DECLASSIFIED

CONFIDENTIAL

UNCLASSIFIED

NRL REPORT 3593

FR-3593

A LIMITED ANALYTICAL SURVEY OF PULSE TIME MODULATION WITH A FIXED REFERENCE

DECLASSIFIED by NRL Contract
Declassification Team
Date: 19 JAN 2017
Reviewer's name(s): [redacted]
Declassification authority: NAVY DECLASS GUIDE / NAVY DECLASS MANUAL, 11 DEC 2012

DEPARTMENT OF THE NAVY OFFICE OF NAVAL RESEARCH

DISTRIBUTION STATEMENT A APPLIES
Further distribution authorized by UNLIMITED only.

NAVAL RESEARCH LABORATORY

WASHINGTON, D.C.

DECLASSIFIED

[redacted]

UNCLASSIFIED

CONFIDENTIAL

NRL REPORT 3593

A LIMITED ANALYTICAL SURVEY OF PULSE TIME MODULATION WITH A FIXED REFERENCE

Roger C. Jones

December 14, 1949

Approved by:

Mr. H. L. Flowers, Head, Command and Report Links Branch
Dr. R. M. Page, Superintendent, Radio Division III

NAVAL RESEARCH LABORATORY
CAPTAIN F. R. FURTH, USN, DIRECTOR
WASHINGTON, D.C.

DISTRIBUTION

CNO	1
ONR Attn: Code 470	1
BuAer Attn: Mr. H. Petrowitz, Code EL-72	1
BuOrd Attn: Mr. W. A. Lewis, Code Re9b	1
BuShips Attn: Mr. S. Levine, Code 927	1
Dir., USNEL	2
CDR, USNOTS Attn: Reports Unit	2
Supt., USNPGS	1
BAGR, CD, Wright-Patterson AFB Attn: CADO-D1	1
OCSigO Attn: Ch. Eng. & Tech. Div., SIGTM-S	1
CO, SCEL Attn: Dir. of Eng.	2
CG, AMC, Wright-Patterson AFB Attn: Eng. Div., Electronics Subdiv., MCREEO-2	1
CO, Watson Labs., AMC, Red Bank Attn: ENR	1
CO, Air Force Cambridge Res. Labs., Attn: ERRS	1
NBS Attn: CRPL	1
RDB Attn: Library Attn: Navy Secretary	2 1
Naval Res. Sec., Science Div., Library of Congress Attn: Mr. J. H. Heald	2
ANAF/GM Mailing List No. 9 Part A	

[redacted]

UNCLASSIFIED

CONTENTS

[redacted]

FOREWORD

The literature concerning pulse time modulation is almost devoid of information dealing with the special case where a fixed reference is employed. This report discusses and analyzes certain aspects of this special case. On the general subject of pulse time modulation many works exist which are applicable to a study of the fixed-reference type of modulation. A major object of this report is to consolidate this information and apply it to the analysis of this special case. Certain analyses are surveyed in connection with the transient phenomena involved. The practical application stressed throughout is a Command Guidance System.

PROBLEM STATUS

This report concludes one phase of a continuing study being made on pulse type command systems. Work on other phases of research, design, and development is continuing.

AUTHORIZATION

NRL Problem R05-14D

A LIMITED ANALYTICAL SURVEY OF PULSE TIME MODULATION WITH A FIXED REFERENCE

INTRODUCTION

In many applications of pulse time modulation the system used is termed "pulse time modulation without a fixed reference." When pulse time modulation is applied to a command guidance system, however, a number of considerations make it advisable to use a system called "pulse time modulation with a fixed reference." Since, by the principle of superposition, the latter type of modulation can be considered as the sum of an unmodulated wave and a wave modulated in time with no fixed reference, it is obvious that a consideration of the nonfixed-reference case is desirable.

Concerning application to a particular control system, the required specifications in order of importance are: reliability, simplicity, and security. Although pulse time modulation with a fixed reference is extremely secure, particularly when the reference pulses are random-modulated, it is by no means simple, and this lack of simplicity causes some reduction in the factor of reliability. Future developments, however, are expected to make the system quite satisfactory in both simplicity and reliability. In connection with this system, some attention must also be given to the problem of proportional control. It is basically because of this problem of proportional control that an analysis of pulse time modulation with a fixed reference is necessary.

A TYPICAL COMMAND CONTROL SYSTEM

In order to decide what analytical procedures are necessary regarding the fixed-reference case, it is desirable to be acquainted with the general manner in which a typical command system functions. A block diagram of a command control system using pulse time modulation with a fixed reference is shown in Figure 1. A short explanation of the system will show the manner in which pulse time modulation enters into its operation.

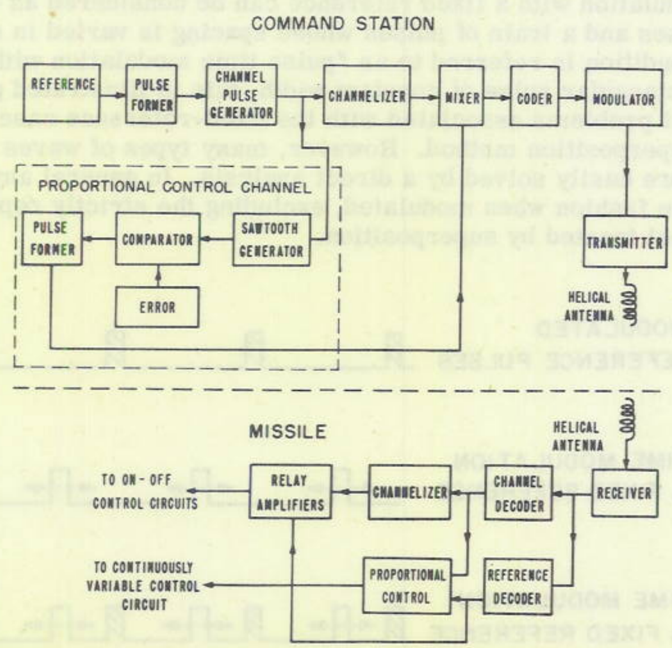

Fig. 1: A Typical Command Control System

A sine-wave generator forms the reference frequency signal which is sent through appropriate pulse-forming networks. The resultant series of reference pulses periodically excites an oscillator which produces the correct number of channel pulses per excitation. The resultant train of pulses is then run through a channelizer which allows pulses to be cut on or off at will. In the meantime, the reference pulses and the on-off pulses are also sent to proportional control units. In these units the reference pulses produce sawtooth pulses, and comparators select small top sections of these pulses in accordance with an input error voltage. The pulse sections are differentiated, and the proportional control pulses are formed. The reference pulse, the on-off channel pulses, and the proportional control pulses are mixed and coded. They then modulate a transmitter which may, for echo-rejection reasons, feed a helical antenna or some other circularly polarized array.

In the missile the action is somewhat similar. Briefly, the receiver, which is of fairly conventional design, feeds its output of coded pulses to the decoders. Because the reference pulse is coded differently from the other pulses, two decoders are needed. The resultant train of decoded single pulses is fed through a channelizer to a series of independent relay amplifiers. The contacts of the associated relays then go to the appropriate control circuits. The proportional control units are fed from the proper points and produce sawtooth pulses whose amplitude and pulsewidth are variable and whose leading edge occurs in a strictly repetitive manner. The variable width sawtooth pulse builds up a step-wave approximation to the intelligence function. This stepped approximation, when adequately filtered, is sent to the appropriate, continuously variable control circuits.

It may be seen from this discussion, and from the associated block diagram, that many circuits must handle rectangular and sawtooth pulses which are recurrent, nonrecurrent, and modulated in a complex fashion. In order to avoid a completely empirical approach to the design of these circuits, it is desirable to be acquainted with the frequency spectra of these various waves.

Pulse Time Modulation Without a Fixed Reference

In all linear systems the principle of superposition applies. Since this condition exists, the case of pulse time modulation with a fixed reference can be considered as the sum of an unmodulated train of pulses and a train of pulses whose spacing is varied in some specified manner. The latter condition is referred to as "pulse time modulation without a fixed reference." For the case of the rectangular pulse of constant width, this is illustrated graphically by Figure 2. Many of the difficult problems associated with the fixed-reference case can be treated most conveniently by this superposition method. However, many types of waves involved in the fixed-reference case are more easily solved by a direct analysis. In general any train of pulses which occur in a nonrepetitive fashion when modulated, excluding the strictly repetitive reference pulses of course, is best treated by superposition.

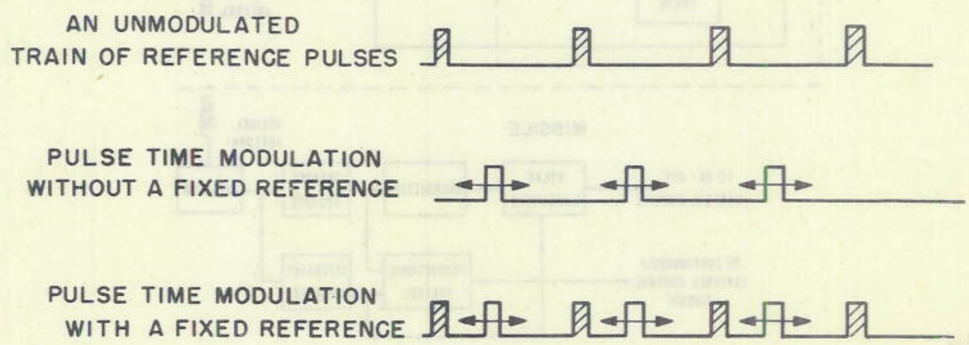

Fig. 2: The Relation Between Fixed and Non-Fixed Reference Cases

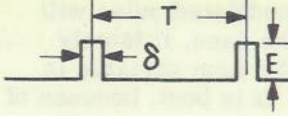

Fig. 3: An Unmodulated Train of Pulses

The unmodulated, rectangular pulse train (Figure 3) has the following well-known equation (1):

$$f(t) = \frac{E\delta}{T} + \frac{2E\delta}{T} \cdot \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}} \cos n\omega t.$$

If the spacing between a set of pulses is varied, then it is reasonable to assume that many new frequencies will be generated. The equation representing the new frequency spectrum is dependent upon the manner in which the modulation is accomplished. Excluding pulsewidth modulation, two important cases exist. These are frequency modulation and phase modulation, sometimes referred to as pulse numbers modulation and pulse position modulation respectively (2).

In pulse frequency modulation the number of pulses per second, evaluated at any point since the width is constant, is directly proportional to the amplitude of the signal voltage. In pulse phase modulation, however, the displacement of a pulse is directly proportional to the signal amplitude. From elementary angular considerations it is known that frequency is equal to a constant times time rate of change of angle. Mathematically,

$$f = \frac{1}{2\pi} \cdot \frac{d\theta}{dt}.$$

Therefore if θ (the pulse position) is proportional to I(t) (the intelligence function) for phase modulation, then $\frac{d\theta}{dt}$ is proportional to I(t) for frequency modulation. Therefore θ is proportional to $\int I(t)dt$ for the case of frequency modulation.

In most cases the difference between the phase- and frequency-modulated cases is of little importance. In some instances, however, it is necessary to use one or the other for special reasons. In command guidance, for example, it is desirable to use pulse phase modulation, because the position of the channel pulse with respect to the reference pulse should be a direct function of the signal amplitude.

There are two main approaches to the analysis of pulse time modulation without a fixed reference. One, which might be termed the direct method, is to consider the nonrepetitive, rectangular pulse train as it actually exists and to carry out a straight-forward, though difficult, analysis. The other method is to attempt to express the nonperiodic pulse train in terms of some periodic function and thus simplify the analysis.

If the unmodulated repetition rate is very much greater than the modulating frequency, then the exact time of the pulse occurrence may be neglected. With this approximation, the direct analysis has been carried out in a satisfactory manner (1). If the repetition rate is not considerably greater than the modulating frequency, however, this approximation produces considerable error. Then, if it is still desired to use this method, it is necessary to solve a transcendental equation to specify the exact instant of pulse occurrence.

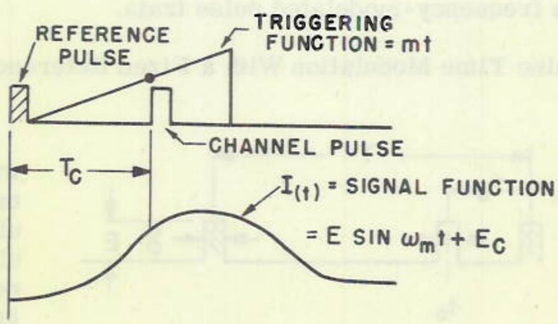

Fig. 4: Proportion Control Channel Pulse Generation.

Referring again to the special case of a command guidance system, normally the triggering function (usually a sawtooth) is balanced out by the intelligence voltage (assumed to be sinusoidal) in a comparator circuit to produce the modulated pulse. This is shown in Figure 4. In this particular case, the modulated pulse will occur when the two functions involved are of equal amplitude. This being the case, it is only necessary to equate the two functions and solve for t. But if this value of the time variable is inserted in the equations involved, their expansion becomes quite difficult. It is best, because of this reason, to proceed with the second method.

The second method, which may be called the synthesis method, has been carried out by at least two separate investigations (2, 3a). A modified analysis, more suited to the nomenclature of this report, appears on page 18 of the Appendix. It is derived by recognizing that the unmodulated, rectangular pulse train represents, mathematically, the sum of two sawtooth waves and a direct component. Because the sawtooth waves are periodic, although of constantly shifting phase, it is not necessary to resort to any approximations. The following equation thus represents accurately the frequency spectrum of a phase modulated, rectangular pulse train.

$$g(t) = \frac{E\delta}{T} + \frac{E_{m_p}}{T}\left[\sin\omega_m\left(t+\frac{\delta}{2}\right) - \sin\omega_m\left(t-\frac{\delta}{2}\right)\right]$$

$$+\frac{E}{\pi}\sum_{n=1}^{\infty}\cdot\frac{1}{n}\left[\begin{array}{l} J_o(n\omega\cdot m_p)\left\{\sin n\omega\left(t+\frac{\delta}{2}\right) - \sin n\omega\left(t-\frac{\delta}{2}\right)\right\} \\ +\sum_{m=1}^{\infty} J_m(n\omega\cdot m_p)\left\{\begin{array}{l} \sin\left[n\omega\left(t+\frac{\delta}{2}\right) + m\omega_m\left(t+\frac{\delta}{2}\right)\right] \\ +(-1)^m\sin\left[n\omega\left(t+\frac{\delta}{2}\right) - m\omega_m\left(t+\frac{\delta}{2}\right)\right] \\ -\sin\left[n\omega\left(t-\frac{\delta}{2}\right) + m\omega_m\left(t-\frac{\delta}{2}\right)\right] \\ -(-1)^m\sin\left[n\omega\left(t-\frac{\delta}{2}\right) - m\omega_m\left(t-\frac{\delta}{2}\right)\right] \end{array}\right\} \end{array}\right]$$

It is obvious from this equation that the term containing the intelligence will be very small in magnitude if the pulsewidth is very small. Since this is the case, it is not desirable to extract the signal by filtering. When combined with an unmodulated wave in such a fashion as to be equivalent to pulse time modulation with a fixed reference, the intelligence is easily obtained by a demodulation process involving a sawtooth generator (see page 6). It is also apparent from the equation that a type of sideband distortion takes place, because the carrier and all of its harmonics form sidebands with all harmonics of the intelligence frequency. Because of the method of demodulation, however, this effect is not important in a command guidance system. By recognizing that the displacement is proportional to the integral of the signal function for the frequency-modulated case, appropriate modifications of the above equation yield the solution for the frequency-modulated pulse train.

Pulse Time Modulation With a Fixed Reference

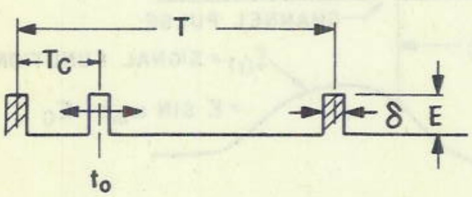

Fig. 5: Pulsed Time Modulation with a Fixed Reference

It has been shown that pulse time modulation with a fixed reference is equivalent to pulse time modulation without a fixed reference plus an unmodulated wave. An arbitrarily chosen pulse in the modulated train must obviously be noncoincident with its reference pulse in the unmodulated train. Since this is the case, a time phase shift must be added to the equation of the unmodulated train. Figure 5 shows the situation that exists. The following equation must therefore be added to the one developed for the nonfixed-reference case in order to obtain the equation of the frequency spectrum for the fixed-reference case:

$$f(t) = \frac{E\delta}{T} + \frac{2E\delta}{T}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}} \cdot \cos n\omega(t + T_c).$$

Certain parts of the system must handle waves whose form is not rectangular and whose width and amplitude vary during the modulating cycle. To permit an analysis of these waves, it is first necessary to learn more of their physical aspect. This may be done by considering a present method of demodulation. In this system, the incoming reference pulse starts a sawtooth wave which is terminated by the channel pulse. Obviously then, since the sawtooth wave is made to be of constant slope, any variation in reference to channel pulse-spacing will produce a variation in the amplitude, width, and lagging-edge time of the sawtooth pulse. The peak of this sawtooth wave is compared with a d-c reference voltage, and the difference is passed on as a control signal. Regardless of any variation in reference-pulse starting time, the sawtooth peak voltage will be determined only by the reference to channel pulse-spacing. Since this is the case, the control voltage is practically unaffected by changes in the reference p.r.f. This is graphically illustrated in Figure 6.

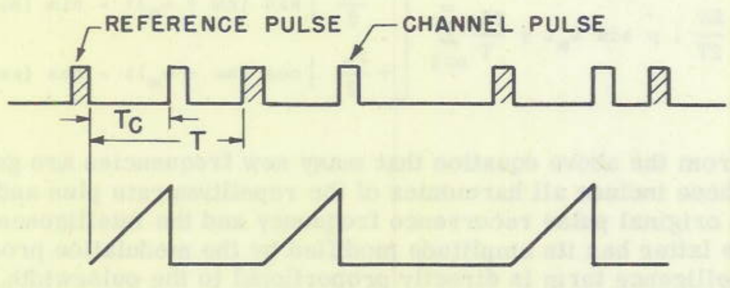

Fig. 6: Random Modulation of the Reference Pulses

When the reference to channel pulse-spacing is varied, then the sawtooth peak voltage varies proportionately. This is shown in Figure 7, from which it is obvious that a standard amplitude modulation analysis may be substituted for the pulse time modulation analysis. The reason that this may be done is that time, amplitude, and pulsewidth are all proportional in this constant slope, sawtooth wave. Furthermore, the magnitudes of time, amplitude, and pulsewidth have a steady-state value, and this value is modulated by the variation in reference to channel pulse-spacing. The analysis of the clipped sawtooth wave in its unmodulated and modulated state is

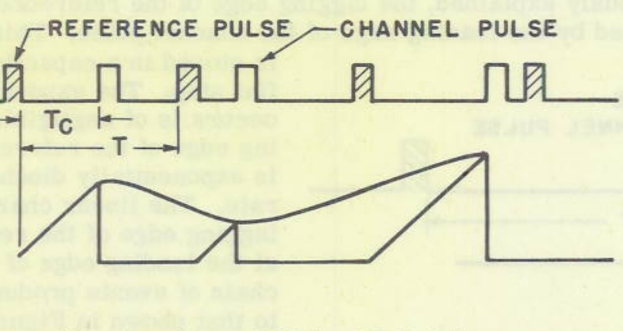

Fig. 7: Channel Pulse Modulation

carried out by a Fourier Series development on page 13 of the Appendix. The equations are as follows:

Unmodulated case:

$$f(t) = \frac{E\delta}{2T} + \frac{2E}{T}\sum_{n=1}^{\infty}\left[\left(\frac{\sin n\pi\delta}{n\omega} + \frac{\cos n\omega\delta - 1}{n^2\omega^2\delta}\right) \cdot \cos n\omega t + \left(\frac{\sin n\omega\delta}{n^2\omega^2\delta} - \frac{\cos n\omega\delta}{n\omega}\right) \cdot \sin n\omega t\right]$$

$$= \frac{E\delta}{2T} + \frac{2E}{T}\sum_{n=1}^{\infty}\left[\alpha_n \cos n\omega t + \beta_n \cdot \sin n\omega t\right]$$

Modulated case:

$$g(t) = \frac{E\delta}{2T} + \frac{2E}{T}\sum_{n=1}^{\infty}\left[\alpha_n \cdot \cos n\omega t + \beta_n \cdot \sin n\omega t\right]$$

$$+ \frac{E\delta}{2T} \cdot \rho \sin \omega_m t + \frac{2E\rho}{T}\sum_{n=1}^{\infty}\left[\frac{\alpha_n}{2}\left\{\sin (n\omega + \omega_m)t - \sin (n\omega - \omega_m)t\right\} + \frac{\beta_n}{2}\left\{\cos (n\omega - \omega_m)t - \cos (n\omega + \omega_m)t\right\}\right]$$

It is apparent from the above equation that many new frequencies are generated by the modulation process. These include all harmonics of the repetition rate plus and minus the modulating frequency. The original pulse recurrence frequency and the intelligence frequency also appear, although the latter has its amplitude modified by the modulation process. Because the amplitude of the intelligence term is directly proportional to the pulsewidth, the signal function can be recovered by a simple filtering process in those systems where the pulsewidth is fairly large relative to the repetition time. If the signal is to be recovered by a filtering process, then f_m, the modulation frequency, must be less than $f - f_m$, where f represents the pulse recurrence frequency. If this is true, then the interfering sideband $f - f_m$, will not be troublesome if an infinitely steep filter is employed. Because of practical filter design, however, $f >> 2f_m$ in order to avoid interference difficulties.

Because pulse modulation always involves a sampling process, the complex intelligence wave is neither transmitted not reconstructed in an exact fashion. The modulation instants are discrete, and the reconstruction sampling times are also discrete. In order to know what filtering problems exist, it is necessary to determine quantitatively the demodulated frequency spectrum.

As has been previously explained, the lagging edge of the reference pulse starts a sawtooth wave which is terminated by the leading edge of the channel pulse. This peak sawtooth voltage is stored in a capacitor as an approximately flat step. The exponential decay which actually occurs is of negligible amplitude. At the leading edge of the reference pulse the capacitor is exponentially discharged at a very rapid rate. The linear charging then starts at the lagging edge of the reference pulse and stops at the leading edge of the channel pulse. This chain of events produces a wave form similar to that shown in Figure 8.

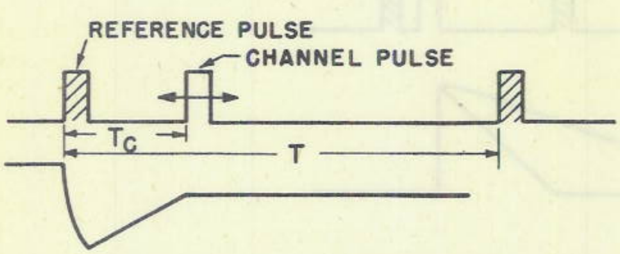

Fig. 8: Demodulator Charging Cycle

Although a wave form of this type could be analyzed, it is more convenient, since it yields adequate information, to proceed with some simplifying assumptions. The assumptions include no exponential decay on the flat top, a straight step in place of the actual discharge pulse, and no change in duration of the flat top with modulation. If it is recognized that the reference frequency (and certain associated harmonics) will actually appear because of the discharge pulse, then little error is introduced by these assumptions. Figure 9 shows this stepped approximation. The reason that the reference frequency does not appear in the analysis is that the steps are assumed to have absolutely straight sides. The energy associated with the amplitude jump is therefore zero.

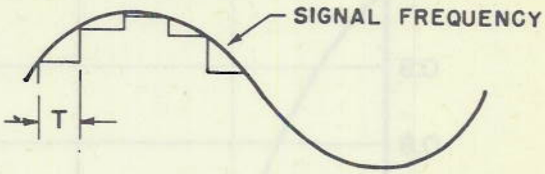

Fig. 9: Stepped Approximation

The simplest way of analyzing a stepped approximation has already been carried out with sufficient generality (3b). However, to correspond with the nomenclature of this report a slightly modified analysis is carried out on page 16 of the Appendix. The resultant equation is

$$g(t) = \frac{E}{\pi}\left[\frac{\omega}{\omega_m}\left\{\sin\frac{\pi\omega_m}{\omega} \cdot \sin \omega_m\left(t - \frac{\pi}{\omega}\right)\right\} - \sum_{n=1}^{\infty}\left\{\frac{\sin\pi\left(n - \frac{\omega_m}{\omega}\right)}{n - \frac{\omega_m}{\omega}} \cdot \sin\,(n\omega - \omega_m)\left(t - \frac{\pi}{\omega}\right) - \frac{\sin\pi\left(n + \frac{\omega_m}{\omega}\right)}{n + \frac{\omega_m}{\omega}} \cdot \sin\,(n\omega + \omega_m)\left(t - \frac{\pi}{\omega}\right)\right\}\right].$$

It is obvious from an inspection of the intelligence term that its amplitude is dependent upon both the repetition and modulation frequencies. Although no quantitative statements can be made, since the analysis was based on a fixed repetition frequency and single-tone modulation, it is apparent that random modulation of the reference pulses will cause a disturbance in the amplitude of the intelligence output. If the repetition frequency is very much greater than the modulation frequency, then the variation in amplitude will not be great. Of course a type of phase modulation takes place whenever the repetition rate is varied, and this introduces new frequencies also.

If the previous criterion of the p.r.f. being much greater than the signal frequency is retained, then the variation in magnitude will be so small that the linearity of the system will be within reasonable bounds. The sideband filter problem will also be of little importance.

FLASH OPERATION

Security of the highest order may be obtained by operating the command control system in the so-called "flash" condition. In this type of operation the only time a command is sent is when an on-off change is required at the receiver. Because of the transient nature of this type of operation, it is virtually impossible to determine the nature of the intelligence being transmitted. Naturally the intelligence transmitted per unit time is extremely low, or the security of the system would be lessened. For purely on-off functions which do not have to be operated too frequently or in a periodic manner, this type of operation is very satisfactory.

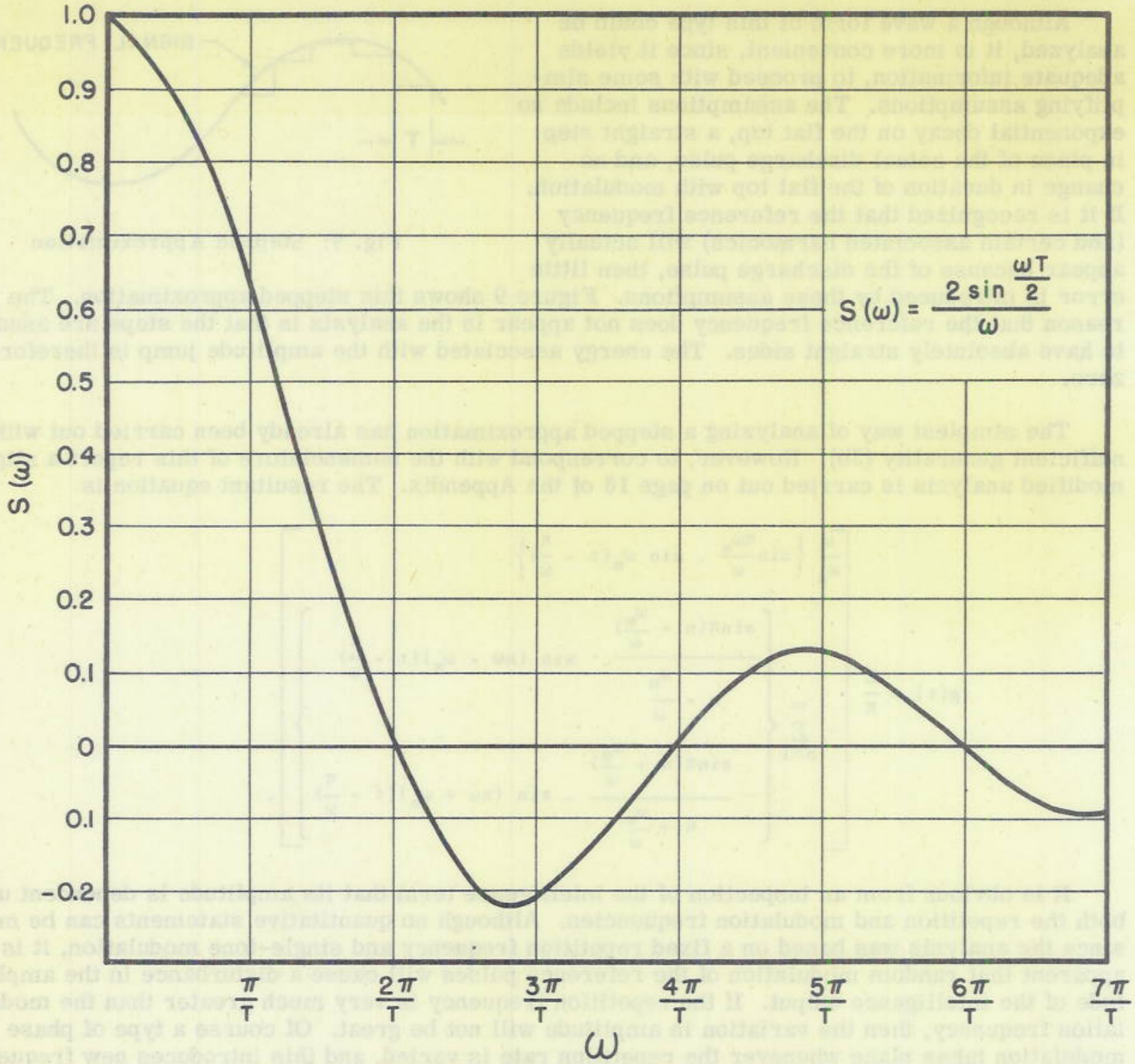

Fig. 10: The Spectrum of a Single Rectangular Pulse

As one approaches completely nonrecurrent operation, the lines of the spectrum of the pulse under consideration (rectangular, sawtooth, etc.) move closer and closer together. In the limit, of course, the nonrecurrent pulse would have a continuous rather than a discrete spectrum. The spectrum of the single, rectangular pulse which has been determined many times before is shown in Figure 10. The spectrum of the single, sawtooth pulse, on the other hand, is less well known. It is desirable, therefore, to undertake an analysis of its spectrum. This is accomplished by means of the Fourier Integral on page 15 of the Appendix. The result is shown in Figure 11, the following equation representing the spectrum:

$$S(\omega) = \frac{m}{\omega^2} \; \omega^2 T^2 + 2[1 - (\cos \omega T + \omega T \sin \omega T)] \quad .$$

The graph of the spectrum shows that a considerable bandwidth is necessary to pass a sawtooth pulse. A variation in the slope of the pulse does not change the frequency spectrum, but it does increase the magnitude of every component in the spectrum. The same statement holds true for the pulsewidth, since the pulsewidth and the slope of a constant amplitude pulse are

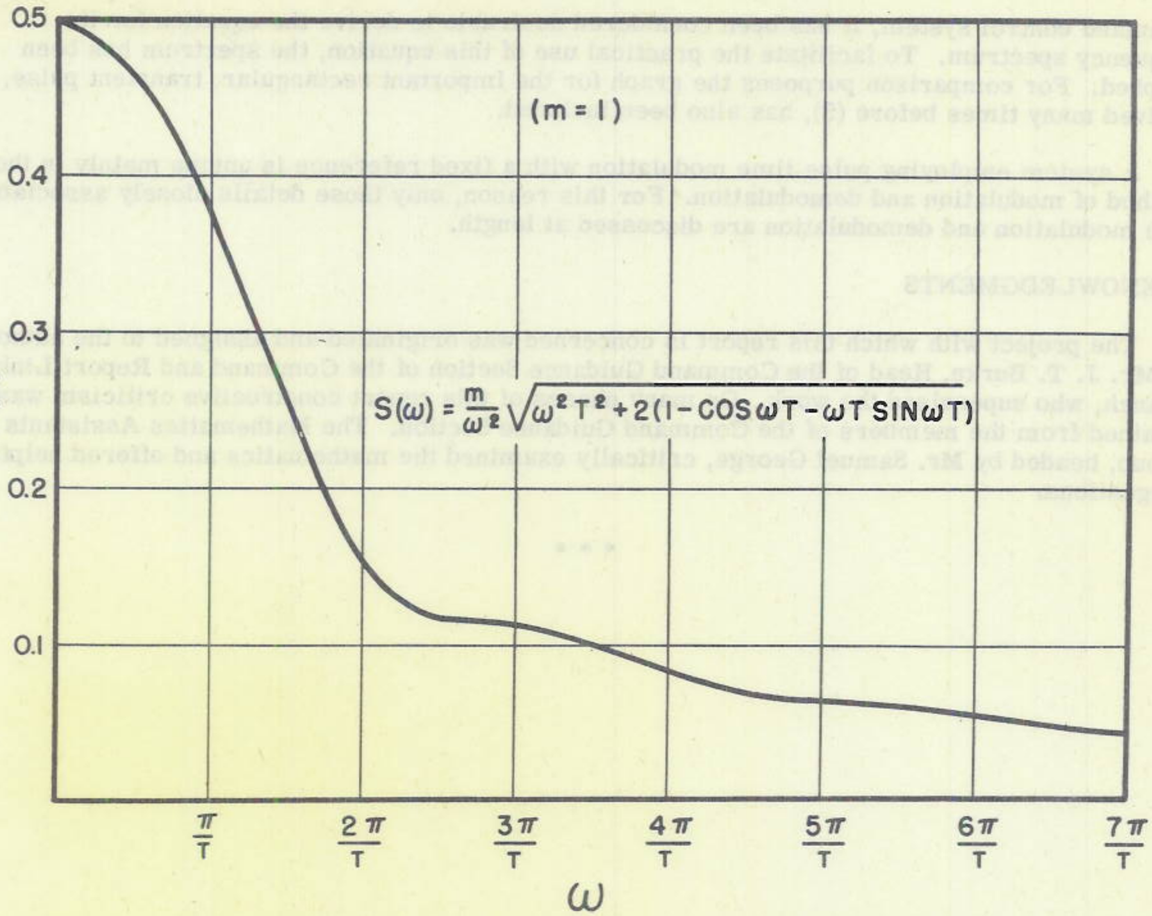

Fig. 11: The Spectrum for a Single Sawtooth Pulse

proportional. The problem of determining the necessary bandwidth is rather difficult despite the availability of a graph showing the frequency distribution. Certain rough considerations, such as examining the area under the curve, show that angular frequencies up to $\frac{2\pi}{T}$ and above play an important part in the determination of the pulse shape.

CONCLUSIONS

The most direct application of this analysis is to the subject of command control systems of the pulse-time type. In this connection it has been shown that the minimum pulse recurrence frequency must be considerably greater than twice the intelligence frequency. Knowing the characteristics of a given missile control loop system, and therefore the maximum intelligence frequency, it is thus possible to determine the minimum p.r.f. for both proportional control and on-off functions.

Equations have been derived which express the frequency spectrum for the important types of waves involved in the fixed-reference case. With this information available, it is possible to proceed with the design of all the necessary filter circuits in a quantitative manner, rather than by the empirical approach.

The transient case, so important in a consideration of "flash" operation, has received some special attention. Because the transient sawtooth pulse plays an important part in a pulse time command control system, it has been considered desirable to derive the equation for its frequency spectrum. To facilitate the practical use of this equation, the spectrum has been graphed. For comparison purposes the graph for the important rectangular, transient pulse, derived many times before (5), has also been included.

A system employing pulse time modulation with a fixed reference is unique mainly in the method of modulation and demodulation. For this reason, only those details closely associated with modulation and demodulation are discussed at length.

ACKNOWLEDGMENTS

The project with which this report is concerned was originated and assigned to the author by Mr. J. T. Burke, Head of the Command Guidance Section of the Command and Report Links Branch, who supervised the work. On many phases of this report constructive criticism was obtained from the members of the Command Guidance Section. The Mathematics Assistants Group, headed by Mr. Samuel George, critically examined the mathematics and offered helpful suggestions.

* * *

UNCLASSIFIED

REFERENCES

(1) Cheng, David K., "Spectrum Analyses of Pulse-Modulated Transmission," USAF Cambridge Field Station. Report E5039 (Unclassified), March 1949.

(2) Ordung, Philip F., "Pulse Time and Amplitude Modulation," Yale University. Research report No. 8, Contract W-36-039-SC-33649 (Unclassified), 3 May 1949.

(3) Bloch, A., "Discussion on Pulse Communication," J. Inst. Elec. Engrs. 94, Part IIIA, 586, 1947.

(4) McGuire, J. H. and Nowacki, P. J., "A Common-Wave Duplex Pulse-Communication System," J. Inst. Elec. Engrs. 94, Part IIIA, 528-532, 1947.

(5) Goldman, S., "Frequency Analysis, Modulation and Noise," McGraw-Hill, New York, 1948.

* * *

SYMBOL NOTATIONS

T = Repetition Time, Transient Sawtooth Pulse Width

t = Time Variable

δ = Rectangular Pulsewidth, Static Sawtooth Pulse Width

ρ = Depth of Amplitude Modulation

m_p = Depth of Phase Modulation

ΔT = Total Time Deviation of a Pulse

n, m, k = Integers

m = Slope of Sawtooth

E = Peak Pulse Voltage

$J_m(n\omega \cdot m_p)$ = Bessel Root of mth Order and Argument $+(n\omega \cdot m_p)$

$f(t)$ = Unmodulated Function of t

$g(t)$ = Modulated Function of t

ω = Angular Repetition Frequency

ω_n = Angular Modulation Frequency

$|f|$ = Absolute Value of the Function of f

θ = Pulse Displacement Function

$\frac{d\theta}{dt}$ = Rate of Change of Displacement

f = Repetition Frequency

f_m = Modulation Frequency

Δf = Change in Repetition Frequency

$I(t)$ = Intelligence Function

$$\alpha_n = \frac{\sin n\omega\delta}{n\omega} + \frac{\cos n\omega\delta - 1}{n^2\omega^2\delta}$$

$$\beta_n = \frac{\sin n\omega\delta}{n^2\omega^2\delta} - \frac{\cos n\omega\delta}{n\omega}$$

APPENDIX

Mathematical Analysis of the Fixed-Reference Case

If, in a command guidance system, it is desired to have a given on-off channel operating continuously, then the problem of a periodic, clipped sawtooth wave is encountered. The solution of this problem is accomplished in a simple manner by a Fourier Series development. Upon examining Figure 12, it is apparent that the following series holds:

$$y = f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t).$$

The coefficients for this series are determined in the usual manner. Therefore,

$$a_o = \frac{2}{T}\int_o^T f(t)dt = \frac{2}{T}\left[\int_o^\delta m.t.dt + \int_\delta^T o.dt\right] = \frac{m\delta^2}{T}\,.$$

But

$$E = m\delta,$$

therefore

$$\frac{a_o}{2} = \frac{E\delta}{2T}. \qquad a_n = \frac{2}{T}\int_o^T \cos n\omega t\cdot f(t)\cdot dt$$

$$= \frac{2}{T}\left[\int_o^\delta mt\cdot\cos n\omega t\cdot dt + \int_\delta^T o\cdot\cos n\omega t\cdot dt\right] = \frac{2m}{T}\left[\frac{\delta \sin n\omega\delta}{n\omega} + \frac{\cos n\omega\delta - 1}{n^2\omega^2}\right]$$

$$= \frac{2E}{T}\left[\frac{\sin n\omega\delta}{n\omega} + \frac{\cos n\omega\delta - 1}{n^2\omega^2\delta}\right].$$

Defining

$$\alpha_n = \frac{\sin n\omega\delta}{n\omega} + \frac{\cos n\omega\delta - 1}{n^2\omega^2\delta},$$

the coefficient a_n becomes equal to $\alpha_n \cdot \frac{2E}{T}$.

$$b_n = \frac{2}{T}\int_o^T \sin n\omega t.f(t).dt = \frac{2}{T}\left[\int_o^\delta m.t.\sin n\omega t dt + \int_\delta^T o\cdot\sin n\omega t\cdot dt\right]$$

$$= \frac{2E}{T}\left[\frac{\sin n\omega\delta}{n^2\omega^2\delta} - \frac{\cos n\omega\delta}{n\omega}\right].$$

Defining

$$\beta_n = \frac{\sin n\omega\delta}{n^2\omega^2\delta} - \frac{\cos n\omega\delta}{n\omega},$$

the coefficient b_n becomes equal to $\beta_n \cdot \frac{2E}{T}$.

The complete trigonometric series may now be expressed in the form:

$$f(t) = \frac{E\delta}{2T} + \frac{2E}{T}\sum_{n=1}^{\infty}(\alpha_n\cdot\cos n\omega t + \beta_n\cdot\sin n\omega t).$$

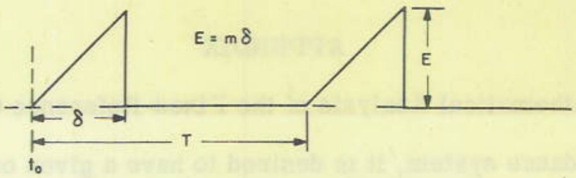

Fig. 12: Clipped Sawtooth Wave

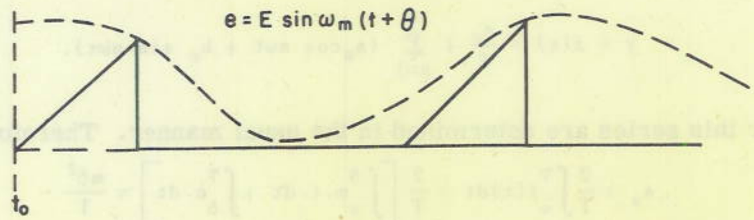

Fig. 13: Modulated Clipped Sawtooth Wave

In a proportional control system, a clipped sawtooth wave exists in which both the pulse-width and amplitude are varied in correspondence to an input error voltage. Figure 13 shows the simplest case. Here this input error voltage varies in a sinusoidal manner. Because the pulse-width, amplitude, and position of the lagging edge all vary simultaneously and proportionately, it is possible to analyze this type of wave by means of a standard amplitude modulation analysis. The modulating signal may be considered to have the form $e = E \sin \omega_m t$. Defining ρ as the percentage of amplitude modulation, the modulated function, $g(t)$, has the form $f(t)\cdot(1 + \rho \sin \omega_m t)$. This equals

$$\frac{E\delta}{2T} + \frac{2E}{T}\sum_{n=1}^{\infty}(\alpha_n \cdot \cos n\omega t + \beta_n \cdot \sin n\omega t) + \frac{E\delta}{2T}\cdot\rho\cdot\sin\omega_m t$$

$$+ \frac{2E}{T}\cdot\rho \sin\omega_m t \sum_{n=1}^{\infty}(\alpha_n \cdot \cos n\omega t + \beta_n \cdot \sin n\omega t).$$

Upon expansion of all trigonometric products and collection of terms, the following equation results:

$$g(t) = \frac{E\delta}{2T} + \frac{2E}{T}\sum_{n=1}^{\infty}(\alpha_n \cdot \cos n\omega t + \beta_n \cdot \sin n\omega t)$$

$$+ \frac{E\delta}{2T}\cdot\rho\cdot\sin\omega_m t + \frac{2E\rho}{T}\sum_{n=1}^{\infty}\left[\begin{array}{l}\frac{\alpha_n}{2}\cdot\left\{\sin(n\omega + \omega_m)t - \sin(n\omega - \omega_m)t\right\} \\ + \frac{\beta_n}{2}\cdot\left\{\cos(n\omega - \omega_m)t - \cos(n\omega + \omega_m)t\right\}\end{array}\right].$$

It is apparent from this equation that, in addition to the original components, several new ones have been generated. These consist of both sine and cosine terms representing upper and lower sidebands of all harmonics of the pulse recurrence frequency. In transient on-off operation, usually referred to as "flash" operation, the duty cycle is extremely low and of an aperiodic nature. This being so, it is informative to consider the limiting case of a single pulse. Both rectangular and sawtooth pulses are important in the system, but the frequency spectrum for the rectangular pulse is well known, having been derived many times before. Its spectrum is shown in Figure 10.

[redacted]

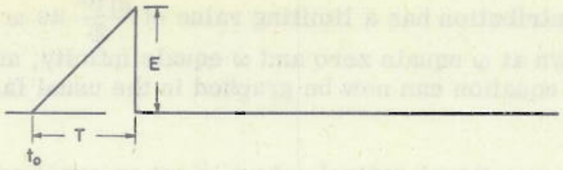

Fig. 14: Transient Sawtooth Wave

The single sawtooth pulse, on the other hand, has received very little attention. By the use of the Fourier Integral, its frequency spectrum may easily be obtained. Applying the standard Fourier Integral to Figure 14,

$$F(f) = \int_{-\infty}^{\infty} G(t)\cdot e^{-j\omega t}dt = \int_{-\infty}^{0} 0\cdot e^{-j\omega t}dt + \int_{0}^{T} mt\cdot e^{-j\omega K}dt + \int_{T}^{\infty} 0\cdot e^{-j\omega t}dt = \int_{0}^{T} m\cdot t\cdot e^{-j\omega t}dt.$$

Integrating by parts, $\int u dv = uv - \int v du$.

Let $t = u$, and $e^{-j\omega t}dt = dv$. Therefore, $F(f) = m\left[\frac{-te^{-j\omega t}}{j\omega} - \int \frac{e^{-j\omega t}}{-j\omega}dt\right]_{0}^{T}$

$$= \frac{m}{j^2\omega^2}\left[(-j\omega T\cdot e^{-j\omega T} - e^{-j\omega T}) - (0 - 1)\right] = \frac{m}{\omega^2}\left[e^{-j\omega t}(1 + j\omega T) - 1\right]$$

$$= \frac{m}{\omega^2}\left[(\cos\omega T - j\sin\omega T)(1 + j\omega T) - 1\right] = \frac{m}{\omega^2}\left[\cos\omega T + j\omega T\cos\omega T - j\sin\omega T + \omega T\sin\omega T - 1\right].$$

Therefore, $S(\omega) = |F(f)| = \frac{m}{\omega^2}\cdot\sqrt{(\cos\omega T + \omega T\sin\omega T - 1)^2 + (\omega T\cos\omega T - \sin\omega T)^2}$

$$= \frac{m}{\omega^2}\sqrt{\omega^2T^2 + 2[1 - (\cos\omega T + \omega T\sin\omega T)]}.$$

In order to make practical use of this equation, it is desirable to present it in the form of a graph. By differentiating the expression $S(\omega)$ and setting it equal to zero, it is found that no maximum or minimum points exist. This is different from the spectrum of the single, rectangular pulse where there are an infinite order of minimum and maximum points. A simple examination of $S(\omega)$ shows that, as ω approaches infinity, its value approaches zero in the limit. Although it may appear that the function would approach infinity as ω approaches zero, the following analysis shows this to be incorrect:

$$S(\omega) = \frac{m}{\omega^2}\sqrt{\omega^2T^2 + 2[1 - (\cos\omega T + \omega T\sin\omega T)]}$$

$$= \frac{m}{\omega^2}\sqrt{\omega^2T^2 + 2\left[1 - \left\{\left(1 - \frac{(\omega T)^2}{2!} + \frac{(\omega T)^4}{4!}\ldots\right) + \omega T\left(\omega T - \frac{(\omega T)^3}{3!} + \frac{(\omega T)^5}{5!}\ldots\right)\right\}\right]}$$

$$= m\sqrt{\frac{T^2}{\omega^2} + 2\left(\frac{T^2}{2\omega^2} - \frac{T^4}{4!} - \frac{T^2}{\omega^2} + \frac{T^4}{3!} - \omega^2T^6\ldots\right)}.$$

Therefore, $\lim_{\omega\to 0} = m\sqrt{2\left(\frac{T^4}{6} - \frac{T^4}{24}\right)} = \frac{mT^2}{2}$.

[redacted]

This shows that the distribution has a limiting value of $\frac{mT^2}{2}$ as ω approaches zero. Because the values of $S(\omega)$ are known at ω equals zero and ω equals infinity, and because no maximum or minimum points exist, the equation can now be graphed in the usual fashion. Its spectrum is shown in Figure 11.

Before filtering, the proportional control output is not an exact reproduction of the original function. Instead, for reasons which have been considered in the text of this report, it is a stepped approximation to this function. If the reference pulses are assumed unmodulated so that a constant-length set of steps exists, then it is possible to analyze this problem with sufficient rigor by elementary mathematical methods.

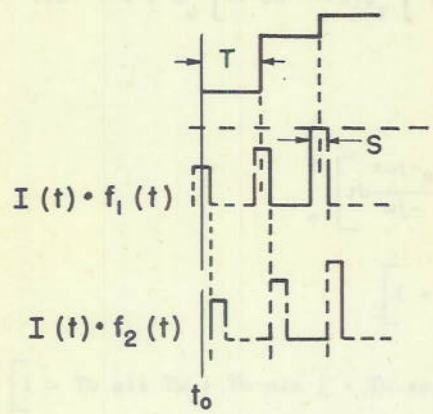

Fig. 15: Synthesis of the Stepped Approximation

Assume, as in Figure 15, that an infinite train of pulses exists whose center lines coincide with the vertical discontinuity of each step. Then assume this set to be modulated by a sine-wave signal in such a fashion that the pulse height is directly proportional to the signal amplitude. The equation of this train of pulses before they have been modulated is obviously

$$f_1(t) = \frac{E\delta}{T}\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}}\cdot\cos n\omega T\right].$$

If the set is modulated, then its equation becomes

$$g_1(t) = \frac{E\delta}{T}\cdot\sin\omega_m t\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}}\cdot\cos n\omega t\right].$$

If the pulsewidth approaches zero, then a large number of pulse trains may be assumed to exist. That is

$$f_2(t) = \frac{E\delta}{T}\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}}\cdot\cos n\omega(t-\delta)\right]$$

or finally

$$f_k(t) = \frac{E\delta}{T}\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}}\cdot\cos n\omega(t-k\delta)\right].$$

The pulsewidth, δ, times k, the number of pulses in a step, equals T, the reference period. Therefore k varies from 0 to $\frac{T}{\delta}$. This being the case, the general modulated function assumes the form

$$g(t) = \frac{E\delta}{T}\sum_{k=0}^{\frac{T}{\delta}}\sin\omega(t-k\delta)\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}}\cdot\cos n\omega(t-k\delta)\right].$$

Consider the quantity $\frac{K\delta}{T}$. As K varies from 0 to $\frac{T}{\delta}$, the quantity varies from 0 to 1. Therefore, it would be convenient to set this quantity equal to a variable, x. Then $K\delta = XT$ and $dX = \frac{\delta}{T}$. Now

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \lim_{\frac{n\pi\delta}{T} \to 0} \frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}} = 1 \, .$$

Therefore

$$g(t) = E\left[\int_0^1 \sin\omega_m(t - XT)\,dX + 2\int_0^1 \sum_{n=1}^{\infty} \sin\omega_n(t - XT)\cdot\cos n\omega(t - XT)\,dX\right].$$

Integrating the first expression,

$$\int_0^1 \sin\omega_m(t - XT)\,dX = \frac{2}{\omega_m T}\left[\sin\frac{\omega_m T}{2}\cdot\sin\omega_m\left(t - \frac{T}{2}\right)\right].$$

But $T = \frac{2\pi}{\omega}$; therefore the expression becomes

$$\frac{\omega}{\pi\omega_m}\left[\sin\frac{\omega_m\pi}{\omega}\cdot\sin\omega_m\left(t - \frac{\pi}{\omega}\right)\right].$$

Integrating the second expression,

$2\int_0^1 \sin\omega_m(t - XT)\bullet\cos n\omega(t - XT)\,dX$, equals, letting $\omega_m t = \theta$, $n\omega t = \varphi$,

$-\omega_m T = A$, and $-n\omega T = B$,

$$-2\left[\frac{\cos\{(A - B) + (\theta - \varphi)\} - \cos(\theta - \varphi)}{2(A - B)} + \frac{\cos\{(A + B) + (\theta + \varphi)\} - \cos(\theta + \varphi)}{2(A + B)}\right].$$

Let $A - B = \alpha$, $\theta - \varphi = \beta$, $A + B = \gamma$, and $\theta + \varphi = \xi$.

The expression then becomes

$$\frac{\cos\beta - \cos(\alpha + \beta)}{\alpha} + \frac{\cos\xi - \cos(\gamma + \xi)}{\gamma}$$

which simplifies to

$$2\left[\frac{\sin\frac{\alpha}{2}\cdot\sin\frac{\alpha + 2\beta}{2}}{\alpha} + \frac{\sin\frac{\gamma}{2}\cdot\sin\frac{\gamma + 2\xi}{2}}{\gamma}\right]$$

Resubstituting the original letters and simplifying, the expression for the second integration becomes

$$-\frac{\sin\pi\left(n - \frac{\omega_m}{\omega}\right)}{\pi\left(n - \frac{\omega_m}{\omega}\right)}\cdot\sin(n\omega - \omega_m)\left(t - \frac{\pi}{\omega}\right) + \frac{\sin\pi\left(n + \frac{\omega_m}{\omega}\right)}{\pi\left(n + \frac{\omega_m}{\omega}\right)}\cdot\sin(n\omega + \omega_m)\left(t - \frac{\pi}{\omega}\right).$$

The complete expression becomes

$$g(t) = \frac{E}{\pi}\left[\frac{\omega}{\omega_m}\left\{\sin\frac{\pi\omega_m}{\omega}\cdot\sin\omega_m\left(t-\frac{\pi}{\omega}\right)\right\} - \sum_{n=1}^{\infty}\left\{\frac{\sin\pi\left(n-\frac{\omega_m}{\omega}\right)}{n-\frac{\omega_m}{\omega}}\cdot\sin(n\omega-\omega_m)\left(t-\frac{\pi}{\omega}\right) - \frac{\sin\pi\left(n+\frac{\omega_m}{\omega}\right)}{n+\frac{\omega_m}{\omega}}\cdot\sin(n\omega+\omega_m)\left(t-\frac{\pi}{\omega}\right)\right\}\right].$$

In considering pulse time modulation with a fixed reference, it is also necessary to be acquainted with the case where a rectangular set of pulses has its spacing varied in some specified manner. In command guidance the spacing should be a direct function of the signal amplitude. This condition is here called phase-modulation.

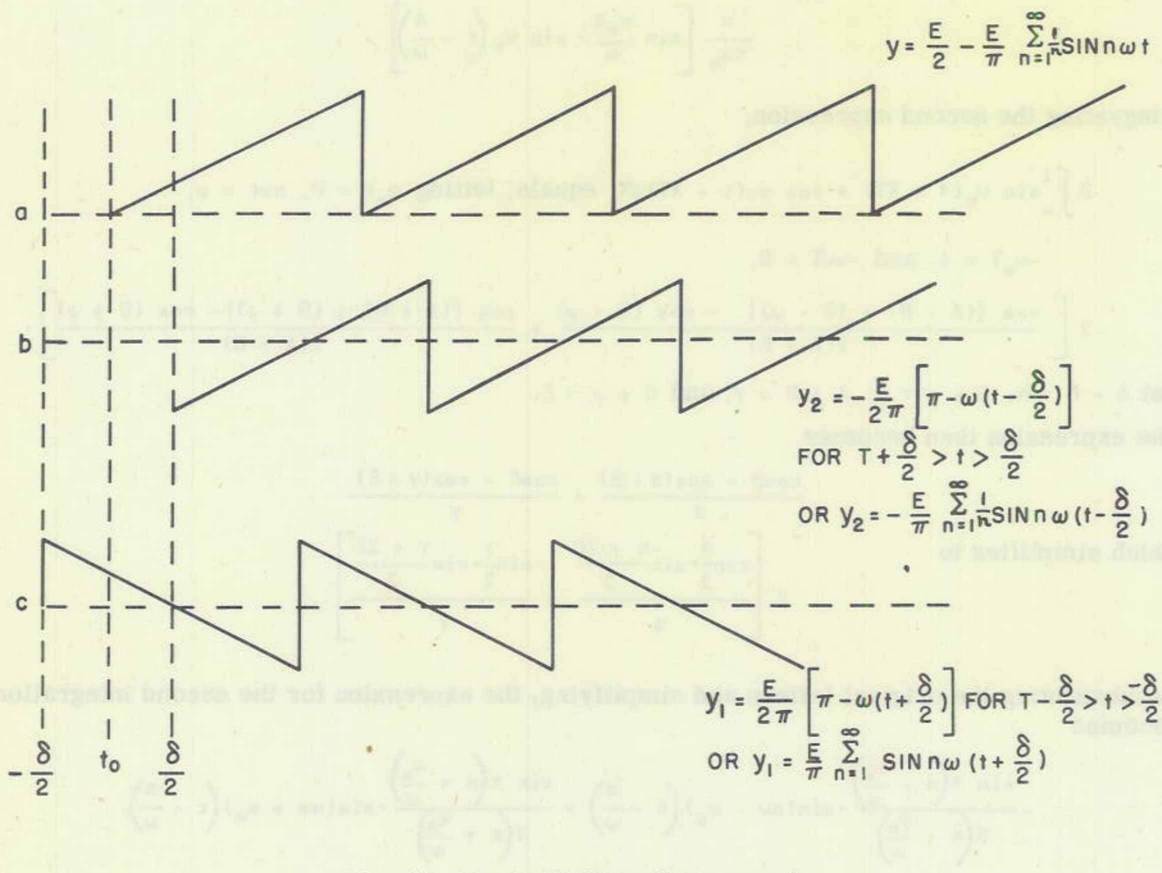

Fig. 16: Sawtooth Wave Components

For an unmodulated, rectangular pulse train, the following equation applies:

$$f(t) = \frac{E\delta}{T} + \frac{2E\delta}{T}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\delta}{T}}{\frac{n\pi\delta}{T}}\cdot\cos n\omega t.$$

This expands into

$$\frac{E\delta}{T} + \frac{E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\sin n\omega\left(t + \frac{\delta}{2}\right) - \sin n\omega\left(t - \frac{\delta}{2}\right) \right] .$$

Now, consider an ordinary sawtooth wave. Referring to Figure 16a, it is obvious that the coefficients for the Fourier Series are obtained as follows:

$$\frac{a_o}{2} = \frac{1}{T}\int_o^T m\cdot t\cdot dt = \frac{E}{2}, \quad a_n = \frac{2}{T}\int_o^T \cos n\omega t\cdot mt\cdot dt = 0$$

$$b_n = \frac{2}{T}\int_o^T \sin n\omega t\cdot mt\cdot dt = -\frac{E}{n\pi}. \text{ Therefore } y = \frac{E}{2} - \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\omega t$$

represents the above sawtooth. Hence, the two sawtooths representing the rectangular wave are as shown in Figures 16b, 16c, and 17.

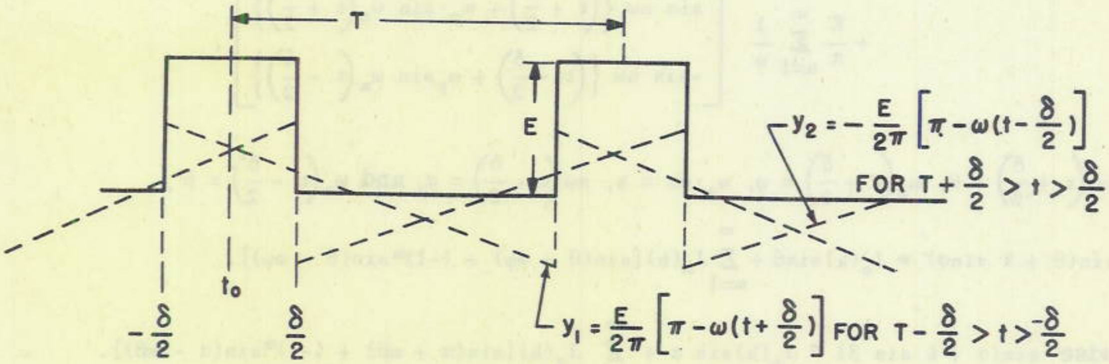

Fig. 17: Synthesis of a Rectangular Wave

Although the unmodulated frequency spectrum is adequately expressed by the expanded form of the original equation, it does not satisfy the condition of

$$f(t) = 0 \quad \text{for} \quad T - \frac{\delta}{2} > t > \frac{\delta}{2}$$

which obviously must be satisfied. When representing the rectangular pulse train by two sawtooth waves, therefore, it is necessary to add a correction factor to bring f(t) to zero for the required length of time. As shown by Ordung (2), this factor is obtained by expressing the two sawtooth waves in a discontinuous form and equating the entire sum of correction factor, sawtooths, and d-c component to zero.

$$\text{Thus } c + y_1 + y_2 + \frac{E\delta}{T} = 0.$$

In discontinuous form

$$y_1 = E\left[\frac{1}{2} - \frac{\left(t + \frac{\delta}{2}\right)}{T}\right] = \frac{E}{2\pi}\left[\pi - \omega\left(t + \frac{\delta}{2}\right)\right], \text{ and}$$

$$y_2 = -E\left[\frac{1}{2} - \frac{\left(t - \frac{\delta}{2}\right)}{T}\right] = -\frac{E}{2\pi}\left[\pi - \omega\left(t - \frac{\delta}{2}\right)\right].$$

By assuming the modulating function to be $e_m = E \sin \omega_m t$, the modulated equations become

$$y_1 = \frac{E}{2\pi}\left[\pi - \omega\left\{\left(t + \frac{\delta}{2}\right) + m_p \sin \omega_m\left(t + \frac{\delta}{2}\right)\right\}\right]$$

and

$$y_2 = -\frac{E}{2\pi}\left[\pi - \omega\left\{\left(t - \frac{\delta}{2}\right) + m_p \sin \omega_m\left(t - \frac{\delta}{2}\right)\right\}\right] .$$

The correction factor becomes

$$-\left[y_1 + y_2 + \frac{E\delta}{T}\right] = \frac{Em_p}{T}\left[\sin \omega_m\left(t + \frac{\delta}{2}\right) - \sin \omega_m\left(t - \frac{\delta}{2}\right)\right].$$

$$\therefore g(t) = \frac{E\delta}{T} + \frac{Em_p}{T}\left[\sin \omega_m\left(t + \frac{\delta}{2}\right) - \sin \omega_m\left(t - \frac{\delta}{2}\right)\right]$$

$$+ \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\begin{array}{l}\sin n\omega\left\{\left(t + \frac{\delta}{2}\right) + m_p \cdot \sin \omega_m\left(t + \frac{\delta}{2}\right)\right\} \\ -\sin n\omega\left\{\left(t - \frac{\delta}{2}\right) + m_p \sin \omega_m\left(t - \frac{\delta}{2}\right)\right\}\end{array}\right].$$

Let $n\omega\left(t + \frac{\delta}{2}\right) = \theta$, $\omega_m\left(t + \frac{\delta}{2}\right) = \varphi$, $m_p \cdot n\omega = k$, $n\omega\left(t - \frac{\delta}{2}\right) = \alpha$, and $\omega_m\left(t - \frac{\delta}{2}\right) = \beta$.

Now $\sin(\theta + k \sin\varphi) \equiv J_o(k)\sin\theta + \sum_{m=1}^{\infty} J_m(k)[\sin(\theta + m\varphi) + (-1)^m \sin(\theta - m\varphi)]$.

Likewise $\sin(\alpha + k \sin \beta) \equiv J_o(k)\sin \alpha + \sum_{m=1}^{\infty} J_m(k)[\sin(\alpha + m\beta) + (-1)^m \sin(\alpha - m\beta)]$.

The complete expression therefore becomes

$$g(t) = \frac{E\delta}{T} + \frac{Em_p}{T}\left[\sin \omega_m\left(t + \frac{\delta}{2}\right) - \sin \omega_m\left(t - \frac{\delta}{2}\right)\right]$$

$$+ \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\begin{array}{l} J_o(n\omega \cdot m_p) \cdot \sin n\omega\left(t + \frac{\delta}{2}\right) + \\ \sum_{m=1}^{\infty} J_m(n\omega . m_p)\left\{\sin\left[n\omega\left(t + \frac{\delta}{2}\right) + m\omega_m\left(t + \frac{\delta}{2}\right)\right] + (-1)^m \sin\left[n\omega\left(t + \frac{\delta}{2}\right) - m\omega_m\left(t + \frac{\delta}{2}\right)\right]\right\} \\ -J_o(n\omega \cdot m_p) \cdot \sin n\omega\left(t - \frac{\delta}{2}\right) - \\ \sum_{m=1}^{\infty} J_m(n\omega \cdot m_p)\left\{\sin\left[n\omega\left(t - \frac{\delta}{2}\right) + m\omega_m\left(t - \frac{\delta}{2}\right)\right] + (-1)^m \sin\left[n\omega\left(t - \frac{\delta}{2}\right) - m\omega_m\left(t - \frac{\delta}{2}\right)\right]\right\}\end{array}\right]$$