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14. ABSTRACT We report on the development of simulation and data-driven analysis tools for investigating the structure of turbulence and sound in forced high-speed jets. Such techniques are critical for the development of noise reduction strategies for military-style aircraft. Both canonical, axisymmetric geometries and a more complex twin-rectangular jet configuration are considered. Spectral Proper Orthogonal Decomposition is generalized to address time-periodic forced flows, and to exploit the D2-symmetry group associated with the twin-rectangular nozzle. The turbulence structure of these forced flows is then analyzed with the newly developed tools.					
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Collaborative theoretical and computational study of jet noise control

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1 Introduction

1.1 Motivation

Reducing the jet noise produced by tactical aircraft is difficult due to the low-bypass-ratio engines required by these aircraft. Thus, employing the successfully utilized commercial aviation approach of increasing the bypass ratio is not possible. Instead, several methods have been developed to subtly alter the flow. Both passive (chevrons, tabs, etc.) and active (fluidic injection, plasma actuators, etc.) control devices have been employed to successfully reduce peak aft-angle radiation by up to 5dB [Casalino et al. \[2008\]](#). Of particular interest are active control devices as they avoid potential performance losses in portions of the flight envelope where noise mitigation is not of primary concern. However, these control devices are empirically designed and we typically lack a mechanistic understanding that would enable them to be scaled and improved.

Amongst studies seeking to characterize mechanisms, [Koenig et al. \[2016\]](#) studied the effect of unsteady forcing on the noise produced by a subsonic turbulent jet. Two forcing frequencies $St_f = 0.23, 0.46$ at an azimuthal mode number $m = 2$ were studied. They found that the lower forcing frequency resulted in a broadband increase in noise production which, by using the parabolized stability equations, they linked to the turbulent mean flow strongly amplifying disturbances at the lower forcing frequency (and azimuthal mode number). Consequently, the lower frequency forcing produced, in addition to a modification of the mean, a strong tonal response that led to a considerable increase in the noise. In contrast, the high-frequency forcing resulted in a broadband noise reduction which was linked to the forcing deforming the mean flow to one that was more stable. Similarly, [Sinha et al. \[2018\]](#) investigated the effect of steady and periodic mass injection on the noise produced by a Mach 1.5 supersonic axisymmetric jet. They showed that jet noise reduction was uniquely linked to the steady component of the forcing applied. Applying unsteady forcing in addition to the steady component resulted in the creation of highly energetic tones at the forcing frequency and its harmonics which overwhelmed the broadband noise reduction created by the steady forcing. These experimental studies suggest that jet noise reduction does not result from any direct interaction between the forcing and the background turbulence. Instead, it is a by-product of Reynolds stresses that are generated by the tonal disturbances (which are independent from those of the background turbulence) that deform the turbulent mean flow of the jet and thus alter the entire turbulence spectrum.

A fundamental assumption frequently evoked to study natural jets is statistical stationarity, meaning that the statistical characteristics of the system are invariant in time. This is a valid assumption for unforced jets; however, in cases where forcing is applied, this fundamental assumption breaks down as the flow, and thus statistics, are expected to be correlated to the forcing. Thus, to analyze the full impact of the forcing on the flow, one must depart from the statistically stationary framework. Flows where the statistics vary periodically, as is expected in a turbulent jet when subjected to a harmonic forcing, can be studied using cyclostationary analysis. Cyclostationary analysis, popularized and heavily developed by [Hurd \[1969\]](#), [Gardner \[1972\]](#), among many others [Boyles and Gardner \[1983\]](#), [Gardner \[1986b\]](#), [Brown III \[1987\]](#), [Gardner \[1986c\]](#), is an extension of statistically stationary analysis to processes with periodically varying statistics. While it is well known and employed in the signal processing and mechanics community, it is relatively unknown to the turbulence/fluid dynamics community. All commonly used statistically stationary descriptors (cross-correlation, cross-spectral density, etc) have equivalent cyclostationary forms that collapse back to their stationary counterpart when analyzing stationary flows, making it ideal for studying these processes.

Among the myriad of tools used to study turbulent jets, spectral proper orthogonal decomposition (SPOD) [Lumley \[1967b, 2007\]](#), [Towne et al. \[2018b\]](#) has recently been employed successfully to study the instability mechanisms present in natural (unforced) turbulence jets, see for example [Pickering et al. \[2020\]](#), [Schmidt and Towne \[2019\]](#), [Cavalieri et al. \[2019\]](#), [Nogueira et al. \[2019\]](#), [Nekkanti and](#)

Schmidt [2021], Kaplan et al. [2021]. By determining the energy spectra and decomposing the flow into energy-ranked coherent structures, these studies could identify and analyze the mechanisms, such as the Kelvin-Helmholtz, Orr, and lift-up, present. Recent studies [Heidt et al., 2021] have investigated a forced turbulent jet while naively assuming a statistically stationary assumption. However, since SPOD requires a statistically stationary assumption, this is not valid for forced turbulent jets. To study turbulent flows that exhibit periodic statistics, we develop an extension to SPOD called cyclostationary SPOD (CS-SPOD), through an extension to Kim et al. [1996]. This extends SPOD to the case with cyclostationary statistics. CS-SPOD shares similar properties to SPOD and naturally collapses back to SPOD when analyzing a statistically stationary process making it ideal for studying forced turbulent flows.

Methods used to model coherent structures are also considered. Specifically, we consider resolvent analysis (also known as input/output analysis), where one seeks forcing modes that give rise to the most amplified response modes with respect to their energetic gain. When applied to turbulent fluid flows, the nonlinear modal interactions are regarded as forcing terms to the linearized time-averaged turbulent mean [McKeon and Sharma, 2010]. Resolvent analysis has been used to study a wide range of transitional and turbulent flows (including jets) [Cossu et al., 2009, McKeon and Sharma, 2010, Meliga et al., 2012, Sharma and McKeon, 2013, Oberleithner et al., 2014, Jeun et al., 2016, Schmidt et al., 2018b], amongst others. Towne et al. [2018c] provided a theoretical connection between SPOD and resolvent, showing that resolvent output modes equal SPOD modes when the resolvent forcing modes are mutually uncorrelated. This provides a theoretical basis to use resolvent analysis to develop models of the space-time statistics of a turbulent flow [Moarref et al., 2013, Towne et al., 2020, Amaral et al., 2021] and the development of various methods [Morra et al., 2019, Pickering et al., 2021] to help whiten the forcing coefficients, thereby improving these models. Resolvent analysis was extended to flows with a time-periodic mean flow in Padovan et al. [2020] and Padovan and Rowley [2022] and is termed *harmonic resolvent analysis*. This leads to a system of frequency-couple equations that provide the ability to study the first-order triadic interactions present in these time-periodic flows. Analogous to the relationship between SPOD and resolvent analysis, in the report, we establish a theoretical connection between CS-SPOD and harmonic resolvent analysis.

The analysis of unforced but non-axisymmetric turbulent jets using SPOD is computationally costly; in particular, their non-axisymmetry precludes Fourier decomposition of the three-dimensional flow field into two-dimensional azimuthal modes. Jets in the dihedral group D_2 , including rectangular, elliptic, and twin jets, are invariant under reflection about their major and minor axes. For these jets, we propose an SPOD workflow that exploits their D_2 geometrical symmetry to obtain a reduction in computational effort, accelerate statistical convergence, and improve the interpretability of results. We decompose the three-dimensional snapshots into four symmetry components. For each symmetry component, we independently perform SPOD on one quadrant of the domain. We demonstrate an application of this D_2 -symmetric SPOD workflow on a large-eddy simulation of a twin-rectangular supersonic jet. Our analysis indicates that the twin jet exhibits screech at Strouhal number 0.3, and that this screech is dominated by components which are anti-symmetric along the minor axis. These observations are consistent with the results from companion experiments at Ohio State University. Furthermore, the same workflow is used to analyze the symmetry-dependence of far-field acoustics at low and high frequencies.

1.2 Objectives

Based on the discussion above, the objectives of this work are broadly to develop the theory, tools, and methodologies required to investigate non-axisymmetric turbulent jets and forced jets. The specific objectives are:

1. Develop and validate data-driven analysis tool for coherent turbulence structures in forced turbulent jets an

2. Investigate how idealized acoustic forcing alters turbulence structure in axisymmetric subsonic and supersonic jets
3. Extend the analysis to more realistic, military-style nozzles by performing GPU-accelerated LES of natural and plasma-actuated jets issuing from a twin rectangular nozzle.
4. Develop techniques that exploit the D_2 -symmetry group of the twin rectangular nozzle to characterize the coherent structures in this complex flow.

2 Data-driven analysis of forced turbulent flows

Periodic and quasi-periodic forced turbulent flows are ubiquitous in engineering and nature. Such flows include those in turbomachinery, weather and climate, and flow control with harmonic actuation (such as forced turbulent jets). In cases where the forcing is slow compared to the turbulence time scales, the statistics may be modeled as quasi-stationary (comprising a series of stationary states). However, in many cases, the forcing is at frequencies commensurate with the turbulence, and the turbulence structure is not only modulated by, but also altered by, the forcing. In such, a key goal is to identify coherent structures that can be compared and contrasted to their occurrence in similar but unforced flows but that are otherwise mutually uncorrelated.

The most commonly used technique to identify coherent structures in turbulence is proper orthogonal decomposition [Lumley, 1967a, 1970, Aubry et al., 1988, Sirovich, 1989, Aubry, 1991], which represents flow data as mutually orthogonal modes whose amplitudes optimally reconstruct the correlation tensor. When applied in its typical space-only form, the modes are not coherent in time, leading many researchers to apply DMD and its variants [Rowley et al., 2009, Schmid, 2010, Schmid et al., 2011]. However, for statistically stationary flows, spectral POD (SPOD) [Lumley, 1967a, 1970, Citriniti and George, 2000, Picard and Delville, 2000, Towne et al., 2018c] leads to an optimal reconstruction of the space-time statistics and results in modes that oscillate at a single frequency. A fundamental assumption required in both space-only POD and SPOD is statistical stationarity, meaning that the statistics are time-invariant. This assumption is appropriate for many unforced flows. However, when forced, this fundamental assumption is no longer valid as the flow, and its statistics, are now correlated to the forcing. Several works have developed extensions to SPOD to study forced turbulent flows. Franceschini et al. [2022] studied flows where a high-frequency turbulent component develops on a low-frequency periodic motion. Subsequently, a quasi-steady assumption is made, and conditionally fixed coherent structures at each phase are determined. Glezer et al. [1989] developed an extended POD method for flows with periodic statistics by summing an ensemble of time series. However, since this method is based on POD, it still contains the shortcomings present in POD. Heidt et al. [2021] applied SPOD to the residual component of the triply decomposed fields [Hussain and Reynolds, 1970, 1972] to isolate the impact of the forcing on the turbulence but still required a stationary assumption. Clearly, SPOD and the aforementioned extensions are not sufficient to study forced turbulent flows. This motivates an extension of SPOD to these flows, which is the primary focus of this report which we achieve by leveraging cyclostationary analysis.

Cyclostationary analysis is an extension to statistically stationary analysis to processes with periodic statistics that has been applied in a range of fields [Gardner, 2018], from economics to physics and mechanics. Initially developed by Gudzenko [1959], Lebedev [1959], and Gladyshev [1963], it was then extensively studied and popularized in Hurd [1969] and Gardner [1972]. The theory of second-order cyclostationary processes was further developed by Boyles and Gardner [1983] and Gardner [1986b], while Brown III [1987] and Gardner [1986c] furthered the theory of complex-valued processes. Cyclostationary analysis provides a robust statistical theory to study these processes, and tools analogous to those used to study stationary processes (e.g. the mean, cross-correlation,

cross-spectral density, etc) have been developed which naturally collapse back to their stationary counterparts when analyzing a stationary process.

Kim et al. [1996] developed cyclostationary empirical orthogonal-functions (CSEOFs) that essentially extends SPOD to cyclostationary processes for one-dimensional data. Kim and North [1997] modified this technique to include multi-dimensional data by reducing the computational cost through several approximations. However, due to a lack of clarity in the literature regarding the derivation, properties, interpretation, and computation of these techniques, their use has been limited. Furthermore, despite the aforementioned approximations, both formulations are computationally intractable for high-dimensional data. In this report, we extend SPOD to flows with time-periodic statistics through an extension to the exact form of CSEOFs [Kim et al., 1996] to include large multi-dimensional data. We hereafter refer to this method as cyclostationary SPOD (CS-SPOD for short).

We then relate CS-SPOD to harmonic resolvent analysis [Padovan et al., 2020, Padovan and Rowley, 2021, 2022] which is an extension of resolvent analysis to flows with a periodic mean flow. Analogous to the relationship between SPOD and resolvent analysis [Towne et al., 2018c], in the present report, we establish a theoretical connection between CS-SPOD and harmonic resolvent analysis. This provides a theoretical basis to use harmonic resolvent analysis to develop models of the space-time statistics (including the turbulent and noise) of forced turbulent flow as has been done with similar natural flows [Moarref et al., 2013, Towne et al., 2020, Amaral et al., 2021].

This chapter is organized as follows. Section 2.1 introduces and outlines the theory of cyclostationary processes and reviews an algorithm to compute their statistics. In §2.2, CS-SPOD is derived, its properties explored, and an efficient computational algorithm is proposed. After validating the method in §2.3, we demonstrate its utility of CS-SPOD in §2.4. Finally, in §2.5, we explore the relationship between CS-SPOD and the harmonic resolvent analysis. Section 6 concludes the section and summarizes the main points.

2.1 Cyclostationary theory

This section provides an overview of the theory of cyclostationary analysis and the tools used to study them, with a focus on fluid dynamics. Comprehensive reviews can be found in Gardner et al. [2006], Antoni [2009], and Napolitano [2019].

A complex-valued scalar process $q(t)$ at time t is cyclostationary in the wide sense if its mean and autocorrelation function are periodic with period T_0 [Gardner, 1986b], giving

$$E\{q(t)\} = E\{q(t + T_0)\}, \quad (1a)$$

$$R(t, \tau) = R(t + T_0, \tau), \quad (1b)$$

where $E\{\cdot\}$ is the expectation operator, R is the autocorrelation function, and τ is a time-delay. Since the mean and autocorrelation are time-periodic, they can be expressed as a Fourier series

$$E\{q(t)\} = \sum_{k_\alpha = -\infty}^{\infty} \hat{q}_{k_\alpha \alpha_0} e^{i2\pi(k_\alpha \alpha_0)t}, \quad (2a)$$

$$R(t, \tau) \equiv E\{q(t + \tau/2)q^*(t - \tau/2)\} = \sum_{k_\alpha = -\infty}^{\infty} \hat{R}_{k_\alpha \alpha_0}(\tau) e^{i2\pi(k_\alpha \alpha_0)t},$$

where $k_\alpha \in \mathbb{Z}$ and the Fourier series coefficients are given by

$$\hat{q}_{k_\alpha \alpha_0} \equiv \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} E\{q(t)\} e^{-i2\pi(k_\alpha \alpha_0)t} dt, \quad (3a)$$

$$\hat{R}_{k_\alpha \alpha_0}(\tau) \equiv \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} R(t, \tau) e^{-i2\pi(k_\alpha \alpha_0)t} dt, \quad (3b)$$

where $\alpha_0 = 1/T_0$ is the fundamental cycle frequency. The Fourier coefficients $\hat{R}_{k_\alpha\alpha_0}(\tau)$ are known as the cyclic autocorrelation functions of $q(t)$ at cycle frequency $k_\alpha\alpha_0$. If a process contains non-zero $\hat{q}_{k_\alpha\alpha_0}$ and/or $\hat{R}_{k_\alpha\alpha_0}(\tau)$, it is said to exhibit first- and second-order cyclostationarity at cycle frequency $k_\alpha\alpha_0$, respectively. Wide-sense stationary processes are the special case for which $\hat{R}_{k_\alpha\alpha_0}(\tau) \neq 0$ for $k = 0$ only.

If the process $q(t)$ contains a deterministic periodic component at cycle frequency $k_\alpha\alpha_0$, it would exhibit both first-order and second-order (and any higher-order) cyclostationarity at cycle frequency $k_\alpha\alpha_0$. Thus, a deterministic component results in a pure first-order component and an impure (i.e. made up from components of a lower-order) second-order (or higher) component [Antoni et al., 2004]. Antoni et al. [2004] and Antoni [2009] showed that in physical systems, it is crucial to analyze the first- and second-order components separately, where the second-order component $q''(t)$ is defined as

$$q''(t) \equiv q(t) - E\{q(t)\}, \quad (4)$$

such that $q(t) = E\{q(t)\} + q''(t)$ and the mean $E\{q(t)\} = E\{q(t+T_0)\}$ is T_0 periodic. This approach makes physical sense considering that the first-order component is the deterministic tonal component that originates from the forcing, while the second-order component is a stochastic component that represents the underlying turbulence that is modified by the forcing. The sequential approach is analogous to the triple decomposition [Hussain and Reynolds, 1970, 1972] where the underlying flow is separated into the first-order (phase-averaged) and second-order (turbulent/residual) components.

In this manuscript, we assume that all processes analyzed using second-order analysis tools are zero-mean processes (or have had their first-order component removed). Thus, by stating that a process exhibits second-order cyclostationarity at cycle frequency $k_\alpha\alpha_0$, we mean that the process exhibits *pure* second-order cyclostationarity at $k_\alpha\alpha_0$.

2.1.1 Second-order cyclostationary analysis tools

In fluid dynamics, we are frequently interested in the correlation between two quantities. Thus, we will now consider the complex-valued process $\mathbf{q}(\mathbf{x}, t)$ at time t and independent variables (or spatial locations) $\mathbf{x}$ instead of the scalar process $q(t)$. Two processes are jointly cyclostationary if their cross-correlation function can be expressed as a Fourier series, such that

$$\mathbf{R}(\mathbf{x}, \mathbf{x}', t, \tau) \equiv E\{\mathbf{q}(\mathbf{x}, t + \tau/2)\mathbf{q}^*(\mathbf{x}', t - \tau/2)\} = \sum_{k_\alpha=-\infty}^{\infty} \hat{\mathbf{R}}_{k_\alpha\alpha_0}(\mathbf{x}, \mathbf{x}', \tau) e^{i2\pi(k_\alpha\alpha_0)t}, \quad (5)$$

where the Fourier series coefficients are given by

$$\hat{\mathbf{R}}_{k_\alpha\alpha_0}(\mathbf{x}, \mathbf{x}', \tau) \equiv \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \mathbf{R}(\mathbf{x}, \mathbf{x}', t, \tau) e^{-i2\pi(k_\alpha\alpha_0)t} dt, \quad (6)$$

and are known as the cyclic cross-correlation functions of between $\mathbf{q}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x}')$ at cycle frequency $k_\alpha\alpha_0$ with $(\cdot)^*$ being the complex conjugate of $(\cdot)$. If the only non-zero cycle frequency is $k_\alpha\alpha_0 = 0$, then $\mathbf{q}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x}')$ are jointly wide-sense stationary. Similar to the common assumption in stationary analysis, we assume that all processes are separately and jointly cyclostationary.

A cyclostationary process can be analyzed in the dual-frequency domain via the cyclic cross-spectral density (CCSD). The CCSD is the generalization of the cross-spectral density (CSD) for cyclostationary processes and is related to the cyclic cross-correlation function via the cyclic Wiener-Khinchin relation [Gardner and Robinson, 1989]

$$\mathbf{S}_{k_\alpha\alpha_0}(\mathbf{x}, \mathbf{x}', f) = \int_{-\infty}^{\infty} \hat{\mathbf{R}}_{k_\alpha\alpha_0}(\mathbf{x}, \mathbf{x}', \tau) e^{-i2\pi f\tau} d\tau. \quad (7)$$

The CCSD can also be written as

$$\begin{aligned} \mathbf{S}_{k_\alpha \alpha_0}(\mathbf{x}, \mathbf{x}', f) \equiv \\ \lim_{\Delta f \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \Delta f E \left\{ \hat{\mathbf{q}}_{1/\Delta f}(\mathbf{x}, t, f + \frac{1}{2}k_\alpha \alpha_0) \hat{\mathbf{q}}_{1/\Delta f}^*(\mathbf{x}', t, f - \frac{1}{2}k_\alpha \alpha_0) \right\} dt, \end{aligned} \quad (8)$$

where $\hat{\mathbf{q}}_W(\mathbf{x}, t, f) \equiv \int_{t-\frac{W}{2}}^{t+\frac{W}{2}} \mathbf{q}(\mathbf{x}, t') e^{-i2\pi f t'} dt'$ is the short-time Fourier transform of $\mathbf{q}(\mathbf{x}, t)$, f is the spectral frequency, and $k_\alpha \alpha_0$ is the cycle frequency. This shows that the CCSD represents the time-averaged statistical correlation (with zero lag) of two spectral components at frequencies $f + \frac{1}{2}k_\alpha \alpha_0$ and $f - \frac{1}{2}k_\alpha \alpha_0$ as the bandwidth approaches zero [Napolitano, 2019]. For $k_\alpha = 0$, the CCSD naturally reduces to the CSD, i.e. $\mathbf{S}_0(\mathbf{x}, \mathbf{x}', f)$. Correlation between spectral components in cyclostationary processes is critical in the derivation of CS-SPOD, and for stationary processes, the lack of correlation between spectral components is why SPOD can analyze each frequency independently.

The Wigner-Ville (WV) spectrum [Martin, 1982, Martin and Flandrin, 1985, Antoni, 2007] shows the spectral information of the process as a function of time (or phase) and, for a cyclostationary process, is given by

$$\mathbf{WV}(\mathbf{x}, t, f) = \sum_{k_\alpha = -\infty}^{\infty} \mathbf{S}_{k_\alpha \alpha_0}(\mathbf{x}, f) e^{i2\pi(k_\alpha \alpha_0)t}, \quad (9)$$

where $\mathbf{S}_{k_\alpha \alpha_0}(\mathbf{x}, f)$ is the cyclic power-spectral density (i.e. $\mathbf{S}_{k_\alpha \alpha_0}(\mathbf{x}, \mathbf{x}, f)$). While nonphysical, the WV spectrum may contain negative energy densities due to the negative interaction terms in the WV spectrum [Antoni, 2007, Flandrin, 1998]. However, Antoni [2007] showed this could be arbitrarily reduced with increasing sampling time. The CCSD and WV spectrum can be integrated with respect to frequency [Gardner, 1994, Randall et al., 2001], which results in the instantaneous variance and the cyclic distribution of the instantaneous variance, respectively

$$\mathbf{m}(\mathbf{x}, t) = E\{\mathbf{q}(\mathbf{x}, t) \mathbf{q}^*(\mathbf{x}, t)\} = \int_{-\infty}^{\infty} \mathbf{WV}(\mathbf{x}, t, f) df, \quad (10a)$$

$$\hat{\mathbf{m}}_{k_\alpha \alpha_0}(\mathbf{x}) = \int_{-\infty}^{\infty} \mathbf{S}_{k_\alpha \alpha_0}(\mathbf{x}, f) df, \quad (10b)$$

where $\mathbf{m}(\mathbf{x}, t)$ is the mean-variance of the process and $\hat{\mathbf{m}}_{k_\alpha \alpha_0}(\mathbf{x})$ quantifies the mean-variance contribution from each cycle frequency $k_\alpha \alpha_0$.

So far, we have assumed that the cycle frequencies are known, but this may not always be the case. To determine the cycle frequencies present in the system, all possible cycle frequencies α are explored by rewriting the CCSD as

$$\begin{aligned} \mathbf{S}(\mathbf{x}, \mathbf{x}', \alpha, f) \equiv \\ \lim_{\Delta f \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \Delta f E \left\{ \mathbf{q}_{1/\Delta f}(\mathbf{x}, t, f + \frac{\alpha}{2}) \mathbf{q}_{1/\Delta f}^*(\mathbf{x}', t, f - \frac{\alpha}{2}) \right\} dt. \end{aligned} \quad (11)$$

A process exhibits cyclostationarity at cycle frequency α when $\mathbf{S}(\mathbf{x}, \mathbf{x}', \alpha, f) \neq 0$. The range of possible cycle frequencies is $\alpha = [-0.5/\Delta t, 0.5/\Delta t]$, which must be searched over with a resolution $\Delta\alpha = 1/T$ [Gardner, 1986a] to ensure all cycle frequencies present are captured. For cyclostationary processes, because the cross-correlation function is periodic, the spectral correlation becomes discrete in α such that

$$\mathbf{S}(\mathbf{x}, \mathbf{x}', \alpha, f) = \sum_{k_\alpha = -\infty}^{\infty} \mathbf{S}_{k_\alpha \alpha_0}(\mathbf{x}, \mathbf{x}', f) \delta(\alpha - k_\alpha \alpha_0). \quad (12)$$

The cyclic distribution of the instantaneous variance is rewritten as

$$\hat{\mathbf{m}}(\mathbf{x}, \alpha) = \int_{-\infty}^{\infty} \mathbf{S}(\mathbf{x}, \alpha, f) df, \quad (13)$$

which similarly becomes discrete for a cyclostationary process.

We must clarify one point of terminology. Considering stationary processes are a subset of cyclostationary processes, all stationary processes are also cyclostationary. We use the most restrictive description, i.e. stationary processes are referred to as stationary and not cyclostationary. By stating that a process exhibits cyclostationarity, we imply that at least one cycle frequency $k_\alpha \alpha_0$, $k_\alpha \neq 0$ exists.

2.1.2 Cycloergodicity

In fluid dynamics, it is laborious to require multiple realizations of a single process, and we often invoke ergodicity in stationary processes to equate the ensemble average with a long-time average of a single realization. We can similarly leverage the concept of cycloergodicity as described in [Boyles and Gardner \[1983\]](#), allowing us to replace the expectation operator with a suitable time average, specifically, the cycle-averaging operator [[Braun, 1975](#)]

$$\tilde{\mathbf{q}}(\mathbf{x}, t) = E\{\mathbf{q}(\mathbf{x}, t)\} = \lim_{P \rightarrow \infty} \frac{1}{P} \sum_{p=0}^P \mathbf{q}(\mathbf{x}, t + pT_0), \quad (14)$$

where $\tilde{\mathbf{q}}(\mathbf{x}, t)$ is the mean. The cycle-averaging operator is used when the data is phase-locked to the forcing (i.e. sampled at an integer number of samples per cycle) and is identical to the phase-average used in the triple decomposition [[Hussain and Reynolds, 1970](#), [Reynolds and Hussain, 1972](#)]. As the cycle average operator is periodic, it can be expressed as a Poisson sum

$$\tilde{\mathbf{q}}(\mathbf{x}, t) = E\{\mathbf{q}(\mathbf{x}, t)\} = \sum_{k_\alpha=-\infty}^{\infty} e^{i2\pi(k_\alpha \alpha_0)t} \lim_{s \rightarrow \infty} \frac{1}{s} \int_{-s/2}^{s/2} \mathbf{q}(\mathbf{x}, t) e^{-i2\pi(k_\alpha \alpha_0)t} dt. \quad (15)$$

This definition is employed for non-phase-locked data or to filter out first-order components which are assumed to be statistical noise [[Franceschini et al., 2022](#), [Sonnenberger et al., 2000](#)] and is identical to the harmonic-averaging procedure used by [Mezić \[2013\]](#) and [Arbabi and Mezić \[2017\]](#) when restricted to a temporally periodic average.

2.1.3 Computing the CCSD

There are practical considerations and nuances to computing the CCSD from discrete data that we discuss in this section. Let the vector $\mathbf{q}_k \in \mathbb{R}^N$ represent a flow snapshot, i.e. the instantaneous state of the process $\mathbf{q}(\mathbf{x}, t)$ at time t_k on a set of points in a spatial domain Ω . The length of the vector N is equal to the number of spatial points multiplied by the number of state variables. We assume that this data is available for M equispaced snapshots, with $t_{k+1} = t_k + \Delta t$. In addition, we assume that this data is phase-locked, meaning that there are an integer number of time steps in the fundamental period, T_0 , and define $N_\theta = T_0/\Delta t$ ¹. Adopting similar notation to [Towne et al. \[2018c\]](#), we estimate the CCSD tensor $\mathbf{S}(\mathbf{x}, \mathbf{x}', \alpha, f)$, which represents the spectral correlation between $\mathbf{q}(\mathbf{x}, t)$ and $\mathbf{q}(\mathbf{x}', t)$ at cycle frequency α and spectral frequency f . For a cyclostationary process, $\mathbf{S}(\mathbf{x}, \mathbf{x}', \alpha, f)$ is non-zero for $\alpha = k_\alpha \alpha_0$ only, and therefore is written as $\mathbf{S}_{k_\alpha \alpha_0}(\mathbf{x}, \mathbf{x}', f)$ or

¹This restriction simplifies and reduces the computational expense of the calculations but can in principle be relaxed by using the Poisson sum time-average as in (15) and the non-computationally-efficient form of CS-SPOD shown in algorithm 2. Alternatively, non-phased-locked data can be temporally interpolated to be phase-locked.

equivalently $\mathbf{S}_{k_\alpha/T_0}(\mathbf{x}, \mathbf{x}', f)$. The space-time data can now be represented as the data matrix $\mathbf{Q}$ and time vector $\mathbf{T}$

$$\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_M] \in \mathbb{R}^{N \times M}, \quad \mathbf{T} = [t_1, t_2, \dots, t_M] \in \mathbb{R}^M. \quad (16, 17)$$

Although we have a formula for the CCSD as seen in (8 and 11), this does not result in a consistent estimator of the CCSD, as the variance of the *estimate* of the CCSD does not tend to zero as the amount of available data becomes large [Jenkins, 1968, Antoni, 2007, Napolitano, 2019]. Instead, this results in an estimate where the variance in the estimate is equal to the squared value of the estimate itself. A consistent estimate of the CCSD can be obtained by employing an appropriate averaging technique. The most common technique is the time-averaging Welch method [Welch, 1967] due to its high computational efficiency. The Welch method averages a number of CCSDs to obtain a consistent *estimate* of the CCSD. From (11), we see that to compute the CCSD the Welch procedure is performed on two frequency-shifted versions of the data, given by

$$\mathbf{Q}_{\pm\alpha/2} = \mathbf{Q}e^{-i2\pi(\pm\alpha/2)\mathbf{T}} = [\mathbf{q}_{1,\pm\alpha/2}, \mathbf{q}_{2,\pm\alpha/2}, \dots, \mathbf{q}_{M,\pm\alpha/2}], \quad (18)$$

where $\mathbf{q}_{k,\pm\alpha/2}$ are the $\pm\frac{1}{2}\alpha$ frequency-shifted data matrices corresponding to the k^{th} snapshot, i.e. $\mathbf{q}_{k,\pm\alpha/2} = \mathbf{q}_k e^{-i2\pi(\pm\alpha/2)t_k}$. Next, we split the two frequency-shifted data matrices into a number of, possibly overlapping, blocks. Each block is written as

$$\mathbf{Q}_{\pm\alpha/2}^{(n)} = [\mathbf{q}_{1,\pm\alpha/2}^{(n)}, \mathbf{q}_{2,\pm\alpha/2}^{(n)}, \dots, \mathbf{q}_{N_f,\pm\alpha/2}^{(n)}] \in \mathbb{C}^{N \times N_f}, \quad (19)$$

where N_f is the number of snapshots in each block and the k^{th} entry of the n^{th} block is $\mathbf{q}_{k,\pm\alpha/2}^{(n)} = \mathbf{q}_{k+(n-1)(N_f-N_0),\pm\alpha/2}$. The total number of blocks, N_b , is given by $N_b = \lfloor \frac{N-N_0}{N_f-N_0} \rfloor$, where $\lfloor \cdot \rfloor$ represents the floor operator and N_0 is the number of snapshots that each block overlaps. The cycloergodicity hypothesis states that each of these blocks is considered to be a single realization in an ensemble of realizations of this cyclostationary flow. Subsequently, the DFT of each block for both frequency-shifted matrices is computed using a window w , giving

$$\hat{\mathbf{Q}}_{\pm\alpha/2}^{(n)} = [\hat{\mathbf{q}}_{1,\pm\alpha/2}^{(n)}, \hat{\mathbf{q}}_{2,\pm\alpha/2}^{(n)}, \dots, \hat{\mathbf{q}}_{N_f,\pm\alpha/2}^{(n)}], \quad (20)$$

where

$$\hat{\mathbf{q}}_{k,\pm\alpha/2}^{(n)} = \frac{1}{\sqrt{N_f}} \sum_{j=1}^{N_f} w_j \mathbf{q}_{j,\pm\alpha/2}^{(n)} e^{-i2\pi(k-1)[(j-1)/N_f]}, \quad (21)$$

for $k = 1, \dots, N_f$ and $n = 1, \dots, N_b$ where $\hat{\mathbf{q}}_{k,\pm\alpha/2}^{(n)}$ is the k^{th} Fourier component of the n^{th} block of the $\pm\alpha/2$ frequency-shifted data matrix, i.e. $\mathbf{f}_{k,\pm\alpha/2}$. The nodal values w_j of a window function are utilized to mitigate spectral and cyclic leakage arising from the non-periodicity of the data within each block. Due to the $\pm\alpha/2$ frequency-shifting applied, the k^{th} discrete frequencies of the $\pm\alpha/2$ frequency-shifted data matrices represent a frequency of

$$f_{k,\pm\alpha/2} = f_k \pm \alpha/2 = \begin{cases} \frac{k-1}{N_f \Delta t} & \text{for } k \leq N_f/2, \\ \frac{k-1-N_f}{N_f \Delta t} & \text{for } k > N_f/2. \end{cases} \pm \frac{\alpha}{2} \quad (22)$$

This shows that the frequency components $f_k + \alpha/2$ and $f_k - \alpha/2$, as required by (11), have the same index k in the shifted frequency vectors $\mathbf{f}_{k,\pm\alpha/2}$, respectively. The CCSD tensor $\mathbf{S}(\mathbf{x}, \mathbf{x}', \alpha, f)$ is then estimated at cycle frequency α and spectral frequency f_k by

$$\mathbf{S}_{f_k,\alpha} = \frac{\Delta t}{sN_b} \sum_{n=1}^{N_b} \hat{\mathbf{q}}_{k,\alpha/2}^{(n)} (\hat{\mathbf{q}}_{k,-\alpha/2}^{(n)})^*, \quad (23)$$

where $s = \sum_{j=1}^{N_f} w_j^2$ is the normalization constant that accounts for the difference in power between the windowed and non-windowed signal. This is written compactly by arranging the Fourier coefficients at the same index k into new frequency-data matrices

$$\hat{\mathbf{Q}}_{f_k, \pm\alpha/2} = \sqrt{\kappa} [\hat{\mathbf{q}}_{k, \pm\alpha/2}^{(1)}, \hat{\mathbf{q}}_{k, \pm\alpha/2}^{(2)}, \dots, \hat{\mathbf{q}}_{k, \pm\alpha/2}^{(N_b-1)}, \hat{\mathbf{q}}_{k, \pm\alpha/2}^{(N_b)}] \in \mathbb{C}^{N \times N_b}, \quad (24)$$

where $\kappa = \frac{\Delta t}{s N_b}$. $\mathbf{S}_{f_k, \alpha}$ is then estimated by

$$\mathbf{S}_{f_k, \alpha} = \hat{\mathbf{Q}}_{f_k, \alpha/2} (\hat{\mathbf{Q}}_{f_k, -\alpha/2})^*. \quad (25)$$

This estimate converges, i.e. the bias and variance become zero, as N_b and N_f are increased together [Welch, 1967, Bendat and Piersol, 2011, Antoni, 2007]. The algorithm to compute the CCSD from data snapshots is outlined in algorithm 1, from which all other second-order cyclostationary analysis tools can be computed. For efficient memory management, variables assigned with ‘ $\leftarrow$ ’ can be deleted after each iteration in their respective loop. Similar to the Welch estimate of the CSD, the estimate of the CCSD suffers from the standard bias-variance trade-off, and caution should be taken to ensure sufficiently converged statistics. In the CCSD, a phenomenon similar to spectral leakage is present and is called cyclic leakage [Gardner, 1986a] that results in erroneous cycle frequencies. Using 67% overlap when using a Hanning or Hamming window results in excellent cyclic leakage minimization and variance reduction [Antoni, 2007]. To reduce the variance sufficiently, $T\Delta f \gg 1$ is required [Antoni, 2009].

Algorithm 1 Algorithm to compute CCSD using frequency-shifted data matrices.

- 1: **for** Each data block, $n = 1, 2, \dots, N_b$ **do**
 - ▷ Compute the frequency-shifted block data matrices
 - 2: $\mathbf{Q}_{\pm\alpha/2}^{(n)} \leftarrow [\mathbf{q}_{1+(n-1)(N_f-N_0), \pm\alpha/2}, \mathbf{q}_{2+(n-1)(N_f-N_0), \pm\alpha/2}, \dots, \mathbf{q}_{N_f+(n-1)(N_f-N_0), \pm\alpha/2}]$
 - ▷ Using a (windowed) fast Fourier transform, calculate and store the row-wise DFT for each frequency-shifted block data matrix
 - 3: $\hat{\mathbf{Q}}_{\pm\alpha/2}^{(n)} = \text{FFT}(\mathbf{Q}_{\pm\alpha/2}^{(n)}) = [\hat{\mathbf{q}}_{1, \pm\alpha/2}^{(n)}, \hat{\mathbf{q}}_{2, \pm\alpha/2}^{(n)}, \dots, \hat{\mathbf{q}}_{N_f, \pm\alpha/2}^{(n)}]$
 - ▷ The column $\hat{\mathbf{q}}_{k, \pm\alpha/2}^{(n)}$ contains the n^{th} realization of the Fourier mode at the k^{th} discrete frequency $f_{k, \pm\alpha/2}$
 - 4: **end for**
 - 5: **for** Each frequency $k = 1, 2, \dots, N_f$ (or some subset of interest) **do**
 - ▷ Assemble the matrices of Fourier realizations from the k^{th} column of each $\hat{\mathbf{Q}}_{\pm\alpha/2}^{(n)}$
 - 6: $\hat{\mathbf{Q}}_{f_k, \pm\alpha/2} \leftarrow \sqrt{\kappa} [\hat{\mathbf{q}}_{k, \pm\alpha/2}^{(1)}, \hat{\mathbf{q}}_{k, \pm\alpha/2}^{(2)}, \dots, \hat{\mathbf{q}}_{k, \pm\alpha/2}^{(N_b-1)}, \hat{\mathbf{q}}_{k, \pm\alpha/2}^{(N_b)}]$
 - ▷ Compute the CCSD at spectral frequency f_k and cycle frequency α
 - 7: $\mathbf{S}_{f_k, \alpha} = \hat{\mathbf{Q}}_{f_k, \alpha/2} (\hat{\mathbf{Q}}_{f_k, -\alpha/2})^*.$
 - 8: **end for**
-

2.2 Cyclostationary spectral proper orthogonal decomposition

2.2.1 Derivation

The objective of CS-SPOD is to find deterministic functions that best approximate, on average, a zero-mean stochastic process. For clarity, we derive CS-SPOD using an approach and notation analogous to the SPOD derivation presented in Towne et al. [2018c] and refer the reader to Brereton and Kodal [1992], Towne et al. [2018c], and Schmidt and Colonius [2020b] for detailed discussions

on POD and SPOD. Like SPOD, we seek deterministic modes that depend on both space and time such that we can optimally decompose the space-time statistics of the flow. Thus, we assume that each realization of the stochastic process belongs to a Hilbert space with an inner product

$$\langle \mathbf{q}_1, \mathbf{q}_2 \rangle_{x,t} = \int_{-\infty}^{\infty} \int_{\Omega} \mathbf{q}_2^*(\mathbf{x}, t) \mathbf{W}(\mathbf{x}) \mathbf{q}_1(\mathbf{x}, t) d\mathbf{x} dt, \quad (26)$$

where $\mathbf{q}_1(\mathbf{x}, t)$, $\mathbf{q}_2(\mathbf{x}, t)$ are two realizations of the flow, $\mathbf{W}(\mathbf{x})$ is a positive-definite weighting tensor, and Ω denotes the spatial domain of interest. We then seek to maximize

$$\lambda = \frac{E\{|\langle \mathbf{q}(\mathbf{x}, t), \boldsymbol{\phi}(\mathbf{x}, t) \rangle_{x,t}|^2\}}{\langle \boldsymbol{\phi}(\mathbf{x}, t), \boldsymbol{\phi}(\mathbf{x}, t) \rangle_{x,t}}, \quad (27)$$

which leads to

$$\int_{-\infty}^{\infty} \int_{\Omega} \mathbf{R}(\mathbf{x}, \mathbf{x}', t, t') \mathbf{W}(\mathbf{x}') \boldsymbol{\phi}(\mathbf{x}', t') d\mathbf{x}' dt' = \lambda \boldsymbol{\phi}(\mathbf{x}, t), \quad (28)$$

where $\mathbf{R}(\mathbf{x}, \mathbf{x}', t, t') \equiv E\{\mathbf{q}(\mathbf{x}, t) \mathbf{q}^*(\mathbf{x}', t')\}$ is the two-point space-time correlation tensor. Until this stage, no assumptions about the flow has been made and is therefore identical to the derivation of SPOD [Lumley, 1967a, 1970, Towne et al., 2018c].

Since cyclostationary flows persist indefinitely, they have infinite energy in the space-time norm, as shown in (26). Consequently, the eigenmodes of (28) do not possess any of the useful quantities relied upon in POD or SPOD. To solve this, a new eigenvalue decomposition is obtained in the spectral domain from which modes with the desired properties are determined. We employ a solution ansatz of

$$\boldsymbol{\phi}(\mathbf{x}, t) = \sum_{m \in \mathcal{A}_m} \boldsymbol{\psi}(\mathbf{x}, \gamma + m\alpha_0) e^{i2\pi(\gamma + m\alpha_0)t}. \quad (29)$$

The set of frequencies present in the solution ansatz $\boldsymbol{\phi}(\mathbf{x}, t)$, is called the γ set of solution frequencies $\Omega_{\gamma} = \{\dots, \gamma - 2\alpha_0, \gamma - \alpha_0, \gamma, \gamma + \alpha_0, \gamma + 2\alpha_0, \dots\}$.

To derive the eigenvalue problem given by (42), we rewrite $\mathbf{R}(\mathbf{x}, \mathbf{x}', t, t') \rightarrow \mathbf{R}(\mathbf{x}, \mathbf{x}', t, \tau) \equiv E\{\mathbf{q}(\mathbf{x}, t + \tau/2) \mathbf{q}^*(\mathbf{x}', t - \tau/2)\}$, where $\tau = t - t'$. Recalling that for a cyclostationary process, the two-point space-time correlation density is a periodic function in time and can be expressed as a Fourier series

$$\mathbf{R}(\mathbf{x}, \mathbf{x}', t, \tau) = \sum_{n \in \mathcal{A}_n} \tilde{\mathbf{R}}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', \tau) e^{i2\pi n\alpha_0 t}, \quad (30)$$

where $\tilde{\mathbf{R}}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', \tau)$ are the cyclic autocorrelation functions of $\mathbf{R}(\mathbf{x}, \mathbf{x}', t, \tau)$ at cycle frequency $n\alpha_0$ and $\mathcal{A}_n = \{\dots, -1, 0, 1, \dots\}$ is, in general, the infinite set of harmonics of the fundamental cycle frequency present in the flow. One can also decompose the two-point space-time correlation density as the following phase-shifted Fourier series

$$\mathbf{R}(\mathbf{x}, \mathbf{x}', t, \tau) = \sum_{n \in \mathcal{A}_n} \hat{\mathbf{R}}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', \tau) e^{-i\pi n\alpha_0 \tau} e^{i2\pi n\alpha_0 t}, \quad (31)$$

where the two Fourier coefficients are related by

$$\tilde{\mathbf{R}}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', \tau) e^{i\pi n\alpha_0 \tau} = \hat{\mathbf{R}}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', \tau). \quad (32)$$

Although somewhat unusual, this simply applies a phase shift to the resulting Fourier series coefficients that, after Fourier transforming, shifts the center frequency of the CCSD. This is identical to the phase shift that relates the symmetric and asymmetric definitions of the cyclic cross-correlation functions and CCSD. Due to this, one can derive CS-SPOD using the symmetric definitions and a phase shift or using the asymmetric definition. We choose the former as it results in a simpler

derivation later. This phase shift is required to ensure the resulting eigensystem is Hermitian and positive semi-definite. Substituting the cyclic Wiener-Khinchin relation from (7) into (31) and then into the Fredholm eigenvalue problem (28) results in

$$\int_{-\infty}^{\infty} \int_{\Omega} \int_{-\infty}^{\infty} \sum_{n \in \mathcal{A}_n} \mathbf{S}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', f) e^{i2\pi n\alpha_0 t} e^{i2\pi(f - \frac{1}{2}n\alpha_0)\tau} \mathbf{W}(\mathbf{x}') \boldsymbol{\phi}(\mathbf{x}', t') df d\mathbf{x}' dt' = \lambda \boldsymbol{\phi}(\mathbf{x}, t). \quad (33)$$

Since $\tau = t - t'$, this leads to the following simplifications,

$$\int_{-\infty}^{\infty} \int_{\Omega} \sum_{n \in \mathcal{A}_n} \mathbf{S}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', f) e^{i2\pi n\alpha_0 t} e^{i2\pi(f - \frac{1}{2}n\alpha_0)t} \mathbf{W}(\mathbf{x}') \int_{-\infty}^{\infty} \left[\boldsymbol{\phi}(\mathbf{x}', t') e^{-i2\pi(f - \frac{1}{2}n\alpha_0)t'} dt' \right] df d\mathbf{x}' = \lambda \boldsymbol{\phi}(\mathbf{x}, t), \quad (34)$$

$$\int_{-\infty}^{\infty} \int_{\Omega} \sum_{n \in \mathcal{A}_n} \mathbf{S}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', f) e^{i2\pi(f + \frac{1}{2}n\alpha_0)t} \mathbf{W}(\mathbf{x}') \hat{\boldsymbol{\phi}}(\mathbf{x}', f - \frac{1}{2}n\alpha_0) df d\mathbf{x}' = \lambda \boldsymbol{\phi}(\mathbf{x}, t), \quad (35)$$

where $\hat{\boldsymbol{\phi}}(\mathbf{x}', f)$ is the temporal Fourier transform of $\boldsymbol{\phi}(\mathbf{x}', t')$. Similar to SPOD, we must choose a solution ansatz. In SPOD, we can solve a single frequency at a time as there is no correlation between different frequency components. However, since cyclostationary processes have spectral components that are correlated, we are unable to solve for each frequency component separately. Instead, we solve multiple coupled frequencies together by choosing our solution ansatz as

$$\boldsymbol{\phi}(\mathbf{x}, t) = \sum_{m \in \mathcal{A}_m} \boldsymbol{\psi}(\mathbf{x}, \gamma + m\alpha_0) e^{i2\pi(\gamma + m\alpha_0)t}, \quad (36)$$

giving,

$$\hat{\boldsymbol{\phi}}(\mathbf{x}, f) = \sum_{m \in \mathcal{A}_m} \boldsymbol{\psi}(\mathbf{x}, \gamma + m\alpha_0) \delta(f - (\gamma + m\alpha_0)), \quad (37)$$

where $\mathcal{A}_m = \{\dots, -1, 0, 1, \dots\}$ gives, in general, the infinite set of frequencies present in the solution (all separated by α_0). The frequency-shifted version of $\hat{\boldsymbol{\phi}}(\mathbf{x}, f)$ is given by

$$\hat{\boldsymbol{\phi}}(\mathbf{x}, f - \frac{1}{2}n\alpha_0) = \sum_{m \in \mathcal{A}_m} \boldsymbol{\psi}(\mathbf{x}, \gamma + m\alpha_0) \delta(f - (\gamma + (m + \frac{1}{2}n)\alpha_0)). \quad (38)$$

Substituting these expressions into (35) and integrating with respect to f results in

$$\int_{\Omega} \sum_{n \in \mathcal{A}_n} \sum_{m' \in \mathcal{A}_m} \mathbf{S}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma + (m' + \frac{1}{2}n)\alpha_0) e^{i2\pi(\gamma + (n+m')\alpha_0)t} \mathbf{W}(\mathbf{x}') \boldsymbol{\psi}(\mathbf{x}, \gamma + m'\alpha_0) d\mathbf{x}' = \lambda \sum_{m \in \mathcal{A}_m} \boldsymbol{\psi}(\mathbf{x}, \gamma + m\alpha_0) e^{i2\pi(\gamma + m\alpha_0)t}. \quad (39)$$

For this equation to hold over all time, we perform a harmonic balance where each frequency component must hold separately. This gives $\gamma + (m' + n)\alpha_0 = \gamma + m\alpha_0 \rightarrow m' + n = m$. An equation for each frequency component of our ansatz is formed as

$$\int_{\Omega} \sum_{n \in \mathcal{A}_n} \sum_{m' \in \mathcal{A}_m} \mathbf{S}_{n\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma + (m' + \frac{1}{2}n)\alpha_0) \mathbf{W}(\mathbf{x}') \boldsymbol{\psi}(\mathbf{x}, \gamma + m'\alpha_0) d\mathbf{x}' \delta_{n+m', m} = \lambda \boldsymbol{\psi}(\mathbf{x}, \gamma + m\alpha_0). \quad (40)$$

Substituting $n = m - m'$, this expression simplifies to

$$\int_{\Omega} \sum_{m' \in \mathcal{A}_m} \mathbf{S}_{(m-m')\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma + \frac{1}{2}(m+m')\alpha_0) \mathbf{W}(\mathbf{x}') \boldsymbol{\psi}(\mathbf{x}, \gamma + m'\alpha_0) d\mathbf{x}' = \lambda \boldsymbol{\psi}(\mathbf{x}, \gamma + m\alpha_0), \quad (41)$$

where we ignore $m - m' \notin \mathcal{A}_n$. Expanding (41) gives the final CS-SPOD eigenvalue problem, written compactly as

$$\int_{\Omega} S(\mathbf{x}, \mathbf{x}', \gamma) \mathbf{W}(\mathbf{x}') \boldsymbol{\Psi}(\mathbf{x}', \gamma) d\mathbf{x}' = \lambda \boldsymbol{\Psi}(\mathbf{x}, \gamma), \quad (42)$$

where

$$S(\mathbf{x}, \mathbf{x}', \gamma) = \quad (43a)$$

$$\begin{bmatrix} \ddots & & & \ddots & & \ddots & \\ & \ddots & & & & & \\ \ddots & \mathbf{S}_0(\mathbf{x}, \mathbf{x}', \gamma - \alpha_0) & \mathbf{S}_{-\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma - \frac{\alpha_0}{2}) & \mathbf{S}_{-2\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma) & & \ddots & \\ \ddots & \mathbf{S}_{\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma - \frac{\alpha_0}{2}) & \mathbf{S}_0(\mathbf{x}, \mathbf{x}', \gamma) & \mathbf{S}_{-\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma + \frac{\alpha_0}{2}) & \ddots & & \\ \ddots & \mathbf{S}_{2\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma) & \mathbf{S}_{\alpha_0}(\mathbf{x}, \mathbf{x}', \gamma + \frac{\alpha_0}{2}) & \mathbf{S}_0(\mathbf{x}, \mathbf{x}', \gamma + \alpha) & & \ddots & \\ \ddots & & & & \ddots & & \ddots \end{bmatrix},$$

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} \ddots & & & \\ & \mathbf{W}(\mathbf{x}) & & \\ & & \mathbf{W}(\mathbf{x}) & \\ & & & \mathbf{W}(\mathbf{x}) \\ & & & & \ddots \end{bmatrix}, \quad (43b) \quad \boldsymbol{\Psi}(\mathbf{x}, \gamma) = \begin{bmatrix} \vdots \\ \boldsymbol{\psi}(\mathbf{x}, \gamma - \alpha_0) \\ \boldsymbol{\psi}(\mathbf{x}, \gamma) \\ \boldsymbol{\psi}(\mathbf{x}, \gamma + \alpha_0) \\ \vdots \end{bmatrix}. \quad (43c)$$

$S(\mathbf{x}, \mathbf{x}', \gamma)$ is the CS-SPOD decomposition tensor, $\mathbf{W}(\mathbf{x})$ is the concatenated weight tensor, and $\boldsymbol{\Psi}(\mathbf{x}, \gamma)$ are the CS-SPOD eigenvectors. The CS-SPOD eigenvectors $\boldsymbol{\phi}(\mathbf{x}, t)$ have Fourier series coefficients, at each $f \in \Omega_{\gamma}$, of $\boldsymbol{\psi}(\mathbf{x}, f)$.

This coupling of frequencies in CS-SPOD occurs because frequency components separated by $n\alpha_0$ are correlated to each other, as shown in (8). In contrast, stationary processes do not exhibit correlation between frequencies, and thus each frequency can be solved independently via SPOD. Due to this coupling, CS-SPOD performed at γ and $\gamma + \alpha_0$ solve the same problem, i.e. giving $\Omega_{\gamma} = \Omega_{\gamma+z\alpha_0}$, where $z \in \mathbb{Z}$, meaning that CS-SPOD only contains unique solutions for the frequency sets corresponding to $\gamma \in \Gamma$, where $\Gamma = (-\alpha_0/2, \alpha_0/2]$.

In practice, the infinite-dimensional problem is not solved, and we restrict our solution frequencies by limiting $\mathcal{A}_m$ to a_1 harmonics, giving $\mathcal{A}_m = \{-a_1, -a_1 + 1, \dots, 0, \dots, a_1 - 1, a_1\}$ and $\Omega_{\gamma} = \{-a_1\alpha_0 + \gamma, (-a_1 + 1)\alpha_0 + \gamma, \dots, \gamma, \dots, (a_1 - 1)\alpha_0 + \gamma, a_1\alpha_0 + \gamma\}$. In addition, the flow may only exhibit cyclostationarity at a_2 harmonics of the fundamental cycle frequency giving $\mathcal{A}_n = \{-a_2, -a_2 + 1, \dots, 0, \dots, a_2 - 1, a_2\}$. We employ identical notation to restrict the harmonics used to compute various second-order tools, such as the Wigner-Ville spectrum. These limits result in $2a_1 + 1$ coupled equations, resulting in a $2a_1 + 1 \times 2a_1 + 1$ block eigensystem that is $2a_2 + 1$ banded-block-diagonal. In practice, a_1 should be chosen such that Ω_{γ} encompasses all frequencies of interest, a_2 should be chosen to encompass all the cycle frequencies present in the flow, and $a_2 < a_1$. An example for $a_1 = 2, a_2 = 1$ is (for compactness, we have dropped the explicit dependence on $\mathbf{x}$ in

this equation)

$$S(\gamma) = \begin{bmatrix} \mathbf{S}_0(\gamma - 2\alpha_0) & \mathbf{S}_{-\alpha_0}(\gamma - \frac{3}{2}\alpha_0) & 0 & 0 & 0 \\ \mathbf{S}_{\alpha_0}(\gamma - \frac{3}{2}\alpha_0) & \mathbf{S}_0(\gamma - \alpha_0) & \mathbf{S}_{-\alpha_0}(\gamma - \frac{1}{2}\alpha_0) & 0 & 0 \\ 0 & \mathbf{S}_{\alpha_0}(\gamma - \frac{1}{2}\alpha_0) & \mathbf{S}_0(\gamma) & \mathbf{S}_{-\alpha_0}(\gamma + \frac{1}{2}\alpha_0) & 0 \\ 0 & 0 & \mathbf{S}_{\alpha_0}(\gamma + \frac{1}{2}\alpha_0) & \mathbf{S}_0(\gamma + \frac{1}{2}\alpha_0) & \mathbf{S}_{-\alpha_0}(\gamma + \frac{3}{2}\alpha_0) \\ 0 & 0 & 0 & \mathbf{S}_{\alpha_0}(\gamma + \frac{3}{2}\alpha_0) & \mathbf{S}_0(\gamma + 2\alpha_0) \end{bmatrix}. \quad (44)$$

In the limiting case that $a_2 = 0$, we obtain a block-diagonal CS-SPOD decomposition matrix where each diagonal block is the standard SPOD eigenvalue problem.

2.2.2 CS-SPOD properties

Since $S(\mathbf{x}, \mathbf{x}', \gamma)$ is compact and finite, Hilbert–Schmidt theory guarantees a number of properties analogous to those for POD and SPOD [Lumley, 1967a, 1970, Towne et al., 2018c]. There are a countably infinite set of eigenfunctions $\Psi_j(\mathbf{x}, \gamma)$ at each unique frequency set Ω_γ that are orthogonal to all other modes at the same frequency set Ω_γ in the spatial inner norm $\langle \mathbf{q}_1, \mathbf{q}_2 \rangle_x = \int_\Omega \mathbf{q}_2^*(\mathbf{x}, t) \mathbf{W}(\mathbf{x}) \mathbf{q}_1(\mathbf{x}, t) d\mathbf{x}$, i.e. $\langle \Psi_j(\mathbf{x}, \gamma), \Psi_k(\mathbf{x}, \gamma) \rangle_x = \delta_{j,k}$. The following concatenated vector of each flow realization at the solution frequencies is optimally expanded as

$$\hat{\mathbf{Q}}(\mathbf{x}, \gamma) = \begin{bmatrix} \vdots \\ \hat{\mathbf{q}}(\mathbf{x}, \gamma - \alpha_0) \\ \hat{\mathbf{q}}(\mathbf{x}, \gamma) \\ \hat{\mathbf{q}}(\mathbf{x}, \gamma + \alpha_0) \\ \vdots \end{bmatrix}, \quad \hat{\mathbf{Q}}(\mathbf{x}, \gamma) = \sum_{j=1}^{\infty} a_j(\gamma) \Psi_j(\mathbf{x}, \gamma), \quad (45a, b)$$

where $\hat{\mathbf{q}}(\mathbf{x}, f)$ is the temporal Fourier decomposition of each flow realization $\mathbf{q}(\mathbf{x}, t)$ at frequency f and $a_j(\gamma) = \langle \hat{\mathbf{Q}}(\mathbf{x}, \gamma), \Psi_j(\mathbf{x}, \gamma) \rangle_x$ are the expansion coefficients, which are uncorrelated i.e.

$$E\{a_j(\gamma) a_k^*(\gamma)\} = \lambda_j(\gamma) \delta_{j,k}.$$

$S(\mathbf{x}, \mathbf{x}', \gamma)$ is positive semi-definite meaning that $S(\mathbf{x}, \mathbf{x}', \gamma)$ has the following unique diagonal representation

$$S(\mathbf{x}, \mathbf{x}', \gamma) = \sum_{j=1}^{\infty} \lambda_j(\gamma) \Psi_j(\mathbf{x}, \gamma) \Psi_j^*(\mathbf{x}', \gamma), \quad (46)$$

in which the CS-SPOD modes are its principal components. This shows that CS-SPOD determines the modes that optimally reconstruct the second-order statistics, one frequency set Ω_γ at a time.

CS-SPOD modes are optimal in terms of their total energy reconstruction of $S(\mathbf{x}, \mathbf{x}', \gamma)$ *only*. Thus, although each of the CCSDs present in $S(\mathbf{x}, \mathbf{x}', \gamma)$ have a diagonal representation, the individual components of $\Psi_j(\mathbf{x}, \gamma)$ are, in general, *not orthogonal* in the space norm, i.e.

$\langle \psi_j(\mathbf{x}, f), \psi_k(\mathbf{x}, f) \rangle_x \neq \delta_{j,k}$. One exception is for stationary processes where the correlation between different frequency components is zero, resulting in a block-diagonal matrix where $\Psi_j(\mathbf{x}, \gamma)$ contains just a single non-zero $\psi_j(\mathbf{x}, \gamma)$ component, with $\langle \psi_j(\mathbf{x}, \gamma), \psi_k(\mathbf{x}, \gamma) \rangle_x = \delta_{j,k}$.

Transforming the eigenvectors $\Psi_j(\mathbf{x}, \gamma)$ back into the time domain, noting the ansatz defined in (29), gives $\phi_{\gamma,j}(\mathbf{x}, t) = \sum_{m \in \mathcal{A}_m} \psi_j(\mathbf{x}, \gamma + m\alpha_0) e^{i2\pi(\gamma + m\alpha_0)t}$, which are orthogonal in the space-time inner product integrated over a complete period. Thus, every mode occurring at each frequency set Ω_γ can be viewed as a unique space-time mode.

The two-point space-time correlation tensor can be written as

$$\mathbf{R}(\mathbf{x}, \mathbf{x}', t, t') = \int_{-\alpha_0/2}^{\alpha_0/2} \sum_{j=1}^{\infty} \lambda_j(\gamma) \phi_{\gamma,j}(\mathbf{x}, t) \phi_{\gamma,j}^*(\mathbf{x}', t') d\gamma. \quad (47)$$

Substituting in the frequency expansion of $\phi_{\gamma,j}(\mathbf{x}, t)$ and applying $t' = t - \tau$ gives

$$\begin{aligned} \mathbf{R}(\mathbf{x}, \mathbf{x}', t, \tau) = & \int_{-\alpha_0/2}^{\alpha_0/2} \sum_{j=1}^{\infty} \lambda_j(\gamma) \sum_{m \in \mathcal{A}_m} \sum_{m' \in \mathcal{A}_m} \\ & \psi_j(\mathbf{x}, \gamma + m\alpha_0) \psi_j^*(\mathbf{x}', \gamma + m'\alpha_0) e^{i2\pi(m-m')\alpha_0 t} e^{i2\pi(\gamma+m'\alpha_0)\tau} d\gamma, \end{aligned} \quad (48)$$

resulting in a reconstruction that is time-periodic due to $e^{i2\pi(m-m')\alpha_0 t}$, which is why the ansatz defined by (29) was chosen.

In summary, for cyclostationary flows, CS-SPOD leads to modes that oscillate at a set of frequencies (Ω_γ) and optimally represent the second-order space-time flow statistics.

2.2.3 Computing CS-SPOD modes in practice

We now detail how to compute CS-SPOD modes from data along with a technique to reduce the cost and memory requirements to levels similar to those of SPOD. Since the dimension of the CCSD is $N \times N$, the overall eigensystem $\mathbf{S}_{\gamma_k}$ (which is the discrete approximation of $\mathcal{S}(\mathbf{x}, \mathbf{x}', \gamma)$) becomes $(2a_1 + 1)N \times (2a_1 + 1)N$ in size. For common fluid dynamics problems, this can become a dense matrix $O(10^6 - 10^9) \times O(10^6 - 10^9)$ in size, which is computationally intractable to store in memory, let alone compute its eigendecomposition. This is also the dimension of the inversion required in the CSEOF methods by Kim et al. [1996] and Kim and North [1997]. Thus, we derive a method-of-snapshots approach similar to the technique employed in POD [Sirovich, 1987b] and SPOD [Cittriniti and George, 2000, Towne et al., 2018c] that reduces the size of the eigenvalue problem from $(2a_1 + 1)N \times (2a_1 + 1)N$ to $(2a_1 + 1)N_b \times (2a_1 + 1)N_b$. Since $N_b \ll N$, the method-of-snapshots technique makes the eigenvalue problem computationally tractable.

To determine CS-SPOD with a finite amount of discrete data, we substitute in the Welch computational procedure for the CCSD into each term of the frequency-limited version of (43a). We numerically evaluate this as

$$\mathbf{S}_{\gamma_k} = \tilde{\mathbf{Q}}_{\gamma_k} \tilde{\mathbf{Q}}_{\gamma_k}^*, \quad \tilde{\mathbf{Q}}_{\gamma_k} = \begin{bmatrix} \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0} \\ \vdots \\ \hat{\mathbf{Q}}_{\gamma_k} \\ \vdots \\ \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0} \end{bmatrix}, \quad (49a, b)$$

where

$$\hat{\mathbf{Q}}_{\gamma_k, m\alpha_0} = \sqrt{\kappa} [\hat{\mathbf{q}}_{k, m\alpha_0}^{(1)}, \hat{\mathbf{q}}_{k, m\alpha_0}^{(2)}, \dots, \hat{\mathbf{q}}_{k, m\alpha_0}^{(N_b-1)}, \hat{\mathbf{q}}_{k, m\alpha_0}^{(N_b)}] \in \mathbb{C}^{N \times N_b}. \quad (50)$$

$\tilde{\mathbf{Q}}_{\gamma_k}$ is called the concatenated frequency-data matrix at the discrete Ω_{γ_k} set of solution frequencies and $\hat{\mathbf{q}}_{k, m\alpha_0}^{(n)}$ is the k^{th} DFT component of the n^{th} block of the $m\alpha_0$ frequency-shifted data matrix. As stated previously, the solution frequency sets are only unique for $\gamma \in \Gamma$, thus the corresponding DFT frequencies are

$$\gamma_k = \begin{cases} \frac{k-1}{N_f \Delta t} & \text{for } k \leq \lfloor \frac{\alpha_0 N_f \Delta t}{2} \rfloor + 1, \\ \frac{k-1-N_f}{N_f \Delta t} & \text{for } N_f - \lceil \frac{\alpha_0 N_f \Delta t}{2} \rceil + 1 < k \leq N_f, \end{cases} \quad (51)$$

which forms the elements $\gamma_k \in \Gamma_k$. Expanding (49) gives

$$\mathbf{S}_{\gamma_k} = \begin{bmatrix} \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0} \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0}^* & \cdots & \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0} \hat{\mathbf{Q}}_{\gamma_k}^* & \cdots & \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0} \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0}^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{\mathbf{Q}}_{\gamma_k} \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0}^* & \cdots & \hat{\mathbf{Q}}_{\gamma_k} \hat{\mathbf{Q}}_{\gamma_k}^* & \cdots & \hat{\mathbf{Q}}_{\gamma_k} \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0}^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0} \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0}^* & \cdots & \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0} \hat{\mathbf{Q}}_{\gamma_k}^* & \cdots & \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0} \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0}^* \end{bmatrix}. \quad (52)$$

This expression shows that $\mathbf{S}_{\gamma_k}$ contains off-diagonal terms that represent spectral correlations that are not present in the process (i.e. not present in $\mathcal{A}_n$). However, as N_b and N are increased together, this system converges and becomes a consistent estimate of (43a). Thus, all terms that represent spectral correlations not present in $\mathcal{A}_n$ converge to zero. Furthermore, the estimate is numerically positive semi-definite resulting in CS-SPOD modes that will inherit the desired properties. We note the restriction of cycle frequencies to $\mathcal{A}_n$ is not required for the numerical computation, and only a_1 is chosen.

Equation (49) shows that the final eigenvalue problem can be compactly written as

$$\mathbf{S}_{\gamma_k} \mathbf{W} \Psi_{\gamma_k} = \Lambda_{\gamma_k} \Psi_{\gamma_k}, \quad (53a)$$

$$\tilde{\mathbf{Q}}_{\gamma_k} \tilde{\mathbf{Q}}_{\gamma_k}^* \mathbf{W} \Psi_{\gamma_k} = \Lambda_{\gamma_k} \Psi_{\gamma_k}. \quad (53b)$$

The spatial inner weight

$$\langle \mathbf{q}_1, \mathbf{q}_2 \rangle_x = \int_{\Omega} \mathbf{q}_1^*(x, t) \mathbf{W}(x) \mathbf{q}_2(x, t) dx \quad (54)$$

is approximated as $\langle \mathbf{q}_1, \mathbf{q}_2 \rangle_x = \mathbf{q}_1^* \mathbf{W} \mathbf{q}_2$ where $\mathbf{W} \in \mathbb{C}^{N \times N}$ is a positive-definite Hermitian matrix that accounts for both the weight and the numerical quadrature of the integral on the discrete grid and $\mathbf{W} \in \mathbb{C}^{(2a_1+1)N \times (2a_1+1)N}$ is the block-diagonal matrix of $\mathbf{W}$ (similar to 43b). The CS-SPOD modes are then given by the columns of Ψ_{γ_k} and are ranked by their corresponding eigenvalues given by the diagonal matrix Λ_{γ_k} . These discrete CS-SPOD modes hold analogous properties to all those previously discussed, including that they are discretely orthogonal $\Psi_{\gamma_k}^* \mathbf{W} \Psi_{\gamma_k} = I$ and optimally decompose the estimated CS-SPOD decomposition matrix $\mathbf{S}_{\gamma_k} = \Psi_{\gamma_k} \Lambda_{\gamma_k} \Psi_{\gamma_k}^*$ (i.e. the second-order statistics).

At most, $\min(N, N_b)$ number of non-zero eigenvalues can be obtained. Thus, it is possible to show that the following $N_b \times N_b$ eigenvalue problem

$$\tilde{\mathbf{Q}}_{\gamma_k}^* \mathbf{W} \tilde{\mathbf{Q}}_{\gamma_k} \Theta_{\gamma_k} = \tilde{\Lambda}_{\gamma_k} \Theta_{\gamma_k}, \quad (55)$$

contains the same non-zero eigenvalues as (53). This approach is known as the method-of-snapshots [Sirovich, 1987b]. The corresponding eigenvectors are exactly recovered as

$$\tilde{\Psi}_{\gamma_k} = \tilde{\mathbf{Q}}_{\gamma_k} \Theta_{\gamma_k} \tilde{\Lambda}_{\gamma_k}^{-1/2}. \quad (56)$$

Other than the simple weighting matrix $\mathbf{W}$, only the concatenated data matrix $\tilde{\mathbf{Q}}_{\gamma_k}$ must be determined, which is easily achieved by computing each term $(\hat{\mathbf{Q}}_{\gamma_k, m\alpha_0})$ in $\tilde{\mathbf{Q}}_{\gamma_k}$ using algorithm 1. Once $\tilde{\mathbf{Q}}_{\gamma_k}$ is determined, one computes $\tilde{\mathbf{Q}}_{\gamma_k}^* \mathbf{W} \tilde{\mathbf{Q}}_{\gamma_k}$ and then performs the eigenvalue decomposition. Typically, only the first few modes are of physical interest, which allows us to employ a truncated decomposition where we determine a limited number of the most energetic CS-SPOD modes using randomized linear algebra methods [Martinsson and Tropp, 2020]. The total energy can be efficiently evaluated by taking the trace of $\tilde{\mathbf{Q}}_{\gamma_k}^* \mathbf{W} \tilde{\mathbf{Q}}_{\gamma_k}$.

Algorithm 2 implements the CS-SPOD in a practical, but computationally inefficient, manner. The algorithm requires computing $2a_1 + 1$ CCSDs, and those the cost is approximately $2a_1 + 1$ times that of the SPOD. The memory requirement scales similarly. This can be prohibitive when analyzing large data sets.

However, significant savings are realized since all the terms in $\tilde{\mathbf{Q}}_{\gamma_k}$ are in the form of $\hat{\mathbf{Q}}_{\gamma_k, m\alpha_0}$, which represent the k^{th} frequency component of the temporal Fourier transform of the $m\alpha_0$ frequency-shifted data matrix. The temporal Fourier transform of the n^{th} realization of the $m\alpha_0$ frequency-shifted data is given by

$$\hat{\mathbf{q}}_{k, m\alpha_0}^{(n)} = \frac{1}{\sqrt{N_f}} \sum_{j=1}^{N_f} w_j \mathbf{q}_{j, m\alpha_0}^{(n)} e^{-i2\pi(k-1)[\frac{j-1}{N_f}]}, \quad (57a)$$

$$= \frac{1}{\sqrt{N_f}} \sum_{j=1}^{N_f} w_j \mathbf{q}_j^{(n)} e^{-i2\pi(m\alpha_0 \Delta t)[(j-1)+(n-1)(N_f-N_0)]} e^{-i2\pi(k-1)[\frac{j-1}{N_f}]}, \quad (57b)$$

where $e^{-i2\pi(m\alpha_0 \Delta t)[(j-1)+(n-1)(N_f-N_0)]}$ is the frequency-shifting operation. We separate these components into a phase-shifting component and a zero-phase-shift frequency-shifting component, by

$$\hat{\mathbf{q}}_{k, m\alpha_0}^{(n)} = e^{-i2\pi(m\alpha_0 \Delta t)[(n-1)(N_f-N_0)]} \frac{1}{\sqrt{N_f}} \sum_{j=1}^{N_f} w_j \mathbf{q}_j^{(n)} e^{-i2\pi(m\alpha_0 \Delta t N_f + k-1)[\frac{j-1}{N_f}]}, \quad (58a)$$

$$\hat{\mathbf{q}}_{k, m\alpha_0}^{(n)} = e^{-i2\pi(m\alpha_0 \Delta t)[(n-1)(N_f-N_0)]} \hat{\mathbf{q}}_{\ell(k, m)}^{(n)}, \quad (58b)$$

where $\ell(k, m)$ is the ℓ^{th} frequency that is a function of k, m . This shows that the f_k discrete frequency of the $m\alpha_0$ -frequency-shifted data matrix ($f_{k, m\alpha_0}$) can be exactly computed as a phase-shifted version of the $f_{\ell(k, m)}$ discrete frequency component of the non-frequency-shifted data matrix. To employ this method, $m\alpha_0 \Delta t N_f \in \mathbb{Z}$. This ensures that the change in frequency due to the applied frequency-shifting operator is equal to an integer change in the index of the frequency vector. Since $\alpha_0 \Delta T = 1/N_\theta$, this gives $\frac{m N_f}{N_\theta} \in \mathbb{Z}$, which requires $N_f = N_{osc} N_\theta$, $N_{osc} \in \mathbb{Z}$. With this restriction, the frequency spectrum of the DFT of a N_f length record is

$$f_k = \begin{cases} \frac{(k-1)\alpha_0}{N_{osc}} & \text{for } k \leq \frac{N_{osc} N_\theta}{2}, \\ \frac{(k-1 - N_{osc} N_\theta)\alpha_0}{N_{osc}} & \text{for } k > \frac{N_{osc} N_\theta}{2}, \end{cases} \quad (59)$$

and the unique frequency sets become

$$\gamma_k = \begin{cases} \frac{(k-1)\alpha_0}{N_{osc}} & \text{for } k \leq \lfloor \frac{N_{osc}}{2} \rfloor + 1, \\ \frac{(k-1 - N_{osc} N_\theta)\alpha_0}{N_{osc}} & \text{for } N_f - \lceil \frac{N_{osc}}{2} \rceil + 1 < k \leq N_f. \end{cases} \quad (60)$$

This demonstrates that a frequency shift of $m\alpha_0$ corresponds to an integer change in the frequency index, i.e. the k^{th} frequency component of the $m\alpha_0$ -frequency-shifted data matrix corresponds to the phase-shifted version of the $\ell(k, m)^{th}$ frequency component ($f_{\ell(k, m)}$) of the non-frequency-shifted data matrix, i.e. $f_{k, m\alpha_0} = f_{\ell(k, m)}$, where

$$\ell(k, m) = \begin{cases} \begin{cases} k + m N_{osc} & \text{for } m \geq 0 \\ k + m N_{osc} + N_f & \text{for } m < 0 \end{cases} & \text{for } k \leq \lfloor \frac{N_{osc}}{2} \rfloor + 1, \\ \begin{cases} k + m N_{osc} - N_f & \text{for } m \geq 0 \\ k + m N_{osc} & \text{for } m < 0 \end{cases} & \text{for } N_f - \lceil \frac{N_{osc}}{2} \rceil + 1 < k \leq N_f. \end{cases} \quad (61)$$

Algorithm 2 Naive algorithm to compute CS-SPOD.

- 1: **for** Each data block, $n = 1, 2, \dots, N_b$ **do**
 - ▷ Construct the block data matrix
 - 2: $\mathbf{Q}^{(n)} = [\mathbf{q}_{1+(n-1)(N_f-N_0)}, \mathbf{q}_{2+(n-1)(N_f-N_0)}, \dots, \mathbf{q}_{N_f+(n-1)(N_f-N_0)}]$
 - ▷ Construct the block time matrix
 - 3: $\mathbf{T}^{(n)} = [t_{1+(n-1)(N_f-N_0)}, t_{2+(n-1)(N_f-N_0)}, \dots, t_{N_f+(n-1)(N_f-N_0)}]$
 - 4: **end for**
 - 5: **for** Each $m \in \mathcal{A}_m$, where $\mathcal{A}_m = \{-a_1, -a_1 + 1, \dots, 0, \dots, a_1 - 1, a_1\}$ **do**
 - 6: **for** Each data block, $n = 1, 2, \dots, N_b$ **do**
 - ▷ Compute the frequency-shifted block data matrices
 - 7: $\mathbf{Q}_{m\alpha_0}^{(n)} \leftarrow \mathbf{Q}^{(n)} e^{-i2\pi(m\alpha_0)\mathbf{T}^{(n)}}$
 - ▷ Using a (windowed) fast Fourier transform, calculate and store the row-wise DFT for each frequency-shifted block data matrix
 - 8: $\hat{\mathbf{Q}}_{m\alpha_0}^{(n)} = \text{FFT}(\mathbf{Q}_{m\alpha_0}^{(n)}) = [\hat{\mathbf{q}}_{1,m\alpha_0}^{(n)}, \hat{\mathbf{q}}_{2,m\alpha_0}^{(n)}, \dots, \hat{\mathbf{q}}_{N_w,m\alpha_0}^{(n)}]$

where, the column $\hat{\mathbf{q}}_{k,m\alpha_0}^{(n)}$ contains the n^{th} realization of the Fourier mode at the k^{th} discrete frequency of the $m\alpha_0$ frequency-shifted block-data matrix
 - 9: **end for**
 - 10: **end for**
 - 11: **for** Each $\gamma_k \in \Gamma_k$ (or some subset of interest) **do**
 - ▷ Assemble the concatenated frequency-data matrix for frequency set Ω_{γ_k}
 - 12:
$$\tilde{\mathbf{Q}}_{\gamma_k} \leftarrow \begin{bmatrix} \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0} \\ \vdots \\ \hat{\mathbf{Q}}_{\gamma_k, 0} \\ \vdots \\ \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0} \end{bmatrix},$$

where $\hat{\mathbf{Q}}_{\gamma_k, m\alpha_0} \leftarrow \sqrt{\kappa}[\hat{\mathbf{q}}_{k,m\alpha_0}^{(1)}, \hat{\mathbf{q}}_{k,m\alpha_0}^{(2)}, \dots, \hat{\mathbf{q}}_{k,m\alpha_0}^{(N_b-1)}, \hat{\mathbf{q}}_{k,m\alpha_0}^{(N_b)}]$ is the matrix of Fourier realizations corresponding to the k^{th} column of the $m\alpha_0$ frequency-shifted block-data matrix $\hat{\mathbf{Q}}_{m\alpha_0}^{(n)}$
 - 13: Compute the matrix $\mathbf{M}_{\gamma_k} \leftarrow \tilde{\mathbf{Q}}_{\gamma_k}^* \mathbf{W} \tilde{\mathbf{Q}}_{\gamma_k}$
 - 14: Compute the eigenvalue decomposition $\mathbf{M}_{\gamma_k} = \mathbf{\Theta}_{\gamma_k} \tilde{\mathbf{\Lambda}}_{\gamma_k} \mathbf{\Theta}_{\gamma_k}^*$
 - 15: Compute and save the CS-SPOD modes $\tilde{\mathbf{\Psi}}_{\gamma_k} = \tilde{\mathbf{Q}}_{\gamma_k} \mathbf{\Theta}_{\gamma_k} \tilde{\mathbf{\Lambda}}_{\gamma_k}^{-1/2}$ and energies $\tilde{\mathbf{\Lambda}}_{\gamma_k}$ for the γ_k frequency set Ω_{γ_k}
 - 16: **end for**
-

This means that all the data required for CS-SPOD (for all frequency sets Ω_{γ_k}) is contained within the Fourier transform of the original data matrix.

Algorithm 3 incorporates these savings and requires only a single DFT of the data matrix, making it similar in computational cost and memory requirement to SPOD. The memory usage to compute CS-SPOD for complex input data is $\approx (\frac{1}{1-N_0/N_f} + 1) \times \text{mem}(\mathbf{Q})$, which is the memory required to store the, possibly overlapping, block data matrix and the original data matrix. Additional memory is required to store the temporary matrix $\tilde{\mathbf{Q}}_{\gamma_k}$, although the size of this matrix is minimal as typically $2a_1 + 1 \ll N_f$. In extreme cases where only a single snapshot can be loaded at a time, a streaming CS-SPOD algorithm could be developed analogous to the streaming SPOD method by Schmidt and Towne [2019].

Algorithm 3 Efficient algorithm to compute CS-SPOD.

- 1: **for** Each data block, $n = 1, 2, \dots, N_b$ **do**
 - ▷ Construct the block data matrix
 - 2: $\mathbf{Q}^{(n)} \leftarrow [\mathbf{q}_{1+(n-1)(N_f-N_0)}, \mathbf{q}_{2+(n-1)(N_f-N_0)}, \dots, \mathbf{q}_{N_f+(n-1)(N_f-N_0)}]$
 - ▷ Using a (windowed) fast Fourier transform, calculate and store the row-wise DFT for each frequency-shifted block data matrix
 - 3: $\hat{\mathbf{Q}}^{(n)} = \text{FFT}(\mathbf{Q}^{(n)})$ ▷ Discard any frequency components that are not required to compute $\tilde{\mathbf{Q}}_{\gamma_k}$ (if one is not computing $\tilde{\mathbf{Q}}_{\gamma_k}$ over all $\gamma_k \in \Gamma_k$)
 - 4: **end for**
 - 5: **for** Each $\gamma_k \in \Gamma_k$ (or some subset of interest) **do**
 - ▷ Assemble the concatenated frequency-data matrix for frequency set Ω_{γ_k}
 - 6: $\tilde{\mathbf{Q}}_{\gamma_k} \leftarrow \begin{bmatrix} \hat{\mathbf{Q}}_{\gamma_k, -a_1\alpha_0} \\ \vdots \\ \hat{\mathbf{Q}}_{\gamma_k, 0} \\ \vdots \\ \hat{\mathbf{Q}}_{\gamma_k, a_1\alpha_0} \end{bmatrix}$

where $\hat{\mathbf{Q}}_{\gamma_k, m\alpha_0} \leftarrow \sqrt{\kappa}[\hat{\mathbf{q}}_{k, m\alpha_0}^{(1)}, \hat{\mathbf{q}}_{k, m\alpha_0}^{(2)}, \dots, \hat{\mathbf{q}}_{k, m\alpha_0}^{(N_b-1)}, \hat{\mathbf{q}}_{k, m\alpha_0}^{(N_b)}]$ is the matrix of Fourier realizations corresponding to the k^{th} column of the $m\alpha_0$ frequency-shifted block-data matrix $\hat{\mathbf{Q}}_{m\alpha_0}^{(n)}$, evaluated efficiently by $\hat{\mathbf{q}}_{k, m\alpha_0}^{(n)} = e^{-i2\pi(m\alpha_0\Delta t)[(n-1)(N_f-N_0)]} \hat{\mathbf{q}}_{\ell(k, m)}^{(n)}$, where the index $\ell(k, m)$ is given by (61)
 - 7: Compute the matrix $\mathbf{M}_{\gamma_k} \leftarrow \tilde{\mathbf{Q}}_{\gamma_k}^* \mathbf{W} \tilde{\mathbf{Q}}_{\gamma_k}$
 - 8: Compute the eigenvalue decomposition $\mathbf{M}_{\gamma_k} = \Theta_{\gamma_k} \tilde{\mathbf{\Lambda}}_{\gamma_k} \Theta_{\gamma_k}^*$
 - 9: Compute and save the CS-SPOD modes $\tilde{\Psi}_{\gamma_k} = \tilde{\mathbf{Q}}_{\gamma_k} \Theta_{\gamma_k} \tilde{\mathbf{\Lambda}}_{\gamma_k}^{-1/2}$ and energies $\tilde{\mathbf{\Lambda}}_{\gamma_k}$ for the γ_k frequency set Ω_{γ_k}
 - 10: **end for**
-

2.3 Validation of our CCSD and CS-SPOD algorithms

We validate our implementation of the CCSD and CS-SPOD using a model problem that has an analytical solution. Let $n(x, t)$ be a zero-mean, complex-valued, stationary random process with uniformly distributed phase (between 0 and 2π), normally distributed unit variance, and a covariance

kernel $c(x, x') = E\{n(x, t)n^*(x', t)\}$, of

$$c(x, x') = \frac{1}{\sqrt{2\pi}\sigma_\eta} \exp \left[-\frac{1}{2} \left(\frac{x - x'}{\sigma_\eta} \right)^2 \right] \exp \left[-i2\pi \frac{x - x'}{\lambda_\eta} \right], \quad (62)$$

where $\sigma_\eta = 4$ is the standard deviation of the envelope, $\lambda_\eta = 20$ is the wavelength of the filter, and $x_0 = 1.5$ is the center off-set distance. This covariance kernel is identical to the one used by Towne et al. [2018c] as its structure is qualitatively similar to statistics present in real flows (e.g. a turbulent jet). The filtered process $\tilde{n}(x, t)$ is defined as the convolution between a filter $f_\ell(x, t)$ and $n(x, t)$, given by

$$\tilde{n}(x, t) = f_\ell(x, t) \otimes n(x, t). \quad (63)$$

We sinusoidally modulate $\tilde{n}(x, t)$ to create a cyclostationary process

$$g(x, t) = \tilde{n}(x, t) \cos(2\pi f_0 t + \phi_0), \quad (64)$$

where $f_0 = 0.5$ is the modulation frequency and $\phi_0 = \frac{1}{3}2\pi$ is a phase offset. Using the theory developed in §2.1, the CCSD of $g(x, t)$ is analytically determined as

$$S_g(x, x', \alpha, f) = \begin{cases} \frac{1}{4} e^{\pm i2\theta} S_{\tilde{n}}(x, x', 0, f) & \text{for } \alpha = \pm 2f_0 \\ \frac{1}{4} S_{\tilde{n}}(x, x', 0, f + f_0) + \frac{1}{4} S_{\tilde{n}}(x, x', 0, f - f_0) & \text{for } \alpha = 0 \\ 0 & \text{otherwise} \end{cases} \quad (65)$$

where $S_{\tilde{n}}(x, x', 0, f)$ is the CCSD of $\tilde{n}(x, t)$ at cycle frequency $\alpha = 0$ (thus equaling the CSD). The fundamental and only non-zero cycle frequency present is $\alpha_0 = \pm 2f_0$, indicating that this process exhibits cyclostationarity. The CSD of $\tilde{n}(x, t)$ is given by

$$S_{\tilde{n}}(x, x', 0, f) = c(x, x') F_\ell(x, f) F_\ell^*(x', f), \quad (66)$$

where $F_\ell(x, f)$ is the temporal Fourier transform of the filter $f_\ell(x, t)$. The filter employed is a 5th-order finite-impulse-response filter with a cutoff frequency f_{co} , that varies as a function of the spatial location $f_{co} = 0.2|x - x_0|/\max(x) + 0.2$. This results in a filter exhibiting a more rapid spectral decay at x_0 and a flatter spectrum moving away from this location. A domain $x \in [-10, 10]$ is employed and is discretized using 2001 equispaced grid points resulting in a grid spacing of $\Delta x = 0.01$. All estimates of the CCSD and CS-SPOD are performed using a Hamming window with $L_w = 10N_\theta$ and an overlap of 67%. Snapshots are saved in time with $\Delta t = 0.04$, resulting in $N_\theta = 25$ time steps per period of the fundamental cycle frequency, $T_0 = 1/\alpha_0 = 1/(2f_0)$. Data is saved for $t_{end} = 2000T_0$, resulting in 50000 snapshots and 593 blocks (realizations) of the process.

Sample paths of the process at $x = 0$, as a function of the phase of the fundamental cycle frequency, are shown in figure 1. As theoretically predicted, we observe a modulation in the amplitude of the process as a function of the phase. Since $\alpha_0 = 2f_0$, the phase offset $\phi_0 = \frac{1}{3}2\pi$ applied to the sinusoidal modulation results in a phase offset of $\frac{1}{6}2\pi$ in the sample paths. This modulation is observed in figure 2, where we plot the analytical WV spectrum computed using (9 and 65) at $x = x' = 0$. This shows the sinusoidal modulation of the PSD as a function of the phase and a decay in the amplitude of the spectrum with increasing $|f|$ due to the applied filter. In figure 3, we compare the magnitude of the analytical and numerical CCSD at $f = 0.1$ and $\alpha = 0, \pm 2f_0$. Here, we observe the aforementioned key structures of the covariance kernel along with the excellent agreement between the numerical and analytical CCSDs, which would further improve with an increasing number of realizations, thereby validating our CCSD implementation (algorithm 1).

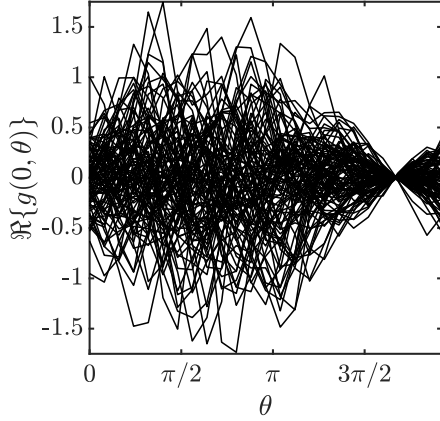


Figure 1: Model process sample paths at $x = 0$.

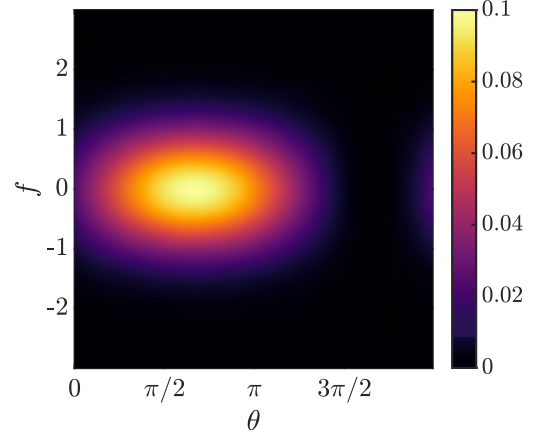


Figure 2: Analytical WV spectrum at $x = x' = 0$ for the model process.

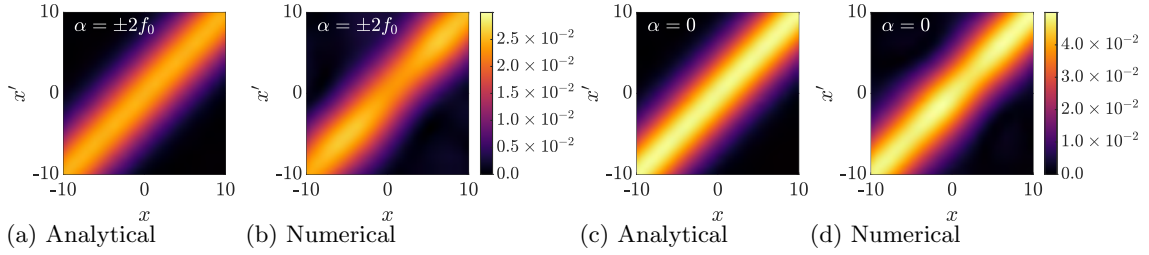


Figure 3: Plot of the magnitude of the CCSD for $f = 0.1$ of the dummy process for the analytical and numerically generated results.

Next, we validate our efficient algorithm to compute CS-SPOD (algorithm 3) and determine its convergence with increasing data by comparing the numerical results to the analytical results. The analytical solution is determined by forming the CS-SPOD eigensystem defined via (43a) through evaluating the analytical CCSDs (given by (65)) and then numerically evaluating the final eigenvalue problem. To encompass the range of relevant frequencies we use $a_1 = 10$ to construct $\mathcal{A}_m$, resulting in $\Omega_\gamma = [-10, 10] + \gamma$. Figure 4 shows a comparison of the analytical and numerical CS-SPOD eigenspectrums (averaged over 10000 realizations of the process), at $\gamma = 0.2$ for $t_{end} = 100T_0, 400T_0$, and $2000T_0$, which corresponds to 27, 117, and 593 blocks, respectively. As the duration of the process increases, we observe an increasingly converged estimate of the eigenspectrum. This is reflected in the percentage error between the averaged numerical eigenvalues and the analytical eigenvalues for the three most dominant CS-SPOD modes, which we show in figure 5. We see that these eigenvalues linearly converge to the true value as the duration of the process increases, which is theoretically expected due to the linear reduction in the variance of the Welch estimate of the CCSD with increasing realizations [Antoni, 2007]. Overall, we obtain a consistent estimate of the CS-SPOD eigenvalues and conclude that our implementation of CS-SPOD is correct.

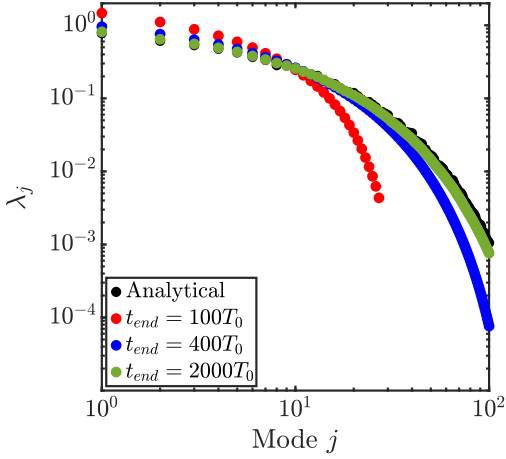


Figure 4: Plot of the analytical and numerical CS-SPOD eigenspectrum at $\gamma = 0.2$ for the dummy problem at multiple signal durations.

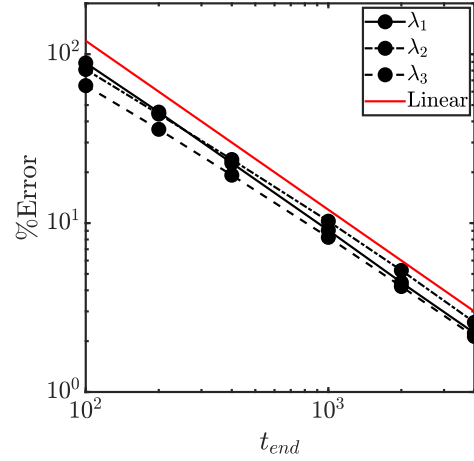


Figure 5: Convergence of CS-SPOD eigenvalues as a function of the total signal duration of the dummy problem.

2.4 Example problems

2.4.1 Application to a modified linearized complex Ginzburg-Landau equation

Our first example is the simple and well-understood linearized complex Ginzburg-Landau equation, which has been used as a model for a convectively unstable flow that exhibits non-modal growth [Chomaz et al., 1988, Cossu and Chomaz, 1997, Hunt and Crighton, 1991]. It can be written in the form of a generic linear forced system

$$\frac{\partial q(x, t)}{\partial t} - L(x, t)q(x, t) = f(x, t), \quad (67)$$

where $q(x, t)$ and $f(x, t)$ represent the state and forcing, respectively, with $|q(x \rightarrow \pm\infty, t)| \rightarrow 0$, and $L(x, t)$ is the linear operator

$$L(x, t) = -\nu_1 \frac{\partial}{\partial x} + \nu_2 \frac{\partial^2}{\partial x^2} - \mu(x, t). \quad (68)$$

We use the commonly used form $\mu(x) = \mu_0 - c_\mu^2 + \mu_2 \frac{x^2}{2}$ [Hunt and Crighton, 1991, Bagheri et al., 2009, Chen and Rowley, 2011, Towne et al., 2018c], resulting in time-invariant dynamics. All constants in (67, 68), except for μ_0 , use the values in Bagheri et al. [2009]. Similar to Franceschini et al. [2022], we construct periodic dynamics by using $\mu_0 = \bar{\mu}_0 + A_{\mu_0} \sin(2\pi f_0 t)$, where $\bar{\mu}_0$ is the average value of μ_0 , A_{μ_0} is the amplitude of the periodic modulation of μ_0 , and f_0 is the frequency of the periodic modulation. For $A_{\mu_0} = 0$ the system has time-invariant dynamics, while for $|A_{\mu_0}| > 0$ the system has time-periodic dynamics, resulting in a stationary and cyclostationary response, respectively. By varying A_{μ_0} , we modify the degree to which the system is cyclostationary. We choose $f_0 = 0.1$, which is substantial compared to the frequencies of interest ($\approx [-0.5, 0.5]$), meaning that the quasi-steady approach of Franceschini et al. [2022] can not be employed. Like Towne et al. [2018c], we use $\bar{\mu}_0 = 0.23$, which for $A_{\mu_0} = 0$ strongly amplifies external noise due to the non-normality of $L(x, t)$ and results in a degree of low-rankness typically present in turbulent flows. As per Franceschini et al. [2022], we confirm the stability of the system using Floquet analysis (results not shown). To demonstrate the utility of CS-SPOD and to facilitate its interpretation, we compare CS-SPOD performed at several levels of cyclostationarity $A_{\mu_0} = 0.0, 0.2$, and 0.4 .

A pseudo-spectral approach utilizing Hermite polynomials is employed to discretize the equations [Bagheri et al., 2009, Chen and Rowley, 2011], where the collocation points $[x_1, x_2, \dots, x_{N_H}]$ correspond to the first N_H Hermite polynomials with scaling factor $\Re\{(-\mu_2/(2\nu_2))^{\frac{1}{4}}\}$. Following Bagheri et al. [2009] and Towne et al. [2018c], we use $N_H = 221$, leading to a computational domain $x \in [-85.19, 85.19]$, which is large enough to mimic an infinite domain. The boundary conditions are implicitly satisfied through the use of Hermite polynomials [Bagheri et al., 2009]. For CS-SPOD, the value of the weighting matrix at x_i is determined as the distance between the midpoints of the neighbouring grid points. Temporal integration is performed using the embedded 5th order Dormand–Prince Runge-Kutta method [Dormand and Prince, 1980, Shampine and Reichelt, 1997]. After the initial transients have decayed, a total of 40000 solution snapshots are saved with $\Delta t = 0.5$, giving a Nyquist frequency of $f_{\text{Nyquist}} = 1$.

To mimic a turbulent system, similar to Towne et al. [2018c], we force our system using spatially correlated band-limited noise. This is performed by constructing spatially correlated noise with the following covariance kernel

$$g(x, x') = \frac{1}{\sqrt{2\pi}\sigma_\eta} \exp\left[-\frac{1}{2}\left(\frac{x-x'}{\sigma_\eta}\right)^2\right] \exp\left[-i2\pi\frac{x-x'}{\lambda_\eta}\right], \quad (69)$$

where σ_η is the standard deviation of the envelope and λ_η is the wavelength of the filter. Spatial correlation is introduced by multiplying white noise by the Cholesky decomposition of the covariance kernel. The white noise has a uniformly distributed phase, normally distributed amplitude with unit variance, and is generated as in Towne et al. [2018c]. The forcing is spatially restricted to an interior portion of the domain via the window $\exp[-(x/L)^p]$, where $L = 60, p = 10$. The spatially correlated noise is low-pass filtered using a 10th-order finite-impulse-response filter with a cutoff frequency equal to $0.6f_{\text{Nyquist}}$. This results in a stationary forcing that is approximately constant in amplitude up to the cutoff frequency (−6dB in amplitude at the cutoff frequency) but has non-zero spatial correlation as defined by (69). The forcing is then linearly interpolated to the temporal locations required by the temporal integration. To compute the WV spectrum, SPOD, and CS-SPOD, we employed a window length $N_w = 10N_\theta$ and an overlap 67%, resulting in $N_b = 595$ (realizations) of the process and a frequency discretization of $\Delta f = 0.01$.

In analyzing the fabricated data, we must first determine those frequencies, if any, where the system exhibits cyclostationarity. To do this, we compute the CCSD and search over all possible values of α in the range of possible cycle frequencies $\alpha \in [-1, 1]$, noting the α discretization required as discussed in §2.1 to ensure no possible cycle frequencies are missed. Figure 6 shows the CCSD and integrated CCSD for the three values of A_{μ_0} at $x = 0$, and confirms that the system is cyclostationary when $A_{\mu_0} > 0$ as high values of the CCSD and the integrated CCSD are seen at $\alpha = 0$, the modulation frequency (f_0), and an increasing number of harmonics as A_{μ_0} is further increased.

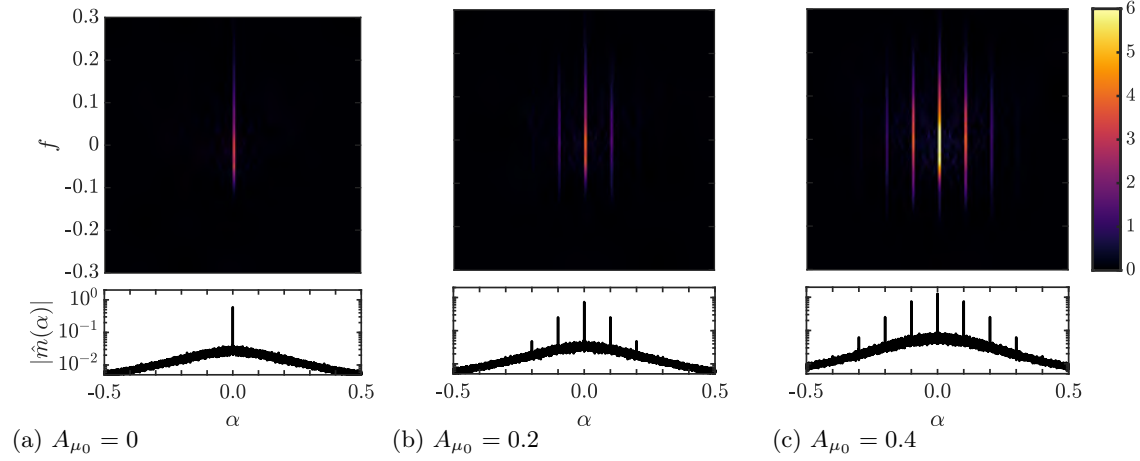


Figure 6: CCSD (top) and integrated CCSD (bottom) for the Ginzburg-Landau system at $x = 0$.

We show 100 realizations of the process for each A_{μ_0} along with the WV spectrum at $x = x' = 0$ as a function of the phase of α_0 in figure 7. The WV spectrum is computed using $a_2 = 5$ to encompass all cycle frequencies present. Figure 7 (a) shows that the statistics are almost constant as a function of phase for $A_{\mu_0} = 0$, which is expected given the time-invariant dynamics. The small degree of modulation observed is due to statistical error. In figures 7 (b, c), we observe increasing levels of modulation in the statistics as A_{μ_0} increases. Furthermore, the peak value of the spectrum also increases due to the increasing non-normality of the system with increasing μ_0 . Given that the largest value of μ_0 occurs at $\theta = 0.5\pi$ and the peak of the WV spectrum occurs at $\theta \approx 0.95\pi$, there is a phase delay of $\approx 0.45\pi$ between when the dynamics of the system are the least stable and when the perturbations are, on average, the largest.

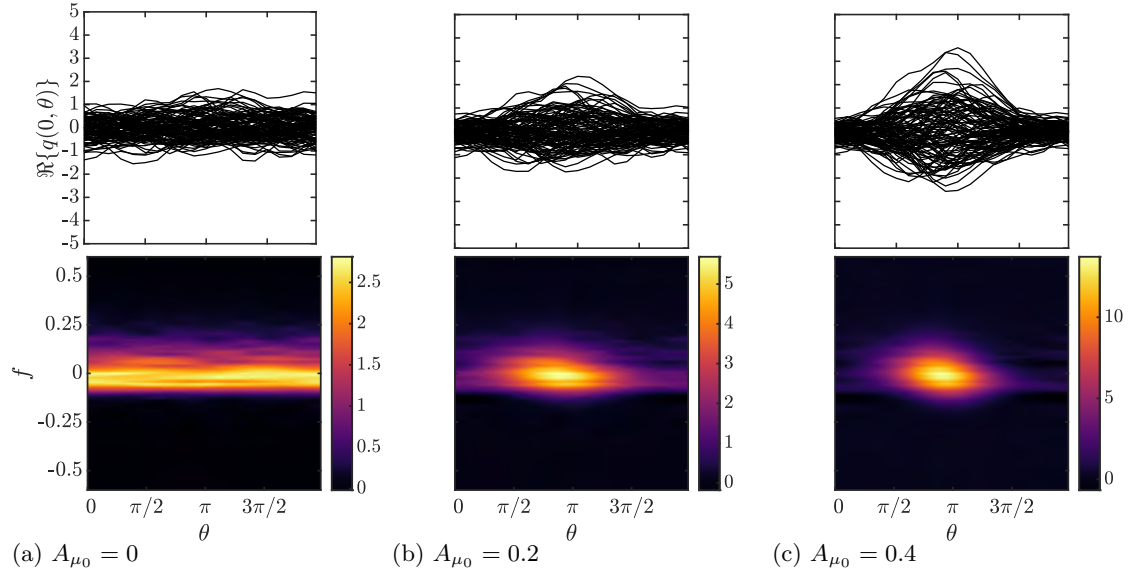


Figure 7: Example Ginzburg-Landau sample paths (top) and WV spectrum at $x = 0$ (bottom).

Based on the preceding analysis and to ensure we encompass all frequencies of interest, we compute CS-SPOD using $a_1 = 5$, resulting in a frequency range of $\Omega_\gamma = [-0.5, 0.5] + \gamma$. We

first consider the stationary process with $A_{\mu_0} = 0.0$. Although CS-SPOD modes are theoretically equivalent to SPOD for the stationary case, finite data length leads to differences.

Figure 8 shows the SPOD eigenspectrum for $A_{\mu_0} = 0.0$. Note that the spectrum is not symmetric in f because the Ginzburg-Landau system is complex. We superpose on the SPOD spectra the set of frequencies $f \in \Omega_\gamma$ for $\gamma = 0.05$, and mark and rank the 6 intersections with the highest energy. Based on the plot, we should find that the 4 most dominant CS-SPOD modes correspond to the dominant SPOD mode at a frequency of $\gamma - \alpha_0, \gamma, \gamma + \alpha_0$, and $\gamma + 2\alpha_0$, respectively. Similarly, the 5th and 6th CS-SPOD modes should correspond to the first subdominant SPOD modes at a frequency of γ and $\gamma + \alpha_0$, respectively. Figure 9 makes comparisons between SPOD and CS-SPOD (performed assuming a fundamental cycle frequency of $\alpha_0 = f_0$) for the energy and eigenfunctions for each of these six modes. While the results are quite similar in each case, there are differences associated with statistics convergence, and this, as expected, occurs when there is a small energy separation between two distinct modes (e.g. modes 5 and 6).

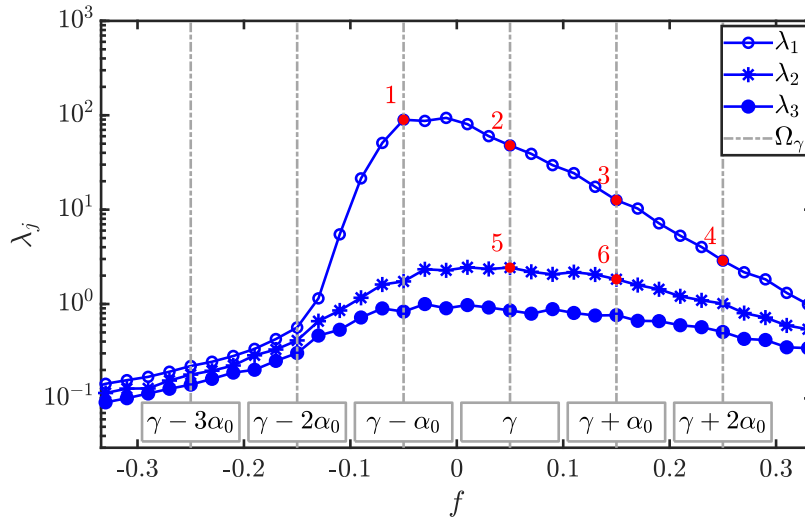


Figure 8: SPOD eigenspectrum for the Ginzburg-Landau system at $A_{\mu_0} = 0.0$ showing the three most energetic modes at each discrete frequency f . Every other eigenvalue has been omitted to improve readability. The 6 highest-energy modes occurring at the frequencies present in the CS-SPOD solution frequencies, i.e. $f \in \Omega_\gamma$, are depicted with the red dots.

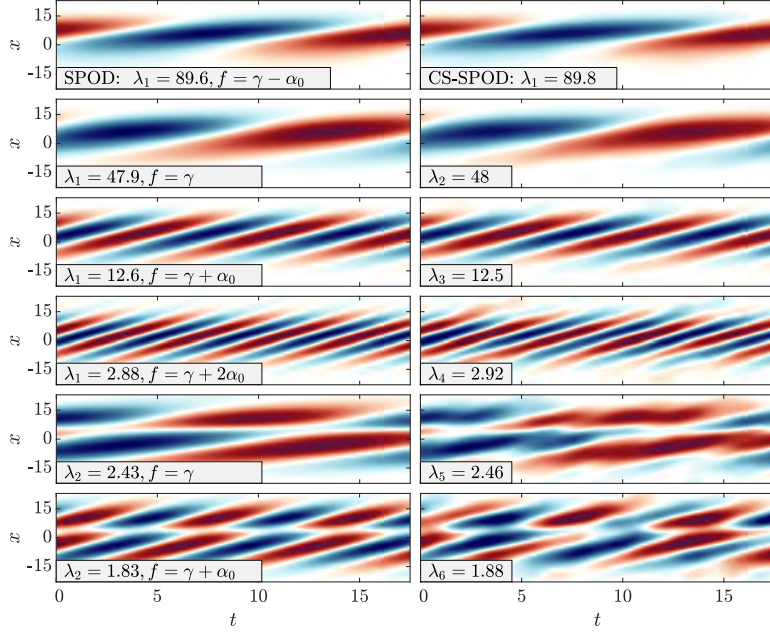


Figure 9: Comparison of SPOD (left) and CS-SPOD modes (right) for the Ginzburg-Landau system at $A_\mu = 0$. From top to bottom are the six most dominant CS-SPOD modes and the six points identified in figure 8. The contour limits for the CS-SPOD eigenfunctions are set equal to corresponding SPOD mode $\pm \|\psi_j(\mathbf{x}, t)\|_\infty$.

In figure 10, we now compare the CS-SPOD eigenspectrum for all $\gamma_k \in \Omega_\gamma$ for the three different values of A_{μ_0} . As A_{μ_0} increases, so does the energy, as the disturbances are increasingly amplified by the increasing non-normality of the linear operator at phases corresponding to positive $A_{\mu_0} \sin(2\pi f_0 t)$, consistent with the trend shown previously in figure 7. A large energy separation between the dominant and sub-dominant CS-SPOD modes is observed, which increases for greater A_{μ_0} , indicating that the process is increasingly low rank. In figure 11, for $\gamma = 0.05$, we show the fraction of the total energy ($\lambda_T = \sum_j \lambda_j$) that the first J CS-SPOD or SPOD modes recover. As theoretically expected for $A_{\mu_0} = 0$, CS-SPOD and SPOD result in almost identical energy distribution. In contrast, with increasing A_{μ_0} , CS-SPOD captures an increasingly greater amount of energy than SPOD. For example, at $A_{\mu_0} = 0.4$, the first CS-SPOD mode captures 64% of the total energy, while the first SPOD mode captures just 45%. Furthermore, the first three CS-SPOD modes capture 92% of the total energy, while seven SPOD modes are required to capture a similar amount of energy. As theoretically expected, the energy captured by SPOD does not exceed the energy captured by CS-SPOD (since SPOD modes are a subset of CS-SPOD modes). Thus, as the statistics become increasingly cyclostationary (i.e. more phase-dependent), CS-SPOD is able to capture an increasingly larger fraction of the phase-dependent statistics present in the process, which SPOD, due to the fundamentally flawed assumption of statistical stationarity, is unable to achieve.

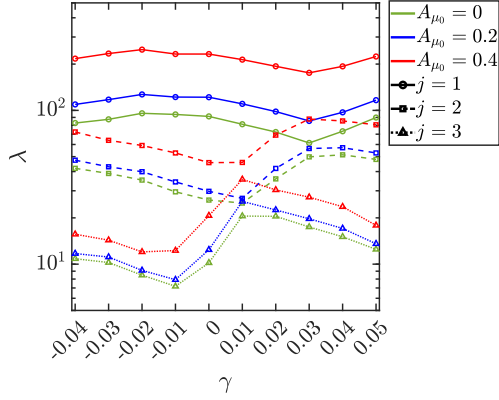


Figure 10: CS-SPOD energy spectrum of the three Ginzburg-Landau systems.

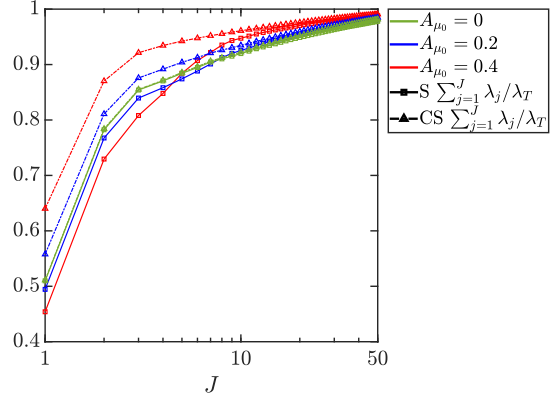


Figure 11: Total fractional energy captured by a truncated set of CS-SPOD (CS) and SPOD (S) modes for the three Ginzburg-Landau systems at $\gamma = 0.05$.

We now investigate how A_{μ_0} modifies the dominant CS-SPOD modes, at $\gamma = 0.05$, by showing the real component and the magnitude of the temporal evolution of the modes $\phi_j(\mathbf{x}, t)$ in figures 12 and 13, respectively. We note that due to the multiple frequency components (Ω_γ) present in $\phi_j(\mathbf{x}, t)$, $\phi_j(\mathbf{x}, t)$ can, unlike SPOD, no longer be completely represented by a single snapshot and instead must be displayed as a function of time. Similarly, the amplitude of the mode is periodic in time with period $T_0 = 1/\alpha_0$, unlike SPOD where the amplitude is constant in time. Thus, the amplitude is displayed as a function of phase θ . Similar results are observed for other values of γ not shown here. Overall, across all values of A_{μ_0} , the real component of the CS-SPOD modes shows a similar structure. However, as A_{μ_0} is increased, an additional modulation is seen that results in increasingly time/phase-dependent magnitudes.

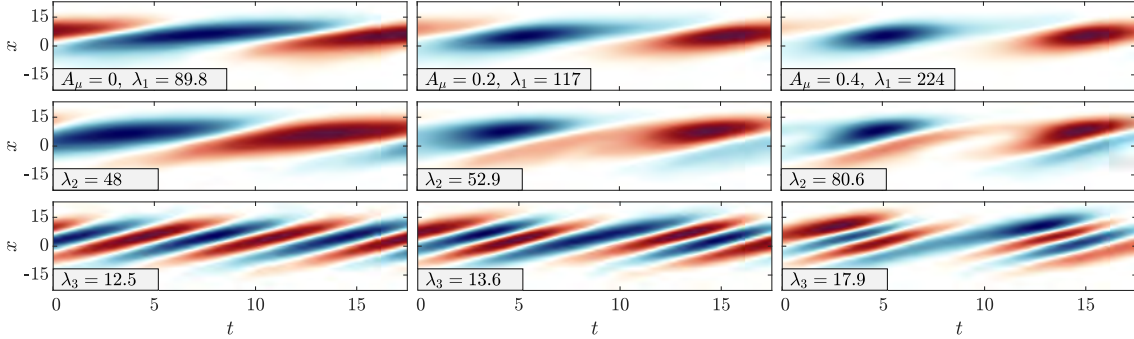


Figure 12: Real component of the three dominant CS-SPOD modes at $\gamma = 0.05$ for the three Ginzburg-Landau systems. The contour limits for each CS-SPOD mode is $\pm |\Re\{\psi_j(\mathbf{x}, t)\}|_\infty$.

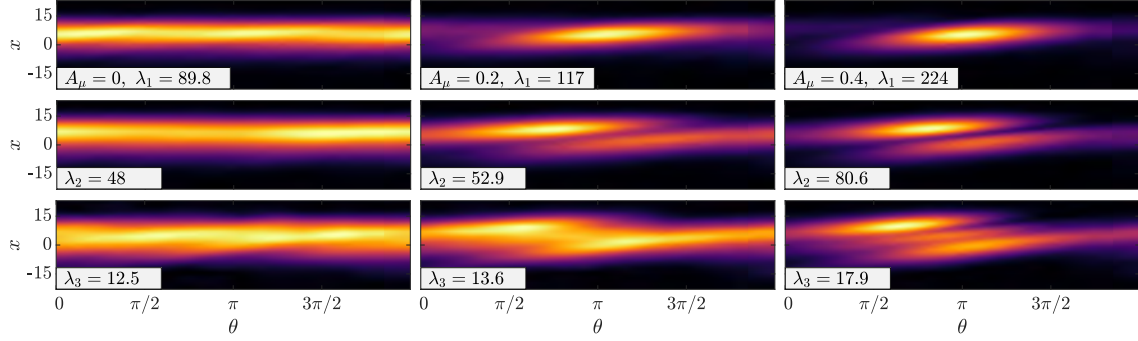


Figure 13: Magnitude of the three dominant CS-SPOD modes at $\gamma = 0.05$ for the three Ginzburg-Landau systems. The contour limits for each CS-SPOD mode is $[0, \|\psi_j(\mathbf{x}, t)\|_\infty]$.

Finally, in figure 14, we investigate which frequency components are most energetic via the fractional energy of each frequency component $f \in \Omega_\gamma$ for each CS-SPOD mode, defined as $E_{f,j} \equiv \psi_j(\mathbf{x}, f)^* \mathbf{W}(\mathbf{x}) \psi_j(\mathbf{x}, f)$, where $\sum_{f \in \Omega_\gamma} E_{f,j} = 1$. As A_{μ_0} increases, the CS-SPOD modes are constructed from an increased number of non-zero-energy frequency components and at higher energy levels. For example, at $\gamma = 0.05$, the dominant frequency component, $f = 0.05$, contains $\approx 100\%$, 83% , and 64% of the total energy of the corresponding CS-SPOD mode for $A_{\mu_0} = 0, 0.2$, and 0.4 , respectively. This occurs because of the increasing amount of correlation present between different frequency components as A_{μ_0} increases. Alternatively, this phenomenon can be understood as the following; as A_{μ_0} increases, the statistics become more time-dependent, and thus, the amount of interaction between frequency components in Ω_γ increases such that the summation of these frequency components result in CS-SPOD modes that capture the time-periodic modulation experienced by the flow.

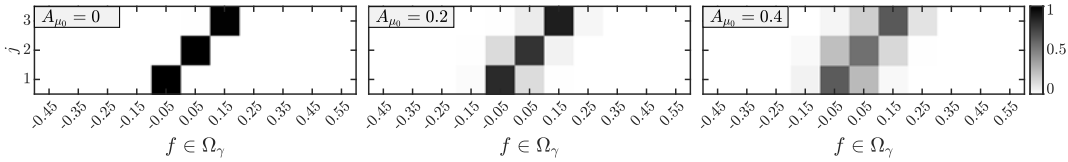


Figure 14: Fractional CS-SPOD modal energy, $E_{f,j}$, at $\gamma = 0.05$ for the Ginzburg-Landau systems.

2.5 Harmonic resolvent analysis and its relationship to CS-SPOD

Harmonic resolvent analysis [Padovan and Rowley \[2022\]](#) extends resolvent analysis to time-periodic mean flows. Starting with the nonlinear governing equations

$$\frac{\partial \mathbf{g}(t)}{\partial t} = \mathbf{H}(\mathbf{g}(t)), \quad (70)$$

where $\mathbf{H}$ is the time-independent continuity, momentum, and energy equations and $\mathbf{g}(t) \in \mathbb{C}^N$ is the state vector of flow variables, we decompose the state as $\mathbf{g}(\mathbf{x}, t) = \tilde{\mathbf{g}}(\mathbf{x}, t) + \mathbf{g}''(\mathbf{x}, t)$, where $\tilde{\mathbf{g}}(t) = \tilde{\mathbf{g}}(t + T_0)$ is the T_0 periodic mean flow component (first-order component) and $\mathbf{g}''(t)$ is the turbulent component (second-order component). Since $\tilde{\mathbf{g}}(t)$ is periodic, it can be expressed as a Fourier series, giving $\tilde{\mathbf{g}}(t) = \sum_{n \in \mathcal{A}_n} \hat{\mathbf{g}}_n e^{i2\pi(n\alpha_0)t}$, where $\hat{\mathbf{g}}_n$ are harmonics of the fundamental frequency $\alpha_0 = 1/T_0$ of the mean flow (i.e. the Fourier series components), T_0 is the period of oscillation of the mean flow, and $n \in \mathcal{A}_n$ is defined as previous. The cycle frequencies, which in the context of linear analysis must be the frequencies present in the mean flow, are $n\alpha_0$. By substituting

this decomposition into (70), we obtain

$$\frac{\partial \mathbf{g}''(t)}{\partial t} = D_{\mathbf{g}}(\mathbf{H}(\tilde{\mathbf{g}}(t))\mathbf{g}''(t) + \mathbf{f}''(t), \quad (71)$$

where $\mathbf{f}''(t)$ contains higher-order terms in $\mathbf{g}''(t)$. The Jacobian $\mathbf{A}(t) = D_{\mathbf{g}}(\mathbf{H}(\tilde{\mathbf{g}}(t)))$ is also a periodic function in time, which, following the discussion in Padovan and Rowley [2022], we assume is a differentiable function of time thereby guaranteeing a unique solution of (71). Subsequently, it is also expanded as a Fourier series $\mathbf{A}(t) = \sum_{n \in \mathcal{A}_n} \hat{\mathbf{A}}_n e^{i2\pi(n\alpha_0)t}$. Inserting this expansion into (71), gives

$$\frac{\partial \mathbf{g}''(t)}{\partial t} = \sum_{n \in \mathcal{A}_n} \hat{\mathbf{A}}_n e^{i2\pi(n\alpha_0)t} \mathbf{g}''(t) + \mathbf{f}''(t), \quad (72)$$

which we Fourier transform in time and then separate by frequency, obtaining

$$i2\pi(\gamma + m\alpha_0)\hat{\mathbf{g}}_{\gamma+m\alpha_0} = \sum_{n \in \mathcal{A}_n} \hat{\mathbf{A}}_n \hat{\mathbf{g}}_{\gamma+(m-n)\alpha_0} + \hat{\mathbf{f}}_{\gamma+m\alpha_0}, \quad (73)$$

where $\hat{\mathbf{g}}_{\gamma}$ and $\hat{\mathbf{f}}_{\gamma}$ are the γ -frequency components of $\mathbf{g}''(t)$ and $\mathbf{f}''(t)$, respectively. Equation (72) represents a system of coupled equations where perturbations at frequency γ are coupled to perturbations at frequency $\gamma - n\alpha_0$ through the $n\alpha_0$ frequency component of the mean flow. In general, this results in an infinite-dimensional problem similar to the infinite-dimensional CS-SPOD eigenvalue problem. In practice, identically to CS-SPOD, we restrict the perturbation frequencies to $[\gamma - a_1\alpha_0, \gamma + a_1\alpha_0]$ and thus, we seek time-periodic perturbations of $\mathbf{g}''(t) = \sum_{m \in \mathcal{A}_m} \hat{\mathbf{g}}_{\gamma+m\alpha_0} e^{i2\pi(\gamma+m\alpha_0)t}$, where $\mathcal{A}_m = \{-a_1, \dots, -1, 0, 1, \dots, a_1\}$. This results in a solution frequency set of $\Omega_{\gamma} = \{-a_1\alpha_0 + \gamma, (-a_1 + 1)\alpha_0 + \gamma, \dots, \gamma, \dots, (a_1 - 1)\alpha_0 + \gamma, a_1\alpha_0 + \gamma\}$.

We also limit the mean flow frequencies to $[-a_2, a_2]$ with $a_2 \leq a_1$. The final problem is compactly written as

$$(i2\pi\gamma I - \hat{\mathbf{T}})\hat{\mathbf{G}} = \hat{\mathbf{F}}, \quad (75)$$

where

$$\hat{\mathbf{T}} = \begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \hat{\mathbf{R}}_{-\alpha_0} & \hat{\mathbf{A}}_{-\alpha_0} & \hat{\mathbf{A}}_{-2\alpha_0} & \ddots \\ & & \hat{\mathbf{A}}_{\alpha_0} & \hat{\mathbf{R}}_0 & \hat{\mathbf{A}}_{-\alpha_0} & \ddots \\ & & \hat{\mathbf{A}}_{2\alpha_0} & \hat{\mathbf{A}}_{\alpha_0} & \hat{\mathbf{R}}_{\alpha_0} & \ddots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \quad \hat{\mathbf{G}} = \begin{bmatrix} \vdots \\ \hat{\mathbf{g}}_{\gamma-\alpha_0} \\ \hat{\mathbf{g}}_{\gamma} \\ \hat{\mathbf{g}}_{\gamma+\alpha_0} \\ \vdots \end{bmatrix}, \quad \hat{\mathbf{F}} = \begin{bmatrix} \vdots \\ \hat{\mathbf{f}}_{\gamma-\alpha_0} \\ \hat{\mathbf{f}}_{\gamma} \\ \hat{\mathbf{f}}_{\gamma+\alpha_0} \\ \vdots \end{bmatrix}, \quad (76a, b, c)$$

$\hat{\mathbf{R}}_{k\alpha_0} = (-i2\pi k\alpha_0 \mathbf{I} + \hat{\mathbf{A}}_0) \in \mathbb{C}^{N \times N}$, and $\mathbf{I} \in \mathbb{R}^{(2a_1+1)N \times (2a_1+1)N}$ is the identity operator. The harmonic resolvent operator is then defined as $\hat{\mathbf{H}} = (i2\pi\gamma I - \hat{\mathbf{T}})^{-1} \in \mathbb{C}^{(2a_1+1)N \times (2a_1+1)N}$ and has $(2a_1+1)$ coupled equations and is $(2a_2+1)$ banded-block-diagonal due to the periodicity of the mean flow. If the flow is time-invariant, then all off-diagonal blocks are zero, i.e. there is no cross-frequency coupling, and the system becomes block-diagonal where each diagonal block is the standard resolvent problem at frequency $\gamma + k\alpha_0, k \in \mathbb{Z}$. As detailed by Padovan and Rowley [2022], the singularity in the harmonic resolvent operator must be removed to avoid numerical difficulties.

Similar to CS-SPOD, harmonic resolvent analysis is periodic in γ , and thus we must only solve over the range $\gamma \in \Omega_{\gamma} = (-\alpha_0/2, \alpha_0/2]$. We then seek to solve the forcing mode $\hat{\mathbf{F}}$ that results in the most energetic response $\hat{\mathbf{G}}$, expressed as the following optimization problem

$$\sigma^2 = \frac{\langle \hat{\mathbf{G}}, \hat{\mathbf{G}} \rangle_G}{\langle \hat{\mathbf{F}}, \hat{\mathbf{F}} \rangle_F}, \quad (77)$$

where $\langle \hat{G}_j, \hat{G}_k \rangle_G$ and $\langle \hat{F}_j, \hat{F}_k \rangle_F$ are inner products on the output and input spaces, respectively, and are given by

$$\langle \hat{G}_j, \hat{G}_k \rangle_G = \int_{\Omega} \hat{G}_k^*(\mathbf{x}, f) W_G(\mathbf{x}) \hat{G}_j(\mathbf{x}, f) d\mathbf{x}, \quad (78a)$$

$$\langle \hat{F}_j, \hat{F}_k \rangle_F = \int_{\Omega} \hat{F}_k^*(\mathbf{x}, f) W_F(\mathbf{x}) \hat{F}_j(\mathbf{x}, f) d\mathbf{x}. \quad (78b)$$

The solution to this optimization problem is given by the singular value decomposition of the weighted harmonic resolvent operator

$$\tilde{H} = W_G^{1/2} \hat{H} W_F^{-1/2} = \tilde{U} \Sigma \tilde{V}^*, \quad (79)$$

where the diagonal matrix $\Sigma = \text{diag}[\sigma_1^2, \sigma_2^2, \dots]$ contains the ranked gains and the columns of $\hat{V} = W_F^{-1/2} \tilde{V}$ and $\hat{U} = W_G^{1/2} \tilde{U}$ contain the forcing and response modes, respectively. These modes have an analogous structure to $\hat{F}$ or $\hat{G}$, and the j^{th} forcing and response modes ($\hat{U}_j$, $\hat{V}_j$) can be reconstructed in the time-domain as

$$U_j = U_j(\mathbf{x}, t) = \sum_{m \in \mathcal{A}_m} \hat{u}_{j, \gamma + m\alpha_0} e^{i2\pi(\gamma + m\alpha_0)t} \quad (80a)$$

$$V_j = V_j(\mathbf{x}, t) = \sum_{m \in \mathcal{A}_m} \hat{v}_{j, \gamma + m\alpha_0} e^{i2\pi(\gamma + m\alpha_0)t}. \quad (80b)$$

respectively. These modes are orthonormal in their respective spatial norms $\langle \hat{U}_j, \hat{U}_k \rangle_G = \langle \hat{V}_j, \hat{V}_k \rangle_F = \delta_{j,k}$ and the temporal modes are orthogonal in their respective space-time norms $\langle U_j, U_k \rangle_{G(x,t)}, \langle V_j, V_k \rangle_{F(x,t)}$, where

$$\langle U_j, U_k \rangle_{U(x,t)} = \int_{\Omega} U_k^*(\mathbf{x}, t) W_G(\mathbf{x}) U_j(\mathbf{x}, t) d\mathbf{x} dt, \quad (81a)$$

$$\langle V_j, V_k \rangle_{V(x,t)} = \int_{\Omega} V_k^*(\mathbf{x}, t) W_F(\mathbf{x}) V_j(\mathbf{x}, t) d\mathbf{x} dt, \quad (81b)$$

The decomposition is complete, allowing the output to be expanded as

$$\hat{G}(\mathbf{x}, \gamma) = \sum_{j=1}^{\infty} \hat{U}_j(\mathbf{x}, \gamma) \sigma_j(\gamma) \beta_j(\gamma), \quad (82)$$

where

$$\beta_j(\gamma) = \langle \hat{F}(\mathbf{x}, \gamma), \hat{U}_j(\mathbf{x}, \gamma) \rangle_F. \quad (83)$$

A connection between harmonic resolvent analysis and CS-SPOD is obtained using an approach analogous to that of Towne et al. [2018c] and is similar to relationship between resolvent analysis and SPOD. In §2.1, it was shown that $\mathbf{S}(\mathbf{x}, \mathbf{x}', \alpha, f)$ can be compactly written as

$$\mathbf{S}(\mathbf{x}, \mathbf{x}', \alpha, f) = E\{\hat{\mathbf{q}}(\mathbf{x}, f - \alpha/2) \hat{\mathbf{q}}^*(\mathbf{x}', f + \alpha/2)\}, \quad (84)$$

where $\hat{\mathbf{q}}(\mathbf{x}, f)$ is the short-time Fourier transform of $\mathbf{q}(\mathbf{x}, t)$. Similarly, the CS-SPOD decomposition tensor for the process $\mathbf{q}(\mathbf{x}, t)$ can be written as

$$S(\mathbf{x}, \mathbf{x}', \gamma) = E\{\hat{\mathbf{Q}}(\mathbf{x}, \gamma) \hat{\mathbf{Q}}^*(\mathbf{x}', \gamma)\}. \quad (85)$$

To develop a relationship between CS-SPOD and harmonic resolvent analysis, we equate the CS-SPOD and harmonic resolvent expansions of the CS-SPOD decomposition matrix and set all norms

to be equal, i.e. $\langle \cdot \rangle = \langle \cdot \rangle_G = \langle \cdot \rangle_F = \langle \cdot \rangle_x$, giving

$$S(\mathbf{x}, \mathbf{x}', \gamma) = \sum_{j=1}^{\infty} \lambda_j(\gamma) \boldsymbol{\Psi}_j(\mathbf{x}, \gamma) \boldsymbol{\Psi}_j^*(\mathbf{x}', \gamma) \quad (86a)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \hat{U}_j(\mathbf{x}, \gamma) \hat{U}_k^*(\mathbf{x}', \gamma) \sigma_j(\gamma) \sigma_k(\gamma) S_{\beta_j \beta_k}(\gamma), \quad (86b)$$

where $S_{\beta_j \beta_k}(\gamma) = E\{\beta_j(\gamma) \beta_k^*(\gamma)\}$ is the scalar CSD between the j^{th} and k^{th} expansion coefficients. Identical to Towne et al. [2018c], the output harmonic resolvent modes and singular values were moved outside of the expectation operator since they are deterministic quantities. Conversely, the expansion coefficients depend on the forcing $\hat{F}(\mathbf{x}, \gamma)$, which is stochastic due to the random nature of turbulent flows and thus is described by the CSD. In the case of a stationary process, $S(\mathbf{x}, \mathbf{x}', \gamma)$ is block-diagonal, meaning that $\boldsymbol{\Psi}_j(\mathbf{x}, \gamma)$ and $\hat{U}_j(\mathbf{x}, \gamma)$ contain only a single non-zero frequency component per mode, and this relationship simplifies to that in Towne et al. [2018c]. For uncorrelated expansion coefficients $S_{\beta_j \beta_k}(\gamma) = \mu_j(\gamma) \delta_{jk}$, the relationship simplifies to

$$S(\mathbf{x}, \mathbf{x}', \gamma) = \sum_{j=1}^{\infty} \lambda_j(f) \boldsymbol{\Psi}_j(\mathbf{x}, \gamma) \boldsymbol{\Psi}_j^*(\mathbf{x}', \gamma), \quad (87a)$$

$$= \sum_{j=1}^{\infty} \hat{U}_j(\mathbf{x}, \gamma) \hat{U}_j^*(\mathbf{x}', \gamma) \sigma_j^2(f) \mu_j(\gamma). \quad (87b)$$

Since orthogonal diagonalizations are unique, this shows that CS-SPOD modes and harmonic resolvent modes are identical, and the k^{th} most energetic CS-SPOD mode corresponds to the resolvent mode with the k^{th} greatest $\sigma_j^2(\gamma) \mu_j(\gamma)$. If $\mu_j = 1$ for all j , then $\sigma_j^2(\gamma) = \lambda_j(\gamma)$ and $\boldsymbol{\Psi}_j(\mathbf{x}, \gamma) = \hat{U}_j(\mathbf{x}, \gamma)$ showing that the ranked CS-SPOD eigenvalues equal the ranked harmonic resolvent gains. To determine the conditions when the expansion coefficients are uncorrelated, we perform identical manipulation to Towne et al. [2018c], and show that

$$S_{\beta_j \beta_k}(\gamma) = \langle \langle S_{FF}(\mathbf{x}, \mathbf{x}', \gamma), \hat{U}_j(\mathbf{x}, \gamma) \rangle^*, \hat{U}_k(\mathbf{x}, \gamma) \rangle^*, \quad (88)$$

where $S_{FF}(\mathbf{x}, \mathbf{x}', \gamma) = E\{\hat{F}(\mathbf{x}, \gamma) \hat{F}^*(\mathbf{x}', \gamma)\}$ is the CS-SPOD decomposition tensor of $\hat{F}(\mathbf{x}, \gamma)$. Since harmonic resolvent modes are orthogonal, if $\langle S_{FF}(\mathbf{x}, \mathbf{x}', \gamma), \hat{U}_j(\mathbf{x}, \gamma) \rangle^* = \mu_j(\gamma) \hat{U}_j(\mathbf{x}, \gamma)$ then $S_{\beta_j \beta_k}(\gamma) = \mu_j(\gamma) \delta_{jk}$. This can be written as

$$\int_{\Omega} S_{\eta\eta}(\mathbf{x}, \mathbf{x}', \gamma) W_{\eta}(\mathbf{x}') \hat{V}_j(\mathbf{x}', \gamma) d\mathbf{x}' = \mu_j(\gamma) \hat{V}_j(\mathbf{x}, \gamma), \quad (89)$$

which is identical to the CS-SPOD of the input. One can then show that the expansion coefficients are uncorrelated if and only if the harmonic resolvent input modes correspond exactly with the CS-SPOD modes of the input. Thus, we conclude that the relationship between CS-SPOD and harmonic resolvent analysis is identical to that of SPOD and resolvent analysis.

We can then specialize for $\mu_j = 1$, giving

$$S_{\eta\eta}(\mathbf{x}, \mathbf{x}', \gamma) W_{\eta}(\mathbf{x}') = I \delta(\mathbf{x} - \mathbf{x}'), \quad (90)$$

which for $W_{\eta}(\mathbf{x}') = I$ results in $S_{\eta\eta}(\mathbf{x}, \mathbf{x}', \gamma) = I \delta(\mathbf{x} - \mathbf{x}')$, i.e. the forcing is unit-amplitude white noise. This results in identical harmonic resolvent and CS-SPOD modes along with equal identical energies/gains, i.e. $\sigma_j^2 = \lambda_j$.

We demonstrate this result by comparing the CS-SPOD and harmonic resolvent analysis results for the modified forced Ginzburg-Landau for $A_{\mu} = 0.4$. For both CS-SPOD and harmonic resolvent

analysis, we employ $a_1 = 5$ resulting in a frequency range of $\Omega_\gamma = [-0.5, 0.5] + \gamma$. To compute CS-SPOD, we force the system with unit variance band-limited white noise. This is constructed similarly to the spatially correlated case previously considered in §2.4.1 without the step to introduce the spatial correlation. We employ identical computational parameters to those used in §2.4.1.

As demonstrated in §2.5, because the forcing is white CS-SPOD modes and harmonic resolvent analysis modes are theoretically identical. Furthermore, since the inner product has unit weight, the CS-SPOD eigenvalues equal the harmonic resolvent analysis gains. Figure 15 shows the first six CS-SPOD eigenvalues and harmonic resolvent gains. Overall, excellent agreement is observed between the CS-SPOD eigenvalues and harmonic resolvent gains. The small amount of jitter present in the CS-SPOD eigenvalues is due to statistical convergence. The minor overshoot or undershoot is associated with spectral and cycle leakage, which can be reduced by increasing the frequency resolution of the estimate. As with any spectral estimate, increasing the length of the blocks reduces the number of blocks leading to the well-known bias-variance tradeoff. Improved control over the bias-variance tradeoff in SPOD was achieved using multi-taper methods [Schmidt, 2022] and could similarly be used for CS-SPOD.

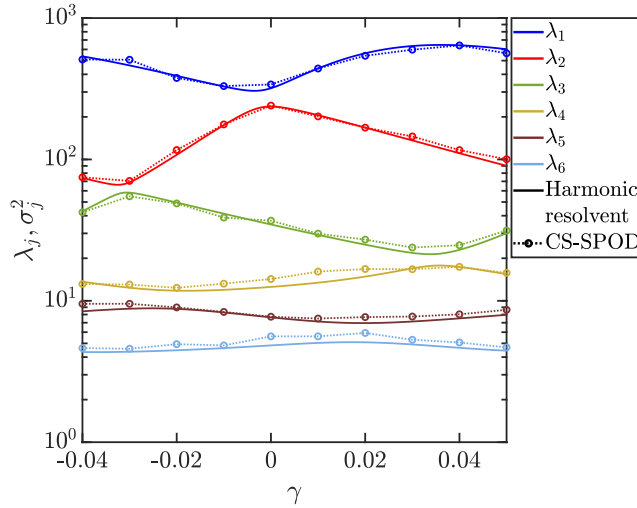


Figure 15: Comparison of the first six harmonic resolvent gains σ_j^2 and CS-SPOD eigenvalues λ_j as a function of γ for the white noise forced Ginzburg-Landau system with $A_\mu = 0.4$.

Figure 16 shows the magnitude of the time evolution of the three most energetic CS-SPOD and harmonic resolvent modes at $\gamma = 0.05$, which we see are almost indistinguishable. The similarity between the CS-SPOD and harmonic resolvent modes is quantified using the projection $\xi_{jk}(f) = \langle \psi_j(\gamma), \hat{U}_k(\gamma) \rangle_x$ and the harmonic-resolvent-mode expansion-coefficient CSD $S_{\beta_j \beta_k}(\gamma)$ given by (88). To compute $S_{\beta_j \beta_k}(f)$, we take two inner products with respect to $\hat{U}_j(\gamma)$ and $\hat{U}_k(\gamma)$ and then divide by $\sigma_j(\gamma)$ and $\sigma_k(\gamma)$, obtaining

$$S_{\beta_j \beta_k}(\gamma) = \sum_{n=1}^{\infty} \frac{\lambda_n(\gamma)}{\sigma_j(\gamma) \sigma_k(\gamma)} \xi_{nj}(\gamma) \xi_{nk}^*(\gamma). \quad (91)$$

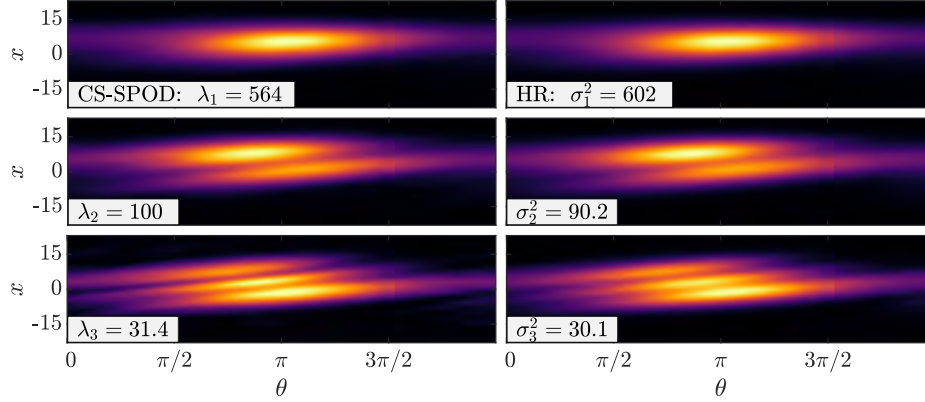


Figure 16: Comparison of the magnitude of the three most energetic CS-SPOD (left) and harmonic resolvent (right) modes at $\gamma = 0.05$ for the Ginzburg-Landau system at $A_\mu = 0.4$. The contour limits for the CS-SPOD modes are set equal to the corresponding harmonic resolvent modes $[0, |\psi_j(\mathbf{x}, t)|_\infty]$.

The projection ξ_{jk} and $\frac{|S_{\beta_j \beta_k}|}{|S_{\beta_j \beta_k}|_\infty}$ are shown in figure 17 for $\gamma = 0.05$. $|S_{\beta_j \beta_k}|$ is, by construction, diagonal, and this should result in a diagonal ξ_{jk} . This is verified for the first eight modes, but for increasingly subdominant modes, off-diagonal terms become increasingly apparent, which is owing to a lack of full statistical convergence.

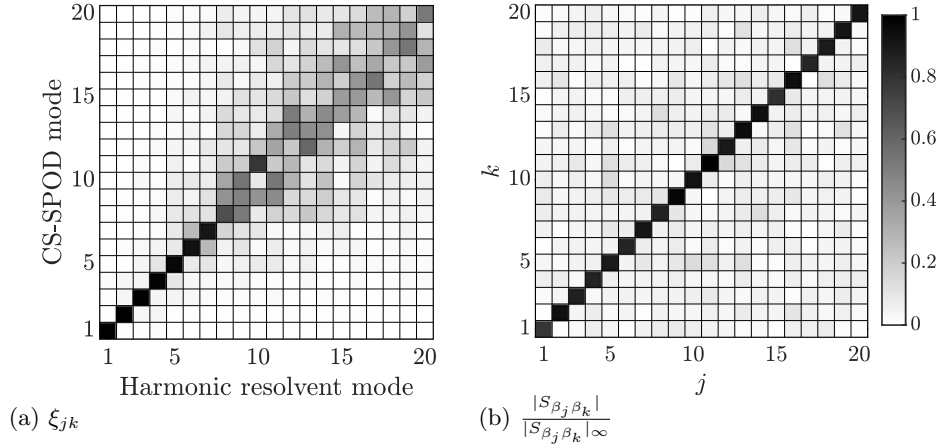


Figure 17: CS-SPOD and harmonic resolvent analysis mode projection coefficient (a) and magnitude of the normalized harmonic-resolvent-mode expansion-coefficient CSD (b) at $\gamma = 0.05$ for the Ginzburg-Landau system with white noise forcing.

Finally, to demonstrate the necessity of using harmonic resolvent and CS-SPOD to model and reduce structures for time-periodic mean flows, we compare our results with a naive application of SPOD and standard resolvent to the time-periodic GL system. Figure 18 compares the (standard) resolvent gains and SPOD eigenvalues for $A_\mu = 0, 0.2$, and 0.4 . When $A_\mu = 0$, the system is stationary, and the resolvent gains and SPOD energies agree (as expected), but there are significant and growing discrepancies as $A_\mu \neq 0$ is increased and the base flow is increasingly oscillatory. For systems with periodic statistics, CS-SPOD and harmonic resolvent analysis must be used to analyze these flows.

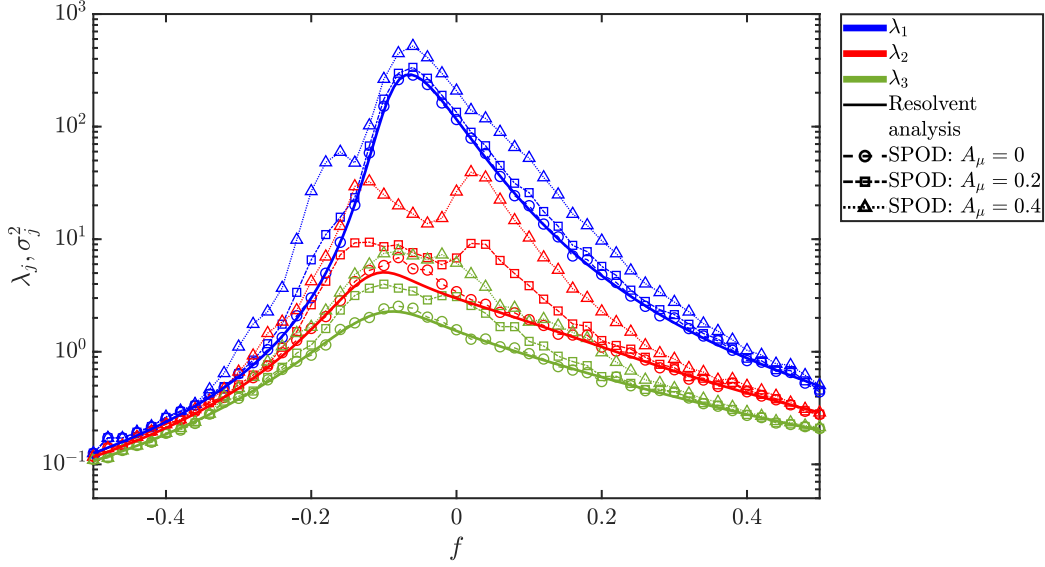


Figure 18: Comparison of the first three resolvent analysis gains σ_j^2 and SPOD eigenvalues λ_j as a function of frequency f for the white noise forced Ginzburg-Landau system at $A_\mu = 0.0, 0.2$, and 0.4 . For clarity, every second SPOD eigenvalue has been omitted.

2.6 Low-frequency and high-frequency forcing limits

In many flows, the frequency of the forcing may be either low or high with respect to the dynamics of interest. In both cases, simplifications can be made to the analysis.

For low-frequency forcing, CS-SPOD and HR tend towards systems that link all frequency components together, thereby making the analysis of the resulting system impractical. However, in many cases, we are interested in frequencies that are much larger than the forcing frequency. [Franceschini et al. \[2022\]](#) showed that high-frequency structures evolving on a low-frequency periodic motion could be analyzed using a quasi-steady approach which they named phase-conditioned localized SPOD (PCL-SPOD) and quasi-steady (QS) resolvent analysis. These methods require $f \gg f_0$ and that at each fixed time (or phase) t , the cross-correlation tensor, around that phase, only depends on the time lag τ . At each phase, all standard SPOD and resolvent analysis properties are satisfied in PCL-SPOD and QS resolvent analysis, and we refer the reader to [Franceschini et al. \[2022\]](#) for a detailed discussion. Although PCL-SPOD was developed without reference to cyclostationary theory and computational methods, by employing a similar derivation to [Franceschini et al. \[2022\]](#), PCL-SPOD can be written as

$$\int_{\Omega} \mathbf{WV}(\mathbf{x}, \mathbf{x}', f, t) \mathbf{W}(\mathbf{x}') \boldsymbol{\psi}(\mathbf{x}', f, t) d\mathbf{x}' = \lambda \boldsymbol{\psi}(\mathbf{x}, f, t), \quad (92)$$

where $\mathbf{WV}(\mathbf{x}, \mathbf{x}', f, t)$ is the Wigner-Ville spectrum and $\boldsymbol{\psi}(\mathbf{x}', f, t)$ are the PCL-SPOD eigenvectors that only contain a single frequency component f and are independent over time. This is analytically identical to the PCL-SPOD shown in [Franceschini et al. \[2022\]](#), but is numerically determined using a different computational procedure. QS resolvent analysis is similarly written as

$$(i2\pi f I - \mathbf{A}(t)) \hat{\mathbf{g}}_f = \hat{\boldsymbol{\eta}}_f, \quad (93)$$

where $\mathbf{R}(t) = (i2\pi f I - \mathbf{A}(t))$ is the QS resolvent operator, and the solution at each time-instance t is independent of the solution at any other time-instance. We then seek to solve the forcing

mode $\hat{\eta}_f$ that results in the most energetic response $\hat{g}_f$, which is determined via the singular value decomposition of the weighted harmonic resolvent operator

$$W_g^{1/2} \mathbf{R}(t) W_\eta^{-1/2} = \tilde{\mathbf{U}}_\dagger \mathbf{\Sigma}_\dagger \tilde{\mathbf{V}}_\dagger^*, \quad (94)$$

where W_η and W_g are the norms on the input and output space, respectively, and are defined similarly to equation 78. The diagonal matrix $\mathbf{\Sigma}_\dagger = \text{diag}[\sigma_1^2, \sigma_2^2, \dots]$ contains the ranked gains and the columns of $\tilde{\mathbf{V}}_\dagger = W_F^{-1/2} \hat{\mathbf{V}}_\dagger$ and $\tilde{\mathbf{U}}_\dagger = W_G^{1/2} \hat{\mathbf{U}}_\dagger$ contain the forcing and response modes, respectively.

Using equation 9, algorithm 1, and a procedure similar to that of regular SPOD, we compute PCL-SPOD of the Ginzburg-Landau systems with white-noise forcing for several different forcing frequencies $f_0 = 0.01, 0.04$, and 0.1 at $A_\mu = 0.2$. Due to the substantially lower forcing frequency, 2×10^5 snapshots are saved instead of 4×10^4 . We then compare the PCL-SPOD and QS resolvent results in figure 19 where we see excellent agreement for small f_0 . We see that as f_0 increases, the PCL-SPOD and QS resolvent results increasingly deviate as the two aforementioned assumptions are increasingly violated.

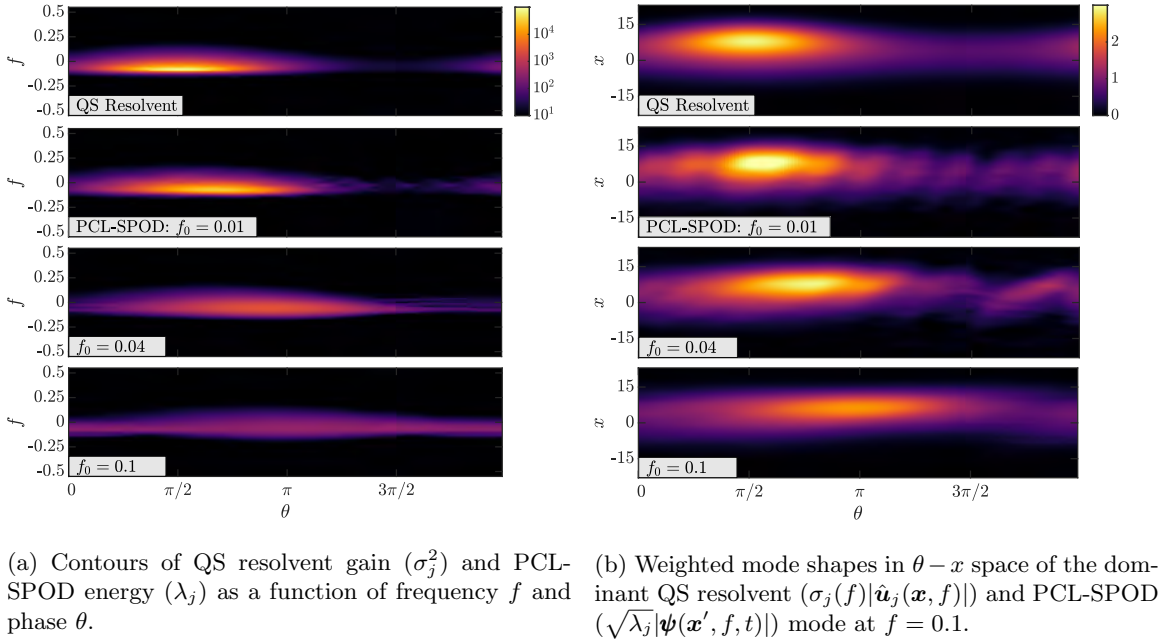


Figure 19: Contours of the gain and weight modes shapes of the white noise forced Ginzburg-Landau system at $A_\mu = 0.2$ and $f_0 = 0.01, 0.04$, and 0.1 .

Many physical systems exhibit some form of spectral peak. If the forcing frequency is sufficiently large, such that the energy contained at $f + k\alpha_0, k \in \mathbb{Z} \ \& \ k \neq 0$ is substantially lower than at f , one can see that the CS-SPOD and harmonic resolvent systems (given by equations 42, 75, respectively) can be approximated by the block diagonal term that corresponds to f (i.e. the most energetic component in Ω_γ). Furthermore, for many systems, the impact of a high-frequency forcing on the low-frequency dynamics is not direct, instead, the low-frequencies are modified as a result of nonlinear interaction that modifies mean flow. Thus, for a large forcing frequency, CS-SPOD and harmonic resolvent analysis approach SPOD and standard resolvent analysis, respectively. In figure 20, we show the SPOD and CS-SPOD eigenspectrum of the white-noise forced Ginzburg-Landau system at $A_\mu = 0.8$ for $f_0 = 0.1, 0.2, 0.4$. To assess the convergence of CS-SPOD to SPOD for large forcing frequencies, the CS-SPOD modes have been mapped to the SPOD mode of greatest

alignment (computing over the same set of frequencies Ω_γ). This is similar to what was performed in §2.4.1 during the comparison between SPOD and CS-SPOD modes. We see that as the forcing frequency increases, the CS-SPOD and SPOD eigenvalues begin to converge in the region where the energy at $f + k\alpha_0 \ll f, k \in \mathbb{Z} \ \& \ k \neq 0$.

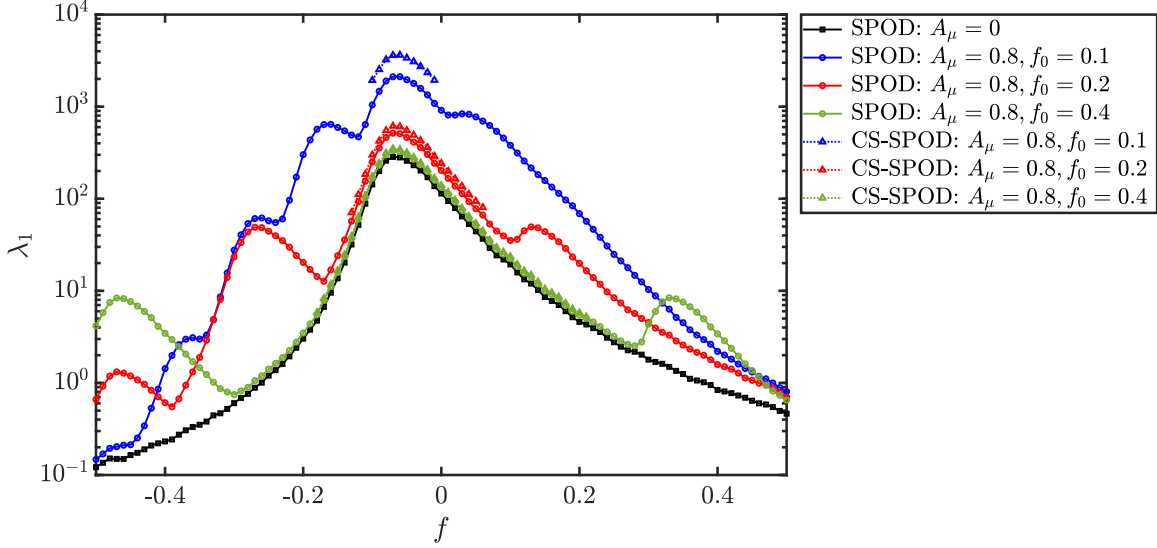


Figure 20: Comparison of the dominant SPOD and CS-SPOD eigenvalues λ_1 as a function of frequency f for the white noise forced Ginzburg-Landau system at $A_\mu = 0.8$ for $f_0 = 0.1, 0.2, 0.4$. The SPOD eigenvalues for $A_\mu = 0$ are overlaid to show the impact of the forcing on the spectrum.

3 The effect of harmonic forcing on subsonic and supersonic axisymmetric turbulent jets

To study forced turbulent jets, we perform a series of high-fidelity large-eddy simulations of turbulent subsonic and supersonic jets subjected to a periodic acoustic forcing. We then study these jets with the tools described and developed in §2. We first study the subsonic jet and investigate the impact of the forcing on the mean flow and how the turbulence present in the jet is correlated to (i.e. modulated by) the applied forcing. Next, we investigate the energy transfer between the applied forcing and turbulence via a cyclostationary decomposition of the Navier-Stokes equation. We then study the impact of the forcing on the coherent structures present in the flow through CS-SPOD. Lastly, we then conduct an analogous study on a similar supersonic jet and then summarize our results.

3.1 Large-eddy simulations of turbulent jets

Large-eddy simulations of isothermal and perfectly expanded Mach 0.4 and Mach 1.5 forced jets are performed using the compressible flow solver “Charles” by Cascade Technologies Brès et al. [2017, 2018a]. Charles solves the spatially filtered compressible Navier-Stokes equations on unstructured grids using a density-based finite-volume method. Computations performed on an unforced Mach 0.9 turbulent jet have shown excellent agreement between the computational and experimental mean, first-order statistics, and sub-decibel agreement between the computational and experimental noise spectra Brès et al. [2018a] at all relevant angles and frequencies.

As in previous computations Brès et al. [2018a], the jet nozzle is explicitly included in the computational domain, and synthetic turbulence is applied inside the nozzle to obtain a fully turbulent boundary layer. The reader can find details on the numerical methods, subgrid-models, and meshing in Brès et al. [2017]. To account for the forcing, a slightly refined grid is employed compared to the natural jet simulations investigated previously [Brès et al., 2018a, 2017], and a total of 19.2 and 42.7 million control volumes are employed for the subsonic and supersonic jets, respectively. A representative schematic of the simulation set up is shown in Fig. 21. The subsonic and supersonic jets have a Reynolds number, $Re_j = \rho_j U_j D / \mu_j = 4.5 \times 10^5, 1.76 \times 10^6$, respectively, and both jets have a Prandtl number of 0.7. The subscripts j and ∞ represent the jet and free-stream conditions, respectively. ρ is the density, μ is the viscosity, $M_j = U_j / c_j$ is the Mach number where c_j and U_j are the speed of sound, and natural jet mean flow velocity magnitude, respectively. Throughout this report, the flow is non-dimensionalized by the jet exit values, lengths by the jet diameter D , and pressure by $\rho_j U_j^2$. Frequencies are expressed with respect to the Strouhal number $St = fD / U_j$, where f is the frequency. For numerical stability, the simulations contain a weak $M_\infty = 0.009, 0.1$ coflow for the subsonic and supersonic jets, respectively.

The forcing is applied as planar acoustic waves in an annular region around the nozzle with a non-dimensional frequency $St_f = f_f D / U_j$ and amplitude a_0 , where f_f is the forcing frequency. The magnitude of the acoustic forcing applied along the co-flow boundary is scaled by the function $c(r)$, which results in the acoustic forcing decaying in an error-function manner about $r = 5$. Nekkanti Nekkanti et al. [Osaka, Japan (Online)] validated a similar forcing strategy for a similar turbulent jet against the experimental data by Maia Maia et al. [2020, 2021, 2022], showing excellent agreement. Along with the natural unforced jet, simulations are run at $St_f = 0.3, 1.5$ ($a_0 / U_j = 1\%, 10\%$) for the subsonic jet and $St_f = 0.3, 1.5$ ($a_0 / U_j = 1\%$) for the supersonic jet. A forcing frequency of $St_f = 0.3$ was chosen to roughly match the forced jet experiments of Crow and Champagne [1971], where $St_f = 0.3$ was observed as the frequency that led to the largest amplification by the flow (i.e. the *jet preferred mode*). The $St_f = 1.5$ case was chosen as experimental studies Samimy et al. [2007, 2010] have shown a broadband reduction in jet noise for both subsonic and supersonic jets at this frequency. The acoustic forcing is defined by the following relations:

$$\begin{aligned} c(r) &= 0.5 [1 - \text{erf}(2(r - 5))], \\ u_f(r, t) &= c(r) \sin(2\pi f_f t), \\ u_x(r, t) &= u_\infty + a_0 u_f(r, t), \\ u_r(r, t) &= u_\theta(r, t) = 0, \\ \rho(r, t) &= \rho_\infty + \rho_\infty (u_x(r, t) - u_\infty) / a_\infty, \\ p(r, t) &= p_\infty + a_\infty \rho_\infty (u_x(r, t) - u_\infty). \end{aligned}$$

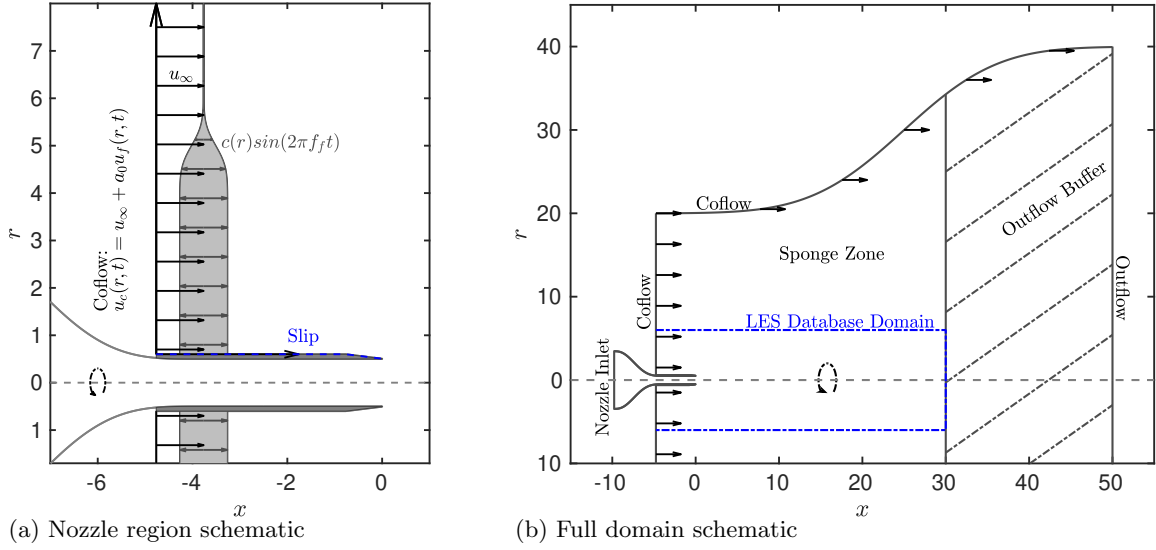


Figure 21: LES simulation schematic, adapted from Brès et al. [2018a].

The simulations are run with a computational time step $\Delta t D/c_\infty = 0.001, 0.000225$ and a total simulation time $t_{sim} D/c_\infty = 4000, 1000$ (after the initial transient, due to the forcing, has passed), for the subsonic and supersonic jets respectively. The unstructured LES data for the subsonic and supersonic jets were interpolated onto structured cylindrical grids that mimics the LES resolution of $(n_x \times n_r \times n_\theta = 656 \times 138 \times 128, 698 \times 136 \times 128)$, respectively. The forced cases were saved at N_{θ_f} phases per forcing oscillation resulting in a temporal spacing between snapshots of $\frac{\Delta t c_\infty}{D}$. A summary of the simulations performed is shown in Table 1, and an instantaneous snapshot of the natural and forced subsonic and supersonic jets is shown in Fig. 22.

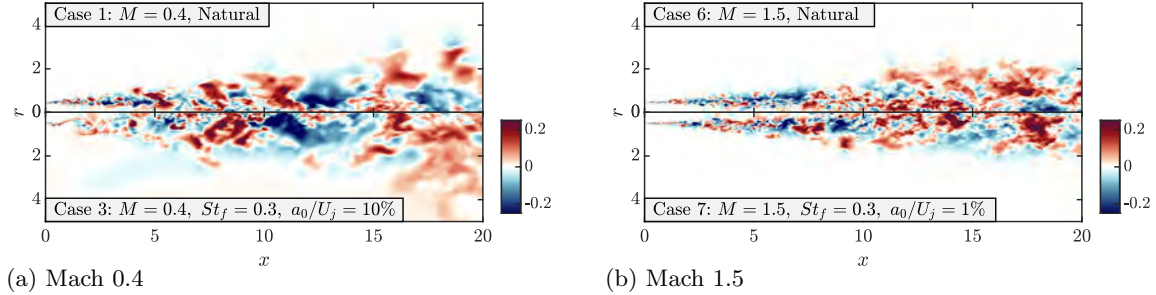


Figure 22: Instantaneous snapshot of u_x'' for the Mach 0.4 and Mach 1.5 turbulent jets.

3.2 Cyclostationary compressible Navier-Stokes equations

To determine how the forcing interacts with the turbulence, we perform a cyclostationary decomposition of the governing equations. In particular, we wish to determine how the turbulence is modulated by (interacts with) the forcing and to what degree there is energy transfer between the mean flow field and the turbulent field.

Since we are studying a compressible flow, we will perform a Favre-averaging of the governing equations. Let $f(x, t)$ be a real-valued cyclostationary process and let $\rho(x, t)$ represent the density. We employ two averages, the ensemble average $\bar{f}(x, t) = \langle f(x, t) \rangle$ and the density-weighted (Favre)

Case	M_j	St_f	a_0/U_j	$\frac{\Delta t_{c\infty}}{D}$	N_{θ_f}
Case 1: $M = 0.4$, Natural	0.4	—	—	0.2	—
Case 2: $M = 0.4$, $St_f = 0.3$, $a_0/U_j = 0.01$	0.4	0.3	0.01	0.174	48
Case 3: $M = 0.4$, $St_f = 0.3$, $a_0/U_j = 0.1$	0.4	0.3	0.1	0.174	48
Case 4: $M = 0.4$, $St_f = 1.5$, $a_0/U_j = 0.01$	0.4	1.5	0.01	0.138	12
Case 5: $M = 0.4$, $St_f = 1.5$, $a_0/U_j = 0.1$	0.4	1.5	0.1	0.138	12
Case 6: $M = 1.5$, Natural	1.5	—	—	0.045	—
Case 7: $M = 1.5$, $St_f = 0.3$, $a_0/U_j = 0.01$	1.5	0.3	0.01	0.0463	48
Case 8: $M = 1.5$, $St_f = 1.5$, $a_0/U_j = 0.01$	1.5	1.5	0.01	0.0371	12

Table 1: Parameters of the simulations performed.

ensemble average $\tilde{f}(x, t) = \{f(x, t)\} = \langle \rho(x, t)f(x, t) \rangle / \langle \rho(x, t) \rangle$. Assuming cycloergodicity, these averages can be computed using the phase average (as shown in Eq. 14) instead of the expectation operator. These averages, for a cyclostationary process, become periodic functions in time and naturally become the long-time average for a statistically-stationary process. We denote the 0th frequency component of the mean as the period-average since the long-time average does not converge for a periodic function. As demonstrated in Eq. (4), we can now decompose a process into its first-order $\bar{f}(x, t)$, $\tilde{f}(x, t)$ and second-order $f'(x, t)$, $f''(x, t)$ components with reference to the ensemble and Favre-ensemble averages, respectively, such that

$$f(x, t) = \bar{f}(x, t) + f'(x, t), \quad (95a)$$

$$f(x, t) = \tilde{f}(x, t) + f''(x, t), \quad (95b)$$

noting that, $\bar{f}' = 0$, $\overline{\rho f''} = 0$, and $\bar{f}'' \neq 0$. We decompose the compressible Navier-Stokes equations in a manner similar to Huang et al. [1995]. The relevant variables are decomposed as follows

$$\rho' = \rho - \bar{\rho}, \quad (96a)$$

$$u_i'' = u_i - \tilde{u}_i, \quad (96b)$$

$$T'' = T - \tilde{T}, \quad (96c)$$

$$\bar{p} = \bar{\rho} R \tilde{T} = \bar{\rho} R (\bar{T} - \bar{T}''), \quad (96d)$$

$$p' = \rho' R \tilde{T} + \bar{\rho} R T'' + \rho' R T'', \quad (96e)$$

$$\widetilde{u_i'' u_j''} = \{u_i'' u_j''\} = \overline{\rho u_i'' u_j''} / \bar{\rho}, \quad (96f)$$

where we have omitted the explicit spatial and temporal dependence for clarity and u_i is the velocity in the i, j, k directions, T is the temperature, R is the gas constant, and the ideal gas assumption has been made. The derivation of the governing equations presented in Huang et al. [1995] is invariant to the type of ensemble average (i.e., applied to statistically stationary or cyclostationary flows), thus, we obtain similar equations and refer the reader to Huang et al. [1995] for a detailed discussion. Critically, the energy transfer between the mean and turbulent component is given by

$$\tilde{E} = -\overline{\rho u_i'' u_j''} \frac{\partial \tilde{u}_i}{\partial x_j}, \quad (97)$$

and is a source in the turbulent kinetic energy equation and a sink in the mean kinetic energy equation. Unlike the Reynolds/triple decomposition [Hussain and Reynolds, 1970, Reynolds and Hussain, 1972], we obtain a phase-dependent energy transfer that allows us to investigate how the energy transfer between the mean field and turbulent field varies as a function of phase.

3.3 Subsonic jet

3.3.1 Mean flow

We begin with an analysis of the impact of the forcing on the mean flow. It is known that high-Reynolds number turbulent jets are receptive to a wide range of forcing frequencies and azimuthal mode numbers, for example, [Crow and Champagne \[1971\]](#), [Samimy et al. \[2007, 2010\]](#), [Sinha et al. \[2018\]](#). Generally, forcing results in a shorter jet as well as a thickening of the shear layer, which can be seen in Figs. 23 - 24.

In Fig. 23a, we display the mean axial velocity for the subsonic jets corresponding to a forcing phase $\theta_f = 0^\circ$. For the $St_f = 0.3$ forcing, a large modulation of the mean is observed with the largest impact along the potential core region. As the forcing amplitude increases, we clearly observe a greater impact on the mean field with regions of increasingly low and high axial velocity resulting in vortex roll-up as seen in Fig. 22a. In contrast, the $St_f = 1.5$ has a more compact spatial impact upon the mean and is spatially limited to the near-nozzle region. Due to the large amplitude of forcing applied, the $St_f = 1.5, a_0/U_j = 10\%$ acoustic waves have steepened as they propagate downstream and are beginning to exhibit shock-like structures resulting in the kinking of the mean velocity profile. In Fig. 23b, we display the 0^{th} frequency component of the phase average (i.e. the period-average). Despite the jet's strong harmonic response to the forcing, we see that the impact on the period-averaged field is minor, with just a small reduction in the length of the 25% contour line across all cases.

In Fig. 24, we display the period-averaged axial velocity $\bar{u}_{x,0}(x)$ and the standard deviation of the mean velocity $\sigma(\bar{u}_x(x))$ along the centerline and lipline, and the momentum thickness δ_θ . Similar to previous studies [Schmidt et al. \[2018b\]](#), δ_θ is defined as

$$\delta_\theta(x) = \int_0^{r_{0.05}} \frac{\bar{\rho}_0(x, r) \bar{u}_{x,0}(x, r)}{\bar{\rho}_0(x, 0) \bar{u}_{x,0}(x, 0)} \left(1 - \frac{\bar{u}_{x,0}(x, r)}{\bar{u}_{x,0}(x, 0)} \right) dr,$$

where $\bar{u}_{x,0}(x, r)$ is the time- and azimuthally averaged streamwise velocity. The radial bound on the integral, $r_{0.05}$, corresponds to the radial location where $\bar{u}_{x,0}(x, r_{0.05}) - U_\infty = 0.05 \bar{u}_{x,0}(x, 0)$. As seen in Fig. 23a, for all forcing frequencies and amplitudes, the jet becomes shorter as it spreads more rapidly, as seen through a decrease in the centerline velocity and an increase in the momentum thickness. We also see that the $St_f = 0.3$ forcing has a substantially greater and more global impact on the mean field than the $St_f = 1.5$ forcing. The $St_f = 0.3$ forcing results in large modulations of the centerline and lipline velocities from the nozzle exit until ≈ 8 diameters downstream, after which the effect of the forcing rapidly decreases. We see that the $St_f = 0.3, a_0/U_j = 10\%$ forcing, due to the high amplitude employed and the resulting strong nonlinear interactions, results in a non-monotonically decreasing period-averaged centerline velocity in contrast to all other cases. The modulation of the mean is also increasingly affected, with a maximum standard deviation (as a ratio of the jet velocity) along the centerline of 15%, 33% and along the lipline of 9.8%, 35% for the $a_0/U_j = 1\%$ and $a_0/U_j = 10\%$ forcing, respectively. In contrast, the $St_f = 1.5$ forcing is more localized to the near-nozzle exit, with the only substantial modulation of the lipline velocity profile occurring within the first 0.15 diameters downstream from the nozzle exit with a standard deviation of 7.0%, 18% for the $a_0/U_j = 1\%$ and $a_0/U_j = 10\%$ forcing, respectively. The effect of the $St_f = 1.5$ forcing along the centerline is minimal, with a maximum modulation of just 3.5% for the $a_0/U_j = 10\%$ forcing, demonstrating the localized receptivity of the forcing. Overall, both frequencies and amplitudes result in a substantial modulation of the mean as a function of phase, but the impact on the period-averaged values is minor, consistent with the trends seen in the literature.

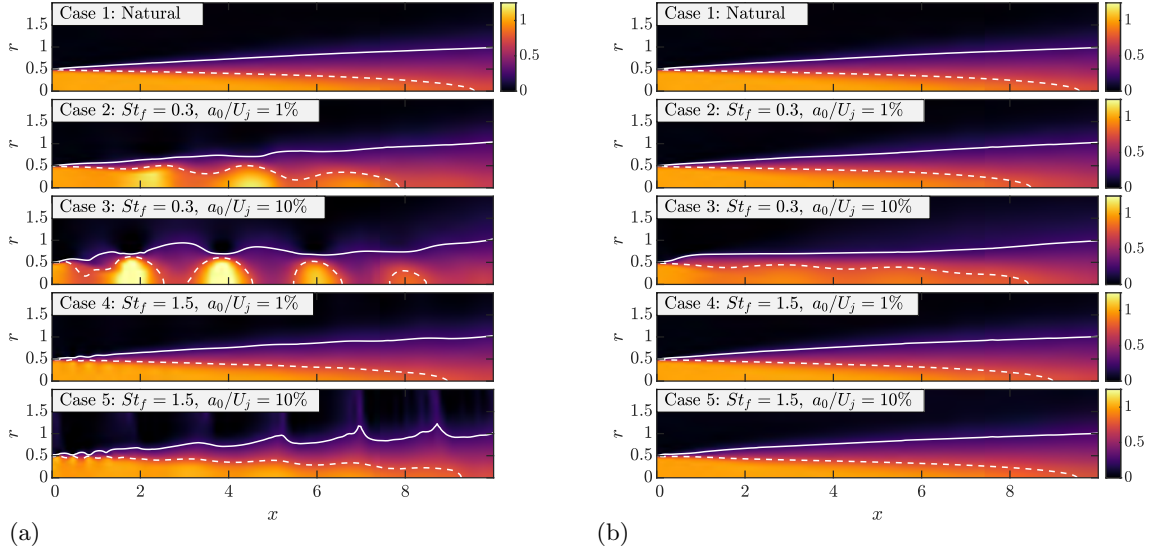


Figure 23: Contours of the mean (phase-averaged) axial velocity $\bar{u}_x$ at $\theta_f = 0$ (a) and the $f = 0$ component (period-averaged component) of the axial velocity $\bar{u}_{x,0}$ (b) for the subsonic jet. The solid and dashed white lines correspond to contours of $\bar{u}_x = 0.25, 0.75$ and $\bar{u}_{x,0} = 0.25, 0.75$ for (a) and (b), respectively.

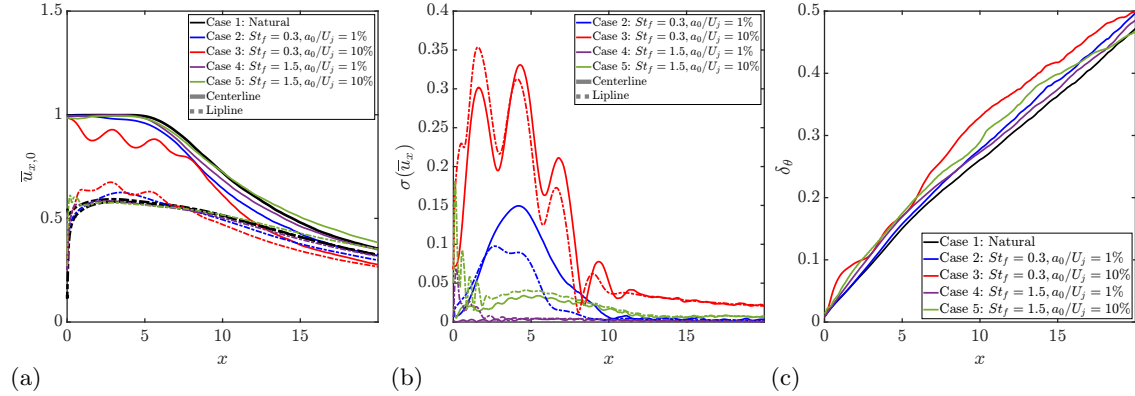


Figure 24: Period-averaged axial velocity $\bar{u}_{x,0}$ (a) and standard-deviation of the mean axial velocity $\sigma(\bar{u}_x)$ (b) along the centerline and lipline, and momentum thickness δ_θ (c) of the subsonic jet.

3.3.2 Second-order statistics

In Fig. 25a, we display contours of the azimuthally averaged root-mean-square (RMS) of the axial velocity $\{u_x''u_x''\}$ at $\theta_f = 0$. The $St_f = 0.3$ forcing results in a modulation of the RMS that has a maximum at the end of the potential core, which corresponds to the maximum modulation of the mean. The effect for $St_f = 0.3$, $a_0/U_j = 1\%$ is mild, whereas for $St_f = 0.3$, $a_0/U_j = 10\%$, a large amount of turbulence is present within the advecting vortices. Similar to the mean field, the effect of the $St_f = 1.5$ forcing is limited to the near-nozzle region with no strong phase-dependency observed after around 2 diameters downstream for both forcing amplitudes.

In Fig. 25b, we plot the period-averaged axial velocity RMS along the centerline and lipline. We see that both $St_f = 0.3$ forcings result in a minor upstream shift of the centerline RMS which

is expected given the minor shortening of the jet previously observed. Additionally, an increase in the peak centerline and lipline RMS is observed, indicating that the period-averaged intensity of the turbulence has been increased. A more substantial effect on the lipline RMS is seen, with a large decrease in the RMS for the first 4 diameters and then a substantial increase between $x \in [6, 11]$ diameters downstream. This decrease in the period-averaged RMS is due to the mean field (and thus the shear layer) being modulated towards and away from the centerline as a function of the phase of the forcing, thereby resulting in a lower period-averaged value. The substantial increase in the RMS at $x \in [5, 11]$ is likely caused by the highly energetic vortices generated by the forcing. These vortices advect downstream while breaking down, thereby creating a substantial amount of turbulent energy. For the $St_f = 1.5$, $a_0/U_j = 10\%$ forcing, the RMS contours clearly show the presence of vortex roll-up. However, in contrast to the $St_f = 0.3$ case, these vortices are smaller in size and are spatially restricted to $x \in [0, 2]$. Subsequently, these vortices more rapidly break down and do not result in an increase in peak RMS along the centerline. Instead, a slight decrease in the peak centerline RMS is observed (and a slight shift upstream for the $a_0/U_j = 10\%$ forcing). Along the lipline, an increase is observed for $x \in [0, 2]$ corresponding to where the modulation of the $St_f = 1.5$ forcing is dominant.

In Fig. 26, we display the azimuthally averaged Wigner-Ville spectrum of the axial velocity at $x = 1.25, 6$, $r = 0.5$, which shows how the PSD of u_x'' at these locations varies as a function of the phase of the forcing. These locations were chosen to approximately correspond to the location where the RMS is most phase-dependent for the $St_f = 0.3, 1.5$ forcing. The natural jet is known to be statistically stationary, resulting in a power spectrum that is, on average, constant. On the other hand, based on our preceding analysis, it is known that the forcing results in phase-dependent statistics. For the natural jet, we compute the Wigner-Ville spectrum *assuming* a forcing frequency of $St_f = 0.25$ to determine the level of statistical convergence. To compute the spectral estimates, a block length of N_{fft} is used such that $\Delta St \approx 0.05$ for each of the cases. To minimize the variance and reduce cyclic leakage, a phenomenon similar to spectral leakage, an overlap of 67% is used, which is slightly higher than previous studies of natural turbulent jets [Towne et al., 2018b, Heidt et al., 2021] and that used in SPOD-based data reconstruction [Nekkanti and Schmidt, 2023]. In general, α_0 may not equal St_f thus, similar to Heidt and Colonius [2023], we scan over all possible values of α_0 and confirm that the fundamental cycle frequency present is indeed the forcing frequency, giving $\alpha_0 = St_f$. We consider cycle frequencies up to $\alpha_{max} = 4.5$, resulting in $k_{max} = 15, 3$ for the $St_f = 0.3, 1.5$ forcing, respectively.

For both locations, we see that the natural jet has no phase dependency indicating that a sufficient level of statistical convergence has been achieved. We clearly observe the localization of the $St_f = 1.5$ forcing and the more global impact of the $St_f = 0.3$ forcing. The $St_f = 0.3$ forcing is seen to have a substantial impact at both locations, while the $St_f = 1.5$ forcing has a minimal impact at $x = 6$, $r = 0.5$, agreeing to the RMS and mean flow figures shown previously. Looking at the mean axial velocity at $x = 6, r = 0.5$ in Fig. 23 for the $a_0/U_j = 1\%, 10\%$ cases, we see that at $\theta_f = 0^\circ$ the $a_0/U_j = 1\%$ forcing case is currently low-velocity and the $a_0/U_j = 10\%$ forcing case is currently high-velocity, showing that the high-velocity regions correspond to the more energetic regions in the Wigner-Ville spectrum. A substantial increase in the amount of energy present at the higher frequencies during these high-energy phases is also seen, which is linked to the modulation of the shear layer and passage of the vortices. This clearly demonstrates the benefit of the cyclostationary framework since we are able to explore the statistical decomposition of phase-dependent turbulent structures.

Overall, it is seen that the $a_0/U_j = 1\%$ cases result in a mild modulation, and a large amplitude is required ($a_0/U_j \gg 1\%$) to substantially modulate the statistics. Additionally, except for the $St_f = 0.3$, $a_0/U_j = 10\%$ case, the effect on the statistics was minor.

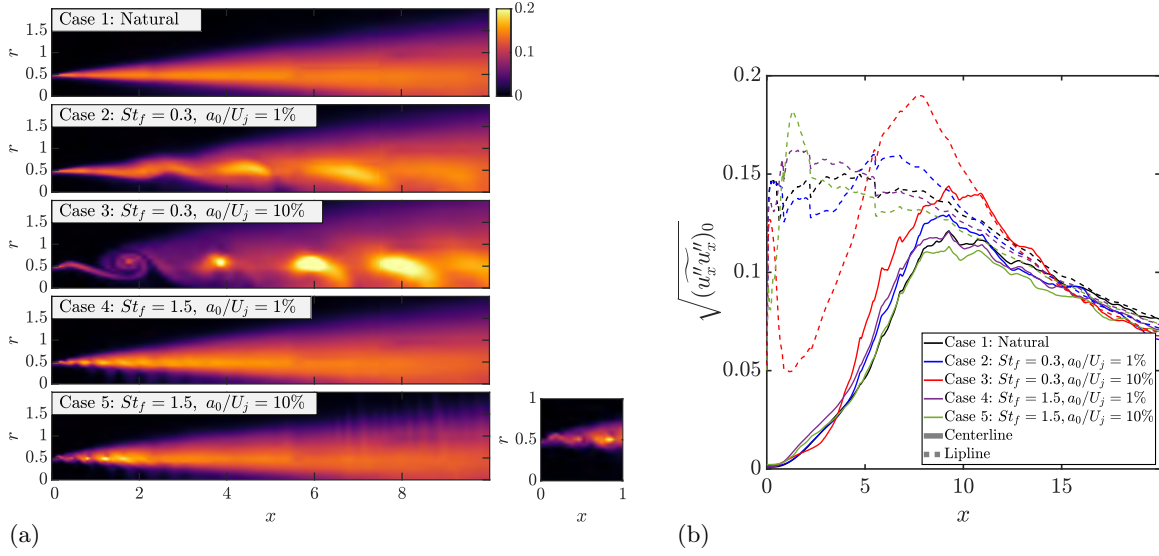


Figure 25: RMS contours $\sqrt{\langle u''_x u''_x \rangle}$ at $\theta_f = 0$ (a) and period-averaged RMS $\sqrt{\langle u''_x u''_x \rangle_0}$ along the centerline and lipline (b) for the subsonic jets.

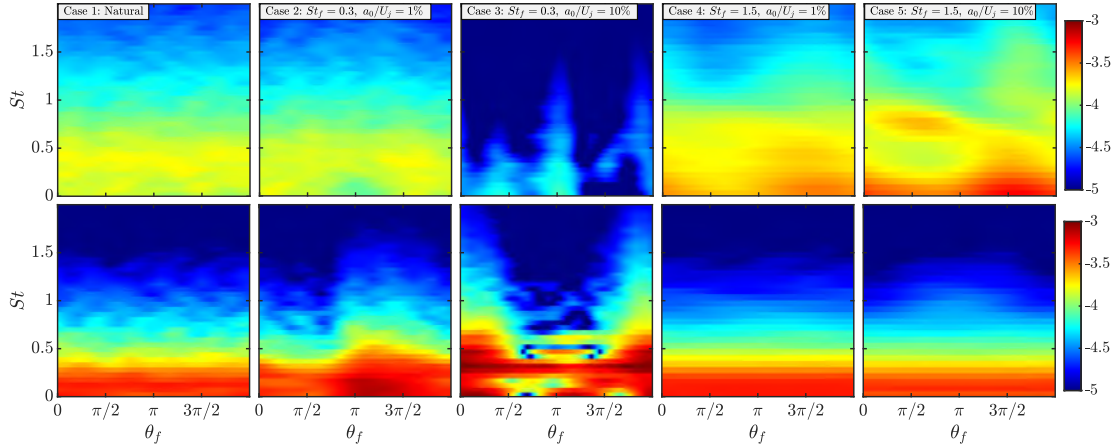


Figure 26: Wigner-Ville spectrum of u''_x at $x = 1.5, r = 0.5$ (top) and $x = 5, r = 0.5$ (bottom) for the subsonic jets, Log10 scale.

3.3.3 Energy transfer

We now quantify the interaction between the mean field and the turbulent field. To study this, we compute the energy transfer term

$$\tilde{E}(x, r, \theta, t) = -\overline{\rho u''_i u''_j} \frac{\partial \tilde{u}_i}{\partial x_j}, \quad (98)$$

which represents the production of turbulent kinetic energy by the action of the mean field $\tilde{u}_i$ against the turbulent Reynolds stresses $\overline{u''_i u''_j}$. In Fig. 27, we display the azimuthally averaged energy transfer at a phase $\theta_f = 0, \pi/3, 2\pi/3$, i.e. $\tilde{E}(x, r, \theta, \theta_f)$. As expected, the $St_f = 0.3$ forcing

results in energy transfer towards the end of the potential core, while the $St_f = 1.5$ is localized to $x \in [0, 2]$. Similar to the RMS contours, the modulation of the energy transfer for the $a_0/U_j = 1\%$ forcing cases is relatively minor. For the $a_0/U_j = 10\%$ forcing cases, the modulation is substantial, and the energy transfer from the vortex breakdown is clearly observed.

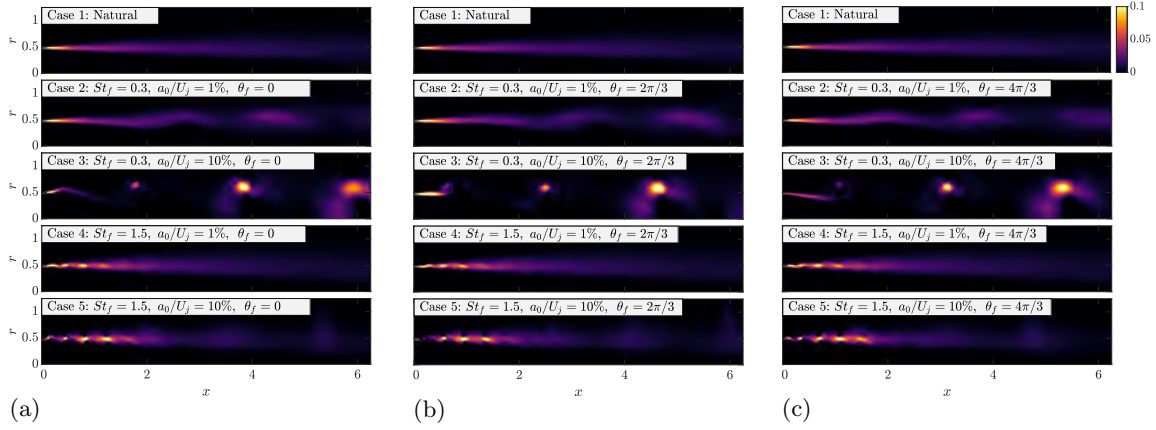


Figure 27: Energy transfer contours $\tilde{E}(x, r, \theta, t)$ for the subsonic jets at $\theta_f = 0$ (a), $\theta_f = 2\pi/3$ (b), $\theta_f = 4\pi/3$ (c).

Next, we seek to determine the global impact that the forcing has on the energy transfer. We define the cumulative energy transfers by integrating in r and θ ,

$$\tilde{E}_r(x, t) = \int_r \int_\theta \tilde{E}(x, r, \theta, t) r \partial r \partial \theta, \quad (99)$$

and then, additionally, in x ,

$$\tilde{E}_{rx}(x, t) = \int_{x'=0}^x \int_r \int_\theta \tilde{E}(x', r, \theta, t) r \partial r \partial \theta \partial x'. \quad (100)$$

In Fig. 28, we display both $\tilde{E}_r(x, t)$ and $\tilde{E}_{rx}(x, t)$. Note that for the natural jet, the vertical lines occurring at $x \approx 0.6, 2.2, 5.5$ are artifacts of transitions in the LES grid resolution. For the $St_f = 0.3$ cases, at $x \in [0, 10]$, we clearly observe a strong phase-dependent energy transfer. We also observe the axial width and magnitude of the phase-dependent high-energy transfer region increase and then decrease as the degree of mean flow and mean RMS modulation increases and then decreases further downstream. For the $St_f = 1.5$ cases, a similar phenomenon occurs but is spatially restricted to the first 2 diameters downstream, in line with our mean flow observations. After this, another phase-dependent energy transfer phenomenon occurs with a spatial length of ≈ 1.66 , which is exactly the distance that the $St_f = 1.5$ acoustic waves travel in a quiescent media over one forcing cycle. This is linked to a direct nonlinear interaction between the forced acoustic waves and the natural perturbations present in the jet on the outer regions of the jet (where minimal shear is present, resulting in unit phase speed).

Now looking at $\tilde{E}_{rx}(x, t)$, we see that despite the strong local modulation of energy transfer, the integrated energy transfer is remarkably similar for all cases. Only the strongest forcing ($St_f = 0.3$, $a_0/U_j = 10\%$) results in modest phase-dependence of the energy transfer. In Fig. 29, we display for $\tilde{E}_{rx}(x, t)$ both the period-averaged minimum and maximum values over the period. We see a similar result, in that only the 10% forcing results in any appreciable modulation in the total energy transfer, with the 1% forcing cases exhibiting minimal phase dependency. Even with the 10%

forcing, the variation as a function of phase is minor after a region extending to about $x \approx 1$ for $St_f = 1.5$ and $x \approx 5$ for $St_f = 0.3$.

These results imply that at a given phase, the higher energy transfer linked to the high-velocity regions is mostly canceled out by the lower energy transfer linked to the low-velocity regions. We also observe that after $x = 2, 0.7$ for the $St_f = 0.3, 1.5$ forcing, the change in the period-averaged energy transfer for the forced jets is large in comparison to the maximum variation as a function of the phase of the forcing. This supports the hypothesis that the primary change in turbulence is driven by a corresponding change to the period-averaged mean, agreeing with previous literature [Koenig et al., 2016, Sinha et al., 2018]. Furthermore, we note that the impact of the 10% forcing is not an order of magnitude greater than the impact of the 1% forcing (except in the near-nozzle region). This is due to nonlinear interaction inhibiting the growth rate of the 10% wave, thereby resulting in a reduced impact on the mean field and energy transfer. This demonstrates that a naive increase in the forcing amplitude is not substantially beneficial in modifying the mean/turbulence of a turbulent jet, especially when contending with tonal actuator noise that has reduced the effectiveness of active control devices Samimy et al. [2010], Sinha et al. [2018], Koenig et al. [2016].

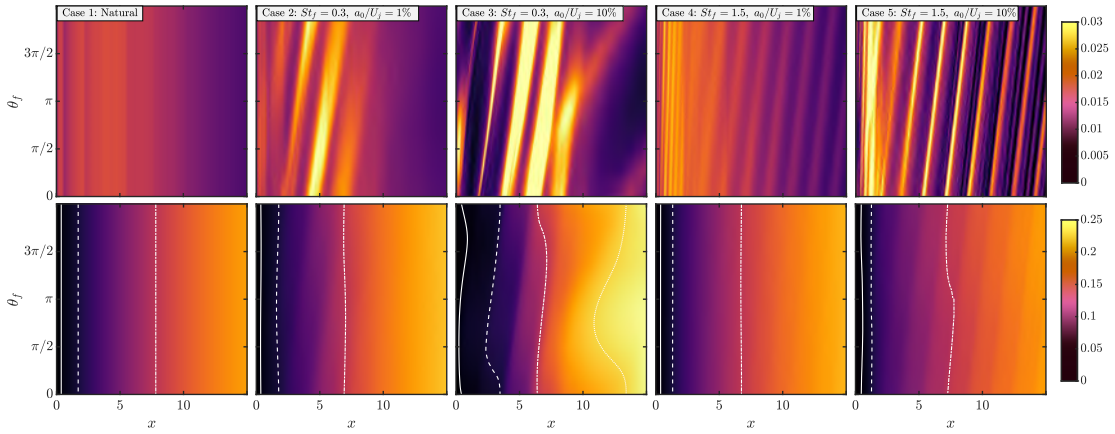


Figure 28: Contours of $\tilde{E}_r(x,t)$ (top) and $\tilde{E}_{rx}(x,t)$ (bottom) as a function of the phase of the forcing θ_f and axial location x for the subsonic jets. The white lines overlaid on the $\tilde{E}_{rx}(x,t)$ contours represent lines of $[2, 10, 50, 90]\% \times 0.25$ energy transfer.

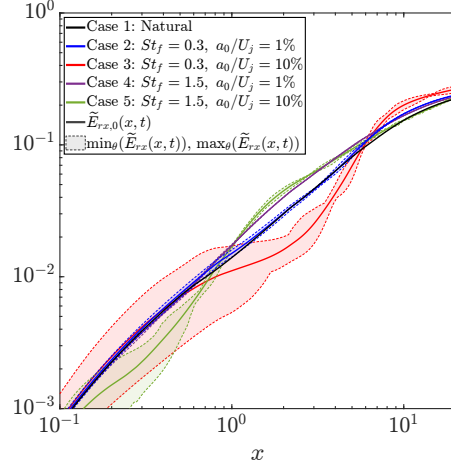


Figure 29: Comparison of $\tilde{E}_{rx,0}(x,t)$ and the maximum fluctuation of $\tilde{E}_{rx}(x,t)$ for the subsonic jets.

3.3.4 CS-SPOD

Lastly, we explore the evolution of the coherent structures in the turbulent (non-phase-locked) response of these forced jets. While the preceding analysis has indicated that changes to the turbulence are minor, alteration of the coherent structures could have an outsized impact on the radiated acoustic field [Jordan and Colonius, 2013b]. This means that a reduction in the strength of these structures, even if the rest of the turbulence is relatively unimpeded (or even amplified), could lead to substantial reductions in jet noise.

Similar to the computation of the Wigner-Ville spectrum, for each case, we employ a block length of N_{fft} such that $\Delta St \approx 0.025$, an overlap of 67%, and we set a_2 such that $f_{max} = 4.5$, where f_{max} is the maximum frequency that CS-SPOD captures (i.e. the maximum frequency in Ω_γ). Due to the cyclostationary nature of the forced jets, the comparison between different forcing frequencies is more difficult. Each CS-SPOD mode contains the set of frequencies $\Omega_\gamma = \{ \dots, \gamma - 2\alpha_0, \gamma - \alpha_0, \gamma, \gamma + \alpha_0, \gamma + 2\alpha_0, \dots \}$, and thus, when the forcing frequency changes (i.e. α_0 changes), the frequencies contained within Ω_γ also change. This makes a direct frequency-by-frequency comparison, as often performed when using SPOD, impossible. Thus, to keep the analysis between the natural and forced jets as direct as possible, we also compute CS-SPOD of the natural jets, by *assuming* a forcing frequency. One way to think of this is as the limit of forcing at a particular frequency as the amplitude of forcing approaches zero. Theoretically, CS-SPOD modes performed on statistically stationary data (i.e. the natural jet) are identical to SPOD modes where CS-SPOD ranks the energy of all modes computed over the frequency set Ω_γ . However, this is not true numerically due to imperfect statistical convergence.

We first investigate the statistical convergence of CS-SPOD. Thus, we compute CS-SPOD on the natural jet with an assumed forcing frequency of $St_f = 0.3, 1.5$ as well as regular SPOD. For the natural jet, since SPOD and CS-SPOD are theoretically identical for a statistically stationary flow, we would expect that the CS-SPOD modes contain just a single frequency component at a frequency equal to the component in Ω_γ with maximum energy. Since the energy spectrum of a jet is approximately monotonically decaying for increasing f , we would expect the frequency maximum energy component in Ω_γ to be $f = \gamma$. For example, for $St_f = 0.3$ and $\gamma = 0.15$, $\Omega_\gamma = [\dots, -0.75, -0.45, -0.15, 0.15, 0.45, 0.75, \dots]$, thus, we would expect the mode to be at a frequency of $f = \pm 0.15$. Similarly, for $St_f = 1.5$ and $\gamma = 0.15$, we would expect the mode to again be at $f = \pm 0.15$.

To assess this proposition, in Fig. 30a, we display the eigenspectrum of the dominant mode as a function of γ, f for the natural jet computed with CS-SPOD assuming a forcing frequency

of $St_f = 0.3, 1.5$ and computed using SPOD. Note that only $\gamma = [0, \alpha_0/2]$ is needed since the $m = 0$ data is real, resulting in a symmetric spectrum. Overall, we see good agreement between the CS-SPOD eigenspectrum with the two assumed forcing frequencies and SPOD. The $St_f = 0.3$ assumed forcing frequency results in a larger deviation from the SPOD eigenspectrum than the $St_f = 1.5$ assumed forcing frequency. This is for two reasons. First, more frequency components are linked together for $St_f = 0.3$, increasing the opportunity for statistical noise to influence the results. Second, since the frequency components are separated by St_f , they are closer in energy separation for $St_f = 0.3$ than $St_f = 1.5$ and thus have a greater effect on the statistical convergence. This is also seen in Fig. 30b, where we display the pressure component of the dominant CS-SPOD ($\phi_p(\theta_f)$) at $\gamma, f = 0.15$ for the same cases. It is seen that the $St_f = 0.3$ assumed forcing frequency CS-SPOD mode is more polluted with spectral content from other frequencies than the $St_f = 1.5$ case. Thus, care should be taken when interpreting these modes to ensure that statistical artifacts are not misinterpreted.

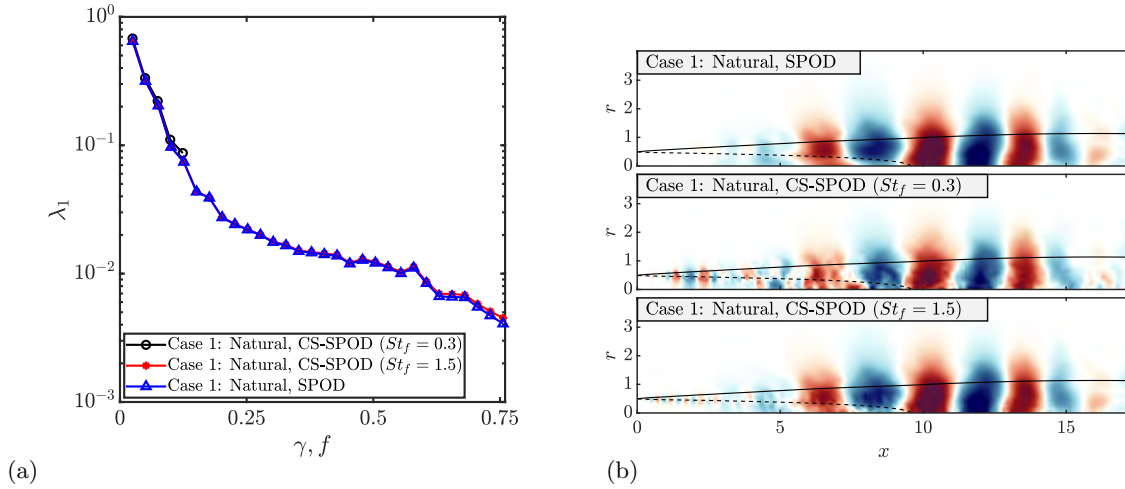


Figure 30: Comparison of the dominant mode eigenspectrum (a) and real component of the pressure ($\phi_p(\theta_f = 0)$) at $\gamma, f = 0.15$ at $\theta_f = 0$ (b) of the natural subsonic jet computed using SPOD, CS-SPOD ($St_f = 0.3$), and CS-SPOD ($St_f = 1.5$). Contour limits are given by $\pm 0.75|\phi_p(\theta_f)|_\infty$, where $\phi_p(\theta_f)$ is the current mode being displayed.

Moving to the results for the forced jets, we first investigate the impact of the $St_f = 0.3$ forcing. In this study, we will focus on the effect of the forcing on the axisymmetric $m = 0$ mode only. In Fig. 31, we display the CS-SPOD eigenspectrum of the two $St_f = 0.3$ forced jets and the natural jet where a $St_f = 0.3$ has been artificially imposed as discussed above. We observe, for all γ , an increasing amount of energy in the dominant modes for increasing levels of forcing. In order to determine how the forcing is impacting the generated coherent structures, we display the pressure component of the dominant CS-SPOD mode for the same cases for $\gamma = 0.05, 0.15$ in Fig. 32. Since CS-SPOD modes have periodic amplitude in time, in Fig. 32, we display all modes at $\theta = 0$, and then in Fig. 33, we show the temporal evolution of the dominant CS-SPOD for the $St_f = 0.3$, 10% forced jet at $\gamma = 0.15$. We have also overlaid contours of $\tilde{u}_x = 0.25, 0.75$ to demonstrate how the coherent structures are modulated by the phase-dependent mean velocity.

By looking at the CS-SPOD modes, we note a few salient features. First, all modes are Orr-type modes, which is consistent with the literature for this $m-St$ pair [Pickering et al., 2020], which shows that the dominant mechanism has not changed. Second, we see an upstream shift in the location of the Orr mode corresponding to the change in the time-averaged field. Third, as the forcing

amplitude is increased, we observed a strong phase-dependent modulation of the dominant Orr-type mode where the highest-amplitude region of the mode is slaved to the greatest velocity/shear present in the jet. This phase-dependent modulation of the dominant mode is clearly observed in Fig. 33.

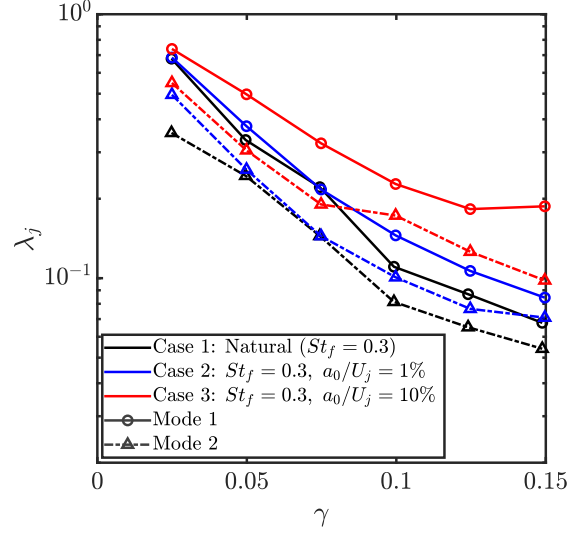


Figure 31: Comparison of the CS-SPOD eigenspectrum of the subsonic natural jet and the two forced jets with $St_f = 0.3$.

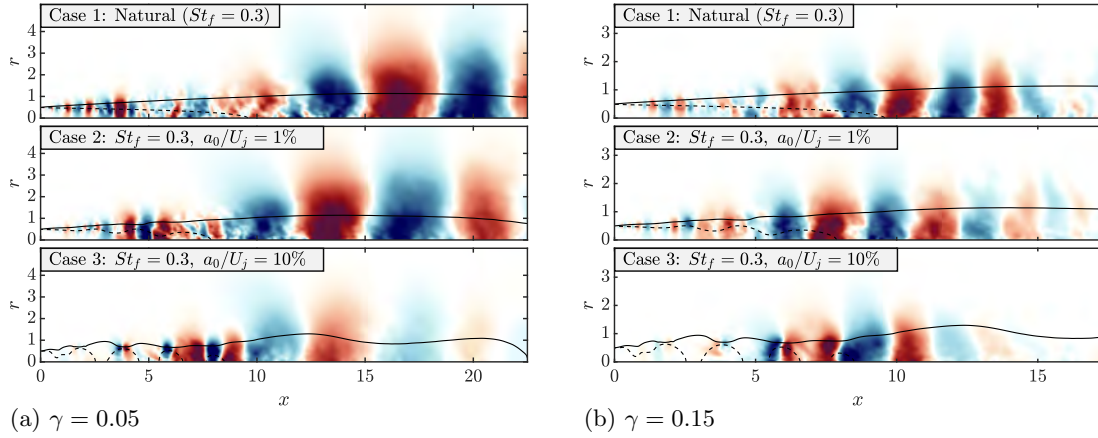


Figure 32: Comparison of the real component of pressure component of the dominant mode at $\gamma = 0.05, 0.15$ of the natural subsonic jet and the two $St_f = 0.3$ forced jets at $\theta_f = 0$. Contour limits are given by $\pm 0.75|\phi_p(\theta_f = 0)|_\infty$.

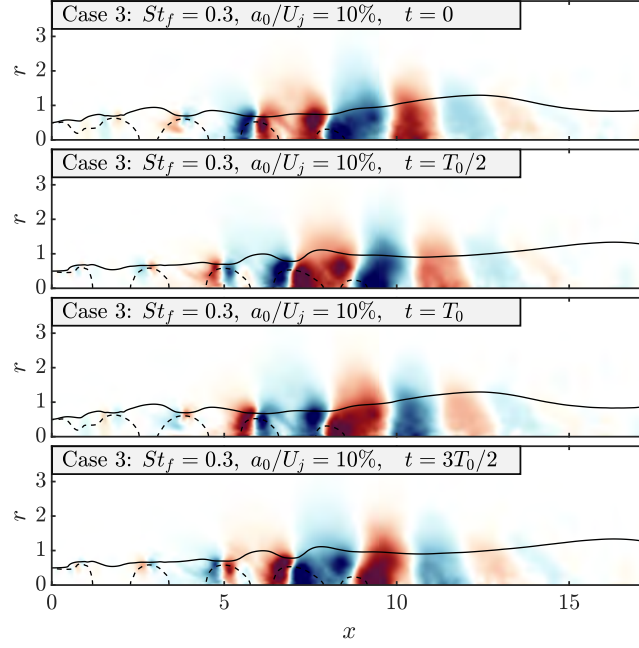


Figure 33: Comparison of the real component of pressure component of the dominant mode at $\gamma = 0.15$ of the $St_f = 0.3$, $a_0/U_j = 10\%$ forced subsonic jet at multiple several instances. Contour limits are given by $\pm 0.75|\phi_p(\theta_f = 0)|_\infty$.

Next, we investigate the impact of the $St_f = 1.5$ forcing. In Fig. 34, we display the CS-SPOD eigenspectrum of the two $St_f = 1.5$ forced jets and the natural jet where a $St_f \approx 1.5$ has been assumed. The spectrum is relatively unaffected by the forcing for approximately $\gamma \in [0, 0.2]$, with a minor decrease in the spectrum for the $a_0/U_j 10\%$ forcing for $\gamma \in [0, 0.05]$. For $\gamma \in [0.2, 0.6]$, we observe a decrease in the eigenspectrum, which continues to decrease for an increased forcing amplitude. Lastly, for approximately $\gamma \in [0.6, 0.75]$, we observe a substantial increase in the energy. Thus, similar to previous studies, our results indicate that high-frequency forcing results in a more stable flow that reduces the amplification of coherent structures. The exception to this is for $\gamma \in [0.6, 0.75]$, where a strong amplification is seen for both forcing amplitudes. A previous study [Heidt et al., 2021] linked this to a direct non-linear phenomenon (i.e. a change in the turbulence that is not attributable to a change in the mean).

In Fig. 35, we display the pressure component of the dominant CS-SPOD mode for the same cases at $\gamma = 0.15$, 0.475 , and 0.75 , which approximately corresponds to the location of no-change, maximum reduction, and maximum amplification seen in the eigenspectrum. We expect that the dominant mechanics at $\gamma = 0.475$, 0.75 to be the Kelvin-Helmholtz mechanism [Pickering et al., 2020]. Overall, minor changes to the dominant modes are observed. For $\gamma = 0.15$, the forcing results in a slight increase in the length of the Orr mode. The direct nonlinear interaction between the forcing and naturally coherent structures is described in Secs. 3.3.1 and 3.3.3 is clearly observed for the $a_0/U_j = 10\%$ forcing. For $\gamma = 0.475$, no substantial change in the mode is observed, while for $\gamma = 0.75$, a reduction in the length of the Kelvin-Helmholtz mode is observed for the forced jet. For $\gamma = 0.15$, 0.475 , no strong phase-dependent modulation in the dominant modes is observed even for the 10% forcing, while for $\gamma = 0.75$, some phase-dependency is observed for the 10% forcing in the near-nozzle region. This is likely due to the dominant wavelength present in the $\gamma = 0.15$, 0.475 modes being much greater than the wavelength of the mean flow deformation caused by the $St_f = 1.5$. This mismatch in the wavelength means that the coherent structures do not react to this modulation and thus evolve independently to this phase-dependent modulation.

In contrast, for $\gamma = 0.75$, the primary wavelength is much more similar, resulting in a phase-dependent modulation. Additionally, the spatial support of these longer wavelength structures is much further downstream than the spatial support of the mean flow deformation. Furthermore, [Heidt and Colonius \[2023\]](#) showed that when there is a large energy difference at the different frequencies in Ω_γ , CS-SPOD can be approximated as SPOD. This is precisely the case for the current case for low γ , meaning that the modes can be well approximated by their SPOD versions.

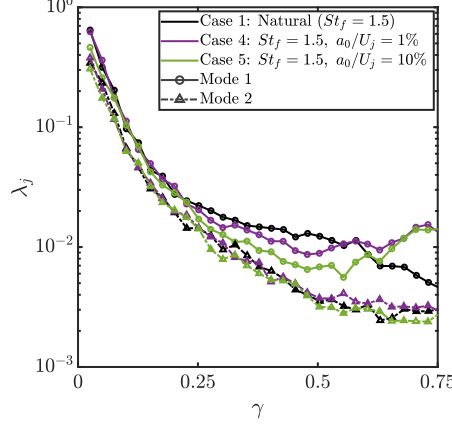


Figure 34: Comparison of the CS-SPOD eigenspectrum of the subsonic natural jet and the two forced jets with $St_f = 1.5$.

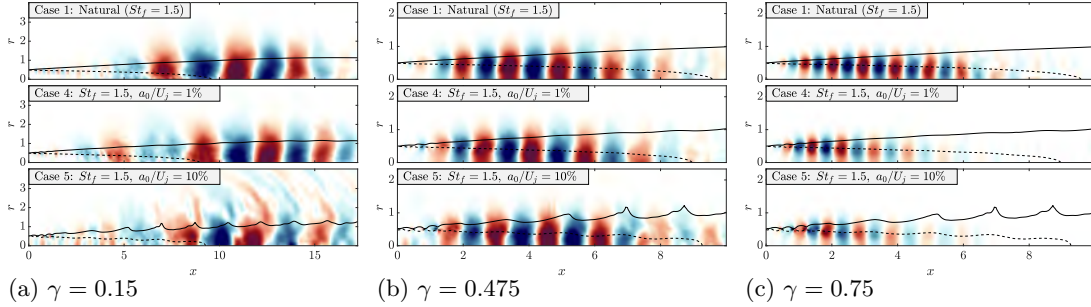


Figure 35: Comparison of the real component of pressure component of the dominant mode at $\gamma = 0.15, 0.475, 0.75$ of the natural subsonic jet and the two $St_f = 1.5$ forced subsonic jets at $\theta_f = 0$. Contour limits are given by $\pm 0.75|\phi_p(\theta_f = 0)|_\infty$.

3.4 Supersonic jet

To determine the effect of Mach number, we perform a similar analysis on a Mach 1.5 supersonic jet. We have limited our forcing amplitude to 1% to study a realistic forcing amplitude only. In Fig. 36, we display the mean axial velocity corresponding to a forcing phase $\theta_f = 0^\circ$ and the period-averaged axial velocity for the supersonic jets. Visually, we see that the impact of the $St_f = 0.3$ forcing on the centerline mean axial velocity and period-averaged axial velocity is lower for the supersonic jet than the subsonic. Quantitatively, in Fig. 37, the supersonic jet subjected to the $St_f = 0.3$ forcing results in a standard deviation of 10% and 9.5% along the lipline and centerline, respectively. In contrast, the subsonic jet subjected to the same forcing results in a standard deviation of 9.8% and 15%. For the $St_f = 1.5$ forcing, a slight increase in the lipline response is seen with the subsonic and

supersonic jets having a standard deviation of 7.0% and 8.5%, respectively. It is also observed that for the supersonic jet, the $St_f = 0.3$ forcing results in a delayed decrease in the centerline velocity and a delayed increase in the momentum thickness, while a rapid decrease in the centerline velocity and rapid increase in the momentum thickness is seen for the $St_f = 1.5$ forcing. This is in contrast to the subsonic jet where the $St_f = 0.3$ forcing resulted in a rapid decrease in the centerline velocity and a delayed increase in the momentum thickness, while the $St_f = 1.5$ forcing resulted in a delayed decrease in the centerline velocity and rapid increase in the momentum thickness.

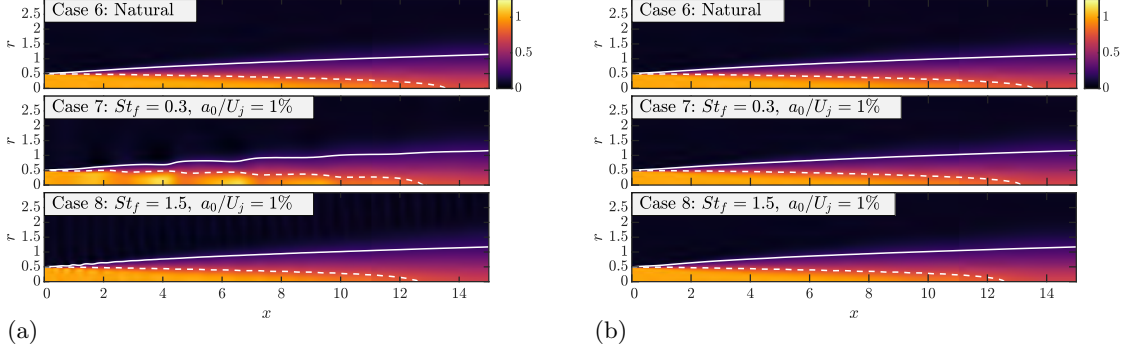


Figure 36: Contours of the mean axial velocity $\bar{u}_x$ at $\theta_f = 0$ (a) and the $f = 0$ component (period-averaged component) of the axial velocity $\bar{u}_{x,0}$ (b) for the supersonic jet. The solid and dashed white lines correspond to contours of $\bar{u}_x = 0.25, 0.75$ and $\bar{u}_{x,0} = 0.25, 0.75$ for (a) and (b), respectively.

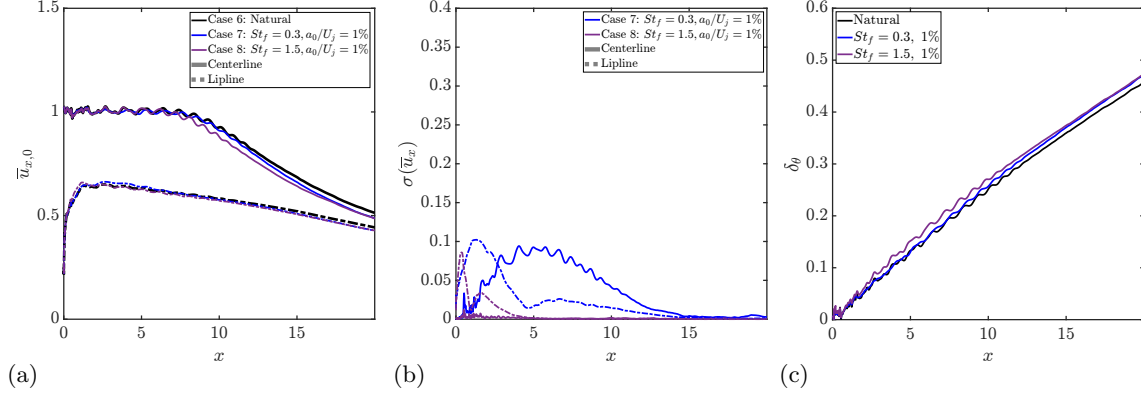


Figure 37: Period-averaged axial velocity $\bar{u}_{x,0}$ (a) and standard-deviation of the mean axial velocity $\sigma(\bar{u}_x)$ (b) along the centerline and lipline, and momentum thickness δ_θ (c) of the supersonic jet.

We display the RMS at $\theta_f = 0$ for the supersonic jet in Fig. 38a. We see that the impact on the RMS is similar to the subsonic jet. The $St_f = 0.3$ forcing results in a modulation of the RMS that increases in magnitude until approximately the end of the potential core, while the $St_f = 1.5$ forcing is more localized to the nozzle exit at $x \in [0, 2]$. In Fig. 38b, we see that the $St_f = 0.3$ forcing results in a slight decrease in the period-averaged RMS in the near-nozzle lipline region and a slight increase in the peak RMS towards the end of the potential core. The $St_f = 1.5$ forcing results in a substantial increase in the RMS in the near-nozzle lipline region but a similar peak RMS profile along the centerline that has been shifted upstream by about 1 diameter.

The impact on the energy transfer between the mean flow and turbulence is also similar to that of the subsonic jet, as seen in Figs. 39a and 39b. Like the RMS contours, the greatest modulation

of the energy transfer is towards the end of the potential core for the $St_f = 0.3$ forcing and near the nozzle exit for the $St_f = 1.5$ forcing. Similar to the subsonic jet, the $St_f = 0.3$ has minimal effect on the period-averaged integrated energy transfer, while the $St_f = 1.5$ forcing results in a slight increase in period-averaged integrated energy transfer in $x \in [1, 10]$. Furthermore, the maximum modulation of the integrated energy transfer as a function of phase is minimal. This again indicates that any increased regions of energy transfer are canceled out by a corresponding region of reduced energy transfer and that the primary driver of change to the turbulence is a change in the mean field.

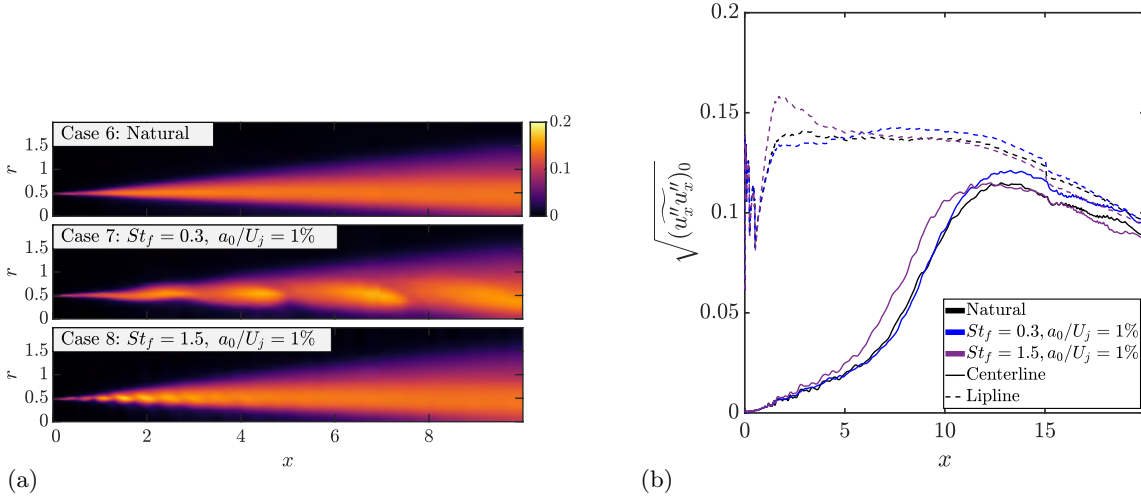


Figure 38: RMS contours $\sqrt{\widetilde{(u''_x u''_x)}}$ at $\theta_f = 0$ (a) and period-averaged RMS $\sqrt{\widetilde{(u''_x u''_x)_0}}$ along the centerline and lipline (b) for the supersonic jets.

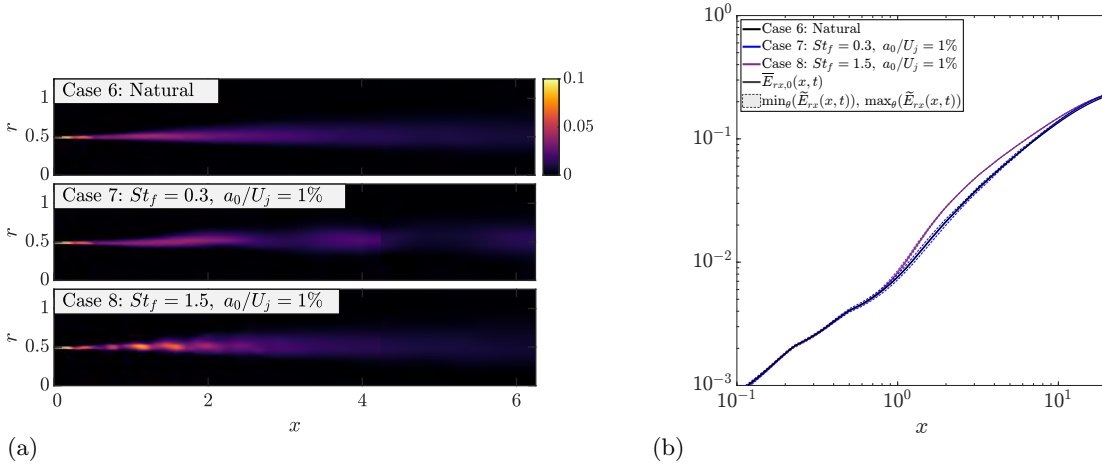


Figure 39: Energy transfer contours $\widetilde{E}(x, r, \theta, t)$ at $\theta_f = 0$ (a) and comparison of $\widetilde{E}_{rx,0}(x, t)$ and the maximum fluctuation of $\widetilde{E}_{rx}(x, t)$ (b) for the supersonic jets.

Lastly, we investigate the impact of the forcing on the coherent structures present in the supersonic jet using CS-SPOD. In Fig. 40, we display the CS-SPOD eigenspectrum for the forced jets. Both the $St_f = 0.3$ and $St_f = 1.5$ forcing result in a minor change to the spectrum over all values of

γ . Unlike the subsonic jet, no general modification of the spectrum is seen. This is also seen in Fig. 41, where we display the pressure component of the dominant CS-SPOD mode at $\gamma = 0.15$ for the $St_f = 0.3$ forced jet and $\gamma = 0.75$ for the $St_f = 1.5$ forced jet. For the $St_f = 0.3$ forcing, we observe almost no change to the mode, and for $St_f = 1.5$, we observe a contraction in the length of the mode. For $St_f = 1.5$, in the far-field, we can clearly see the direct non-linear interaction between the forcing and the super directivity waves. Overall, it is clear that the differences are smaller for the supersonic jet than the subsonic jet for the $St_f = 0.3$ forcing. In contrast, the $St_f = 1.5$ forcing has a similar impact to the subsonic jet.

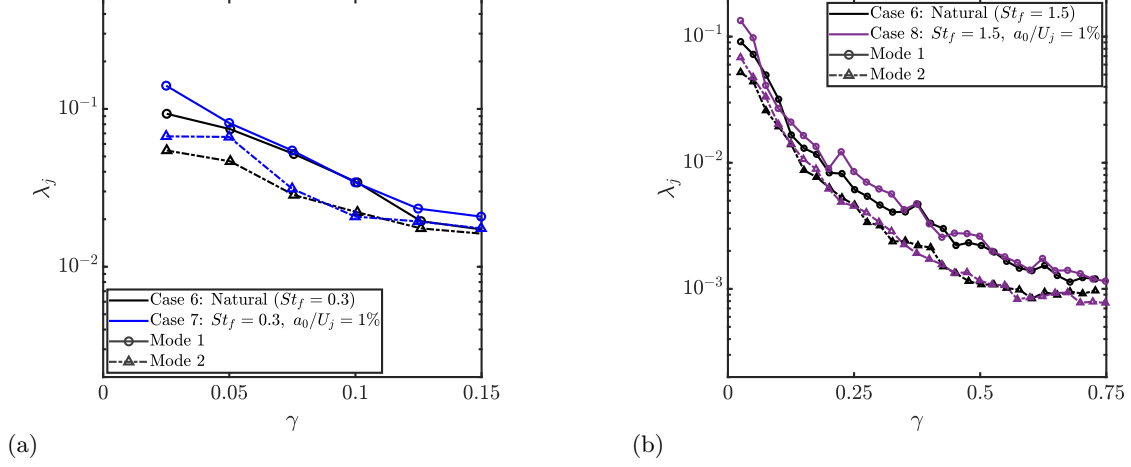


Figure 40: Comparison of the CS-SPOD eigenspectrum of the supersonic natural jet and the $St_f = 0.3$ (a) and $St_f = 1.5$ (b) forced supersonic jets.

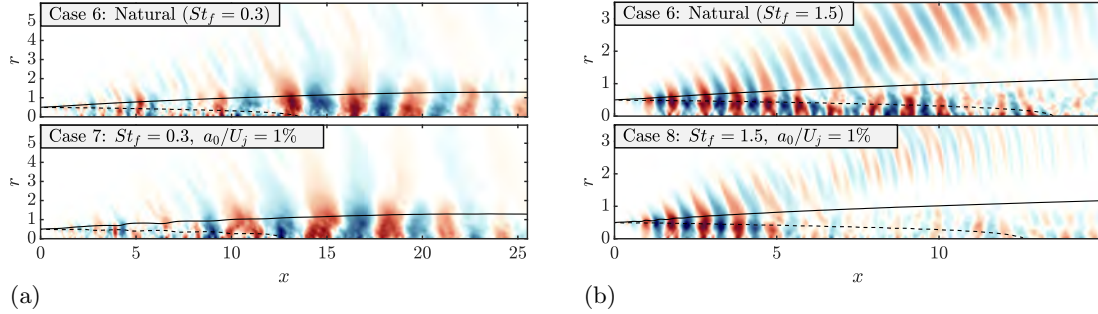


Figure 41: Comparison of the real component of pressure of the dominant mode at $\gamma = 0.15$ for the natural and $St_f = 0.3$ forced supersonic jets (a) and at $\gamma = 0.75$ for the natural and $St_f = 1.5$ forced supersonic jets. Contour limits are given by $\pm 0.75|\phi_p(\theta_f = 0)|_\infty$.

4 GPU-accelerated large-eddy simulations of supersonic jets from twin rectangular nozzle

Large eddy simulations of supersonic jets from a twin rectangular nozzle are performed with the GPU-accelerated version of Cascade’s “charLES” compressible flow solver. Comparisons of the numerical noise predictions with near-field and far-field microphone measurements show good agreement for the screech tone frequency, broadband spectra and overall sound pressure levels. For the

screech tone amplitude, the agreement is further improved when the reflective surfaces upstream of the nozzle exit are modified in the computational domain to account for the thick acoustic foam covering these surfaces in the wind-tunnel. Such modifications and efficient exploration of the nozzle design space are made possible by the increased computational throughput with the GPU-accelerated solver: for the present $\mathcal{O}(100)$ million cell mesh and relatively long noise data collection of 2000 acoustic time units, the simulation results are obtained in about 12 hours on 30 standard GPUs.

4.1 Introduction

Despite intensive research, the reduction of jet engine noise remains a theoretically and technologically challenging problem. Detailed measurements in full-scale engines are costly and difficult, and most lab facilities are limited to smaller scale jets at lower temperatures. While simulations can now accurately predict the acoustic radiation from complex nozzles at realistic conditions, reducing time-to-solution and computational costs are a priority to bring nozzle design optimization within reach [Brès and Lele \[2019\]](#).

One potential path forward is to utilize recent developments in high performance computing (HPC) where accelerated architectures (e.g., graphics processing units (GPUs), co-processing units) have become increasingly prevalent in contemporary HPC systems. Some jet simulations have already been performed on GPUs by Markesteijn, Semiletov & Karabasov [Markesteijn, A. P., and Semiletov, V.A. and Karabasov, S. A. \[2015\]](#), [Markesteijn, A. P., Semiletov, V.A. and Karabasov, S. A. \[2016\]](#), [Markesteijn, A. P., and Karabasov, S. A. \[2017, 2018\]](#), and showed promising results in terms of performance. Recently, Cascade Technologies has been developing a GPU-accelerated version of its flagship large eddy simulation (LES) software “charLES” that optimally leverages new GPU-accelerated HPC architectures for predictive, fast, and cost-efficient simulations of turbulent flows. Compared to traditional CPU-based HPC approach, a significant reduction in computational cost (or increase in computational throughput) has been observed [Goc et al. \[2021\]](#), up to more than an order of magnitude, depending on the configuration.

This present study is part of a larger collaborative effort with Caltech, University of San Diego (UCSD) and Ohio State University (OSU) to exploit experimental measurements, large eddy simulation and data-driven techniques, in pursuit of fundamental understanding of jet noise actuation as well as high-fidelity modeling of complex nozzle geometries with actuation. The LES work is a continuation of previous (CPU-based) simulation of Brès *et al.* [Brès et al. \[2021\]](#) and focuses on supersonic jet from the twin rectangular nozzle shown in figure 42 at the baseline conditions (i.e., without plasma actuation) matching the experiment at OSU [Esfahani et al. \[2021\]](#), [Leahy et al. \[2022\]](#). The configuration is complex, with a large flow region of interest and an extensive transient LES database required for data-driven analysis. It is therefore demanding in terms of computational resources and a logical candidate to demonstrate the benefits of GPU computing to speed up the turnaround time of the simulation and LES data acquisition.

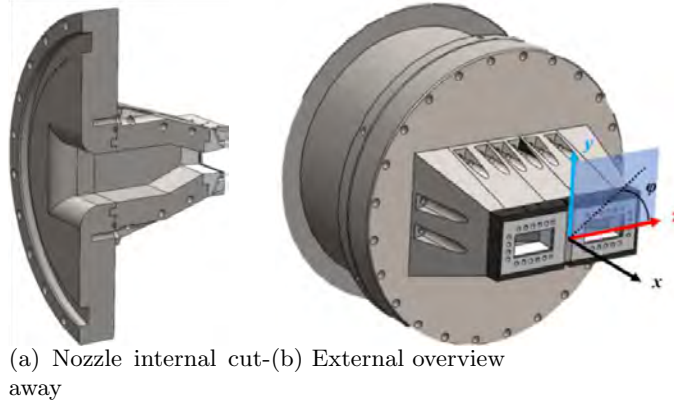


Figure 42: Twin nozzles assembly showing minor axis y , major axis z , and azimuthal angle ϕ (from Ref. Leahy et al. [2022])

Another important feature in this configuration with imperfectly expanded supersonic jets is the presence in the pressure spectra of high-intensity tones at distinct frequencies associated with screech. As originally described by Powell [1953b], jet screech is a resonance phenomenon characterized by a flow-acoustics feedback loop with four components: Kelvin-Helmholtz instability waves in the shear layer near the nozzle exit grow as they propagate downstream and interact with shock cells in the jet plume. This interaction generates strong acoustic waves that travel upstream and in turn interact with the shear layer and nozzle lip, thus exciting further instabilities. Supersonic jet screech has been extensively studied over the years (see reviews in Refs. Raman [1998], Edgington-Mitchell [2019b]) and is still a topic of active research, in particular in terms of the upstream-traveling component of the feedback loop [Edgington-Mitchell et al. [2018b], Wu et al. [2021]]. While the screech frequencies can generally be well estimated based on phase criteria for the closed feedback loop, the screech amplitude is less well understood and remains challenging to predict. Here, the reduced computational cost of the GPU-accelerated solver provided an opportunity to start exploring some of the sensitivities of the screech amplitude to nozzle design features.

4.2 Overview of jet configuration and numerical approach

4.2.1 Experimental setup

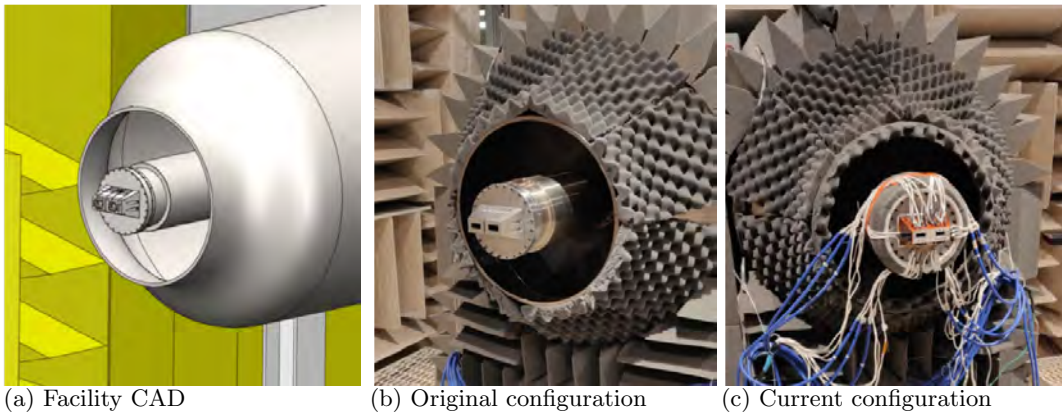


Figure 43: Schematic and pictures of the nozzle assembly installation in the OSU wind-tunnel facility.

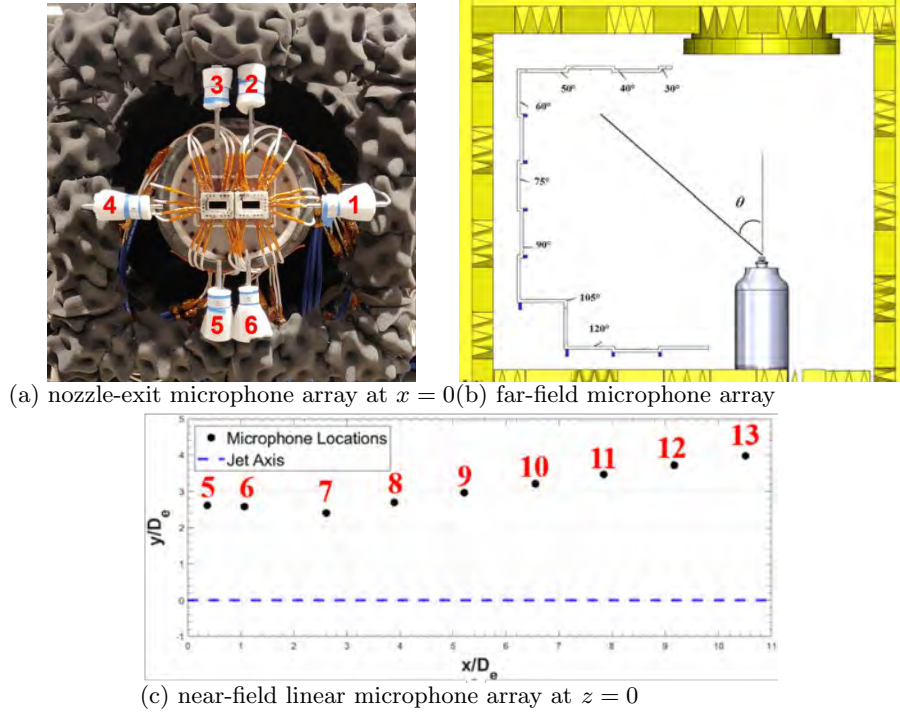


Figure 44: Description of the experimental microphones (from Refs. [Esfahani et al. \[2021\]](#) and [Leahy et al. \[2022\]](#)).

The experimental setup has been designed and fabricated at the Gas Dynamics and Turbulence Laboratory at OSU to study flow physics, aeroacoustics, and active flow control of closely-spaced supersonic twin rectangular jets. The setup is briefly summarized here and more details of the facility, experimental configuration and measurements can be found in [Esfahani et al. \[2021\]](#) and [Leahy et al. \[2022\]](#). As shown in Figure 42, each rectangular nozzle is a bi-conical nozzle with a sharp throat and design Mach number $M_d = 1.5$, height $h = 12.065$ mm and aspect ratio width-to-height $w/h = 2$. The area-based equivalent nozzle diameter D_e is defined as the diameter of a circular jet whose exit area is the same as the rectangular cross-section. The value is $D_e = 19.25$ mm, or $D_e/h = 1.6$, which is also the twin jet spacing, i.e., the inner distance between the nozzle sidewalls at the exit. The center-to-center spacing between the nozzles is therefore $2.25D_e$.

Figure 43 shows the installation of the twin nozzles in the wind-tunnel facility. Several experimental campaigns have been conducted at OSU, in particular acoustic measurements in the near-nozzle exit and near-field region to study jet coupling & growth/decay of wavepackets/instability waves, and in the far-field to investigate radiated noise. Figure 44 shows schematics of the different microphone arrays. For the current configuration, the cabling required for the plasma actuation is present in all the baseline measurements and all the upstream reflective surfaces (including the pipe) have been covered with thick open cell acoustic foam. Experimentally, it was found that covering the co-flow channel tip had a significant effect on screech amplitude. Similarly, [Leahy et al. \[2022\]](#) reported that the screech feedback loop and amplitude can be affected by ambient temperature and moisture levels as well as any variations in jet stagnation temperature. These sensitivities further highlight the challenges of accurately predicting screech amplitude.

4.2.2 Large-eddy simulation methodology

The computations are performed using the LES framework and unstructured compressible flow solver “charLES” developed at Cascade Technologies. The mesh is generated through the computation of Voronoi diagrams: Cascade’s Voronoi-based meshing technology computes a grid from the specification of the relevant surface geometry and the set of generating points where the solution is to be sampled. For this complex rectangular nozzle configuration, the approach drastically simplifies the meshing process and the generation of a high-quality mesh well-suited for large-eddy simulations does not present any particular challenge, even with the inclusion in the computational domain of all the groove, electrodes and small geometrical details inside the nozzle, which are expected to be important for the flow physics and plasma modeling. For the present version of the charLES software, the flow solver is GPU-accelerated, with multi-GPU capabilities on both NVIDIA and AMD hardware. A mixed precision implementation was used to increase solver throughput without adversely affecting the accuracy of the acoustic predictions.

In this study, the simulations correspond to the baseline conditions without plasma actuation. The numerical setup is described in more details in Ref. Brès et al. [2021] and is only briefly summarized here for completeness. The LES nondimensionalization is based on the nozzle height h and the ambient speed of sound $c_\infty = \sqrt{\gamma p_\infty / \rho_\infty}$. The resulting form of the ideal gas law is $p = \rho T / \gamma$, with constant specific heat ratio $\gamma = 1.4$. The nozzle pressure ratio and nozzle temperature ratio are defined as $NPR = p_t / p_\infty = 3.67$ and $NTR = T_t / T_\infty = 1$, where the subscript t and ∞ refer to the stagnation (total) property and free-stream conditions, respectively. For the present nominally ideally-expanded conditions, the jet is cold ($T_j / T_\infty = 0.69$) and the jet Mach number and acoustic Mach number are defined as $M_j = U_j / c_j = M_a = 1.5$ and $M_a = U_j / c_\infty = 1.25$, where U_j is the mean (time-averaged) streamwise jet velocity and the subscript j refer to the (equivalent) fully-expanded jet properties. The experimental Reynolds number is $Re_j = \rho_j U_j D_e / \mu_j \approx 10^6$ and is matched in the simulations. In the wind-tunnel, the twin-jet nozzle assembly is installed inside a co-flow channel which is not active for this configuration and not included in the computational domain (see figure 45). The midpoint between the nozzle spacing is at $(x, y, z) = (0, 0, 0)$, such that the two nozzle exits are centered at $(0, 0, \pm 1.8h)$.

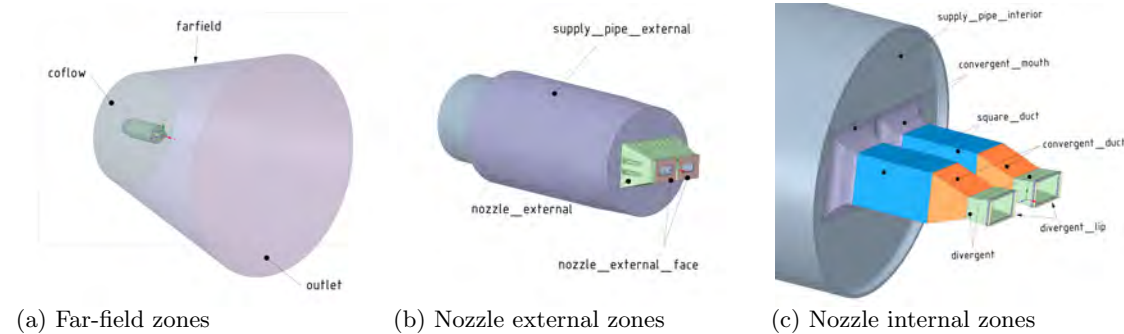


Figure 45: Overview of the computational domain and the different zones for boundary conditions.

The rest of the numerical setup is similar to previous jet studies with the (CPU-based) charLES solver Brès et al. [2017, 2018a,b, 2019]. A very slow coflow at Mach number $M_\infty = 0.01$ is imposed outside the nozzle in the simulation, to prevent any spurious recirculation and facilitate flow entrainment. Sponge layers and damping functions are applied to avoid spurious reflections at the boundary of the computational domain Freund [1997], Mani [2012]. The Vreman Vreman [2004] sub-grid model is used to account for the physical effects of the unresolved turbulence on the resolved flow. The far-field noise is computed using the frequency-domain permeable formulation Lockard [2000] of the Ffowcs Williams & Hawkins Ffowcs Williams and Hawkins [1969] (FW-H) equations. For the

NPR	NTR	M_j	M_a	T_j/T_∞	Re_j	$dt c_\infty/h$	$t_{sim} c_\infty/h$
3.67	1.0	1.5	1.25	0.69	1E6	0.0025	2000

Table 2: Summary of operating conditions and computational parameters, including simulation time step dt and total simulation time t_{sim} for the collection of data and statistics.

treatment of the FW-H outflow disk, the method of “end-caps” of Shur *et al.* Shur *et al.* [2005] is used. To try to capture the thin internal boundary layer and nozzle-exit turbulent state Brès *et al.* [2018a], wall-stress modeling based on the equilibrium boundary layer assumption Kawai and Larsson [2012], Larsson *et al.* [2016], Bose and Park [2018] and near-wall grid refinement are used for the interior surface of the nozzle, in the convergent duct and divergent sections (see figure 45(c)). The other solid surfaces are treated as adiabatic no-slip walls. Table 2 summarizes the operation conditions, along with details of the computations. For noise predictions, the FW-H surface data is sampled every 50 time steps (i.e., $DSt = 10.24$)

4.2.3 Acoustics postprocessing

All the noise results are reported in terms of power spectral density (PSD, in dB/St) as a function of frequency in Strouhal $St = f D_e / U_j$ and overall sound pressure levels (OASPL, in dB), relative to $p_{ref} = 20 \times 10^{-6}$ Pa. All the FW-H calculations are performed at the exact location of the experimental microphones, assuming lossless propagation. For the far-field microphones, the effects of atmospheric absorption Bass *et al.* [1995] is then included in the FW-H results for comparison with the as-measured data, using the ambient temperature and relative humidity in the wind-tunnel the day of the measurements. For the microphone locations and relatively high frequencies considered in this scale model configuration (i.e., the distance of propagation is $\mathcal{O}(3)$ m and $St = 3$ corresponds to approximately 66kHz), the atmospheric absorption is not negligible and leads to a reduction to 1.2 – 3.3 dB in the $St = 1 - 3$ frequency range.

For reference, the experimental time-history of pressure data is recorded as 100 blocks of 32,768 samples for the near-field microphones and 400 blocks of 8,192 samples for the far-field microphones, with sampling frequency $Df_{exp} = 200$ kHz (i.e., $DSt_{exp} = 8.9$). The samples within a block are time-resolved, but not time-resolved between blocks. The duration of each experimental block is $t_{exp} c_\infty / h \approx 4522$ and 1131 acoustic time units, for the near-field and far-field microphones respectively, compared to the simulation data for a single block of 2000 acoustic time units in the LES. For each block, the narrowband PSD is computed, and for the experimental data, average and rms values over all the blocks are then calculated. The rms can be useful as a measure of the block-to-block variation in the short-time-averaged spectra of the long experimental measurements, to reflect on the differences in statistical averaging between the experiment and the LES.

For both the experimental and numerical data, a (raw) bin-average PSD is computed from the narrowband PSD, with a relatively small bin size $\Delta St = 0.01$ to properly represent the screech tones. Following the procedure by Leahy *et al.* Leahy *et al.* [2022], a smoothed (de-toned) PSD is also computed by applying a moving median filter to the narrowband PSD with a window size of 1000 Hz. This filter removes sharp peaks associated with the jet screech and highlights the broadband characteristics of the noise spectra without the uncertainties associated with the screech tone amplitudes. The raw and de-toned OASPL are then calculated by integrating the corresponding PSD over the specified frequency range.

4.2.4 Previous work & present numerical setup

In previous work Brès *et al.* [2021], a single initial (CPU-based) LES was performed at the baseline conditions (i.e., no plasma actuation) without prior knowledge of the experimental data. The mesh

used for the initial simulation contained approximately 65 million cv, with emphasized refinement along the walls and within the shear layer of the jet. The finest resolution was $\Delta = 0.01h$ in the near wall region, including the plasma groove, such that there were approximately 8 cv and 4cv in the width and depth of the groove, respectively. After the initial transient, the runtime for collection of statistics was $t_{sim}c_\infty/h = 611$ and the computational cost was approximately 240 kCPUh (i.e., time-to-solution of 36h on 6600 cores). Overall, the preliminary noise predictions were in good agreement with the microphone data for most relevant frequencies and angles.

To investigate some of the small discrepancies observed in the comparisons with the initial LES, additional simulations are performed in the present work with the GPU-accelerated solver. The key differences between the two sets of computation can be summarized as follows. First, additional grid refinement is applied in the near-wall region inside the nozzle and in the jet plume to further improve noise predictions at higher frequencies. The finest resolution in the near-wall region is now $\Delta = 0.005h$, effectively doubling the number of cells across the width and depth of the groove. As shown in figure 46, the resolution is $\Delta = 0.01h$ to $0.02h$ in the near-nozzle exit region (up to $x = 8h$) and then increased to $\Delta = 0.04h$ in the bulk of jet plume up to $32h$ downstream of the nozzle. The total mesh size is increased to approximately 97 million cv.

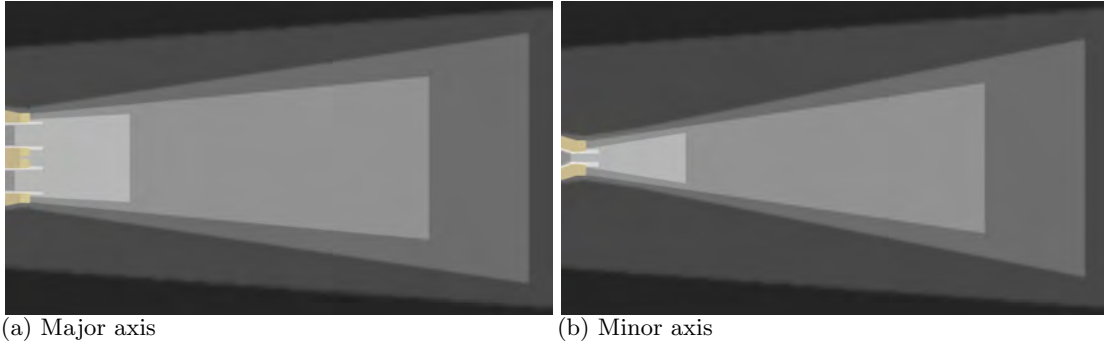


Figure 46: Description of the grid refinement up to $45h$ downstream of the nozzle exit. The gray scale contours represent mesh resolution levels from $\Delta = 0.005h$ (light) to $0.32h$ (dark).

Second, the simulation runtime for collection of statistics (after the initial transient removed) is increased to $t_{sim}c_\infty/h = 2000$ acoustic time units. The present single time block of LES data is therefore the same order of magnitude in duration than a block of experimental data.

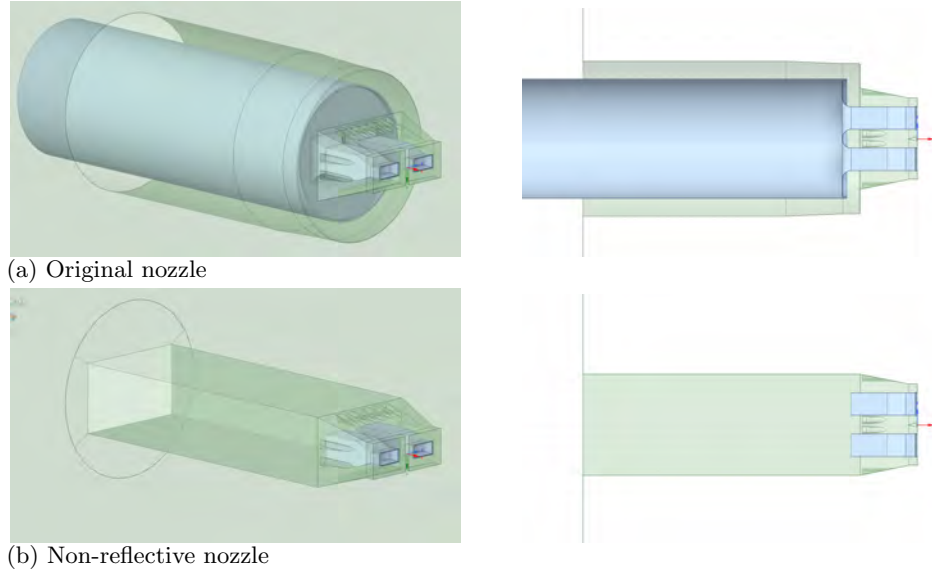


Figure 47: Overview (left) and top view (right) of the two nozzle configurations considered in the simulations.

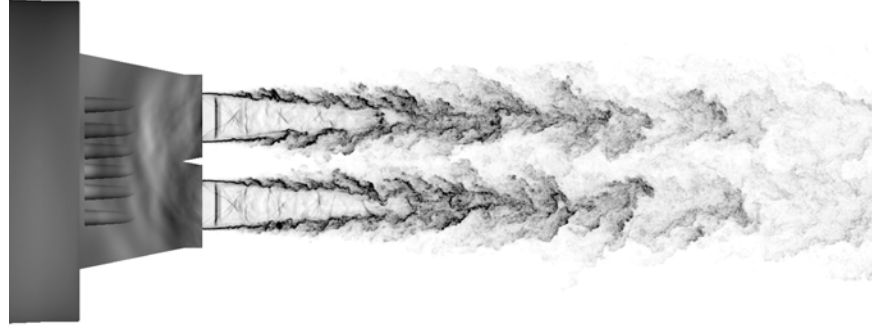
Third, the installed configuration in the OSU wind-tunnel facility includes thick open cell acoustic foam and cabling on the supply pipe surfaces upstream of the nozzle exit (see figure 43(c)) which are challenging to model in the LES and can be expected to have an impact on the feedback loop and screech tone amplitude. To investigate these effects, a second “non-reflective” nozzle is considered in which all the reflective surfaces associated with the supply pipe are removed. Figure 47 shows a comparison of the original and non-reflective nozzles. The changes were done in Cascade’s pre-processing tool without the need of a CAD package. Aside from the removal of the supply pipe, all the details of the twin nozzles are retained in the computational domain, including the external shape and the internal groove and electrodes. The total mesh size is slightly reduced to 94 million cv. The operating conditions, numerical setup, mesh resolution and runtime are kept identical for the two independent simulations with original and non-reflective nozzle (see table 2).

Finally, the present simulations are performed with the GPU-accelerated version of the charLES flow solver. The computational cost for the collection of 2000 acoustics time unit of data on the 97 million cv mesh was approximately 360 GPU hours (i.e., time-to-solution of 12h on 30 GPUs). Thanks to the GPU acceleration, the time-to-solution was therefore cut down by a factor of 3 compared to the previous initial (CPU-based) LES, even though the mesh size was increased by about 50% and simulation runtime was more than tripled. Additional details about the GPU-accelerated solver performance and scalability are presented in section 4.4.

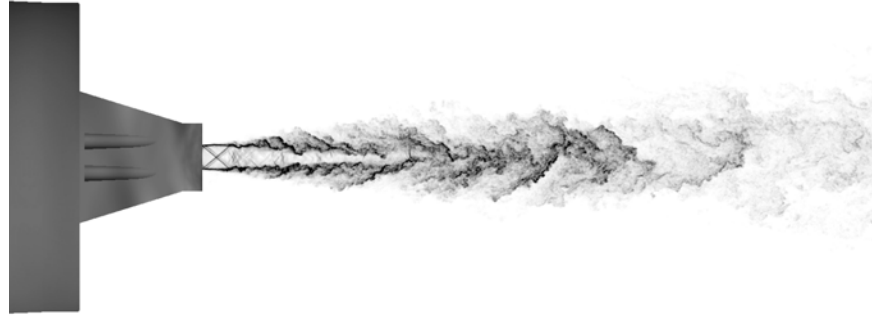
4.3 Results and analysis

4.3.1 Flow field

As a qualitative visualization of the jet flow, numerical schlieren of the instantaneous density are shown in figure 48 in planar cuts through the centerlines long the major and minor axis. Surface pressure fluctuations are also visualized on the original nozzle geometry. Since the configuration features bi-conical nozzles with sharp throat, there are shock cells in the potential cores even for the present nominally ideally-expanded conditions. As discussed in more detail in the next section, the imprints of upstream-traveling acoustics waves are visible on the nozzle external surfaces upstream of the nozzle exit, which can have an impact on aircraft structural fatigue.



(a) $y = 0$ (i.e., through centerline along major axis)



(b) $z = 1.8h$ (i.e., through centerline along minor axis)

Figure 48: Visualization of the pressure fluctuations ($-0.01 \leq (p - p_\infty)/p_\infty \leq 0.01$) on the twin rectangular nozzle external surfaces and numerical schlieren ($0.75 \leq \exp(-5|\nabla\rho|/\max(|\nabla\rho|)) \leq 0.99$) in the jet plume for the original nozzle.

Figure 49 shows the instantaneous pressure fluctuation $(p - p_\infty)/p_\infty$ in the same planar cuts for the full computational domain. The effects of the sponge layers applied away from the nozzle and jet plume can clearly be seen in the figure as acoustics waves are damped towards the boundary of the domain without spurious reflections. Aside from the typical broadband noise radiation pattern toward the aft angles (i.e., polar angles $\theta \approx 30 - 55^\circ$) associated with the Mach waves, the main features in the pressure field are the strong upstream-traveling acoustic waves from the jet screech feedback loop. The waves have a well defined wavelength $\lambda/h \approx 4.27$ related to the main screech frequency at $St \approx 0.3$ and are more intense along minor axis for both nozzles (see images on the right in figure 49). With the original nozzle, there is an additional downstream radiation pattern near the nozzle exit associated with the reflections of the waves off the supply pipe surfaces. These reflected waves are not present for the non-reflective nozzle with the supply pipe removed.

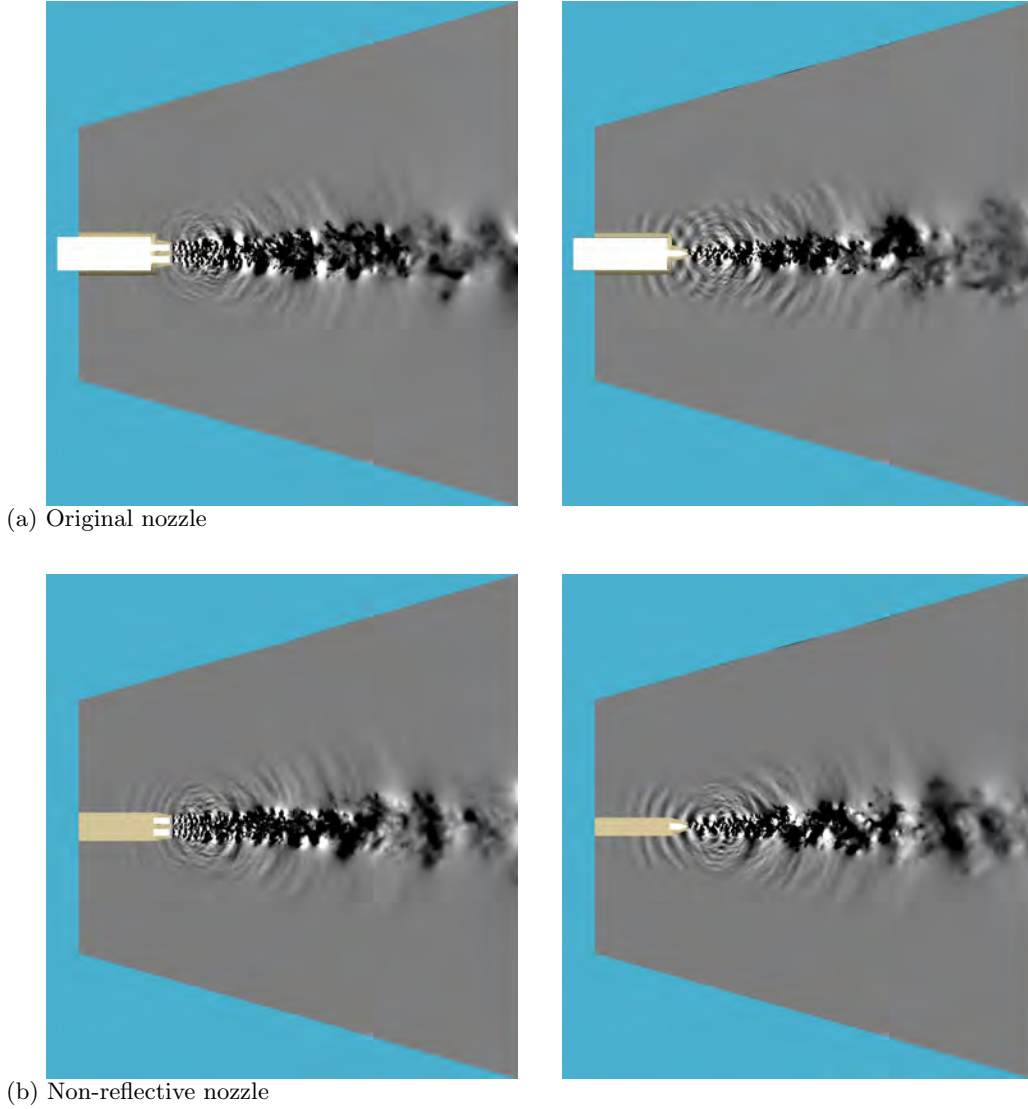


Figure 49: Planar cuts of the instantaneous pressure (left) in the $y = 0$ plane (i.e., through centerline along major axis) and (right) in the $z = 1.8h$ plane (i.e., through centerline along minor axis) for the original and non-reflective nozzles: $(p - p_\infty)/p_\infty$ from -0.01 (black) to 0.01 (white). The solid nozzle surfaces are shown in golden color.

4.3.2 Near-field noise

More quantitative noise results are presented in this section for the nozzle-exit microphone array and the linear microphone array described in figure 44(a) and (c), respectively. As discussed in Refs. Esfahani et al. [2021], Leahy et al. [2022], the nozzle-exit microphone array was designed specifically to provide information on screech frequency and on twin jets coupling modes. Similarly, the linear microphone array was setup specifically to investigate growth and decay of large-scale wavepackets/instability waves along the jet plume and its location in the $z = 0$ plane between the two nozzle was chosen to minimize the effects of probes on the screech and coupling. For these near-field measurements, a protection grid cap is used to protect the microphone from getting damaged, and the acoustics at higher frequencies is challenging to capture. Therefore, the near-field spectra

comparisons are limited to frequencies up to $St \approx 1 - 1.5$. Both raw and de-toned OASPL are computed through integration of the corresponding PSD over $0.04 \leq St \leq 1.5$.

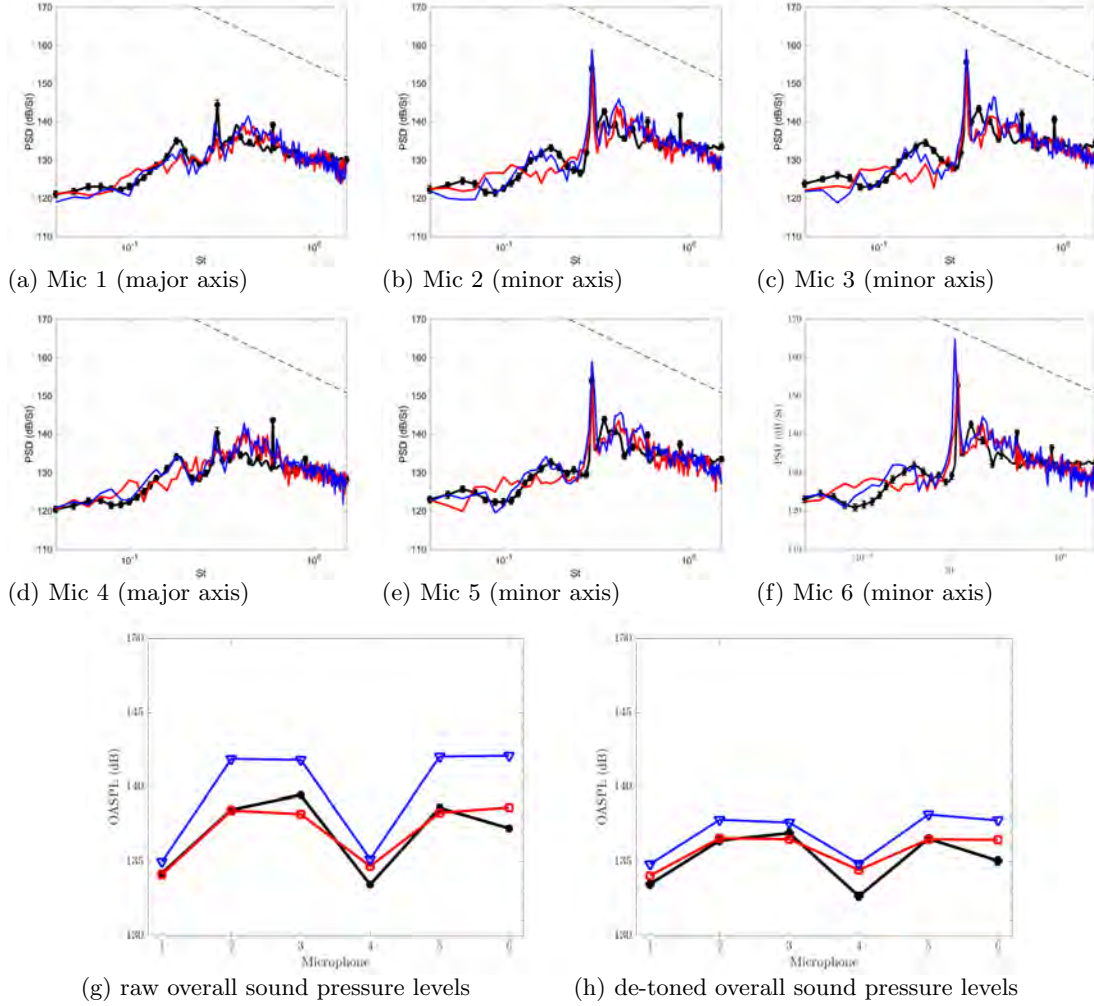


Figure 50: Noise comparisons at the nozzle-exit microphone array (see figure 44(a)) between the experimental measurements (—•) and the GPU-accelerated LES predictions for the original nozzle (—△) & non-reflective nozzle (—□). The vertical bars show the block-to-block variations in the short-time-averaged spectra of the long experimental data, compared to the 1-block LES data. The dashed line corresponds to the theoretical decay of pressure fluctuations in isotropic turbulence with $-7/3$ slope.

Experimental measurements and numerical predictions of the near-field noise are presented in figure 50 for the nozzle-exit microphone array. Given the nozzle symmetries, similar results can be expected for the statistically-equivalent microphones 1 & 4 along the major axis, and 2, 3, 5 & 6 along the minor axis (see photograph of nozzle-exit microphone locations in figure 44(a)). Overall, the agreement is good over most of the frequency range and the main screech tone frequency at $St \approx 0.3$ is well predicted. The jet screech is more intense along the minor axis (i.e., mic 2, 3, 5 & 6) than along the major axis (i.e., mic 1 & 4), consistence with the visualization of the pressure fluctuations in figure 49. For $St < 0.25$, the lobes in the low-frequency spectra are well captured for

the simulation with the original nozzle and the agreement gets slightly worse with the non-reflective nozzle. Here, the large supply pipe geometry upstream for the nozzle exit seems to play a key role in setting up the acoustic waves with long wavelengths. The main difference between the two LES predictions is the screech tone amplitude: for the original nozzle, the amplitude along the minor axis is over-predicted by 3 – 5dB. With the non-reflective nozzle, the screech amplitude now is within 1 – 2 dB for most microphones. Similarly, some of the additional lobes around $St \approx 0.42$ and 0.55 have been reduced for the non-reflective nozzle for a closer match with the experimental spectra.

These differences in screech amplitude can clearly be seen in the comparisons of the raw and de-toned OASPL in figures 50(g) & (h). For the raw OASPL with all the tones included, the LES results for the non-reflective nozzle (i.e., upstream supply pipe reflective surfaces removed) match the experimental results (i.e., surfaces covered by acoustic foam) within 0.5 – 1dB while the original nozzle produces higher raw OASPL levels because of the stronger screech tone long the minor axis. For the de-toned OASPL with the screech contributions removed, all levels are reduced and now similar in magnitude.

Similar conclusions holds for the near-field noise shown in figure 51 at the linear microphone array. Again, with the original nozzle, the low-frequency lobes are remarkably captured, in particular at the microphones 5 & 6 closer to the nozzle exit, but the amplitude of the screech primary tone at $St \approx 0.3$ is over-predicted. With the non-reflective nozzle, the amplitude is reduced and the simulation recovers the experimental trend that the screech tone first harmonic at $St \approx 0.6$ is the main feature in the near-field spectra in the $x/D_e = 4 - 8$ region (i.e., microphones 8 to 11). For $St > 1$, some small discrepancies are observed because of the challenges of the high-frequency measurements previously discussed. The LES spectra are similar for both nozzles, with a more uniform $-7/3$ slope corresponding to the traditional decay rate of pressure fluctuation in isotropic turbulence. Inspection of the raw and de-toned OASPL in figures 51(g) & (h) confirms that most of the discrepancies are due to the differences in screech amplitude predictions. To quantify growth and decay of large-scale wavepackets/instability waves, the de-toned levels are more relevant and the predictions matches the measurements within 0.5 – 1 dB for most microphones.

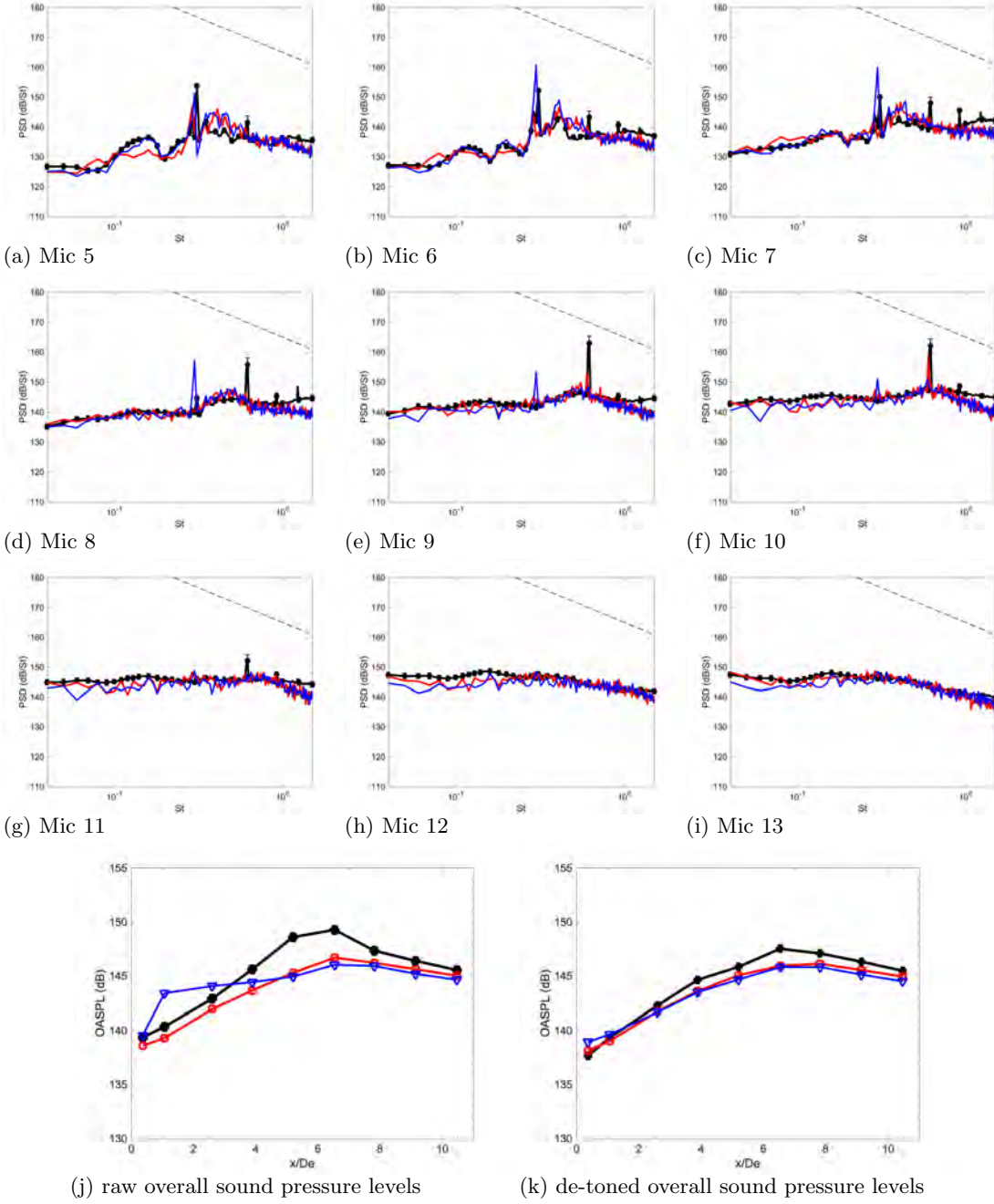


Figure 51: Noise comparisons at the near-field linear microphone array (see figure 44(c)) between the experimental measurements (— •) and the GPU-accelerated LES predictions for the original nozzle (— $\triangle$) & non-reflective nozzle (— $\square$). The vertical bars show the block-to-block variations in the short-time-averaged spectra of the long experimental data, compared to the 1-block LES data. The dashed line corresponds to the theoretical decay of pressure fluctuations in isotropic turbulence with $-7/3$ slope.

4.3.3 Far-field noise

Finally, the far-field noise comparisons are shown in figure 52. For these microphones, the protection grid cap is not used: the spectra comparisons are reported up to $St = 3$ and the raw OASPL is computed through integration of the PSD over $0.04 \leq St \leq 3$. Further inspection of the experimental setup pointed out that part of the field of views for the normal and upstream microphones in figure 44(b)) could be affected by the thick nozzle lip design to accommodate the plasma actuators. Based on these discussions with the experimentalists at OSU, the comparisons are presented up polar angle $\theta = 75^\circ$.

Overall, the agreement is good for all the microphones over the full frequency range. For most relevant angles and frequencies, the LES predictions are within the small block-to-block variations of the experimental data (i.e., vertical bars in figure 52), of order ± 0.5 to 1.5 dB. In the far-field, the dominant component of the radiated noise is the broadband mixing noise towards the aft angles (i.e., $\theta \approx 30 - 50^\circ$) associated with the large-scale wavepackets/instability waves, rather than the tonal screech noise. The de-toned OASPL is therefore nearly identical to the raw OASPL, and is not reported. The simulations for both original and non-reflective nozzle also show similar results in terms of far-field radiated noise. The only difference is that the screech primary tone at $St \approx 0.3$ is visible in the spectra along the minor axis for the original nozzle and not for the non-reflective nozzle & experiment. This is likely related to the enhanced contributions from reflected waves bouncing off the upstream supply pipe surface and propagating towards the aft angles, as shown in the instantaneous field of pressure fluctuations in figure 49(a) for the original nozzle. However, these contributions are small and the OASPL directivity is well captured for both simulations: the predicted OASPL levels are again within 0.5 – 1 dB of the experimental measurements.

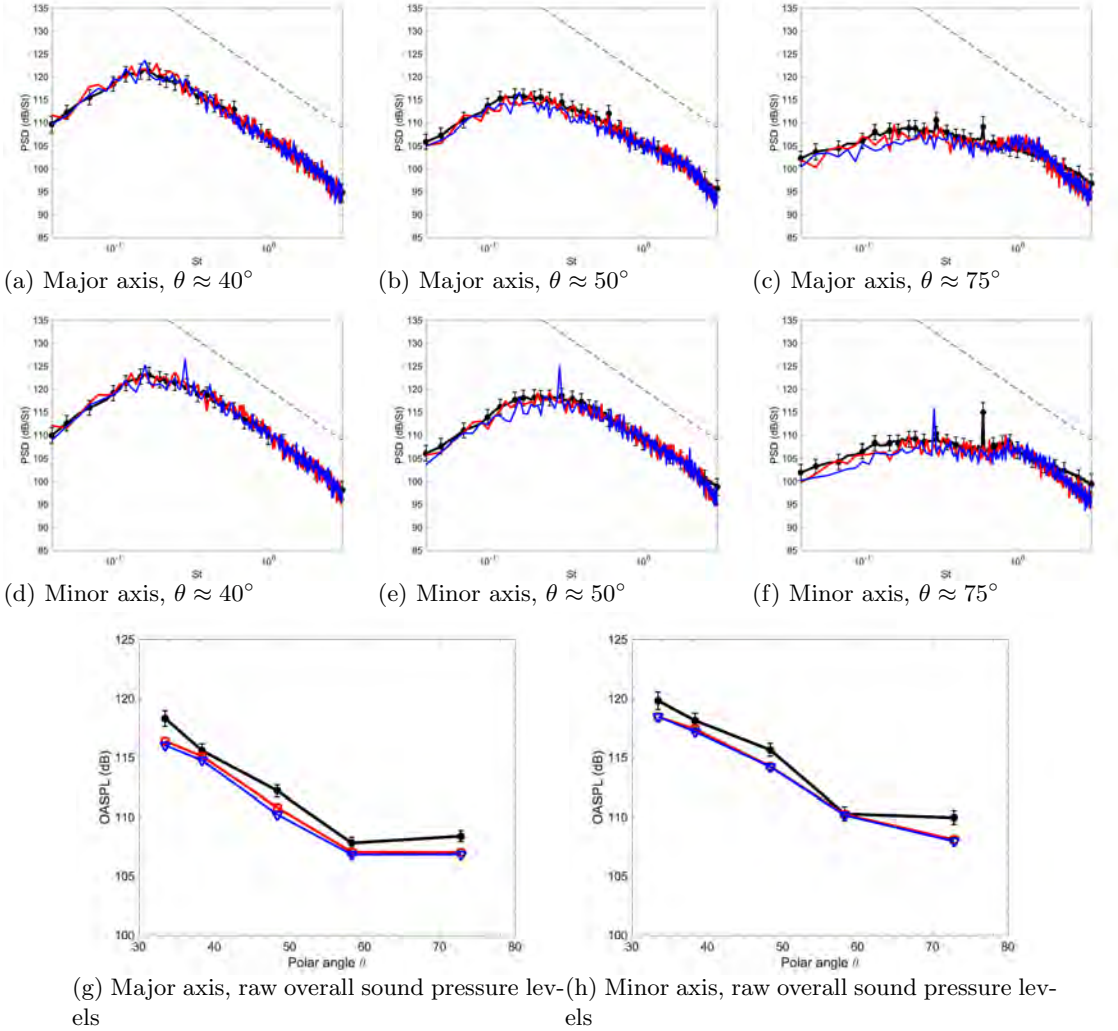


Figure 52: Noise comparisons at the far-field microphone array (see figure 44(b)) between the experimental measurements (—•) and the GPU-accelerated LES predictions for the original nozzle (— $\triangle$) & non-reflective nozzle (— $\square$). The vertical bars show the block-to-block variations in the short-time-averaged spectra of the long experimental data, compared to the 1-block LES data. The dashed line corresponds to the theoretical decay of pressure fluctuations in isotropic turbulence with $-7/3$ slope.

4.4 Scalability study of the GPU-accelerated solver

As mentioned in section 4.2.4.2.4, the present simulations were performed on 30 GPUS, corresponding to 15 MLA-2 accelerated nodes of the HPE Cray EX system “Narwhal” located at the US Navy DoD Supercomputing Resource Center (DSRC). Each node has 2 NVIDIA V100 PCIe GPUs with 32GB of memory per GPU and HPE Slingshot interconnect.

For the 97 million cv mesh, a strong scalability study was also conducted on the Narwhal system, from 5 to 30 nodes (i.e., nearly the full set of available GPUs). Figure 53 shows the solver speedup compared to ideal speedup, as well as the loading (in Mcv per GPUs) as a function of the GPU count. The GPU-accelerated solver scales perfectly up to 30 GPUs, corresponding to a loading of

approximately 3.2M cv per GPU, and remains at about 90% scalability on 60 GPUs, for a loading as low as 1.6M cv per GPU.

For comparison, the performance of the CPU-based version of the solver was tested for the same jet case on Narwhal standard HPC nodes, with 128 AMD 7H12 Rome processors per nodes. Here, each V100 GPU is found to be equivalent in throughput to $\mathcal{O}(400)$ cores, which is similar to the equivalence observed for the ideal-gas compressible flow solver on other HPC systems. This acceleration enables simulation turn-around times of order hours on just 10s of GPUs for mesh sizes relevant to practical engineering applications.

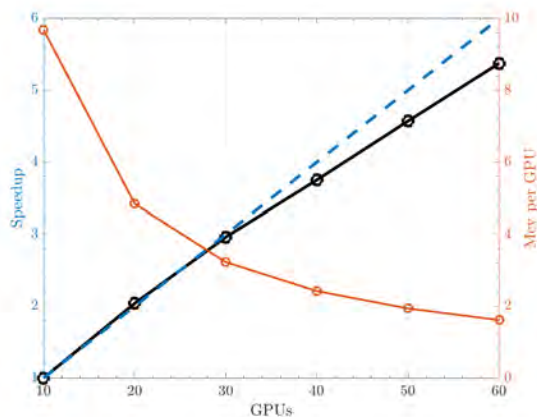


Figure 53: Strong scalability of the GPU-accelerated ideal-gas compressible flow solve charLES (—) for a supersonic jet on a 97 million cv mesh.; Ideal speedup (- - -); loading in Mcv per GPU (—○—).

5 Plasma actuation modeling and SPOD of Supersonic Twin-Rectangular Jet Flow

The statistical analysis of non-axisymmetric turbulent jets using spectral proper orthogonal decomposition (SPOD) is computationally costly; in particular, their non-axisymmetry precludes Fourier decomposition of the three-dimensional flow field into two-dimensional azimuthal modes. Jets in the dihedral group D_2 , including rectangular, elliptic, and twin jets, are invariant under reflection about their major and minor axes. For these jets, we propose an SPOD workflow that exploits their D_2 geometrical symmetry to obtain a reduction in computational effort, accelerate statistical convergence, and improve the interpretability of results. We decompose the three-dimensional snapshots into four symmetry components. For each symmetry component, we independently perform SPOD on one quadrant of the domain. We demonstrate an application of this D_2 -symmetric SPOD workflow on a large-eddy simulation of a twin-rectangular supersonic jet. Our analysis indicates that the twin jet exhibits screech at Strouhal number 0.3, and that this screech is dominated by components which are antisymmetric along the minor axis. These observations are consistent with the results from companion experiments at Ohio State University. Furthermore, the same workflow is used to analyze the symmetry-dependence of far-field acoustics at low and high frequencies.

5.1 Introduction

Spectral proper orthogonal decomposition (SPOD) is a statistical technique which has been employed in numerical studies of turbulent jets (see e.g. Towne et al. [2018a] and Schmidt et al. [2018a]). Part

of the appeal of SPOD stems from its ability to educe from turbulent flows structures which are temporally as well as spatially coherent [Schmidt et al. \[2018a\]](#). Recent applications of SPOD include the identification of Kelvin-Helmholtz (KH) type wavepackets within jet shear layers [Schmidt et al. \[2018a\]](#), and the examination of far-field acoustic radiation from jets [Nekkanti and Schmidt \[January 2021\]](#).

To date, the SPOD analysis of turbulent jets has been dominated by axisymmetric configurations. Their axisymmetry can be exploited by performing a Fourier decomposition of the three-dimensional flow fields into two-dimensional azimuthal wavenumber components: a significant simplification. Non-axisymmetric jets, in contrast, cannot be azimuthally decomposed. SPOD of the large data set of a non-axisymmetric jet containing three-dimensional snapshots is thus computationally challenging and—to our knowledge—has not been demonstrated. Although they lack axisymmetry, rectangular, elliptic, and twin jets each possess two reflection symmetries: one about the major axis, another about the minor axis. It is natural to consider whether the same benefits of symmetry decomposition can be extended to these nozzle geometries as well. In this paper we present an SPOD workflow that respects the two mirror symmetries, and decomposes the three-dimensional jet flow into symmetry components.

We demonstrate this workflow on a time-resolved database collected from large-eddy simulation (LES) of the supersonic twin-rectangular jet experimentally investigated by [Esfahani et al. \[January 2021\]](#) at Ohio State University (OSU). Their results have shown that the jet emits screech at the experimentally-determined, nominally ideally-expanded condition. This same condition is matched by our LES. The earliest model for screech in imperfectly expanded supersonic jets dates back to [Powell \[1953a\]](#), who proposed that screech is a resonance phenomenon involving interactions between the shear layer and shocks. Though the existence of the screech feedback loop is now widely accepted, the production and nature of the waves that sustain the loop continue to be subjects of active research. In particular, whereas the downstream-propagating waves are generally accepted to be KH instabilities [\[Edgington-Mitchell, 2019a\]](#), universal consensus regarding the upstream-propagating waves that close the loop remains elusive. Powell’s original model described the upstream-propagating waves as acoustic waves traveling in the free stream [\[Powell, 1953a\]](#). However, recent findings by [Gojon et al. \[2018\]](#) and [Edgington-Mitchell et al. \[2018a\]](#) suggest the upstream-propagating component is more likely to be a neutrally-stable acoustic mode of the mean flow, which has also been identified by [Towne et al. \[2017\]](#) in subsonic jets. A comprehensive review of screech has been provided by [Edgington-Mitchell \[2019a\]](#).

Section 5.2.1 describes our LES of the supersonic twin-rectangular jet. The details of the symmetry-enforcing SPOD workflow are discussed in Section 5.2.2. We demonstrate this workflow on our database in Section 5.3, then offer concluding remarks in Section ??.

5.2 Numerical setup

5.2.1 Large-eddy simulation

The LES is carried out using the unstructured compressible flow solver ‘Charles’ [Brès et al. \[2017\]](#), to simulate the twin-rectangular jet of aspect ratio two at OSU [Esfahani et al. \[January 2021\]](#). The twin-nozzle geometry is shown in Fig. 54. Fig. 54(b) highlights the nozzle lip, which includes a groove designed to house plasma actuators. For more details on the OSU facility, we refer the reader to [Esfahani et al. \[January 2021\]](#).

The jet is nominally ideally-expanded and cold, at $M_j = U_j/c_j = 1.5$ and $T_j/T_\infty = 0.69$. The density, temperature, and velocities are nondimensionalized by ρ_∞ , T_∞ , and $c_\infty = \sqrt{\gamma p_\infty/\rho_\infty}$, respectively. Lengths are nondimensionalized by h , giving $D_e/h = 1.6$. The simulation Reynolds number, $Re_j = \rho_j U_j D_e/\mu_j = 1.07 \times 10^6$, approximately matches the OSU experiments. The nozzle pressure ratio (NPR) and nozzle temperature ratio (NTR) are $p_t/p_\infty = 3.671$ and $T_t/T_\infty = 1$, respectively.

Table 3: LES parameters.

Case	M_j	M_a	Re_j	p_t/p_∞	p_j/p_∞	T_t/T_∞	T_j/T_∞	N_{cv}	$dt c_\infty/h$	$t_{sim} c_\infty/h$
Baseline	1.5	1.25	1.07×10^6	3.671	1	1	0.69	77×10^6	0.002	500

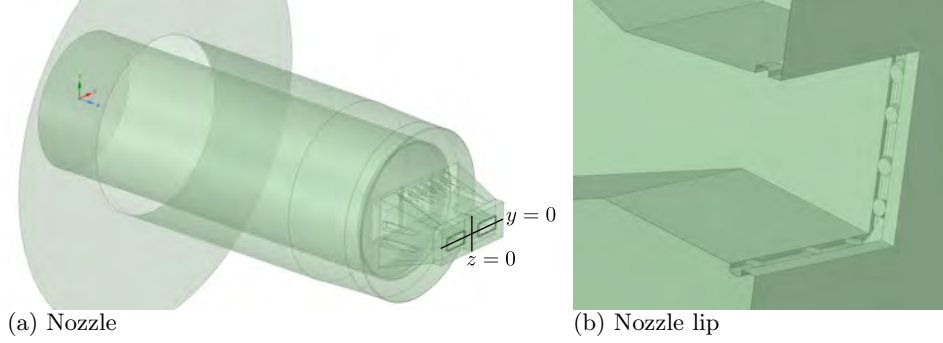


Figure 54: Twin-rectangular nozzle geometry Brès et al. [2021], which is explicitly included in the computational domain: (a) nozzle external and internal surfaces; (b) cutaway view of the groove near the nozzle exit.

An LES of this setup was recently performed by Brès et al. Brès et al. [2021] and validated against experimental data from OSU. To demonstrate the symmetry-enforcing SPOD workflow, we carry out an LES to collect a time-resolved flow database. Table 3 summarizes the simulation parameters.

Figure 55 visualizes the magnitude of the gradient of the instantaneous density, $|\nabla \rho|$. Fig. 55(a) is taken from the $z = 1.8h$ plane, which intersects the center line of a single nozzle; the major-axis view in Fig. 55(b) is taken from the $y = 0$ plane, which intersects the center lines of both nozzles. Although the jets are nominally ideally-expanded at the experimentally-determined design Mach number, $M_j = 1.5$, the rectangular nozzles, sharp throats, and the groove immediately upstream of each exit lead to shock cells that are visible within the potential cores.

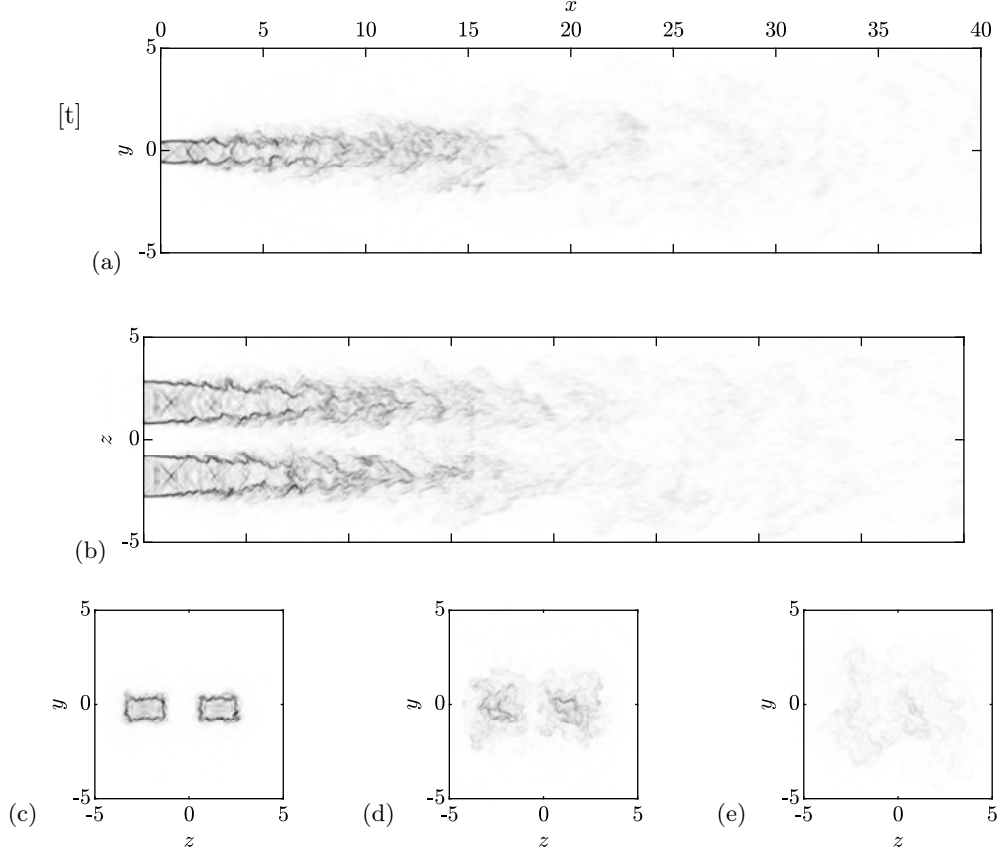


Figure 55: Numerical schlieren ($\square \blacksquare 0 \leq |\nabla \rho| \leq 6$): (a) $z = 1.8h$ plane through the center line of a single nozzle; (b) $y = 0$ plane through the center lines of both nozzles; (c-e) transverse planes through $x = 2h$, $x = 10h$, and $x = 20h$, respectively.

5.2.2 Three-dimensional, D_2 -symmetric SPOD

Unlike an axisymmetric jet, a twin-rectangular jet does not enjoy azimuthal symmetry; as such, its three-dimensional flow field cannot be Fourier decomposed into two-dimensional flow fields, each with a different azimuthal wavenumber. It does, however, belong in the dihedral group D_2 , i.e., its geometry is invariant under reflection about the y - and z -axes. By accounting for these mirror symmetries, we propose a D_2 symmetry decomposition. Other common D_2 nozzle geometries include twin-round, single rectangular, and elliptic jets, to all of which the procedure below directly generalizes.

For the twin-rectangular jet, D_2 geometrical symmetry permits a total of four symmetric and antisymmetric components [Tam and Thies \[1993\]](#), which the flow must recover in a statistical sense. To describe these symmetry components unambiguously, we adopt the nomenclature proposed by [Rodríguez et al. \[July 2018\]](#) and subsequently used for single rectangular jets [Rodríguez et al. \[August 2021\]](#) as well as twin-round jets [Rodríguez \[August 2021\]](#), [Stavropoulos et al. \[August 2021\]](#). Each symmetry component is denoted by two letters: SS, SA, AS, or AA. The first letter indicates symmetry (S) or antisymmetry (A) about $y = 0$; the second indicates symmetry (S) or antisymmetry (A) about $z = 0$ (see [Fig. 54\(a\)](#)).

To compute the D_2 -symmetric SPOD, we first compute the long-time mean, $\bar{\mathbf{q}}$. Exploiting D_2

symmetry, we enforce SS symmetry on the estimated mean by averaging its four quadrants,

$$\bar{\mathbf{q}}_{\text{SS}}(x, y, z) = \frac{1}{4} \begin{bmatrix} \bar{\rho}(x, y, z) + \bar{\rho}(x, -y, z) + \bar{\rho}(x, y, -z) + \bar{\rho}(x, -y, -z) \\ \bar{u}(x, y, z) + \bar{u}(x, -y, z) + \bar{u}(x, y, -z) + \bar{u}(x, -y, -z) \\ \bar{v}(x, y, z) - \bar{v}(x, -y, z) + \bar{v}(x, y, -z) - \bar{v}(x, -y, -z) \\ \bar{w}(x, y, z) + \bar{w}(x, -y, z) - \bar{w}(x, y, -z) - \bar{w}(x, -y, -z) \\ \bar{T}(x, y, z) + \bar{T}(x, -y, z) + \bar{T}(x, y, -z) + \bar{T}(x, -y, -z) \end{bmatrix}. \quad (101)$$

Note the sign changes in the $\bar{v}$ and $\bar{w}$ components. If $\bar{v}$ is symmetric about $y = 0$, then $\bar{v}(x, y, z) = -\bar{v}(x, -y, z)$; similarly, if $\bar{w}$ is symmetric about $z = 0$, then $\bar{w}(x, y, z) = -\bar{w}(x, y, -z)$. Using the improved estimate of the mean, we apply the Reynolds decomposition to the instantaneous flow field to obtain the fluctuations,

$$\mathbf{q}'(x, y, z, t) = \mathbf{q}(x, y, z, t) - \bar{\mathbf{q}}_{\text{SS}}(x, y, z). \quad (102)$$

Next we enforce D_2 symmetry, and decompose $\mathbf{q}'$ into four symmetry components:

$$\mathbf{q}'_{\text{SS}}(x, y, z, t) = \frac{1}{4} \begin{bmatrix} \rho'(x, y, z, t) + \rho'(x, -y, z, t) + \rho'(x, y, -z, t) + \rho'(x, -y, -z, t) \\ u'(x, y, z, t) + u'(x, -y, z, t) + u'(x, y, -z, t) + u'(x, -y, -z, t) \\ v'(x, y, z, t) - v'(x, -y, z, t) + v'(x, y, -z, t) - v'(x, -y, -z, t) \\ w'(x, y, z, t) + w'(x, -y, z, t) - w'(x, y, -z, t) - w'(x, -y, -z, t) \\ T'(x, y, z, t) + T'(x, -y, z, t) + T'(x, y, -z, t) + T'(x, -y, -z, t) \end{bmatrix}, \quad (103)$$

$$\mathbf{q}'_{\text{SA}}(x, y, z, t) = \frac{1}{4} \begin{bmatrix} \rho'(x, y, z, t) + \rho'(x, -y, z, t) - \rho'(x, y, -z, t) - \rho'(x, -y, -z, t) \\ u'(x, y, z, t) + u'(x, -y, z, t) - u'(x, y, -z, t) - u'(x, -y, -z, t) \\ v'(x, y, z, t) - v'(x, -y, z, t) - v'(x, y, -z, t) + v'(x, -y, -z, t) \\ w'(x, y, z, t) + w'(x, -y, z, t) + w'(x, y, -z, t) + w'(x, -y, -z, t) \\ T'(x, y, z, t) + T'(x, -y, z, t) - T'(x, y, -z, t) - T'(x, -y, -z, t) \end{bmatrix}, \quad (104)$$

$$\mathbf{q}'_{\text{AS}}(x, y, z, t) = \frac{1}{4} \begin{bmatrix} \rho'(x, y, z, t) - \rho'(x, -y, z, t) + \rho'(x, y, -z, t) - \rho'(x, -y, -z, t) \\ u'(x, y, z, t) - u'(x, -y, z, t) + u'(x, y, -z, t) - u'(x, -y, -z, t) \\ v'(x, y, z, t) + v'(x, -y, z, t) + v'(x, y, -z, t) + v'(x, -y, -z, t) \\ w'(x, y, z, t) - w'(x, -y, z, t) - w'(x, y, -z, t) + w'(x, -y, -z, t) \\ T'(x, y, z, t) - T'(x, -y, z, t) + T'(x, y, -z, t) - T'(x, -y, -z, t) \end{bmatrix}, \quad (105)$$

and

$$\mathbf{q}'_{\text{AA}}(x, y, z, t) = \frac{1}{4} \begin{bmatrix} \rho'(x, y, z, t) - \rho'(x, -y, z, t) - \rho'(x, y, -z, t) + \rho'(x, -y, -z, t) \\ u'(x, y, z, t) - u'(x, -y, z, t) - u'(x, y, -z, t) + u'(x, -y, -z, t) \\ v'(x, y, z, t) + v'(x, -y, z, t) - v'(x, y, -z, t) - v'(x, -y, -z, t) \\ w'(x, y, z, t) - w'(x, -y, z, t) + w'(x, y, -z, t) - w'(x, -y, -z, t) \\ T'(x, y, z, t) - T'(x, -y, z, t) - T'(x, y, -z, t) + T'(x, -y, -z, t) \end{bmatrix}. \quad (106)$$

This decomposition is analogous to the Fourier decomposition of a round jet, which belongs in the orthogonal group $O(2)$, into azimuthal wavenumber components. Like azimuthal Fourier modes, D_2 symmetry components are mutually orthogonal. As an example, by linearity of inner products,

$$\begin{aligned} \langle \mathbf{q}'_{\text{SS}}(x, y, z, t), \mathbf{q}'_{\text{SA}}(x, y, z, t) \rangle &= \langle \mathbf{q}'_{\text{SS}}(x, y, z > 0, t), \mathbf{q}'_{\text{SA}}(x, y, z > 0, t) \rangle \\ &\quad + \langle \mathbf{q}'_{\text{SS}}(x, y, z < 0, t), \mathbf{q}'_{\text{SA}}(x, y, z < 0, t) \rangle, \end{aligned} \quad (107)$$

where we have used the fact that $\mathbf{q}'_{\text{SA}}(x, y, 0, t) = 0$ due to antisymmetry. Recognizing that

$$\mathbf{q}'_{\text{SS}}(x, y, z < 0, t) = \mathbf{q}'_{\text{SS}}(x, y, z > 0, t) \quad \text{and} \quad \mathbf{q}'_{\text{SA}}(x, y, z < 0, t) = -\mathbf{q}'_{\text{SA}}(x, y, z > 0, t), \quad (108)$$

we get

$$\langle \mathbf{q}'_{\text{SS}}(x, y, z, t), \mathbf{q}'_{\text{SA}}(x, y, z, t) \rangle = 0. \quad (109)$$

Therefore $\mathbf{q}'_{\text{SS}}$ and $\mathbf{q}'_{\text{SA}}$ are orthogonal, and the same can be shown for any pair of components. Furthermore, D_2 decomposition permits the exact reconstruction of the original flow field, $\mathbf{q}'$, by summing the four components,

$$\mathbf{q}'(x, y, z, t) = \mathbf{q}'_{\text{SS}}(x, y, z, t) + \mathbf{q}'_{\text{SA}}(x, y, z, t) + \mathbf{q}'_{\text{AS}}(x, y, z, t) + \mathbf{q}'_{\text{AA}}(x, y, z, t), \quad (110)$$

with *no loss of generality*. In each component, all four quadrants now contain the same information. As we seek structures which are coherent over the *three*-dimensional domain, having to process just one quadrant leads to substantial memory savings for the SPOD algorithm, or indeed for any modal decomposition technique. More importantly, explicitly imposing symmetry on the fluctuations significantly accelerates the convergence of the SPOD; we will elaborate on this point in Section 5.3.

Without loss of generality, we proceed to perform SPOD on each symmetry component independently. For each component, after taking the temporal Fourier transform of the first quadrant, at each frequency we construct the data matrix,

$$\hat{\mathbf{Q}} = [\hat{\mathbf{q}}^{(1)} \quad \hat{\mathbf{q}}^{(2)} \quad \dots \quad \hat{\mathbf{q}}^{(N_{\text{blk}})}]. \quad (111)$$

The eigenvalue decomposition of the cross-spectral density (CSD) matrix,

$$\hat{\mathbf{C}} = \frac{1}{N_{\text{blk}} - 1} \hat{\mathbf{Q}} \hat{\mathbf{Q}}^H, \quad (112)$$

is the SPOD. The weighted eigenvectors and eigenvalues of the CSD correspond to the SPOD modes and energies, respectively. In practice, we compute a smaller eigenvalue decomposition using the method of snapshots [Sirovich \[1987a\]](#). The weight matrix, $\mathbf{W}$, allows the inner product, $\langle \mathbf{q}_1, \mathbf{q}_2 \rangle = \mathbf{q}_1^H \mathbf{W} \mathbf{q}_2$, and by extension the eigenvalues, to be physically interpreted as mode energies. Two choices of weights will be used in this work. For $\mathbf{q}' = p'(x, y, z, t)$, we choose

$$\mathbf{W} = \int_0^{z_{\text{max}}} \int_0^{y_{\text{max}}} \int_{x_{\text{min}}}^{x_{\text{max}}} dx dy dz, \quad (113)$$

such that the SPOD eigenvalues represent the volume-integrated variance of the pressure fluctuations; for $\mathbf{q}' = [\rho, u, v, w, T]^T(x, y, z, t)$, we choose

$$\mathbf{W} = \int_0^{z_{\text{max}}} \int_0^{y_{\text{max}}} \int_{x_{\text{min}}}^{x_{\text{max}}} \text{diag} \left(\frac{\bar{T}}{\gamma \bar{\rho} M_j^2}, \bar{\rho}, \bar{\rho}, \bar{\rho}, \frac{\bar{\rho}}{\gamma(\gamma - 1) \bar{T} M_j^2} \right) dx dy dz, \quad (114)$$

such that the eigenvalues represent the volume-integrated compressible energy density [Chu \[September 1965\]](#).

The D_2 -symmetric SPOD workflow is illustrated in Fig. 56. For an in-depth discussion of the SPOD algorithm itself, we direct the reader to [Schmidt and Colonius \[2020a\]](#).

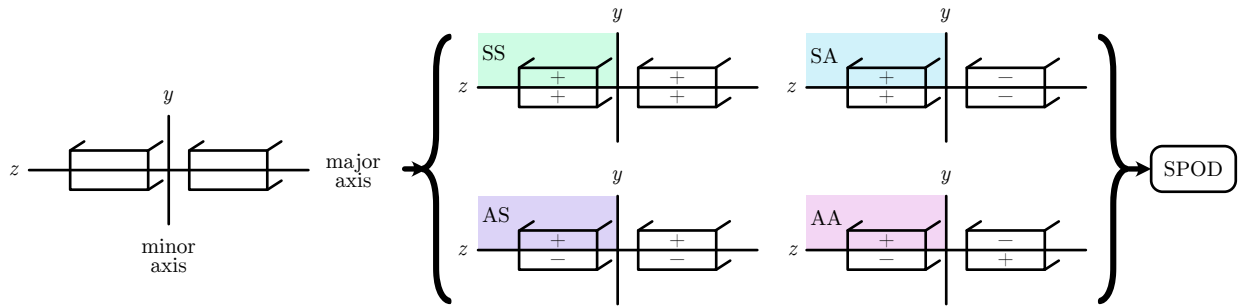


Figure 56: Schematic of the D_2 -symmetric SPOD workflow, which enforces D_2 symmetry, both minimizing memory requirement and accelerating statistical convergence.

Table 4: Interpolation and SPOD parameters.

Case	Database								Parameters		
	N_x	N_y	N_z	$[x_{\min}, x_{\max}]$	$[y_{\min}, y_{\max}]$	$[z_{\min}, z_{\max}]$	N_t	St_s	N_{FFT}	N_{ovlp}	N_{blk}
Baseline	469	202	254	$[0, 85h]$	$[-16h, 16h]$	$[-16h, 16h]$	5000	12.8	256	128	38

5.3 Results

To perform D_2 -symmetric SPOD, the unstructured LES data are first interpolated onto a Cartesian grid; the interpolation parameters are summarized in Table 4. The data are then Reynolds- and D_2 -decomposed according to Eqs. (102) and (110), respectively. To exemplify D_2 decomposition, Fig. 57 shows representative snapshots of the SS, SA, AS, and AA components of the instantaneous streamwise velocity fluctuations, u' . In Fig. 57(a,c,e,f), for instance, u'_{SS} is symmetric about both $y = 0$ and $z = 0$. By analogy, u'_{SA} , u'_{AS} , and u'_{AA} are symmetric or antisymmetric about $y = 0$ or $z = 0$.

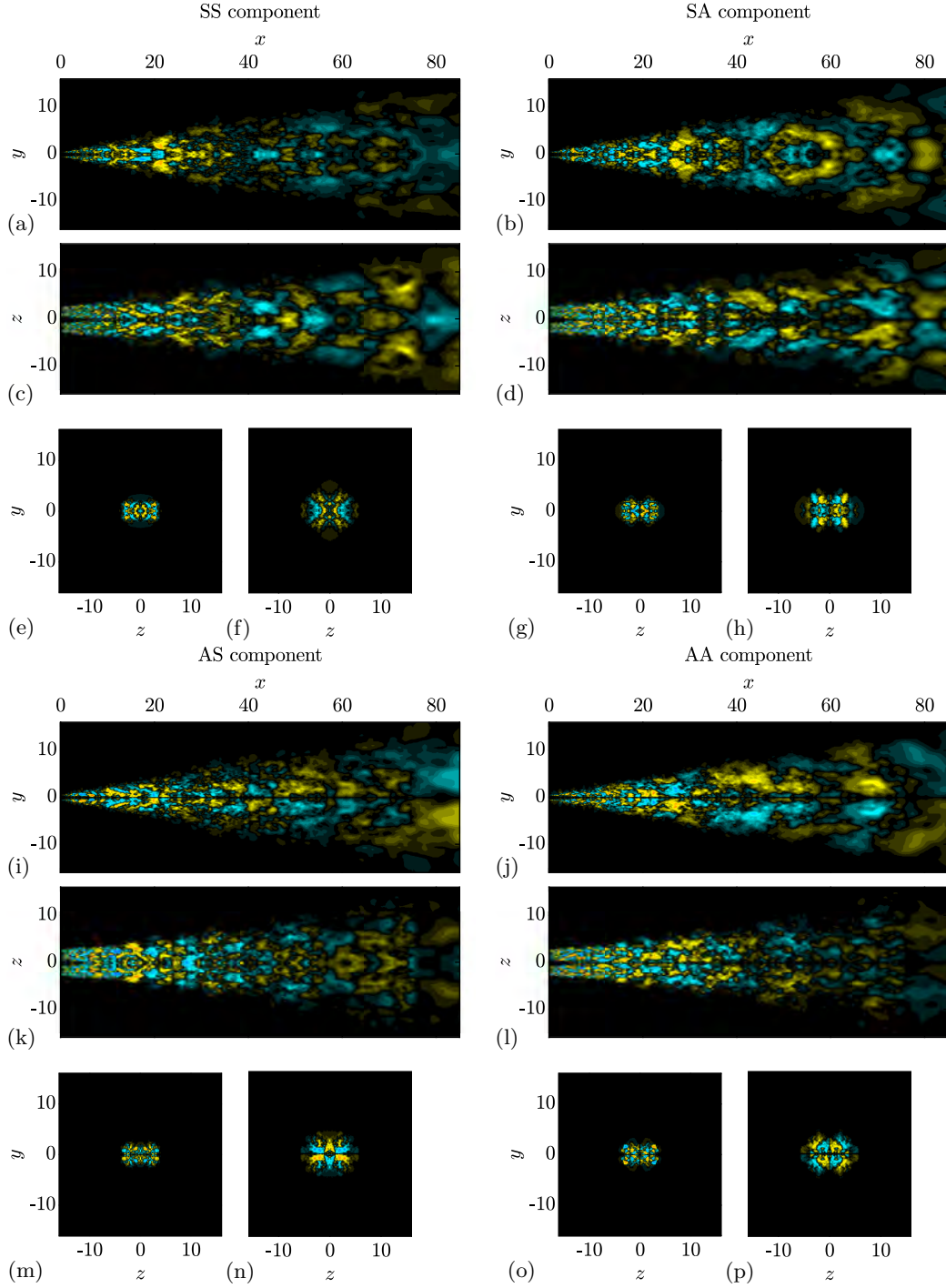


Figure 57: Symmetry-decomposed instantaneous streamwise velocity fluctuations: (a,c,e,f) SS component ($\begin{smallmatrix} + & + \\ \blacksquare & \blacksquare \end{smallmatrix}$; $-0.47 \leq u'_{SS} \leq 0.47$); (b,d,g,h) SA component ($\begin{smallmatrix} + & - \\ \blacksquare & \blacksquare \end{smallmatrix}$; $-0.38 \leq u'_{SA} \leq 0.38$); (i,k,m,n) AS component ($\begin{smallmatrix} + & + \\ \blacksquare & \blacksquare \end{smallmatrix}$; $-0.37 \leq u'_{AS} \leq 0.37$); (j,l,o,p) AA component ($\begin{smallmatrix} + & - \\ \blacksquare & \blacksquare \end{smallmatrix}$; $-0.38 \leq u'_{AA} \leq 0.38$). (a,b,i,j) show the $z = 1.8h$ plane through the center line of a single nozzle; (c,d,k,l) show the $y = 0.5h$ plane through the lip lines of both nozzles; (e,g,m,o) show the $x = 10h$ plane; (f,h,n,p) show the $x = 20h$ plane.

For each symmetry component, we independently compute the SPOD using the parameters listed in Table 4. The eigenvalue spectra from the SPOD of the pressure are shown in Fig. 58(b–e). The blue curves represent the volume-integrated power spectral densities (PSD), reconstructed by summing all the eigenvalues at each frequency. Highlighted in red are the separations between the leading and first suboptimal eigenvalues. In the AA and AS components (Fig. 58(b,c)), there is a distinct peak at $St = 0.3$. At this frequency, the large separation brought about by the peak in turn indicates the dominance of a physical instability Schmidt et al. [2018a], in this case the screech tone Esfahani et al. [January 2021], Brès et al. [2021]. The absence of this low-rank behavior from the SS or SA components (Fig. 58(d,e)) suggests the AA and AS components are primarily responsible for screech.

For comparison only, we also compute the SPOD of the pressure over the entire flow field, without D_2 decomposition. The resulting spectra are shown in Fig. 58(a). The leading and first suboptimal eigenvalues show a prominent peak at $St = 0.3$, corresponding to the peak at the same frequency in the leading eigenvalues of the AA and AS components. By energy conservation, the PSD is four times the sum of the PSDs of the individual symmetry components, which reflects the fact that SPOD of the non- D_2 -decomposed data is carried out over all four quadrants, while SPOD of each symmetry component is carried out over only a single quadrant.

Given that SPOD seeks orthogonal modes, and that different symmetry components are mutually orthogonal, the SPOD of non- D_2 -decomposed data must *eventually* recover all four symmetry components, *provided* the number of realizations is sufficiently large to converge them. This means that in general, given a finite sample, without D_2 decomposition there is no guarantee SPOD will recover all four symmetries. In the case of our twin jet, however, the four leading modes of the full data correspond to the leading mode of each symmetry component, with the caveat that the fourth mode cannot be unambiguously characterized due to lack of convergence. Figures 59, 60, 61, and 62 compare the four leading SPOD modes at $St = 0.3$, with (right column) and without (left column) D_2 decomposition. Recall that when D_2 symmetry is enforced, the eigenvectors need only be computed and stored in the first quadrant, i.e., $x, y, z \in [0, 85h] \times [0, 16h] \times [0, 16h]$; here these eigenvectors have been unfolded into the remaining three quadrants for visual clarity. The first, second, and third SPOD modes of the non- D_2 -decomposed pressure clearly resemble the first SPOD mode of the AA, AS, and SS components, respectively. However, with D_2 decomposition, the mode shapes appear smoother, implying better statistical convergence. They are also perfectly (anti-)symmetric by construction. The fourth SPOD mode of the non- D_2 -decomposed pressure is too poorly converged for visual confirmation that it is equivalent to the first SPOD mode of the SA component. Nevertheless it is noteworthy that both modes show the wavenumber has doubled in the z -direction, relative to the three preceding pairs of modes.

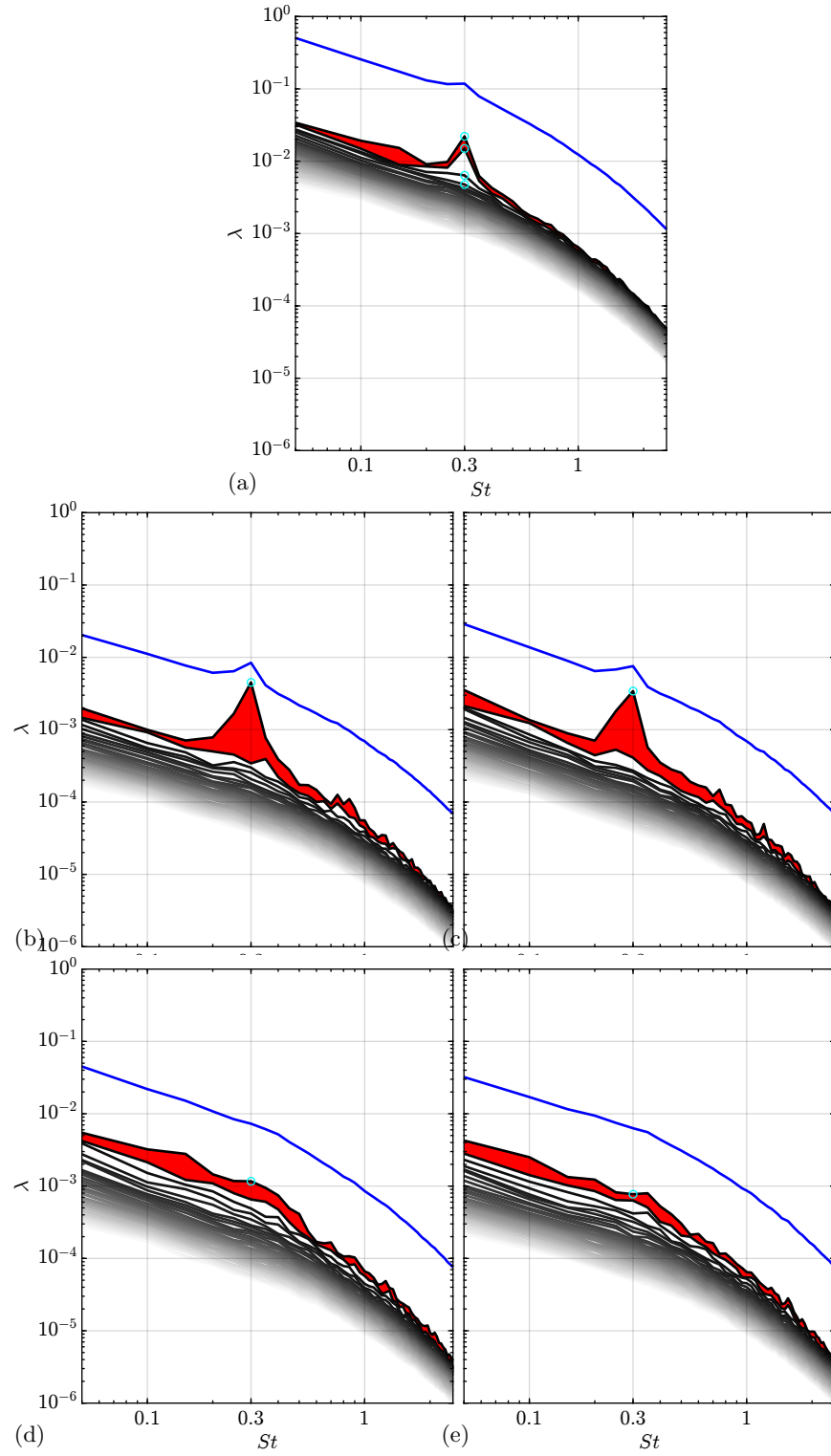


Figure 58: Eigenvalue spectra of the SPOD of pressure: (a) non- D_2 -decomposed fluctuations; (b) AA component ($\pm \pm$); (c) AS component ($\pm \pm$); (d) SS component ($\pm \pm$); (e) SA component ($\pm \mp$). The separation between the leading and first suboptimal eigenvalues is highlighted in red. The volume-integrated PSD, reconstructed by summing the eigenvalues at each frequency, is shown in blue. The modes marked with cyan circles are shown in Fig. 59, 60, 61, and 62.

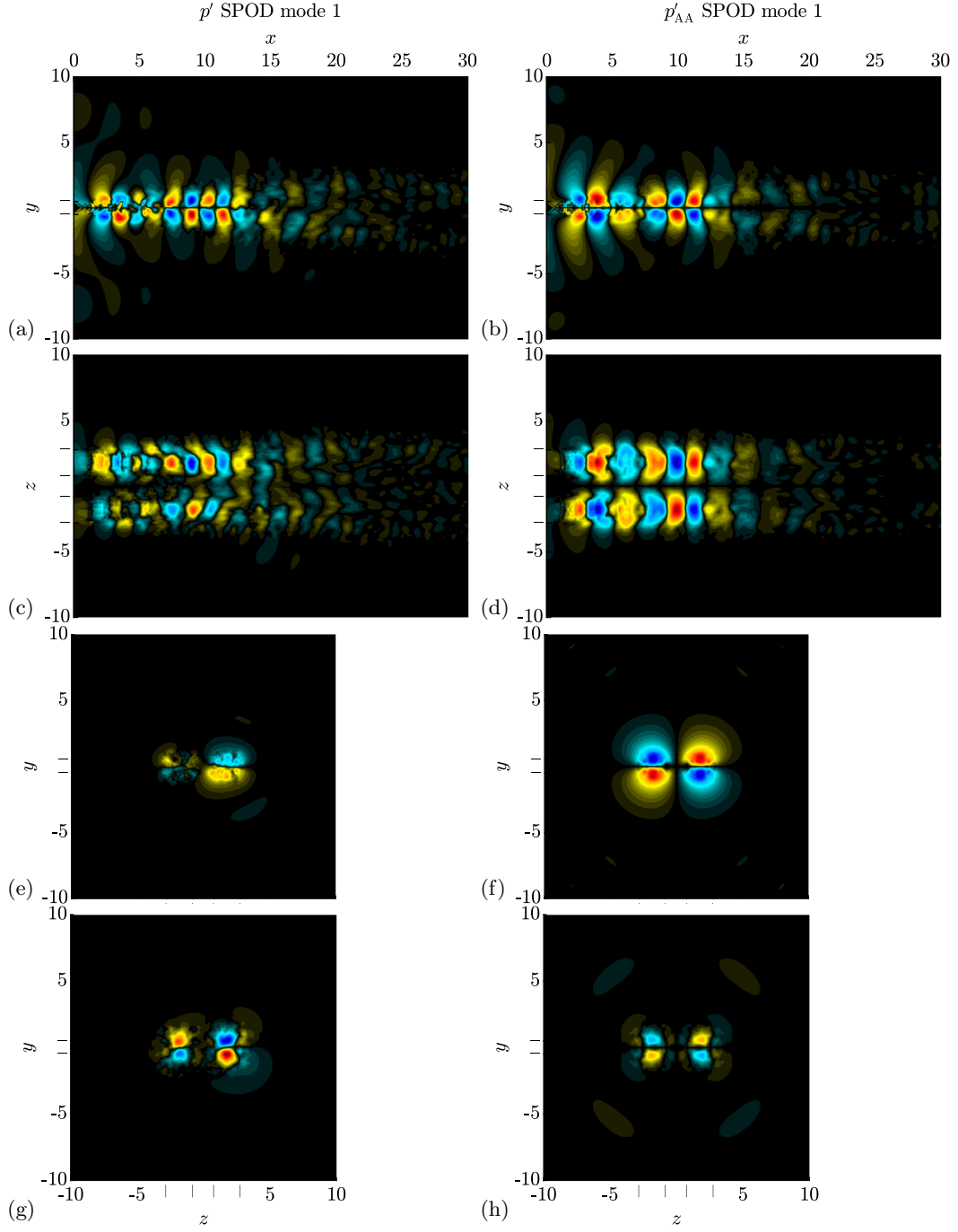


Figure 59: Comparison between SPOD mode 1 of the non- D_2 -decomposed pressure fluctuations (a,c,e,g) and SPOD mode 1 of the AA component ($\begin{smallmatrix} + \\ - \end{smallmatrix}$) of the pressure fluctuations (b,d,f,h), at $St = 0.3$. (a,b) show the $z = 1.8h$ plane through the center line of a single nozzle; (c,d) show the $y = 0.5h$ plane through the lip lines of both nozzles; (e,f) show the $x = 4h$ plane; (g,h) show the $x = 9h$ plane. Modes have been normalized ($\begin{smallmatrix} \text{blue} & \text{red} \\ -1 & 1 \end{smallmatrix} \leq \phi / \|\phi\|_\infty \leq 1$). The corresponding eigenvalues are marked with cyan circles in Fig. 58. Nozzle lip lines are marked with black lines.

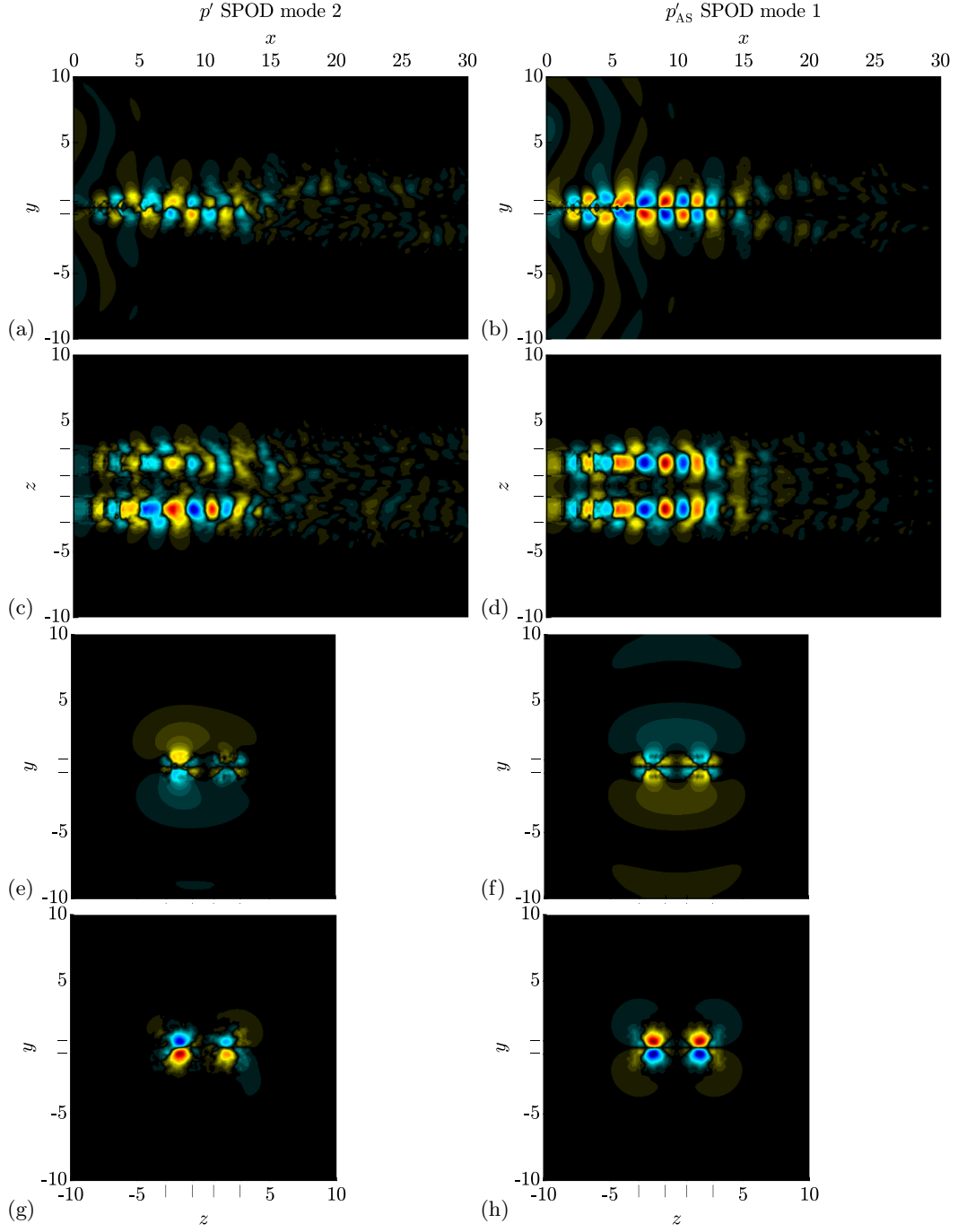


Figure 60: Comparison between SPOD mode 2 of the non- D_2 -decomposed pressure fluctuations (a,c,e,g) and SPOD mode 1 of the AS component ($\pm$ $\pm$) of the pressure fluctuations (b,d,f,h), at $St = 0.3$. (a,b) show the $z = 1.8h$ plane through the center line of a single nozzle; (c,d) show the $y = 0.5h$ plane through the lip lines of both nozzles; (e,f) show the $x = 4h$ plane; (g,h) show the $x = 9h$ plane. Modes have been normalized ($\blacksquare \blacksquare \blacksquare -1 \leq \phi / \|\phi\|_\infty \leq 1$). The corresponding eigenvalues are marked with cyan circles in Fig. 58. Nozzle lip lines are marked with black lines.

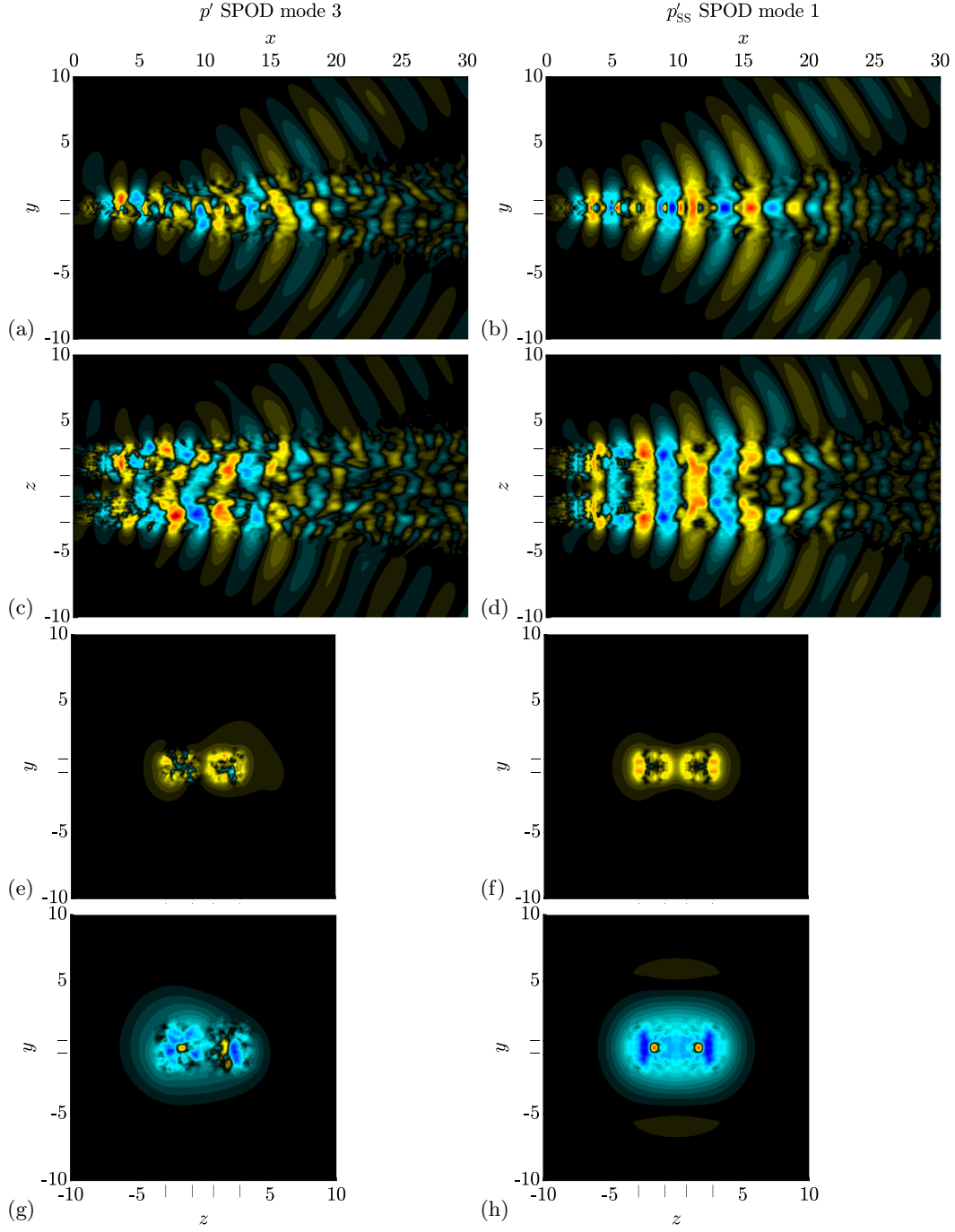


Figure 61: Comparison between SPOD mode 3 of the non- D_2 -decomposed pressure fluctuations (a,c,e,g) and SPOD mode 1 of the SS component ($\begin{smallmatrix} + \\ + \end{smallmatrix}$) of the pressure fluctuations (b,d,f,h), at $St = 0.3$. (a,b) show the $z = 1.8h$ plane through the center line of a single nozzle; (c,d) show the $y = 0.5h$ plane through the lip lines of both nozzles; (e,f) show the $x = 4h$ plane; (g,h) show the $x = 9h$ plane. Modes have been normalized ($\begin{smallmatrix} \text{blue} & \text{red} \end{smallmatrix} -1 \leq \phi / \|\phi\|_\infty \leq 1$). The corresponding eigenvalues are marked with cyan circles in Fig. 58. Nozzle lip lines are marked with black lines.

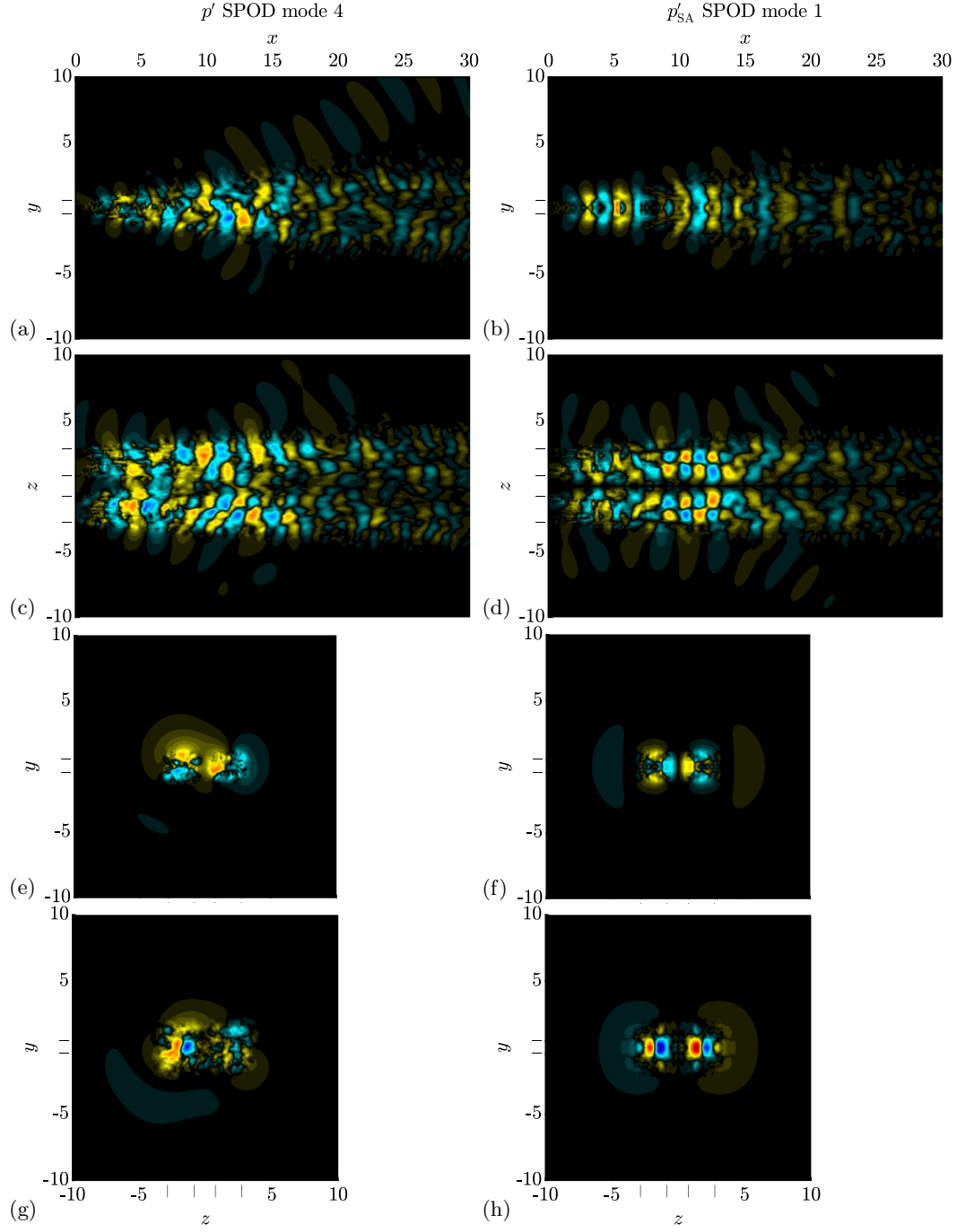


Figure 62: Comparison between SPOD mode 4 of the non- D_2 -decomposed pressure fluctuations (a,c,e,g) and SPOD mode 1 of the SA component ($\pm$) of the pressure fluctuations (b,d,f,h), at $St = 0.3$. (a,b) show the $z = 1.8h$ plane through the center line of a single nozzle; (c,d) show the $y = 0.5h$ plane through the lip lines of both nozzles; (e,f) show the $x = 4h$ plane; (g,h) show the $x = 9h$ plane. Modes have been normalized ($\frac{\phi}{\|\phi\|_\infty} \in [-1, 1]$). The corresponding eigenvalues are marked with cyan circles in Fig. 58. Nozzle lip lines are marked with black lines.

Figure 63 shows the eigenvalues of the SPOD of the primitive variables vector, $[\rho, u, v, w, T]^\top$, decomposed into symmetry components. As with the pressure SPOD spectra in Fig. 58, we observe a peak corresponding to the screech tone at $St = 0.3$, only in the AA and AS components, not SS or SA. For each symmetry component, the normalized pressure components of the leading SPOD modes at $St = 0.3$ and 0.9 are visualized as 3D isosurfaces of $\phi/\|\phi\|_\infty = \pm 0.3$ in Fig. 64. We select this isovalue, ± 0.3 , to highlight the near-field hydrodynamic structures. KH-type wavepackets, previously observed in axisymmetric jets Schmidt et al. [2018a], are evident here in the shear layers. At higher frequency, the maximum amplitude of the modes is localized further upstream, and the modes do not extend as far downstream.

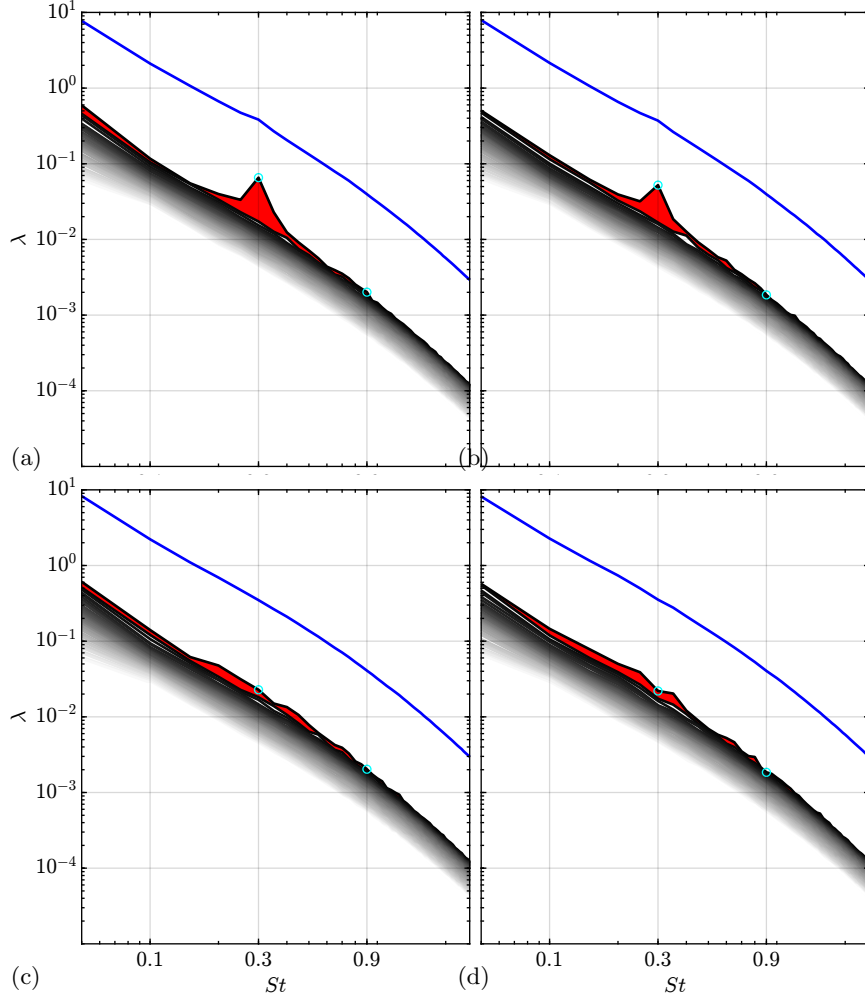


Figure 63: Eigenvalue spectra of the SPOD of the primitive variables vector, $[\rho, u, v, w, T]^\top$: (a) AA component ($\begin{smallmatrix} + & - \\ - & + \end{smallmatrix}$); (b) AS component ($\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}$); (c) SS component ($\begin{smallmatrix} + & + \\ + & + \end{smallmatrix}$); (d) SA component ($\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}$). The separation between the leading and first suboptimal eigenvalues is highlighted in red. The volume-integrated PSD, reconstructed by summing the eigenvalues at each frequency, is shown in blue. The modes marked with cyan circles are shown in Fig. 64 and 65.

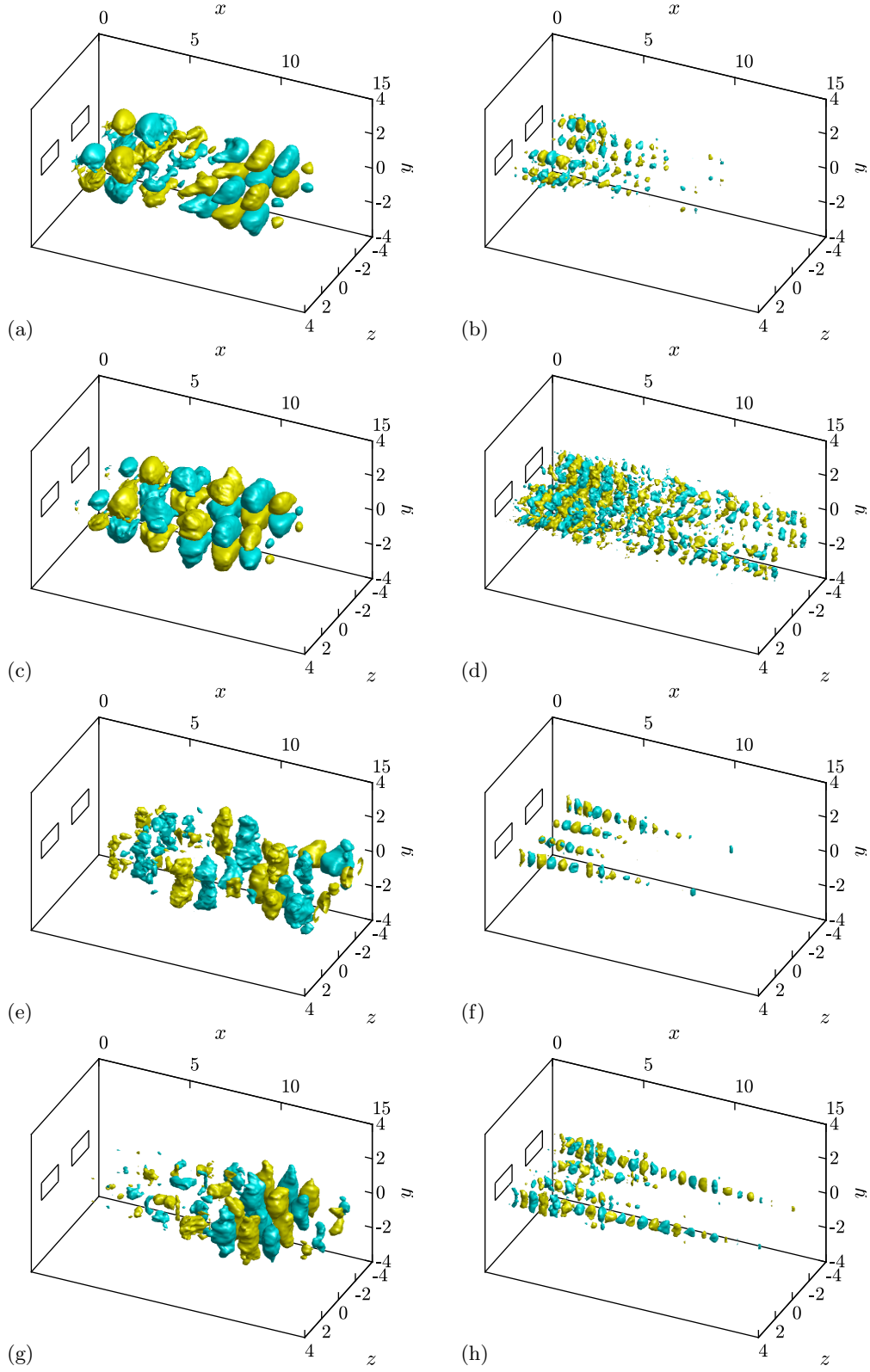


Figure 64: The pressure components of the leading SPOD modes of the primitive variables vector, decomposed into AA (a,b), AS (c,d), SS (e,f), and SA (g,h) components, at $St = 0.3$ (a,c,e,g) and $St = 0.9$ (b,d,f,h). Isosurfaces of $\phi/\|\phi\|_\infty = \pm 0.3_4$ (■) are shown. The corresponding eigenvalues are marked with cyan circles in Fig. 63.

Figure 65 visualizes the 3D isosurfaces of $\phi/\|\phi\|_\infty = \pm 0.025$. The smaller isovalue highlights the acoustic radiation into the far field. The acoustical structures are very well-defined, and are distinctly characterized by a single wavenumber, which is consistent with the observation that the most coherent structures in the jet are the ones that most efficiently radiate sound into the far field [Jordan and Colonius \[2013a\]](#). At $St = 0.3$ (Fig. 65(a,c,e,g)), only the AA, AS, and SS components produce large acoustic perturbations. Such perturbations are much less visible in the SA component. Note that each mode is normalized with its maximum amplitude, not scaled by energy. Among the AA, AS, and SS components, SS demonstrates increasing amplitude with downstream distance, whereas the amplitudes of AA and AS decrease with downstream distance. This should come as no surprise: at $St = 0.3$, AA and AS exhibit screech (Fig. 58 and 63), which originates in the interaction between the shear layers and the shocks [Edgington-Mitchell \[2019a\]](#). As the shocks weaken with downstream distance, so does the shock-associated far-field sound. At $St = 0.9$, the AA, AS, and SA components show distinct far-field acoustic fields (Fig. 65(b,d,h)), whereas the SS component does not (Fig. 65(f)).

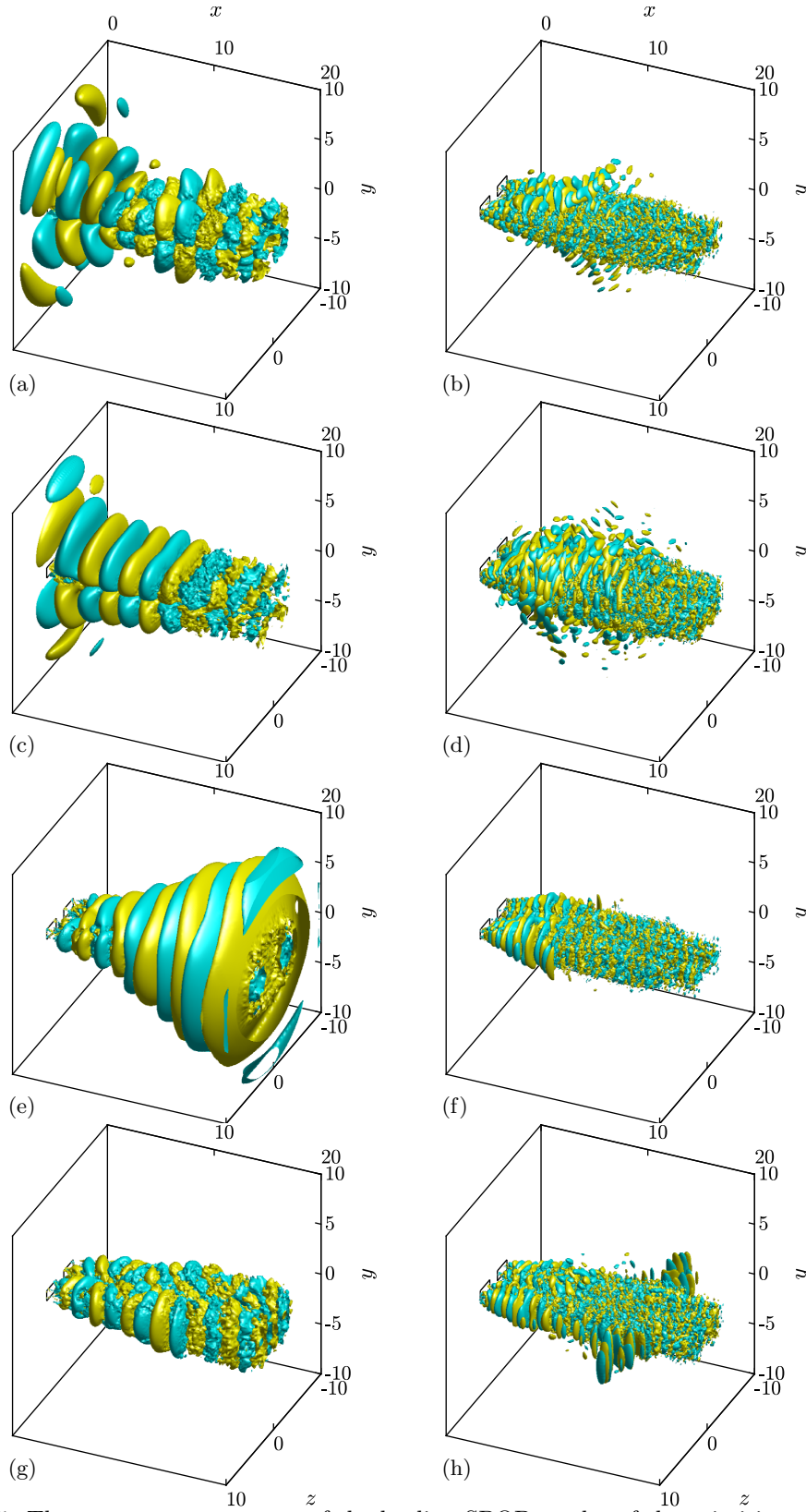


Figure 65: The pressure components of the leading SPOD modes of the primitive variables vector, decomposed into AA (a,b), AS (c,d), SS (e,f), and SA (g,h) components, at $St = 0.3$ (a,c,e,g) and $St = 0.9$ (b,d,f,h). Isosurfaces of $\phi/\|\phi\|_{\infty} = \pm 0.025$ (cyan/yellow) are shown. The corresponding eigenvalues are marked with cyan circles in Fig. 63.

5.4 Modeling of plasma actuation

This section describes the plasma actuation model, which is identical to the one proposed by Kim et al. [Kim et al. \[2009\]](#). Representing the plasma filament that develops between pairs of electrodes, and added to the right-hand side of the energy equation, is a volumetric heat source

$$S(x, y, z, t) = n(t) l(r) m(z) \frac{VI}{\pi r_0^2 L}, \quad (115)$$

with spatial distribution functions

$$l(r) = \frac{1}{2} \left[\tanh \left(-\sigma_{xy} \left(r - \frac{1}{2} \right) \right) + 1 \right], \quad (116)$$

and

$$m(z) = -\frac{1}{2} \left[\tanh \left(-\sigma_z \frac{z - z_l}{r_0} \right) + 1 \right] + \frac{1}{2} \left[\tanh \left(-\sigma_z \frac{z - z_r}{r_0} \right) + 1 \right]. \quad (117)$$

This heat source has the shape of a cylinder of radius r_0 , centered in the groove. It is bounded by $z = z_l$ and $z = z_r$ in the axial dimension, and centered at $(x_{\text{act}}, y_{\text{act}})$ so that $r = \sqrt{((x - x_{\text{act}})/r_0)^2 + ((y - y_{\text{act}})/r_0)^2}$. σ_{xy} and σ_z control the sharpness of the boundary that encloses the heat source.

The square-wave signal is given by

$$n(t) = \frac{1}{2} \left[\tanh \left(\frac{t - t_i}{t_{r,i}} \right) - \tanh \left(\frac{t - t_f}{t_{r,f}} \right) \right], \quad (118)$$

where t_i, t_f are the times at which the signal turns ‘on’ and ‘off’, and $t_{r,i}, t_{r,f}$ are the corresponding rise times.

The plasma actuation model is validated within a flat-plate turbulent boundary layer. The actuator is situated in a groove across the plate, as it is intended to be in the twin jet nozzle. Although the plasma filaments in the nozzle have finite length, here—for simplicity—the test domain has been made spanwise-periodic. This geometry is illustrated in Fig. 66, with dimensions normalized by the groove width. The freestream Mach number is defined as $M_\infty = 1.5$. The Reynolds number based on momentum thickness is defined as $Re_\theta = 31000$. The strength of the actuation is loosely based on values from Utkin et al. [Utkin et al. \[2007\]](#): $V = 400$ and $I = 0.25$. The actuator has a radius of $r_0 = 0.25$, and spans the domain such that $z_l = -1.5$ and $z_r = 1.5$. The sharpness parameters are defined as $\sigma_{xy} = \sigma_z = 5$. The actuation period is defined as $P_{\text{act}} = 2$, while the signal ‘on’, ‘off’, and rise times are $t_i = 0.4$, $t_f = 0.8$, and $t_{r,i} = t_{r,f} = 0.04$.

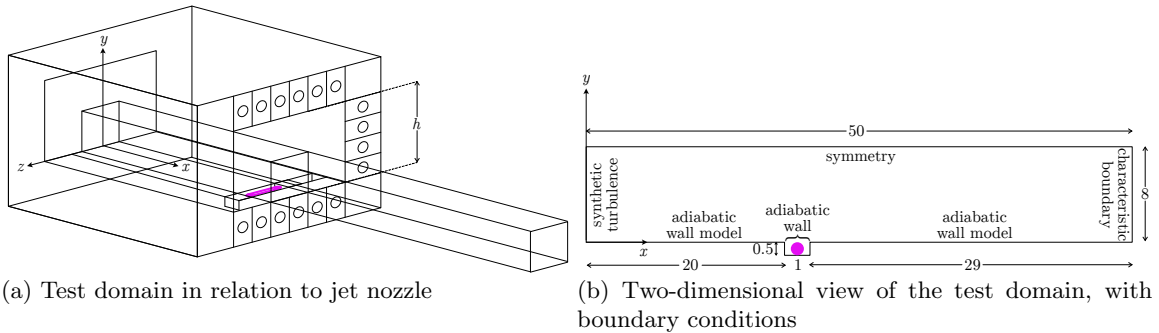


Figure 66: Computational domain used to analyze the implementation of the plasma model. The magenta cylinder represents a plasma filament. Dimensions are normalized by the groove width.

5.5 Verification of the plasma model

From the test domain (Fig. 66) used to verify the plasma actuation model, 3 successively refined meshes are constructed. The first mesh, $\Delta_{\min} \approx 0.12$, matches the near-wall resolution of the twin jet simulation, and contains about 0.15 million cells. The second and third, $\Delta_{\min} \approx 0.06$ and $\Delta_{\min} \approx 0.03$, each doubles the resolution of the previous mesh, and have about 0.62 million and 2.5 million cells, respectively. A small region near the groove in each mesh is visualized in Fig. 67.

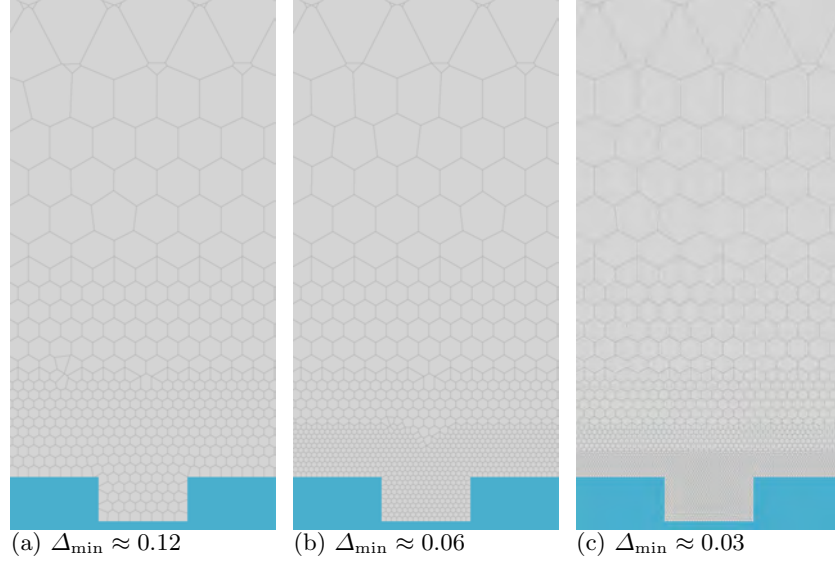


Figure 67: Computational meshes used in the grid resolution study of the simplified test domain. Dimensions are normalized by the groove width. Only a small region enclosing the groove is shown here.

Without the benefit of experimental boundary-layer measurements for direct validation, the $\Delta_{\min} \approx 0.03$ case will be treated as an approximation to the true solution. Although an in-depth investigation of cavity flows is beyond the scope of this work, it should be noted that in the baseline simulation, this resolution is sufficiently high for pressure waves emanating from the trailing edge of the groove—corresponding to Rossiter mode—as well as recirculating vortices within the groove, to be observed, as shown in Fig. 68.

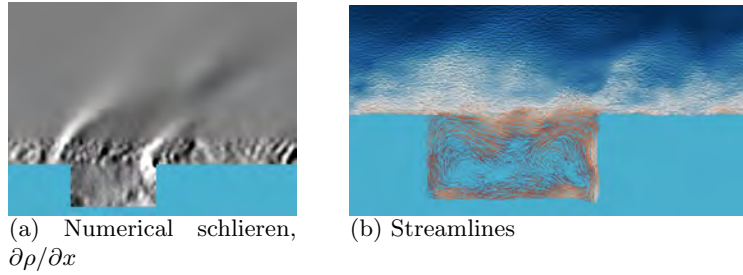


Figure 68: Details of the baseline simulation in the $\Delta_{\min} \approx 0.03$ test case, in and around the groove. Streamlines are colored by velocity magnitude.

After the initial transients, each simulation is run for 12 actuation periods, at the end of which the first pulse from the actuator has approximately reached the domain outlet. In the highest-resolution

case, i.e. $\Delta_{\min} \approx 0.03$, the instantaneous temperature field after 12 periods, averaged along the spanwise direction, is visualized in Fig. 69.

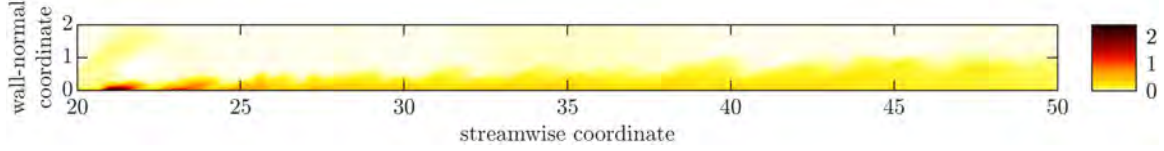


Figure 69: Log of temperature in the highest-resolution test case ($\Delta_{\min} \approx 0.03$) averaged along the spanwise direction, after 12 actuation periods.

Figure 70(a) plots the spanwise-average temperature and turbulent kinetic energy (TKE) after 12 actuation periods, integrated over wall-normal distance. As expected, both temperature and TKE show the higher-resolution meshes capturing more of the energy injected into the boundary layer by the plasma actuation, with lower dispersive error. In particular, near the actuator, the $\Delta_{\min} \approx 0.03$ and $\Delta_{\min} \approx 0.06$ cases predict prominent peaks in TKE, whereas $\Delta_{\min} \approx 0.12$ shows no peaks in that region. Far downstream of the groove, $\Delta_{\min} \approx 0.12$ remains the lowest in temperature; however, the $\Delta_{\min} \approx 0.03$ and $\Delta_{\min} \approx 0.06$ cases predict very similar temperatures. As an attempt to isolate some of the effects of grid resolution on the plasma model itself, Fig. 70(b) plots the temperature and TKE, now with the baseline quantities subtracted. As before, close to the actuator, the higher-resolution meshes capture higher temperatures. Away from the actuator however, the temperatures predicted by all 3 cases show close agreement. This suggests that the energy present in the baseline turbulent boundary layer is better captured by finer grids—again as anticipated—but the *additional* energy injected by the plasma, measured in terms of temperature far from the actuator, is much less sensitive to grid resolution. Though the physics near the groove is significantly modified by the larger grid spacings, as long as the region of interest is far downstream of the actuator, these results tentatively suggest that the current refinement level within the nozzle-exit boundary layer of the twin jet case is sufficient. A detailed comparison of temperature and TKE between successively refined meshes requires fully converged solutions, and will be included as part of future work.

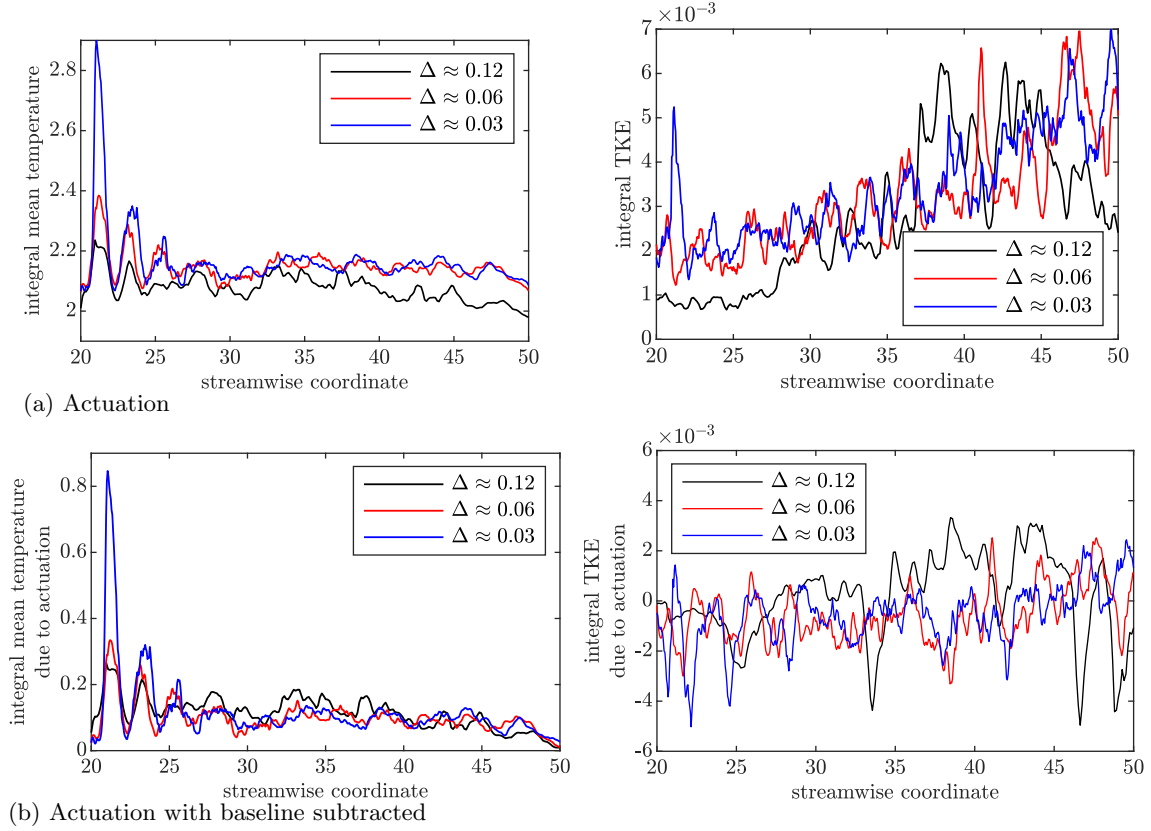


Figure 70: Spanwise-average temperature and turbulent kinetic energy after 12 actuation periods, integrated over wall-normal distance.

6 Conclusions

In this work, we have developed simulation and data-driven analysis tools for forced, turbulent jets relevant to noise reduction on military-style aircraft, and we have applied these tools to canonical round subsonic and supersonic nozzles, as well as to a supersonic rectangular twin jet nozzle. We have used these tools to examine how the fundamental turbulence structure in these flows is modified by forcing.

First, we proposed a cyclostationary variant of the SPOD algorithm, which we term CS-SPOD, for the extraction of the most energetic coherent structures from complex turbulent flows whose statistics vary periodically in time. This is achieved by an extension of the one-dimensional technique developed by [Kim et al. \[1996\]](#) to large high-dimensional data through the use of the method-of-snapshots to make the algorithm computationally feasible for large data. The orthogonality/optimality properties of the modes generated by CS-SPOD are shown, where, similar to SPOD analysis of stationary flows, CS-SPOD determines the set of orthogonal modes that optimally reconstruct the statistics of these flows in terms of the space-time norm.

In contrast to SPOD, where the modes oscillate at a single frequency and have a constant amplitude in time, CS-SPOD modes oscillate at a set of frequencies separated by the fundamental cycle frequency (typically the frequency of modulation), have a periodic amplitude in time, and optimally reconstruct the second-order statistics. We show that CS-SPOD naturally becomes SPOD when analyzing a stationary process, allowing the CS-SPOD results to be interpreted in a familiar

manner. Furthermore, we develop an efficient computational algorithm to compute CS-SPOD with a computational cost and memory requirement similar to SPOD, thus allowing CS-SPOD to be computed on a wide range of problems. Lastly, similar to the relationship that exists between SPOD and standard resolvent analysis [Towne et al., 2018c], CS-SPOD modes are identical to harmonic resolvent modes in the case where the harmonic-resolvent-mode expansion coefficients are uncorrelated. We also discuss simplifications that can be made when forcing at a low or high frequency.

Our results show that CS-SPOD successfully extends SPOD to flows with cyclostationary statistics. apart from forced jets, the tool can be used to study a wide range of flows with time-periodic statistics such as turbomachinery, weather and climate, and wake flows. Although we focused on strictly cyclostationary processes, further generalizations are possible to almost periodic flows and flows forced with several non-commensurate frequencies.

Next, we applied CS-SPOD to time-periodic acoustic forcing of turbulent axisymmetric subsonic and supersonic jets. We performed a series of large-eddy simulations subjected to periodic forcing at $St_f = 0.3$, which corresponds to the jet-preferred mode [Crow and Champagne, 1971], and at $St_f = 1.5$, which corresponds to forcing that previous studies have shown results in a broadband noise reduction Samimy et al. [2007, 2010], at an amplitude of 1%, 10% of the jet velocity. Both the low-frequency $St_f = 0.3$ and high-frequency $St_f = 1.5$ forcing generated an energetic tonal response. The $St_f = 0.3$ forcing exhibited the largest effect towards the end of the potential core while the $St_f = 1.5$ forcing was localized to $x \in [0, 2]$. Despite the strong response, all forcing cases had a limited effect on the period-averaged mean, and a forcing amplitude greater than 1% was required to achieve a small change.

The phase-dependency of the statistics was found to follow a similar, but more subtle, trend. The RMS, Wigner-Ville spectrum, and energy transfer between the mean and turbulent fields were found to be mildly modulated by forcing, with only a substantial modification coming from the $St_f = 0.3$, $a_0/U_j = 10\%$ forcing case. Despite the modulation of the phase-dependent energy transfer, the domain-integrated energy transfer was found to be surprisingly constant as a function of phase, indicating that regions of locally greater energy transfer are canceled out by corresponding region of locally lower energy transfer. Furthermore, the change in the period-averaged energy transfer was found to be substantially larger than the phase-dependent modulation, in contrast to the mean flow modulation where the phase-dependent modulation is much greater. This suggests that the impact of the forcing on the turbulence is minor and that any substantial change to the turbulence needs to be driven by a corresponding change to the period-averaged mean.

An examination of the CS-SPOD modes shows how the dominant coherent structures are modulated by external forcing. For $St_f = 0.3$, a broadband increase in the energy of these structures was found. Furthermore, a phase-dependent modulation of the low-frequency coherent structures was seen, with the high-energy region of the mode being slaved to the high-velocity/high-shear regions of the mean flow. The $St_f = 1.5$ was found to have lower energy over almost the entire spectrum, suggesting that the new period-averaged mean flow is more stable. No phase-dependent modulation of the low-frequency coherent structures was seen due to a large difference in the wavelength and spatial support between the coherent structures and the mean field. Over a small range of γ , the $St_f = 1.5$ forcing was found to result in a substantial increase in the energy, which a previous study Heidt et al. [2021] had linked to vortex pairing. However, further investigation is required to prove this. A similar investigation on a $M_j = 1.5$ supersonic jet showed a similar impact of the forcing for the $St_f = 1.5$ forcing and a reduced impact for the $St_f = 0.3$ forcing.

Theses results largely confirm conclusions drawn from two earlier studies Sinha et al. [2018], Koenig et al. [2016], namely that while the jet is strongly receptive to unsteady forcing, it produces a phase-locked response that has only a modest impact on the period-averaged mean flow and that does not result in any significant reorganization of the turbulence unless the forcing amplitude is substantially high (on the order of 10% of the jet velocity for the present study). Future work includes determining effective forcing strategies based principally on optimizing the deformation of

the mean flow field while minimizing the directly radiating component.

In order to extend our understanding to more complex geometries, large-eddy simulations were performed for Mach 1.5 jets issuing from a twin rectangular nozzle of aspect ratio 2, using the GPU-accelerated version of Cascade’s compressible flow solver “charLES”. The operating condition and complex nozzle geometry match the experimental configuration from Ohio State University, which was designed to investigate the effects of plasma actuation [Esfahani et al. \[2021\]](#), [Leahy et al. \[2022\]](#). In this initial study, the LES were conducted at the baseline conditions (i.e., no plasma actuation) but all the nozzle internal details are included in the computational domain, i.e., the electrodes and small groove upstream of the nozzle exit to shelter the arc discharge between electrodes during actuation.

Comparisons of the experimental measurements and LES predictions at near-field and far-field microphones show good overall agreement in terms of screech tone frequencies, broadband levels, and main far-field noise radiation towards the aft angles. In contrast, the screech feedback loop and tone amplitudes are more sensitive to the precise operating conditions & nozzle geometrical details, and therefore more challenging to measure and predict. To account for the thick open cell acoustic foam installed in the wind-tunnel on the supply pipe surfaces upstream of the nozzle exit, a second “non-reflective” nozzle was simulated in which all the reflective surfaces associated with the supply pipe are removed. For the original nozzle with the reflective surfaces included, the screech tone amplitude at the near-field microphones along the minor axis is over-predicted by 3 – 5 dB. With the non-reflective nozzle, the screech amplitude is reduced and within 1 – 2 dB of the measurements for most microphones.

In terms of performance with the GPU-accelerated solver, the computational cost for the collection of the relatively long noise database of 2000 acoustics time unit on the 97 million cv mesh was approximately 360 GPU hours. This represents a time-to-solution of 12h on 30 Nvidia V100 GPUs. For this case, a strong scalability study was also conducted on “Narwhal” (HPE Cray EX system) at Navy DSRC. The GPU-accelerated version of charLES displayed 90% scalability on 60 GPUs (i.e., loading of 1.6M cv per GPU), with a throughput of $\mathcal{O}(400)$ equivalent cores per V100 GPU. This reduction in computational cost will make possible the exploration of more nozzle designs and/or actuation cases in future work, along with the collection of longer LES databases for statistical (SPOD) analysis [Yeung et al. \[2022\]](#).

Finally, we analyzed the turbulent structures in the twin rectangular nozzle. We applied a novel symmetry decomposition to jets in the D_2 symmetry group, i.e., nozzle geometries that possess two planes of reflection symmetry. Snapshots from LES are divided into four symmetry components, termed SS, SA, AS, and AA. For each component, we performed an independent SPOD analysis, without loss of generality. A comparison between SPOD with and without D_2 decomposition confirms that D_2 decomposition both accelerates statistical convergence and reduces memory requirement—crucial to the efficient use of high-performance computing and local post-processing resources. In terms of flow physics, the eigenvalue spectra confirm the screech tone at $St = 0.3$ observed in LES by [Brès et al. \[2021\]](#) as well as in companion experiments by [Esfahani et al. \[January 2021\]](#). Microphone measurements from [Esfahani et al. \[January 2021\]](#) also showed that the screech mode is antisymmetric along the minor axis. This is again confirmed by our analysis: D_2 -symmetric SPOD identifies the AA and AS components, both of which are antisymmetric along the minor axis, as the primary sources of screech. Our SPOD, which is carried out in 3D, further suggests the screech mode can be either symmetric or antisymmetric along the major axis. 3D isosurfaces of the SPOD modes reveal the symmetry-dependence of the far-field acoustics. At low frequency, the AA, AS, and SS components produce clear acoustic structures. At higher frequency, however, AA, AS, and SA are the components that radiate well-defined acoustic fields. These observations suggest that D_2 decomposition offers more than computational convenience: it improves the interpretability of results.

The twin-rectangular jet’s natural preference for the AA and AS components at the screech frequency opens up the possibility for the dominant instabilities of the jet to be controlled through ex-

ternal forcing. We plan to apply D_2 -symmetric SPOD to the simulation of a forced, twin-rectangular jet. As part of future work, we will also apply D_2 decomposition to other modal decomposition techniques to analyze the twin jet.

As a first step towards modeling plasma actuation in supersonic jets, a Mach 1.5 cold jet issuing from a twin rectangular nozzle of aspect ratio 2 is investigated using large eddy simulation. The operating condition and geometries match the experimental configuration from Ohio State University, including the electrodes and small groove upstream of the nozzle exit to shelter the arc discharge between electrodes during actuation. In the present study, a single initial LES was performed at the baseline conditions (i.e., no plasma actuation) without prior knowledge of the experimental data. The preliminary noise predictions are in good agreement with the near-field and far-field microphone data. The broadband levels, the screech tone frequencies and amplitudes and the main noise radiation towards the aft angles are well predicted. Additional analysis is ongoing to investigate some of the discrepancies for upstream angles and higher frequencies. Nevertheless, the overall sound pressure level directivity from the LES is within 1dB of the experimental value for most relevant frequencies and angles.

In parallel, the modeling of plasma actuation is investigated in the canonical configuration of a flat-plate turbulent boundary layer, with the actuator situated in a groove across the plate, as it is intended to be in the twin jet nozzle. A grid resolution study was conducted to verify the model implementation, and guide the minimum grid resolution required to correctly predict energy transport within the nozzle-exit boundary layer. Preliminary results tentatively suggest that the current resolution in the twin jet case is sufficient.

In future work, the baseline LES will be further analyzed and validated against additional experimental data (including PIV), as it becomes available. Validation of the plasma actuator model in the settings of the twin jet configuration will be performed with the OSU data. For both baseline and actuation cases, full LES databases will then be collected for statistical (SPOD, CS-SPOD) analysis.

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