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PASSIVE BEARING FINDER
REPORT NO. 2

SIGNAL-TO-NOISE RATIO OF THE R-C COUPLED BAND PASS AMPLIFIER

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SIGNAL-TO-NOISE RATIO OF THE R-C COUPLED BAND PASS AMPLIFIER

J. W. Tucker

June 19, 1950

Approved by:

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ABSTRACT

A theoretical analysis was carried out to evaluate the possibility of appreciably improving the signal-to-noise ratio of the Passive Bearing Finder amplifier by simple adjustment of the R-C time constants.

It is shown that the frequency response of the R-C coupled amplifier may be closely approximated by a simple function completely determined by the values of two frequency parameters. These two parameters are determined by the specification of a significant pair of operating characteristics (such as the cut-off frequencies) or may be derived from three circuit constants (the two stage-time-constants and the number of stages).

Expressions are derived for integrated noise output and steady-state, signal-to-noise ratio in terms of the noise and signal input and the two parameters of the response function. The types of noise considered are white noise, inverse frequency noise, and a combination of the two. A procedure is indicated for combining the results of a transient analysis with the expressions for noise output to obtain peak-to-peak-signal to RMS noise output.

Application of results to the present periodic PBF amplifier shows that circuit constants are sufficiently close to the values for optimum signal-to-noise ratio.

PROBLEM STATUS

This is an interim report; work is continuing.

AUTHORIZATION

NRL Problem N07-09R
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PASSIVE BEARING FINDER

REPORT NO. 2

SIGNAL-TO-NOISE RATIO OF THE R-C COUPLED BAND PASS AMPLIFIER

INTRODUCTION

The Passive Bearing Finder is a periscope-mounted thermal radiation detector for use on submarines against surface vessels. In an effort to increase the nominal operating range of the equipment, NRL has been engaged in a systematic investigation of the various factors which now limit the system's operation. One basic drawback of the present equipment is the very high noise level in the associated signal amplifier. In one of the best amplifiers built to date,[1] the total output noise is 70 percent flicker noise from the input tube. In other words, the signal-to-noise ratio of the system is determined primarily by amplifier noise rather than by the Johnson noise from the sensitive receiving element employed.

Time gating was considered as a possible method of improving the signal-to-noise ratio of the system but a theoretical and experimental analysis of this scheme[2] has shown that little can be gained through its use.

Tuning of the signal amplifier has been given consideration and is the subject of this report. The low signal frequencies involved make it impractical to use L-C circuits for tuning, and resistance capacity tuning is ordinarily employed. Because of the many applications of this type of amplifier, the analysis will be carried out in general terms and certain of the results applied to the specific PBF problem. These results are particularly significant with regard to the PBF amplifier since consideration has been given to inverse frequency noise (flicker noise, semiconductor noise, contact noise, etc.) as well as to white noise (shot effect, Johnson noise, etc.).

---

[1] *Clark, H.L., "A 5 CPS Amplifier for the Passive Bearing Finder," NRL Report H-2895, July 12, 1946 (Confidential)*

[2] *Tucker, J. W., "Passive Bearing Finder, Report No. 1 – Effect of Synchronized Time Gating on the Signal-to-Noise Ratio of a Broadband Amplifier," NRL Report N-3463, September 15, 1949 (Confidential)*

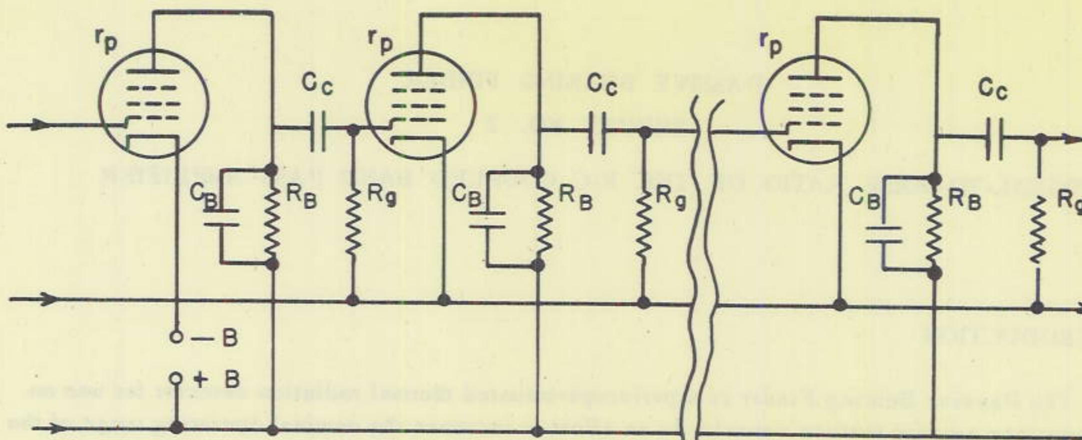

Figure 1 - Simplified circuit diagram of the R-C amplifier

REPRESENTATION OF THE AMPLIFIER RESPONSE AS THE PRODUCT OF TWO MONOTONIC FUNCTIONS

The response of the N-stage R-C coupled amplifier (Figure 1) may be symbolized by:

$$\frac{e_o}{e_i} = G\alpha(f)\,p(f) \tag{1}$$

where e_o is the harmonic output voltage due to the harmonic input voltage e_i, G is a constant gain factor, f is the frequency of e_i and e_o, and $\alpha(f)$ and p(f) are respectively the low- and high-frequency response functions (Figure 2). $\alpha(f)$ and p(f) may be further described by:

$$\left.\begin{aligned} \lim_{f\to 0} \alpha(f) &= 0 \\ \lim_{f\to\infty} \alpha(f) &= 1 \\ \lim_{f\to 0} p(f) &= 1 \\ \lim_{f\to\infty} p(f) &= 0 \end{aligned}\right\} \tag{2}$$

For the simple R-C amplifier under consideration, $\alpha(f)$ and p(f) will be seen to be monotonic functions so that low and high K-value frequencies, f_L and f_H, may be defined by:

$$\left.\begin{aligned} \alpha(f_L) &= K \\ p(f_H) &= K \\ 0 &< K < 1 \end{aligned}\right\} \tag{3}$$

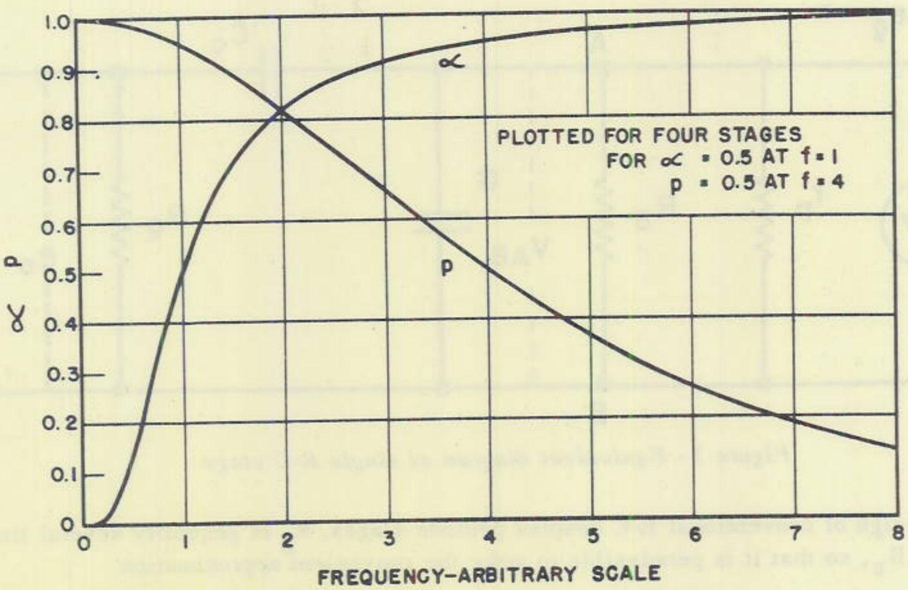

Figure 2 - The low and high pass functions

The constant K is arbitrary within its limits and may be defined in any convenient manner. It will later be defined so that when $f_H \gg f_L$ the K-value frequencies, f_L and f_H, will approximate the nominal low and high cut-off frequencies of the over-all amplifier response characteristic.

The constant gain factor, G, has no effect on relative signal, noise, or signal-to-noise ratios in any given amplifier, and for the purpose of this analysis may be normalized to unity by considering e_o to represent output voltage divided by gain factor. With this definition for e_o, equation (1) can be written

$$e_o = \alpha(f)\, p(f)\, e_i \, . \tag{4}$$

To put equation (4) in more explicit form, it is necessary to examine the functions α and p. An equivalent diagram of a single R-C coupled pentode stage is given in Figure 3. The pentode is considered as a constant current generator producing a current

$$i_p = g_m e_g \tag{5}$$

where g_m is the effective transconductance of the pentode with it's degenerative cathode resistor and e_g is the applied grid voltage, in this case the input voltage, e_i, of equation (4). The pentode plate resistance, r_p, is in parallel with the plate load resistance, R_B, and the equivalent resistance of this combination will be designated by R_1.

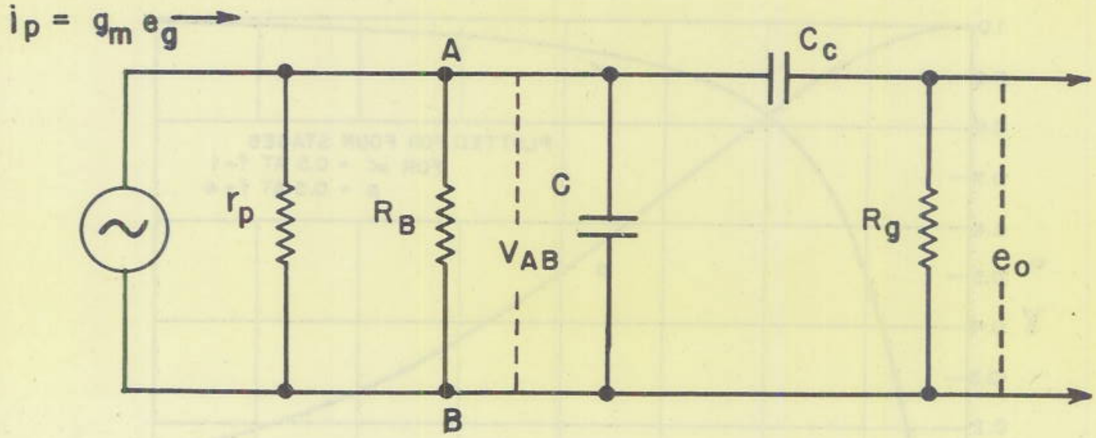

Figure 3 - Equivalent diagram of single R-C stage

In the design of conventional R-C coupled pentode stages, R_g is generally several times greater than R_B, so that it is permissible to make the convenient approximation

$$V_{AB} = i_p \left| Z_1 \right|, \tag{6}$$

where

$$\left| Z_1 \right| = \frac{R_1}{\sqrt{\omega^2 C_B^2 R_1^2 + 1}} \tag{7}$$

and is the parallel impedance of R_1 and C_B. It is now seen that

$$\left| e_o \right| = \frac{\omega C_c R_g}{\sqrt{\omega C_c^2 R_g^2 + 1}} \left| V_{AB} \right|, \tag{8}$$

and in consequence of equations (5), (6), and (7),

$$\left| e_o \right| = g_m R_1 \frac{\omega C_c R_g}{\sqrt{\omega^2 C_c^2 R_g^2 + 1}} \frac{1}{\sqrt{\omega^2 C_B^2 R_1^2 + 1}} \left| e_1 \right| . \tag{9}$$

Considering the relations (2), it is seen that (9) is in the form of equation (1) with

$$\left.\begin{aligned} G &= g_m R_1 \\ \alpha(f) &= \frac{2\pi f C_c R_g}{\sqrt{4\pi^2 f^2 C_c^2 R_g^2 + 1}} \\ p(f) &= \frac{1}{\sqrt{4\pi^2 f^2 C_B^2 R_1^2 + 1}} \\ 2\pi f &= \omega \end{aligned}\right\} . \tag{10}$$

To simplify the notation, the following substitutions will be made

$$\left.\begin{aligned} L &= 2\pi C_c R_g = 2\pi\tau_L \\ H &= 2\pi C_B R_1 = 2\pi\tau_H \end{aligned}\right\} \tag{11}$$

and τ_L and τ_H will be referred to as the low and high frequency time constants. Also e_o will be defined as in equation (4) so that G = 1. With these substitutions, equations (9) and (10) become

$$\left.\begin{aligned} e_o &= \frac{Lf}{\sqrt{L^2 f^2 + 1}} \frac{1}{\sqrt{H^2 f^2 + 1}} e_1 \\ G &= 1 \\ \alpha(f) &= \frac{Lf}{\sqrt{L^2 f^2 + 1}} \\ p(f) &= \frac{1}{\sqrt{H^2 f^2 + 1}} \end{aligned}\right\} . \tag{12}$$

The above results derived for the single stage amplifier may immediately be extended to the N-stage amplifier by repeated application of equation (12). The result is seen to be

$$
\left.
\begin{aligned}
e_o &= \left[\frac{Lf}{\sqrt{L^2 f^2 + 1}}\right]^N \left[\frac{1}{\sqrt{H^2 f^2 + 1}}\right]^N e_1 \\
G &= 1 \\
\alpha(f) &= \left[\frac{Lf}{\sqrt{L^2 f^2 + 1}}\right]^N \\
p(f) &= \left[\frac{1}{\sqrt{H^2 f^2 + 1}}\right]^N
\end{aligned}
\right\} \qquad (13)
$$

Behavior of the Response Functions

In order that the physical behavior of α and p may be more apparent, the constants L and H will be expressed in terms of the lower and upper K-value frequencies as defined by equation (3),

$$
\left.
\begin{aligned}
\alpha(f_L) &= \left[\frac{Lf_L}{\sqrt{L^2 f_L^2 + 1}}\right]^N = K \\
p(f_H) &= \left[\frac{1}{\sqrt{H^2 f_H^2 + 1}}\right]^N = K
\end{aligned}
\right\} . \qquad (14)
$$

Solving for L^2 and H^2,

$$
\left.
\begin{aligned}
L^2 &= \frac{K^{2/N}}{f_L^2 (1-K^{2/N})} = \frac{1}{f_L^2 (K^{-2/N}-1)} \\
H^2 &= \frac{1-K^{2/N}}{f_H^2 K^{2/N}} = \frac{K^{-2/N}-1}{f_H^2}
\end{aligned}
\right\} . \qquad (15)
$$

Substituting equation (15) in equation (13),

$$\left.\begin{aligned} \alpha(f) &= \left[\frac{1}{\sqrt{\left(\frac{f_L}{f_H}\right)^2 \left(K^{-2/N}-1\right)+1}}\right]^N = \mu\left(\frac{f_L}{f}\right) = \mu(x_\alpha) \\ p(f) &= \left[\frac{1}{\sqrt{\left(\frac{f}{f_H}\right)^2 \left(K^{-2/N}-1\right)+1}}\right]^N = \mu\left(\frac{f}{f_H}\right) = \mu(x_p) \end{aligned}\right\}, \qquad (16)$$

where

$$x_\alpha = \frac{f_L}{f}, \qquad x_p = \frac{f}{f_H}. \qquad (17)$$

This parametric expression for α and p is useful in that $\mu(x)$ may be investigated, and the behavior of $\alpha(f)$ and p(f) immediately determined by the relations (17). Alternatively, the inverse similarity between (16) allows $\alpha(f)$ to be investigated and the behavior of p(f) deduced therefrom by merely substituting variables in accordance with

$$p(f) = \alpha\left(\frac{f_L f_H}{f}\right). \qquad (18)$$

The latter procedure will be followed, for $\alpha(f)$ is much the more important factor in the present consideration of the effect of inverse frequency noise. In the following, primary attention will be given to $\alpha(f)$, and the behavior of p(f) will be understood to be implied by equation (18).

Inverse Frequency Noise

When considering noise it is often convenient to classify it as being either random noise or non-random noise. Goldman[3] has given the following definition of random noise: " Any stochastic function of time of specified length whose Fourier series components each have a

[3] *Goldman, S., "Frequency Analysis, Modulation and Noise," p. 326, McGraw-Hill Book Co., New York, 1948*

two-dimensional normal distribution and random phase will be called random noise, provided that the quadratic content of no single component is an appreciable percentage of the total." Noise which does not fit this definition sufficiently well for the purpose at hand is called non-random noise. Whether a particular kind of noise should be considered as random or non-random is in practice often determined by the experimental conditions. Power supply hum and microphonics are examples of non-random noise. Vacuum tube shot noise, flicker noise, and thermal noise are common examples of random noise. In this report only random noise will be considered.

Random noise may be specified as to type by its source amplitude-frequency spectrum, that is its spectrum before passage through frequency selective systems. Two very important types of random noise are white noise and inverse frequency noise. When the amplitude spectrum is flat over the frequency interval concerned, the noise will be considered to be white noise at least for that particular frequency interval. When the spectrum follows an inverse frequency law, the noise will be referred to as inverse frequency noise.

It has been found experimentally[4,5] that the noise most troublesome at frequencies below 100 cps (flicker noise, semiconductor noise, contact noise, etc.) often has a spectrum which may be represented by

$$e(f) = Af^{-n} \tag{19}$$

volts/square root cycle, where A is a constant determined by the noise power per cycle at a specified frequency and where n has been found to have values in the neighborhood of unity. When n is unity, the noise may properly be called inverse frequency noise. Actually, n must vary with frequency, as may be seen by considering the integral

$$\int_0^\infty e^2 df = A^2 \int_0^\infty f^{-2n} df \,. \tag{20}$$

For this integral (representing the square of the noise voltage) to be convergent requires that $n > 0.5$ at the upper limit and that $n < 0.5$ at the lower limit. However, experiments on particular kinds of noise have shown the inverse frequency law to be approximately true even at very low frequencies. In the case of flicker noise these spectrum measurements have been extended to frequencies at low as 0.01 cps.

If g(f) is the response function of any linear frequency selective system and e(f) is the amplitude frequency spectrum of random input noise voltage, then the over-all RMS noise voltage output will be given by

$$E = \sqrt{\int_0^\infty [g(f)]^2 [e(f)]^2 df} \quad \text{volts.} \tag{21}$$

---

[4] *Clark, H. L., "Flicker Noise in Vacuum Tubes," NRL Report H-2894, August 28, 1946 (Unclassified)*

[5] *MacFarlane, G. G., "A Theory of Flicker Noise in Valves and Impurity Semiconductors," Proc. Phys. Soc. 59:366-75, May 1947*

This result follows directly from the definition of random noise. Since it is the area under the g^2e^2 curve that is significant so far as output noise voltage is concerned, attention will be given in the following to $[g(f)]^2$ and to the product function $[g(f)]^2\,[e(f)]^2$ as well as to the response function itself.

Effect of Cascading Stages on α, α^2, and $\alpha^2 f^{-2}$

Equations (15) and (11) show that, when stages are cascaded, it is necessary to change the time constants of the individual amplifier stages if the same K-value frequencies are to be maintained. The effect of not making this adjustment is shown in Figure 4 where α^2 is plotted as a function of f/f_{L1} for 1, 2, 3, 4, and 5 stages of amplification, the single stage lower K-factor frequency, f_{L1}, being defined by

$$\alpha(f_{L1}) = K = 0.5 \quad \text{for} \quad N = 1 \tag{22}$$

and with the low frequency time constant, so determined, being the same for each curve. The addition of successive stages results in a modification of such an arbitrarily defined cut-off frequency.

By combining equations (15) and (11) with $\alpha = K = 0.5$ at $f = f_L$ the following expressions are obtained:

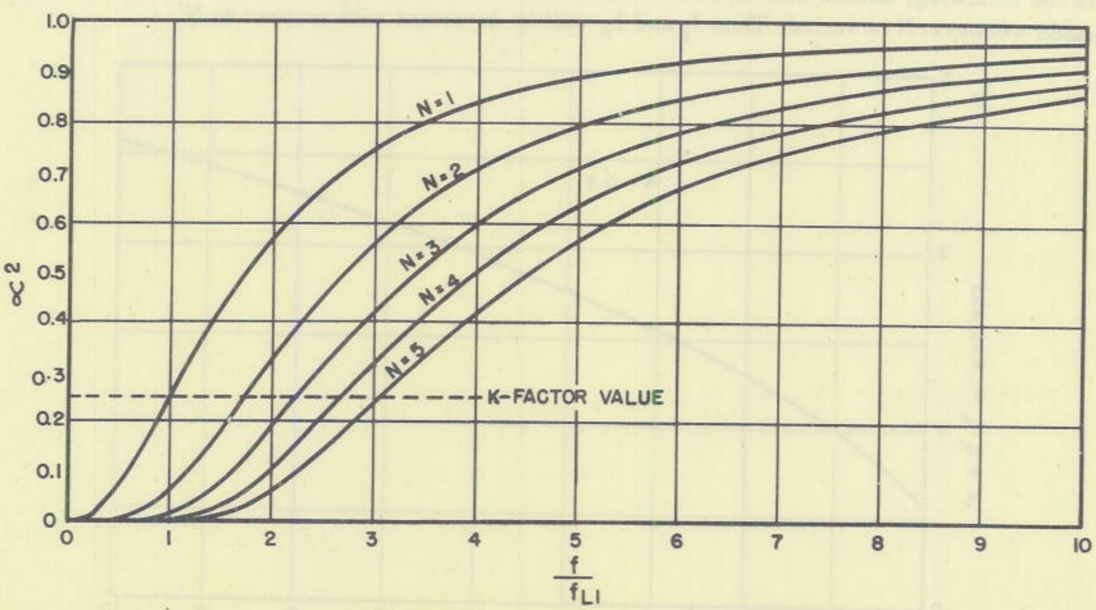

Figure 4 - Effect of N on α^2 for constant τ_L

$$\left.\begin{aligned} f_L &= \frac{1}{2\pi\tau_L\sqrt{4^{1/N}-1}}, \quad \tau_L = R_g C_c \\ \\ f_H &= \frac{\sqrt{4^{1/N}-1}}{2\pi\tau_H}, \quad \tau_H = R_1 C_B \end{aligned}\right\} \qquad (23)$$

In Figure 5, $2\pi\tau_L f_L$ is plotted as a function of N with K = 0.5. This curve shows the variation of f_L with N for constant τ_L and also the adjustment of τ_L necessary to maintain the same f_L for various N's. Since $2\pi\tau_H f_H = 1/2\pi\tau_L f_L$, the relation between τ_H, f_H, and N may also be inferred from the curve of Figure 5.

In general, f_L and f_H will be determined by the particular amplifier application, and the necessary time constants appropriately adjusted for the particular number of stages in the amplifier. The curve in Figure 5 may be used for this purpose if f_L and f_H are defined by

$$\alpha(f_L) = 0.5, \; p(f_H) = 0.5 \; .$$

In the following, unless otherwise stated, it will be assumed that this adjustment of τ will be made whenever N is varied. Thus f_L and f_H will be invariant with respect to N.

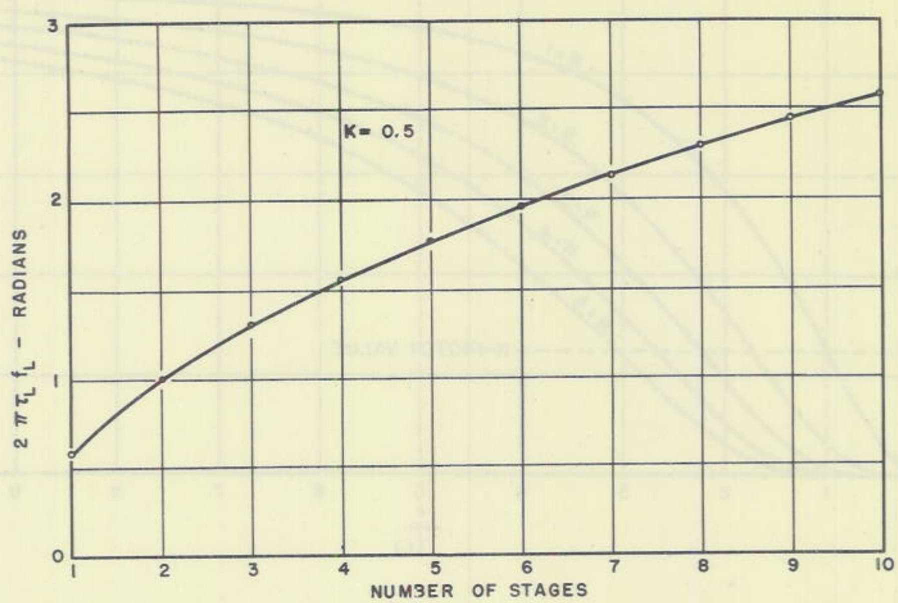

Figure 5 - Curve of $2\pi\tau_L f_L$ as a function of N

When α^2 is plotted as a function of f for N = 1, 2, 3, . . . , (Figure 6), it is seen that the variation in α^2 with N decreases rapidly with increasing N, and it is natural to investigate the limit of $\alpha^2(f)$ as $N \to \infty$. By equation (16)

$$\alpha^2(f) = \left[\frac{1}{\left(\frac{f_L}{f}\right)^2\left(K^{-2/N}-1\right)+1}\right]^N$$

$$= \epsilon^{N \log_\epsilon \left[\frac{1}{\left(\frac{f_L}{f}\right)\left(K^{-2/N}-1\right)+1}\right]}$$

$$= \epsilon^{\frac{\log_\epsilon \left[\frac{1}{\left(\frac{f_L}{f}\right)^2\left(K^{-2/N}-1\right)+1}\right]}{N^{-1}}}$$

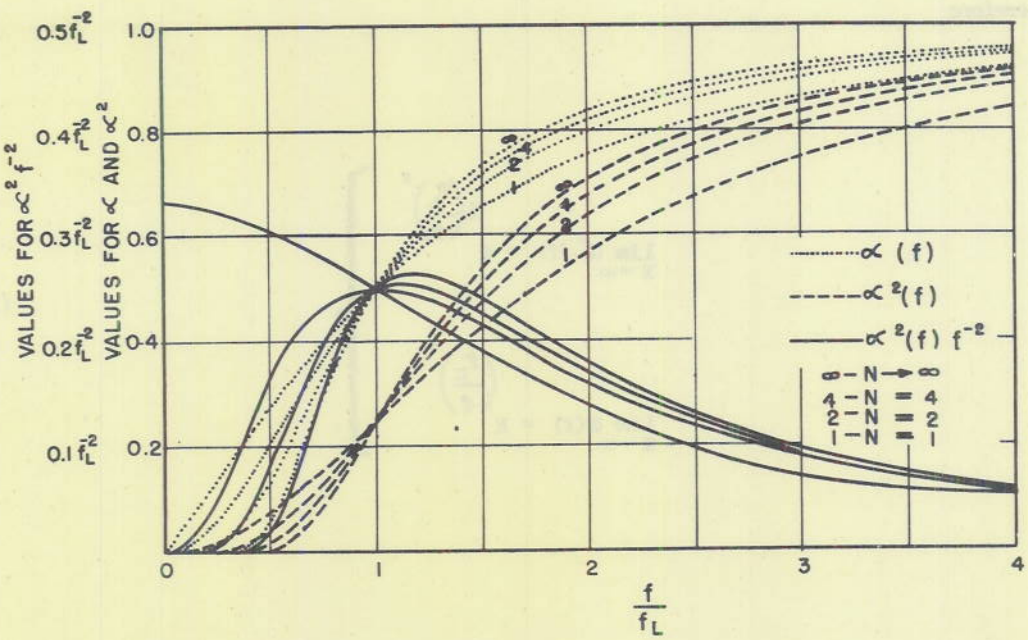

Figure 6 - Curves of $\alpha(f)$, $\alpha^2(f)$ and, $\alpha^2(f)\,f^{-2}$ for N = 1,2,4, and $N \to \infty$

Considering N as a continuous variable,

$$
\left.
\begin{aligned}
\lim_{N\to\infty} \alpha^2(f) &= \lim_{N\to\infty} \epsilon^{\dfrac{\dfrac{\partial}{\partial N}\left[\log_\epsilon \dfrac{1}{\left(\frac{f_L}{f}\right)^2\left(K^{-2/N}-1\right)+1}\right]}{\dfrac{\partial}{\partial N}\left(N^{-1}\right)}} \\
&= \lim_{N\to\infty} \epsilon^{2K^{-2/N}\left(\frac{f_L}{f}\right)^2\left[\log_\epsilon K\right]\left[\left(\frac{f_L}{f}\right)\left(K^{-2/N}-1\right)+1\right]^{-1}} \\
&= \epsilon^{2\left(\frac{f_L}{f}\right)^2 \log_\epsilon K}
\end{aligned}
\right\} . \qquad (24)
$$

Therefore

$$
\left.
\begin{aligned}
\lim_{N\to\infty} \alpha^2(f) &= K^{2\left(\frac{f_L}{f}\right)^2} \\
\lim_{N\to\infty} \alpha(f) &= K^{\left(\frac{f_L}{f}\right)^2}
\end{aligned}
\right\} \qquad (25)
$$

and by equation (18)

$$\left.\begin{aligned} \lim_{N\to\infty} p^2(f) &= K^{2\left(\frac{f}{f_H}\right)^2} \\ \lim_{N\to\infty} p(f) &= K^{\left(\frac{f}{f_H}\right)^2} \end{aligned}\right\} . \qquad (26)$$

$\alpha^2(f)$ for $N\to\infty$ is plotted in Figure 6 and it is seen that this limiting function is approached very rapidly as the number of stages is increased. Curves of $\alpha^2 f^{-2}$ are also plotted in Figure 6 and it is further seen that this product function also converges rapidly to a limiting function with increasing N. The rapid convergence to the limiting functions suggests that, when several stages of amplification are used, the exact expression for $\alpha(f)$ may be approximated by $\lim_{N\to\infty} \alpha(f)$. The curves in Figure 6 show this approximation to be good when $f/f_L \geq 1$.
To evaluate the effect of this approximation on the inverse frequency noise contribution in the region $f/f_L \ll 1$, it is necessary to investigate the behavior of the integral

$$\int_0^{f_L} \alpha^2 f^{-2} df$$

with increasing N.

Making use of equation (13),

$$\int_0^{f_L} \alpha^2 f^{-2} df = \int_0^{f_L} \left[\frac{L^2 f^2}{L^2 f^2 + 1}\right]^N f^{-2} df \qquad (27)$$

$$= L\int_{\frac{1}{Lf_L}}^{\infty} \frac{dv}{(1+v^2)^N} , \qquad (28)$$

where $v = \frac{1}{Lf}$

and $L = \frac{1}{f_L\sqrt{K^{-2/N}-1}}$ for finite values of N.

For $N \to \infty$, equation (25) gives

$$\int_0^{f_L} \alpha^2 f^{-2} df = \int_0^{f_L} K^{2\left(\frac{f_L}{f}\right)^2} f^{-2} df, \tag{29}$$

$$\int_0^{f_L} \alpha^2 f^{-2} df = \frac{1}{f_L \sqrt{\log_\epsilon \frac{1}{K^2}}} \int_{\sqrt{\log_\epsilon \frac{1}{K^2}}}^{\infty} \epsilon^{-\mu^2} d\mu \tag{30}$$

where

$$\mu = \frac{f_L}{f} \sqrt{\log_\epsilon \frac{1}{K^2}} \ .$$

In Table 1 values of $\int_0^{f_L} \alpha^2 f^{-2} df$, with $K = 0.5$, are given for $N = 1, 2, 3, 4$ and $N \to \infty$. Values of

$$\int_0^{\infty} \alpha^2 f^{-2} df$$

are given for comparison and it is seen that both integrals rapidly approach a limit with increasing N.

It may similarly be shown that the integrals

$$\int_{f_H}^{\infty} p^2 f^{-2} df \qquad \text{and} \qquad \int_0^{\infty} p^2 f^{-2} df$$

rapidly approach a limiting value for increasing N. A qualitative evaluation may now be made of the effect, on the total output noise and its general frequency distribution, of using $\lim_{N \to \infty} \alpha$

TABLE 1

Values* of $\int_0^{f_L} \alpha^2 f^{-2} df$ and of $\int_0^{\infty} \alpha^2 f^{-2} df$

for N = 1, 2, 3, 4, and $N \to \infty$

N	$\int_0^{f_L} \alpha^2 f^{-2} df$	$\int_0^{\infty} \alpha^2 f^{-2} df$
1	$0.302 f_L^{-1}$	$0.907 f_L^{-1}$
2	$0.143 f_L^{-1}$	$0.785 f_L^{-1}$
3	$0.113 f_L^{-1}$	$0.769 f_L^{-1}$
4	$0.098 f_L^{-1}$	$0.762 f_L^{-1}$
∞	$0.073 f_L^{-1}$	$0.752 f_L^{-1}$

* Values computed by integration and checked graphically.

and $\lim_{N \to \infty} p$ in place of the expressions derived for α and p with finite N. It will be convenient to classify and consider separately the high-pass, the broad-band-pass, and the narrow-band-pass amplifier. The high-pass amplifier frequency response is given by equation (4) with p(f) = 1; the broad-band-pass and narrow-band-pass amplifiers are defined by the response of equation (4) with

$$f_H \geq 4f_L \quad \text{and} \quad 4f_L > f_H$$

respectively. The distinction between the broad-band-pass and narrow-band-pass amplifier is arbitrary but convenient.

The High-Pass Amplifier

In the case of the high-pass amplifier, the frequency response is characterized by $\alpha(f)$ alone so that the lower K-value frequency becomes the nominal amplifier cut-off frequency, f_c, here defined by $\alpha(f_c) = 0.5$. It has been shown that $\lim_{N\to\infty} \alpha(f)$ may be used to approximate $\alpha(f)$ in the multistage amplifier and that total noise output,

$$A\sqrt{\int_0^{\infty} \alpha^2(f)\, f^{-2} df},$$

is little affected by the approximation when four or more interstage couplings are used. Even in the region below cut-off where the deviation of $\lim_{N\to\infty} \alpha$ from α is greatest, the integrated output noise,

$$A\sqrt{\int_0^{f_c} \alpha^2(f) f^{-2}\, df},$$

is within 15 percent of its value with only four stages.

The Broad-Band-Pass Amplifier

The broad-band-pass amplifier is characterized by the fact that, for $f \geq f_H$, α is in the neighborhood of unity and the percentage change of α with f is much less than that of p with f, and that, for $f \leq f_L$, p is in the neighborhood of unity and the percentage change of p with f is much less than that of α with f. The lower and upper K-value frequencies will approximate the nominal lower and upper amplifier cut-off frequencies defined by $R(f_c) = \alpha(f_c)p(f_c) = 0.5R_M$, where $R_M = R(f_M)$ is the maximum value of R(f), and f_M is the frequency at which this maximum occurs. Below lower cut-off, response will be chiefly determined by α, and above upper cut-off by p. Below lower cut-off, the discussion of the last paragraph in regard to inverse frequency noise applies, and the change in integrated noise output in this region, resulting from replacing α for $N = 4$ by $\lim_{N\to\infty} \alpha$, will be about 15 percent. In the region above upper cut-off, relatively unimportant from the standpoint of contribution to total noise, the effect of replacing p by $\lim_{N\to\infty} p$ will be of the same order of magnitude. Within the pass band, the region of greatest noise contribution, the percentage difference of $\lim_{N\to\infty} \alpha$ and $\lim_{N\to\infty} p$ from α and p, for $N = 4$, is seen to be small, less than 3 percent (Figure 6). The amplifier response in this region, αp, will thus be approximated by $\lim_{N\to\infty} \alpha \lim_{N\to\infty} p$. It is to be noted that $\lim_{N\to\infty} \alpha \lim_{N\to\infty} p > \alpha p$ within the pass band, while $\lim_{N\to\infty} \alpha < \alpha$ below f_L, and $\lim_{N\to\infty} p < p$ above f_H. Thus the changes in the contributions to total integrated noise tend to cancel. The result is that $\lim_{N\to\infty} \alpha^2 \lim_{N\to\infty} p^2 f^{-2}$ makes a good approximation to the $\alpha^2 p^2 f^{-2}$ of a multistage amplifier. In the remainder of this report α and p will, unless otherwise stated, refer to $\lim_{N\to\infty} \alpha$ and $\lim_{N\to\infty} p$. Similarly $R = \alpha p$ will refer to $\lim_{N\to\infty} R$.

Since $R_M = R(f_M) < 1$, it is desireable to normalize R so that its maximum value is unity (Figure 7). This will be done by defining the amplifier response by

$$R^* = \frac{1}{R_M}\, \alpha(f)p(f), \tag{31}$$

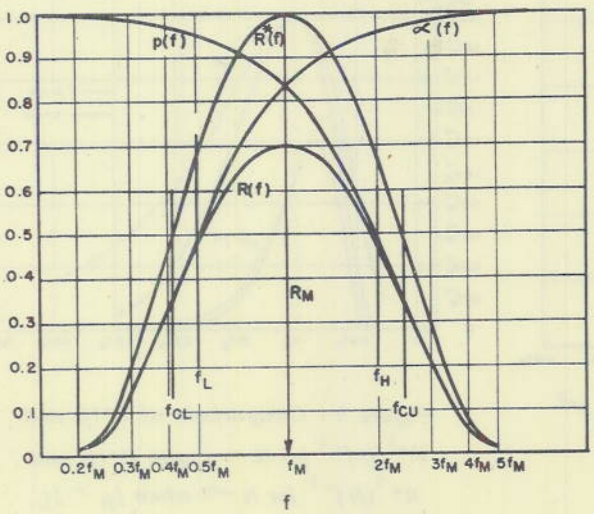

Figure 7 - Curves illustrating relationship between $\alpha(f)$, $p(f)$, $R(f)$, and $R^*(f)$ when $f_H = 4f_L$

where R_M is the maximum value of $R = \alpha(f)p(f)$. The difference between $\lim_{N\to\infty} R_M$ and R_M, for $N \geq 4$, is not great enough to affect substantially the results, discussed above, of using $\lim_{N\to\infty} R$ in place of R for $N \geq 4$. There is some deterioration of the response fit outside the pass band but an improvement of the fit within the pass band (Figure 8).

The Narrow-Band Amplifier

In the case of the narrow-band amplifier, the K-value frequencies cannot, in general, be considered as approximating the cut-off frequencies defined by

$$R(f_c) = \alpha(f_c)\,p(f_c) = 0.5\,R_M. \qquad (32)$$

This is due to the small value of R_M for the narrow-band amplifier. Normalization will be assumed and the response of the N-stage amplifier written

$$R^* = \frac{1}{R_M}\,\alpha(f)\,p(f) \qquad (33)$$

where, unless otherwise stated, the limits for $N \to \infty$ will be implied. Curves are given in Figure 8 of R^* and $R^{*2}f^{-2}$ for $f_H = 4f_L$ and $N = 4$ and compared with the plots of R^* and $R^{*2}f^{-2}$ for $N \to \infty$. The same functions are plotted in Figure 9 for $f_H = f_L$. These examples indicate that R^* for $N \to \infty$ makes a good approximation to R^*, for $N = 4$, within the pass band and that the total integrated noise for $N = \infty$ is nearly the same as that for $N = 4$. Total integrated noise voltage for $N \to \infty$ and for $N = 4$ is given by the square root of the area under $R^{*2}f^{-2}$, $N \to \infty$, and $R^{*2}f^{-2}$, $N = 4$, respectively.

It may be noted that no lower limit has been placed on f_H/f_L and the relations to be derived in the next two sections will indicate that, for a given frequency of maximum response, the bandwidth may be decreased indefinitely by making f_H/f_L sufficiently small. However, as f_H/f_L decreases below unity, R_M decreases so rapidly that it is not ordinarily practical to use values of f_H/f_L much less than unity. For example, when $f_H/f_L = 1$, bandwidth is equal to the frequency of maximum response and $R_M = 1/4$. To halve this bandwidth requires that $f_H/f_L = 1/4$, and under this condition $R_M = 1/256$. In this case, the bandwidth reduction of one half is accompanied by a sixty-four fold reduction in the over-all stage gain.

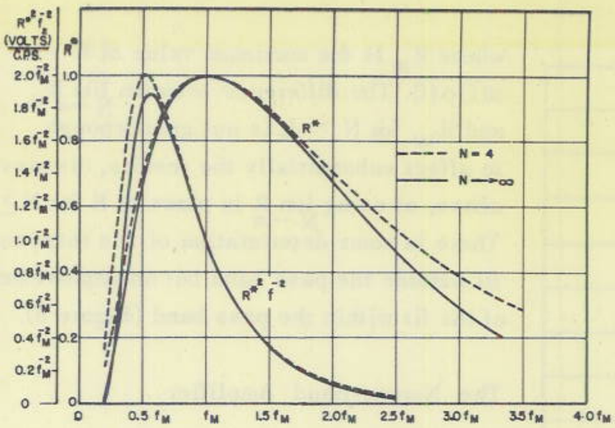

Figure 8 - Comparison of $R^*(f)$ and $R^{*2}(f)f^{-2}$ for $N = 4$ with $R^*(f)$ and $R^{*2}(f)f^{-2}$ for $N \to \infty$ when $f_H = 4f_L$

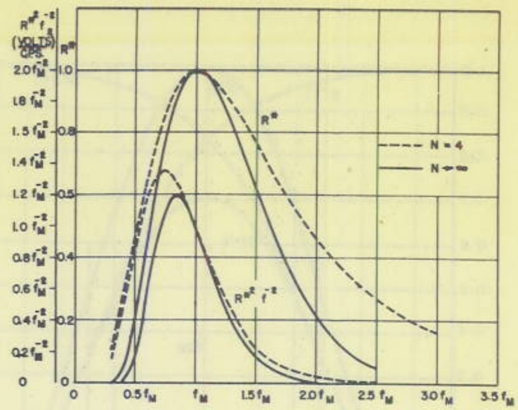

Figure 9 - Comparison of $R^*(f)$ and $R^{*2}(f)f^{-2}$ for $N = 4$ with $R^*(f)$ and $R^{*2}(f)f^{-2}$ for $N \to \infty$ when $f_H = f_L$

In the last three sections, the effects of replacing α and p for $N \geq 4$ by α and p for $N \to \infty$ were examined and the normalized response function R* defined by

$$R^*(f) = \frac{1}{R_M} \alpha(f) p(f) . \tag{34}$$

This definition is seen to apply equally well to the three amplifier classes considered; thus for the high-pass amplifier, $f_H \to \infty$ and $R_M = 1$. The important conclusions reached may be summarized:

(1) R*(f) for $N \to \infty$ closely approximates the R*(f), $N \geq 4$ of the multistage amplifier in the amplifier pass region.

(2) With inverse-frequency-noise input, the total noise output is not much affected by using R*(f), $N \to \infty$ in place of R*(f), $N \geq 4$ for the multistage amplifier.

(3) For the high-pass and broad-band-pass amplifiers, the cut-off frequencies are approximated by the K-value frequencies.

It should be noted that, while statement (2) refers to inverse-frequency noise, it is not true because of any fortunate property of inverse-frequency noise, but is true despite the rise of input noise in the region below cut-off where response representation is relatively poor. Statement (1) indicates that, for less-frequency-dependent types of noise, the total output noise would be even less affected by the proposed response representation.

In the remainder of this report R*, $N \to \infty$ will in general be used to characterize the response of the multistage R-C coupled amplifier. The properties of the representation are suitable for the purposes of this report and even where there is an appreciable deviation on an absolute basis, it may be seen that for many purposes, the relative deviation in results is a second-order effect and therefore negligible. An example of this situation would be where an investigation is made of the variation with center frequency of the noise output of a fixed bandwidth amplifier. Here the exact shape of the response curve would be secondary in importance to the bandwidth and center frequency.

Normalization of the Response Function

In order to normalize the response function to unity it is necessary to determine its maximum value, R_M.

From equations (25) and (26), it is seen that

$$R = \alpha \dot{p} = K^{\frac{f_L^2}{f^2}} K^{\frac{f^2}{f_H^2}} = K^{\frac{f_L^2}{f^2} + \frac{f^2}{f_H^2}} . \qquad (35)$$

Differentiating with respect to f,

$$\frac{dR}{df} = K^{\frac{f_L^2}{f^2} + \frac{f^2}{f_H^2}} (\log_e K) \cdot \left(-2\frac{f_L^2}{f^3} + 2\frac{f}{f_H^2} \right) . \qquad (36)$$

Solving $dR/df = 0$ for f, it is seen that the function is properly asymptotic at $f = 0$ and $f \to \infty$ and that

$$f_M = \sqrt{f_L f_H} \qquad (37)$$

where f_M is the frequency of maximum response.

Substituting in equation (35) and solving for R_M,

$$R_M = K^{2\frac{f_L}{f_H}} . \qquad (38)$$

Normalizing in accordance with equation (34),

$$R^* = K^{-2\frac{f_L}{f_H}} K^{\frac{f_L^2}{f^2} + \frac{f^2}{f_H^2}} = K^{\left(\frac{f_L}{f} - \frac{f}{f_H}\right)^2}$$

or letting $K = 0.5$

$$R^* = [0.5]^{\left(\frac{f_L}{f} - \frac{f}{f_H}\right)^2} \tag{39}$$

Relations Between Parameters

The cut-off frequencies, f_C, of R^* may be determined by placing $R^* = K$ in equation (38). Thus,

$$K = K^{\left(\frac{f_L}{f_C} - \frac{f_C}{f_H}\right)^2} \tag{40}$$

$$\left(\frac{f_L}{f_C} - \frac{f_C}{f_H}\right) = 1 , \tag{41}$$

and solving for the positive roots,

$$f_{CU} = \frac{f_H}{2}\left[\sqrt{1 + 4\frac{f_L}{f_H}} + 1\right]$$

$$f_{CL} = \frac{f_H}{2}\left[\sqrt{1 + 4\frac{f_L}{f_H}} - 1\right] \tag{42}$$

where f_{CU} and f_{CL} are the upper and lower cut-off frequencies (Figure 7). Expanding the radicals in equation (42),

$$\left.\begin{aligned} \frac{f_{CU}}{f_H} &= 1 + \left(\frac{f_L}{f_H}\right) - \left(\frac{f_L}{f_H}\right)^2 + 2\left(\frac{f_L}{f_H}\right)^3 - \ldots \\ \frac{f_{CL}}{f_L} &= 1 - \left(\frac{f_L}{f_H}\right) + 2\left(\frac{f_L}{f_H}\right)^2 - \ldots \end{aligned}\right\} , \tag{43}$$

it is seen that $\frac{f_{CU}}{f_H} \to 1$ and $\frac{f_{CL}}{f_L} \to 1$ as $\frac{f_L}{f_H} \to 0$.

Bandwidth, F, is seen by equation (42) to be

$$F = f_{CU} - f_{CL} = f_H \tag{44}$$

or by equation (37)

$$\left.\begin{aligned} F &= \sqrt{\frac{f_H}{f_L}}\; f_M \\ f_M &= \sqrt{f_H f_L} \end{aligned}\right\} . \tag{45}$$

Center frequency, f_{center}, is given by equation (42) as

$$f_{center} = \frac{f_{CU} + f_{CL}}{2} = \frac{f_H}{2}\sqrt{1 + 4\frac{f_L}{f_H}} , \tag{46}$$

which by equation (37) may be written

$$f_{center} = \frac{f_M}{2}\sqrt{\frac{f_H}{f_L} + 4} \quad . \tag{47}$$

It is often desirable to have f_H, f_L, and R* expressed in terms of f_M and F, f_{center} and F, or f_{CU} and f_{CL}. From equations (44), (45), and (46) it is seen that

$$\left.\begin{aligned} f_H &= F \\ f_L &= \frac{f_M^2}{F} = \frac{4f_{center}^2 - F}{4F} \\ \frac{f_L}{f_H} &= \frac{f_M^2}{F^2} \end{aligned}\right\} \tag{48}$$

and substituting in equation (39)

or letting K = 0.5,

$$\left.\begin{aligned} R^* &= [K]^{\left(\frac{f_M^2 - f^2}{fF}\right)^2} \\ R^* &= [K]^{\left(\frac{4f_{center}^2 - F - 4f^2}{4Ff}\right)^2} \\ R^* &= [0.5]^{\left(\frac{f_M^2 - f^2}{fF}\right)^2} \\ R^* &= [0.5]^{\left(\frac{4f_{center}^2 - F - 4f^2}{4Ff}\right)^2} \end{aligned}\right\} \tag{49}$$

From equations (42) and (48)

$$\left.\begin{aligned} f_{CU} &= \frac{1}{2}\left[\sqrt{F^2 + 4f_M^2} + F\right] \\ f_{CL} &= \frac{1}{2}\left[\sqrt{F^2 + 4f_M^2} - F\right] \end{aligned}\right\}, \qquad (50)$$

and another expression for center frequency is seen to be

$$f_{center} = \sqrt{f_M^2 + \frac{F^2}{4}} . \qquad (51)$$

Solving (50) for f_M and F,

$$\left.\begin{aligned} F &= f_{CU} - f_{CL} \\ f_M^2 &= f_{CU} f_{CL} \end{aligned}\right\}, \qquad (52)$$

so that by equations (48) and (49)

$$\left.\begin{aligned} f_H &= f_{CU} - f_{CL} \\ f_L &= \frac{f_{CU} f_{CL}}{f_{CU} - f_{CL}} \end{aligned}\right\}, \qquad (53)$$

and

$$R^* = [K]\left\{\frac{f_{CU} f_{CL} - f^2}{(f_{CU} - f_{CL}) f}\right\}^2 . \qquad (54)$$

or letting K = 0.5,

$$R^* = [0.5]^{\left\{\frac{f_{CU}f_{CL} - f^2}{(f_{CU} - f_{CL})f}\right\}^2}. \tag{55}$$

Inverse Frequency Noise Output

Let inverse frequency noise input be represented by $e_i^2 = A^2 f^{-2}$ $(\text{volts})^2$/cycle bandwidth where A^2 = the volts squared per cycle bandwidth at a frequency of 1 cps. The resultant noise output, E_n, from the multistage amplifier, normalized for unity gain, will then be

$$E_n = \sqrt{\int_0^\infty (R^* e_i)^2 df}\,, \tag{56}$$

and from equations (55) and (39)

$$E_n^2 = A^2 K^{-4\frac{f_L}{f_H}} \int_0^\infty K^{2\frac{f_L^2}{f^2}} K^{2\frac{f^2}{f_H^2}} f^{-2} df\,. \tag{57}$$

With the variable transformation

$$\mu = \frac{\sqrt{2}\, f_L \sqrt{\log_\epsilon \frac{1}{K}}}{f}, \tag{58}$$

$$E_n^2 = -\frac{A^2 K^{-4\frac{f_L}{f_H}}}{\sqrt{2}\, f_L \sqrt{\log_\epsilon \frac{1}{K}}} \int_0^\infty K^{\frac{\mu^2}{\log \frac{1}{K}}} K^{\frac{4 f_L^2 \log_\epsilon \frac{1}{K}}{f_H^2 \mu^2}} d\mu$$

$$= \frac{A^2 K \, \epsilon^{-4\frac{f_L}{f_H}}}{\sqrt{2}\, f_L \sqrt{\log_\epsilon \frac{1}{K}}} \int_0^\infty \left(\frac{1}{K}\right)^{\frac{1}{\log \frac{1}{K}}\left[-\mu^2 - \left(\frac{2f_L \log \frac{1}{K}}{f_H}\right)^2 \frac{1}{\mu^2}\right]} d\mu \qquad (59)$$

so that

$$E_n^{\,2} = \frac{A^2 K \, \epsilon^{-4\frac{f_L}{f_H}}}{\sqrt{2}\, f_L \sqrt{\log \frac{1}{K}}} \int_0^\infty \epsilon^{\left[-\mu^2 - \left(\frac{2f_L \log \frac{1}{K}}{f_H}\right)^2 \frac{1}{\mu^2}\right]} d\mu \; . \qquad (60)$$

Performing the indicated integration,

$$E_n^{\,2} = \frac{A^2 (K) \, \epsilon^{-4\frac{f_L}{f_H}}}{\sqrt{2}\, f_L \sqrt{\log_\epsilon \frac{1}{K}}} \left(\frac{\sqrt{\pi}}{2}\, \epsilon^{-4\frac{f_L}{f_H} \log_\epsilon \left(\frac{1}{K}\right)}\right) , \qquad (61)$$

$$\frac{\ldots}{E_n} = 1.152 \, \frac{\ldots}{A} \sqrt{\frac{\ldots}{f_{CU} - f_{CL}}} \; . \qquad (71)$$

[redacted]

Equation (69) shows that the lower K-value frequency is sufficient to determine the S/N ratio but, as equation (70) indicates, a constant value for f_L implies a proportional relation between f_M and F. Equation (70) is particularly significant in showing that for constant bandwidth, S/N ratio varies directly with the operating frequency (f_M assumed always made equal to the operating frequency), and also in showing that for any particular operating frequency, S/N ratio varies inversely with the square root of the bandwidth.

Noise Output and S/N Ratio with White Noise Input

Let white noise input be represented by

$$e_i^2 = B^2 \text{ (volts)}^2\text{/ cycle bandwidth} \qquad (72)$$

where B is independent of frequency. Then, output noise voltage will be given by

$$E_n = \sqrt{B^2 K^{-4\frac{f_L}{f_H}} \int_0^{\infty} K^{2\frac{f^2}{f_H^2}} K^{2\frac{f_L^2}{f^2}} df} \,. \qquad (73)$$

Let

$$\mu = \sqrt{2 \log_{\epsilon} \frac{1}{K}} \, \frac{f}{f_H} \,.$$

Then

$$E_n^2 = B^2 K^{-4\frac{f_L}{f_H}} \int_0^{\infty} K^{\frac{\mu^2}{\log_{\epsilon}\frac{1}{K}}} K^{\frac{4 f_L^2 \log_{\epsilon}\frac{1}{K}}{f_H^2 \mu^2}} \frac{f_H}{\sqrt{2 \log \frac{1}{K}}} \, d\mu \,, \qquad (74)$$

[redacted]

$$= \frac{B^2 f_H K^{-4\frac{f_L}{f_H}}}{\sqrt{2 \log_\epsilon \frac{1}{K}}} \int_0^\infty \left(\frac{1}{K}\right)^{\frac{1}{\log_\epsilon \frac{1}{K}}\left[-\mu^2 - \left(\frac{2f_L \log_\epsilon \frac{1}{K}}{f_H}\right)^2 \frac{1}{\mu^2}\right]} d\mu \,,$$

$$= \frac{B^2 f_H K^{-4\frac{f_L}{f_H}}}{\sqrt{2 \log_\epsilon \frac{1}{K}}} \int_0^\infty \epsilon^{-\mu^2 - \left(\frac{2f_L \log_\epsilon \frac{1}{K}}{f_H}\right)^2 \frac{1}{\mu^2}} d\mu \,,$$

$$= \frac{B^2 f_H K^{-4\frac{f_L}{f_H}}}{\sqrt{2 \log_\epsilon \frac{1}{K}}} \left(\frac{\sqrt{\pi}}{2} \epsilon^{-\frac{4f_L \log_\epsilon \frac{1}{K}}{f_H}}\right),$$

$$= \frac{B^2 f_H}{\sqrt{2 \log_\epsilon \frac{1}{K}}} \left(\frac{\sqrt{\pi}}{2}\right), \tag{74}$$

so that

$$E_n^2 = B^2 f_H \left(\frac{\pi}{8 \log_\epsilon \frac{1}{K}}\right)^{\frac{1}{2}}, \tag{75}$$

$$E_n = B\sqrt{f_H} \left(\frac{\pi}{8 \log_\epsilon \frac{1}{K}}\right)^{\frac{1}{4}}, \tag{76}$$

and, on letting K = 0.5 in accordance with the K-value and cut-off convention,

$$E_n = 0.8675\, B \sqrt{f_H}\ , \tag{77}$$

or by using equation (42),

$$E_n = 0.8675\, B \sqrt{F}\ , \tag{78}$$

$$E_n = 0.8675\, B \sqrt{f_{CU} - f_{CL}}\ . \tag{79}$$

Recalling the discussion on S/N ratio in the previous section, equation (68) may be combined with equations (77), (78), and (79) to give these expressions for S/N output ratio with white noise input:

$$\frac{E_{SM}}{E_n} = 1.152\, \frac{e_{SM}}{B}\, \frac{1}{\sqrt{f_H}}\ , \tag{80}$$

$$\frac{E_{SM}}{E_n} = 1.152\, \frac{e_{SM}}{B}\, \frac{1}{\sqrt{F}}\ , \tag{81}$$

$$\frac{E_{SM}}{E_n} = 1.152\, \frac{e_{SM}}{B}\, \frac{1}{\sqrt{f_{CU} - f_{CL}}}\ . \tag{82}$$

It is seen that the S/N ratio is a function only of the bandwidth of the amplifier and the RMS values of the input signal and noise. The S/N ratio is seen to vary inversely as the square root of the amplifier bandwidth.

Output and S/N Ratio with Both Inverse Frequency and White Noise Input

Let the input noise be represented by

$$e_i^2 = B^2 + A^2 f^{-2}\ (\text{volts})^2 / \text{ cycle bandwidth}\ . \tag{83}$$

Output noise will then be represented by the sum of the expressions given in equations (62) and (75). Letting E_n be the resultant output noise voltage,

$$E_n^2 = \frac{A^2}{f_L}\left(\frac{\pi}{8 \log_\epsilon \frac{1}{K}}\right)^{1/2} + B^2 f_H \left(\frac{\pi}{8 \log_\epsilon \frac{1}{K}}\right)^{1/2} , \quad (84)$$

and, on letting K = 0.5 in accordance with the K-value and cut-off convention,

$$E_n = 0.8675 \sqrt{B^2 f_H + A^2 \frac{1}{f_L}} , \quad (85)$$

which, by equations (48) and (53), may also be written

$$E_n = 0.8675 \sqrt{F\left(B^2 + \frac{A^2}{f_M^2}\right)} = 0.8675 \sqrt{F\left(B^2 + \frac{4A^2}{4f_{center}^2 - F}\right)} \quad (86)$$

and

$$E_n = 0.8675 \sqrt{\left(f_{CU} - f_{CL}\right)\left(B^2 + \frac{A^2}{f_{CU} f_{CL}}\right)} . \quad (87)$$

For a given response function, expressions for maximum signal-to-noise ratio will be given by combining equation (68) with equations (85), (86), and (87). Thus,

$$\frac{E_{SM}}{E_n} = 1.152 \frac{e_{SM}}{\sqrt{B^2 f_H + A^2 \frac{1}{f_L}}} , \quad (88)$$

$$\frac{E_{SM}}{E_n} = 1.152 \frac{e_{SM}}{\sqrt{F\left(B^2 + \frac{A^2}{f_M^2}\right)}} = 1.152 \frac{e_{SM}}{\sqrt{F\left(B^2 + \frac{4A^2}{4f_{center}^2 - F}\right)}}, \tag{89}$$

$$\frac{E_{SM}}{E_n} = 1.152 \frac{e_{SM}}{\sqrt{\left(f_{CU} - f_{CL}\right)\left(B^2 + \frac{A^2}{f_{CU}f_{CL}}\right)}}. \tag{90}$$

Amplifier Phase Characteristic

For considerations of harmonic steady state signals and randomly phased noise spectra, only the amplitude of the response function is significant. Even where phase distortion affects results, the influence of phase distortion is generally secondary to that of amplitude distortion and may often be neglected.

The phase factor has been omitted in the expressions derived for $\alpha(f)$, $p(f)$, and $R(f)$. In the following, certain of these expressions will be written in complex form and the arguments will represent the phase factors. The detailed development is analagous to that already given and will not be repeated.

Corresponding to equation (9) and equation (10),

$$e_o = g_m R_B \frac{\omega C R_g}{\sqrt{\omega^2 C_C^2 R_g^2 + 1}} \epsilon^{j \tan^{-1} \frac{1}{\omega C_C R_g}} \frac{1}{\sqrt{\omega^2 C_B^2 R_1^2 + 1}} \epsilon^{-j \tan^{-1} \omega C_B R_1} e_i , \tag{91}$$

$$
\left.
\begin{aligned}
G &= g_m R_B \\[2ex]
\alpha(f) &= \frac{2\pi f C_c R_g}{\sqrt{4\pi^2 f^2 C_c^2 R_g^2 + 1}} \, \epsilon^{\, j \tan^{-1} \frac{1}{2\pi f C_c R_g}} \\[2ex]
p(f) &= \frac{1}{\sqrt{4\pi^2 f^2 C_B^2 R_1^2 + 1}} \, \epsilon^{-j \tan^{-1} 2\pi f C_B R_1} \\[2ex]
2\pi f &= \omega
\end{aligned}
\right\} \qquad (92)
$$

and to equation (13)

$$
\left.
\begin{aligned}
e_o &= \left[\frac{Lf}{\sqrt{L^2 f^2 + 1}}\right]^N \left[\frac{1}{\sqrt{H^2 f^2 + 1}}\right]^N \epsilon^{\, jN\left(\tan^{-1}\frac{1}{Lf} - \tan^{-1} Hf\right)} e_i \\[2ex]
G &= 1 \\[2ex]
\alpha(f) &= \left[\frac{Lf}{\sqrt{L^2 f^2 + 1}}\right]^N \epsilon^{\, jN \tan^{-1} \frac{1}{Lf}} \\[2ex]
p(f) &= \left[\frac{1}{\sqrt{H^2 f^2 + 1}}\right]^N \epsilon^{-jN \tan^{-1} Hf}
\end{aligned}
\right\}, \qquad (93)
$$

where as in equation (15),

$$L = \frac{1}{f_L \sqrt{K^{-2/N} - 1}} , \quad H = \frac{\sqrt{K^{-2/N} - 1}}{f_H} . \tag{94}$$

Unfortunately the function,

$$N\left(\tan^{-1}\frac{1}{Lf} - \tan^{-1} Hf\right),$$

does not converge to a finite limiting function with increasing N and must therefore be considered as a function of both N and f. From a strict point of view, the compatability relations[6] between phase and amplitude characteristics would indicate that the phase factor should be associated with the amplitude characteristic corresponding to the same N. The approximations made in the derivation of both represent physically realizable, though not actual, situations. For instance, the independence of α and p might be achieved by using an isolating stage between the low-frequency and high-frequency time constant circuits. In practice it will be found that a fairly accurate representation may be made by the hybrid association of the phase factor function for actual finite N with the amplitude response function for infinite N. Thus, the normalized complex response function can be written

$$R^* = K^{\left(\frac{f_L}{f} - \frac{f}{f_H}\right)^2} \epsilon^{-jN\left(\tan^{-1}\frac{f}{f_H}\sqrt{K^{-2/N} - 1} - \tan^{-1}\frac{f_L}{f}\sqrt{K^{-2/N} - 1}\right)} , \tag{95}$$

or

$$R^* = K^{\left(\frac{f_M^2 - f^2}{fF}\right)^2} \epsilon^{-jN\left(\tan^{-1}\frac{f}{F}\sqrt{K^{-2/N} - 1} - \tan^{-1}\frac{f_M^2}{fF}\sqrt{K^{-2/N} - 1}\right)} . \tag{96}$$

---

[6] *Gullemin, E. A., "Communication Networks," pp. 501-2, J. Wiley & Sons, New York, 1931-35*

 [redacted]

The behavior of the phase function may be more clearly seen bv writing equation (96) as

$$\left.\begin{aligned} R^* &= K\left(\frac{f_M^2 - f^2}{fF}\right)\epsilon^{-jN\tan^{-1}M\left(\frac{f}{f_M}-\frac{f_M}{f}\right)} \\ M &= \frac{\frac{F}{f_M}\sqrt{4^{1/N}-1}}{\frac{F^2}{f_M^2}+4^{1/N}-1} \end{aligned}\right\} \quad (97)$$

The total phase lag in radians is given by

$$\theta = N\tan^{-1}M\left(\frac{f}{f_M}-\frac{f_M}{f}\right) \quad (98)$$

Phase lag per stage, θ/N, is plotted in Figure 10 as a function of f/f_M for several values of M. The nominal cut-off points for the case when $F = f_M$ are indicated, and it is seen that, in this case, the departure from linearity of the phase characteristic is not severe within the pass band.

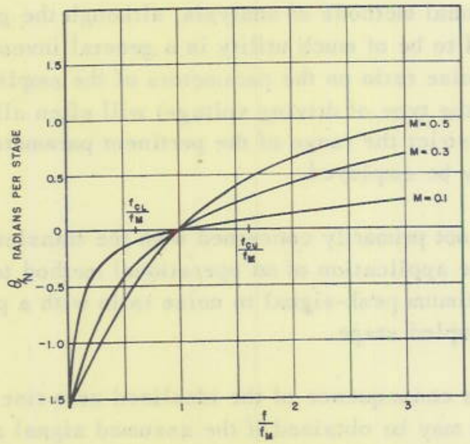

Figure 10 - Variation of phase angle with frequency for several values of the parameter M

Amplifier Build-Up Time

The complex form of the response function theoretically may be used to determine output voltage of the amplifier when the complex frequency spectrum, $g(f)\,e^{j\omega t}$, of the input signal voltage is known. Thus,

$$e_o(t) = \int_o^{\infty} g(f) e^{j\omega t} R^*(f)\, df \tag{99}$$

where $R^*(f)$ is the complex response function as given above.

A detailed discussion of the relative effects of the amplitude and phase factor functions on the output voltage function is beyond the scope of this report. It will only be said and illustrated in the Appendix, that the build-up time or time constant of this type of tuned amplifier is mainly determined by the bandwidth, F, and is not greatly affected by the phase function. Here, build-up time is defined as the time required for the voltage output to rise from zero to its steady-state value when a harmonic signal of center frequency is suddenly applied. This time, T, is roughly given by the relation

$$T = \frac{1}{F} \tag{100}$$

where F is the bandwidth. The time, T, is also approximately the time required for the output of the amplifier to reach its maximum value when a harmonic signal outside of the pass band (the d-c signal is a special case of this) is suddenly applied.

The great difficulty in evaluating definite integrals of the type encountered in equation (99) limits the usefulness of the Fourier integral method for a study of the transient behavior of the R-C coupled amplifier. The response functions that may be conveniently handled by this method are highly idealized and are not usually derived from the physical constants of the amplifier. Transient behavior in terms of these physical constants may be derived by application of classical or operational methods of analysis, although the general formulae so obtained are usually too complicated to be of much utility in a general investigation of the dependence of the transient-signal to noise ratio on the parameters of the amplifier. Fortunately the particular application (such as the type of driving voltage) will often allow suitable approximate representations and also restrict the range of the pertinent parameters so that graphical or trial numerical methods may be employed.

Although this report is not primarily concerned with the transient response of amplifiers, an example will be given of the application of an operational method to the determination of the amplifier parameters for optimum peak-signal to noise ratio with a particularly simple type of signal and only one R-C coupled stage.

The trivial results are a consequence of the idealized zero rise time assumed for the signal voltage. Significant results may be obtained if the assumed signal approximates the actual signal in its rise and decay characteristics. However, any useful representation generally results in equations which are difficult to solve except by numerical methods.

Example of Transient S/N Ratio Analysis

Here the transient-signal to noise ratio is defined as the ratio of the peak output signal voltage in the absence of noise to the RMS noise voltage output caused by inverse frequency noise input. A step function will be assumed for signal input and, as will be evident, the results may be extended to a rectangular pulse of any width. Considering a single amplifier stage and referring to Figure 11,

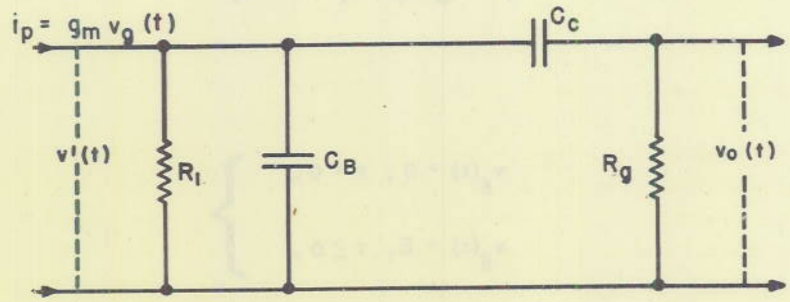

Figure 11 - Equivalent diagram of single R-C stage for transient analysis

$$C_B \frac{d v'(t)}{dt} + \frac{1}{R_1} v'(t) = g_m v_g(t) \, , \qquad (101)$$

$$\frac{dv'(t)}{dt} = \frac{v_o(t)}{R_g C_c} + \frac{dv_o(t)}{dt} \quad , \qquad (102)$$

where equation 102 represents an approximation permissable because R_g is generally several times greater than R_1.

Taking Laplace transforms of (101) and (102),

$$\left(C_B S + \frac{1}{R_1}\right) V'(S) = g_m V_g(S) \qquad (103)$$

$$SV'(S) = \frac{1}{R_g C_c} V_o(S) + S V_o(S) \qquad (104)$$

where capital letters are used to represent the transforms of the functions designated by the corresponding lower case letters. Solving equations (103) and (104) for $V_o(S)$,

$$V_o(S) = \frac{g_m}{C_B} \frac{S}{\left(S + \frac{1}{R_g C_c}\right)\left(S + \frac{1}{R_1 C_B}\right)} V_g(S) \quad . \tag{105}$$

Now, letting

$$\left.\begin{aligned} v_g(t) &= 0\,, \quad t < 0\,, \\ v_g(t) &= E, \quad t \geq 0\,, \end{aligned}\right\} \tag{106}$$

and defining τ_1, τ_2, and r by

$$\left.\begin{aligned} \tau_1 &= R_1 C_B \\ \tau_2 &= R_g C_c \\ r &= \frac{\tau_2}{\tau_1} \end{aligned}\right\} \tag{107}$$

and noting that the transform of $v_g(t)$ as defined by equation (107) is given by

$$V_g(S) = \frac{g_m R_1}{\tau_1} \quad , \tag{108}$$

it is seen that

$$V_o(S) = \frac{g_m R_1}{\tau_1} \frac{E}{\left(S + \frac{1}{\tau_1}\right)\left(S + \frac{1}{\tau_2}\right)} \quad . \tag{109}$$

Now making the inverse transformation,

$$V_o(t) = \frac{g_m R_1}{\tau_1} E \; \frac{\epsilon^{-\frac{t}{\tau_1}} - \epsilon^{-\frac{t}{\tau_2}}}{\frac{1}{\tau_2} - \frac{1}{\tau_1}} \quad . \qquad (110)$$

Equation (110) describes the time response of the amplifier stage to the step function, $v_g(t)$. The response to a rectangular pulse is, by the principle of superposition, given by

$$\left.\begin{array}{ll} V_o(t) , & t < T \\ V_o(t) - V_o(t-T) , & t \geq T \end{array}\right\} \qquad (111)$$

where T is the duration of the signal pulse.

Equating the time derivative of $V_o(t)$ to zero gives

$$\frac{\tau_1}{\tau_2} = \epsilon^{\left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right) t} , \qquad (112)$$

and the time of peak response as

$$t_p = \frac{\tau_1 \tau_2}{\tau_2 - \tau_1} \log_\epsilon \frac{\tau_2}{\tau_1} = \tau_2 \frac{\log_\epsilon r}{r - 1} \qquad (113)$$

Substituting this expression for t_p in equation (108) gives the peak output voltage as

$$(V_o)_p = g_m E R_1 \left(\frac{\tau_2}{\tau_1}\right)^{\frac{\tau_1}{\tau_1 - \tau_2}} = g_m E R_1 (r)^{\frac{1}{1-r}} \quad . \qquad (114)$$

The range of variation of $(V_o)p$ with r is seen to be from 0 to $g_m ER_1$ as r varies from 0 to ∞. It is evident that a lower limit must be placed upon r if a lower limit is to be placed on the gain of the stage. If, for example, the peak gain is to be ε^{-1} of the maximum possible gain, then r must be ≥ 1.

For calculating the absolute noise output in terms of the circuit parameters, the unnormalized response function must be used. Equation (61), when adjusted by the normalization factor, becomes

$$E_n = 0.8675\,(4)^{-\frac{f_L}{f_H}} \frac{1}{\sqrt{f_L}}\,, \tag{115}$$

but

$$\left.\begin{aligned} f_L &= \frac{1}{2\pi\sqrt{3}R_g C_c} = \frac{1}{2\pi\sqrt{3}\,\tau_2} \\ f_H &= \frac{\sqrt{3}}{2\pi R_1 C_B} = \frac{\sqrt{3}}{2\pi\tau_1} \end{aligned}\right\}\,, \tag{116}$$

so that

$$E_n = 0.8675A(4)^{-\frac{\tau_1}{3\tau_2}} \sqrt{2\pi\sqrt{3}\,\tau_2}\,, \tag{117}$$

or

$$E_n = 2.864\,\sqrt{\tau_2}\,(1.59)^{-\frac{1}{r}}\,. \tag{118}$$

The ratio of peak-signal output to RMS noise voltage output is now seen to be given by

$$\frac{(V_o)_p}{E_n} = \frac{g_m ER_1}{2.864\,\sqrt{\tau_2}}\,(r)^{\frac{1}{1-r}}\,(1.59)^{\frac{1}{r}}\,. \tag{119}$$

Maximizing this expression with respect to r gives a maximum ratio for r = 1. With constant r, the ratio is seen to increase with $\frac{1}{\tau_2}$. Substituting r = 1 in equations (113), (114), and (117) gives

$$t_p = \tau_2 \tag{120}$$

$$(V_o)_p = g_m E R_1 \epsilon^{-1}, \tag{121}$$

and

$$\frac{(V_o)_p}{E_n} = \frac{g_m E R_1}{2.864\sqrt{\tau_2}}\, \epsilon^{-1} (1.59)\,. \tag{122}$$

It appears that any desired peak-signal to noise ratio might be attained if τ_2 were made sufficiently small. The peak output is independent of τ_2 (for constant r), but inverse frequency noise naturally decreases with decreasing τ_2. Since t_p decreases with decreasing τ_2, a finite rectangular pulse of any width will offer the same possibility of indefinitely increasing the peak-signal to RMS noise ratio. These results are a consequence of having assumed a zero rise time function for the grid signal. If we assume that equations (120), (121), and (122) are approximately true for steep fronted pulses, with finite rise time, T', it is seen from equation (120) that a lower limit must be placed on τ_2. It is easily shown, by considering input signal functions of the form $v_g(t) = ct$, that the signal output voltage at time, t_p, will be roughly two-thirds of $(V_o)p$ (as given by equation (121)) if t_p is made equal to the input pulse rise time, T'. Where the rate of rise is a function of time, an equivalent rise time may be defined as the time required for the input signal to reach some fraction (appropriate to the shape of the rise curve) of its maximum value. In this example a practical lower limit on τ_2 would seem to be given by

$$\tau_2 = t_p = T', \tag{123}$$

and since r = 1

$$\tau_1 = T. \tag{124}$$

Equations (107), (123), and (124) may then be used to determine approximately the amplifier parameters for optimum peak-signal to RMS noise ratio when the noise input is inverse frequency noise, and the other appropriate equations used to predict amplifier behavior. The example given is artificial in that a generalized response function has been used for a single stage of amplification, and is approximate also because formulae based upon an idealized situation have been assumed useful under modified conditions.

Example of Steady-State S/N Ratio Analysis

The bandwidth of an amplifier designed to amplify steady-state signals is often determined by considerations of time constant; thus, the band must be wide enough so that changes in the amplitude of the signal will be followed with the desired fidelity. Then, equation (100) places a lower limit on F in terms of build-up time. Since noise decreases as $\sqrt{F}$, this lower limit is the value of F that should be chosen (unless other considerations are involved). With F thus determined, f_M may then be chosen for maximum S/N ratio.

Let f_s be signal frequency, and let the noise, E_n, be inverse frequency noise. From equations (49) and (65), S/N ratio may be written as a function of f_M

$$\frac{E_s}{E_n} = \frac{[0.5]^{\left(\frac{f_M^2 - f_s^2}{f_s F}\right)^2}}{0.8675\,\frac{\sqrt{F}}{f_M}} \left(\frac{E_i}{A}\right), \qquad (125)$$

where E_i is the input signal in volts,

A is the RMS noise voltage per square root cycle at one cps.

Now

$$\frac{\partial}{\partial f_M}\left(\frac{E_s}{E_n}\right) = \frac{E_i}{0.8675 A \sqrt{F}} \left[(0.5)^{\left(\frac{f_M^2 - f_s^2}{f_s F}\right)^2} + \frac{4 f_M^2 \left(f_M^2 - f_s^2\right)}{f_s^2 F^2} (0.5)^{\left(\frac{f_M^2 - f_s^2}{f_s F}\right)^2} \log_\epsilon 0.5 \right]. \qquad (126)$$

Maximizing with respect to f_M,

$$1 + \frac{4 f_M^2 \left(f_M^2 - f_s^2\right)}{f_s^2 F^2} \log_\epsilon 0.5 = 0,$$

$$f_M^2 = \frac{f_s^2}{2} + \frac{1}{2} \sqrt{f_s^4 + f_s^2 \frac{F^2}{0.693}},$$

and

$$f_M^2 = f_S^2 \left[\frac{1}{2} + \frac{1}{2} \sqrt{1 + 1.442 \left(\frac{F}{f_s} \right)^2} \right]. \tag{127}$$

As a rather extreme numerical example, if $F = 5.77 f_S$ then $f_M = 2f_S$ for optimum S/N ratio.

Some Applications to the Passive Bearing Finder

Passive bearing finder amplifiers may be divided into two classes, those which must respond to an isolated signal pulse, and those which amplify a periodic signal. In either case, if the amplifier employed is of the type considered in this report, the equations derived for RMS noise output, in terms of the amplifier parameters or operating characteristics, will apply. It is necessary, of course, to determine or assume the magnitude and nature of the equivalent amplifier input noise.

The situations with respect to signal output are, however, quite different for the two classes of amplifiers. The response to a steady-state signal is simply given by the response function while the response to a transient pulse is a function not only of the amplifier stage parameters, but also of the number of stages and of the shape and time spread of the signal pulse.

In the case of the PBF pulse amplifier, the characteristics of the signal to be amplified are known or may be determined. Conventional methods of transient analysis will then give the output signal as a time function involving N and the two stage parameters. Peak-to-peak output may then be plotted as a function of one of these three parameters for various assigned values of the other two. These curves may then be compared to the appropriate noise output curves, plotted from the equations derived in this report, to determine by inspection the parameters for optimum peak-signal to noise ratio. No attempt is made here to determine the parameters for optimum S/N ratio in the PBF transient amplifier, for considerable labor is required to calculate peak-to-peak signal response for the various possible values of the parameters, and such an investigation is beyond the scope of this report.

The periodic or harmonic PBF amplifier is very simply treated analytically, and the results derived in the example of a steady state S/N ratio analysis may be immediately applied to it. Signal frequency, f_S, is 5 cps and the desired time constant of 0.2 second dictates a bandwidth, F, of 5 cps. The other parameter, F_M, should then be adjusted for maximum S/N ratio. Noise is predominately inverse frequency noise. By equation (127),

$$f_M^2 = 25 \left[\frac{1}{2} + \frac{1}{2} \sqrt{1 + 1.442} \right] ,$$

$$= 25 \times 1.281 = 32.05 ,$$

so that

$$f_M = 5.66 \text{ CPS.}$$

By equation (44),

$$f_H = F = 5 \text{ CPS.}$$

By equation (48),

$$f_L = \frac{f_M^2}{F} = \frac{32.05}{5} = 6.4 \text{ CPS.}$$

The number of stages, N, is arbitrary so far as S/N ratio is concerned. Having decided on the N to be used, τ_L and τ_H may be determined from the curves in Figure 5, and C_C, R_g, C_B, R_1 chosen in compliance with (11). Plate resistance, r_p, is known and R_B is then given by the relation

$$R_B = \frac{r_p R_1}{r_p - R_1} .$$

By equation (125),

$$\frac{E_s}{E_n} = \frac{[0.5]^{\left(\frac{32.05 - 25}{25}\right)^2}}{0.8675 \quad \frac{\sqrt{5}}{5.66}} \left(\frac{E_i}{A}\right) ,$$

$$= 5.66 \times .0.946\,(0.515)\left(\frac{E_i}{A}\right) = 2.76\,\frac{E_i}{A} .$$

Were f_M simply made equal to f_S, the S/N ratio would be given by

$$\frac{E_s}{E_n} = \frac{f_M}{0.8675\sqrt{F}}\left(\frac{E_i}{A}\right) = \frac{\sqrt{5}}{0.8675}\left(\frac{E_i}{A}\right) = 2.58\,\frac{E_i}{A},$$

and it is seen that the optimum S/N ratio is 7 percent greater than would be obtained by merely setting f_M equal to f_s.

SUMMARY AND CONCLUSIONS

General

The frequency response of the multistage R-C coupled band pass amplifier, normalized for unity gain at the frequency of maximum response, may be very closely approximated by a simple function completely determined by the specification of but two frequency parameters (equation (39)). The response function is thus independent of the number of amplifier stages when the two frequency parameters are specified.

These frequency parameters are determined by the two nominal cut-off frequencies, by bandwidth and frequency of maximum response, or by bandwidth and center frequency (equations (48) and (53)). Any of these significant pairs of operating characteristics may therefore be considered as the specifying frequency parameters for the response function and it is simply expressed in terms of them (equations (49) and (54)).

The two frequency parameters may be derived from three circuit constants, i.e., the two stage-time-constants and the number of stages (equation 23 and Figure 5). Conversely, if the frequency parameters and the number of stages are specified, the amplifier time constants are determined.

Analytical expressions for integrated noise output and steady-state signal-to-noise ratio in terms of the noise and signal input, and the two defining parameters of the response function show that:

(1) For inverse frequency noise, the output noise voltage is inversely proportional to the frequency of maximum response and directly proportional to the noise input and the square root of the bandwidth (equations (64), (65), (66)).

(2) For white noise, the noise output is independent of the frequency of maximum response and proportional to the noise input and the square root of the bandwidth (equations (77), (78), (79)).

(3) For inverse frequency and white noise, the output noise voltage is given by an expression involving the inverse frequency and the white noise input, the frequency of maximum response, and the bandwidth (equations (85), (86), (87)).

(4) Signal-to-noise ratio for a given amplifier is in each case simply given by the signal input at maximum response divided by the appropriate expression for the noise output (equations (69-71), (80-82), (88-90)).

The complex response function involves an additional parameter, the number of stages in the amplifier (equation (97)).

The difficulty of evaluating Fourier transforms involving the complex response function is so great that the use of the response function in a transient analysis is not recommended. However, the results of a transient analysis may well be combined with the equations for integrated noise output obtained by using the amplitude frequency response function.

Conclusions with Regard to the PBF

A quantitative conclusion with regard to the optimum peak-to-peak signal-to-RMS noise ratio that the simple R-C amplifier might yield when the signal consists of isolated pulses, requires detailed information regarding the transient response of the amplifier to such signal pulses for different numbers of stages and for various values of the two stage parameters. Transient analysis would furnish this information and make it possible to design for peak performance.

The circuit constants of present periodic PBF amplifiers are sufficiently close to the optimum values so that only a small percentage improvement in S/N ratio could be achieved by adjustment of the circuit constants.

Only the simple R-C amplifier has been considered in this report. More elaborate amplifiers, perhaps employing feedback circuits degenerative for signals below signal frequency, might be designed with a much sharper low frequency cut-off, and such amplifiers should give a somewhat greater signal-to-noise ratio. However, it appears that unless inverse frequency noise can be reduced, substantial improvement in S/N ratio will require a higher signal frequency.

Signal-to-noise ratio should increase directly with signal frequency, that is so long as inverse frequency noise predominates over white noise. Signal frequency might be increased by:

(1) Using a higher scan rate or some form of optical chopping, either of which would require faster sensitive elements.

(2) Frequency conversion by chopping the electrical signal or by some other form of modulation.

* * *

APPENDIX

Method Used to Estimate Relative Influence of Amplitude and Phase Characteristics on Amplifier Response Time

Let $\Delta_i e(t)$ represent the contribution to the time response of the band, $\Delta\omega$, of the amplifier output spectrum with center angular frequency, ω_i.

Then

$$\Delta_i e(t) = \int_{\omega_i - \frac{\Delta\omega}{2}}^{\omega_i + \frac{\Delta\omega}{2}} A(\omega)\, d\omega \; \epsilon^{j[\omega t - \theta(\omega)]} , \tag{128}$$

where $A(\omega)$ is the product of the original amplitude spectrum and the amplitude response function; $\theta(\omega)$ is the amplifier phase lag as a function of the angular frequency, ω; and where, for convenience, the components of the input signal are assumed to have zero phase at time zero. Now, if $\Delta\omega$ is sufficiently small, $A(\omega)$ may be considered constant and $\theta(\omega)$ represented by $\alpha_i + t_i\omega$, where α_i is the θ intercept and t_i the slope of the tangent to the $\theta(\omega)$ curve at ω_i. Therefore,

$$\Delta_i e(t) = \text{Re}\; A(\omega_i)\; \epsilon^{-j\alpha_i} \int_{\omega_i - \frac{\Delta\omega}{2}}^{\omega_i + \frac{\Delta\omega}{2}} \epsilon^{j\omega(t - t_i)}\, d\omega ,$$

$$= \text{Re}\; A(\omega_i)\; \epsilon^{-j\alpha_i} \left\{ \frac{\epsilon^{j\left(\omega_i + \frac{\Delta\omega}{2}\right)(t - t_i)} - \epsilon^{j\left(\omega_i - \frac{\Delta\omega}{2}\right)(t - t_i)}}{j(t - t_i)} \right\} , \tag{129}$$

or

$$\Delta_i e(t) = \text{Re}\, A(\omega_i) \frac{\Delta\omega}{2} \frac{\sin \frac{\Delta\omega}{2}(t - t_i)}{\frac{\Delta\omega}{2}(t - t_i)} \epsilon^{j\,[\omega_i(t - t_i) - \alpha_i]} \quad . \tag{130}$$

Now assume for the moment that the input spectrum has zero phase at $t = 0$ and constant amplitude, $A_o d\omega$ (such would be the case for a pulse much shorter than $1/F$ and symmetrical about $t = 0$), and further that the response function has amplitude unity within the pass band, zero amplitude elsewhere, and a phase characteristic representable in the pass band by $\alpha + t'\theta$. It is then seen that the integration indicated in equation (128) may be extended over the entire pass band to give

$$e(t) = \text{Re}\, \pi F A_o \frac{\sin \pi F(t - t')}{\pi F(t - t')} \epsilon^{j\,[\omega_o(t - t') - \alpha]} \tag{131}$$

where ω_o is the center frequency of the pass band. It is seen that with such an idealized amplitude characteristic and no phase distortion, the envelope rise time is

$$T = \frac{1 - Ft'}{F} - \frac{Ft'}{F} = \frac{1}{F} \quad . \tag{132}$$

The effect of phase distortion on the arrival time of the component wave groups is indicated by equation (130). The time of arrival of the wave group around ω_i is thus t_i. The influence on build-up time of phase distortion may be considered as being roughly the difference in time of arrival of the most essential frequencies and the first to arrive. Letting τ represent this difference,

$$\tau = t_o - [t_i]_{min} \quad . \tag{133}$$

If $T > \tau$, it may be said that response time is determined mainly by bandwidth.

Because of the actual shape of the response function for the R-C amplifier, τ was computed by assuming

$$\tau = t_m - t_{CU} \tag{134}$$

where the m subscripts refer to frequency of maximum response and the CU subscripts refer to the upper cut-off frequency.

As an example of the computation, let $F = f_M$ and $N = 4$. Then, by equation (97), $M \approx 0.5$ and t_m and t_{CU} may be obtained from the curve in Figure 11. In this case, remembering that $N = 4$,

$$t_m = \frac{4.2}{2\pi F} \qquad t_{CU} = \frac{2.2}{2\pi F}$$

and

$$\tau = \frac{4.2}{2\pi F} - \frac{2.2}{2\pi F}$$

$$\tau = \frac{1}{\pi F}$$

Since $T = \pi\tau$, it is proper to say that, in this case, F determines the response time of the amplifier.

* * *