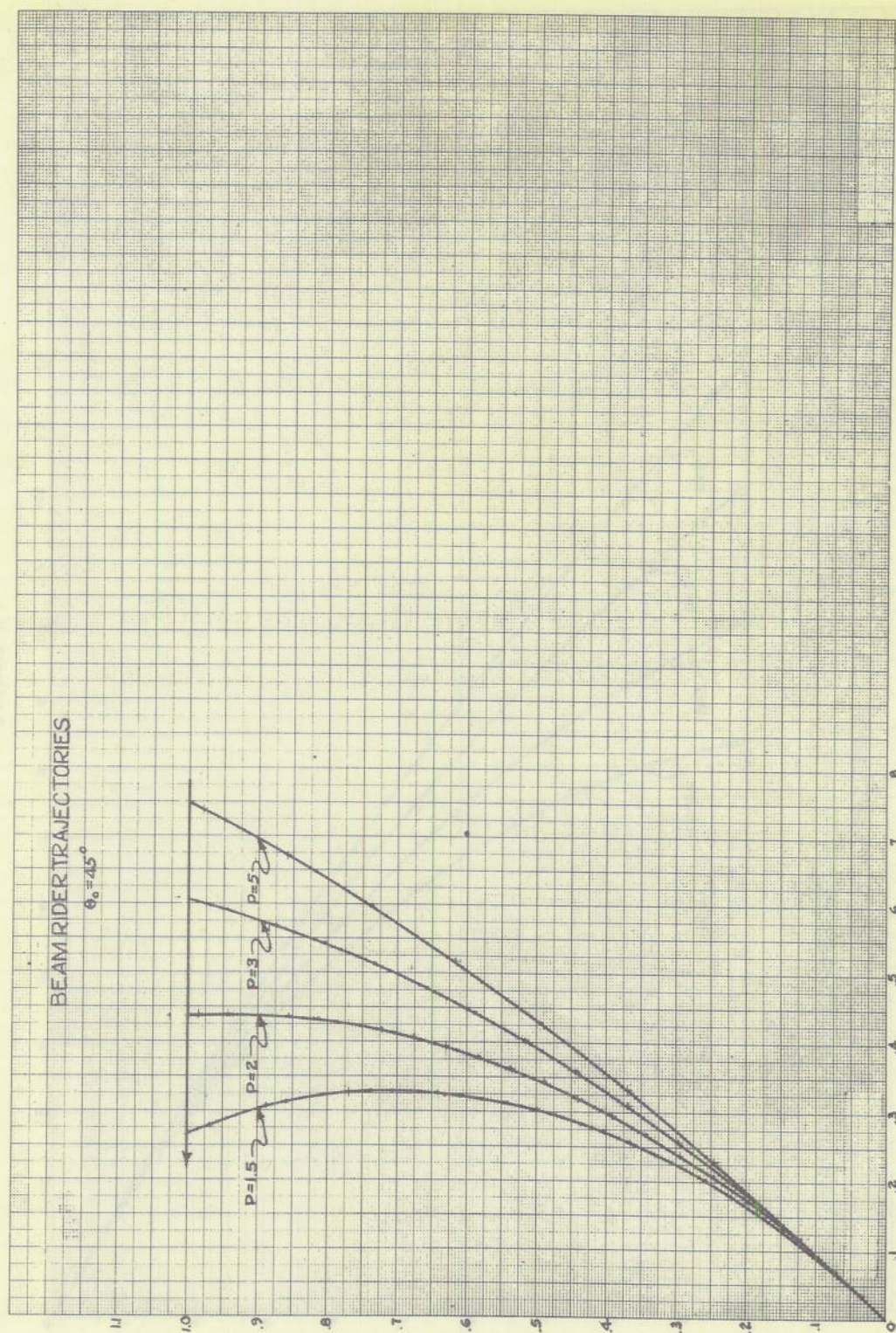


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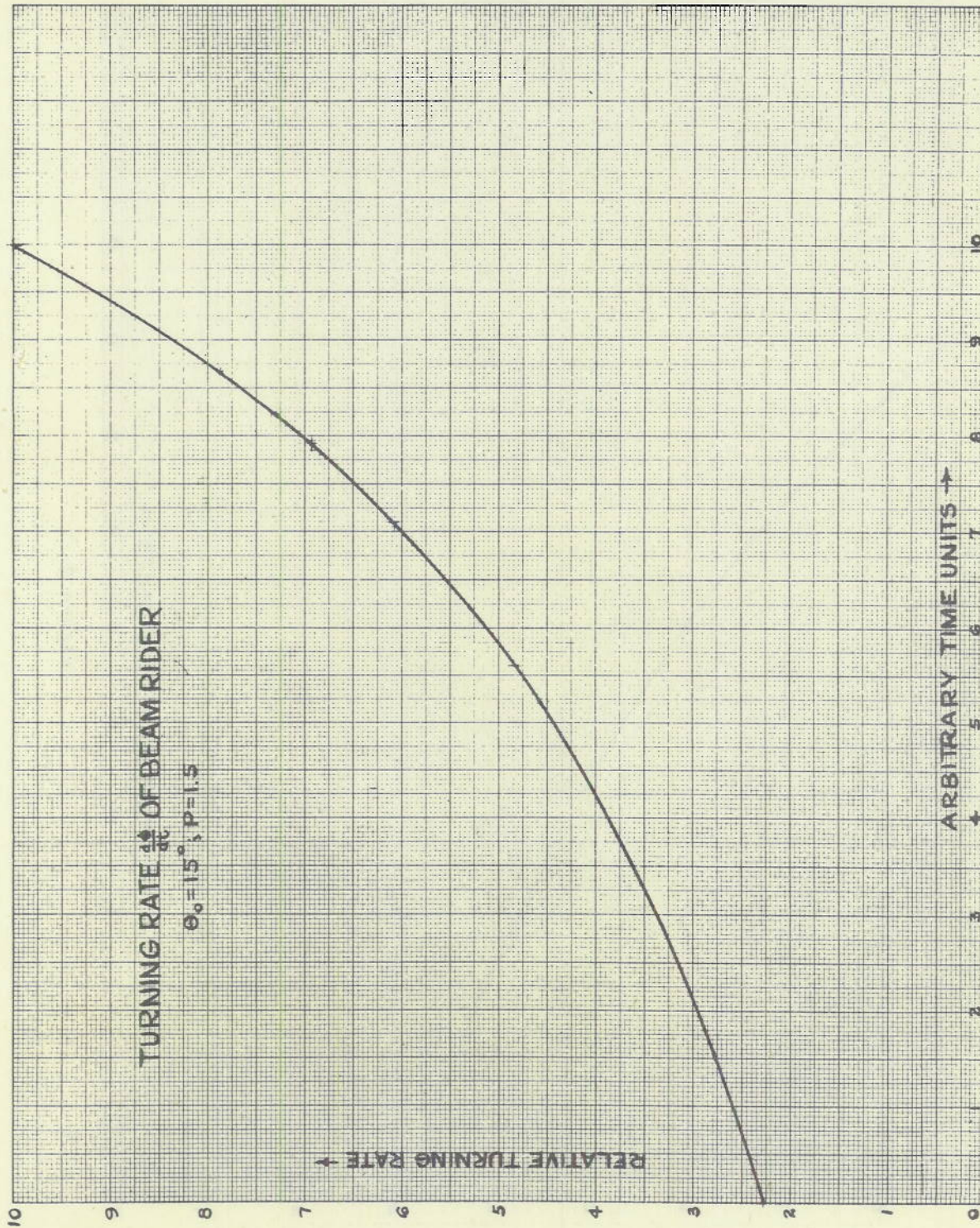


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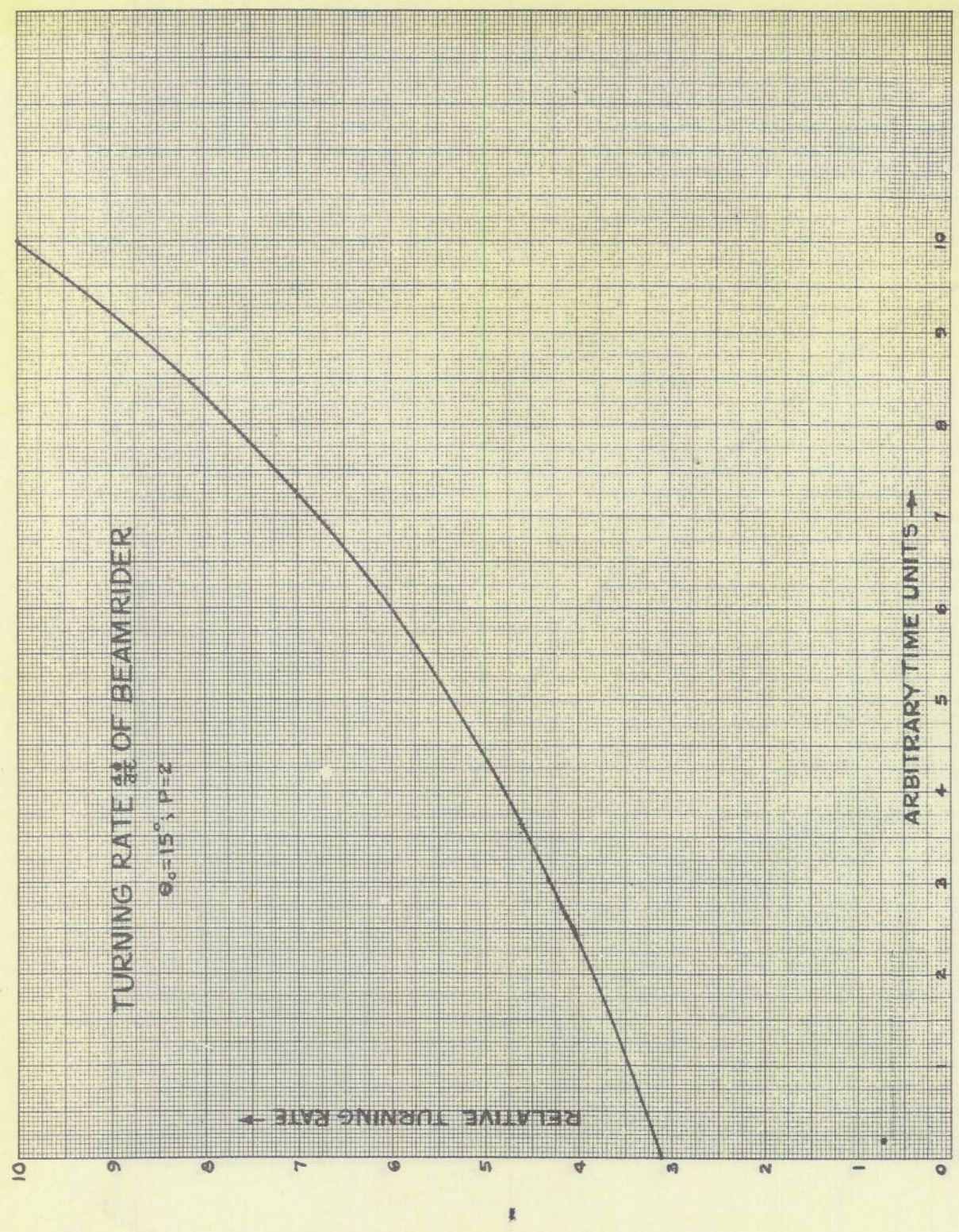


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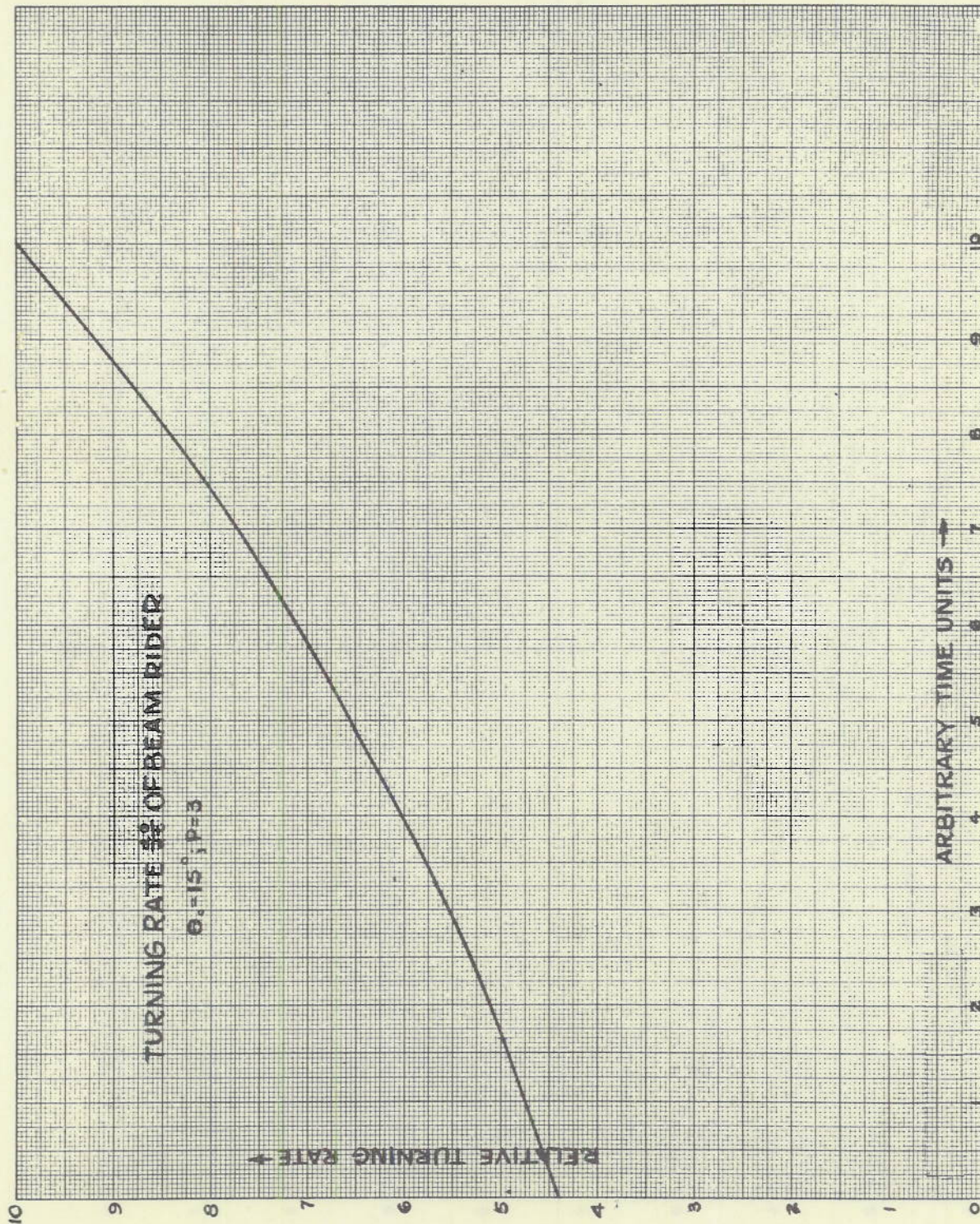


Figure 8

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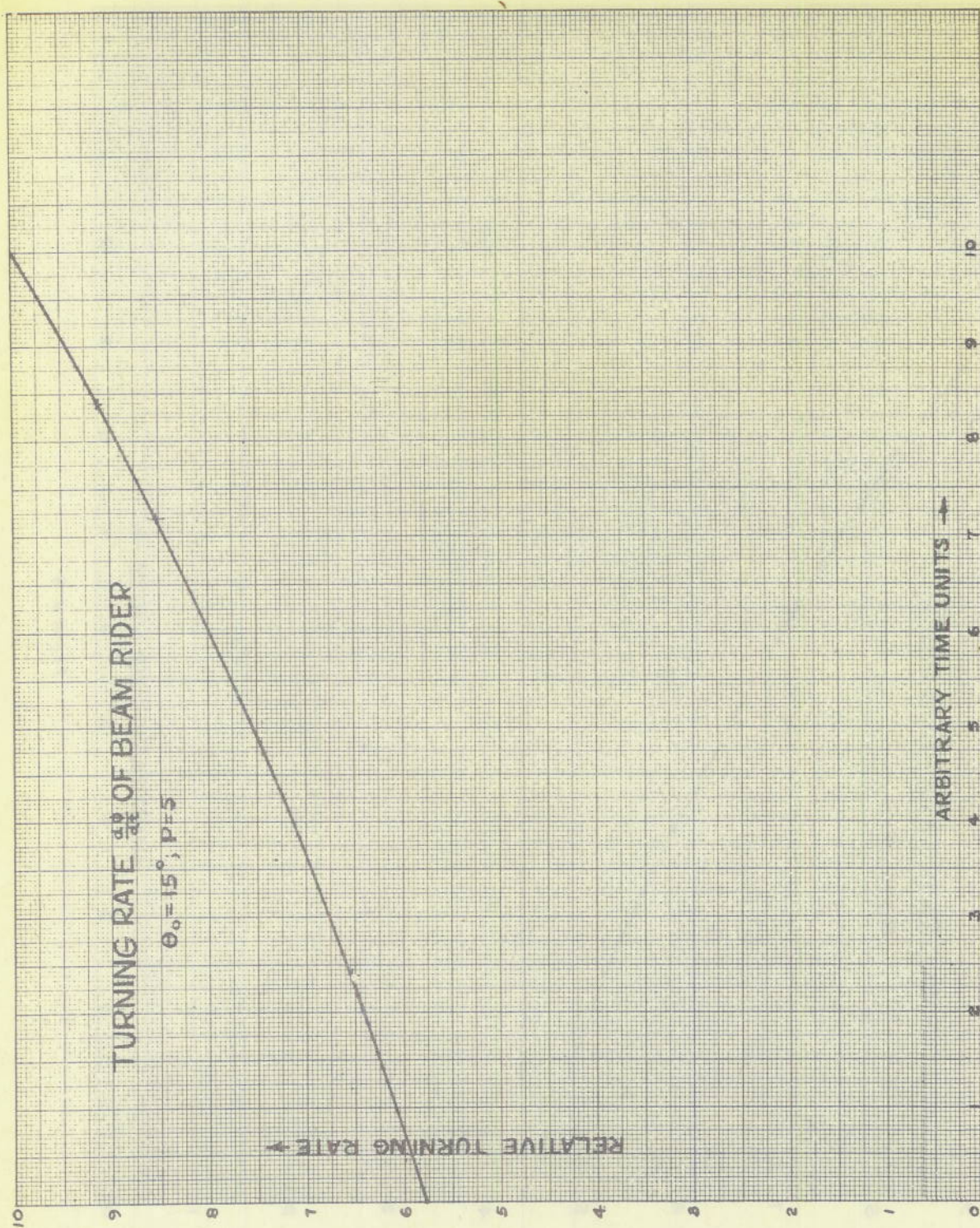


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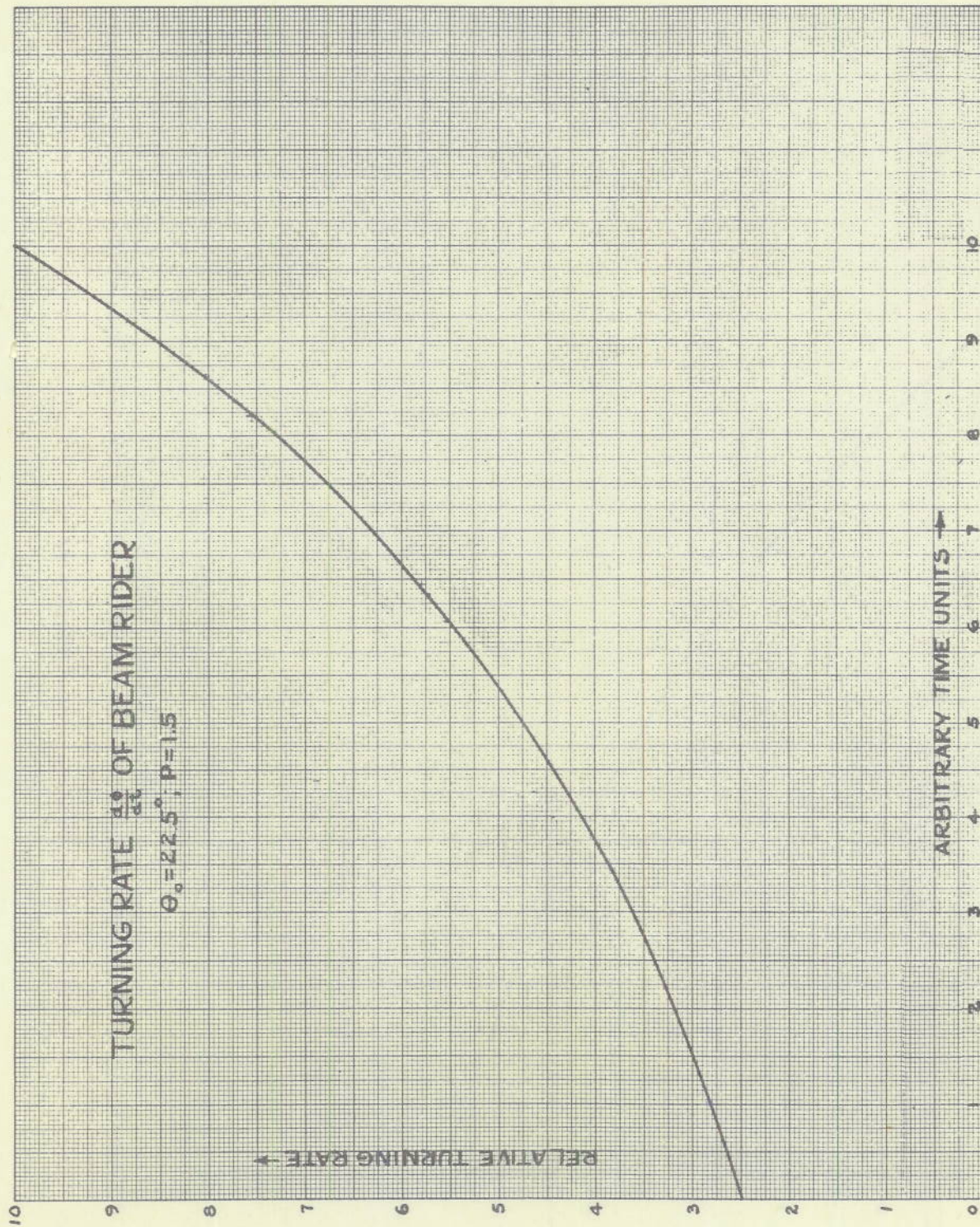


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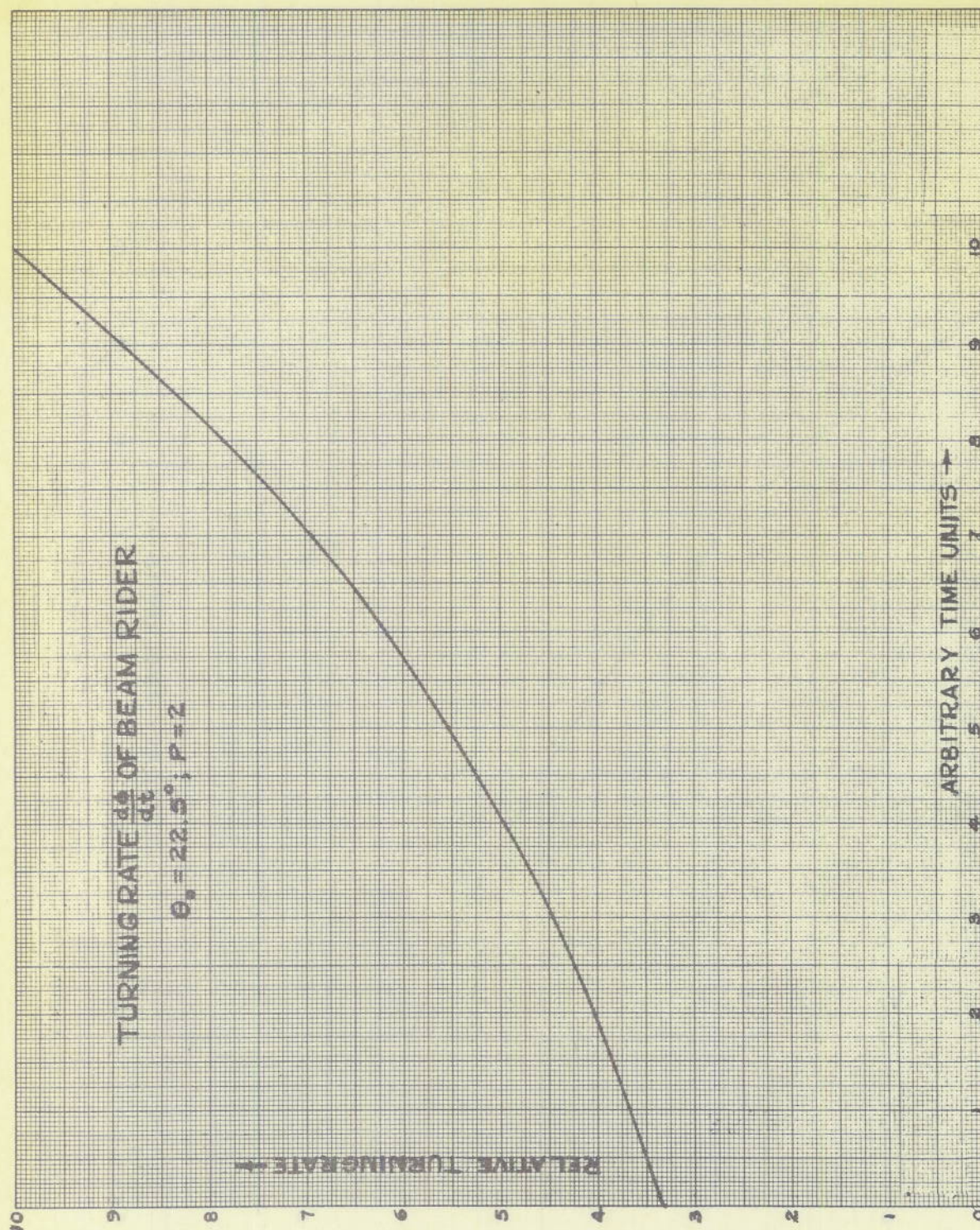


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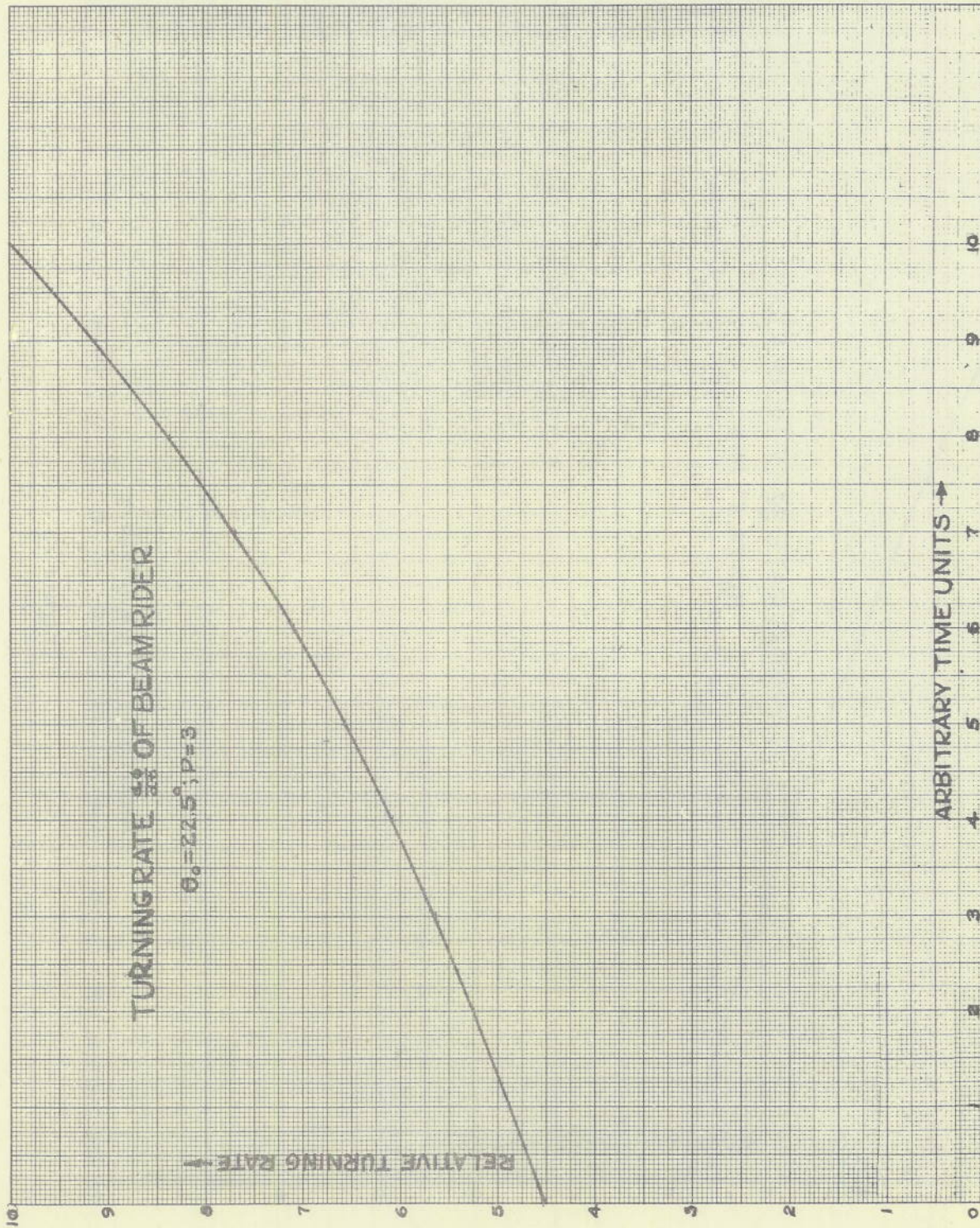


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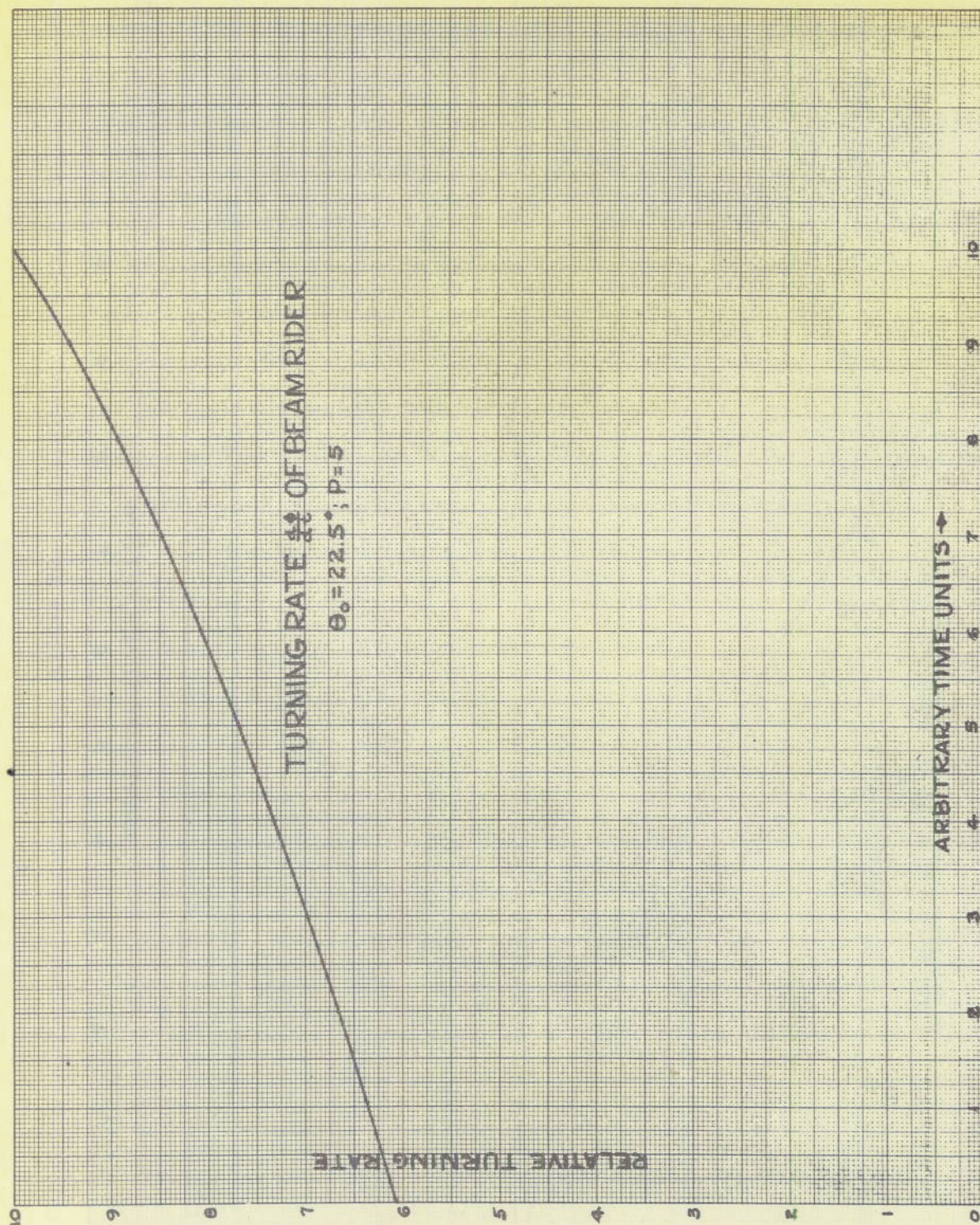


Figure 13

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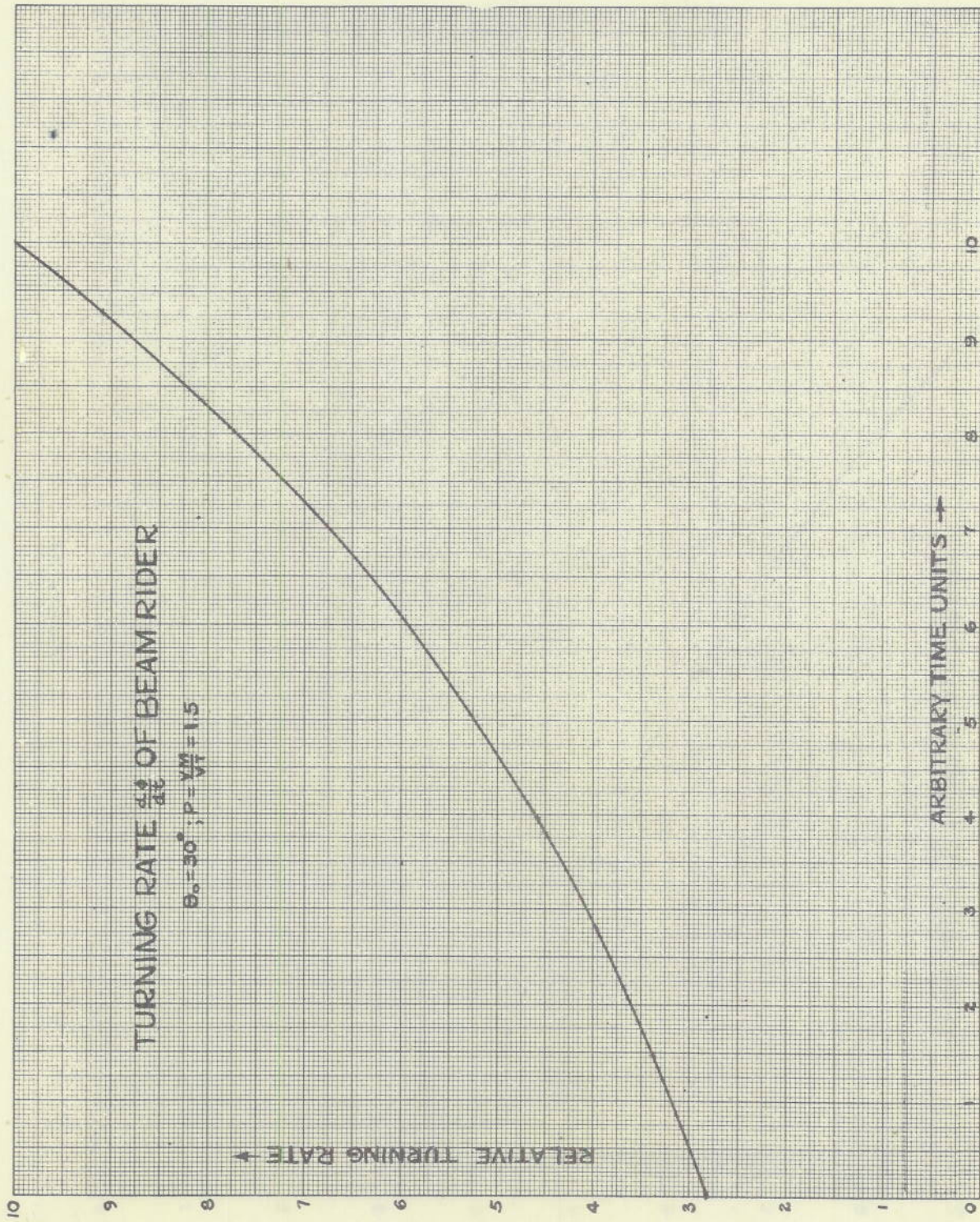


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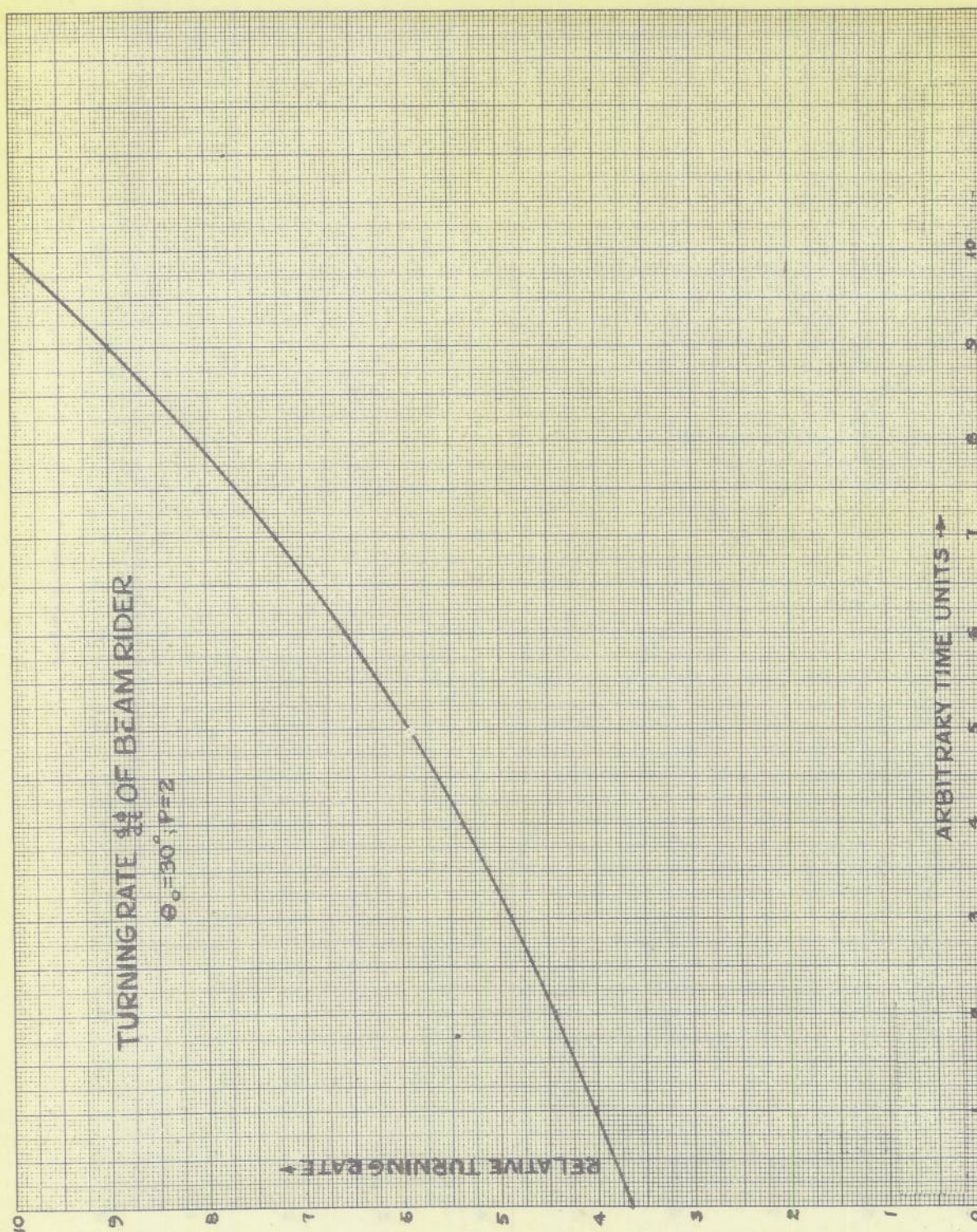


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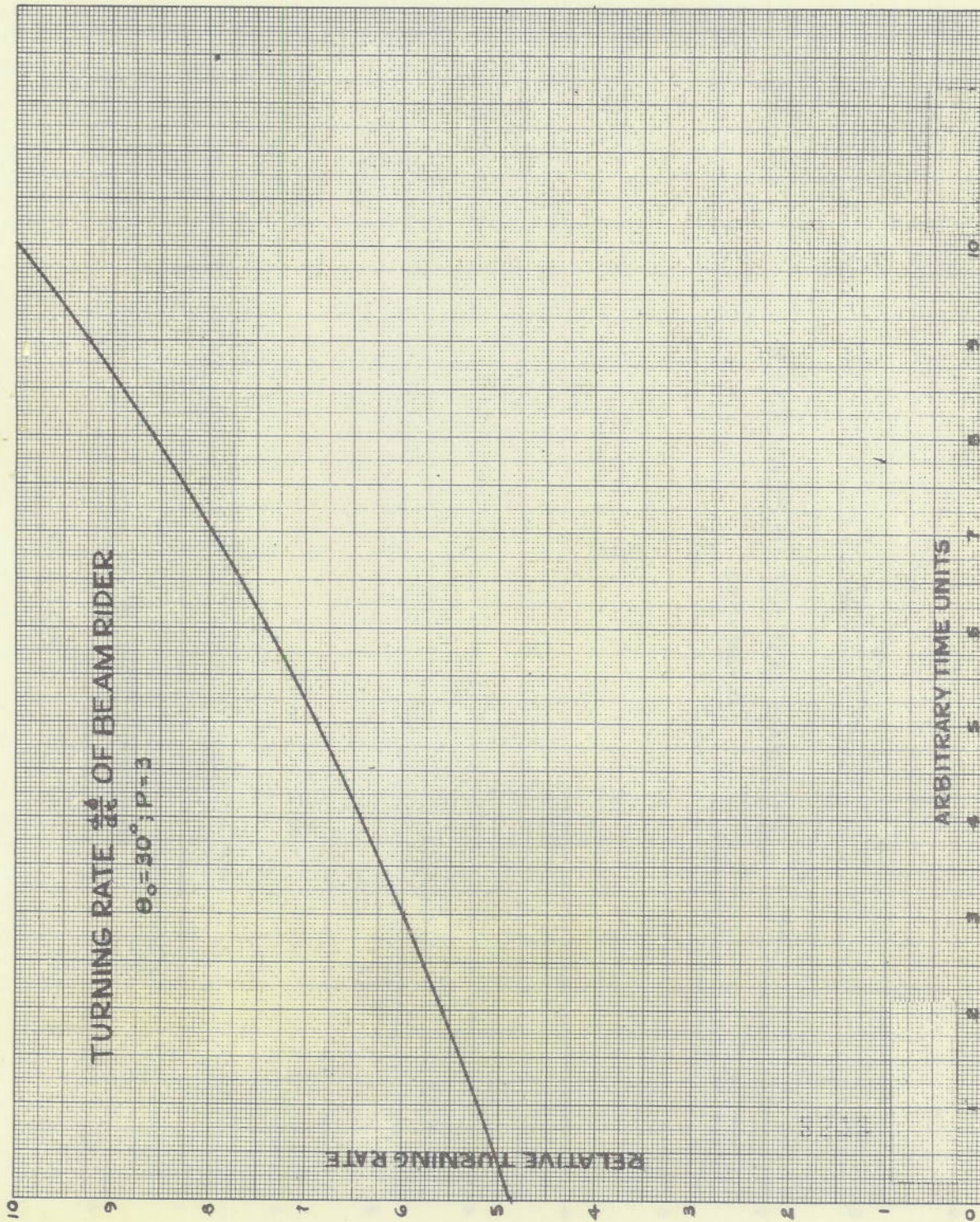


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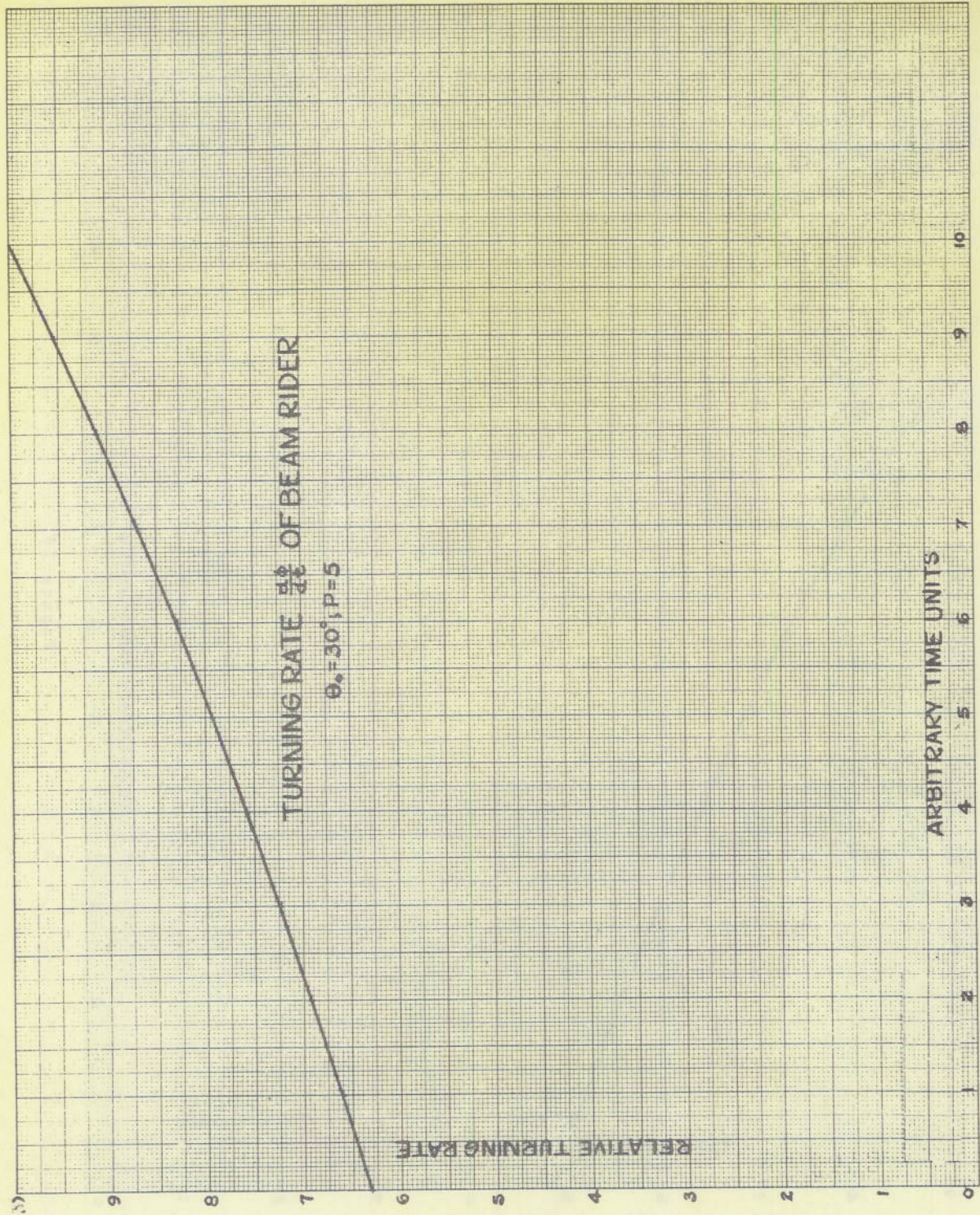


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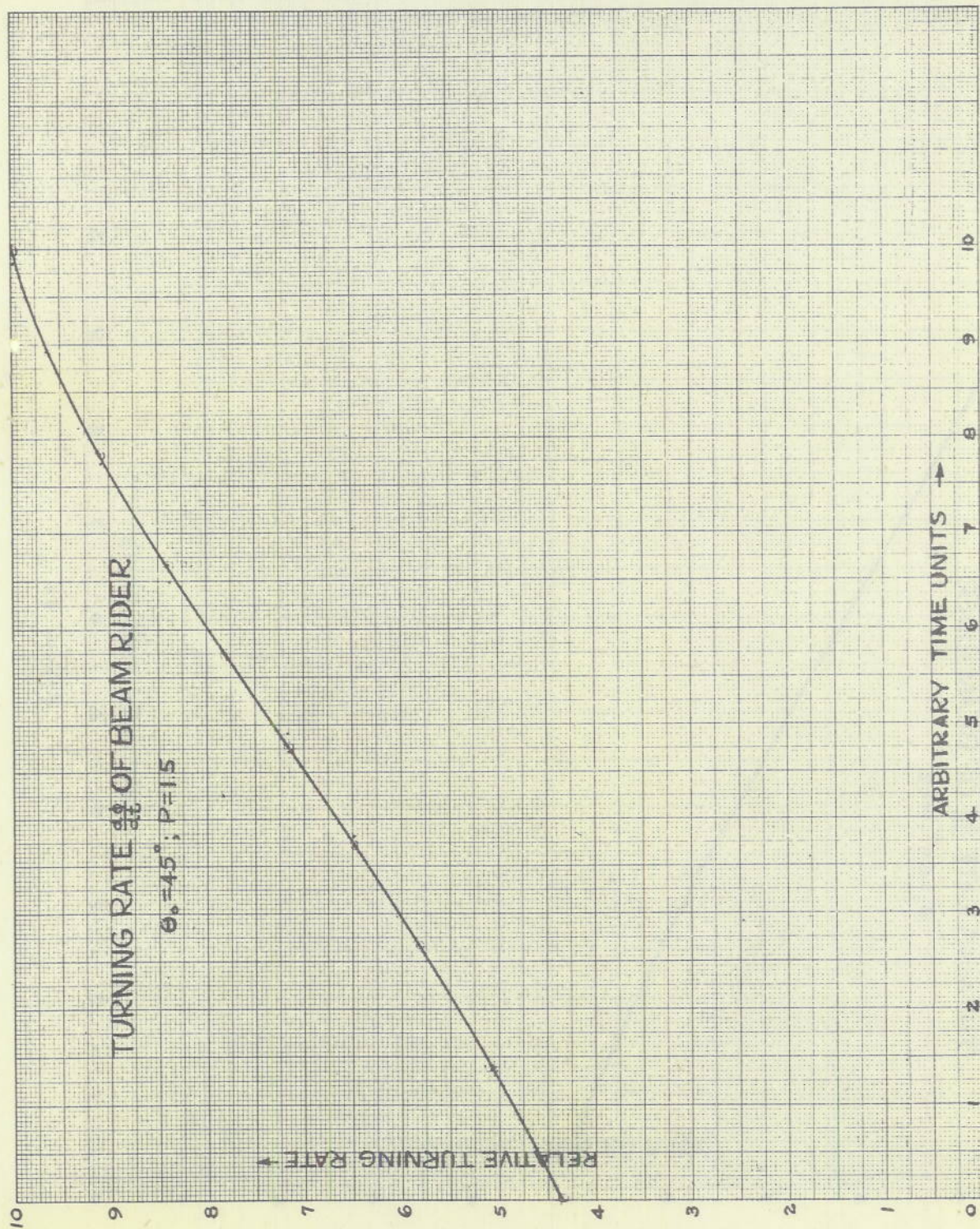


Figure 18

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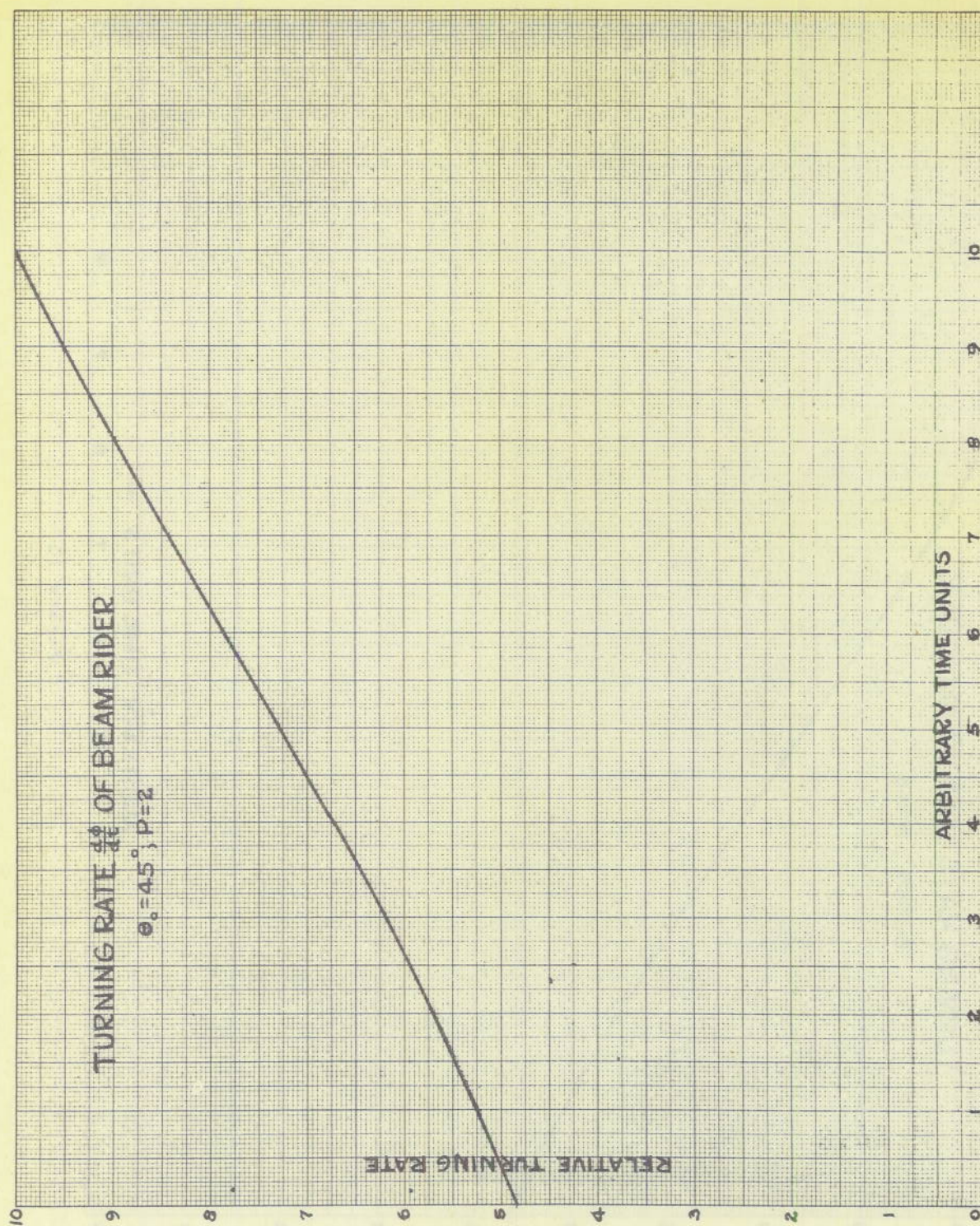


Figure 19

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51

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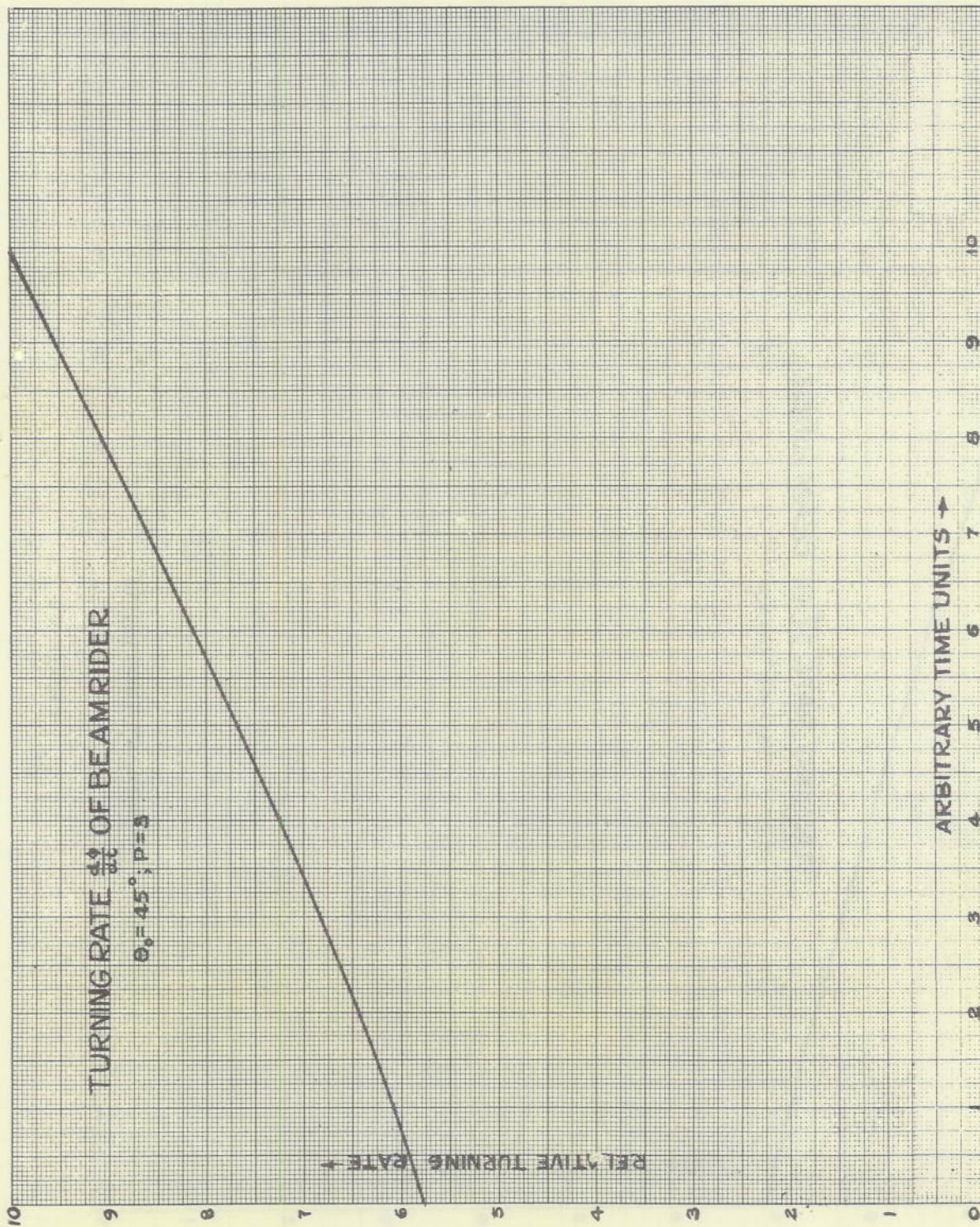


Figure 20

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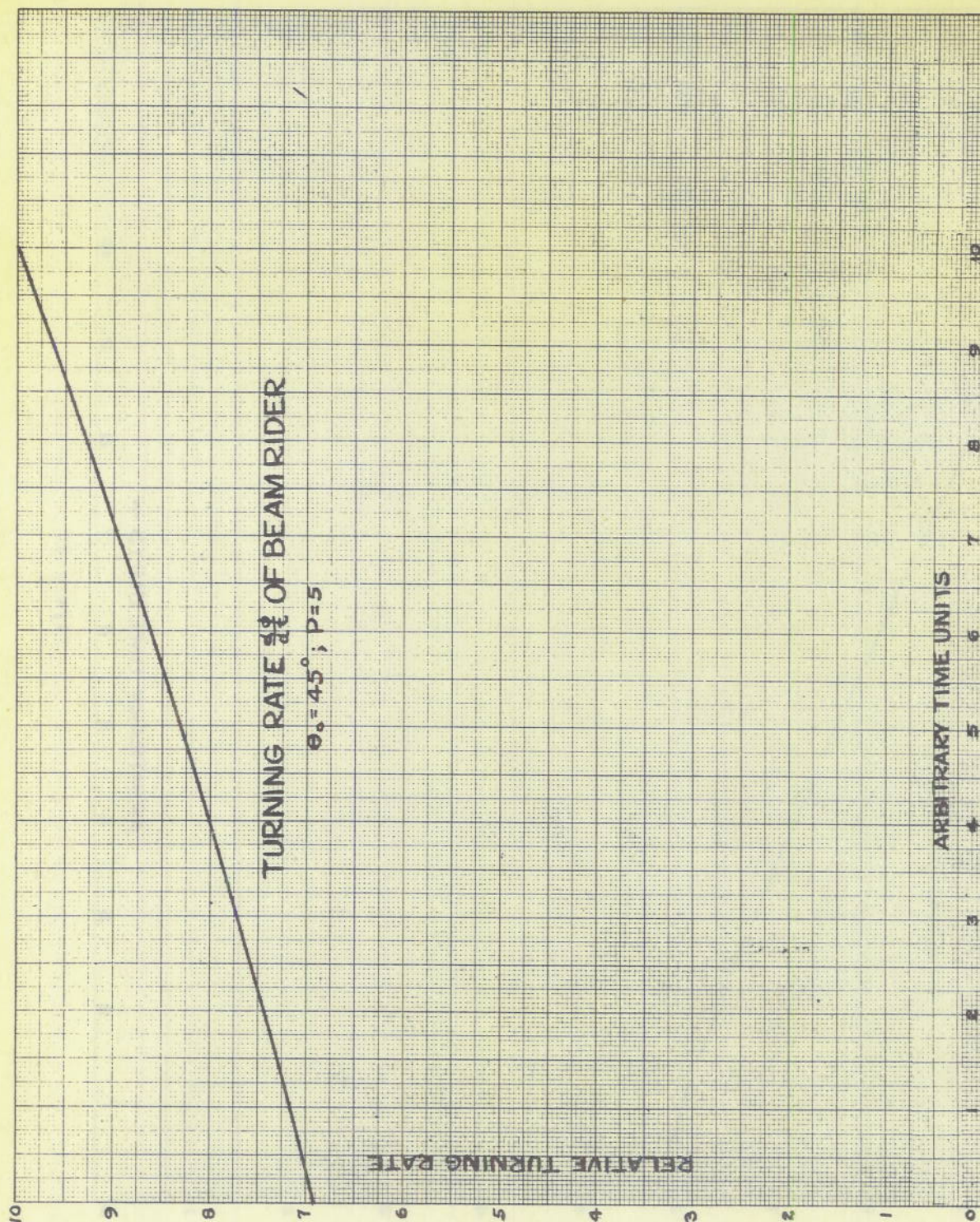


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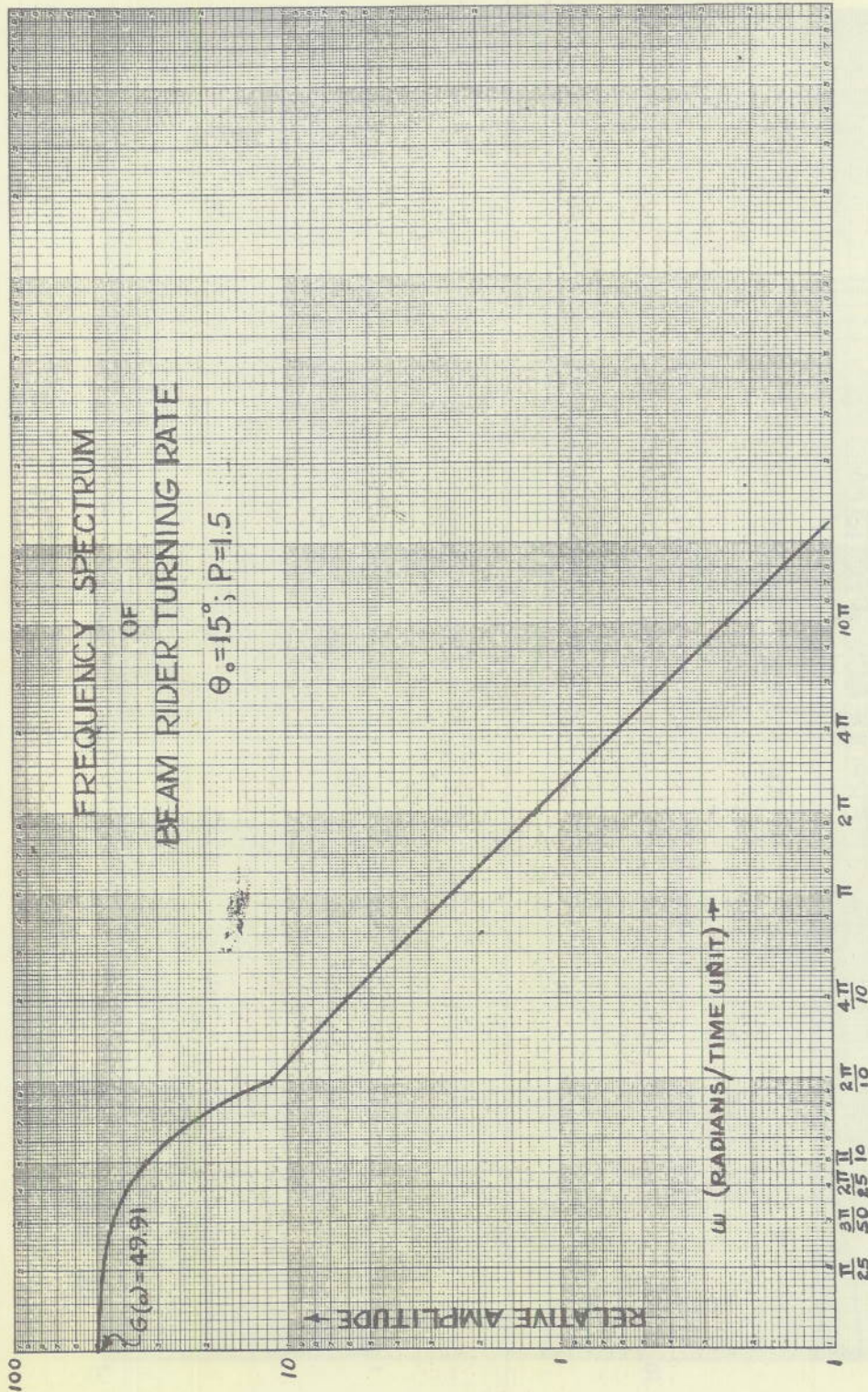
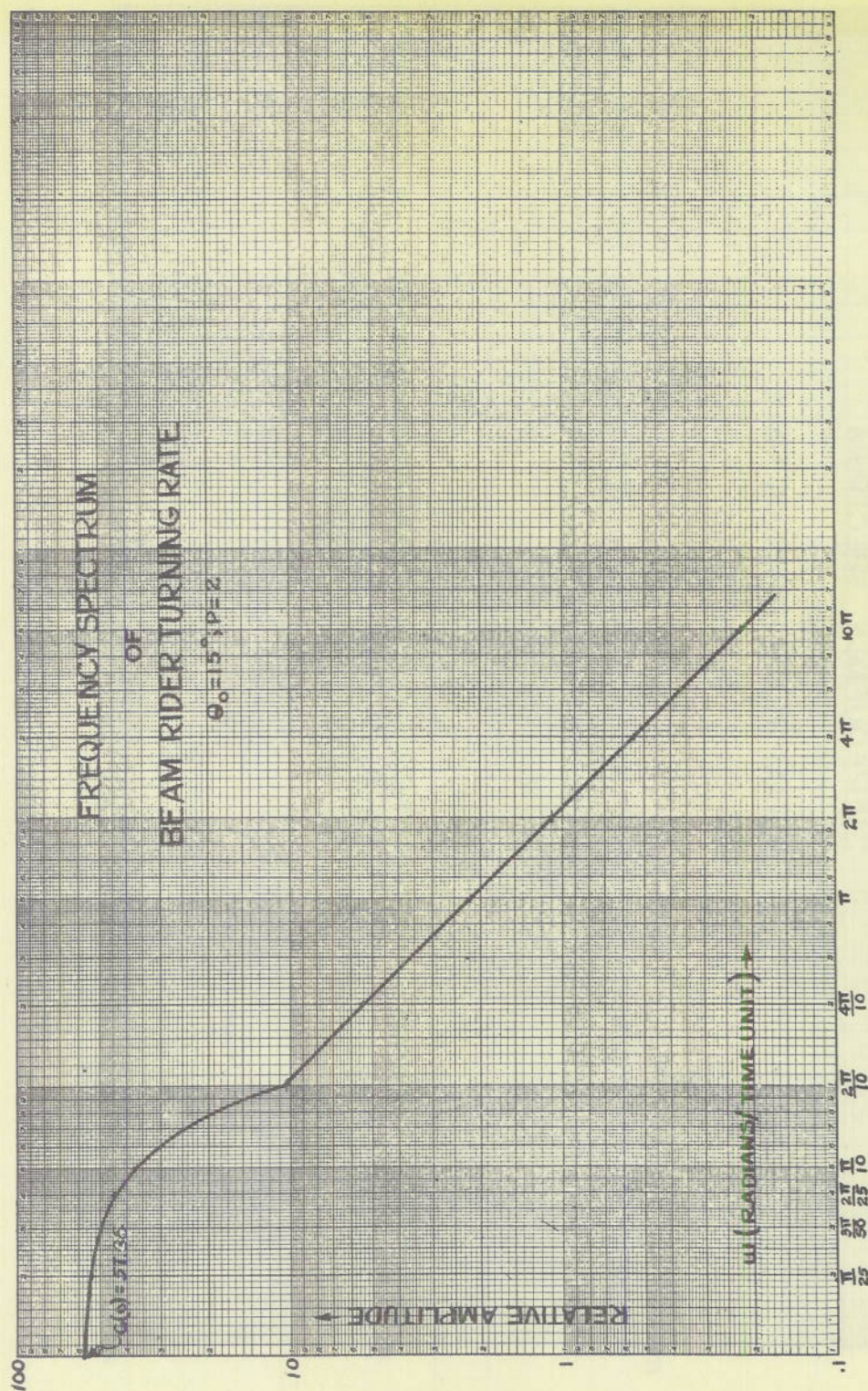


Figure 22

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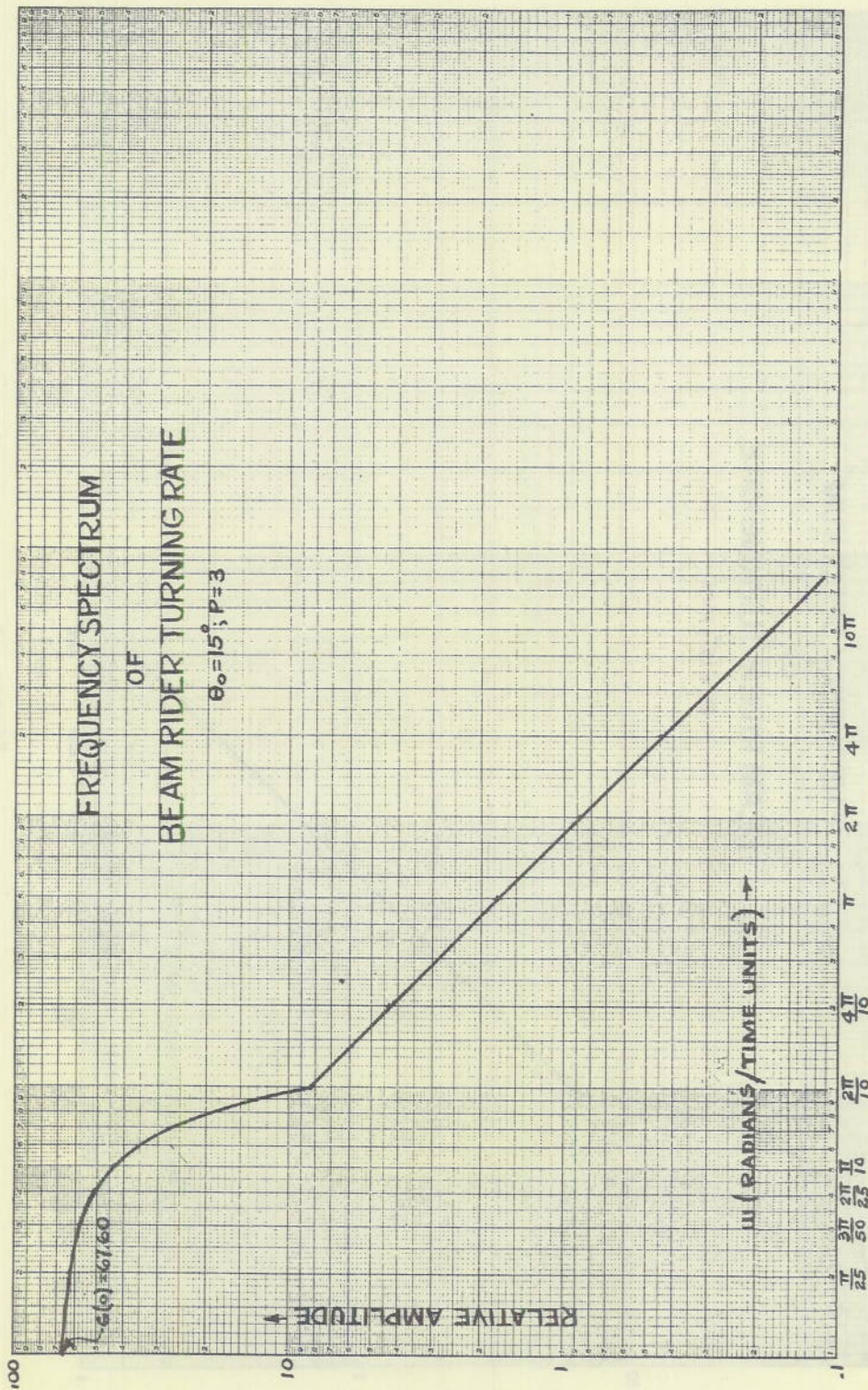


Figure 24

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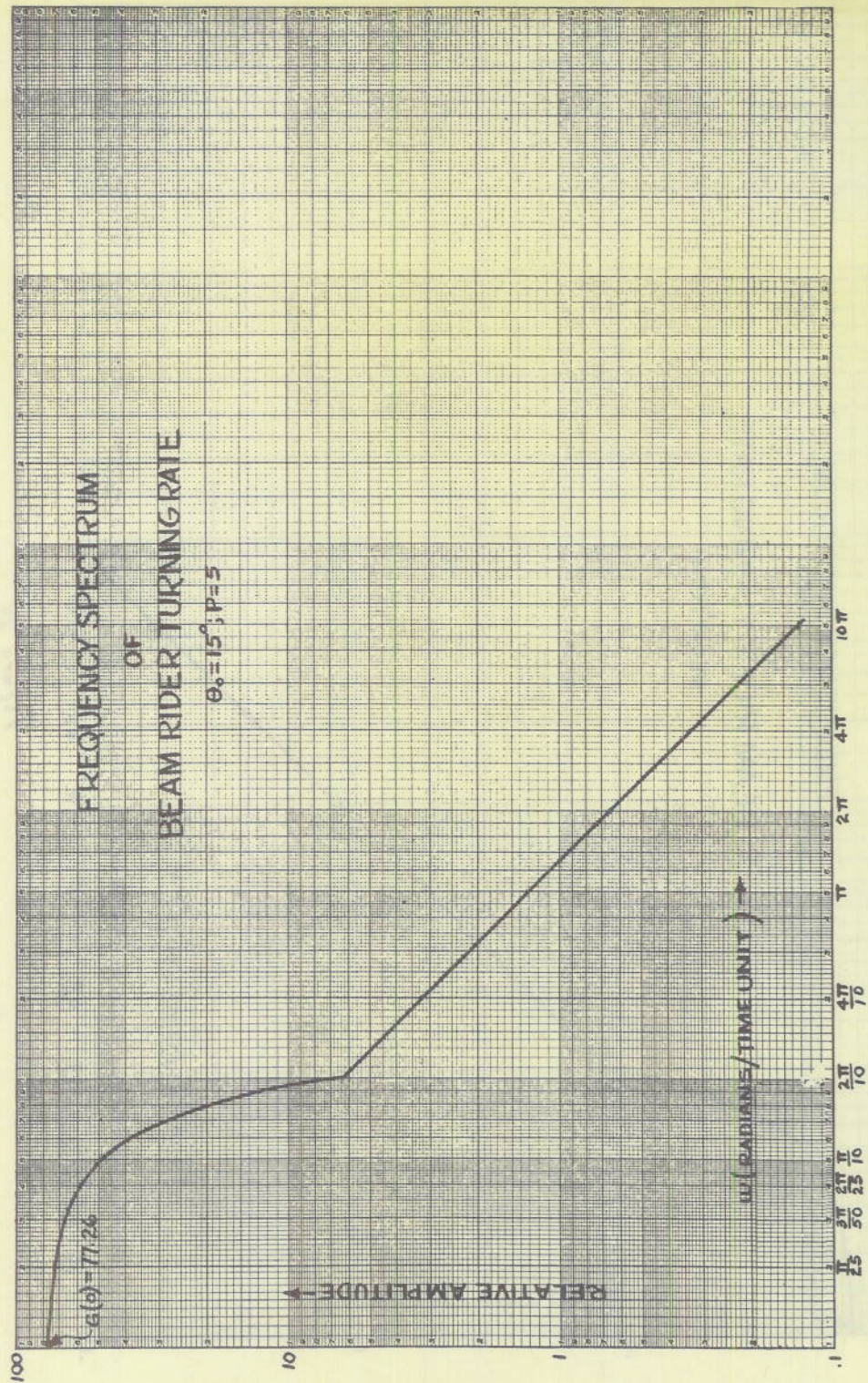


Figure 25

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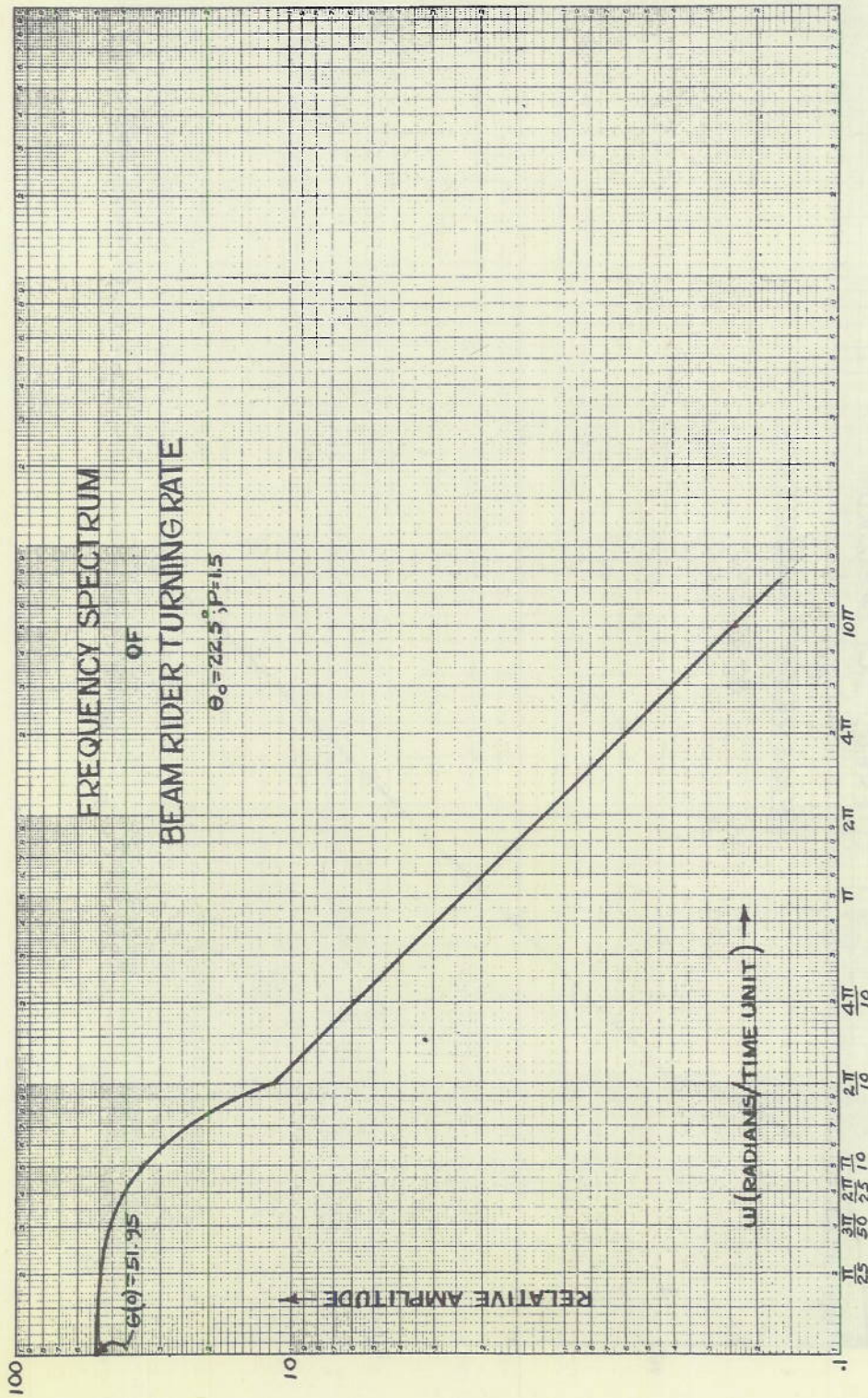


Figure 26

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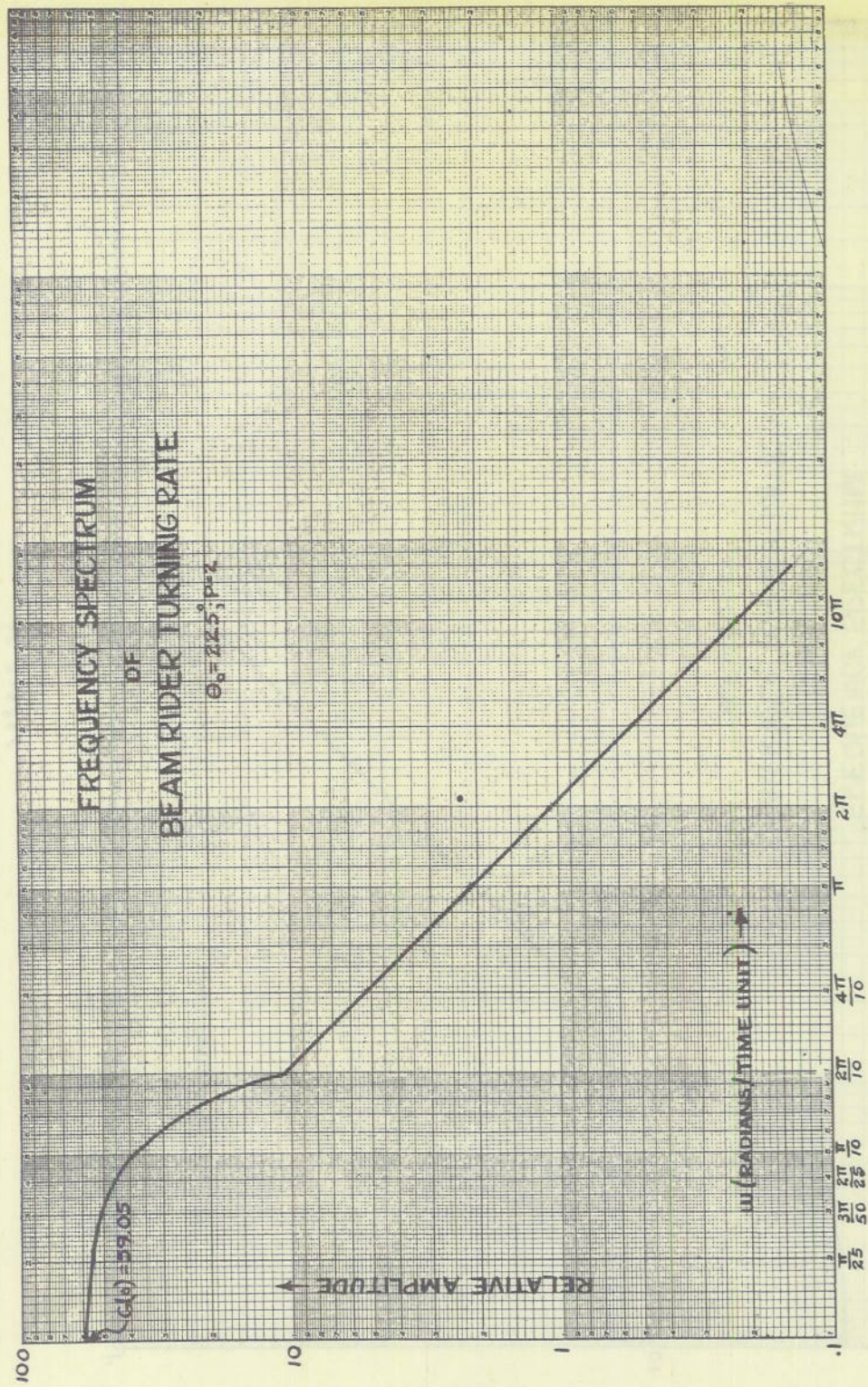


Figure 27

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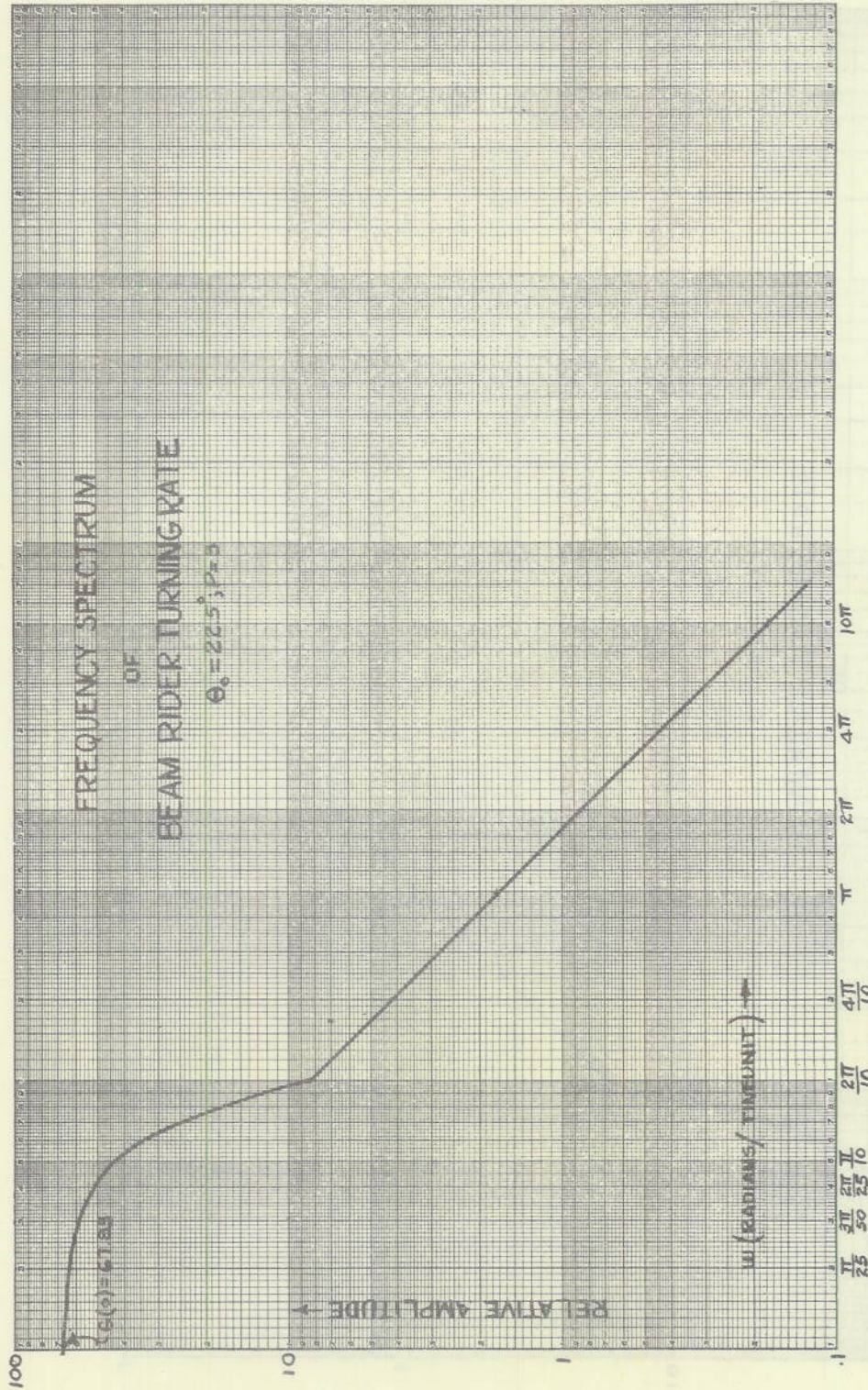


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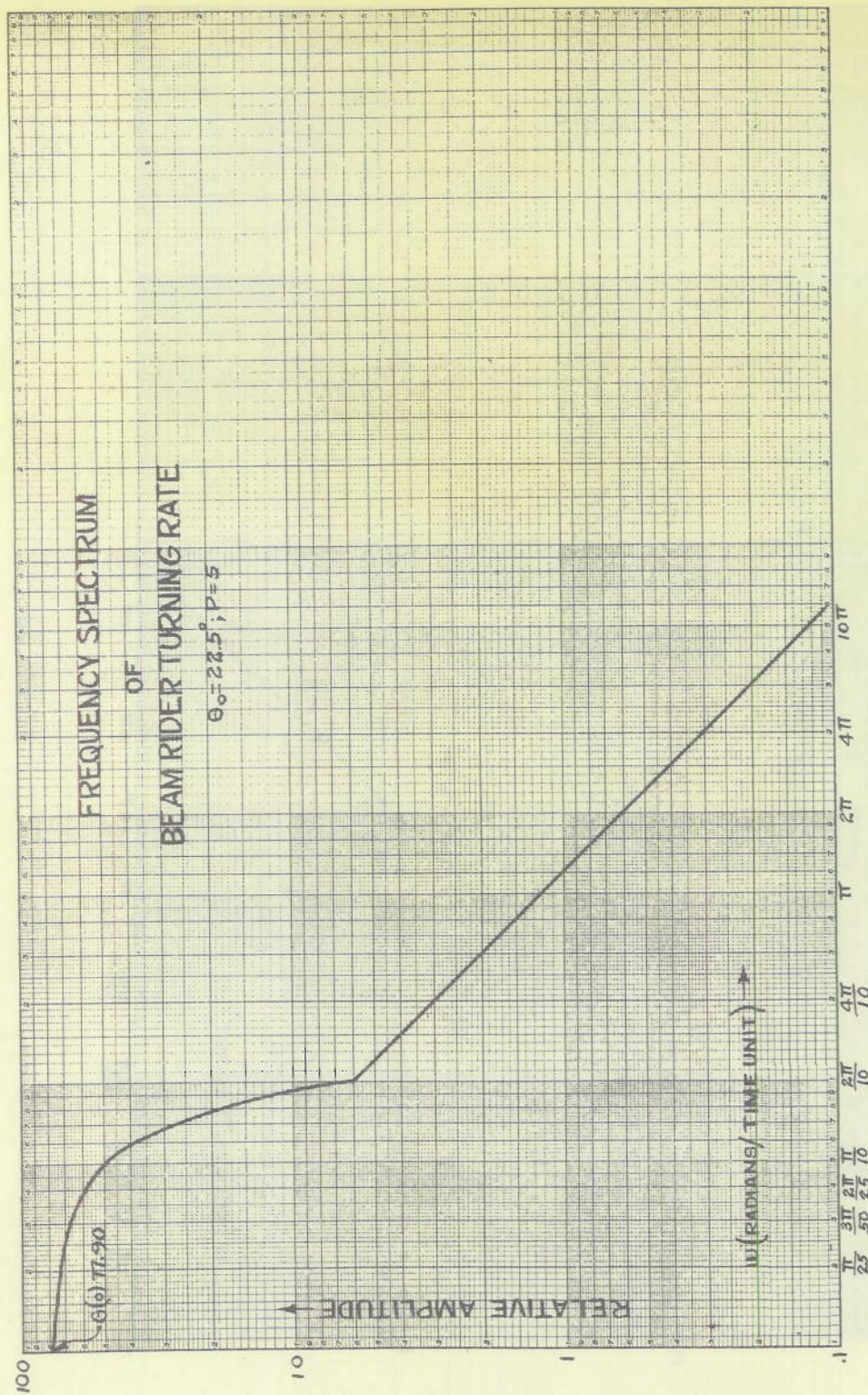


Figure 29

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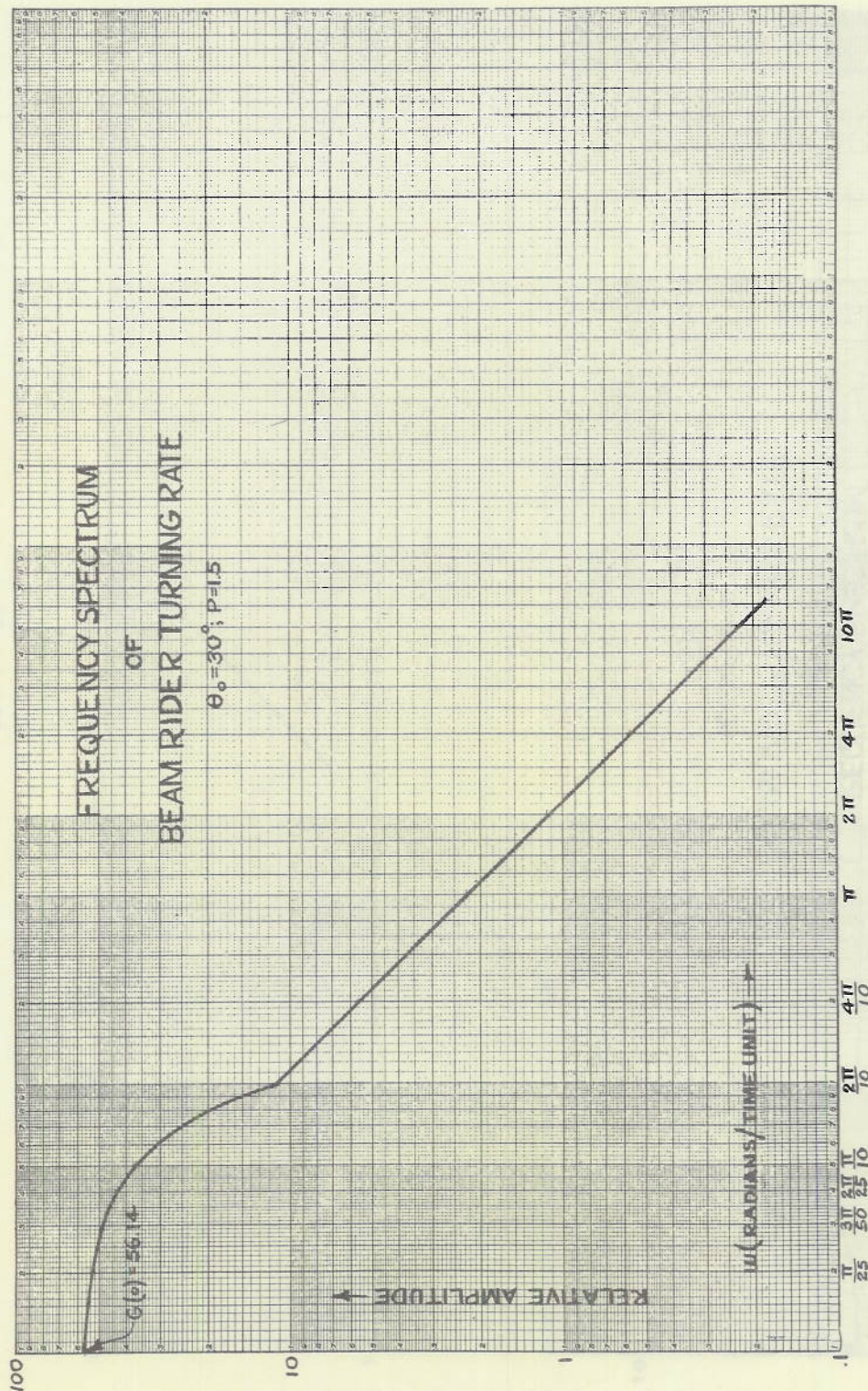


Figure 30

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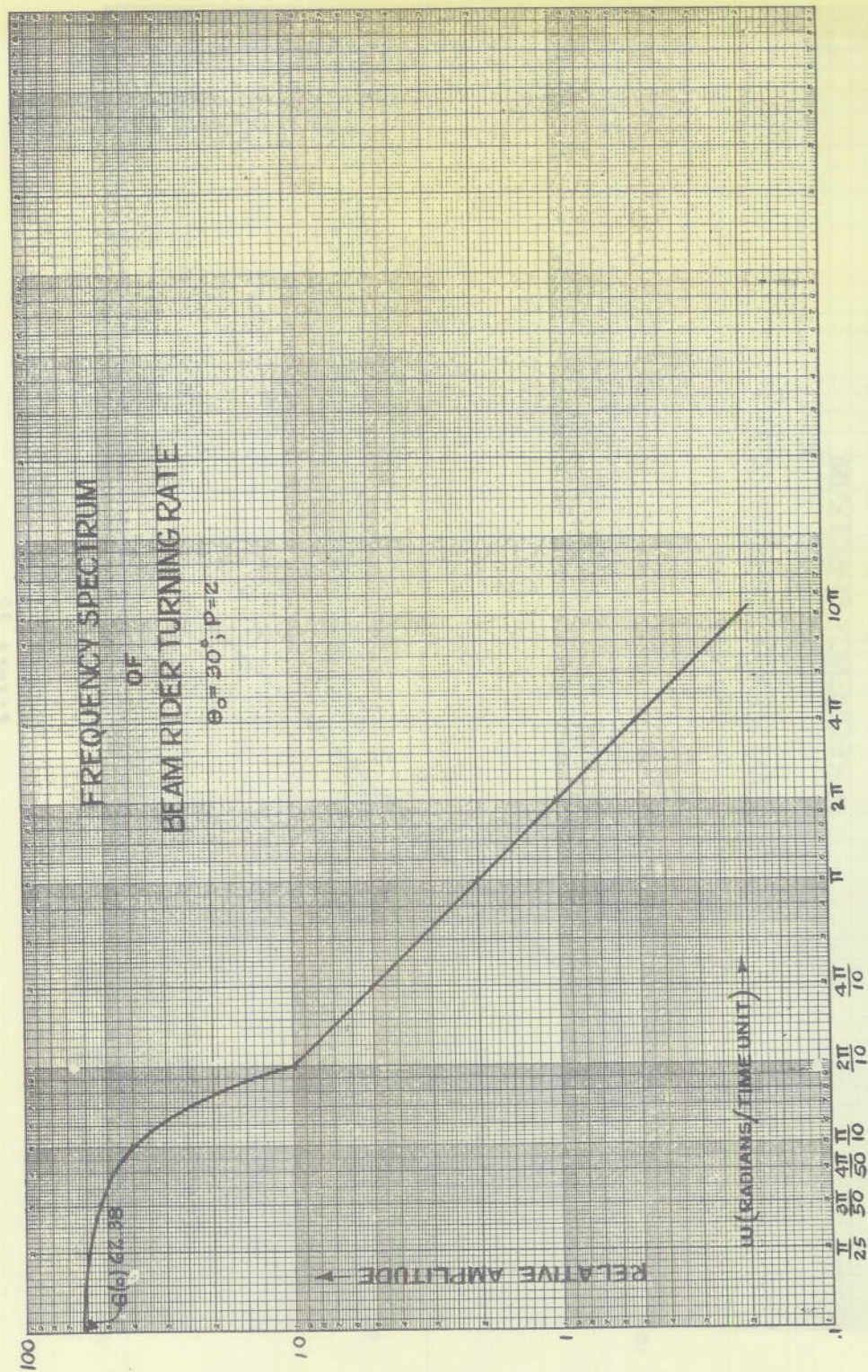


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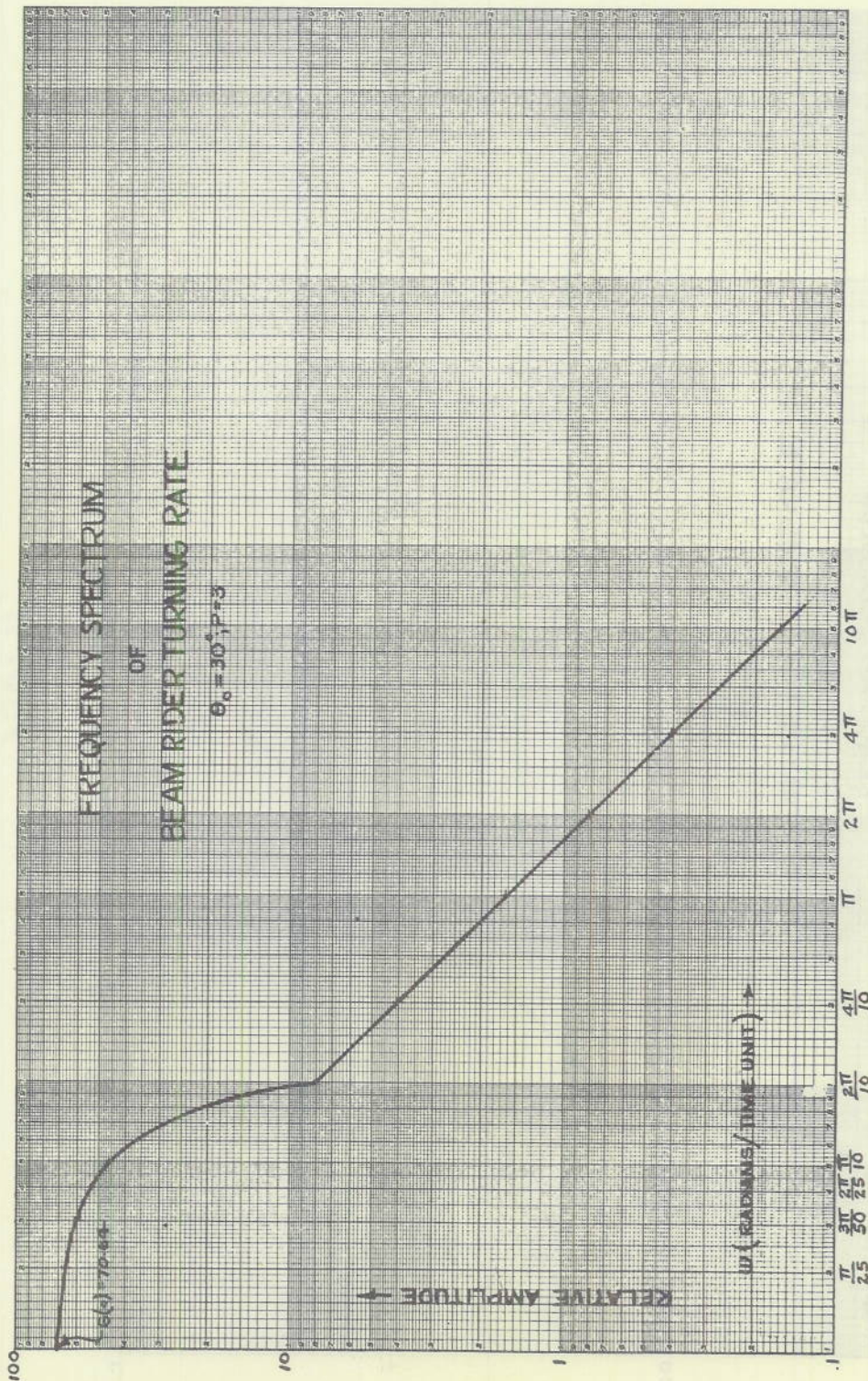


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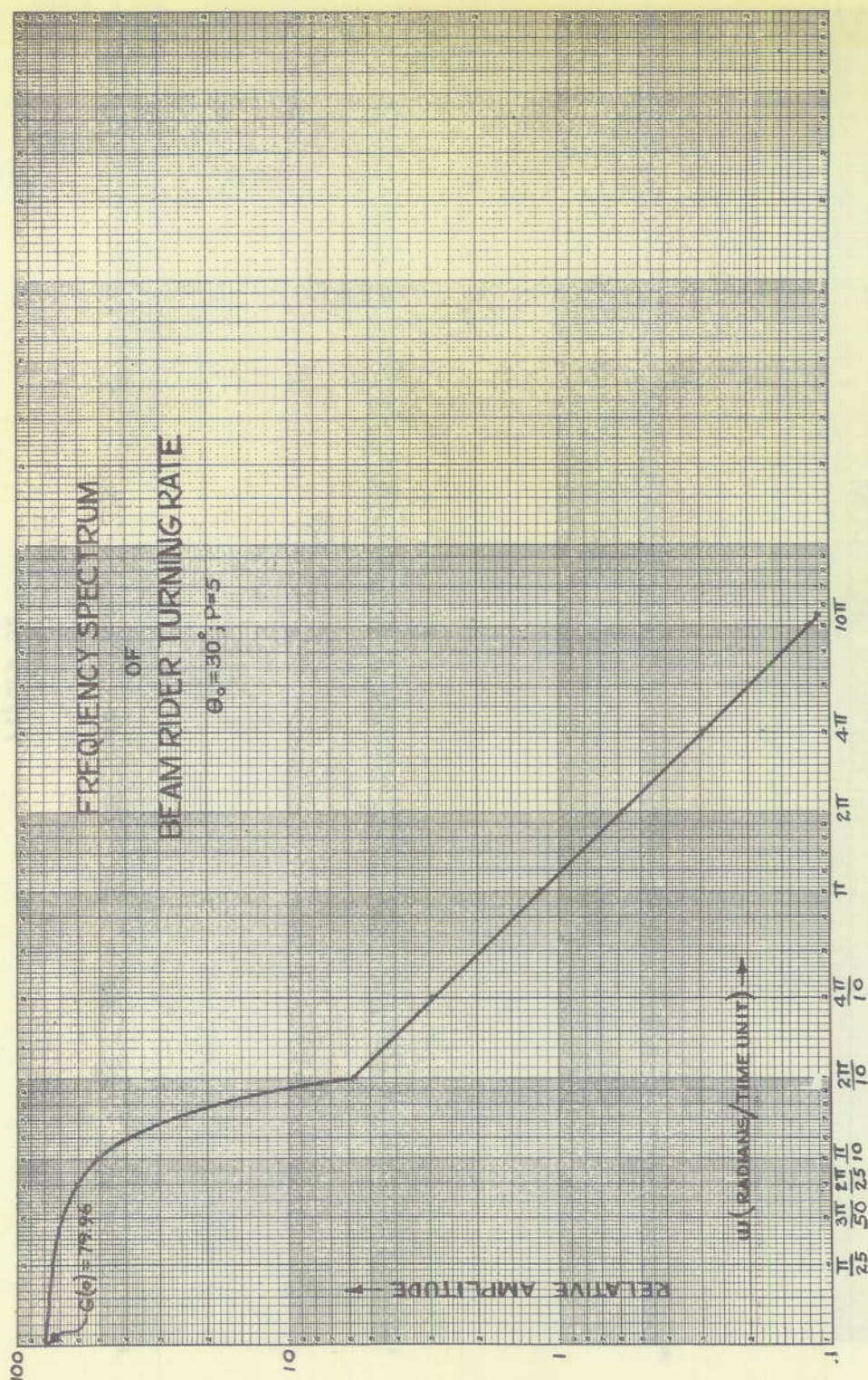


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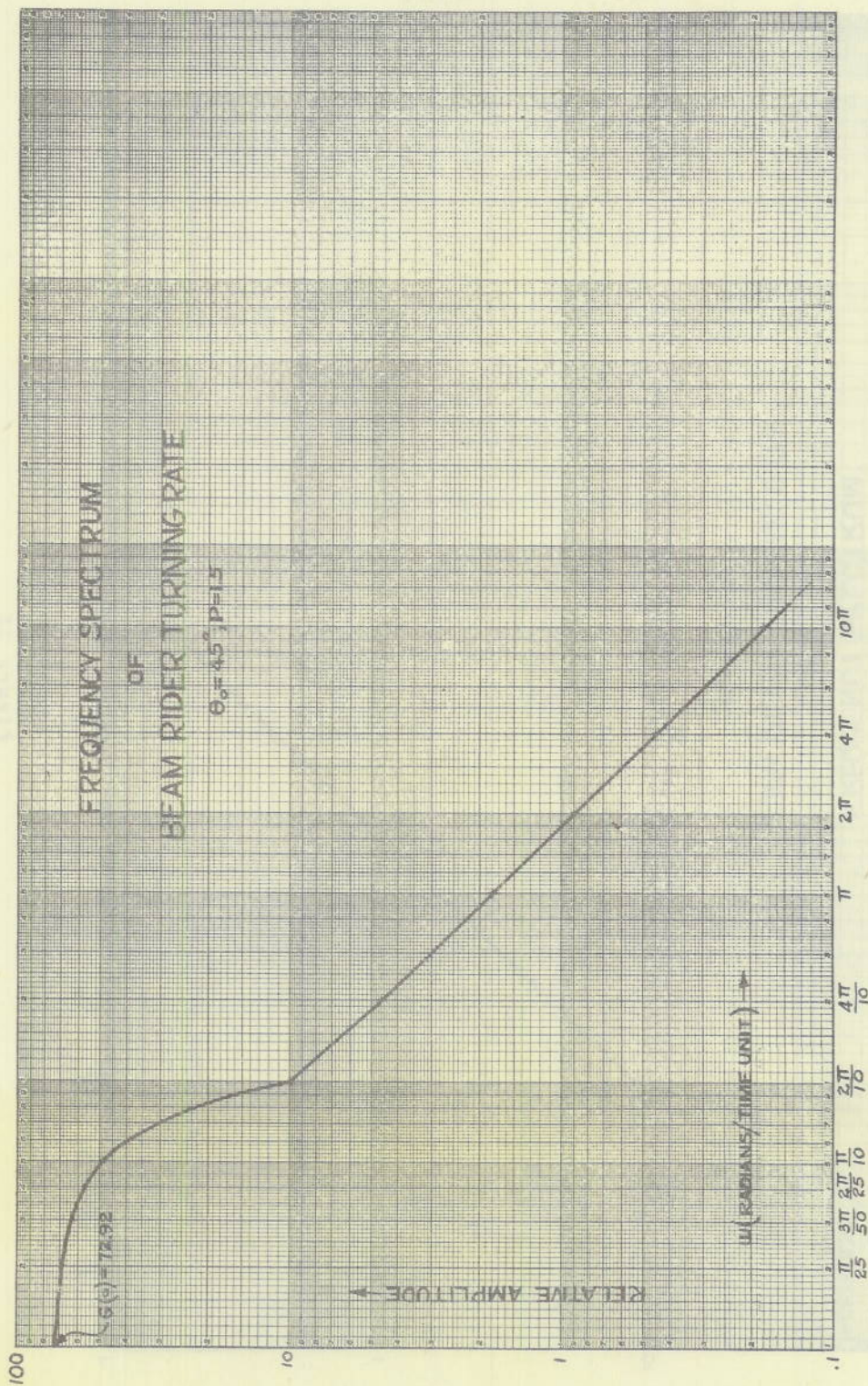


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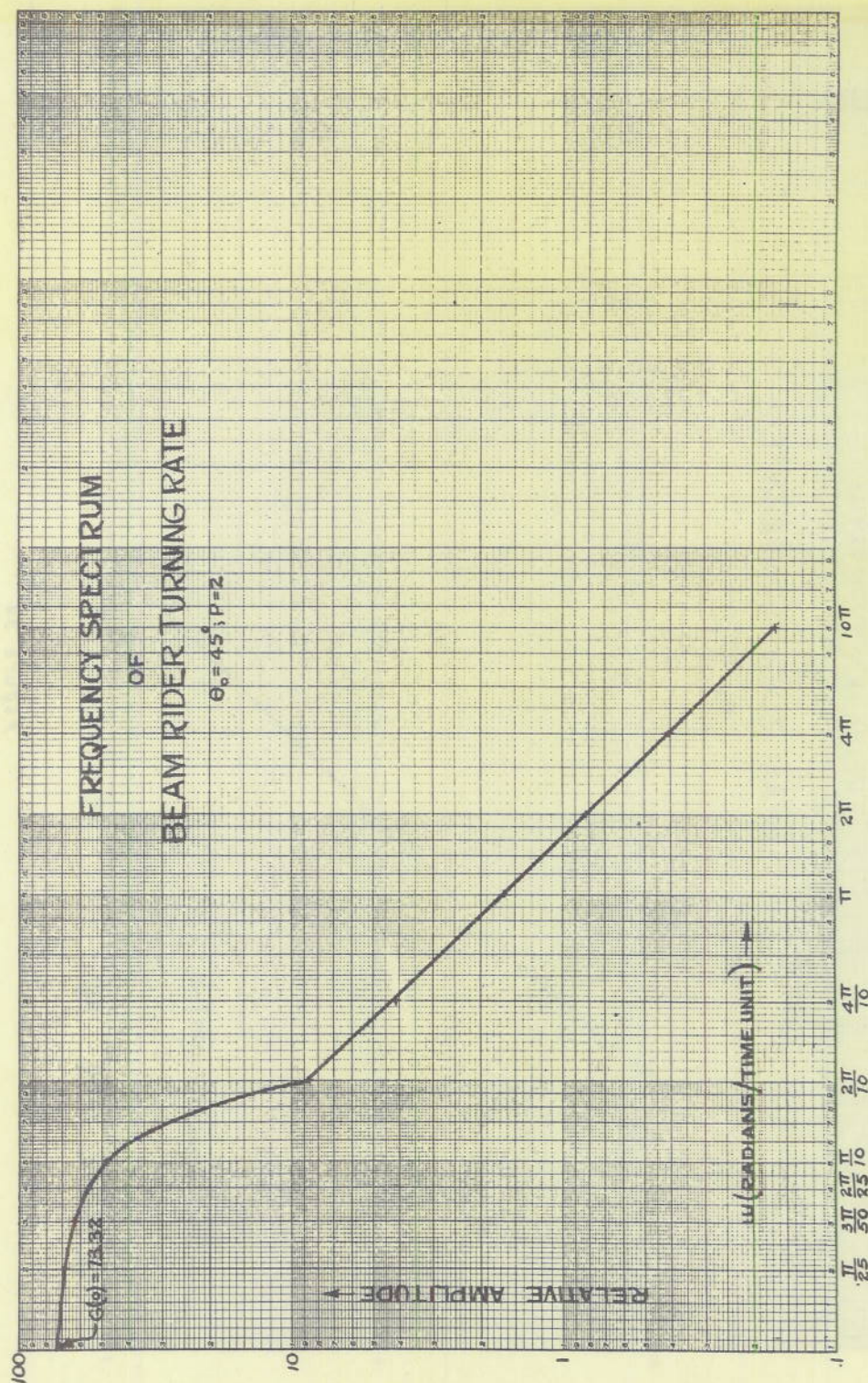


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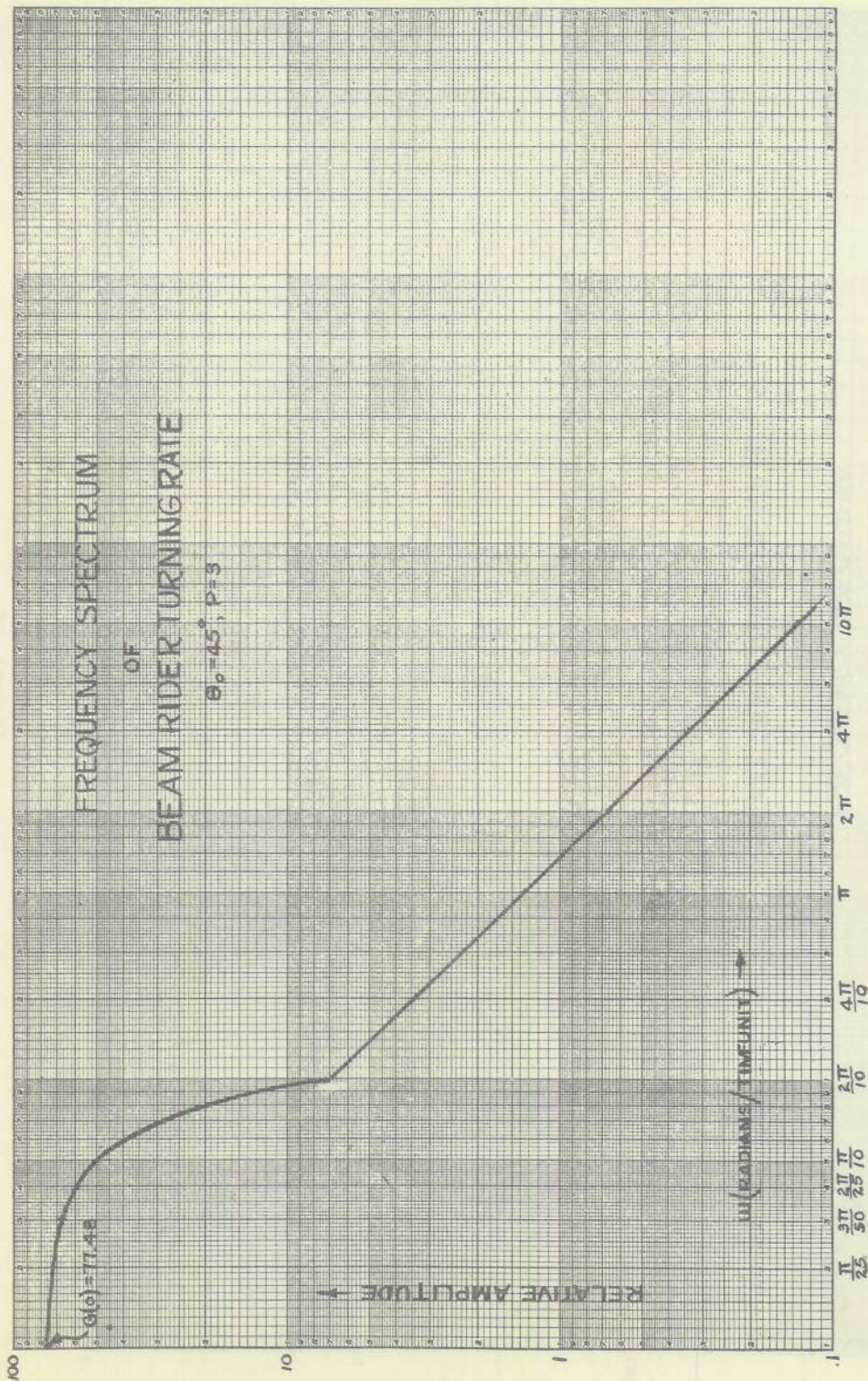


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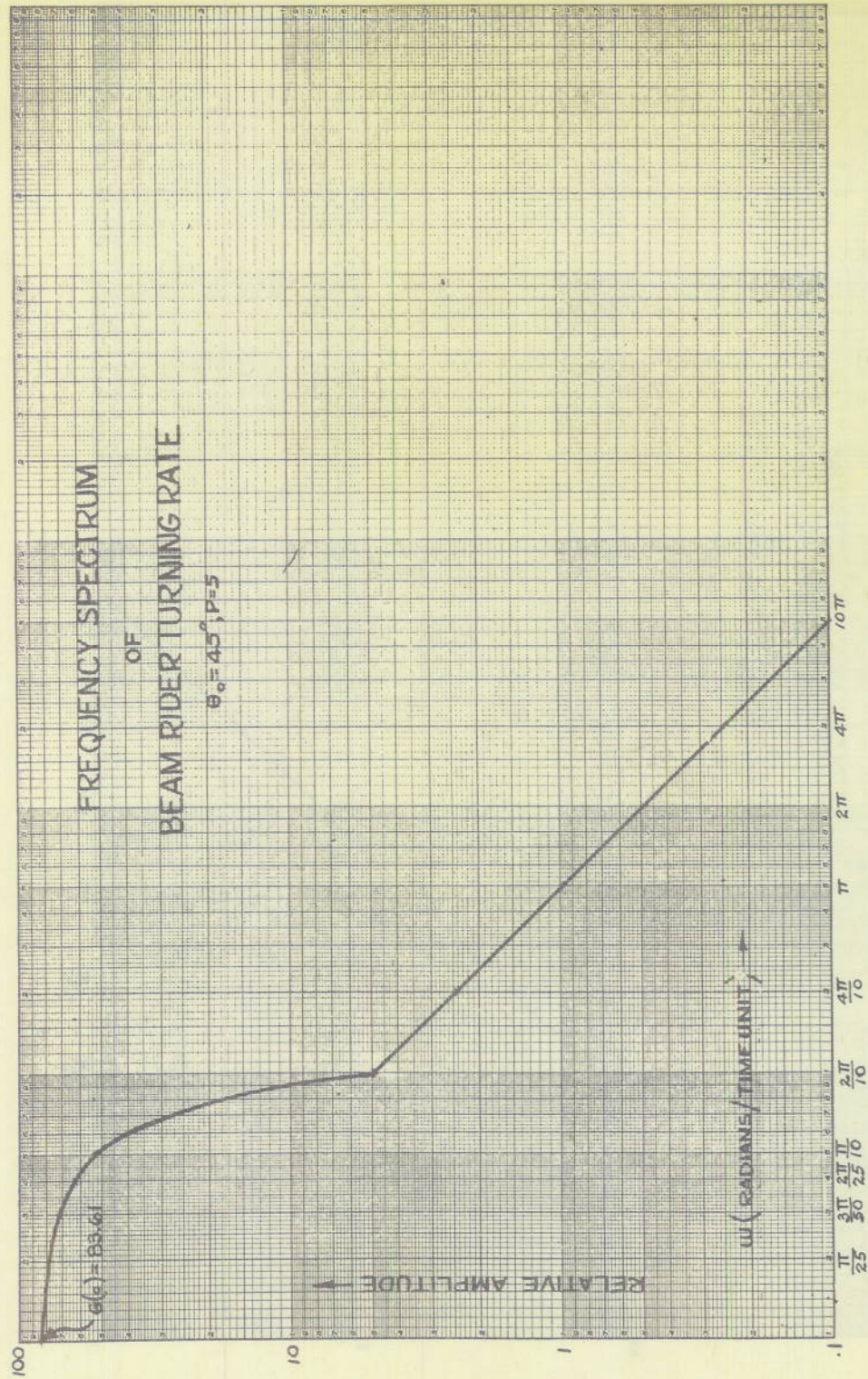


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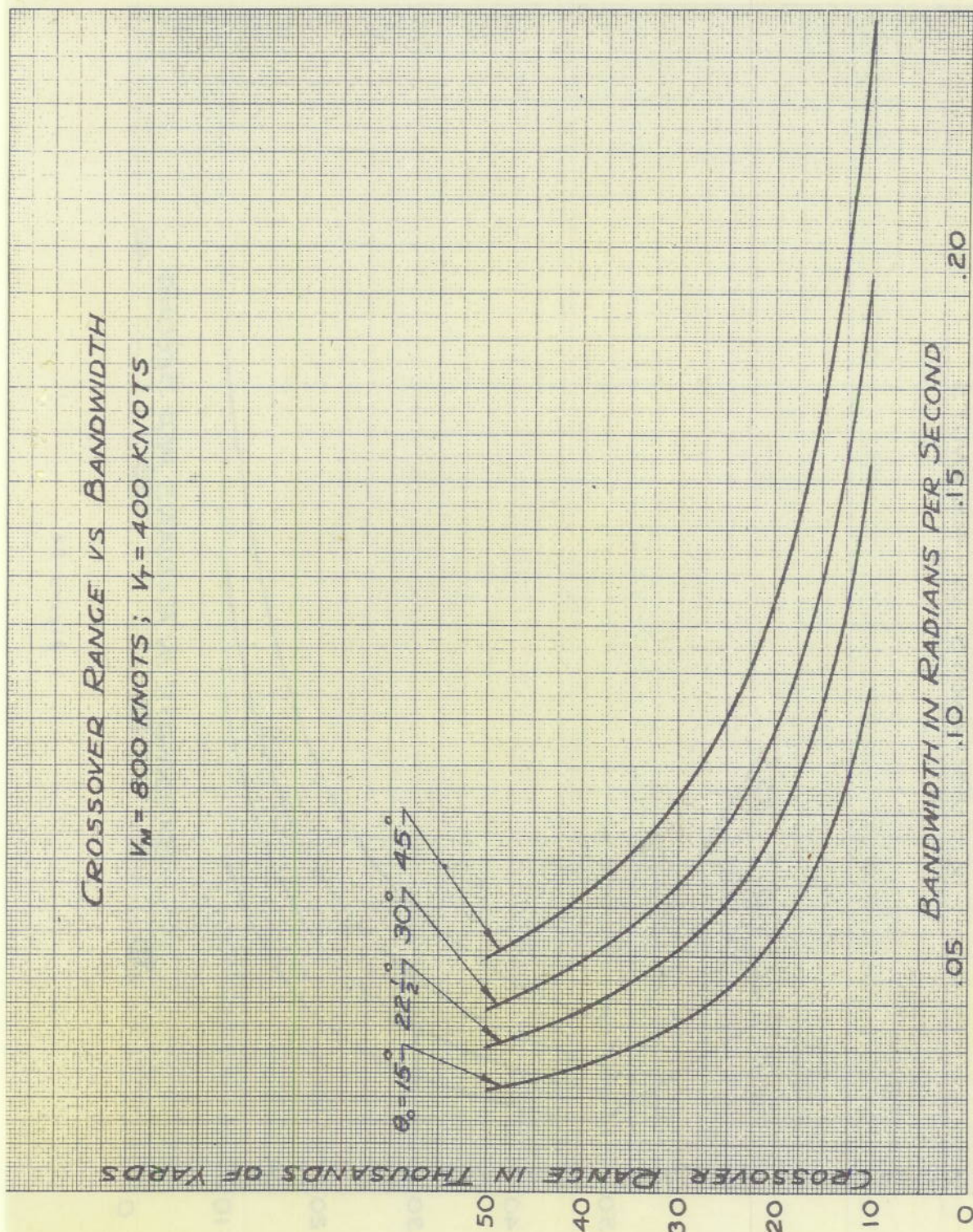


Figure 38

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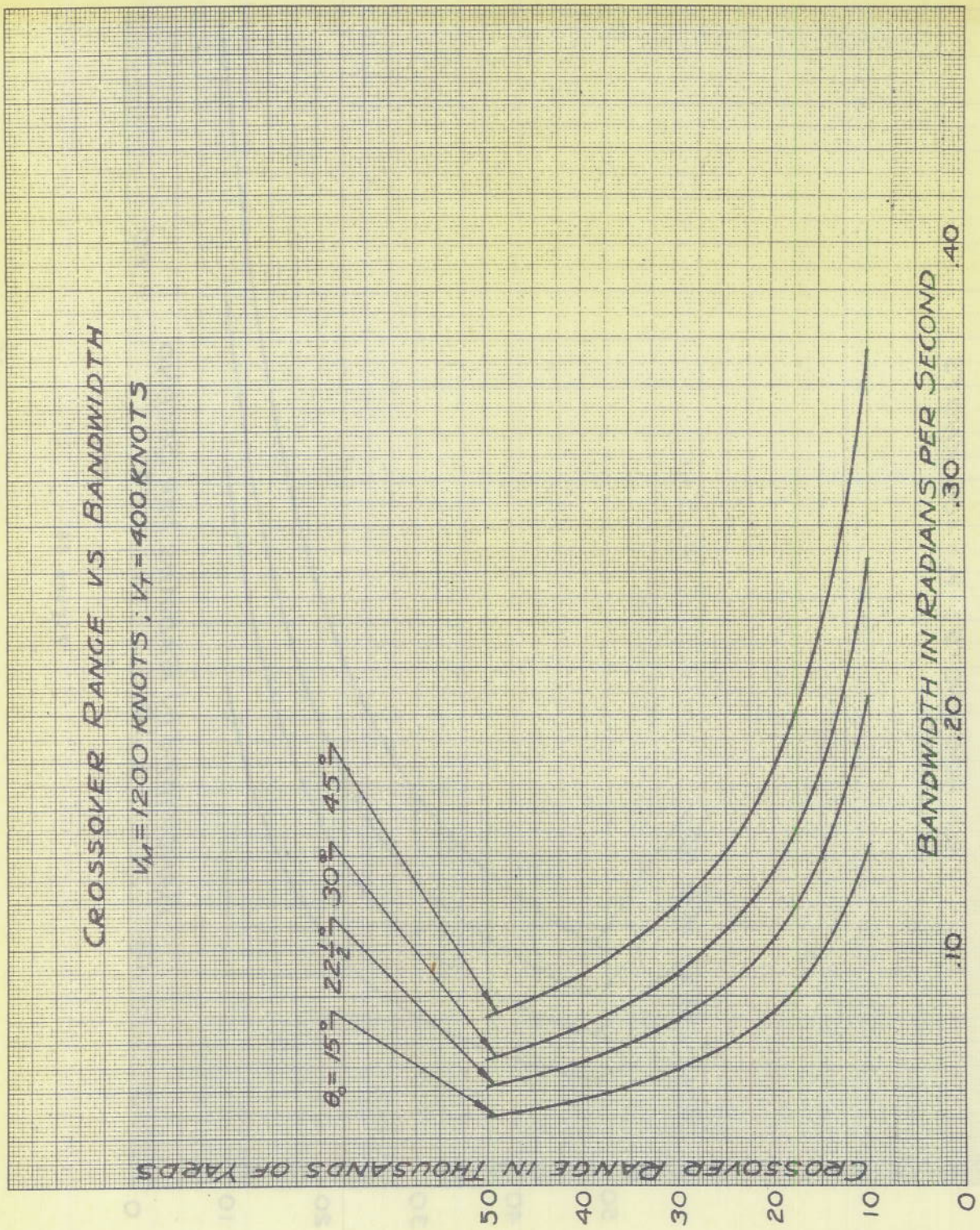
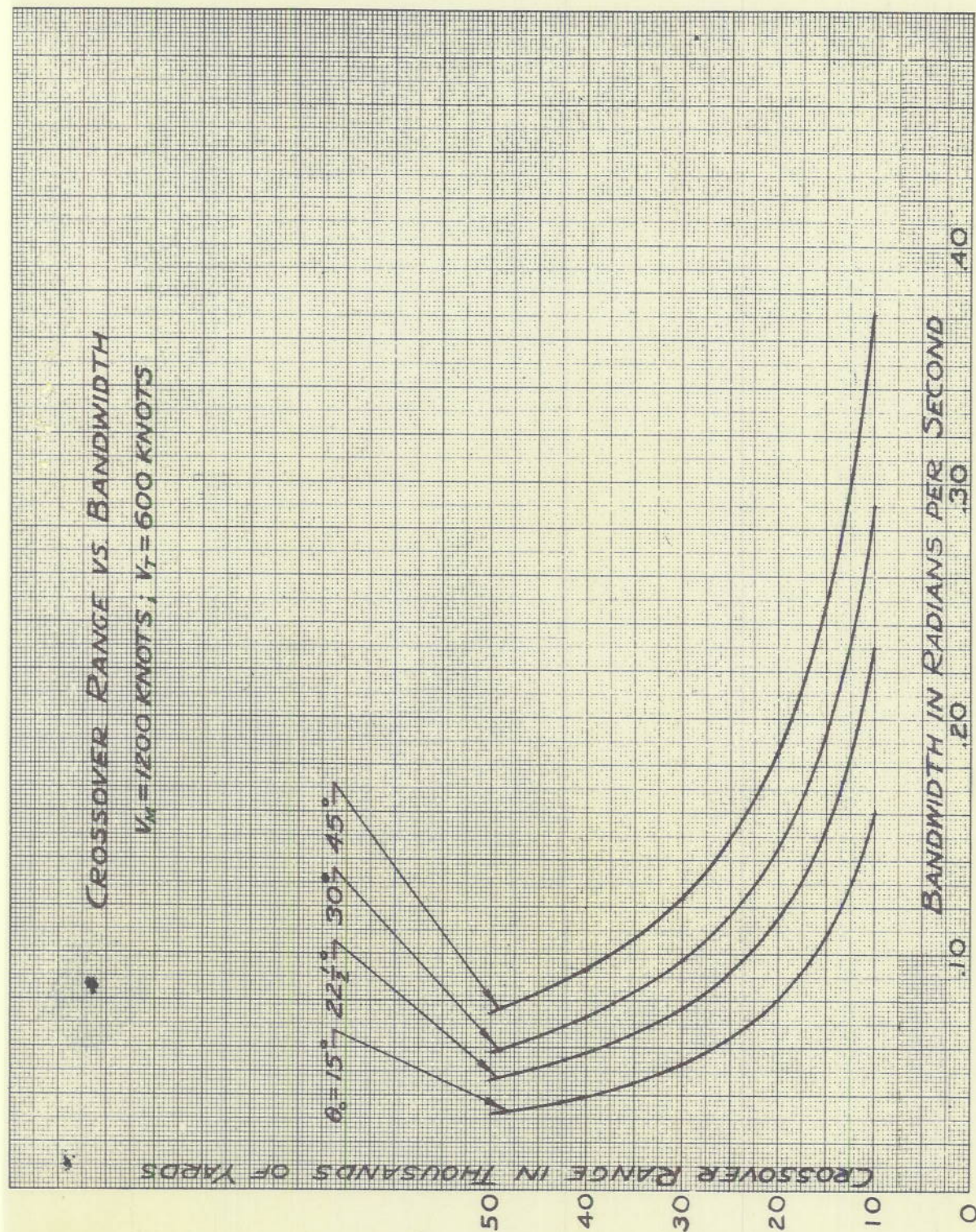
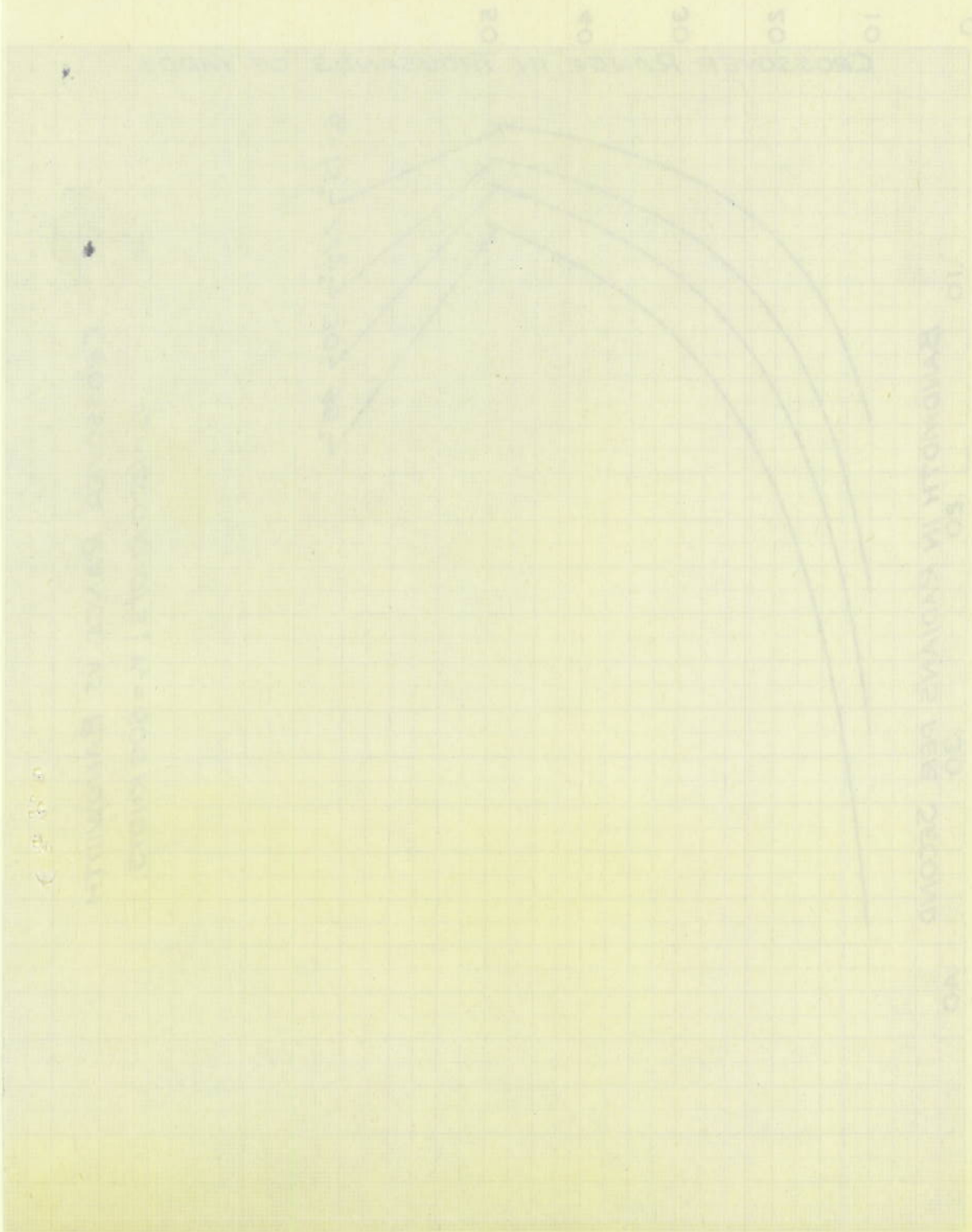


Figure 39

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