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HARMONIC POWER GENERATION USING UNBIASED IDEAL LINEAR RECTIFIERS

R. L. Easton

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Approved by:

Dr. C. E. Cleeton, Head, Security Systems Branch
Dr. J. M. Miller, Superintendent, Radio Division I

NAVAL RESEARCH LABORATORY
CAPTAIN F. R. FURTH, USN, DIRECTOR
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ABSTRACT

Analyses have been made of three half-wave and three full-wave circuits employing unbiased linear rectifiers as harmonic power generators. These three circuits utilize (1) a resistive generator and a resistive load, (2) a resistive generator and a tuned load, and (3) a tuned generator and a tuned load.

The maximum harmonic gain, defined as the ratio of the available harmonic power to the maximum available fundamental power, has been found for each circuit. In all cases the gain decreases as the fourth power of the harmonic. Odd harmonics above the first are not present.

PROBLEM STATUS

This is an interim report; work is continuing on the general problem.

AUTHORIZATION

NRL Problem R03-06R
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HARMONIC POWER GENERATION USING UNBIASED IDEAL LINEAR RECTIFIERS

INTRODUCTION

Microwave power of good frequency stability is required for many applications. It may be obtained by frequency multiplication of lower frequency power from oscillators controlled by quartz crystals through use of the nonlinear characteristic of point-contact rectifiers. Because these have limited power-handling capabilities, consideration of the conditions for maximum harmonic output is important.

Previous treatments of harmonic generation have considered the cases of triode or pentode operation where the reaction of the output circuit on the source can be neglected. Terman and Ferns[1] calculated the output current at various harmonics for the special case of a pentode where the output circuit does not affect the shape of the current pulses, and Scott and Black[2] generalized this to include the effect of a tuned output circuit on the harmonic output currents. Both papers assume a sinusoidal input wave without reference to the internal impedance of the generator.

In this paper the harmonic output power for various diode circuits is calculated in terms of the available power from the source. This ratio is here called the harmonic gain of the network and is significant when a chain of multiplier stages is used, since then the output of each stage is the source for the following multiplier.

This report analyzes several diode rectifier circuits, both half-wave and full-wave, which use ideal unbiased linear rectifiers for harmonic generation. For completeness, the well-known circuit[3] of the ideal half-wave rectifier in series with a resistive load and a resistive generator is treated first. This is then generalized by first replacing the load resistance by a tuned impedance and then by replacing both the generator resistance and the load resistance by a tuned impedance. Finally, each of these cases is treated as a full-wave circuit.

Throughout this treatment the tuned circuits are idealized by assuming that they present a resistive impedance at some harmonic frequency and zero impedance at all other

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[1] *Terman, F. E. and Ferns, J. H. "The Calculation of Class C Amplifier and Harmonic Generator Performance of Screen-Grid and Similar Tubes," Proc. I.R.E., 22, 359-373, 1934.*

[2] *Scott, H. J. and Black, L. J. "Harmonic Generation," Proc. I.R.E., 26, 449-468, 1938.*

[3] *Furst, U. R. "Harmonic Analysis of Overbiased Amplifiers," Electronics, 17, 143-144, Mar. 1944.*

frequencies. The rectifiers are assumed to have a constant resistance ρ in the forward direction and an infinite resistance in the reverse direction. The half angle of conduction ϕ is illustrated for the general case of the second harmonic by Figure 1.

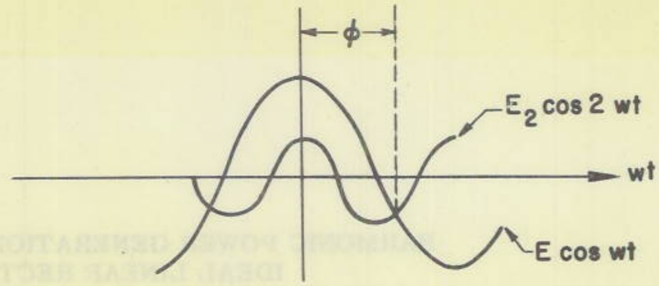

Figure 1 - Definition of ϕ, the half angle of conduction (conduction ceases when the voltage across the load equals the driving voltage)

CASE I: RESISTIVE GENERATOR, RESISTIVE LOAD

For the first case to be analyzed (Figure 2), we may write the equation for the circuit during the period of rectifier conduction as:

$$E \cos \omega t = i(t)\ (r + \rho + R)\ , \tag{1}$$

from which

$$i(t) = \frac{E \cos \omega t}{r + \rho + R}\ . \tag{2}$$

Since i(t) is an even function it can be analyzed into a Fourier cosine series. The peak n^{th}- harmonic current is given by

$$I_n = \frac{1}{\pi} \int_{-\pi}^{\pi} i(t) \cos n\omega t \ d(\omega t)\ , \tag{3}$$

or

$$I_n = \frac{2E}{\pi\ (r + \rho + R)} \int_{o}^{\phi} \cos \omega t \cos n\omega t \, d(\omega t) \tag{4}$$

on substituting (2) in (3) and changing the limits to o to ϕ, since the current is zero for angles greater than ϕ, where ϕ is the half angle of rectifier conduction. For this case we see, by setting i(t) = 0 in (1), that $\cos \phi = 0$ or $\phi = \pi/2$. Then

$$I_n = \frac{E}{(r + \rho + R)} \left[\frac{\sin (n-1)\phi}{n-1} + \frac{\sin (n+1)\phi}{n+1} \right]_0^{\pi/2} ; \tag{5}$$

$$i(t) = \frac{I_0}{2} + I_1 \cos \omega t + I_2 \cos 2\omega t + ... + I_n \cos n\omega t + ...; \tag{6}$$

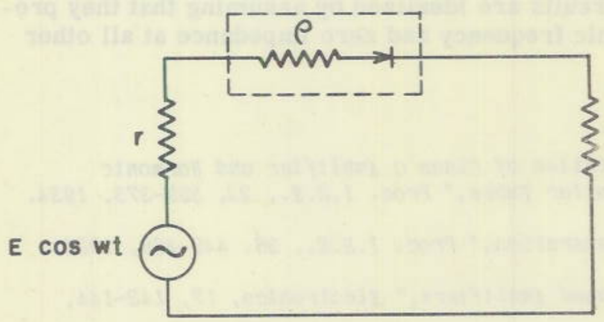

Figure 2 - Resistive generator feeding a resistive load

$$i(t) = \frac{E}{\pi(r + \rho + R)} \left[1 + \frac{\pi}{2} \cos \omega t + \frac{2}{3} \cos 2\omega t - \frac{2}{15} \cos 4\omega t + ... + 2 \frac{(-1)^{(n/2)+1}}{n^2 - 1} \cos n\omega t + ... \right]; \tag{7}$$

where n is even. It will be noticed that there are no odd harmonics present (except the fundamental).

It is convenient to introduce the following function, which will be used throughout this paper:

$$F_{mn}(\phi) = F_{nm}(\phi) = \frac{\sin(n-m)\phi}{n-m} + \frac{\sin(n+m)\phi}{n+m} = F_{mn} = F_{nm}\,. \tag{8}$$

When m = 1,

$$F_{1n} = \frac{\sin(n-1)\phi}{n-1} + \frac{\sin(n+1)\phi}{n+1}\,. \tag{9}$$

The power present at the n^{th} harmonic is P_n, where

$$P_n = \tfrac{1}{2} I_n^2\, R = \frac{\tfrac{1}{2} E^2 R\, F_{1n}^2}{[\pi(r+\rho+R)]^2}. \tag{10}$$

The power that the generator would deliver to a matched load (without the rectifier) is $P = \frac{1}{2}\, E^2/4r$, where R = r and E is the peak voltage. The harmonic power gain of the network is

$$G_n = \frac{P_n}{\dfrac{\frac{1}{2} E^2}{4r}}$$

$$G_n = \frac{4r\, F_{1n}^2}{\pi^2} \frac{R}{[r+\rho+R]^2}\,. \tag{11}$$

Since $\phi \doteq \pi/2$, from (9) we find

$$F_{1n} = \pm\frac{2}{n^2-1} \tag{12}$$

where n is even, and therefore

$$G_n = \frac{16}{\pi^2(n^2-1)^2} \frac{rR}{(r+\rho+R)^2}\,. \tag{13}$$

If R is varied, the maximum gain occurs for $R = r + \rho$ and is

$$G_n^{max(R)} = \frac{4}{\pi^2(n^2-1)^2} \frac{r}{r+\rho}\,, \tag{14}$$

whereas if r is varied, the maximum gain occurs for $r = R + \rho$ and is

$$G_n^{max(r)} = \frac{4}{\pi^2(n^2-1)^2} \frac{R}{R+\rho}\,. \tag{15}$$

These maxima can be equal only when

$$\frac{R}{R+\rho} = \frac{r}{r+\rho}\,,$$

which occurs when R = r. For $R = r + \rho$ and $r = R + \rho$, $\rho = 0$. Then

$$G_n^{max} = \frac{4}{\pi^2(n^2-1)^2}\,. \tag{16}$$

Thus it is seen that, in this case, the gain can always be increased by decreasing the rectifier resistance, or what is equivalent, by increasing the ratios of load resistance to rectifier resistance and of generator resistance to rectifier resistance.

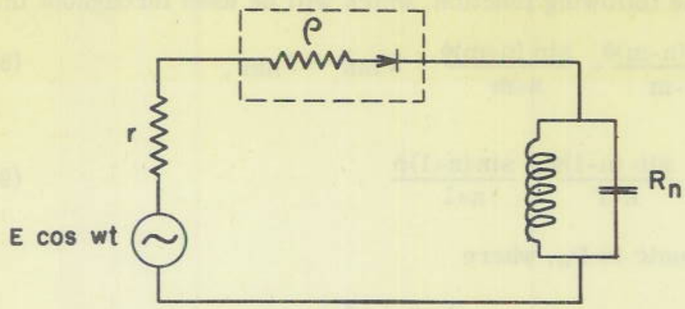

Figure 3 - Resistive generator feeding a tuned load

CASE II: RESISTIVE GENERATOR, TUNED LOAD

In practical cases the load impedance will generally be a tuned circuit. This may be represented by an impedance which is resistive to the n^{th} harmonic and offers no impedance to other frequencies.

For the period of diode conduction in the circuit shown in Figure 3,

$$E \cos\omega t = i(t)\, r + i(t)\, \rho + I_n R_n \cos n\omega t\ . \tag{17}$$

Using the same method as used in Case I we find

$$I_n = \frac{2}{\pi(r+\rho)} \int_0^{\phi} [E\cos\omega t - I_n R_n \cos n\omega t] \cos n\omega t\, d(\omega t), \tag{18}$$

where ϕ is again the half angle of rectifier conduction but in this case need not be $\pi/2$ since the flywheel action of R_n may have some effect.

Integrating and using (8),

$$I_n = \frac{E F_{1n}}{\pi(r+\rho) + R_n F_{nn}} \tag{19}$$

where n is even. The angle ϕ is determined by putting i(t) = 0 in (17). Then

$$I_n = \frac{E\cos\phi}{R_n \cos n\phi}\ . \tag{20}$$

Equating (19) and (20),

$$R_n = \frac{\pi(r+\rho)}{F_{1n}\dfrac{\cos n\phi}{\cos\phi} - F_{nn}}\ . \tag{21}$$

As in case I,

$$G_n = \frac{P_n}{\dfrac{\frac{1}{2}E^2}{4r}} = \frac{8 r P_n}{E^2} = \frac{4 I_n^2\, r R_n}{E^2}\ . \tag{22}$$

Substituting (19) in (22), we find

$$G_n = \frac{4 F_{1n}^2\, r R_n}{[\pi(r+\rho) + R_n F_{nn}]^2}\ . \tag{23}$$

The variable G_n is a function of r, R_n, and ϕ , where ϕ is determined by the implicit equation

$$R_n\left(F_{1n}\frac{\cos n\phi}{\cos\phi} - F_{nn}\right) - \pi(r+\rho) = f(r, R, \phi) = 0\ . \tag{24}$$

We can now find the maxima of G_n by using Lagrange's method of undetermined multipliers[4]:

$$G_n(r, R_n, \phi) + \lambda f(r, R_n, \phi) = 0$$

$$\frac{\partial G_n}{\partial r} + \frac{\partial f}{\partial r} = 0 \qquad \frac{\partial G_n}{\partial R_n} + \lambda \frac{\partial f}{\partial R_n} = 0 \qquad \frac{\partial G_n}{\partial \phi} + \frac{\partial f}{\partial \phi} = 0 \tag{25}$$

$$\frac{\partial G_n}{\partial r} = \frac{[\pi(r+\rho) + R_n F_{nn}]\, 4F_{1n}^2 R_n}{[\pi(r+\rho) + R_n F_{nn}]^3} \tag{26}$$

$$\frac{\partial f}{\partial f} = -\pi \tag{27}$$

$$\lambda = \frac{[\pi(r+\rho) + R_n F_{nn}]\ 4F_{1n}^2 R_n - 8\ F_{1n}^2\, r R_n}{\pi[\pi(r+\rho) + R_n F_{nn}]^3}. \tag{28}$$

Using the second equation of (25),

$$\lambda\left(F_{1n}\frac{\cos n\phi}{\cos\phi} - F_{nn}\right) + \frac{[\pi(r+\rho) + R_n F_{nn}]\ 4F_{1n}^2 r - 8F_{1n}^2 F_{nn}\, r R_n}{[\pi(r+\rho) + F_{nn} R_n]^3} = 0. \tag{29}$$

Substituting (28) in (29), we obtain a solution for $F_{1n} = 0$. Since $F_{1n} = 0$ at $\phi = 0$, we find that the gain again has no absolute maximum.

Substituting (21) into (23) and simplifying,

$$G_n = \frac{4\left(F_{1n}\dfrac{\cos n\phi}{\cos\phi} - F_{nn}\right)}{\dfrac{\cos^2 n\phi}{\cos^2\phi}}\ \frac{r}{r+\rho}\ . \tag{30}$$

If we now let $\rho = 0$,

$$G_n = \frac{4}{\pi}\ \frac{F_{1n}\dfrac{\cos n\phi}{\cos\phi} - F_{nn}}{\dfrac{\cos^2 n\phi}{\cos^2\phi}}. \tag{31}$$

We can find the maximum for this G_n graphically or approximately by making the approximation that $\phi = \pi/2$. Then

$$F_{1n} = \pm\frac{2}{n^2-1},\ \cos n\phi = \mp 1,\ F_{nn} = \pi/2.$$

Then if ϕ differs only slightly from $\pi/2$,

$$\cos\phi = -(\phi - \pi/2) = -\alpha\,.$$

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[4] *Courant, R, "Differential and Integral Calculus," Vol. II, pp. 188-202, Interscience Publishers, Inc., 1936.*

Then

$$G_n = \frac{4}{\pi}\frac{\left(\frac{2}{n^2-1}\,\frac{1}{\alpha} - \frac{\pi}{2}\right)}{\frac{1}{\alpha^2}} = \frac{4}{\pi}\left(\frac{2\,\alpha}{n^2-1} - \frac{\pi\,\alpha^2}{2}\right) . \tag{32}$$

Maximized with respect to α,

$$G_n^{\max\,\alpha} = \frac{8}{\pi^2\,(n^2-1)^2} , \tag{33}$$

assuming $\phi = \pi/2$ and $\rho = 0$. We notice that this value of G_n is double that obtained for the same simplifications used for Case I (16).

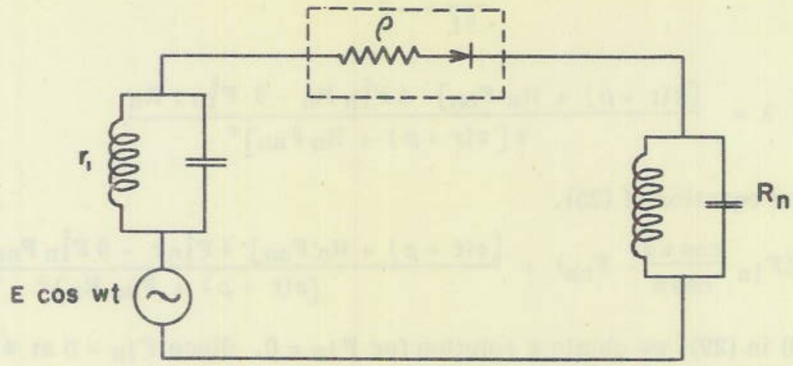

Figure 4 - Tuned generator feeding a tuned load

CASE III: TUNED LOAD, TUNED GENERATOR

Perhaps the most practical circuit using an ideal rectifier as a harmonic generator would utilize both a tuned generator and a tuned load. In this case, Figure 4, we can generalize the solution by solving for the m^{th} harmonic current when the load circuit is tuned to the n^{th} harmonic of the fundamental. As before,

$$E \cos\omega t = I_1 r_1 \cos\omega t + i(t)\,\rho + I_n R_n \cos n\omega t ; \tag{34}$$

$$I_m = \frac{1}{\pi}\int_{-\pi}^{\pi} i(t)\,\cos m\omega t\; d(\omega t) \tag{35}$$

$$= \frac{2}{\pi\rho}\int_0^{\phi}\left[(E - I_1 r_1)\cos\omega t - I_n R_n \cos n\omega t\right]\cos m\omega t\, d(\omega t) \tag{36}$$

$$= \frac{1}{\pi\rho}\left[(E - I_1 r_1)\,F_{1m} - I_n R_n\,F_{nm}\right] . \tag{37}$$

For m = 1,

$$I_1 = \frac{1}{\pi\rho}\left[(E - I_1 r_1)\,F_{11} - I_n R_n F_{1n}\right] . \tag{38}$$

For n = m,

$$I_n = \frac{1}{\pi\rho}\left[(E - I_1 r_1)\,F_{11} - I_n R_n F_{nn}\right] . \tag{39}$$

Solving (38) and (39) simultaneously we find

$$I_1 = E\,\frac{\pi\rho F_{11} + R_n(F_{11}F_{nn} - F_{1n}^2)}{\pi^2\rho^2 + \pi\rho F_{nn}R_n + \pi\rho F_{11}r_1 + r_1 R_n(F_{11}F_{nn} - F_{1n}^2)} \qquad (40)$$

$$I_n = E\,\frac{F_{1n}}{\pi^2\rho^2 + \pi\rho F_{nn}R_n + \pi\rho F_{11}r_1 + r_1 R_n(F_{11}F_{nn} - F_{1n}^2)}\,. \qquad (41)$$

The cutoff half angle of current flow (ϕ) is found from (36) when $i(t) = 0$:

$$(E - I_1 r_1)\cos\phi = I_n R_n \cos n\phi\,. \qquad (42)$$

Substituting (40) and (41) in (42) we obtain

$$R_n = \frac{\pi\rho}{F_{1n}\dfrac{\cos n\phi}{\cos\phi} - F_{nn}}\;; \qquad (43)$$

$$G_n = \frac{\frac{1}{2} I_n^2 R_n}{\dfrac{\frac{1}{2}E^2}{4r_1}} = \frac{4r I_n^2 R_n}{E^2}\,. \qquad (44)$$

Substituting (41) and (43) in (44),

$$G_n = \frac{4\pi\rho r_1\left(F_{1n}\dfrac{\cos n\phi}{\cos\phi} - F_{nn}\right)}{\left[\pi\rho\dfrac{\cos n\phi}{\cos\phi} + F_{11}r_1\dfrac{\cos n\phi}{\cos\phi} - F_{1n}r_1\right]^2}\,. \qquad (45)$$

G_n is seen to be a function of ϕ and r_1, ϕ being a function of R_n but not of r_1 (45). Since it is difficult to maximize G_n with respect to ϕ, we first maximize G_n with respect to r_1, obtaining

$$r_1^{\max G_n} = \frac{\pi\rho + R_n F_{nn}}{F_{11} + \dfrac{R_n}{\pi\rho}(F_{11}F_{nn} - F_{1n}^2)}\,. \qquad (46)$$

Substituting for R_n,

$$r_1^{\max G_n} = \frac{\pi\rho\dfrac{\cos n\phi}{\cos\phi}}{F_{11}\dfrac{\cos n\phi}{\cos\phi} - F_{1n}}\,. \qquad (47)$$

Substituting (47) into (46),

$$G_n^{\max r_1} = \frac{F_{1n}\dfrac{\cos n\phi}{\cos\phi} - F_{nn}}{\dfrac{\cos n\phi}{\cos\phi}\left[F_{11}\dfrac{\cos n\phi}{\cos\phi} - F_{1n}\right]}\,. \qquad (48)$$

This is the exact equation for harmonic gain, maximized with respect to the generator impedance. The maximum G_n as a function of ϕ has been computed numerically for $n = 2, 4, 6,$ and 8; results are shown in Figure 5. The ratio of R_2 to ρ for any angle ϕ

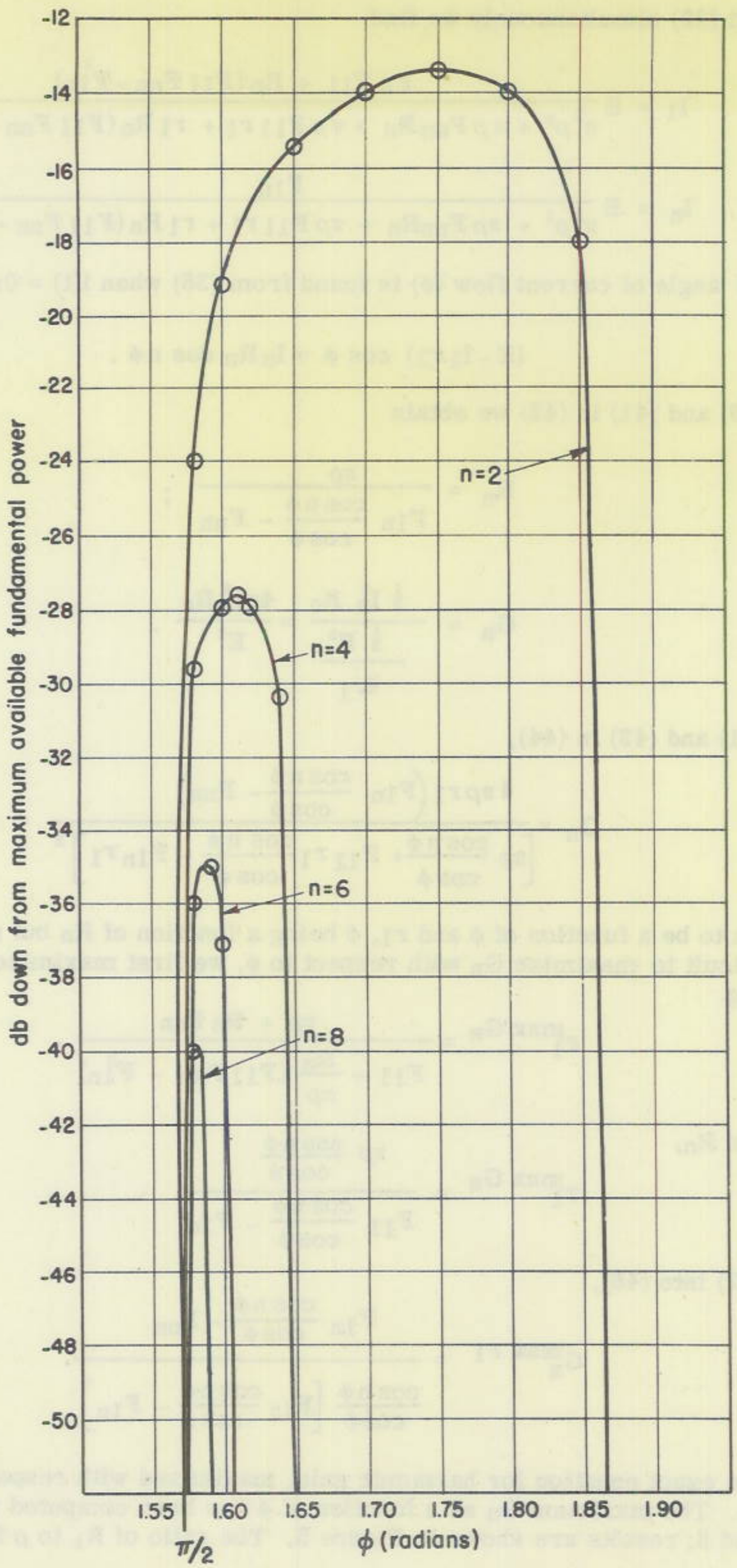

Figure 5 - G_n as a function of ϕ for equation 48

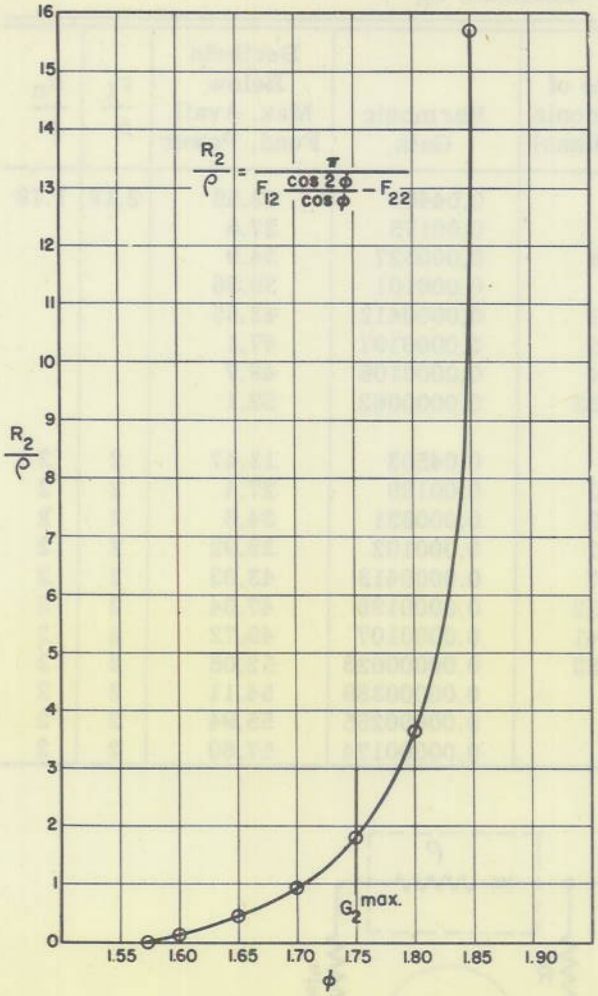

Figure 6 - R_2/ρ vs. ϕ

is shown in Figure 6. An approximate solution can also be found as in the previous case:

$$G_n^{\max r1} = \frac{\frac{2}{n^2-1}\,\frac{1}{\alpha} - \frac{\pi}{2}}{\frac{1}{\alpha}\left(\frac{\pi}{2}\,\frac{1}{\alpha} - \frac{2}{n^2-1}\right)}. \quad (49)$$

Since ϕ is approximately $\pi/2$,

$$\frac{\pi}{2\alpha} \gg \frac{2}{n^2-1} \quad (50)$$

and hence

$$G_n^{\max r1} = \frac{4\alpha}{\pi(n^2-1)} - \alpha^2. \quad (51)$$

Maximizing $G_n^{\max r1}$ with respect to α,

we obtain $\alpha = \frac{2}{\pi(n^2-1)}$, (52)

and

$$G_n^{\max r1} = \frac{4}{\pi^2(n^2-1)^2}. \quad (53)$$

As is shown in Table 1, this approximation for $G_n^{\max}$ gives results close to those obtained by plotting (48). It will be noticed in Table 1 that there is very little difference in going to a high harmonic in steps rather than in one operation. Thus for n = 4, we lose 27.6 db by obtaining the harmonic in one stage and 27.1 db by obtaining the fourth harmonic in two stages.

Equation (53) is seen to be identical with (16) except that in the tuned generator, tuned load case, the approximation that $\rho = 0$ is not made.

If we use the approximations above and substitute (52) in (43), we obtain $R_n^{\max G_n} = 2\rho$. Doing the same in (46), and neglecting $2/(n^2-1)$ as compared to $(\pi^2/4)(n^2-1)$, we obtain $r_1^{\max G_n} = 2\rho$.

CASE IV: FULL-WAVE RESISTIVE GENERATOR, RESISTIVE LOAD

Since the output of a perfect full-wave rectifier contains no fundamental current, it might be assumed that the remaining harmonic powers would be greater than in the full-wave case just described.

The full-wave rectifier circuit may be analyzed by splitting it up as shown in Figure 7. The current through the "A" loop may be considered independent of the "B" circuit since when current flows through the "A" loop none flows through the "B" loop and vice versa.

TABLE 1
Solutions for Maximum G_n

Using Equations:	Harmonic	ϕ Half Angle of Max. Harmonic Gain (Radians)	Harmonic Gain	Decibels Below Max. Avail. Fund. Power	$\frac{r_1}{\rho}$	$\frac{R_n}{\rho}$
(48) (Exact)	2	1.75	0.04400	13.56	2.18	1.79
	4	1.61	0.00175	27.6		
	6	1.588	0.000327	34.9		
	8	1.58	0.000101	39.96		
	10	1.577	0.0000412	43.85		
	12	1.575	0.0000197	47.1		
	14	1.574	0.0000106	49.7		
	16	1.5733	0.0000062	52.1		
(52) and (53) (Approximate)	2	1.78	0.04503	13.47	2	2
	4	1.613	0.00180	27.4	2	2
	6	1.589	0.000331	34.8	2	2
	8	1.581	0.000102	39.92	2	2
	10	1.577	0.0000413	43.83	2	2
	12	1.5752	0.0000198	47.04	2	2
	14	1.5741	0.0000107	49.72	2	2
	16	1.5733	0.00000623	52.06	2	2
	18		0.00000389	54.11	2	2
	20		0.00000255	55.94	2	2
	22		0.00000174	57.60	2	2

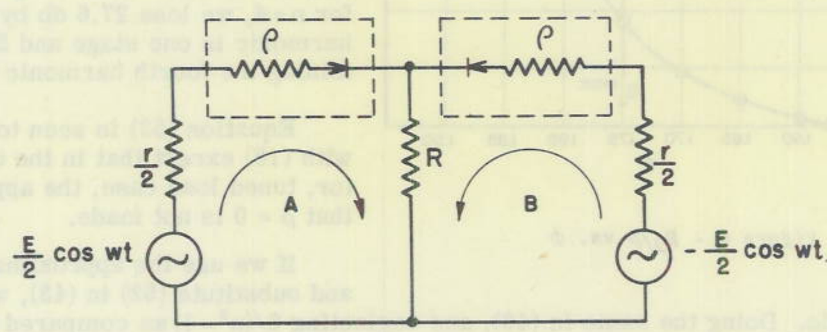

Figure 7 - Full-wave resistive generator feeding a resistive load

Then
$$\frac{E}{2}\cos\omega t = i_a \frac{r}{2} + i_A \rho + i_A R; \tag{54}$$

$$i_A = \frac{E\cos\omega t}{r + 2\rho + 2R}. \tag{55}$$

Likewise

$$i_B = \frac{-E\cos\omega t}{r + 2\rho + 2R}; \tag{56}$$

$$I_{An} = \frac{2}{\pi} \int_0^{\pi/2} i_A \cos n\omega t \, d(\omega t) = \frac{2E}{\pi (r + 2\rho + 2R)} \int_0^{\pi/2} \cos \omega t \cos n\omega t \, d(\omega t) \tag{57}$$

$$I_{A0} = \frac{2E}{\pi (r + 2\rho + 2R)}; \tag{58}$$

$$I_{A1} = \frac{2E}{\pi (r + 2\rho + 2R)} \frac{\pi}{4}; \tag{59}$$

$$I_{An} = \frac{E F_{1n}(\pi/2)}{\pi (r + 2\rho + 2R)}; \tag{60}$$

$$I_{Bm} = \frac{2E}{\pi (r + 2\rho + 2R)} \int_{\pi/2}^{\pi} \cos \omega t \cos m\omega t \, d(\omega t); \tag{61}$$

$$I_{B0} = \frac{2E}{\pi (r + 2\rho + 2R)} = I_{A0}; \tag{62}$$

$$I_{B1} = -\frac{2E}{\pi (r + 2\rho + 2R)} \frac{\pi}{4} = -I_{A1}; \tag{63}$$

$$I_{Bn} = \frac{E F_{1n}(\pi/2)}{\pi (r + 2\rho + 2R)} = I_{An}; \tag{64}$$

$$I_n = I_{An} + I_{Bn} = \frac{2E F_{1n}(\pi/n)}{\pi (r + 2\rho + 2R)}; \tag{65}$$

$$P_n = \frac{1}{2} I_n^2 R = \frac{2R E^2 F_{1n}^2 (\pi/2)}{\pi^2 (r + 2\rho + 2R)^2}; \tag{66}$$

$$G_n = \frac{\dfrac{P_n}{1/2 E^2}}{4r} = \frac{8 r P_n}{E^2} = \frac{16 r R F_{1n}^2 (\pi/2)}{\pi^2 (r + 2\rho + 2R)^2}. \tag{67}$$

Since

$$F_{1n}^2 (\pi/2) = \frac{4}{(n^2 - 1)^2},$$

$$G_n = \frac{64 rR}{\pi^2 (n-1)^2 (r + 2\rho + 2R)^2}. \tag{68}$$

Here again we have no absolute maximum unless $\rho = 0$, in which case

$$G_n = \frac{64 rR}{\pi^2 (n^2 - 1)^2 (r + 2R)^2}. \tag{69}$$

This expression is a maximum if $r = 2R$, in which case

$$G_n = \frac{8}{\pi^2 (n^2 - 1)^2}. \tag{70}$$

We see that the harmonic gain for the full-wave case is double that for the similar half-wave case.

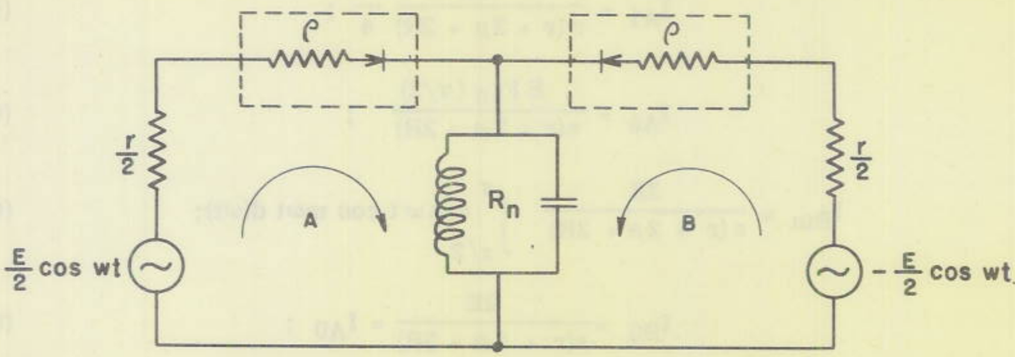

Figure 8 - Full-wave resistive generator feeding a tuned load

CASE V: FULL-WAVE RESISTIVE GENERATOR, TUNED LOAD

For the full-wave case with a resistive generator and a tuned load (Figure 8) we can make the following derivation:

$$\frac{E}{2} \cos\omega t = i_A(t) \left(\frac{r}{2} + \rho\right) + (I_{An} + I_{Bn}) R_n \cos n\omega t; \tag{71}$$

$$-\frac{E}{2} \cos\omega t = i_B(t) \left(\frac{r}{2} + \rho\right) + (I_{An} + I_{Bn}) R_n \cos n\omega t; \tag{72}$$

$$i_A(t) = \frac{E \cos \omega t - 2(I_{An} + I_{Bn}) R_n \cos n\omega t}{r + 2}; \tag{73}$$

where ϕ_A is the half angle of rectifier conduction through the "A" loop. Then

$$I_{Am} = \frac{2}{\pi(r + 2\rho)} \int_0^{\phi_A} \left[E \cos \omega t - 2(I_{An} + I_{Bn}) R_n \cos n\omega t \, d(\omega t)\right] \tag{74}$$

$$= \frac{1}{\pi(r + 2\rho)} \left[E F_{1m}(\phi_A) - 2(I_{An} + I_{Bn}) R_n F_{mn}(\phi_A)\right]; \tag{75}$$

$$I_{A1} = \frac{1}{\pi(r + 2\rho)} \left[E F_{11}(\phi_A) - 2(I_{An} + I_{Bn}) R_n F_{1n}(\phi_A)\right]; \tag{76}$$

$$I_{An} = \frac{1}{\pi(r + 2\rho)} \left[E F_{1n}(\phi_A) - 2(I_{An} + I_{Bn}) R_n F_{nn}(\phi_A)\right]; \tag{77}$$

Likewise, assuming that the two rectifiers conduct for equal periods, and that the current pulse through the "A" loop is symmetrical about the reference axis, the current pulse through the "B" loop will be symmetrical about a point π radians from the reference axis. Thus

$$I_{Bm} = -\frac{2}{\pi(r + 2\rho)} \int_{\pi-\phi_A}^{\pi} [E \cos \omega t + 2(I_{An} + I_{Bn})R_n \cos m\omega t \, d(\omega t)] \tag{78}$$

$$= -\frac{1}{\pi(r + 2\rho)} \Big[E\, F_{1m}(\phi) + 2\,(I_{An} + I_{Bn})\, R_n\, F_{mn}(\phi)\Big]_{\pi-\phi_A}^{\pi} ; \tag{79}$$

$$F_{nn}(\phi)\Big]_{\pi-\phi_A}^{\pi} = \phi + \frac{\sin 2\,n\phi}{2n}\Big]_{\pi-\phi_A}^{\pi} = \phi_A + \frac{\cos 2n\pi \sin 2n\phi_A}{2n}$$

$$= \phi_A + \frac{\sin 2m\ \phi_A}{2n} = F_{nn}(\phi_A); \tag{80}$$

$$F_{1n}(\phi)\Big]_{\pi-\phi_A}^{\pi} = \frac{\sin(n-1)\phi}{n-1} + \frac{\sin(n+1)\phi}{n+1}\Big]_{\pi-\phi_A}^{\pi} \tag{81}$$

$$= -F_{1n}(\phi_A) \text{ for n an even number} \tag{82}$$

$$= +F_{1n}(\phi_A) \text{ for n an odd number.} \tag{83}$$

Assuming only even (except for the fundamental) currents present,

$$I_{B1} = -\frac{1}{\pi(r + 2\rho)}\Big[E\ F_{11}(\phi_A) - 2(I_{An}+I_{Bn})R_n\, F_{1n}(\phi_A)\Big] = -I_{A1}; \tag{84}$$

$$I_{Bn} = -\frac{1}{\pi(r + 2\rho)}\Big[-E\ F_{1n}(\phi_A) + 2(I_{An}+ I_{Bn})R_n\, F_{nn}(\phi_A)\Big] = I_{An}. \tag{85}$$

If we let $I_{A1} = I$, $I_{An} = I_n$, $\phi_A = \phi$, then

$$I_n = \frac{1}{\pi(r + 2\rho)}\Big[E\ F_{1n}(\phi) - 4I_n\Big] R_n\ F_{nn}(\phi)\Big] \tag{86}$$

$$= \frac{E\, F_{1n}(\phi)}{\pi(r + 2\rho) + 4R_n\, F_{nn}(\phi)} . \tag{87}$$

When $i_A(t) = 0$ in equation (71),

$$E \cos \phi = 4\, I_n\, R_n\, \cos n\, \phi ; \tag{88}$$

$$I_n = \frac{E \cos \phi}{4\, R_n \cos n\phi} . \tag{89}$$

Equating (87) and (89),

$$\frac{E \cos \phi}{4\, R_n\ \cos n\phi} = \frac{E\ F_{1n}(\phi)}{\pi(r + 2\rho) + 4\, R_n\, F_{nn}(\phi)} ; \tag{90}$$

$$R_n = \frac{\pi(r + 2\rho)}{4\,F_{1n}\frac{\cos n\phi}{\cos\phi} - 4\,F_{nn}(\phi)}\,, \tag{91}$$

Substituting (91) in (87),

$$I_n = \frac{E(F_{1n}\frac{\cos n\phi}{\cos\phi} - F_{nn})}{\pi(r + 2\rho)\,\frac{\cos n\phi}{\cos\phi}}\,; \tag{92}$$

$$G_n = \frac{8\,P_n\,r}{E^2} = \frac{8\,r\,\frac{1}{2}\,(2I_n^2)^2\,R_n}{E^2} = \frac{16r\,R_n I_n^2}{E^2}\,. \tag{93}$$

Substituting (91) and (92) in (93),

$$G_n = \frac{4}{\pi} = \frac{F_{1n}\,\frac{\cos n\,\phi}{\cos\phi} - F_{nn}}{\frac{\cos^2\,n\phi}{\cos^2\phi}}\;\frac{r}{r + 2\rho}\,. \tag{94}$$

We notice that (94) is equal to (31) when $\rho = 0$ but is slightly less than (30) when $\rho > 0$.

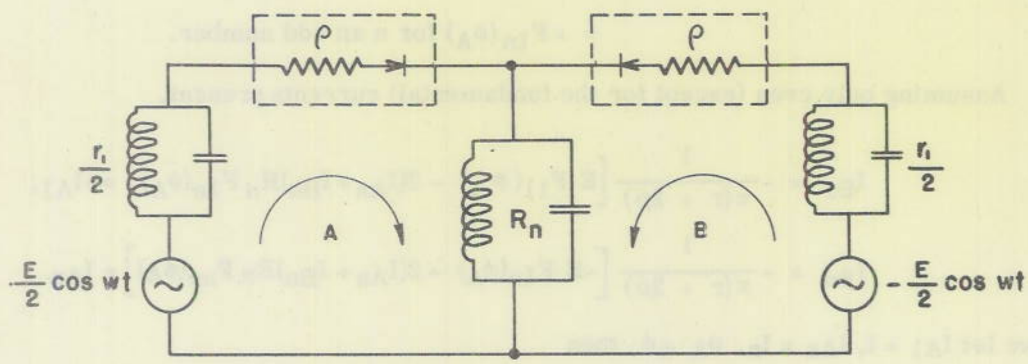

Figure 9 - Full-wave tuned generator feeding a tuned load

CASE VI: FULL-WAVE TUNED GENERATOR, TUNED LOAD

For the tuned generator, tuned load case (Figure 9), we can write the following equations for the period of rectifier conduction. In the "A" loop

$$\frac{E\cos\omega t}{2} = \frac{I_1 r_1}{2}\cos\omega t + i(t)\rho + (I_{An} + I_{Bn})\,R_n\cos n\omega t\,. \tag{95}$$

In the "B" loop

$$\frac{E\cos\omega t}{2} = \frac{I_1 r_1}{2}\cos\omega t + i(t)\rho + (I_{An} + I_{Bn})\,R_n\cos n\omega t\,. \tag{96}$$

As in the previous case, current can be assumed to flow in the "A" loop from $-\phi_A$ to $+\phi_A$ and in the "B" loop from $\pi - \phi_A$ to $\pi + \phi_A$ radians.

So

$$I_{Am} = \frac{2}{\pi\rho}\int_0^{\phi_A}\left[\frac{E-I_{A1}r_1}{2}\cos\omega t - (I_{An}+I_{Bn})R_n\cos n\omega t\right]\cos m\omega t\, d(\omega t); \quad (97)$$

$$I_{Bm} = \frac{2}{\pi\rho}\int_{\pi-\phi_A}^{\pi}\left[-\left(\frac{E+I_{B1}r_1}{2}\right)\cos\omega t - (I_{An}+I_{Bn})R_n\cos n\omega t\right]\cos n\omega t\, d(\omega t)\,. \quad (98)$$

For $m = 1$,

$$I_{A1} = \frac{1}{\pi\rho}\left[\frac{E-I_{A1}r_1}{2}F_{11}(\phi_A) - (I_{An}+I_{Bn})R_n F_{1n}(\phi_A)\right]. \quad (99)$$

Using (80) and (82),

$$-I_{B1} = I_{A1}\,. \quad (100)$$

For $m = n$ and $I_{B1} = -I_{A1}$,

$$I_{An} = \frac{1}{\pi\rho}\left[\frac{E-I_{1a}r_1}{2}F_{1n}(\phi) - (I_{An}+I_{Bn})R_n F_{nn}(\phi)\right]_0^{\phi_A}; \quad (101)$$

$$I_{Bn} = \frac{1}{\pi\rho}\left[-\frac{E-I_{A1}r_1}{2}F_{1n}(\phi) - (I_{An}+I_{Bn})R_n F_{nn}(\phi)\right]_{\pi-\phi_A}^{\pi}. \quad (102)$$

Using (80) and (82),

$$I_{An} = I_{Bn}\,. \quad (103)$$

Let $I_{A1} = I_1, I_{An} = I_n$, $\phi_A = \phi$, and $F_{mn}(\phi) = F_{mn}$. Then

$$I_1 = \frac{1}{\pi\rho}\left[\frac{E-I_1r_1}{2}F_{11} - 2I_nR_nF_{1n}\right]; \quad (104)$$

$$I_n = \frac{1}{\pi\rho}\left[\frac{E-I_1r_1}{2}F_{1n} - 2I_nR_nF_{nn}\right]. \quad (105)$$

From these two equations,

$$I_1 = \frac{EF_{11}(2\pi\rho + 4F_{nn}R_n) - 4EF_{1n}^2R_n}{(2\pi\rho + F_{11}r_1)(2\pi\rho + 4F_{nn}R_n) - 4F_{1n}^2r_1R_n}; \quad (106)$$

$$I_n = \frac{2\pi\rho E F_{1n}}{(2\pi\rho + F_{11}r_1)(2\pi\rho + 4F_{nn}R_n)\, 4F_{1n}r_1R_n}. \quad (107)$$

From (99) the cutoff equation for the "A" loop is

$$(E - I_1r_1)\cos\phi = 2E_n\cos n\phi\,. \quad (108)$$

Substituting (106) and (107) in (108), we obtain

$$R_n = \frac{\pi\rho}{2\left[F_{1n}\dfrac{\cos n\phi}{\cos\phi} - F_{nn}\right]}\,. \quad (109)$$

TABLE 2
Equations for G_n

Half Wave	Full Wave
Gain for Resistive Generator, Resistive Load	
$\frac{16\, r\, R}{\pi^2 (n^2-1)^2 (r+\rho+R)^2}$	$\frac{64\, r\, R}{\pi^2 (n^2-1)^2 (r+2\rho+2R)^2}$
for $\rho = 0, G_n$ maximized	
$\frac{4}{\pi^2 (n^2-1)^2}$	$\frac{8}{\pi^2 (n^2-1)^2}$
Gain for Resistive Generator, Tuned Load	
$\left(\frac{4}{\pi}\right) \frac{\left(F_{1n} \frac{\cos n\phi}{\cos\phi} - F_{nn}\right)}{\left(\frac{\cos^2 n\phi}{\cos^2\phi}\right)} \left(\frac{r}{r+\rho}\right)$	$\left(\frac{4}{\pi}\right) \frac{F_{1n} \frac{\cos n\phi}{\cos\phi} - F_{nn}}{\frac{\cos^2 n\phi}{\cos^2\phi}} \left(\frac{r}{r+2\rho}\right)$
Approximate Maximum Gain	
$\frac{8}{\pi^2 (n^2-1)^2} \left(\frac{r}{r+\rho}\right)$	$\frac{8}{\pi^2 (n^2-1)^2} \left(\frac{r}{r+2\rho}\right)$
Gain for Tuned Generator, Tuned Load	
$\frac{F_{1n} \frac{\cos n\phi}{\cos\phi} - F_{nn}}{\frac{\cos n\phi}{\cos\phi} \left(F_{11} \frac{\cos n\phi}{\cos\phi} - F_{1n}\right)}$	$\frac{F_{1n} \frac{\cos n\phi}{\cos\phi} - F_{nn}}{\frac{\cos n\phi}{\cos\phi} \left(F_{11} \frac{\cos n\phi}{\cos\phi} - F_{1n}\right)}$
Approximate Maximum Gain	
$\frac{4}{\pi^2 (n^2-1)^2}$	$\frac{4}{\pi^2 (n^2-1)^2}$

Substituting (109) in (107),

$$I_n = \frac{E\left(F_{1n}\frac{\cos n\phi}{\cos\phi} - F_{nn}\right)}{2\pi\rho\frac{\cos n\phi}{\cos\phi} + F_{11}r_1\frac{\cos n\phi}{\cos\phi} - F_{1n}r_1}\,; \qquad (110)$$

$$P_n = \frac{1}{2}(2I_n)^2 R_n = 2I_n^2 R_n\;; \qquad (111)$$

$$G_n = \frac{2I_n^2 R_n}{\frac{\frac{1}{2}E^2}{4r_1}} \quad \frac{16\, I_n^2\, r_1 R_n}{E^2}\,. \qquad (112)$$

Substituting (109) and (110) in (112),

$$G_n = \frac{8\pi\rho\, r_1\left(F_{1n}\frac{\cos n\phi}{\cos\phi} - F_{nn}\right)}{2\frac{\cos n\phi}{\cos\phi} + r_1\left(F_{1n}\frac{\cos n\phi}{\cos\phi} - F_{1n}\right)^2}\,. \qquad (113)$$

From (109), ϕ does not depend on r_1 so we can maximize G_n with respect to r_1:

$$G_n^{\max r_1} = \frac{F_{1n}\frac{\cos n\phi}{\cos\phi} - F_{nn}}{\frac{\cos n\phi}{\cos\phi}\left(F_{11}\frac{\cos n\phi}{\cos\phi} - F_{1n}\right.}\,. \qquad (114)$$

We notice that (114) is equal to (48), so for the tuned generator tuned load case the full-wave rectifier has no advantage in harmonic gain over the half-wave rectifier.

CONCLUSIONS

The gain of unbiased linear rectifier circuits as harmonic power generators decreases as $1/[\pi^2(n^2 - 1)^2]$, where n is the order of the harmonic and is an even number. Exact equations are given in Table 2.

High harmonic orders may be obtained by several multiplier stages or by only one with little difference in harmonic gain.

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