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THESIS

STOCHASTIC DUEL WITH THREE OR MORE PLAYERS

by

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Consider a stochastic duel with many players. Each player chooses an opponent to shoot at, makes a hit after a random amount of time that follows an exponential distribution, and is killed as soon as being hit for the first time. The duel continues until all but one player is killed, and the lone survivor is declared the winner. The goal of each player is to decide which opponent to target at any given time in order to maximize their winning probability. We develop an algorithm to compute each player's optimal strategy and winning probability. In particular, the strongest player—the one having the largest kill rate—need not be the most likely to win, and it is not necessarily optimal for each player to shoot at their strongest opponent. We also consider a variation of the game in which players arrive sequentially to select their own kill rates, knowing the selections made by players who arrived earlier. In this sequential-move game with three players, the first player wants to be mediocre and the third player has the best chance to win. Our findings enable further understanding of military conflicts that involve three or more adversaries in the same area of operations.

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STOCHASTIC DUEL WITH THREE OR MORE PLAYERS

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ABSTRACT

Consider a stochastic duel with many players. Each player chooses an opponent to shoot at, makes a hit after a random amount of time that follows an exponential distribution, and is killed as soon as being hit for the first time. The duel continues until all but one player is killed, and the lone survivor is declared the winner. The goal of each player is to decide which opponent to target at any given time in order to maximize their winning probability. We develop an algorithm to compute each player's optimal strategy and winning probability. In particular, the strongest player—the one having the largest kill rate—need not be the most likely to win, and it is not necessarily optimal for each player to shoot at their strongest opponent. We also consider a variation of the game in which players arrive sequentially to select their own kill rates, knowing the selections made by players who arrived earlier. In this sequential-move game with three players, the first player wants to be mediocre and the third player has the best chance to win. Our findings enable further understanding of military conflicts that involve three or more adversaries in the same area of operations.

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Executive Summary

Most combat models in academic literature concern two players in a zero-sum game in which winner takes all. The optimal strategy in these games is well established; by using an optimal strategy, players can guarantee the value of the game regardless of their opponent's actions. Throughout history, however, there are many combats that involve three or more players in a winner takes all context. For example, the war in Syria since 2011 presents a paradigm very different from the typical two-sided, force-on-force engagement in which four players—the Syrian government, Syrian Kurdish forces, Syrian rebel forces, and the Islamic State group—are all in conflict with one another. United States military focus remains on facing a single enemy rather than having to address multiple enemies at the same time. Extension of duels to three or more players will shed light into how multiple players will interact and facilitate further understanding of military conflicts involving multiple adversaries in the same area of operations.

To better understand the dynamics of three or more players, we first develop a stochastic duel game with multiple players. In this game, each player's strength is represented by a kill rate—the rate at which each player kills an opponent of their choosing. All players always have complete knowledge of who is still competing in the game. The game proceeds as players are eliminated and ends when all but one player is killed and the lone survivor is declared the winner. Each player's objective is to choose an opponent to target at any given time in order to maximize their individual probability of winning.

To solve this stochastic duel game with multiple players, we develop an algorithm to compute each player's optimal strategy recursively. When there are only two players left, there is only one choice: Shoot at the other player. When there are three players left, each player would evaluate each potential two-player state and shoot at the player whose elimination would give the shooter the best chance to win. With a similar idea, each player then can determine which opponent to shoot at when there are four players in the game, and so on. It turns out that with three players, it is optimal for each player to shoot at their strongest opponent, but there is no clear pattern when there are four or more players. Depending on the relative strengths between players, anyone—the strongest player or the weakest player or anyone in between—may end up being the player most likely to win.

Given that the stronger player is not always the more likely to win, we next consider how a player would choose a kill rate if given the opportunity. We design a game in which three players choose their kill rates between 0 and 1, sequentially; thereafter they engage in a three-person stochastic duel. Player 1 is the first to arrive and select a kill rate; after observing Player 1's choice, Player 2 selects a kill rate; after observing Player 1's and Player 2's choices, Player 3 selects a kill rate. After selecting their kill rates, the three players engage in the stochastic duel with the same rule. The lone survivor in the stochastic duel is declared the winner.

To solve this sequential-move game, we first derive Player 3's optimal response against Player 1's and Player 2's kill rate selections. We next consider Player 2's optimal response, and then finally the best choice Player 1 can make. It turns out that it is optimal for Player 1 to select a kill rate just shy of 0.5, followed by Player 2 selecting a kill rate of 1, followed by Player 3 selecting a kill rate just shy of that of Player 1's. With these optimal strategies, the winning probabilities are $3/12$, $4/12$, and $5/12$ for Players 1, 2, and 3, respectively. The results demonstrate that knowledge gives players who select kill rates later an advantage. Somewhat surprising, in order to maximize winning probability, Player 1 wants to be the mediocre player rather than the strongest player—which is in stark contrast to the common sense in a two-player conflict in which everyone wants to be as strong as possible.

With the potential for the United States to face multiple adversaries in the same area of operations in the near future, this thesis provides a baseline context to consider how three or more players will interact in the same arena. While being stronger provides a decisive advantage in a two-player combat, with three or more players it becomes much more difficult to win by sheer force. Our findings help shed light into how each player should act optimally in a multilateral combat scenario in order to gain a competitive edge.

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CHAPTER 1: Introduction

Most combat models in academic literature concern two players in a zero-sum game in which winner takes all. In a two-person zero-sum game, the analysis typically focuses on determining the optimal strategy of each player and the value of the game. Throughout history, however, there are many combats that involve three or more players in a winner takes all context. The war in Syria since 2011 presents a paradigm very different from the typical two-sided, force-on-force engagement in which four *players*—the Syrian government, Syrian Kurdish forces, Syrian rebel forces, and the Islamic State group—are all in conflict with one another. The Algerian War from 1954-1962 also presents an example in which three *players*—France, the Algerian National Liberation Front, and the Secret Armed Organization—were all at odds. Studying combat models that involve three or more players will provide a foundation for further understanding of military conflicts involving multiple adversaries in the same area of operations.

In this thesis, we first develop stochastic duel models that involve three or more players in order to shed light into how each player would behave in a complicated combat scenario. We consider a game with n players engaged in a duel. Each player's strength is modeled by a kill rate; regardless of which opponent player i chooses to attack, the amount of time needed to make a kill follows an exponential distribution with rate λ_i, for $i = 1, \ldots, n$. Each player has complete knowledge of who else is still competing. The game ends when all but one player is killed and the surviving player is declared the winner. The goal of each player is to maximize the probability of winning the duel. We formulate the game as a Markov game where each state corresponds to the set of players remaining in the game, and develop an algorithm to determine each player's optimal strategy. It turns out that when $n = 3$, it is optimal for each player to target at his strongest opponent, but there is no clear pattern when $n \geq 4$. Depending on the relative strengths between players, anyone—the strongest player or the weakest player or anyone in between—may end up being the player most likely to win.

Second, we consider a sequential game with three players. The three players choose their kill rates sequentially from the interval $[0, 1]$; thereafter they engage in a three-person stochastic duel. Player 1 arrives first and selects his kill rate. Player 2 arrives second, observes the kill rate selected by Player 1, and then selects his own kill rate. Player 3 arrives last, observes the kill rates selected by Player 1 and Player 2, and then selects his own kill rate. Thereafter, the three players engage in the aforementioned stochastic duel, with each player trying to maximize their own winning probability. While in a two-person duel, it is clear that each player wants to select the highest kill rate possible to maximize their winning probability, in a three-person duel it is not the case. It turns out that it is optimal for Player 1 to select a kill rate just shy of 0.5, followed by Player 2 selecting a kill rate 1, followed by Player 3 selecting a kill rate just shy of that of Player 1's. With these optimal strategies, the winning probabilities are $3/12$, $4/12$, and $5/12$ for Players 1, 2, and 3, respectively.

We next present a literature review and then give an outline of this thesis.

1.1 Literature Review

1.1.1 Combat Models with Two Players

The earlier work on combat models dates back more than one hundred years ago. Lanchester (1916) devises a series of differential equations to demonstrate the power relationships between opposing forces (Wrigge et al. (1995)). Lanchester in his *Square Law* assumes the rate at which a force is depleted is proportional to the size of the enemy force and the individual capability of each enemy entity; he concludes that if two like forces target each other using the same weapons, they will suffer the same rate of casualties until the smaller force is eliminated. Ultimately, Lanchester equations are able to determine the winner of a duel using a fixed fire distribution. There are a lot of works that use differential equations to quantify force attrition similar to the spirit of Lanchester equations. For example, Lin and MacKay (2014) studies optimal fire distribution between homogeneous and heterogeneous forces, Kress and Szechtman (2009) and Kaplan et al. (2010) study insurgent warfare, and Schramm and Gaver (2013) studies cyber warfare. For a general discussion about combat models, please see Kress (2012) and Washburn and Kress (2009).

During World War II, Koopman (1940) extends Lanchester's results and suggests a reformulation in stochastic form. After World War II, Morse and Kimball (1946) derive stochastic models for duels. Additional works extend the Lanchester models to stochastic duel models by introducing randomness to shot outcomes, time between taking shots, and so forth. Initially these models only take into account homogeneous units, so no decision making is required as any strategy that maximizes a players rate of fire is optimal. Friedman (1977) and Kikuta (1983) study a single unit against a heterogeneous unit and determine the optimal fire order, but computing a game's optimal strategy is burdensome because of the game's massive payoff matrix. Lin (2014) creates an iterative algorithm to compute the optimal strategy without having to enumerate the entire payoff matrix. For additional extensions to Lanchester models for one-against-many combats, see Colegrave and Hyde (1993), Jaiswal et al. (1995), and Fowler (1999). The Markov and sequential games developed in this thesis will use a recursive method similar to Lin (2014) in order to determine the optimal strategies of all players.

1.1.2 Combat Models with Three or More Players

Marryat (1836) first provides a description of a three way duel; however, formulation of three player duels is not published until much later. Kilgour and Brams (1997) in an article in *Mathematics Magazine* formalizes three games with different firing rules: sequential (fixed), sequential (random), and simultaneous. Additionally, they introduce both marksmanship (proficiency) and survival probabilities to players, demonstrating how a player's attributes can greatly alter strategies and outcomes. This thesis leverages the basic design of Kilgour and Brams's three player simultaneous duel, but introduces uncertainty by allowing the amount of time needed to make a kill to follow an exponential distribution.

Kress et al. (2018a) extends Lanchester models to include three players, and derives trajectories of force sizes based on initial force strengths and each player's fire allocation between the two opponents. Kress et al. (2018b) extends classical force-on-force Lanchester models to study the attrition dynamics of three-way and multilateral war. They formulate the multilateral war as an nonzero-sum game between many players and use Nash equilibria to predict the outcomes. This thesis will extend these existing, deterministic models to consider a Markov game with three or more players.

1.2 Thesis Outline

The rest of this thesis proceeds as follows: Chapter 2 presents the stochastic duel with three players. Chapter 3 extends the stochastic duel to four or more players. Chapter 4 presents the sequential duel with three players. Chapter 5 concludes this thesis and provides future research directions.

CHAPTER 2:
Stochastic Duel with Three Players

Consider a stochastic duel game with three players. At any time, each player chooses an opponent still in the game to attack. Regardless of which opponent Player i chooses to attack, the amount of time needed to make a kill follows an exponential distribution with rate λ_i, for $i = 1, 2, 3$. Without loss of generality, we label the players such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. At any given time, each player has complete knowledge of who else is still competing and chooses one of these players to attack. The game ends when all but one player is killed, and the surviving player is declared the winner. Each player's objective is to maximize their winning probability.

The rest of this chapter proceeds as follows. Section 2.1 formulates the game as a Markov game. Section 2.2 presents each player's optimal strategy. Section 2.3 derives the probabilities of each player winning the duel when executing optimal strategies. Section 2.4 discusses how altering player kill rates affects the most likely winner of the game.

2.1 Formulation as a Markov Game

Because the amount of time it takes for each player to make a kill follows an exponential distribution, once we know who are still in the game, the history of the game becomes irrelevant. Specifically, if Player 1 has been eliminated from the game, it does not matter if the kill was executed by Player 2 or Player 3; previous kills do not have an effect on transitions from the current state. Therefore, the game can be formulated as a Markov game, in which each state corresponds to the set of players still in the game. The game moves from the current state to a new state when any player makes a kill. Beginning with state $\{1, 2, 3\}$, the state space and potential transitions can be seen in Figure 2.1.

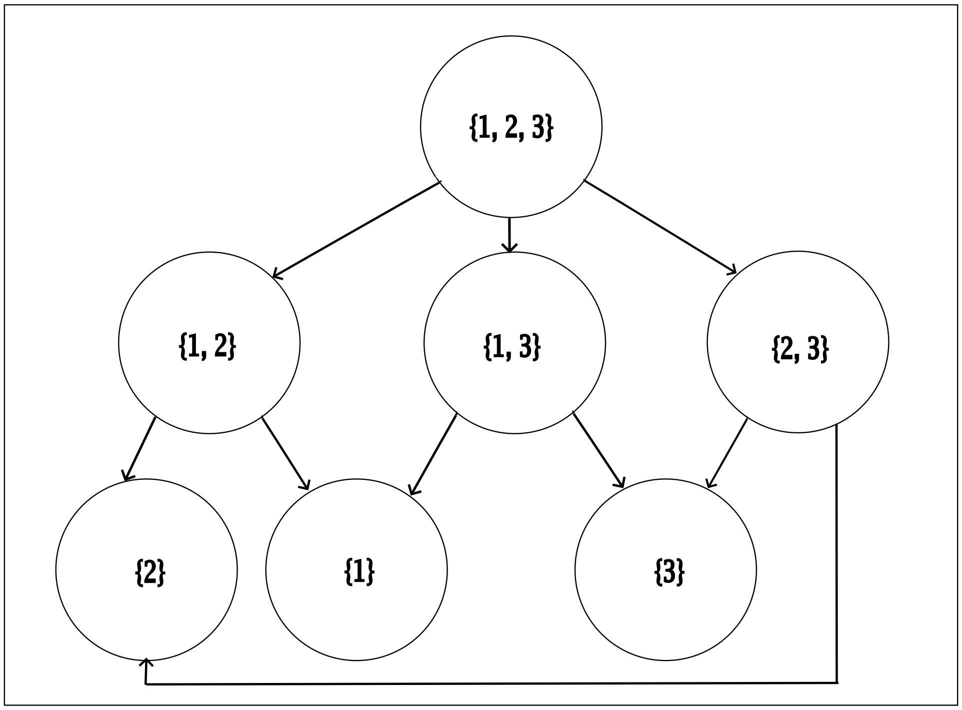

Figure 2.1. Three-Player Stochastic Duel as a Markov Game

As seen in Figure 2.1, there is one state with three players, namely {1,2,3}, at the beginning of the game. As soon as some player makes a kill, the game transitions into a state with two players, namely {1,2}, {1,3}, or {2,3}. After another kill, the game transitions to a state with a single player and the game ends.

2.2 Analysis of the Optimal Strategies

To analyze each player's optimal strategy, we first consider a state with only two players left. Because only two players are still in the game, it is optimal for each player to attack the other player. By writing X_i and X_j for two independent exponential random variables with respective rates λ_i and λ_j, it follows that the probability that Player i will be the winner is

$$P\{X_i < X_j\} = \frac{\lambda_i}{\lambda_i + \lambda_j},$$

for $i, j = 1, 2, 3$.

In state {1,2,3}, each player has the option to target either of their two opponents and must decide which opponent to attack. Because the amount of time it takes for player i to make a kill follows an exponential distribution with rate λ_i, for $i = 1, 2, 3$, the amount of time it takes until *someone* makes a kill follows an exponential distribution with rate $\lambda_1 + \lambda_2 + \lambda_3$. Furthermore, the probability that Player i is the one who makes the first kill is $\lambda_i / \sum_{j=1}^{3} \lambda_j$, for $i = 1, 2, 3$. The player who makes the first kill decides to which state the process next transitions.

First, consider the optimal strategy of Player 1. If it is Player 2 or Player 3 who makes the first kill, then Player 1 has no control over which state the process next transitions to, so Player 1's action in state $\{1, 2, 3\}$ is inconsequential. If it is Player 1 who makes the first kill, then Player 1 can decide whether to enter state $\{1, 2\}$ or $\{1, 3\}$. The probability that Player 1 will be the winner in state $\{1, 2\}$ is:

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}$$

The probability that Player 1 will be the winner in state $\{1, 3\}$ is:

$$\frac{\lambda_1}{\lambda_1 + \lambda_3}$$

Because $\lambda_2 > \lambda_3$, it follows that $\lambda_1/(\lambda_1 + \lambda_2) < \lambda_1/(\lambda_1 + \lambda_3)$, so to maximize the winning probability Player 1 prefers state $\{1, 3\}$. Therefore, it is optimal for Player 1 to attack Player 2 in state $\{1, 2, 3\}$.

Next, consider the optimal strategy of Player 2. With a similar argument, Player 2 has no control which state the game transitions to if either Player 1 or Player 3 makes the first kill. If Player 2 is the player making the first kill, then Player 2 can choose whether to enter state $\{1, 2\}$ or state $\{2, 3\}$. Because $\lambda_1 > \lambda_3$, Player 2 prefers state $\{2, 3\}$ which gives Player 2 a higher winning probability $\lambda_2/(\lambda_2 + \lambda_3)$ than $\lambda_2/(\lambda_1 + \lambda_2)$—Player 2's winning probability in state $\{1, 2\}$. Therefore, it is optimal for Player 2 to attack Player 1 in state $\{1, 2, 3\}$.

Finally, consider the optimal strategy of Player 3. With a similar argument, Player 3 prefers state $\{2, 3\}$ to state $\{1, 3\}$ if Player 3 is the first player to make a kill. Therefore, Player 3 should attack Player 1 in state $\{1, 2, 3\}$.

The optimal action of each player in state {1,2,3} can be summarized below:

- Player 1 attacks Player 2
- Player 2 attacks Player 1
- Player 3 attacks Player 1

With each player using their respective optimal strategies, the state transitions in the Markov game can be modeled by a Markov chain, as displayed in Figure 2.2. Please note that state $\{1,2\}$ will never be visited, because neither Player 1 nor Player 2 will attack Player 3 in state $\{1,2,3\}$.

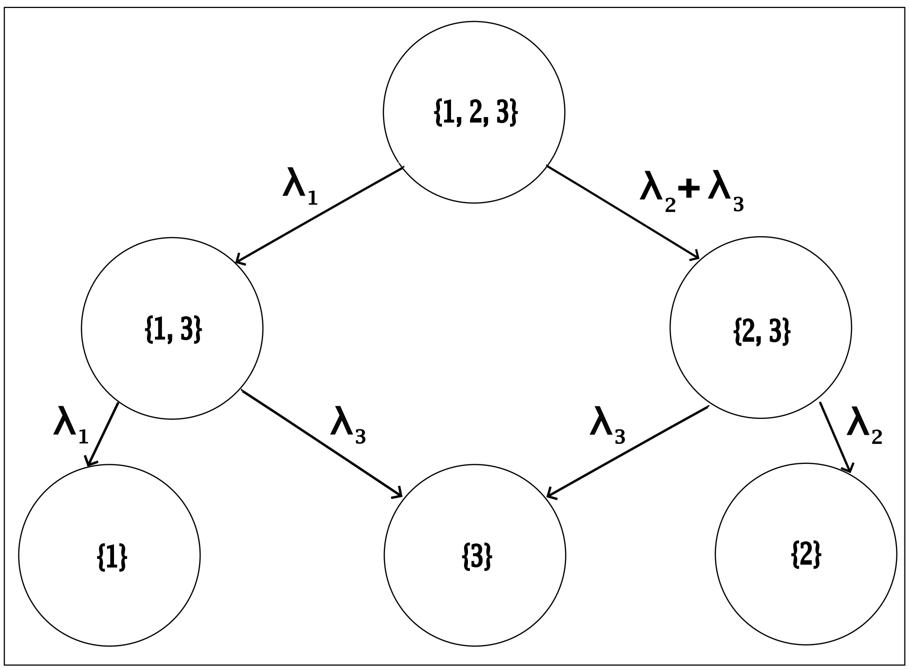

Figure 2.2. Markov Chain for Three-Player Stochastic Duel with Optimal Strategies

2.3 Winning Probability of Each Player

Considering the optimal strategies derived in the previous section, calculating the probability of individual players winning the duel becomes a simple combination of the probabilities of specific transitions occurring. Let $g_i(\lambda_1, \lambda_2, \lambda_3)$ denote the probability that Player i wins the three-player stochastic duel, for $i = 1, 2, 3$, if the three players have respective kill rates $\lambda_1, \lambda_2, \lambda_3$.

Player 1 will win only if he kills Player 2 in state $\{1, 2, 3\}$ and then kills Player 3 in state $\{1, 3\}$. Therefore, the probability of Player 1 winning the duel is

$$g_1(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_1}{\lambda_1 + \lambda_3}\right). \tag{2.1}$$

Player 2 will win only if he or Player 3 kills Player 1 in the {1,2,3} and he then kills Player 3 in state $\{2, 3\}$. Therefore, the probability of Player 2 winning the duel is

$$g_2(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{\lambda_2 + \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 + \lambda_3}\right) = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}. \tag{2.2}$$

Player 3 could win in two ways. First, Player 1 kills Player 2 in state $\{1, 2, 3\}$ and then Player 3 kills Player 1 in state $\{1, 3\}$. Second, Player 2 or Player 3 kills Player 1 in state $\{1, 2, 3\}$ and then Player 3 kills Player 2 in state $\{2, 3\}$. Therefore, the probability of Player 3 winning the duel is

$$\begin{aligned} g_3(\lambda_1, \lambda_2, \lambda_3) &= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_3}{\lambda_1 + \lambda_3}\right) + \left(\frac{\lambda_2 + \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_3}{\lambda_2 + \lambda_3}\right). \\ &= \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\left(\frac{2\lambda_1 + \lambda_3}{\lambda_1 + \lambda_3}\right) \end{aligned} \tag{2.3}$$

Having derived the probabilities of individual players winning when executing optimal strategies, we will now examine the impact of altering kill rates for player in determining the outcome of the duel.

2.4 The Most Likely Winner

In a two player stochastic duel, the player who has the higher kill rate has the higher probability of winning. Is it still the case in a three-player stochastic duel? To answer this question, we set $\lambda_1 = 1$ without loss of generality and vary λ_2 and λ_3 such that $0 \le \lambda_3 \le \lambda_2 \le 1$. Figure 2.3 displays the most likely winner.

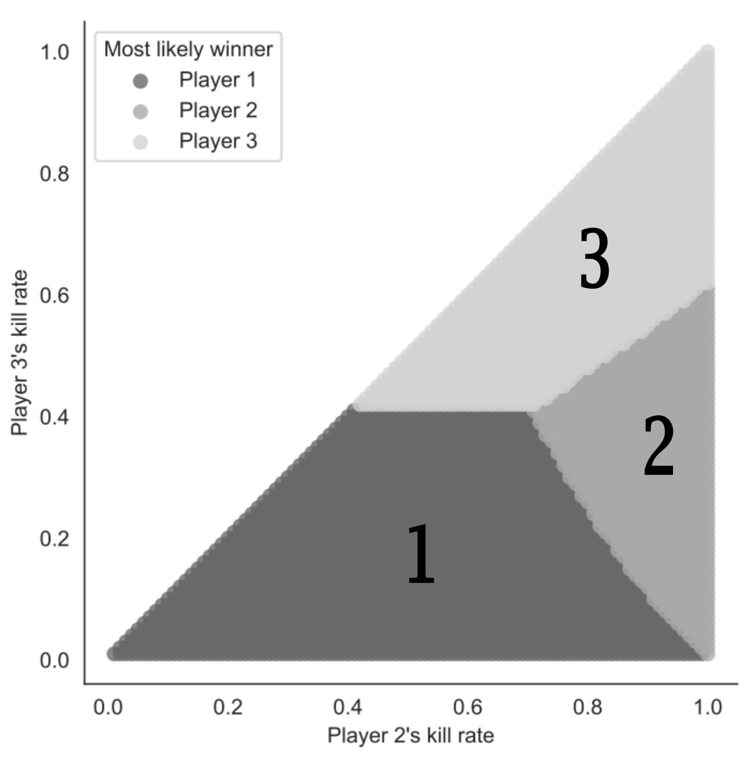

Figure 2.3. Most Likely Winner for Three Player Stochastic Duel

As seen in Figure 2.3, there are three regions, each of which corresponds to a player being the most likely winner. We next examine the boundaries of these regions, and then provide some insights by varying λ_2 and λ_3, respectively.

2.4.1 Boundaries

First, we examine the case where each player is equally likely to win the game, which is exhibited by the intersection of regions 1, 2 and 3 in Figure 2.3. Recall the winning probabilities of Player 1, Player 2, and Player 3 from equations (2.1), (2.2), and (2.3), respectively. We can determine the intersection of the three regions by setting

$$g_1(\lambda_1, \lambda_2, \lambda_3) = g_2(\lambda_1, \lambda_2, \lambda_3) = g_3(\lambda_1, \lambda_2, \lambda_3),$$

which reduces to

$$\frac{\lambda_1^2}{\lambda_1 + \lambda_3} = \lambda_2 = \frac{\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} + \lambda_3. \tag{2.4}$$

Solving for λ_2 and λ_3 in terms of λ_1 produces

$$\frac{\lambda_2}{\lambda_1} = \frac{\sqrt{2}}{2} \approx 0.707,$$
$$\frac{\lambda_3}{\lambda_1} = \sqrt{2} - 1 \approx 0.414.$$

If the kill rates $\lambda_1, \lambda_2, \lambda_3$ satisfy the preceding equations, then each player has the same probability of winning. In other words, if Player 2 has a kill rate that is approximately 70.7% of Player 1 and Player 3 has a kill rate that is approximately 41.4% of Player 1, each player will win the game with probability 1/3.

Next, we examine the boundaries between the regions in Figure 2.3. The boundary between region 1 and region 3 in Figure 2.3 can be found by setting

$$g_1(\lambda_1, \lambda_2, \lambda_3) = g_3(\lambda_1, \lambda_2, \lambda_3),$$

which—according to equation (2.4)—reduces to

$$\frac{\lambda_1^2}{\lambda_1 + \lambda_3} = \frac{\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} + \lambda_3,$$

or equivalently,

$$\frac{\lambda_3}{\lambda_1} = \sqrt{2} - 1.$$

Similarly, we can find the boundary between the region 1 and the region 2 by setting

$$g_1(\lambda_1, \lambda_2, \lambda_3) = g_2(\lambda_1, \lambda_2, \lambda_3),$$

which reduces to

$$\frac{\lambda_1^2}{\lambda_1 + \lambda_3} = \lambda_2.$$

Finally, we can find the boundary between region 2 and region 3 by setting

$$g_2(\lambda_1, \lambda_2, \lambda_3) = g_3(\lambda_1, \lambda_2, \lambda_3),$$

which reduces to

$$\lambda_2 = \frac{\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} + \lambda_3.$$

2.4.2 When λ_2 varies

From the analysis in Section 2.4.1, the left-most point on the boundary between region 1 and region 3 has $\lambda_2/\lambda_1 = \lambda_3/\lambda_1 = \sqrt{2} - 1$. The right-most point on the boundary between region 1 and region 3 has $\lambda_2/\lambda_1 = \sqrt{2}/2$ with λ_3 maintaining the same value.

Along this boundary, if

$$\frac{\lambda_3}{\lambda_1} \leq \sqrt{2} - 1,$$

then Player 1 is the most likely winner, because Players 2 and 3 are too weak to compete against Player 1 even though they both attack Player 1 in state $\{1, 2, 3\}$.

If

$$\frac{\lambda_3}{\lambda_1} \geq \sqrt{2} - 1,$$

then Player 3 is the most likely winner due to the increased kill rate while still avoiding being attacked in state $\{1, 2, 3\}$. However, Player 2 will never be the most likely winner because λ_2 is not large enough and Player 2 will be Player 1's target in state $\{1, 2, 3\}$.

If

$$\frac{\lambda_2}{\lambda_1} \geq \frac{\sqrt{2}}{2},$$

there is the potential for any player to be the most likely to win. Player 3 becomes the most likely winner if

$$\lambda_2 \leq \frac{\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} + \lambda_3;$$

This is again due to Player 3 having an increased kill rate while also avoiding being attacked in state $\{1,2,3\}$.

Player 2 is the most likely winner if

$$\lambda_2 \geq \frac{\lambda_1^2}{\lambda_1 + \lambda_3}$$

This is due to Player 2 and Player 3's combination of kill rates likely eliminating Player 1 in state $\{1,2,3\}$ and Player 2 being stronger than Player 3 in a subsequent duel. Otherwise, Player 1 is the most likely winner because his strength is likely to overpower both Player 2 and Player 3 in state $\{1,2,3\}$ and then eliminate Player 3 in state $\{1,3\}$.

2.4.3 When λ_3 varies

From the analysis in Section 2.4.1, if

$$\frac{\lambda_3}{\lambda_1} \leq \sqrt{2} - 1,$$

then Player 3 will not be the most likely winner.

Player 2 will be the most likely winner if

$$\lambda_2 \geq \frac{\lambda_1^2}{\lambda_1 + \lambda_3}.$$

This is due to Player 2 and Player 3's combination of kill rates likely eliminating Player 1 in state $\{1,2,3\}$ and Player 2 being the stronger player in a duel with Player 3. Otherwise, Player 1 will be the most likely winner.

If

$$\frac{\lambda_3}{\lambda_1} \geq \sqrt{2} - 1,$$

then Player 1 will not be the most likely winner.

Player 2 will be the most likely winner if

$$\lambda_2 \geq \frac{\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} + \lambda_3,$$

This is due to Player 2 and Player 3's combination of kill rates likely eliminating Player 1 in state $\{1, 2, 3\}$ and Player 2 being stronger than Player 3. Otherwise, Player 3 will be the most likely winner. Whether Player 2 will be the most likely winner depends on if λ_2 exceeds the threshold in the preceding equation.

CHAPTER 3:
Stochastic Duel with Four or More Players

Consider n players engaged in a stochastic duel game defined in Chapter 2, with $n \geq 4$. At any time, each player chooses an opponent still in the game to attack. Regardless of which opponent Player i chooses to attack, the amount of time needed to make a kill follows an exponential distribution with rate λ_i, for $i = 1, 2, \ldots, n$. Without loss of generality, we label the players such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, so Player 1 is the strongest and Player n the weakest. At any given time, each player has complete knowledge of who else is still competing and chooses one of these players to attack. The game ends when all but one player is killed and the surviving player is declared the winner. Each player's objective is to maximize their winning probability.

The rest of this chapter proceeds as follows. Section 3.1 presents an analysis for the case with 4 players. Section 3.2 explains how the same methodology can be extended to the case with 5 or more players.

3.1 The Case with Four Players

Identical to the three-person stochastic duel in Chapter 2, because the amount of time it takes for each player to make a kill follows an exponential distribution, once we know who are still in the game, the history of the game becomes irrelevant; the game can be formulated as a Markov game in which each state corresponds to the set of players still in the game. The game moves from the current state to a new state when any player makes a kill. Beginning with states {1, 2, 3, 4}, the state space and potential transitions can be seen in Figure 3.1.

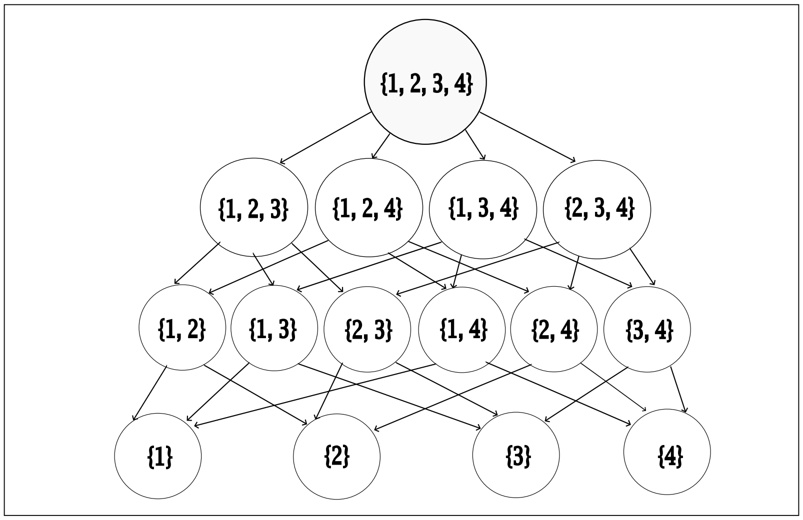

Figure 3.1. Four-Player Stochastic Duel as a Markov Game

As seen in Figure 3.1, there is one state with four players, namely {1, 2, 3, 4}, at the beginning of the game. As soon as a player is killed, the game transitions into a state with three players, namely {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}. After another player is killed, the game transitions into a state with two players, namely {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, or {3,4}. After a final kill, the game transitions to a state with a single player and the game ends.

In state {1, 2, 3, 4}, each of the 4 players must decide which opponent to attack. The decision depends on which of the resulting 3-player state is most preferred by each player. Recall from Chapter 2 that a player's winning probability in a state with 3 players remaining can be computed by equations (2.1), (2.2), and (2.3). We examine each player's situation separately.

3.1.1 Player 1

Player 1, unless eliminated in state {1, 2, 3, 4}, will remain the strongest player throughout the game. To decide which player to attack in state $\{1, 2, 3, 4\}$, Player 1 compares $\{1, 2, 3\}$, $\{1, 2, 3\}$, and $\{1, 3, 4\}$ and chooses the largest probability from $g_1(\lambda_1, \lambda_2, \lambda_3)$, $g_1(\lambda_1, \lambda_2, \lambda_4)$, and $g_1(\lambda_1, \lambda_3, \lambda_4)$.

Given $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, it follows that

$$g_1(\lambda_1, \lambda_3, \lambda_4) \geq g_1(\lambda_1, \lambda_2, \lambda_4) \geq g_1(\lambda_1, \lambda_2, \lambda_3).$$

Player 1's most preferred state with three players is $\{1, 3, 4\}$, so it is optimal for Player 1 to attack Player 2 in state $\{1, 2, 3, 4\}$.

3.1.2 Player 2

To decide which player to attack in state $\{1, 2, 3, 4\}$, Player 2 compares $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$, and chooses the largest probability from $g_2(\lambda_1, \lambda_2, \lambda_3)$, $g_2(\lambda_1, \lambda_2, \lambda_4)$, and $g_1(\lambda_2, \lambda_3, \lambda_4)$.

In either state $\{1, 2, 3\}$ or state $\{1, 2, 4\}$, Player 2 remains the second strongest player.

Because $\lambda_3 \geq \lambda_4$, it follows that $g_2(\lambda_1, \lambda_2, \lambda_3) \leq g_2(\lambda_1, \lambda_2, \lambda_4)$, so Player 2 prefers state $\{1, 2, 4\}$. In other words, Player 2's winning probability by attacking Player 4 is smaller than or equal to the winning probability by attacking Player 3. Consequently, it is sufficient for Player 2 to consider between attacking Player 1 or attacking Player 3. It is optimal for Player 2 to attack Player 1 if $g_1(\lambda_2, \lambda_3, \lambda_4) \geq g_2(\lambda_1, \lambda_2, \lambda_4)$, or to attack Player 3 otherwise.

3.1.3 Player 3

To decide which player to attack in state $\{1, 2, 3, 4\}$, Player 3 compares $\{1, 2, 3\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$, and chooses the largest probability from $g_3(\lambda_1, \lambda_2, \lambda_3)$, $g_2(\lambda_1, \lambda_3, \lambda_4)$, and $g_2(\lambda_2, \lambda_3, \lambda_4)$.

In either state $\{1, 3, 4\}$ or state $\{2, 3, 4\}$, Player 3 becomes the second strongest player. Because $\lambda_1 \geq \lambda_2$, it follows that $g_2(\lambda_1, \lambda_3, \lambda_4) \leq g_2(\lambda_2, \lambda_3, \lambda_4)$, so Player 3 prefers state $\{2, 3, 4\}$. In other words, Player 3's winning probability by attacking Player 2 is smaller than or equal to the winning probability by attacking Player 1. Consequently, it is sufficient for Player 3 to consider between attacking Player 1 or attacking Player 4. It is optimal for Player 3 to attack Player 1 if $g_2(\lambda_2, \lambda_3, \lambda_4) \geq g_3(\lambda_1, \lambda_2, \lambda_3)$, or to attack Player 4 otherwise.

3.1.4 Player 4

Player 4, unless eliminated in state $\{1, 2, 3, 4\}$, will remain the weakest player throughout the game. To decide which player to attack, Player 4 compares $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$ and chooses the largest probability from $g_3(\lambda_1, \lambda_2, \lambda_4)$, $g_3(\lambda_1, \lambda_3, \lambda_4)$, $g_3(\lambda_2, \lambda_3, \lambda_4)$.

Given $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, it follows that

$$g_3(\lambda_2, \lambda_3, \lambda_4) \geq g_3(\lambda_1, \lambda_3, \lambda_4) \geq g_3(\lambda_1, \lambda_2, \lambda_4).$$

Player 4's most preferred state with three players is $\{2, 3, 4\}$, so it is optimal for Player 4 to attack Player 1 in state $\{1, 2, 3, 4\}$.

3.1.5 An Example with Four Players

This section presents an example of four-player duel with the following kill rates:

$$\lambda_1 = 1,$$
$$\lambda_2 = 0.75,$$
$$\lambda_3 = 0.5,$$
$$\lambda_4 = 0.26.$$

As established in Sections 3.1.1 and 3.1.4, given that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, it is optimal for Player 1 to attack Player 2, and for Player 4 to attack Player 1.

To determine Player 2's optimal strategy, compute

$$g_1(\lambda_2, \lambda_3, \lambda_4) = 0.368,$$
$$g_2(\lambda_1, \lambda_2, \lambda_4) = 0.373.$$

Therefore, Player 2 prefers to eliminate Player 3 rather than Player 1. It is optimal for Player 2 to attack Player 3 at the beginning, so that Player 2 can become the middle player when only three players remain.

To determine Player 3's optimal strategy, compute

$$g_2(\lambda_2, \lambda_3, \lambda_4) = 0.331,$$
$$g_3(\lambda_1, \lambda_2, \lambda_3) = 0.370.$$

Therefore, Player 3 prefers to eliminate Player 4 rather than Player 1. It is optimal for Player 3 to attack Player 4 at the beginning, so that Player 3 can become the weak player when only three players remain.

Consequently, when each player acts optimally to maximize individual winning probability, Player 1 targets Player 2, Player 2 targets Player 3, Player 3 targets Player 4, and Player 4 targets Player 1.

3.2 Extension to Five or More Players

Consider the stochastic duel with n players, in which Player i has kill rate λ_i, for $i = 1, 2, \ldots, n$. Without loss of generality, assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, so Player 1 is the strongest and Player n the weakest.

Define $g_i^n(\lambda_1, \lambda_2, \ldots, \lambda_n)$ as the probability that the player having kill rate λ_i will be the eventual winner of the stochastic duel if each player uses their optimal strategy, when there are n players still alive in the game. To decide which opponent to attack in state $\{1, 2, \ldots, n\}$, Player i compares the $n - 1$ potential resulting states $\{1, 2, \ldots, n\} - \{j\}$, for $j = 1, 2, \ldots, i-1, i+1, \ldots, n$, to determine which resulting state is most preferable. If Player $j < i$ is eliminated from state $\{1, 2, \ldots, n\}$, then Player i becomes the $(i - 1)$st strongest player in state $\{1, \ldots, j-1, j+1, \ldots, i, \ldots, n\}$, so Player i's winning probability becomes

$$g_{i-1}^{n-1}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_i, \ldots, \lambda_n), \qquad \text{for } j = 1, 2, \ldots, i-1.$$

If Player $j > i$ is eliminated from state $\{1, 2, \ldots, n\}$, then Player i remains the ith strongest player in state $\{1, \ldots i, \ldots, j-1, j+1, \ldots, n\}$, so Player i's winning probability becomes

$$g_i^{n-1}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n), \qquad \text{for } j = i+1, i+2, \ldots, n.$$

It is optimal for Player i to attack Player j to maximize Player i's winning probability in the resulting state when Player j is removed. In other words, it is optimal for Player i to attack Player k in state $\{1, 2, \ldots, n\}$, where

$$k = \arg\max_{j} \{ \max_{j=1,\ldots,i-1} g_{i-1}^{n-1}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_i, \ldots, \lambda_n), \\ \max_{j=i+1,\ldots,n} g_i^{n-1}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n) \}.$$

To carry out this calculation, we can use equations (2.1), (2.2), and (2.3) to compute $g_i^4(\cdot)$ when 4 players are in the game, which in turn can be used to compute $g_i^5(\cdot)$ when 5 players are in the game, and so on. Consequently, the optimal strategy of each player in the stochastic game with n players can be computed recursively for $n = 3, 4, 5, \ldots$.

CHAPTER 4: Sequential Duel with Three Players

Consider a sequential duel game with three players. The three players choose their kill rates sequentially from the interval $[0, 1]$; thereafter they engage in the three-person stochastic duel defined in Chapter 2. Player 1 is the first person to select a kill rate $\theta_1 \in [0, 1]$. After observing Player 1's choice of θ_1, Player 2 then selects $\theta_2 \in [0, 1]$. Finally, after observing θ_1 and θ_2, Player 3 selects $\theta_3 \in [0, 1]$. The objective of each player is to choose a kill rate in order to maximize their winning probability in the subsequent three-person stochastic duel.

The rest of this chapter proceeds as follows. Section 4.1 offers some preliminary analysis. Section 4.2 derives the optimal strategy for Player 3. Section 4.3 derives the optimal strategy for Player 2. Section 4.4 derives the optimal strategy for Player 1.

4.1 Preliminary Analysis

In this sequential game, let θ_i denote the kill rate selected by Player i, $i = 1, 2, 3$, and let $h_i(\theta_1, \theta_2, \theta_3)$ denote the probability that Player i wins the follow-on stochastic duel, for $i = 1, 2, 3$.

Recall from Chapter 2 that by requiring $\lambda_1 > \lambda_2 > \lambda_3$, we write $g_i(\lambda_1, \lambda_2, \lambda_3)$ for the winning probability of the ith strongest player in the three-person stochastic duel. In the sequential game, each player can choose a kill rate freely in the interval $[0, 1]$, so θ_1, θ_2, and θ_3 do not need to obey any particular order.

The link between the functions $h_i(\theta_1, \theta_2, \theta_3)$, for $i = 1, 2, 3$ and $g_i(\lambda_1, \lambda_2, \lambda_3)$, for $i = 1, 2, 3$ depends on the ordering between θ_1, θ_2, and θ_3. For example, if $\theta_1 > \theta_2 > \theta_3$, their respective winning probabilities are

$$h_1(\theta_1, \theta_2, \theta_3) = g_1(\theta_1, \theta_2, \theta_3),$$
$$h_2(\theta_1, \theta_2, \theta_3) = g_2(\theta_1, \theta_2, \theta_3),$$
$$h_3(\theta_1, \theta_2, \theta_3) = g_3(\theta_1, \theta_2, \theta_3).$$

For another example, if $\theta_1 > \theta_3 > \theta_2$, their respective winning probabilities are

$$h_1(\theta_1, \theta_2, \theta_3) = g_1(\theta_1, \theta_3, \theta_2),$$
$$h_2(\theta_1, \theta_2, \theta_3) = g_3(\theta_1, \theta_3, \theta_2),$$
$$h_3(\theta_1, \theta_2, \theta_3) = g_2(\theta_1, \theta_3, \theta_2).$$

In a sequential game, if another player has selected kill rate θ, then a player who arrived later would not choose the same kill rate θ, because choosing $\theta - \epsilon$ for some $\epsilon > 0$ is strictly better. Intuitively, choosing $\theta - \epsilon$ would make the player a less desirable target than the player having kill rate θ. The next proposition formalizes this idea.

Proposition 1 *A player will not choose a kill rate identical to that already chosen by another player who arrived earlier.*

Proof. Suppose Player 1 has chosen θ_1 and Player 2 has chosen θ_2. Consider three cases:

1. $\theta_1 > \theta_2$: If Player 3 chooses $\theta_3 = \theta_1 - \epsilon$, for some $\epsilon \in (0, \theta_1 - \theta_2)$, then Player 3's winning probability is

$$g_2(\theta_1, \theta_1 - \epsilon, \theta_2) = \frac{\theta_1 - \epsilon}{2\theta_1 + \theta_2 - \epsilon} \tag{4.1}$$

If Player 3 chooses $\theta_3 = \theta_1$, then Player 3's winning probability is

$$\frac{1}{2}(g_1(\theta_1, \theta_1, \theta_2) + g_2(\theta_1, \theta_1, \theta_2)) = \frac{1}{2}\left(\frac{\theta_1}{2\theta_1 + \theta_2}\frac{\theta_1}{\theta_1 + \theta_2} + \frac{\theta_1}{2\theta_1 + \theta_2}\right) = \frac{1}{2}\frac{\theta_1}{\theta_1 + \theta_2} \tag{4.2}$$

After some algebra, one can show that (4.1) is greater than (4.2) if $\epsilon < \theta_1\theta_2/(\theta_1 + 2\theta_2)$.

Consequently, choosing $\theta_3 = \theta_1$ is dominated so it should not be used.

If Player chooses $\theta_3 = \theta_2 - \epsilon$, for some $\epsilon \in (0, \theta_1 - \theta_2)$, then Player 3's winning probability is

$$g_3(\theta_1, \theta_2, \theta_2 - \epsilon) = \left(\frac{\theta_2 - \epsilon}{\theta_1 + 2\theta_2 - \epsilon}\right)\left(\frac{2\theta_1 + \theta_2 - \epsilon}{\theta_1 + \theta_2 - \epsilon}\right) \tag{4.3}$$

If Player 3 chooses $\theta_3 = \theta_2$, the Player 3's winning probability is

$$\frac{1}{2}(g_2(\theta_1, \theta_2, \theta_2) + g_3(\theta_1, \theta_2, \theta_2)) = \frac{1}{2}\left(\frac{\theta_2}{\theta_1 + 2\theta_2} + \left(\frac{\theta_2}{\theta_1 + 2\theta_2}\right)\left(\frac{2\theta_1 + \theta_2}{\theta_1 + \theta_2}\right)\right) \tag{4.4}$$

After some algebra, one can show that (4.3) is greater than (4.4) if $\epsilon < 3\theta_1\theta_2/(2\theta_1 + \theta_2)$. Consequently, choosing $\theta_3 = \theta_2$ is dominated so it should not be used.

2. $\theta_1 = \theta_2$: If Player 3 chooses $\theta_3 = \theta_1 - \epsilon$, for some $\epsilon \in (0, \theta_1)$, then Player 3's winning probability is

$$g_3(\theta_1, \theta_1, \theta_1 - \epsilon) = \frac{\theta_1 - \epsilon}{2\theta_1 - \epsilon} \tag{4.5}$$

If Player 3 chooses $\theta_3 = \theta_1$, then Player 3's winning probability is $1/3$ due to symmetry. After some algebra, one can show that (4.5) is greater than $1/3$ if $\epsilon < \theta_1/2$. Consequently, choosing $\theta_3 = \theta_1$ is dominated so it should not be used.

3. $\theta_1 < \theta_2$: Similar to case 1.

In all three cases, choosing $\theta_3 = \theta_1$ or $\theta_3 = \theta_2$ is dominated by another strategy, which completes the proof for Player 3. With a similar argument, we can show that for Player 2, choosing $\theta_2 = \theta_1$ is dominated by choosing $\theta_2 = \theta_1 - \epsilon$ for some $\epsilon > 0$ regardless of what Player 3 does. The proof is completed. □

4.2 Player 3's Optimal Response

Player 3 arrives to our sequential duel knowing Player 1 and Player 2 have chosen their respective kill rates $\theta_1, \theta_2 \in [0, 1]$. In order to maximize the winning probability, Player 3 does not care whether θ_1 or θ_2 is larger between the two. Without loss of generality, we will label the first two players such that $\theta_1 > \theta_2$. According to Proposition 1, Player 2 would not choose the same kill rate that Player 1 has chosen, so Player 3 would not encounter the case $\theta_1 = \theta_2$ if Player 2 plays optimally, but we will address this case at the end of this subsection.

Having observed $\theta_1 > \theta_2$, Player 3 must select a kill rate with one the following relationships to Player 1 and 2:

1. Become the strong player: $\theta_2 < \theta_1 < \theta_3$
2. Become the middle player: $\theta_2 < \theta_3 < \theta_1$
3. Become the weak player: $\theta_3 < \theta_2 < \theta_1$

We next investigate each case separately.

1. Become the strong player. In this case, we have $\theta_2 < \theta_1 < \theta_3$, which is only possible if $\theta_1 < 1$. Because Player 3 is the strong player, using (2.1) we can compute Player 3's winning probability by

$$h_3(\theta_1, \theta_2, \theta_3) = g_1(\theta_3, \theta_1, \theta_2) = \left(\frac{\theta_3}{\theta_3 + \theta_1 + \theta_2}\right)\left(\frac{\theta_3}{\theta_3 + \theta_2}\right).$$

   Keeping θ_1 and θ_2 constant, the preceding function increases strictly in θ_3. To maximize the winning probability, it is optimal for Player 3 to choose $\theta_3 = 1$. In other words, if $\theta_1 < 1$ and if Player 3 chooses $\theta_3 \in (\theta_1, 1]$, then it is optimal for Player 3 to choose $\theta_3 = 1$.

2. Become the middle player. In this case, we have $\theta_2 < \theta_3 < \theta_1$. Because Player 3 is the middle player, using (2.2) we can compute Player 3's winning probability by

$$h_3(\theta_1, \theta_2, \theta_3) = g_2(\theta_1, \theta_3, \theta_2) = \left(\frac{\theta_3 + \theta_2}{\theta_1 + \theta_3 + \theta_2}\right)\left(\frac{\theta_3}{\theta_3 + \theta_2}\right).$$

   Keeping θ_1 and θ_2 constant, the preceding function increases strictly in θ_3. To maximize the winning probability, Player 3 wants to choose θ_3 as large as possible from the interval (θ_2, θ_1) as large as possible. In other words, Player 3 wants to choose $\theta_3 < \theta_1$

but as close to θ_1 as possible. Loosely speaking, we write this choice as θ_1^-. If it is optimal for Player 3 to be the middle player, then Player 3 does not have an optimal policy. Rather, Player 3 has an ϵ-optimal policy by choosing $\theta_3 = \theta_1^-$, which gives Player 3 a winning probability arbitrarily close to the optimal winning probability.

3. Become the weak player. In this case, we have $\theta_3 < \theta_2 < \theta_1$. Because Player 3 is the weak player, using (2.3) we can compute Player 3's winning probability by

$$h_3(\theta_1, \theta_2, \theta_3) = g_3(\theta_1, \theta_2, \theta_3) = \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3}\left(\frac{2\theta_1 + \theta_3}{\theta_1 + \theta_3}\right).$$

   Keeping θ_1 and θ_2 constant, the preceding function increases in θ_3. To maximize the winning probability, Player 3 wants to choose θ_3 as large as possible from the interval $(0, \theta_2)$ as large as possible. Similar to the discussion in the previous case, we write this choice as θ_2^-.

In summary, after learning about θ_1 and θ_2, Player 3 needs to compare only three viable choices—namely, 1, θ_1^-, and θ_2^-—and choose the one that produces the highest winning probability for Player 3.

If for some reason, Player 2 did not act optimally and chooses $\theta_2 = \theta_1$, then Player 3 can choose to be the strong player by choosing $\theta_3 \in (\theta_1, 1]$, or the weak player by choosing $\theta_3 \in [0, \theta_1)$. According to Proposition 1, it is sub-optimal for Player 3 to choose $\theta_3 = \theta_1 = \theta_2$. With a similar analysis, we can see that Player 3 has two viable choices 1 and θ_1^-. In other words, it is sufficient for Player 3 to compare these two choices to find his optimal strategy.

Figure 4.1 depicts Player 3's optimal choice for each (θ_1, θ_2) pair. For example, if $\theta_1 = 0.1$ and $\theta_2 = 0.2$, then Player 3 will choose to be the strong player by choosing $\theta_3 = 1$. For another example, if $\theta_1 = 0.1$ and $\theta_2 = 0.95$, then Player 3 will choose to be the middle player by choosing $\theta_3 = 0.95^-$.

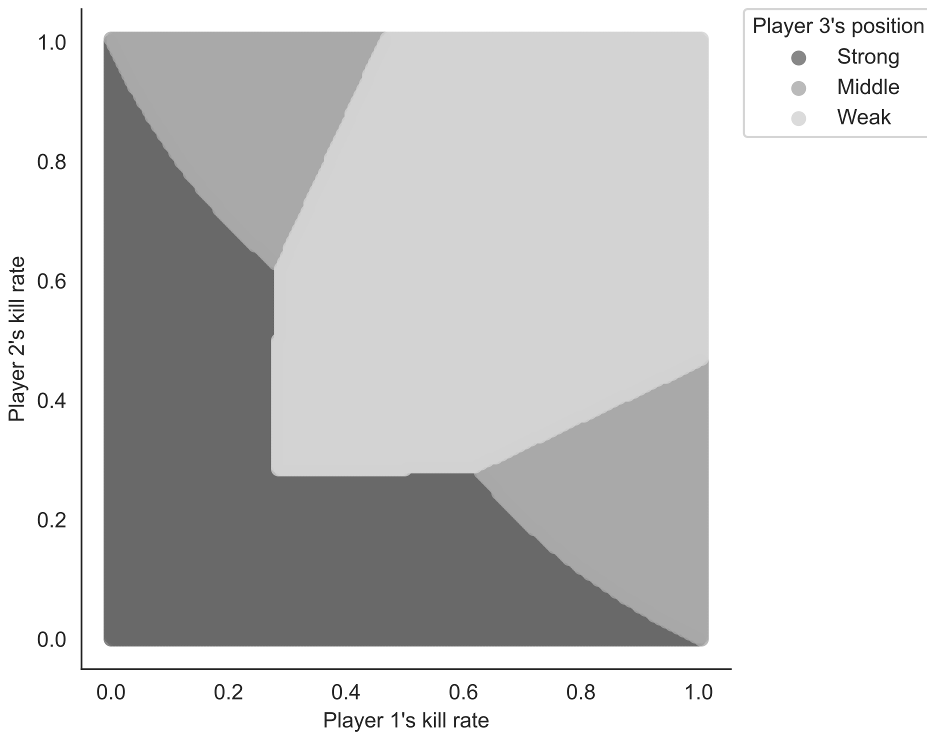

Figure 4.1. Player 3's Optimal Response Against Players 1's and 2's Kill Rates

4.3 Player 2's Optimal Response

Player 2 arrives second, and needs to choose kill rate θ_2 after observing Player 1's kill rate choice $\theta_1 \in [0, 1]$. The objective of Player 2 is to choose θ_2 that will maximize Player 2's winning probability, knowing that Player 3 will later act optimally by choosing θ_3 among 1, θ_1^-, and θ_2^-, in order to maximize Player 3's winning probability.

For each pair (θ_1, θ_2), Player 2 can look at the problem from Player 3's standpoint and determine the optimal response from Player 3, as seen in Figure 4.1. For given θ_1, Player 2 can then evaluate each potential choice of $\theta_2 \in [0, 1]$ by first determining Player 3's optimal response, then computing Player 2's corresponding winning probability. The optimal choice for Player 2, which is a function of θ_1, is shown in Figure 4.2.

As seen in Figure 4.2, if θ_1 is too small, then it is optimal for Player 2 to choose $\theta_2 = 1$, and Player 3 will subsequently choose to be the middle player. If $\theta_1 \in (0.47, 0.5)$, it is optimal for Player 2 to choose $\theta_2 = 1$ also, but Player 3 will subsequently choose to be the weak player. If $\theta_1 \in (0.5, 1]$, then it is optimal for Player 2 to choose $\theta_2 = \theta_1^-$, and Player 3 will subsequently choose $\theta_3 = \theta_2^-$, in which case the winning probability for Players 1, 2, and 3 are $1/6$, $1/3$, and $1/2$, respectively.

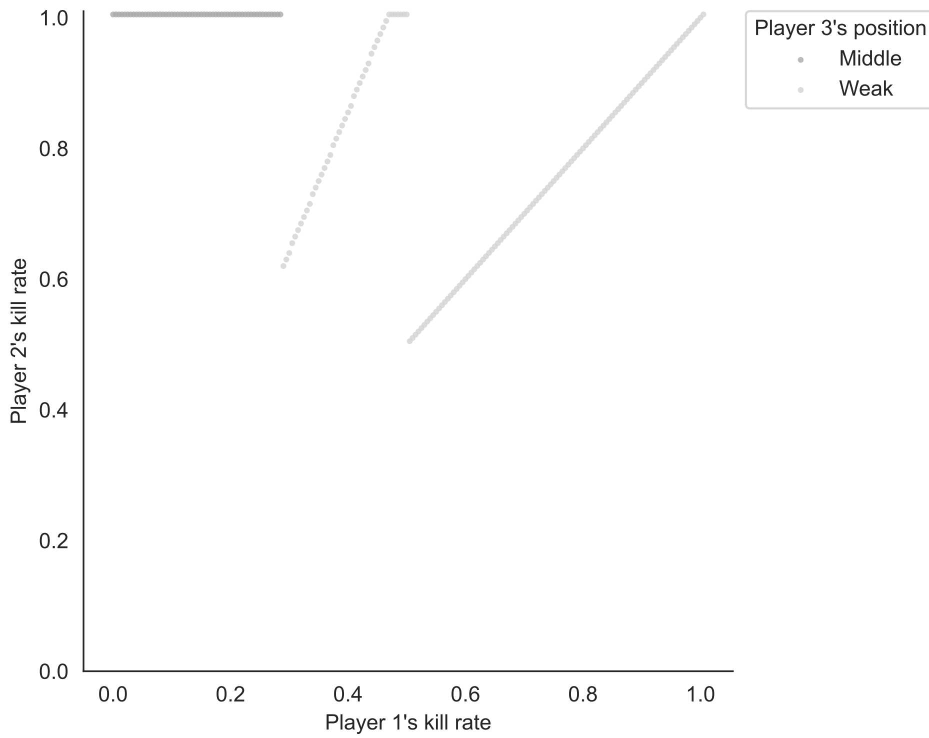

Figure 4.2. Player 2's Optimal Response Against Player 1's Kill Rate

To further investigate the optimal choice of Player 2, we study two cases. The first case concerns $\theta_1 = 1$, and the second case concerns a small region where $\theta_1 < 0.5$. To facilitate the analysis, we need a definition and a lemma.

Definition 2 *Define r to be the unique root in $[0, 1]$ to the polynomial function*

$$r^3 + 2r^2 + r - 1 = 0.$$

One can compute $r \approx 0.46557$.

Lemma 3 *The function*

$$f(x) = \frac{1}{2+x} - \frac{x(2+x)}{(1+2x)(1+x)}$$

is strictly decreasing in x for $x \in [0,1]$, and $f(r) = 0$.

Proof. The statement can be verified by some algebraic and calculus work, or by plotting the function. □

We next present the theorem that concerns Player 2's optimal response if Player 1 chooses $\theta_1 = 1$.

Theorem 4 *If Player 1 chooses $\theta_1 = 1$, then it is optimal for Player 2 to choose $\theta_2 = 1^-$ and for Player 3 to subsequently choose $\theta_3 = \theta_2^-$. The payoffs are $1/6$, $1/3$, and $1/2$ to Players 1, 2, and 3, respectively.*

Proof. First, suppose that Player 2 chooses $\theta_2 \in [0, 1)$, Player 3 has three viable options: choose $\theta_3 = 1$ to be the strong player, choose $\theta_3 = 1^-$ to be the middle player, or choose $\theta_3 = \theta_2^-$ to be the weak player. Player 3's payoffs are

$$\begin{aligned} g_1(1, 1, \theta_2) &= \frac{1}{2}\left(\frac{1}{(2+\theta_2)(1+\theta_2)} + \frac{1}{2+\theta_2}\right), \\ g_2(1, 1^-, \theta_2) &\rightarrow \frac{1}{2+\theta_2}, \\ g_3(1, \theta_2, \theta_2^-) &\rightarrow \frac{\theta_2(2+\theta_2)}{(1+2\theta_2)(1+\theta_2)}, \end{aligned}$$

respectively. Because $g_1(1, 1, \theta_2) < g_2(1, 1^-, \theta_2)$, Player 3 should choose to be either the middle player if

$$\frac{1}{2+\theta_2} > \frac{\theta_2(2+\theta_2)}{(1+2\theta_2)(1+\theta_2)}, \tag{4.6}$$

or the weak player otherwise.

Consider three cases:

1. If $\theta_2 < r$, then according to Lemma 3, the inequality in (4.6) holds. It is optimal for Player 3 to be the middle player, so Player 2's payoff is

$$g_3(1, 1^-, \theta_2) \to \frac{\theta_2}{2+\theta_2}\frac{2+\theta_2}{1+\theta_2} = \frac{\theta_2}{1+\theta_2},$$

   which increases in θ_2. If Player 2 is to choose $\theta_2 \in [0, r)$, then the best choice is $\theta_2 = r^-$, which results in payoff $r/(1+r) \approx 0.3177$.

2. If $\theta_2 > r$, then according to Lemma 3, it is optimal for Player 3 to be the weak player, so Player 2's payoff is

$$g_2(1, \theta_2, \theta_2^-) \to \frac{\theta_2}{1+2\theta_2},$$

   which increases in θ_2. If Player 2 is to choose $\theta_2 \in (r, 1)$, then the best choice is $\theta_2 = 1^-$, which results in payoff $1/3$.

3. If $\theta_2 = r$, then Player 3 feels indifferent between r^- and 1^-, so Player 2 would get either $r/(1+r) \approx 0.3177$ or $r/(1+2r) \approx 0.2411$, depending on Player 3's choice.

By comparing these three cases, we can conclude that for $\theta_2 \in [0, 1)$, it is best for Player 2 to choose $\theta_2 = 1^-$ to get $1/3$. It is optimal for Player 3 to subsequently choose $\theta_3 = \theta_2^-$, so the payoffs are $1/6$ and $1/2$ to Player 1 and Player 3, respectively.

To complete the proof, consider the case $\theta_2 = 1$. With $\theta_1 = \theta_2 = 1$, it is optimal for Player 3 to choose $\theta_3 = 1^-$, so Player 2's payoff is $1/4 < 1/3$, which completes the proof. □

Figure 4.2 confirms our proof in which we observe that if Player 1 selects $\theta_1 \in (0.5, .1)$, it is optimal for Player 2 to choose θ_1^- and for Player 3 to subsequently choose $\theta_3 = \theta_2^-$.

The next theorem concerns Player 2's optimal response if Player 1 chooses $\theta_1 \in (r, 0.5)$, where r is defined in Definition 2.

Theorem 5 *If Player 1 chooses* $\theta_1 \in (r, 0.5)$*, then it is optimal for Player 2 to choose* $\theta_2 = 1$ *and for Player 3 to subsequently choose* $\theta_3 = \theta_1^-$.

Proof. Consider Player 2's choice in three cases:

1. If $\theta_2 > \theta_1$, Player 3 has three viable choices 1, θ_2^-, and θ_1^-, with respective payoffs

$$g_1(1, \theta_2, \theta_1) \to \frac{1}{(1 + \theta_2 + \theta_1)(1 + \theta_1)}, \tag{4.7}$$

$$g_2(\theta_2, \theta_2^-, \theta_1) \to \frac{\theta_2}{2\theta_2 + \theta_1}, \tag{4.8}$$

$$g_3(\theta_2, \theta_1, \theta_1^-) \to \frac{\theta_1}{\theta_2 + 2\theta_1} \frac{2\theta_2 + \theta_1}{\theta_2 + \theta_1}. \tag{4.9}$$

   Because $\theta_1/\theta_2 > r/1$, it follows from Lemma 3 that $g_3(\theta_2, \theta_1, \theta_1^-)$ in (4.9) is greater than $g_2(\theta_2, \theta_2^-, \theta_1)$ in (4.8). In addition, observe that

$$\theta_1(1 + \theta_1) \left(\frac{\theta_1 + 2\theta_2}{\theta_1 + \theta_2}\right) \left(\frac{1 + \theta_1 + \theta_2}{\theta_2 + 2\theta_1}\right) > r(1 + r) \left(\frac{3}{2}\right) (1) > 1,$$

   where the inequality is due to $\theta_1 > r$, $\theta_2 > \theta_1$, and $\theta_1 < 1$. Therefore, $g_3(\theta_2, \theta_1, \theta_1^-)$ in (4.9) is greater than $g_1(1, \theta_2, \theta_1)$ in (4.7). In conclusion, it is optimal for Player 3 to choose $\theta_3 = \theta_1^-$ to be the weak player. Consequently, it is best for Player 2 to choose $\theta_2 = 1$ to get

$$g_1(1, \theta_1, \theta_1^-) \to \frac{1}{(1 + 2\theta_1)(1 + \theta_1)} > \frac{1}{3}. \tag{4.10}$$

   because $\theta_1 < 0.5$.

2. If $\theta_2 < \theta_1$, Player 3 has three viable choices 1, θ_1^-, and θ_2^-, with respective payoffs

$$g_1(1, \theta_1, \theta_2) = \frac{1}{(1 + \theta_1 + \theta_2)(1 + \theta_2)}, \tag{4.11}$$

$$g_2(\theta_1, \theta_1^-, \theta_2) = \frac{\theta_1}{2\theta_1 + \theta_2}, \tag{4.12}$$

$$g_3(\theta_1, \theta_2, \theta_2^-) = \frac{\theta_2}{\theta_1 + 2\theta_2} \frac{2\theta_1 + \theta_2}{\theta_1 + \theta_2}. \tag{4.13}$$

   First, we show that

$$g_1(1, \theta_1, \theta_2) > g_2(\theta_1, \theta_1^-, \theta_2),$$

   which is equivalent to

$$2\theta_1 + \theta_2 > \theta_1(1 + \theta_2)(1 + \theta_1 + \theta_2),$$

which is true because taking the difference yields

$$\begin{aligned}2\theta_1 + \theta_2 - \theta_1(1+\theta_2)(1+\theta_1+\theta_2) &= \theta_1 + \theta_2 - \theta_1\theta_2 - \theta_1(\theta_1+\theta_2) - \theta_1\theta_2(\theta_1+\theta_2)\\ &> \theta_1 + \theta_2 - \theta_1\theta_2 - \theta_1(1) - \theta_1\theta_2(1)\\ &= \theta_2(1-2\theta_1) > 0,\end{aligned}$$

where the inequalities are due to $\theta_2 < \theta_1 < 0.5$. Therefore, it is suboptimal for Player 3 to be the middle player.

To compare (4.11) and (4.13), set $g_1(1, \theta_1, \theta_2) = g_3(\theta_1, \theta_2, \theta_2^-)$ to obtain, after some algebraic work,

$$\theta_2^4 + (3\theta_1 + 2)\theta_2^3 + (2\theta_1^2 + 5\theta_1 - 1)\theta_2^2 + (2\theta_1^2 - \theta_1)\theta_2 - \theta_1^2 = 0,$$

which can been seen as a quartic function in θ_2, whose coefficients are functions of θ_1. For $\theta_1 \in [r, 0.5]$, this quartic function has a unique root in $[0, 1]$. Write $f(\theta_1)$ for that unique root as a function of θ_1. For example, one can compute $f(r) \approx 0.283725$ and $f(0.5) \approx 0.284967$.

Consider three cases:

(a) If $\theta_2 \in [0, f(\theta_1))$, then $g_1(1, \theta_1, \theta_2) > g_3(\theta_1, \theta_2, \theta_2^-)$, so it is optimal for Player 3 to be the strong player. The payoff to Player 2 becomes

$$g_3(1, \theta_1, \theta_2) = \frac{\theta_2}{1+\theta_1+\theta_2}\frac{2+\theta_2}{1+\theta_2},$$

which is maximized by taking $\theta_2 \to f(\theta_1) \le f(0.5) < 0.285$. Therefore,

$$g_3(1, \theta_1, \theta_2) \le \frac{f(\theta_1)}{1+\theta_1+f(\theta_1)}\frac{2+f(\theta_1)}{1+f(\theta_1)} < \frac{0.285}{1.285+r}\frac{2.285}{1.285} < 0.29.$$

(b) If $\theta_2 \in (f(\theta_1), \theta_1)$, then $g_3(\theta_1, \theta_2, \theta_2^-) > g_1(1, \theta_1, \theta_2)$, so it is optimal for Player 3 to be the weak player. The payoff to Player 2 becomes

$$g_2(\theta_1, \theta_2, \theta_2^-) \to \frac{\theta_2}{\theta_1 + 2\theta_2} < \frac{1}{3}, \tag{4.14}$$

because $\theta_2 < \theta_1$.

(c) If $\theta_2 = f(\theta_1)$, then Player 3 feels indifferent between 1 and θ_2^-. In either case, the payoff to Player 2 is strictly less than 1/3.

3. If $\theta_2 = \theta_1$, Player 3 has two viable choices 1 and θ_1^-, with respective payoffs

$$g_1(1, \theta_1, \theta_1) = \frac{1}{(1 + 2\theta_1)(1 + \theta_1)} < \frac{1}{(1 + 2r)(1 + r)} < 0.3534, \tag{4.15}$$

$$g_3(\theta_1, \theta_1, \theta_1^-) \to \frac{\theta_1}{3\theta_1} \frac{3\theta_1}{2\theta_1} = \frac{1}{2}. \tag{4.16}$$

Therefore, it is optimal for Player 3 to choose θ_1^-, so the payoff to Player 2 is

$$\frac{1}{2} \left(g_1(\theta_1, \theta_1, \theta_1^-) + g_2(\theta_1, \theta_1, \theta_1^-) \right) \to \frac{1}{2} \left(\frac{1}{6} + \frac{1}{3} \right) = \frac{1}{4}.$$

Compare the three cases, we can conclude that it is optimal for Player 2 to choose $\theta_2 = 1$, and for Player 3 to subsequently choose $\theta_3 = \theta_1^-$, which completes the proof. □

4.4 Player 1's Optimal Strategy

Player 1 arrives to our sequential duel first and needs to choose a kill rate θ_1, knowing that Players 2 will arrive next, then followed by Player 3, and that both Players 2 and 3 will act optimally to maximize their own respective winning probabilities.

From the analysis in Sections 4.2 and 4.3, we see that Player 1's choice θ_1 drives the subsequent choices of Player 2 and Player 3. For each θ_1, we can compute Player 2's optimal response and Player 3's optimal response, and compute the corresponding winning probability for Player 1, which we plot in Figure 4.3.

As seen in Figure 4.3, it appears that Player 1 should choose $\theta_1 < 0.5$ but as close to 0.5 as possible. The next theorem formalizes this observation.

Theorem 6 *It is optimal for Player 1 to choose* $\theta_1 = 0.5^-$*, for Player 2 to choose* $\theta_2 = 1$*, and for Player 3 to choose* $\theta_3 = \theta_1^-$*. The payoffs are* 1/4, 1/3*, and* 5/12 *to Players 1, 2, and 3, respectively.*

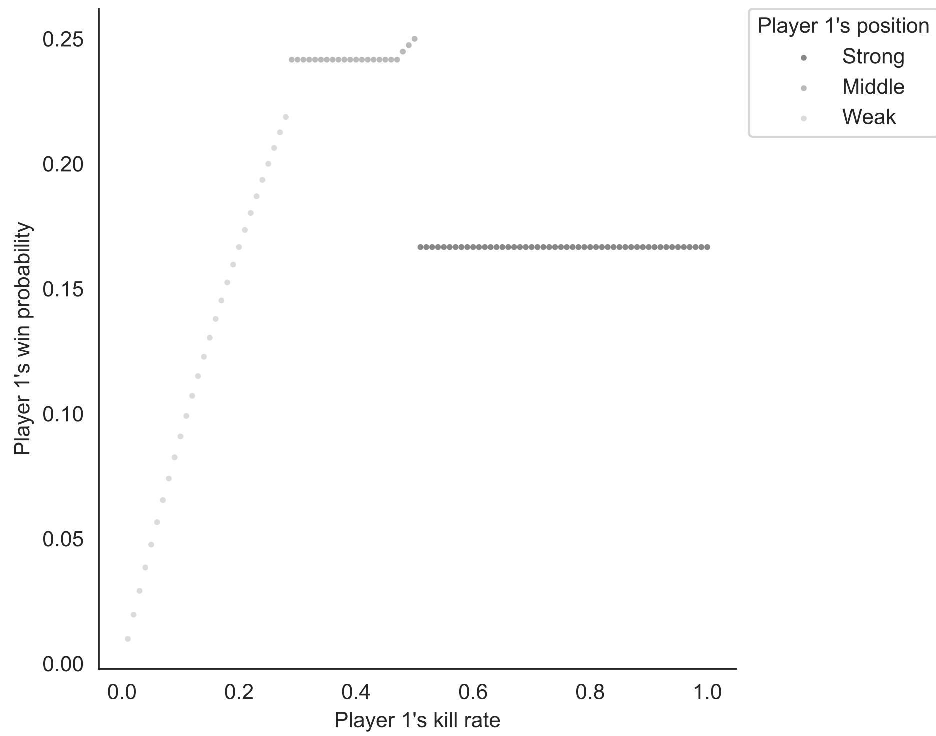

Figure 4.3. Player 1's Winning Probability as a Function of His Kill Rate

Proof. As seen in Figure 4.3, the payoff to Player 1 for $\theta_1 \notin (r, 0.5)$ is strictly smaller than those obtained with $\theta_1 \in (r, 0.5)$. For $\theta_1 \in (r, 0.5)$, according to Theorem 5, it is optimal for Player 2 to choose $\theta_2 = 1$ and for Player 3 to subsequently choose $\theta_3 = \theta_1^-$. Therefore, Player 1's payoff is

$$g_2(1, \theta_1, \theta_1^-) = \frac{\theta_1}{1 + 2\theta_1},$$

which increases strictly in θ_1, so Player 1 can increase it by taking $\theta_1 \uparrow 0.5$. Consequently, Player 1's payoff approaches

$$g_2(1, 0.5^-, 0.5^-) \to \frac{0.5}{1 + 0.5 + 0.5} = \frac{1}{4}.$$

Correspondingly, Player 2's payoff approaches

$$g_1(1, 0.5^-, 0.5^-) \to \left(\frac{1}{1 + 0.5 + 0.5}\right)\left(\frac{1}{1 + 0.5}\right) = \frac{1}{3},$$

and Player 3's payoff approaches

$$g_3(1, 0.5^-, 0.5^-) \to \frac{0.5}{1 + 0.5 + 0.5}\left(\frac{2 + 0.5}{1 + 0.5}\right) = \frac{5}{12},$$

which completes the proof. □

We have solved the three-player sequential game. Player 1—having full understanding of Player 2's optimal response and Player 3's subsequent optimal response—will select a kill rate just shy of 0.5 and end up with $1/4$ probability of winning. The optimal response of Player 2 is to select a kill rate of 1 and end up with $1/3$ probability of winning. With the first two players selecting kill rates 0.5^- and 1, Player 3 will want to be the weak player by selecting a kill rate just below that of Player 1's 0.5^-. Despite having the lowest kill rate, Player 3 will end up with $5/12$ probability of winning—the highest among the three players.

These results demonstrate that knowledge gives players who select kill rates later a distinct advantage. Somewhat surprisingly, the first player to arrive wants to be mediocre in order to maximize their winning probability—a stark contrast to a two-player contest in which each player always wants to select the highest kill rate possible.

CHAPTER 5:
Conclusion

This thesis presents a stochastic duel with three or more players. Our stochastic duels with three or more players demonstrate that being the stronger player does not always result in having a higher winning probability. In addition, it is not necessarily optimal for each player to simply target their strongest opponent. Our sequential duel with three players demonstrate the power of knowledge. A player who selects the kill rate after observing the selections of the other players has a higher probability of winning.

With the potential for the United States to face multiple adversaries in the same area of operations in the near future, a further understanding of how three or more players will interact is critical to shed light into how each player should act optimally to gain a competitive edge. Our extension of duel models provides context to this interaction.

This thesis lays a foundation for a stochastic duel with three or more players. In order to make the mathematics tractable, we make several assumptions. For example, the strength of each player is modeled by a single number—the kill rate. Each hit immediately results in a kill, and each player can shoot at any opponent at any given time. Below we offer a few future research directions that relax some of our model assumptions.

All kill rates in this thesis were treated as universal, regardless of the opponent a player would choose to target. However, there may be several reasons for which the kill rate depends not only on the attacker but also on the attacked. For example, in the case of individuals, if one opponent is wearing armor and another is not, it stands to reason that the kill rate against the armor-less opponent should be higher. In the case of military unit types, if one opponent has an anti-air capability but does not possess an anti-tank capability, it stands to reason that they would have a higher kill rate against an air type opponent when compared to a tank type opponent. This generalization complicates the problem substantially.

Our model assumes that each hit results in a kill. In the real world, a player who is hit may simply become weakened and continues to engage the combat with a weakened fire power. In this scenario, a player does not get killed until being hit for a few times. Allowing a player to continue fighting until getting hit for several times could provide another element or realism to our games.

For this thesis, all players are able to target any opponent still remaining in the game at any given time. There are many combat scenarios that would demonstrate that this will not always be the case. For example, if we consider players with different effective ranges, then depending on each player's real-time location, a player may not be able to attack some other players at certain time simply because they are out of range. Another example would be when considering anti-submarine warfare; some units are able to target a submarine when it is submerged where others are only able to target a submarine when it is surfaced. Altering which opponents specific players could target in specific states could greatly alter the optimal strategies pursued as well as the overall outcome of the game.

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