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Propagation and Interference of Vector Plane Waves

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14. ABSTRACT
The Navy has an interest in the free-space propagation of electromagnetic waves from radio frequencies to optical frequencies. Mathematically, these waves are generally represented as a vector in the full three-dimensional space. In many cases, simplified scalar analyses can be used to successfully model the beam propagation. In other cases, however, especially where the polarization of the beam is important, the scalar approach is insufficient and a more complete treatment - a vector approach - must be applied in order to accurately predict propagation.\par A critical aspect of either approach is to find a proper model for the source of the waves and, further, to find a model for propagation that is both physically accurate and mathematically tractable. To this end a well-known approach is the so-called angular-spectrum representation in which the radiation emitted by a source is modeled as a collection of simple plane waves of various strengths and propagating in various directions. However, in the case of a vector approach the plane waves must also include various states of polarization. Unfortunately, the inclusion of polarization in the angular spectrum approach is not well studied in the literature. \par The purpose of this report is to lay the foundation for a vector angular-spectrum representation theory by calculating and summarizing the relevant properties of vector plane waves. In addition to understanding their propagation properties, it is also necessary to understand how these waves interfere since, in the end, it is total power at a particular downrange position that is often of most interest.While I was unable to detect overwinters with these tools, several important observations were made regarding fire tracking and emergence. From these results it should be possible to automate new andresuming fire detection. Further, I explain how the tools developed here can slot into the operational flow of end-users thanks to GeoIPS status as a tactical decision aide. Future research is suggested, and plans for possible continuance are discussed.

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1 EXECUTIVE SUMMARY

The Navy has an interest in the free-space propagation of electromagnetic waves from radio frequencies to optical frequencies. Mathematically, these waves are generally represented as a vector in the full three-dimensional space. In many cases, simplified scalar analyses can be used to successfully model the beam propagation. In other cases, however, especially where the polarization of the beam is important, the scalar approach is insufficient and a more complete treatment - a vector approach - must be applied in order to accurately predict propagation.

A critical aspect of either approach is to find a proper model for the source of the waves and, further, to find a model for propagation that is both physically accurate and mathematically tractable. To this end a well-known approach is the so-called angular-spectrum representation in which the radiation emitted by a source is modeled as a collection of simple plane waves of various strengths and propagating in various directions. However, in the case of a vector approach the plane waves must also include various states of polarization. Unfortunately, the inclusion of polarization in the angular spectrum approach is not well studied in the literature.

The purpose of this report is to lay the foundation for a vector angular-spectrum representation theory by calculating and summarizing the relevant properties of vector plane waves. In addition to understanding their propagation properties, it is also necessary to understand how these waves interfere since, in the end, it is total power at a particular downrange position that is often of most interest.

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2 Introduction

A standard problem in electromagnetic wave propagation is the determination of the field and intensity at a point (x, y, z) in an observation plane (sometimes called the image plane) downrange from a point $(x_s, y_s, 0)$ in a source in the $z = 0$ plane (sometimes called the object plane) as shown in Fig. 1. The source is assumed to lie entirely in the $x, y-$plane.

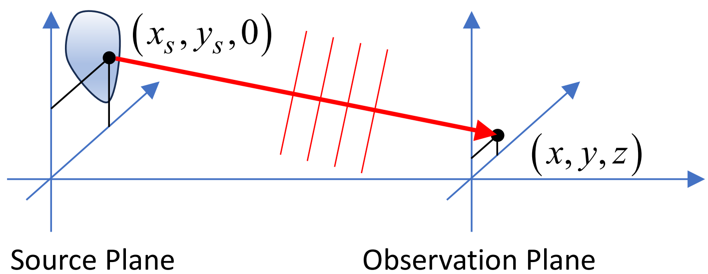

Figure 1: Geometry for propagation of optical waves from a point $(x_s, y_s, 0)$ in the source plane to a point (x, y, z) in the observation plane.

Propagation of the optical field from the source to the observation point must obey Maxwell's equations as well as the resultant wave equations. In the very simple case of a source emitting a single, monochromatic plane wave, the wave retains its shape and amplitude as it propagates. Only the wave's phase changes upon propagation and this is true for any plane wave. That is, *plane waves are modes or eigenfunctions of the monochromatic wave equation in free space.* An especially useful feature of modes is that the collection of all the modes forms a mathematical basis and, hence, any field distribution can be expressed as a linear combination of these basis functions. An approach for solving any free-space electromagnetic propagation problem in free space can be summarized as follows (see, for example, [1, 2]).

1. Write the source E-field distribution $E_s(x_s, y_s, 0)$ as a linear combination of plane-wave modes. This is the so-called angular spectrum representation (or, simply angular representation) of the source. As will be seen below, it turns out that this step is equivalent to taking the two-dimensional Fourier transform of the source distribution.

2. Choose a downrange observation point (x, y, z) . Multiply each term in the linear combination of modes by an appropriate propagation phase factor to account for the distance from $(x_s, y_s, 0)$ to (x, y, z).

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The result is the angular spectrum representation of the field in the image plane. In many cases, approximations can be made at this step to simplify the phase propagation factor. These include the Fresnel approximation and the Fraunhofer approximation.

3. Finally, the downrange E-field is obtained as the inverse Fourier transform of the image plane angular spectrum representation.

In the next section we summarize Maxwell's equations, the wave equations, the Helmholtz equation, and the paraxial Helmholtz equation, all for the case of propagation in free space and we discuss the situations in which each equation is applicable. In the literature various representations of the electric field a re c ommonly u sed d epending o n t he s ituation a nd w e are careful to note explicitly which form of the solution solves which equation exactly, and which form solves an equation only approximately.

Section 4 comprises the main body of results for this report. We develop in depth the characteristics and representations of an arbitrary, monochromatic, uniform *vector* plane wave, that is, a plane wave propagating in an arbitrary direction and having arbitrary, uniform state of polarization. We then apply various constraints to the arbitrary case, most notably, the paraxial approximation, and we derive the form for the representation of a plane wave in this approximation. This result (equation 29) is perhaps the main result of the report. In this section, following Roux [2], we show that the plane wave functions are orthogonal and complete and thus form a basis.

In Section 5 we discuss an important question: "How does one compare the polarization states for two plane wave propagating in different directions?" For co-propagating beams the question is easy to answer both theoretically and practically by means of direct measurement. For non-co-propagating beams the question cannot be answered unambiguously unless certain conventions are first a dopted. H owever, a r elated q uestion, namely, "What is the intensity resulting from the interference between two beams?" *can* be answered unambiguously and in this section we derive the result and provides a few numerical examples.

Finally, Section 6 provides a very brief summary of the concepts of transverse electric (TE) waves, transverse magnetic (TM) waves, and transverse electromagnetic (TEM) waves since they are often encountered in discussions in the literature regarding vector beams.

Appendix 1 provides a brief summary of the relationship between the paraxial approximation and the slowly-varying envelope approximation. Appendix 2 is a brief review of the calculation leading to the wave equation starting from Maxwell's equations.

3 Background

For the purposes of this report we make the following assumptions.

(A) All waves travel in free space and thus

  (a) the propagation is lossless,

  (b) the electric permittivity ϵ_0 and magnetic permeability μ_0 are those of free space, and

  (c) the wave propagates with the speed of light in vacuum $c = 1/\sqrt{\epsilon_0 \mu_0}$.

(B) At certain points in the discussion below we will simplify the analysis by making two important and related approximations, namely, a) the paraxial approximation and b) the slowly-varying envelope approximation (SVEA).

(C) The wave is monochromatic with optical (radian) frequency ω and corresponding wavenumber (propagation constant) $k = \omega/c$. To satisfy the wave equation, any solution must contain a function of the form $f(\omega t \mp kz)$. All other factors in any solution must be functions only of position $\mathbf{r}$. We will assume the waves propagate mainly in the $+z$ direction and we choose the propagation phase term to have the form $\exp i(\omega t - kz)$ in which we adopt the so-called positive "complex sign convention" [3].

(D) The terms "fields" and "waves" are used interchangeably.

(E) To simplify the language we assume the waves are optical waves, however, the results obtained below apply as well to electromagnetic radiation in free space at any frequency.

3.1 Maxwell's Equations and Wave Equations

Let $\mathbf{r}$ denote position in real 3-dimensional space represented in Cartesian coordinates by the vector $\mathbf{r} \equiv \begin{bmatrix} x & y & z \end{bmatrix}$. Maxwell's equations in free space in Cartesian coordinates are

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu_0 \, \partial_t \mathbf{H}(\mathbf{r}, t) \tag{1a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \epsilon_0 \, \partial_t \mathbf{E}(\mathbf{r}, t) \tag{1b}$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0 \tag{1c}$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0. \tag{1d}$$

where the quantities and their SI units are given by

$\mathbf{r}$	position vector $\begin{bmatrix} x & y & z \end{bmatrix}$	(m)
t	time	(s)
$\mathbf{E}(\mathbf{r},t)$	electric field strength vector	(V/m)
$\mathbf{B}(\mathbf{r},t)$	magnetic field induction vector	(V$\cdot$ s/m^2)
$\mathbf{H}(\mathbf{r},t)$	magnetic field strength vector	(A/m)
$\mathbf{D}(\mathbf{r},t)$	electric field displacement vector	(A$\cdot$ s/m^2)
ϵ_0	permittivity of free space	J/($m\cdot$V^2)
μ_0	permeability of free space	J/($m\cdot$A^2)

∂_t denotes partial derivative with respect to time and $\nabla = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix}$ is the del operator. Here

m is meters
t is seconds
V is volts
A is amperes
J is joules.

The constitutive relationships in free-space are given by

$$\mathbf{D}(\mathbf{r},t) = \epsilon_0 \mathbf{E}(\mathbf{r},t), \tag{2a}$$

$$\mathbf{B}(\mathbf{r},t) = \mu_0 \mathbf{H}(\mathbf{r},t). \tag{2b}$$

If we now apply the assumption of monochromatic fields and use the positive complex sign convention, Maxwells' equations become

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -i\omega\mu_0\, \mathbf{H}(\mathbf{r},t) \tag{3a}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = i\omega\epsilon_0\, \mathbf{E}(\mathbf{r},t) \tag{3b}$$

$$\nabla \cdot \mathbf{H}(\mathbf{r},t) = 0 \tag{3c}$$

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0. \tag{3d}$$

The following additional time-average optical quantities can be defined.

$$\text{Energy Density} \quad u\left(\frac{J}{m^3}\right) = \begin{cases} \frac{1}{2}\epsilon_0|\mathbf{E}|^2 + \frac{1}{2}\mu_0|\mathbf{H}|^2, & \text{General} \\ \epsilon_0|\mathbf{E}|^2, & \text{Plane wave} \end{cases}, \tag{4a}$$

$$\text{Poynting Vector } \mathbf{P}\left(\frac{J}{m^2\cdot s}\right) = \begin{cases} (1/2)\mathbf{E}\times\mathbf{H}^*, & \text{General} \\ (1/2)c\epsilon_0|\mathbf{E}|^2\hat{\mathbf{k}} = (1/2)cu\hat{\mathbf{k}} & \text{Plane wave} \end{cases}, \tag{4b}$$

where $\mathbf{k} = (2\pi/\lambda)\hat{\mathbf{n}}$ is the wavevector for a plane wave of wavelength λ propagating in direction $\hat{\mathbf{n}}$. The Poynting vector is the rate at which electromagnetic energy flows in the direction of unit vector $\hat{\mathbf{n}}$ per unit area per unit time, hence, $\mathbf{P}$ is *optical flux*. The magnitude of the Poynting vector is also called the *irradiance* or the *optical intensity* or, simply, the *intensity* of the beam.

Using the standard approach (see Appendix 1), a wave equation for each vector field $\mathbf{E}$ and $\mathbf{H}$ can be obtained from Maxwell's equations. An electric field $\mathbf{E}$ propagating in free space obeys the vector wave equation

$$\left(\nabla^2 - (1/c^2)\partial_{tt}\right)\mathbf{E}(x,y,z,t) = 0 \tag{5}$$

and the corresponding magnetic field $\mathbf{H}$ obeys the same vector wave equation,

$$\left(\nabla^2 - (1/c^2)\partial_{tt}\right)\mathbf{H}(x,y,z,t) = 0 \tag{6}$$

where the operators

$$\begin{aligned} \partial_{tt} &= \partial^2/\partial t^2, \\ \text{and} \quad \nabla^2 &= \partial_{xx} + \partial_{yy} + \partial_{zz}. \end{aligned}$$

Each equation (5) and (6) is compact notation for three equations, one for each component of the field vector.

3.2 Helmholtz and Paraxial Helmholtz Wave Equations

Assume the field is monochromatic with optical frequency ω and thus can be written generally as

$$\mathbf{E}(x,y,z,t) = \mathbf{U}(x,y,z)e^{i\omega t} = \mathbf{V}(x,y,x)e^{i\zeta(x,y,z)}e^{i\omega t}. \tag{8}$$

where the complex amplitude vector $\mathbf{U}(x,y,z)$ can itself be expressed as a phasor with real amplitude $\mathbf{V}(x,y,x)$ and real phase $\zeta(x,y,z)$, both of which depend only on spatial coordinates and not time. Under this assumption the wave equation reduces to the **Helmholtz wave equation** (HE)

$$\left(\nabla^2 + k^2\right)\mathbf{E}(x,y,z) = \left(\nabla^2 + k^2\right)\mathbf{U}(x,y,z) = 0, \tag{9a}$$

The Helmholtz wave equation applies only to a monochromatic beam. It is solved exactly by both the time-independent complex amplitude vector $\mathbf{U}(x, y, z)$ *and also by the full E-field vector* $\mathbf{E}(x, y, z, t)$*.*

We now discuss the modal solutions of the HE - a topic we touched on briefly in the Introduction. Typically, given a set of boundary conditions for a propagating monochromatic wave, there exist multiple solutions. It is well-known (see, for example, [1, 4]) that such a set of solutions forms a basis and each member of the set is called a **mode**. That is, an arbitrary propagating beam can always be represented by a linear combination of the modes. *In Cartesian coordinates in free space the modal solutions of the Helmholtz equation are plane waves.* The great utility of a modal representation is that, upon propagation, the amplitude of each mode remains unchanged while the mode's phase changes as given by the argument of $\exp i(\omega t - k\hat{\mathbf{n}} \cdot \mathbf{r})$. In a collection of modes, each mode accumulates phase at its own rate and, by definition, the modes never interact with each other.

Assume now that all parts of the beam propagates approximately in the z-direction. Write the complex amplitude $\mathbf{U}(x, y, z)$ as the product of a complex vector amplitude function $\mathbf{W}(x, y, z)$ that varies slowly in the $z-$direction and the function $\exp(-ikz)$ that accounts for the rapid changes in phase as the beam propagates downrange,

$$\mathbf{U}(x, y, z)\big|_{SVEA} = \mathbf{W}(x, y, z)e^{-ikz}. \tag{10}$$

We assume the spatial divergence of the beam is small (hence, spherical waves will not be considered.) This assumption is incorporated into the solution by making the so-called *slowly-varying-envelope approximation* (SVEA) in which it is assumed that, over $z-$distances of a wavelength or so, variations in the beam amplitude are small compared to the amplitude itself. With this approximation, the Helmholtz equation reduces to the so-called parabolic wave equation or the *paraxial Helmholtz equation* (PHE)

$$\left(\nabla_T^2 - 2ik\partial_z\right)\mathbf{W}(x, y, z) = 0 \tag{11}$$

where $\nabla_T = \partial_{xx} + \partial_{yy}$ is the transverse Laplacian. (For details of this derivation as well as a general discussion of the relationship between the paraxial approximation and the SVEA, see Appendix 2).

The paraxial Helmholtz equation (PHE) is solved exactly only by the slowly-varying complex amplitude $\mathbf{W}(x, y, z)$*. Neither the complex vector amplitude* $\mathbf{U}(x, y, z)|_{SVEA}$ *nor the full vector field* $\mathbf{E}(x, y, z, t)$ *are exact solutions of the PHE.*

4 Monochromatic, Vector Plane Waves with Uniform Polarization in Free Space

We remarked in the previous section that the modes of electromagnetic propagation in free space in Cartesian coordinates are plane waves and that we can use this fact to describe optical propagation within the Huygens-Fresnel framework. In this section we discuss various details of plane waves and plane-wave propagation in free space.

Scalar plane waves are prevalent in the literature and are useful in problems where a) the entirety of the beam is linearly polarized; b) the coordinate system is chosen so that one of the coordinate axes is oriented along the direction of polarization so that there is only one non-zero component of the field; and c) the polarization state does not change upon propagation.[1] A typical representation of a monochromatic, "scalar" plane wave propagating in the direction of unit vector $\hat{\mathbf{n}}$ is

$$E_p(\overrightarrow{r}) = E_0\, e^{i(\omega t - k\overrightarrow{r}\cdot\hat{\mathbf{n}})}$$

In contrast, a vector plane wave has the general representation

$$\mathbf{E}_p(\overrightarrow{r}) = E_0\, \hat{\xi}(\hat{\mathbf{n}})\, e^{i(\omega t - k\overrightarrow{r}\cdot\hat{\mathbf{n}})}$$

where the complex unit vector $\hat{\xi}(\hat{\mathbf{n}})$ depends on both the state of polarization and the direction of propagation. Note that both scalar and the vector plane waves are unphysical in the sense that they have finite amplitude E_0 and are of infinite extent in the transverse direction. However, they are perfectly good mathematical objects in the same sense that a pure sine wave $\sin 2\pi f t$ is of infinite temporal extent. Just as a collection of sine waves in the form of a Fourier integral can be an accurate representation of a finite, integrable time-domain waveform so too a collection of plane waves can serve as an accurate model of a finite, integrable optical source in space. We next present a summary of the relevant properties of vector plane waves.

A monochromatic vector plane wave propagating in free space where *the entire beam is in the same state of polarization* has the following characteristics.

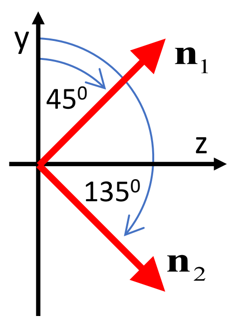
[1]Depending on one's definition of "scalar", the term may be seriously misused in this context. If one uses scalar to mean a quantity whose value is invariant under coordinate transformations then "scalar plane wave" is an oxymoron. However, if one uses scalar to mean just a number, ignoring the coordinate system in which the number is represented, then "scalar plane wave" perhaps has meaning.

4.1 Directionality

The wave propagates in a single direction defined by unit vector $\hat{\mathbf{n}} = \begin{bmatrix}\alpha & \beta & \gamma\end{bmatrix}$ where α, β and γ are the direction cosines of $\hat{\mathbf{n}}$ with respect to the positive axes of the chosen Cartesian coordinate system. Since $\hat{\mathbf{n}}$ is a unit vector the direction cosines are satisfy the constraint

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{12}$$

Since $\gamma^2 = 1 - \alpha^2 - \beta^2$, the propagation direction is specified completely by just two of the three direction cosines. We follow tradition and choose the pair (α, β). **NOTE:** Since direction cosines are always measured in terms of angles with respect to the *positive* axis directions, α and β can be positive or negative depending on whether the angle is less than or greater than $\pi/2$ as illustrated in Fig. 2.



$$\hat{\mathbf{n}}_1 = \begin{bmatrix}\alpha_1 & \beta_1 & \gamma_1\end{bmatrix} = \begin{bmatrix}0 & 0.707 & 0.707\end{bmatrix}$$

$$\hat{\mathbf{n}}_2 = \begin{bmatrix}\alpha_2 & \beta_2 & \gamma_2\end{bmatrix} = \begin{bmatrix}0 & -0.707 & 0.707\end{bmatrix}$$

Figure 2: Illustrating the direction cosines for two beams , both propagating at 45^0 with respect to the z-axis but with one at 45^0 with respect to the positive $y-$axis and the other at 135^0 with respect to the positive $y-$axis.

4.2 Constant Amplitude

The amplitude of a plane wave is constant everywhere in space and remains unchanged during propagation.

4.3 Phase Variation

Consider a geometric plane perpendicular to $\hat{\mathbf{n}}$ at some location $\mathbf{r}_1$ as shown in Fig. 3. The phase of the wave at this location, call it ψ_1, is constant everywhere on this plane. At a fixed time, if we evaluate the wave at a different location along the $\hat{\mathbf{n}}$ direction say, at $\mathbf{r}_2$, we find that the phase has changed by an amount $\Delta\psi = \psi_2 - \psi_1 = -k|\mathbf{r}_2 - \mathbf{r}_1|$. That is, as a function of distance along the propagation direction at a fixed time, the phase retards at a rate k radians per meter.

Correspondingly, at a fixed location, as a function of time, the phase advances at a rate ω radians per second.

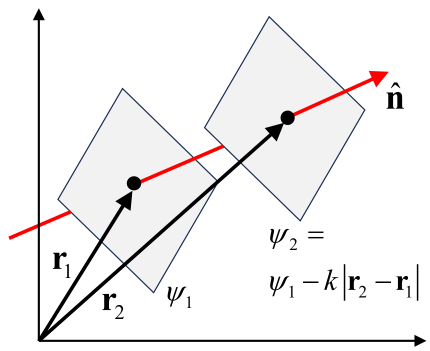


Figure 3: Plane wave propagating in the direction $\hat{\mathbf{n}}$. In the plane transverse to $\hat{\mathbf{n}}$ at $\mathbf{r}_1$ the amplitude is everywhere constant out to infinity and the phase is constant with value, say, ψ_1. At some fixed time (snapshot), for the same plane wave evaluated instead at position $\mathbf{r_2}$ the phase is retarded by the amount $k|\mathbf{r_2} - \mathbf{r_1}|$.

4.4 Transversality

Both the electric field of the plane wave, denoted here by $\mathbf{E}_p$, and the associated magnetic field $\mathbf{H}_p$ lie entirely in the plane transverse to $\hat{\mathbf{n}}$ and are related by the right-hand rule

$$\mathbf{H}_p = Z_0^{-1}\,\mathbf{n} \times \mathbf{E}_p \tag{13}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Thus $\mathbf{E_p}$ and $\mathbf{H}_p$ satisfy the so-called transversality conditions

$$\hat{\mathbf{n}} \cdot \mathbf{E}_p = \hat{\mathbf{n}} \cdot \mathbf{H}_p = \mathbf{E}_p \cdot \mathbf{H}_p = 0. \tag{14}$$

4.5 Representation of an Arbitrary Vector Plane Wave

A plane wave of arbitrary, uniform state of polarization propagating in direction $\hat{\mathbf{n}}$ with direction cosines (α, β, γ), evaluated at position (x, y, z) and at time t, can be represented by a complex vector, that is, a vector in which each component is itself a complex number.

$$\mathbf{E}_p(x,y,z,t;\alpha,\beta) = E_0 \begin{bmatrix} \xi_x \\ \xi_y \\ \xi_z \end{bmatrix} \exp i(\omega t - k\hat{\mathbf{n}}\cdot\mathbf{r}), \tag{15a}$$

$$= E_0 \begin{bmatrix} \xi_x \\ \xi_y \\ \xi_z \end{bmatrix} e^{i\omega t} \exp\big(-ik(\alpha x + \beta y + \gamma z)\big), \tag{15b}$$

$$= E_0 \begin{bmatrix} \xi_x \\ \xi_y \\ \xi_z \end{bmatrix} e^{i\omega t} \exp\big(-i2\pi(ax + by + c(a,b)z)\big). \tag{15c}$$

where E_0 is the wave amplitude, the complex constants ξ_x, ξ_y, ξ_z define the polarization state, and where

$$a = \frac{\alpha}{\lambda}, \qquad b = \frac{\beta}{\lambda}, \qquad c(a,b) = \Big(\frac{1}{\lambda^2} - a^2 - b^2\Big)^{1/2}. \tag{16}$$

An important caveat on notation. In some cases the the direction cosine notation α, β, γ clarifies and eases the calculation such as when we calculate interference between plane waves in Section 5.2. In other cases, such as when the plane-wave angular spectrum representation is used, the spatial frequency components notation a, b, c is more useful since then certain integrals become standard 2D Fourier transform integrals. The reader is thus cautioned that we may switch back and forth between these notation and, in some cases, even mix the notations, depending on the circumstance.

Continuing, from (13), the associated magnetic field is given by

$$\mathbf{H}_p = Z_0^{-1}\mathbf{n} \times \mathbf{E}_p \tag{17a}$$

$$= E_0\sqrt{\frac{\epsilon_0}{\mu_0}} \begin{bmatrix} \beta\xi_z - \gamma\xi_y \\ -\alpha\xi_z + \gamma\xi_x \\ \alpha\xi_y - \beta\xi_x \end{bmatrix} e^{i\omega t} \exp\big(-ik(\alpha x + \beta y + \gamma z)\big). \tag{17b}$$

$$= E_0\sqrt{\frac{\epsilon_0}{\mu_0}} \begin{bmatrix} \beta\xi_z - \gamma\xi_y \\ -\alpha\xi_z + \gamma\xi_x \\ \alpha\xi_y - \beta\xi_x \end{bmatrix} e^{i\omega t} \exp\big(-i2\pi(ax + by + c(a,b)z)\big). \tag{17c}$$

The complex amplitudes in (15) must satisfy the constraint imposed by Maxwell's equation (1d)

$$\nabla \cdot \mathbf{E}_p = 0$$

which implies

$$(-ik)\big(\alpha\xi_x + \beta\xi_y + \gamma\xi_z\big)E_0 = 0. \tag{18}$$

This can be written as a constraint on ξ_z,

$$\xi_z = \frac{-(\alpha\xi_x + \beta\xi_y)}{\gamma}. \tag{19}$$

With this constraint we find

$$|\xi_x|^2 + |\xi_y|^2 + |\xi_z|^2 = \left(1 + \frac{\alpha^2}{\gamma^2}\right)|\xi_x|^2 + \left(1 + \frac{\beta^2}{\gamma^2}\right)|\xi_y|^2 + 2\frac{\alpha\beta}{\gamma^2}\mathcal{R}e(\xi_x^*\xi_y). \tag{20}$$

Hence, the electromagnetic fields comprising a monochromatic plane wave of uniform polarization state and propagating with direction cosines $(\alpha,\, \beta)$ can be represented by the complex vectors

$$\mathbf{E}_p(x, y, z, t; \alpha, \beta) = \frac{E_0}{q}\begin{bmatrix} \xi_x \\ \xi_y \\ \dfrac{-1}{\gamma}\Big(\alpha\xi_x + \beta\xi_y\Big) \end{bmatrix} e^{i\omega t}e^{-ik(\alpha x + \beta y + \gamma z)}, \tag{21a}$$

$$\mathbf{H}_p(x, y, z, t; \alpha, \beta) = \frac{E_0}{q}\sqrt{\frac{\epsilon_0}{\mu_0}}\begin{bmatrix} -\dfrac{\alpha\beta}{\gamma}\xi_x - \left(\dfrac{\beta^2}{\gamma} + \gamma\right)\xi_y \\ \left(\dfrac{\alpha^2}{\gamma} + \gamma\right)\xi_x + \dfrac{\alpha\beta}{\gamma}\xi_y \\ \alpha\xi_y - \beta\xi_x \end{bmatrix} e^{i\omega t}e^{-ik(\alpha x + \beta y + \gamma z)}, \tag{21b}$$

$$q^2 = \left(1 + \frac{\alpha^2}{\gamma^2}\right)|\xi_x|^2 + \left(1 + \frac{\beta^2}{\gamma^2}\right)|\xi_y|^2 + 2\frac{\alpha\beta}{\gamma^2}\mathcal{R}e(\xi_x^*\xi_y), \tag{21c}$$

$$\gamma^2 = 1 - \alpha^2 - \beta^2. \tag{21d}$$

The normalization factor q in the denominator guarantees that the amplitude of $\mathbf{E}_p$ and $\mathbf{H}_p$ are E_0 and E_0/Z_0, respectively. As written here, $\mathbf{E_p}$ and $\mathbf{H_p}$ exactly satisfy all Maxwell's equations.[2]

Now if we had neglected the $z-$component E-field, that is, if we had set $\xi_z = 0$, then the field $\mathbf{E}_p$ would not satisfy (1d) to first order in the direction cosines,

$$\nabla \cdot \mathbf{E}_p = (ik)\,(\alpha\xi_x + \beta\xi_y) \neq 0. \tag{22}$$

However, (1d) *is* satisfied exactly if the beam propagates exactly along the $z-$axis since then $\alpha = \beta = 0$. Otherwise, the error grows linearly with α and β, that is, as the beam direction moves further off the $z-$axis.

We note that the vectors in (21a) and (21b) are members of a Euclidean linear vector space called a 3-dimensional Hilbert space. In this space, for arbitrary vectors $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$, the *norm* of a vector is defined by

$$|\mathbf{u}|^2 = |u_1|^2 + |u_2|^2 + |u_3|^2$$

and the *complex scalar product* is defined by

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) &= \mathbf{u}^* \cdot \mathbf{v} \\ &= \begin{bmatrix} u_1^* & u_2^* & u_3^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \sum_{i=1}^{3} u_i^* v_i. \end{aligned}$$

For the purposes of this report, we assume that all fields are monochromatic at the same frequency ω and, hence, the scalar product will always eliminate the time-dependent term and leave only terms with spatial dependence.

Physically, the fields $\mathbf{E_p}$ and $\mathbf{H_p}$ must be real quantities but for the subsequent calculations we here *choose* to represent the fields in complex form. These quantities *do not* represent the fields in real 3-D space and, consequently, some care must be exercised in interpreting the results of operations — especially nonlinear operations — on complex field quantities.

[2]If we specify the beam to be propagating along the, say, $y-$axis in the $yz-$plane, then $\alpha = 0$ and $\beta = 1$ and q is undefined since $\gamma = 0$. This is an artifact of putting the constraint (19) onto the amplitude of the $z-$component. For a wave propagating along the $y-$axis we would put the constraint instead on the $y-$component in which case the $y-$component will be zero and q will not be undefined.

4.6 Alternate Representation of an Arbitrary Vector Plane Wave

For reasons that will become clear momentarily we now introduce the following notation,

$$\xi_x = e^{i\phi_x} \cos\theta/2, \quad (23a)$$
$$\xi_y = e^{i\phi_y} \sin\theta/2, \quad (23b)$$

$$\phi = \big(\phi_y - \phi_x\big), \quad (23c)$$
$$\overline{\phi} = (1/2)\big(\phi_y + \phi_x\big), \quad (23d)$$

where $-\pi/2 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$. Then (21a), (21b) and (21c) become, respectively,

$$\mathbf{E}_p(x, y, z, t; \alpha, \beta, \theta, \phi, \overline{\phi})$$
$$= \frac{E_0\, e^{i\overline{\phi}}}{q} \begin{bmatrix} e^{-i\phi/2} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi/2} \sin\left(\frac{\theta}{2}\right) \\ \frac{-1}{\gamma}\left(\alpha e^{-i\phi/2} \cos\left(\frac{\theta}{2}\right) + \beta e^{i\phi/2} \sin\left(\frac{\theta}{2}\right)\right) \end{bmatrix} e^{i\omega t} e^{\left(-ik(\alpha x + \beta y + \gamma z)\right)}. \quad (24a)$$

$$\mathbf{H}_p(x, y, z, t; \alpha, \beta, \theta, \phi, \overline{\phi})$$
$$= \frac{E_0\, e^{i\overline{\phi}}}{q} \sqrt{\frac{\epsilon_0}{\mu_0}} \begin{bmatrix} -\frac{\alpha\beta}{\gamma} e^{-i\phi/2} \cos\left(\frac{\theta}{2}\right) - \left(\frac{\beta^2}{\gamma} + \gamma\right) e^{i\phi/2} \sin\left(\frac{\theta}{2}\right) \\ \left(\frac{\alpha^2}{\gamma} + \gamma\right) e^{-i\phi/2} \cos\left(\frac{\theta}{2}\right) + \frac{\alpha\beta}{\gamma} e^{i\phi/2} \sin\left(\frac{\theta}{2}\right) \\ \alpha e^{i\phi/2} \sin\left(\frac{\theta}{2}\right) - \beta e^{-i\phi/2} \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$
$$\cdot\; e^{i\omega t} e^{-ik(\alpha x + \beta y + \gamma z)}, \quad (24b)$$

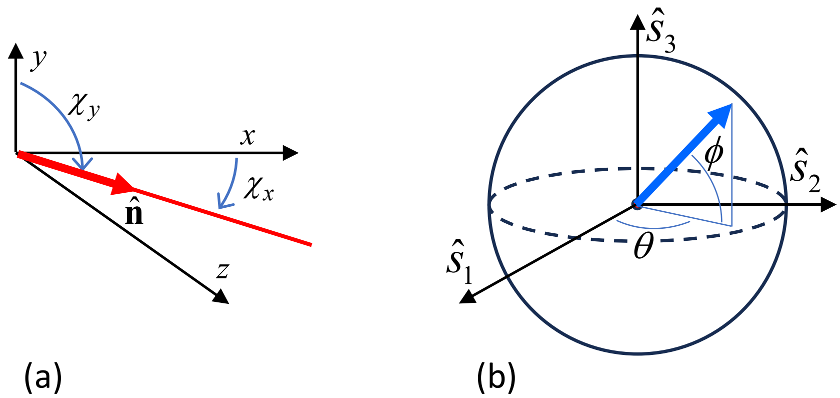


Figure 4: (a) Direction cosines $\alpha = \cos\chi_x$ and $\beta = \cos\chi_y$ in real space define the direction of propagation. (b) Angles on the Poincare sphere corresponding to θ and ϕ in Eq. (24a) define the state of polarization.

$$q^2 = 1 + \frac{1}{\gamma^2}\left[\alpha^2\cos^2\left(\frac{\theta}{2}\right) + \beta^2\sin^2\left(\frac{\theta}{2}\right) + \alpha\beta\sin\theta\cos\phi\right]. \tag{24c}$$

The reason for the notation change is that, in the special case where the beam propagates exactly along the $z-$axis ($\alpha = \beta = 0$), the angles (θ, ϕ) map directly on to the Poincare-sphere representation of polarization states [3] as shown in Fig. 4. The limits on θ and ϕ were thus chosen to guarantee that a) $\cos\theta/2$ and $\sin\theta/2$ are always non-negative amplitudes, and b) the phase covers the full range of possible polarization states on the sphere.

For waves propagating along an arbitrary direction we hesitate to associate θ and ϕ with "linear polarization angle" and "ellipticity", respectively. Instead, we will simply refer to the pair of angles (θ, ϕ) as characterizing the state of polarization of the beam. In the next section we will investigate the interference of two or more beams having arbitrary directions, overall phases, and arbitrary polarization states. Perhaps as expected, we will find that the interference depends generally on *differences* in these quantities.

In summary, for uniform polarization, a plane wave has six degrees of freedom:

[3]The mapping assumes the following ordering of Pauli matrices in defining the axes of the Poincare sphere $\hat{s}_1 : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\hat{s}_2 : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\hat{s}_3 : \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$.

- *amplitude* E_0*,*
- *overall phase* $\bar{\phi}$*,*
- *direction cosines* α *and* β*, that specify the direction of propagation, and*
- *angles* θ *and* ϕ *that specify the state of polarization.*

Note: a) The overall phase $\bar{\phi}$ does not include the linear phase shift accrued upon propagation. The propagation phase is taken into account in the term $\exp(-ik\mathbf{n}\cdot\mathbf{r})$. b) In some cases, the overall phase $\bar{\phi}$ may depend on the polarization state (θ, ϕ). This can occur as a result of the specific manner in which the beam's SOP was prepared in which case it is related to the geometric or Pancharatnam phase.

4.7 Condition Under Which (24a) is an Exact Solution

It is instructive to determine the situations for which (24a) is an exact solution. It is straightforward to show that (24a) satisfies exactly the conditions

- Constant amplitude $|\mathbf{E}_p| = E_0$,
- Maxwell equation $\nabla \cdot \mathbf{E}_p = 0$.

and each component of $\mathbf{E}_p$ exactly satisfies

- Wave equation $\left(\nabla^2 - (1/c^2)\partial_{tt}\right)\mathbf{E}_p = 0$,
- Helmholtz equation $\left(\nabla^2 + k^2\right)\mathbf{U}_p = \left(\nabla^2 + k^2\right)\mathbf{E}_p = 0$.

To test the paraxial Helmholtz equation (PHE) first recall that the full vector field $\mathbf{E}_p$ is not a solution of the PHE. Instead, using (8) and (10), recall that

$$\mathbf{E}_p = \mathbf{W}_p e^{i(\omega t - kz)} \tag{25}$$

and thus we test whether $\mathbf{W}_p$ solves the PHE where

$$\mathbf{W}_p(x,y,z,t;\alpha,\beta,\,\theta,\,\phi,\bar{\phi})$$

$$= \frac{E_0\, e^{i\bar{\phi}}}{q} \begin{bmatrix} e^{-i\phi/2}\cos\left(\dfrac{\theta}{2}\right) \\ e^{i\phi/2}\sin\left(\dfrac{\theta}{2}\right) \\ \dfrac{-\left(\alpha e^{-i\phi/2}\cos\left(\dfrac{\theta}{2}\right)+\beta e^{i\phi/2}\sin\left(\dfrac{\theta}{2}\right)\right)}{\gamma} \end{bmatrix} e^{\left(-ik(\alpha x+\beta y+(\gamma-1)z)\right)} \tag{26}$$

$$= \frac{E_0\, e^{i\bar{\phi}}}{q} \begin{bmatrix} e^{-i\phi/2}\cos\left(\dfrac{\theta}{2}\right) \\ e^{i\phi/2}\sin\left(\dfrac{\theta}{2}\right) \\ \dfrac{-\left(\alpha e^{-i\phi/2}\cos\left(\dfrac{\theta}{2}\right)+\beta e^{i\phi/2}\sin\left(\dfrac{\theta}{2}\right)\right)}{\gamma} \end{bmatrix} e^{\left(-i2\pi(ax+by+\left(c-\frac{1}{\lambda}\right)z)\right)}. \tag{27}$$

and note that $(c-1/\lambda)=\left(1/\lambda^2-a^2-b^2\right)^{1/2}-1/\lambda$.

Unfortunately, this $\mathbf{W}_p$ *does not solve the PHE* in general but the error is second-order in the direction cosines α and β:

- Paraxial Helmholtz Equation
  $\left(\nabla_T^2-2ik\partial_z\right)\mathbf{W}_p=k^2\left[1-(\alpha^2+\beta^2)-(1-\alpha^2-\beta^2)^{1/2}\right]\neq 0.$

However, if $\alpha^2+\beta^2=0$, then $\mathbf{W}_p$ does solve the PHE exactly.

Earlier in (10) we proposed a solution $\mathbf{U}_{SVEA}=\mathbf{W}\exp(-ikz)$ to the Helmholtz equation under the slowly-varying envelope approximation and we found that the complex amplitude $\mathbf{W}$ satisfies the paraxial Helmholtz equation (PHE). Here we have just discovered that *the only plane wave solutions to the PHE are plane waves of arbitrary polarization propagating exactly along the $z-$axis* where $\alpha=\beta=0$ and $\gamma=1$. If we insist on using off-axis plane waves as solutions to the PHE then we will incur an error but the error is second-order in α and β.

In the paraxial regime $(\alpha^2 + \beta^2) << 1$ and the phase term can be approximated as

$$-i2\pi(ax + by + (c - 1/\lambda)z) = -\, i2\pi\Big(ax + by + \Big[\Big(\frac{1}{\lambda^2} - a^2 - b^2\Big)^{1/2} - \frac{1}{\lambda^2}\Big]\, z\Big)$$

$$\approx -\, i2\pi\left(ax + by - \frac{\lambda z}{2}\Big(a^2 + b^2\Big)\right). \tag{28}$$

In this approximation, (26) becomes

$$\mathbf{W}_{p,\,parax}(\mathbf{r}, t; \alpha, \beta,\, \theta,\, \phi, \bar{\phi})$$

$$= \frac{E_0\, e^{i\bar{\phi}}}{q} \begin{bmatrix} e^{-i\phi/2}\cos\left(\dfrac{\theta}{2}\right) \\ e^{i\phi/2}\sin\left(\dfrac{\theta}{2}\right) \\ \dfrac{-\left(\alpha e^{-i\phi/2}\cos\left(\dfrac{\theta}{2}\right) + \beta e^{i\phi/2}\sin\left(\dfrac{\theta}{2}\right)\right)}{\gamma} \end{bmatrix}$$

$$\cdot \exp -i2\pi\left(ax + by - \frac{\lambda z}{2}\Big(a^2 + b^2\Big)\right) \tag{29}$$

and it is straightforward to show that $\mathbf{W}_{p,\,parax}$ does indeed solve the PHE exactly ,

- Paraxial Helmholtz equation $\Big(\nabla_T^2 - 2ik\partial_z\Big)\mathbf{W}_{p,\,parax} = 0$.

Hence, in the paraxial approximation, off-axis plane waves are solutions to the PHE provided the proper phase propagation term is used as in (29).

Numerical Example. Consider a plane wave at $\lambda = 1.55\mu m$ that deviates from the $z-$ axis by 1 meter in the $y-$direction at downrange distance $z = 30$m. Then $\chi_y = \pi/2 - 1/30 = 1.5375\,$rad and $(\alpha = 0, \beta = 0.0333, \gamma = 0.9994)$ giving $(a = 0, b = \beta/\lambda = 0.0333\lambda, c = \sqrt{1/\lambda^2 - b^2} = 0.9994/\lambda)$. Assume the beam is linearly-polarized $\phi = 0$ with $\theta = \pi/2$ giving $q = 1.0006$. Then the exact solution

$\mathbf{W}_{p,parax}$ in (29) to the PHE becomes

$$\mathbf{W}_{p,parax} = \frac{E_0\, e^{i\overline{\phi}}}{1.0006\sqrt{2}} \begin{bmatrix} 1 \\ \\ 1 \\ \\ -0.033 \end{bmatrix} e^{-i2\pi(\dfrac{0.0333}{\lambda}\cdot y-(0.001/2)\cdot z)} \tag{30}$$

which is approximately a beam with zero $z-$component since the power in the $z-$component is 30dB below the power in the other components.

In this same approximation $\mathbf{E}_{p,\,parax} = \mathbf{W}_{p,\,parax} \exp i(\omega t - kz)$ satisfies the requirement for constant amplitude E_0

- Constant amplitude $\left|\mathbf{E}_{p,\,parax}\right| = E_0$.

However, by inserting the paraxial approximation into the phase of $\mathbf{E}_{p,\,parax}$, this total field $\mathbf{E}_{p,\,parax}$ itself no longer satisfies exactly any of the following equations

- Maxwell's equation
  $\nabla \cdot \mathbf{E}_{p,\,parax} = (-ik)\left(1 + \alpha + \beta - (1/2)(\alpha^2 + \beta^2)\right) \neq 0$, and the residual here is first-order in the direction cosines.

- Wave equation
  $\left(\nabla^2 - (1/c^2)\partial_{tt}\right)\mathbf{E}_{p,\,parax} = -(k^2/4)(\alpha^2 + \beta^2) \neq 0,$

- Helmholtz equation
  $\left(\nabla^2 + k^2\right)\mathbf{E}_{p,\,parax} = -(k^2/4)(\alpha^2 + \beta^2) \neq 0,$

In these last two cases, the error is second-order in the direction cosines.

In the special case where the plane wave propagates exactly along the $z-$axis, then $\alpha = \beta = 0$ and (26) becomes simply

$$\mathbf{E}_{p,\,z-axis}(x, y, z, t; 0, 0, \theta, \phi, \overline{\phi}) = e^{i\overline{\phi}} \begin{bmatrix} e^{-i\phi/2}\cos\left(\dfrac{\theta}{2}\right) \\ \\ e^{i\phi/2}\sin\left(\dfrac{\theta}{2}\right) \\ \\ 0 \end{bmatrix} e^{i(\omega t - kz)}$$
$$\equiv \mathbf{W}_{p,\,z-axis}(x, y, z, t; 0, 0, \theta, \phi, \overline{\phi})\, e^{i(\omega t - kz)} \tag{31}$$

where the upper two components in brackets are recognized as comprising the standard Jones vector for a field of arbitrary uniform polarization.

In this very special case the field $\mathbf{E}_{p,\,z-axis} = \mathbf{W}_{p,\,z-axis}e^{i(\omega t-kz)}$ exactly satisfies *all* the equations

- Constant amplitude $\left|\mathbf{E}_{p,\,z-axis}\right| = E_0$.
- Maxwell's equation $\nabla \cdot \mathbf{E}_{p,\,z-axis} = 0$,
- Wave equation $\left(\nabla^2 + (1/c^2)\partial_{tt}\right)\mathbf{E}_p\big|_{z-axis} = 0$,
- Helmholtz equation $\left(\nabla^2 + k^2\right)\mathbf{E}_{p,\,z-axis}e^{-ikz} = 0$.
- Paraxial Helmholtz equation $\left(\nabla_T^2 - 2ik\partial_z\right)\mathbf{W}_{p,\,z-axis} = 0$.

From these results we see that the entire machinery of Jones calculus and the entirety of the associated Stokes calculus assumes the very specific case of a plane wave of uniform polarization propagating exactly along the Cartesian $z-$axis.

A brief summary of vector plane-wave representations and the equations they solve is presented in Fig. 5.

To conclude this section we note that a solitary plane wave travelling along an arbitrary direction can always be analyzed as propagating along a new $z-$axis by a simple transformation of coordinates and all the cool results obtained above apply as well. The challenge begins when we must deal simultaneously with two or more plane waves propagating in arbitrary directions, as discussed in the next section.

Summary:

Monochromatic Plane Wave Propagating with Direction Cosines (α, β)

$$\mathbf{E_p}(x,y,z,t) = \frac{E_0}{q}\begin{bmatrix}\xi_x\\ \xi_y\\ \xi_z\end{bmatrix} e^{i(\omega t - 2\pi(ax+by+cz))} = \mathbf{U_p}(x,y,z)e^{i\omega t} \xrightarrow[Parax,\ SVEA]{} \mathbf{W_p}(x,y,z)e^{i(\omega t - kz)}$$

Description	**Field**	**Solves ...**
A) Full Field	$\mathbf{E_p}(x,y,z,t)$	Maxwell's Equations Wave Equation Helmholtz Equation ~~Paraxial Helmholtz Equation~~
B) Complex Amplitude	$\mathbf{U_p}(x,y,z)$	~~Maxwell's Equations~~ ~~Wave Equation~~ Helmholtz Equation ~~Paraxial Helmholtz Equation~~
C) Slowly-Varying Amplitude, Paraxial Approximation	$\mathbf{W_{p,\,parax}}(x,y,z)$	~~Maxwell's Equations~~ ~~Wave Equation~~ ~~Helmholtz Equation~~ Paraxial Helmholtz Equation
D) Slowly-Varying Envelope, Propagation Along z-axis $(\alpha = \beta = 0)$	$\mathbf{E_{p,\,z\text{-}axis}}(x,y,z)$	Maxwell's Equations Wave Equation Helmholtz Equation Paraxial Helmholtz Equation

Figure 5: Summary of representations of a vector plane wave and the equations they solve. In A), the full field $\mathbf{E_p}$ cannot solve the paraxial Helmholtz equation be-cause there is no restriction on the direction cosines (α, β). In D), the slowly-varying envelope approximation in the very specific case of propagation exactly along the z $-$axis ($a = \alpha/\lambda = 0$, $b = \beta/\lambda = 0$), $\mathbf{E_{p,\ z-axis}}$ solves *everything*. In this case the z $-$component of the field vanishes and the remaining $x-$ and $y-$components comprise what is commonly called the Jones vector. Hence the entirety of Jones and Stokes approaches to polarization analysis are confined to the specific, highly-constrained case D).

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5 Comparison of Polarization States for Vector Plane Waves

5.1 Comparison of Polarization States by Direct Measurement

Suppose there are two independent plane waves both co-propagating along direction $\hat{\mathbf{n}}$ such as would be found in a Michelson or Mach-Zehnder interferometer. Then it is meaningful to compare their states of polarization by simply comparing the (θ, ϕ) values in their representations (31).

Suppose now that the two beams are not co-propagating but, instead, one beam is propagating along the positive $z-$axis while the second beam propagates along arbitrary direction $\hat{\mathbf{n}}$ as shown in Fig. 6. Considered in practical terms, if we wished to actually measure the SOP of the beam along $\hat{\mathbf{n}}$ and compare it to the SOP of the beam propagating along the $z-$axis how, exactly, do we re-orient the polarimeter between the two measurements? Do we keep the polarimeter aligned along the $z-$ axis as in Fig.(6a)? or do we tilt the polarimeter so that it looks directly down into $\hat{\mathbf{n}}$ as in Fig. (6b)? The SOP reported by a polarimeter depends on the relative orientation

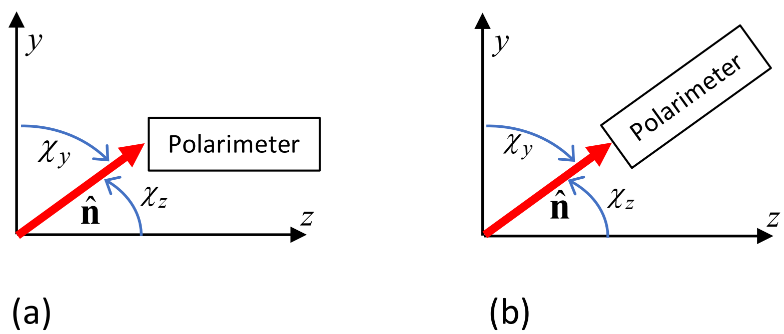

Figure 6: (a) Polarimeter optical axis remains along the $z-$axis for measurement of large angle plane wave. (b) Polarimeter optical axis aligned looking into the direction of the plane wave $\hat{\mathbf{n}}$.

of the coordinate system used internally by the polarimeter and the real space oscillations of the electric field. Hence, it is expected that the two measurements in Figs.(6a) and (6b) may give different results for the same plane-wave beam. There is no unique algorithm to reorient the polarimeter in this case. However, if a standard reorientation algorithm were agreed upon, then meaningful comparisons could indeed be made between plane waves travelling in different directions. One example where such a standard exists is surface ellipsometry [5] where source and polarimeter move in tandem in the plane of incidence with respect to the normal to a sample under investigation (Fig. 7).

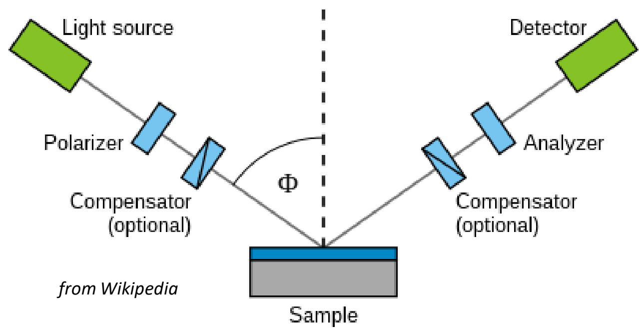

Figure 7: A simplified ellipsometry set-up. Incident and reflected light define a plane of incidence. Polarimetric measurements are taken at each angle Φ and the light source and/or detector angle is varied, always remaining in the plane of incidence. (Source: Wikipedia)

Hence, absent a universally-accepted reorientation algorithm, it is generally not meaningful to compare the SOPs of plane waves propagating in different directions in terms of a polarimeter measurement. However, it is possible to answer unambiguously a related question, namely, what is the intensity resulting form the interference between two plane waves propagating in arbitrary directions and having arbitrary states of polarization. We address this question in the next section.

5.2 Comparison of Polarization States by Interference

5.2.1 Overview

In this section we calculate the intensity resulting from the interference of two arbitrary plane waves. We shall find that, given a complete description of each plane wave, the resulting spatial distribution of intensity can always be determined uniquely. However, the reverse is not true. Given an intensity distribution there are, in general, many different combinations of two (or more) plane waves that are consistent with the observed intensity. For these calculations it is more natural to keep the α, β notation in the phase.

Consider two plane waves from different directions incident on a small ($Area << \lambda^2$) detector as shown in Fig. 8. We assume the two waves

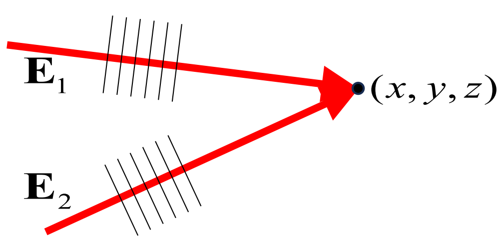

Figure 8: Two monochromatic plane waves of the same optical frequency but different polarization states incident from different directions on a small detector at location (x, y, z).

are monochromatic at the same optical frequency ω but may have different polarization states and may have different overall phases. Using (24a) we represent the fields as follows.

$$\mathbf{E}_1(x, y, z, t; \alpha_1, \beta_1, E_{01}, \bar{\phi}_1,\ \theta_1,\ \phi_1) = \frac{E_{01}\, e^{i\bar{\phi}_1}}{q_1} \begin{bmatrix} e^{-i\phi_1/2} \cos\left(\frac{\theta_1}{2}\right) \\ e^{i\phi_1/2} \sin\left(\frac{\theta_1}{2}\right) \\ \frac{-\left(\alpha_1 e^{-i\phi_1/2} \cos\left(\frac{\theta_1}{2}\right) + \beta_1 e^{i\phi_1/2} \sin\left(\frac{\theta_1}{2}\right)\right)}{\gamma_1} \end{bmatrix} \cdot e^{i\omega t} e^{-ik(\alpha_1 x + \beta_1 y + \gamma_1 z)}, \tag{32a}$$

$$\mathbf{E}_2(x, y, z, t; \alpha_2, \beta_2,\ \theta_2,\ \phi_2, \bar{\phi}_2) = \frac{E_{02}\, e^{i\bar{\phi}_2}}{q_2} \begin{bmatrix} e^{-i\phi_2/2} \cos\left(\frac{\theta_2}{2}\right) \\ e^{i\phi_2/2} \sin\left(\frac{\theta_2}{2}\right) \\ \frac{-\left(\alpha_2 e^{-i\phi_2/2} \cos\left(\frac{\theta_2}{2}\right) + \beta_2 e^{i\phi_2/2} \sin\left(\frac{\theta_2}{2}\right)\right)}{\gamma_2} \end{bmatrix} \cdot e^{i\omega t} e^{-ik(\alpha_2 x + \beta_2 y + \gamma_2 z)} \tag{32b}$$

where

$$q_{1,2} = q(\alpha_{1,2},\ \beta_{1,2}, \theta_{1,2}, \phi_{1,2}) = \left[1 + \frac{1}{\gamma_{1,2}^2}\left(\alpha_{1,2}^2 \cos^2 \frac{\theta_{1,2}}{2} + \beta_{1,2}^2 \sin^2 \frac{\theta_{1,2}}{2} + \alpha_{1,2}\beta_{1,2} \sin\theta_{1,2} \cos\phi_{1,2}\right)\right]^{1/2}$$

$$\gamma_{1,2} = \gamma(\alpha_{1,2}, \beta_{1,2}) = \sqrt{1 - \alpha_{1,2}^2 - \beta_{1,2}^2}.$$

The total field $\mathbf{E}_T$ is simply the vector sum

$$\mathbf{E}_T = \mathbf{E}_1 + \mathbf{E}_2$$

and the intensity is proportional to the magnitude squared of the total field

$$I \propto |\mathbf{E}_T|^2 = |\mathbf{E}_1|^2 + |\mathbf{E}_2|^2 + 2\,\Re e\,(\mathbf{E^*}_1 \cdot \mathbf{E}_2) \tag{34}$$

where $\Re e$ denotes real part. After some algebra we obtain the formidable result

$$\begin{aligned} I(x,y,z) = &\; E_{01}^2 + E_{02}^2 \\ &+ \frac{2E_{01}E_{02}}{q_1q_2}\Bigg[\left(1+\frac{\alpha_1\alpha_2}{\gamma_1\gamma_2}\right)\cos\left(\Delta\sigma-\frac{\Delta\phi}{2}\right)\cos\frac{\theta_1}{2}\cos\left(\frac{\theta_1+\Delta\theta}{2}\right) \\ &\qquad + \left(1+\frac{\beta_1\beta_2}{\gamma_1\gamma_2}\right)\cos\left(\Delta\sigma+\frac{\Delta\phi}{2}\right)\sin\frac{\theta_1}{2}\sin\left(\frac{\theta_1+\Delta\theta}{2}\right) \\ &\qquad + \frac{1}{\gamma_1\gamma_2}\Bigg(\alpha_1\beta_2\cos\left(\Delta\sigma+\frac{\Psi}{2}\right)\cos\frac{\theta_1}{2}\sin\left(\frac{\theta_1+\Delta\theta}{2}\right) \\ &\qquad\qquad + \beta_1\alpha_2\cos\left(\Delta\sigma-\frac{\Psi}{2}\right)\sin\frac{\theta_1}{2}\cos\left(\frac{\theta_1+\Delta\theta}{2}\right)\Bigg)\Bigg] \end{aligned} \tag{35}$$

where

$$\begin{cases} \Delta\bar{\phi} = \bar{\phi}_2 - \bar{\phi}_1 \\ \Delta\phi = \phi_2 - \phi_1 \\ \Delta\theta = \theta_2 - \theta_1 \\ \Psi = \phi_2 + \phi_1 \\ \overrightarrow{r} = \begin{bmatrix} x & y & z \end{bmatrix} \\ \overrightarrow{\Delta\eta} = \begin{bmatrix} \Delta\alpha & \Delta\beta & \Delta\gamma \end{bmatrix} \\ \Delta\varrho = \varrho_2 - \varrho_1 \quad \text{for} \quad \varrho = \alpha, \beta, \gamma \\ \Delta\sigma = \Delta\bar{\phi} - k\overrightarrow{r}\cdot\overrightarrow{\Delta\eta} \end{cases} . \tag{36}$$

This is the most general form of the intensity resulting from the interference between two arbitrary plane waves evaluated at position $\overrightarrow{r} = (x, y, z)$.

We next apply the above result $I(x, y, z)$ to some limiting cases.

5.2.2 Both beams along $z-$axis; same SOP.

In this case

$$\begin{matrix} \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \\ \gamma_1 = \gamma_2 = 1 \end{matrix} \qquad \Longrightarrow \begin{cases} \overrightarrow{\Delta\eta} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \\ q_1 = q_2 = 1 \end{cases}$$

and

$$\begin{matrix} \theta_1 = \theta_2 = \theta \\ \phi_1 = \phi_2 = \phi \\ \bar{\phi}_1 \neq \bar{\phi}_2 \end{matrix} \quad \Longrightarrow \begin{cases} \Delta\theta = 0 \\ \Delta\phi = 0 \\ \Psi = 2\phi \\ \Delta\bar{\phi} \neq 0 \end{cases}, \tag{37}$$

and the two fields simplify to

$$\begin{matrix} E_1 = E_{01} e^{i\bar{\phi}_1} \begin{bmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \\ 0 \end{bmatrix} e^{i\omega t}, \\ E_2 = E_{02} e^{i\left(\bar{\phi}_1 + \Delta\bar{\phi}\right)} \begin{bmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \\ 0 \end{bmatrix} e^{i\omega t} \end{matrix}. \tag{38}$$

The intensity is

$$I(x, y, z) \propto E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos\Delta\bar{\phi} \tag{39}$$

which is just the well-known expression for two-beam interference where $\Delta\bar{\phi}$ is the phase difference between the beams. Note also that this result is independent of the absolute values of θ and ϕ and that there is no spatial variation in the interference pattern. Figure 9 shows the intensity as a function of position y along the $y-$axis for this example for a few different $\Delta\bar{\phi}$ values.

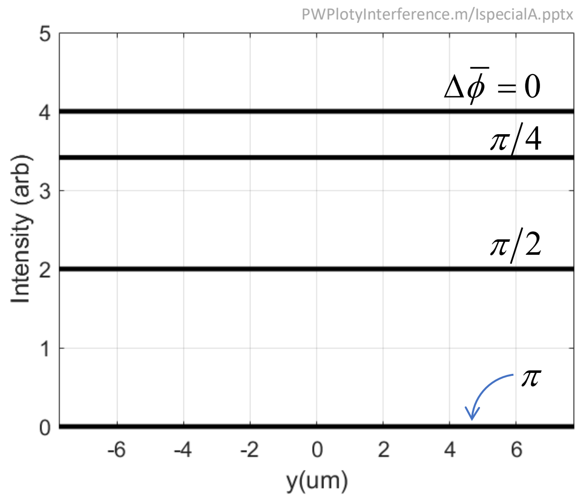

Figure 9: Two-beam intensity as a function of y for various values of $\Delta\bar{\phi}$ when both waves propagate along the $z-$axis and both are in the same polarization state. The results are independent of polarization state.

5.2.3 Both beams along $z-$ axis; different SOPs

In this case,

$$\begin{matrix} \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \\ \gamma_1 = \gamma_2 = \gamma \end{matrix} \quad \Longrightarrow \begin{cases} \overrightarrow{\Delta \eta} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \\ q_1 = q_2 = 1 \end{cases}$$

and

$$\begin{matrix} \theta_1 \neq \theta_2 \\ \phi_1 \neq \phi_2 \\ \bar{\phi}_1 \neq \bar{\phi}_2 \end{matrix} \ ,$$

and the two fields become

$$\begin{aligned} E_1 &= E_{01} e^{i\bar{\phi}_1} \begin{bmatrix} e^{-i\phi_1/2} \cos \frac{\theta_1}{2} \\ e^{i\phi_1/2} \sin \frac{\theta_1}{2} \\ 0 \end{bmatrix} e^{i\omega t}, \\ E_2 &= E_{02} e^{i\left(\bar{\phi}_1 + \Delta\bar{\phi}\right)} \begin{bmatrix} e^{-i(\phi_1+\Delta\phi)/2} \cos\left(\frac{\theta_1+\Delta\theta}{2}\right) \\ e^{i(\phi_1+\Delta\phi)/2} \sin\left(\frac{\theta_1+\Delta\theta}{2}\right) \\ 0 \end{bmatrix} e^{i\omega t} \end{aligned} \quad . \tag{40}$$

The intensity is

$$\begin{aligned} &I(x,y,z) \propto \\ &\quad E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \Big[\cos\frac{\theta_1}{2} \cos\left(\frac{\theta_1+\Delta\theta}{2}\right) \cos\left(\Delta\bar{\phi} - \frac{\Delta\phi}{2}\right) \\ &\qquad + \sin\frac{\theta_1}{2} \sin\left(\frac{\theta_1+\Delta\theta}{2}\right) \cos\left(\Delta\bar{\phi} + \frac{\Delta\phi}{2}\right) \Big] \end{aligned} \tag{41}$$

Plots of I vs y are shown in Fig. 10 for various combinations of $\Delta\bar{\phi}$, $\Delta\theta$ and $\Delta\phi$. The results of any one set of simulations is not particularly noteworthy however, taken as a whole, they illustrate the strong interplay between difference in overall phase and difference in polarization state in determining the outcome of an interference measurement.

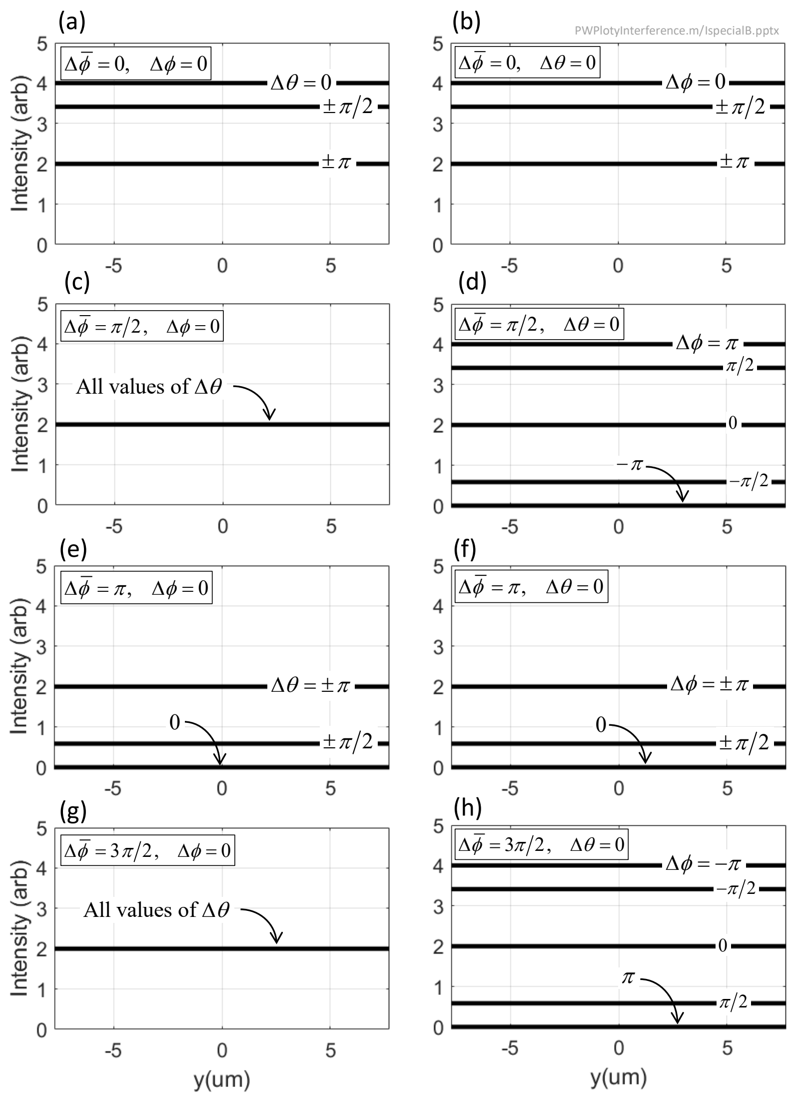

Figure 10: Two-beam intensity as a function of y for various differences in polarization state $\Delta\phi$ and $\Delta\theta$ when both waves propagate along the $z-$axis and for various differences in overall phase $\Delta\bar{\phi}$. These results illustrate the strong interferometric interactions between overall phase difference $\Delta\bar{\phi}$ and polarization state state difference $(\Delta\theta, \Delta\phi)$.

If we restrict the beams to only "linear" polarization states where $\phi_1 = \phi_2 = 0$ and thus $\Delta\phi = 0$, then

$$I(x, y, z) \propto E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos \Delta\bar{\phi} \cos\left(\frac{\Delta\theta}{2}\right). \tag{42}$$

This is similar to the simple two-beam expression (39) but with an extra polarization-dependent factor $\cos(\Delta\theta/2)$ in the interference term. In this case we see explicitly that, *in terms of their effect on interference between linearly-polarized beams, phase differences* $\Delta\bar{\phi}$ *and differences in linear polarization* $\Delta\theta/2$ *cannot be distinguished.* If the beams have the same polarization then $\Delta\theta = 0$ and (42) reduces to (39) as expected. However, if the beams are in relative polarization states such that $\Delta\theta/2$ is an odd multiple of $\pi/2$, then the interference term vanishes regardless of the value of $\Delta\bar{\phi}$. Similarly, when the difference in overall phase $\Delta\bar{\phi}$ is an odd multiple of $\pi/2$ the interference term also vanishes independent of $\Delta\theta$.

5.2.4 Same SOP but different directions.

In this case the geometry is shown in the diagram at the top of Fig. 11,

$$q_1 \neq q_2 \tag{43}$$

and

$$\begin{matrix} \theta_1 = \theta_2 = \theta \\ \phi_1 = \phi_2 \equiv \phi \\ \bar{\phi}_1 \neq \bar{\phi}_2 \end{matrix} \quad \Longrightarrow \quad \begin{cases} \Delta\theta = 0 \\ \Delta\phi = 0 \\ \Psi = 2\phi \\ \Delta\bar{\phi} \neq 0 \\ \Delta\sigma = \Delta\bar{\phi} - k\overrightarrow{r} \cdot \overrightarrow{\Delta\eta} \end{cases} , \tag{44}$$

yielding

$$\begin{aligned} &I(x,y,z) \\ &\propto E_{01}^2 + E_{02}^2 + \frac{2E_{01}E_{02}}{q_1 q_2}\left[\cos\Delta\sigma\left(1 + \frac{\alpha_1\alpha_2}{\gamma_1\gamma_2}\cos^2\frac{\theta}{2} + \frac{\beta_1\beta_2}{\gamma_1\gamma_2}\sin^2\frac{\theta}{2}\right)\right. \\ &\qquad\left. + \frac{\sin\theta}{2\gamma_1\gamma_2}\left(\alpha_1\beta_2\cos\left(\Delta\sigma - \phi\right) + \alpha_2\beta_1\cos\left(\Delta\sigma + \phi\right)\right)\right]. \end{aligned} \tag{45}$$

In this case the interference depends not only on differences in overall phase $\Delta\bar{\phi}$ and arrival direction $\overrightarrow{\Delta\eta}$ but also on the polarization state $(\theta,\ \phi)$. Plots of the interference pattern for two beams arriving symmetrically at arbitrarily-chosen locations $(0, y, 3.1m)$ are shown in Fig. 11 for various angles-of-arrival and for the simple case of $\bar{\phi}_1 = \bar{\phi}_2 = 0$ and $\theta_1 = \theta_2 = \phi_1 = \phi_2 = 0$. (Incidentally, the two-beam geometry shown in Fig. 11 is often used to write Bragg gratings in fiber where the grating pitch is adjusted by adjusting the angle between the beams.)

It is clear from just these few examples that the linear combination of arbitrary vector plane waves provides a rich array of resulting interference pattern. The reader is thus cautioned not to rely on intuition developed for the very special case of co-propagating beams in understanding vector beam interference.

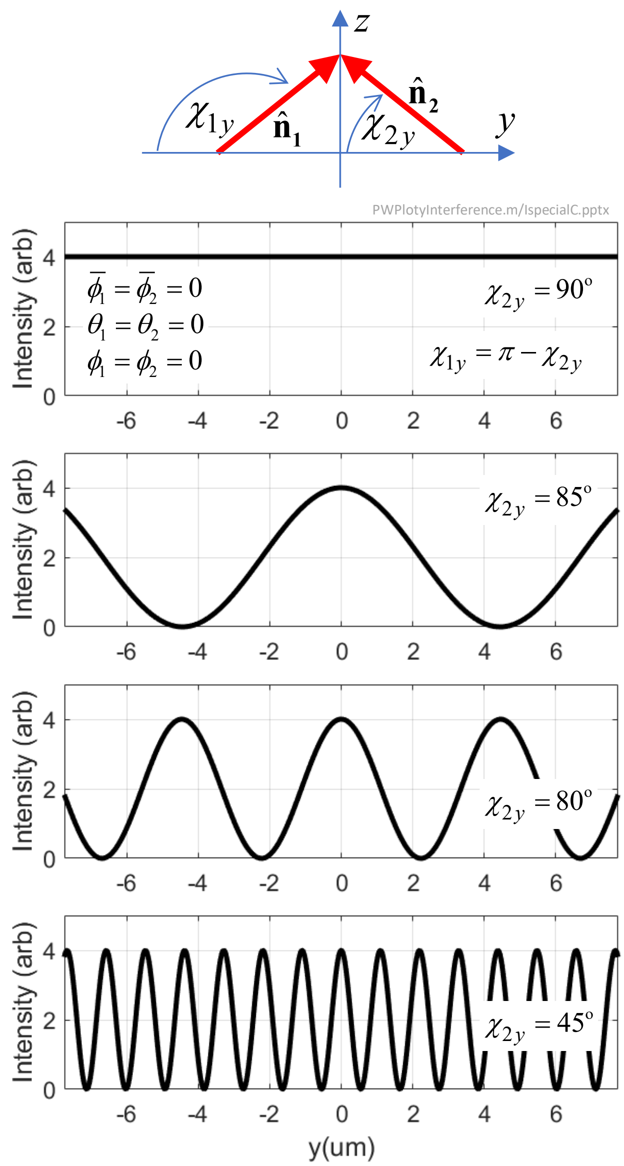

Figure 11: Interference patterns resulting from two beams in the same polarization state and the same overall phase arriving at $(0, y, 3.1m)$ from different angles symmetric about the $z-$axis.

6 Orthonormality and Completeness of the Plane Waves

In this section we show explicitly that unit-amplitude plane waves propagating in different directions are orthonormal in the source plane and plane waves propagating in the same direction but evaluated at different positions in the source plane are complete. Thus the unit-amplitude plane wave functions in the source plane form a basis. The calculation follows Roux [2].

Let

$$\begin{aligned} f(a,b;x,y) &= \exp\left(-i2\pi(ax+by)\right) \\ g(a',b';x,y) &= \exp\left(-i2\pi(a'x+b'y)\right) \end{aligned}$$

denote two plane waves propagating in different directions and evaluates at the same transverse position (x,y). Then the inner product

$$\begin{aligned} &\big(f(a,b;x,y), g(a',b';x,y)\big) \\ &\quad= \int\int_{-\infty}^{\infty} f^*(a,b;x,y)(a',b';x,y)dxdy \\ &\quad= \int\int_{-\infty}^{\infty} \exp\left(-i2\pi\left[(a-a')x+(b-b')y\right]\right)dxdy \\ &\quad= \delta(a-a')\delta(b-b') \end{aligned} \tag{46}$$

proving the orthonormality of unit-amplitude plane waves evaluated in the source plane $z=0$.

Next evaluate the inner product of two plane waves propagating in the same direction but evaluated at different positions in the transverse source plane,

$$\begin{aligned} &\big(f(a,b;x,y), g(a,b;x',y')\big) \\ &\quad= \int\int_{-\infty}^{\infty} f^*(a,b;x,y)(a',b';x,y)dxdy \\ &\quad= \int\int_{-\infty}^{\infty} \exp\left(-i2\pi\left[a(x-x')+b(y-y')\right]\right)dadb \\ &\quad= \delta(x-x')\delta(y-y') \end{aligned} \tag{47}$$

thus proving the completeness requirement.

7 TE, TM and TEM Plane Waves

Although vector beam interference is not frequently discussed in the literature it does appear in places such as the classic text [6] and in [7, 8]. In many of these works the calculation is often performed in Cartesian coordinates as we have done here but with specific reference to so-called TE and TM waves. We conclude this report with a brief review of this topic.

The descriptors TE (transverse electric), TM (transverse magnetic), and TEM (transverse electromagnetic) are ubiquitous in the study of rectangular microwave waveguides [9]. As such, they depend on *both* the intrinsic electromagnetic properties of the wave and the coordinate system chosen in which to represent the wave. In general, these descriptions find utility in cases where the geometry of the problem presents a preferred plane. In the case of a single plane wave propagating in free space there typically is no such preferred plane and the terms TE, TM and TEM do not have any meaning. In the case of two interfering plane waves propagating in different directions, the plane containing the two propagation vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ — called the plane of incidence — defines such a preferred plane. In the case of more than two plane waves propagating in arbitrary directions, there can be no preferred direction or orientation in general so the descriptions TE, TM and TEM have no utility in that case.

7.1 Transvere Electromagnetic (TEM) Plane Waves

A TEM wave is one in which the $z-$component of both the electric and magnetic field is zero. From (21a) we see that this can be strictly true for a plane wave if and only if the wave propagates exactly along the $z-$axis where $\alpha = \beta = 0$. A TEM plane wave thus has Cartesian-coordinate representations

$$\begin{aligned} \mathbf{E}_{TEM} = &\quad E_0 \begin{bmatrix} \xi_x \\ \xi_y \\ 0 \end{bmatrix} e^{i(\omega t - kz)}, \\ \mathbf{H}_{TEM} = &\quad E_0 \sqrt{\frac{\epsilon_0}{\mu_0}} \begin{bmatrix} -\xi_y \\ \xi_x \\ 0 \end{bmatrix} e^{i(\omega t - kz)} \end{aligned} \tag{48}$$

for real E_0, ω, and k and complex ξ_x and ξ_y. Note that, if the coordinate system is transformed such that the new $z-$axis points in a different direction than the original $z-$axis, then this same plane wave can no longer be described as "TEM" in the new coordinate system.

7.2 Transverse Electric (TE) Plane Waves

For TE waves (also called $H-$waves) the E-field is everywhere transverse to the plane of incidence [6, 9]. Assume the plane of incidence is the $yz-$plane as depicted at the top of Fig. 11. Then

$$\mathbf{E}_{TE} = \frac{E_0}{q}\begin{bmatrix}\xi_x\\0\\0\end{bmatrix} e^{i(\omega t - k(\beta y+\gamma z))},$$
$$\mathbf{H}_{TE} = \frac{E_0}{q}\sqrt{\frac{\epsilon_0}{\mu_0}}\begin{bmatrix}0\\ \gamma\xi_x\\ -\beta\xi_x\end{bmatrix} e^{i(\omega t - k(\beta y+\gamma z))} \tag{49}$$

That is, $\mathbf{E}_{TE}$ has only a nonzero $x-$component and thus is entirely transverse to the $yz-$plane while $\mathbf{H}_{TE}$ has no $x-$component and lies entirely in the $yz-$plane as shown in Fig. 12(a).

7.3 Transverse Magnetic (TM) Plane Waves

For TM waves (also called $E-$waves) it is the H-field that is everywhere transverse to the plane of incidence [6, 9]. Again, assume the plane of incidence is the $yz-$plane as shown at the top of Fig. 11. Then

$$\mathbf{E}_{TM} = \frac{E_0}{q}\begin{bmatrix}0\\ \xi_y\\ (-\beta/\gamma)\xi_y\end{bmatrix} e^{i(\omega t - k(\beta y+\gamma z))},$$
$$\mathbf{H}_{TM} = \frac{E_0}{q}\sqrt{\frac{\epsilon_0}{\mu_0}}\begin{bmatrix}\gamma\xi_y\\0\\0\end{bmatrix} e^{i(\omega t - k(\beta y+\gamma z))} \tag{50}$$

Here $\mathbf{H}_{TM}$ has only a nonzero $x-$component and thus is entirely transverse to the $yz-$plane while $\mathbf{E}_{TM}$ has no $x-$component but both $y-$ and $z-$components and lies entirely in the plane of incidence as shown in Fig. 12(b).

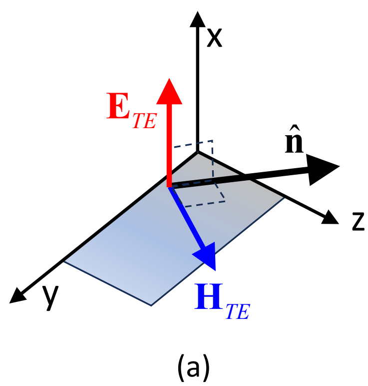

Figure 12: Comparison of the field orientations for (a) TE waves, and (b) TM waves where $\hat{\mathbf{n}}$ is the direction of propagation in the $yz-$plane.

8 Acknowledgment

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9 Appendix 1

In this Appendix we review the paraxial approximation and the slowly-varying envelope approximation (SVEA). As practical matters, we will see that i) invoking the paraxial approximation *requires* the SVEA to be valid, and ii) the SVEA is consistent with the paraxial approximation but, iii) in and of itself, the SVEA does not require the paraxial assumption to be made.

Typically, the paraxial approximation is made because of the geometrical optics of the problem. That is, provided that any dimension of the downrange observation area is small compared to the distance z_0 between the source plane and observation plane, then for any angle θ between any point on the finite source aperture and the finite observation aperture the following approximations are valid, $\sin\theta \approx \tan\theta \approx \theta$ and $\cos\theta \approx 1$. Equivalently, for any geometrical ray in the problem along unit propagation vector $\hat{n}$, the scalar product $\hat{n} \cdot \hat{z} = \cos\theta \approx 1$, where $\hat{z}$ is the unit vector along the $z-$axis. Said yet another way, in the paraxial approximation the spectrum (distribution) of θ values associated with the beam is very strongly peaked around $\theta = 0$.

For the plane wave $A_0 \exp(-ikz)$ the $\theta-$spectrum is clearly a delta function $\delta(0)$. If we now allow the amplitude to vary with position $A_0 \rightarrow A(x, y, z)$ then the $\theta-$spectrum must broaden and the amount of broadening depends on how strongly $A(x, y, z)$ varies with position.

For example, suppose that somewhere along the path of the beam the amplitude of the beam changes abruptly. By the Fourier theorem, this abrupt change in amplitude must broaden the spectrum of $\theta-$values by introducing components with θ values much larger than were present in the original spectrum. Weak variations in amplitude as a function of propagation distance z can be tolerated while remaining within the paraxial approximation but strong, rapid amplitude variations can push the θ-spectrum out beyond the paraxial approximation. A useful rule of thumb is the following [10].

> *Over distances of a wavelength λ or so, the variation in amplitude ΔA should be small compared to A itself, $\Delta A \ll A$.*

With this assumption and since $\Delta A = (\partial_z A/\Delta z)\Delta z \approx (\partial_z A/\partial z)\lambda$ we can say

$$\left| \frac{\partial A}{\partial z} \right| \ll kA.$$

Taking the derivative of both sides and again setting $\Delta z \approx \lambda$ we can further say

$$\left|\frac{\partial^2 A}{\partial z^2}\right| \ll k\left|\partial_z A\right| \tag{51}$$

which is the inequality applied to the Helmholtz equation (9a) to yield the paraxial Helmholtz equation (11) after making the substitution (10).

As an example of a case where the SVEA can be applied outside the paraxial approximation consider a spherical wave $(A/r)\exp(-ikr)$. By definition this wave does not satisfy the paraxial assumption for *any* value of r. However, if desired, we could still impose the SVEA and require that changes in A be small compared to A for changes in radial distance of a wavelength or so and for changes in azimuthal angle φ and polar angle θ of less than 0.1 rad or so. In spherical coordinates the Laplacian operator is

$$\begin{aligned}\nabla^2 A &= \left[\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2 \sin\theta}\partial_\theta \sin\theta\,\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\right]A \\ &= \left[\frac{2}{r}\partial_r + \partial_r^2 + \frac{\cot\theta}{r^2}\partial_\theta + \frac{1}{r^2}\sin\theta\,\partial_\theta^2 + \frac{\sin\theta}{r^2}\partial_\varphi^2\right]A\end{aligned}$$

Following the same reasoning used above,

$$\begin{aligned}\Delta A = \frac{\partial A}{\partial r}\Delta r = \frac{\partial A}{\partial r}\lambda &\Rightarrow \frac{\partial A}{\partial r} \ll kA \\ &\Rightarrow \frac{\partial^2 A}{\partial r^2} \ll k^2 A\end{aligned}$$

$$\begin{aligned}\Delta A = \frac{\partial A}{\partial \theta}\Delta \theta = 0.1\frac{\partial A}{\partial \theta} &\Rightarrow \frac{\partial A}{\partial \theta} \ll 10A \\ &\Rightarrow \frac{\partial^2 A}{\partial \theta^2} \ll 100A\end{aligned}$$

$$\begin{aligned}\Delta A = \frac{\partial A}{\partial \varphi}\Delta \varphi = 0.1\frac{\partial A}{\partial \varphi} &\Rightarrow \frac{\partial A}{\partial \varphi} \ll 10A \\ &\Rightarrow \frac{\partial^2 A}{\partial \varphi^2} \ll 100A\end{aligned}$$

and the only guaranteed benefit of the SVEA in this case is elimination of the ∂_r^2 term in comparison to the $k\partial_r$ term leading to a spherical Helmholtz equation in the slowly-varying amplitude approximation

$$\left(\nabla^2 + k^2\right) A \longrightarrow \left[\frac{2}{r}\partial_r + \frac{\cot\theta}{r^2} + \frac{\sin\theta}{r^2}\partial_\theta^2 + \frac{\sin\theta}{r^2}\partial_\varphi^2\right] A = 0,$$

a result that may be of significance in some applications.

10 Appendix 2

For reference purposes, in this Appendix we obtain the wave equations in Cartesian coordinates from Maxwell's equations for a wave in free space.

Take the curl of (1a) and use (1b) to obtain

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \partial_t \epsilon_0 \partial_t \mathbf{E} = -\epsilon_0 \mu_0 \partial_{tt} \mathbf{E}. \tag{52}$$

where, since $\nabla\times$ and ∂_t are both linear operations, we could exchange their order of application.

Use the Cartesian vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla \left(\nabla \cdot \mathbf{E} \right) - \nabla^2 \mathbf{E}$$

and invoke (3d) to eliminate $\nabla \cdot \mathbf{E}$. Recall that $\epsilon_0 \mu_0 = 1/c^2$ to obtain, finally, the wave equation

$$\nabla^2 \mathbf{E} - 1/c^2 \partial_{tt} \mathbf{E} = 0$$

In the special case where the wave is monochromatic with optical radian frequency ω, the wave equation becomes the Helmholtz equation

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$

where the propagation constant (wavenumber) $k = \omega/c$.

An analogous procedure is used to obtain the wave equation and Helmholtz equation for the **H** field.

$$\nabla^2 \mathbf{H} - 1/c^2 \partial_{tt} \mathbf{H} = 0$$

and

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0.$$

References

[1] J.W. Goodman. Introduction to Fourier Optics. 1968.

[2] F.S. Roux. "Optical Waves in". Laser Beam Propagation*, A. Forbes (ed)* (2014), pp. 3–40.

[3] N.J. Frigo, F. Bucholtz, G.A. Cranch, and J.M. Singley. "On the choice of conventions in polarization evolution calculations and representations". *J. Lightwave Technol.* **40**(1) (2022), pp. 179–190.

[4] A. Forbes (Ed.) Laser Beam Propagation. 2014.

[5] R.M.A. Azzam and N.M. Bashara. Ellipsometry and Polarized Light. 1988.

[6] R.H. Clarke and J. Brown. Diffraction Theory and Antennas. 1988.

[7] D.R. Rhodes. "On a fundamental principle in the theory of planar antennas". *Proc. IEEE* **52**(9) (1964), pp. 1013–1021.

[8] W.H. Carter. "Electromagnetic field of a Gaussian beam with an elliptical cross section". *J. Opt. Soc. Am.* **62**(10) (1972), pp. 1195–1201.

[9] D.M. Pozar. Microwave Engineering, 4th ed. 2012.

[10] B.E.A. Saleh and M.C. Teich. Fundamentals of Photonics. 1991.