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Modeling Uncertainty in Reliability Growth Plans

by Martin Wayne

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14. ABSTRACT
This report outlines an updated approach for reliability growth planning that considers statistical uncertainty on the parameters that are traditionally used to construct reliability growth planning curves. The approach uses a more detailed treatment of the uncertainty, which provides a more complete result when compared to past efforts in this area. The approach can also be used in combination with a reliability assurance test at the end of a test program, which provides improved reliability growth plans that may be more appropriate for the constrained test programs that are common in acquisition.

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1. INTRODUCTION

Reliability growth planning is centered on the development of a reliability growth planning curve that depicts how reliability is expected to grow throughout a developmental test program. The curve reflects the discovery and subsequent corrective actions of failure modes during the developmental testing of a system. Development of the planned growth curve generally relies on experience from previous programs to model the growth that can be expected. Reliability data collected during testing are then compared with the planned values to determine if the reliability is improving in a satisfactory manner, and management can use the planning curves to aid in future decision-making for the program.

The reliability growth planning curve is based on several input parameters that describe the quantity and duration of reliability growth test phases, desired reliability goal, initial reliability, Fix Effectiveness Factor (FEF), and Management Strategy (MS). The Planning Model Based on Projection Methodology (PM2) is the standard model currently used by the U.S. Army, and the model assumptions are found in Ellner and Hall.[1] The resulting failure rate after some amount of testing, t, can be expressed as

$$\lambda_G = (1 - MS)\lambda_I + \frac{(1-\mu_d)MS\lambda_I\beta t}{(1+\beta t)} + \frac{MS\lambda_I}{(1+\beta t)} , \tag{1}$$

where λ_I is the initial failure rate prior to testing; MS is the assumed proportion of the initial failure rate that will be addressed via corrective action during the test-fix-test process; and μ_d is the planned average FEF, which is the fractional reduction in the failure rate for a failure mode after a corrective action is applied. Finally, β is a parameter associated with the distribution of the underlying failure rates for failure modes in the system; it impacts the overall shape of the planning curve.

By setting t in Eq. 1 to the amount of reliability growth test time, T, the planning curve can be fully determined. The final parameter, β, can be expressed in terms of the previously defined parameters. It is given in Ellner and Hall[1] as

$$\beta = \left(\frac{1}{T}\right)\left(\frac{1 - \frac{M_I}{M_G}}{\mu_d MS - \left(1 - \frac{M_I}{M_G}\right)}\right), \tag{2}$$

where M_I and M_G are the reciprocals of the initial and goal failure rates, respectively.

2. INCLUDING UNCERTAINTY IN PLANNING CURVES

One issue with the PM2 planning approach is the lack of uncertainty in the resulting reliability growth plan. The parameters are each treated as fixed values, and the resulting planning curve is just the expected value of the process. But variability in the process is to be expected. The corrective action process involving the FEF and MS will have uncertainty, and the underlying model uses probabilistic assumptions on the distribution of the failure rates for the underlying failure modes. Using the same probabilistic assumptions on the mode failure rates and choosing appropriate distributions for the MS and FEF can provide a fuller treatment of the problem. It can also help to better inform decisions and manage expectations within the reliability growth program.

To develop the model while considering these uncertainties, start by decomposing the system failure rate, λ_s. The system failure rate can be split into the failure rate from failure modes observed during testing, λ_{obs}, and the failure from modes not observed during testing, λ_{unobs}.

$$\lambda_s = \lambda_{obs} + \lambda_{unobs} \tag{3}$$

The observed failure rate comprises failure modes that may have corrective actions applied to mitigate their failure rates when observed. This failure rate is further broken into two parts: one comprising modes that will have corrective actions, and another comprising those modes that will not have corrective actions. The split is determined by the MS, and the part comprising failure modes that will have corrective actions will also have the FEF applied. This results in the definition for the system failure rate after corrective actions as shown in Eq. 4.

$$\lambda_s = (1 - MS)\lambda_{obs} + MS(1 - \mu_d)\lambda_{obs} + \lambda_{unobs} \tag{4}$$

Simplifying the first two terms then yields

$$\lambda_s = (1 - MS\mu_d)\lambda_{obs} + \lambda_{unobs.} \tag{5}$$

The observed and unobserved failure rates can then be further defined using individual mode failure rates, λ_i, which is shown in Eq. 6; K is the total number of failure modes assumed to be present in the system.

$$\lambda_s = (1 - MS\mu_d)\sum_{i=1}^{K} I_i(t)\lambda_i + \sum_{i=1}^{K}[1 - I_i(t)]\lambda_i \tag{6}$$

The indicator function in Eq. 6 is defined as

$$I_i(t) = \begin{cases} 1, & mode\ i\ occurs\ by\ time\ t \\ 0, & otherwise \end{cases}, \tag{7}$$

and the expected value and variance of the expression in Eq. 6 can then be used to find the overall distribution on the system failure rate.

2.1 Expected Value of System Failure Rate

The expected value of the system failure rate in Eq. 6 is

$$E[\lambda_s] = E\left[(1 - MS\mu_d)\sum_{i=1}^{K} I_i(t)\lambda_i\right] + E\left[\sum_{i=1}^{K}[1 - I_i(t)]\lambda_i\right]. \tag{8}$$

The mode failure rates use the same assumptions as PM2, where the individual failure rates, λ_i, follow a common Gamma distribution given by

$$p(\lambda_i|\alpha,\tau) = \frac{\lambda_i^{\alpha-1}\left(\frac{1}{\beta}\right)^{\alpha}}{\Gamma(\alpha)} exp\left(-\frac{1}{\beta}\lambda_i\right). \tag{9}$$

The expected value of the second term in Eq. 8 then simplifies to

$$\begin{aligned} E\left[\sum_{i=1}^{K}[1 - I_i(t)]\lambda_i\right] &= \sum_{i=1}^{K}\int_0^{\infty} \lambda_i exp(-\lambda_i t)\frac{\lambda_i^{\alpha-1}\left(\frac{1}{\beta}\right)^{\alpha}}{\Gamma(\alpha)} exp\left(-\frac{1}{\beta}\lambda_i\right)\partial\lambda_i \\ &= \sum_{i=1}^{K}\frac{\alpha\left(\frac{1}{\beta}\right)^{\alpha}}{\left(\frac{1}{\beta}+t\right)^{\alpha+1}} \\ &= \frac{K\alpha\beta}{(1+\beta t)^{\alpha+1}}. \end{aligned} \tag{10}$$

Substituting $\lambda_I = K\alpha\beta$ and taking the limit as the number of failure modes, K, becomes large yields

$$\lim_{K\to\infty} E\left[\sum_{i=1}^{K}[1 - I_i(t)]\lambda_i\right] = \frac{\lambda_I}{(1+\beta t)}. \tag{11}$$

The expected value of the first term in Eq. 8 is given by

$$E\left[(1-MS\mu_d)\sum_{i=1}^{K} I_i(t)\lambda_i\right] = (1-E[MS]E[\mu_d])E\left[\sum_{i=1}^{K} I_i(t)\lambda_i\right]. \quad (12)$$

Beta distributions are natural choices for distributions on the FEF and MS due to their support being the (0,1) interval. Assume $\mu_d \sim Beta(a,b)$, with density function defined as

$$p(\mu_d) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} {\mu_d}^{a-1}(1-\mu_d)^{b-1}. \quad (13)$$

For the MS, assume $MS \sim Beta(a_1, b_1)$ with density function defined as

$$p(MS) = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)} MS^{a_1-1}(1-MS)^{b_1-1}. \quad (14)$$

Expanding Eq. 12 using the distributions in Eqs. 13 and 14 yields

$$(1-E[MS]E[\mu_d])E\left[\sum_{i=1}^{K} I_i(t)\lambda_i\right] = \left[1-\left(\frac{a_1}{a_1+b_1}\right)\left(\frac{a}{a+b}\right)\right]\left[\sum_{i=1}^{K}\int_0^\infty \lambda_i[1-exp(-\lambda_i t)]\frac{\lambda_i^{\alpha-1}\left(\frac{1}{\beta}\right)^{\alpha}}{\Gamma(\alpha)} exp\left(-\frac{1}{\beta}\lambda_i\right)\partial\lambda_i\right] \quad (15)$$

Simplifying the left term and solving the integral in Eq. 15 results in the expected value of

$$(1-E[MS]E[\mu_d])E\left[\sum_{i=1}^{K} I_i(t)\lambda_i\right] = \left[\frac{ab_1+a_1b+b_1b}{(a_1+b_1)(a+b)}\right]K\alpha\beta\left[1-\frac{1}{(1+\beta t)^{\alpha+1}}\right] \quad (16)$$

Substituting $\lambda_I = K\alpha\beta$ and taking the limit as the number of failure modes becomes large yields

$$\lim_{K\to\infty}(1-E[MS]E[\mu_d])E\left[\sum_{i=1}^{K} I_i(t)\lambda_i\right] = \left[\frac{ab_1+a_1b+b_1b}{(a_1+b_1)(a+b)}\right]\frac{\lambda_I\beta t}{(1+\beta t)}. \quad (17)$$

Combining Eqs. 11 and 17 provides the system-level expected failure rate given by

$$\lim_{K\to\infty} E[\lambda_s] = \left[\frac{ab_1 + a_1 b + b_1 b}{(a_1 + b_1)(a + b)}\right]\frac{\lambda_I \beta t}{(1+\beta t)} + \frac{\lambda_I}{(1+\beta t)}. \tag{18}$$

2.2 Variance of System Failure Rate

The variance of the system failure rate can be found similarly to the expected value and is defined from Eq. 6 as

$$Var[\lambda_s] = Var\left[(1 - MS\mu_d)\sum_{i=1}^{K} I_i(t)\lambda_i\right] + Var\left[\sum_{i=1}^{K}[1 - I_i(t)]\lambda_i\right]. \tag{19}$$

The first term in Eq. 19 is the variance of failure rate for modes observed during testing, and the second term is the variance of the failure rate for modes not observed during testing. Assuming mode failure rates are independent, the first term in Eq. 19 can be expressed as

$$Var\left[(1 - MS\mu_d)\sum_{i=1}^{K} I_i(t)\lambda_i\right] = \sum_{i=1}^{K} Var[(1 - MS\mu_d)I_i(t)\lambda_i]. \tag{20}$$

The variance term inside the summation can then be expressed as

$$Var[(1 - MS\mu_d)I_i(t)\lambda_i] = E\left[\left((1 - MS\mu_d)I_i(t)\lambda_i\right)^2\right] - E[(1 - MS\mu_d)I_i(t)\lambda_i]^2, \tag{21}$$

which allows for a straightforward solution. Using the same techniques as in the expected value, the second term in Eq. 21 is then

$$E[(1 - MS\mu_d)I_i(t)\lambda_i]^2 = \left(\left[\frac{ab_1 + a_1 b + b_1 b}{(a_1 + b_1)(a + b)}\right]\alpha\beta\left[1 - \frac{1}{(1+\beta t)^{\alpha+1}}\right]\right)^2. \tag{22}$$

The first term can be split into a product defined by

$$E\left[\left((1 - MS\mu_d)I_i(t)\lambda_i\right)^2\right] = E[(1 - MS\mu_d)^2]E[(I_i(t)\lambda_i)^2]. \tag{23}$$

The individual elements in the right-hand side of Eq. 23 are given in Eqs. 24 and 25.

$$E[(1-MS\mu_d)^2] = 1 - 2E[MS]E[\mu_d] + E[MS^2]E[{\mu_d}^2]$$

$$= 1 - 2\left(\frac{a_1}{a_1+b_1}\right)\left(\frac{a}{a+b}\right) + \left[\frac{{a_1}^2(a_1+b_1+1)+a_1b_1}{(a_1+b_1)^2(a_1+b_1+1)}\right]\left[\frac{a^2(a+b+1)+ab}{(a+b)^2(a+b+1)}\right] \tag{24}$$

$$E[(I_i(t)\lambda_i)^2] = \alpha(\alpha+1)\beta^2\left(1-\frac{1}{(1+\beta t)^{\alpha+2}}\right) \tag{1}$$

The expected value in Eq. 23 is therefore

$$E\left[\left((1-MS\mu_d)I_i(t)\lambda_i\right)^2\right] = \left[1-2\left(\frac{a_1}{a_1+b_1}\right)\left(\frac{a}{a+b}\right) + \left[\frac{{a_1}^2(a_1+b_1+1)+a_1b_1}{(a_1+b_1)^2(a_1+b_1+1)}\right]\left[\frac{a^2(a+b+1)+ab}{(a+b)^2(a+b+1)}\right]\right]\alpha(\alpha+1)\beta^2\left(1-\frac{1}{(1+\beta t)^{\alpha+2}}\right) \tag{26}$$

Substituting Eqs. 22 and 26 into Eq. 21 and combining with Eq. 20 gives the finite form of the variance of the observed failure rate in Eq. 27.

$$Var[(1-MS\mu_d)\textstyle\sum_{i=1}^{K} I_i(t)\lambda_i] = \left[1-2\left(\frac{a_1}{a_1+b_1}\right)\left(\frac{a}{a+b}\right) + \left[\frac{{a_1}^2(a_1+b_1+1)+a_1b_1}{(a_1+b_1)^2(a_1+b_1+1)}\right]\left[\frac{a^2(a+b+1)+ab}{(a+b)^2(a+b+1)}\right]\right]K\alpha\beta(\alpha+1)\beta\left(1-\frac{1}{(1+\beta t)^{\alpha+2}}\right) - K\left(\left[\frac{ab_1+a_1b+b_1b}{(a_1+b_1)(a+b)}\right]\alpha\beta\left[1-\frac{1}{(1+\beta t)^{\alpha+1}}\right]\right)^2 \tag{27}$$

Taking the limit similarly to the expected value yields

$$\lim_{K\to\infty} Var[(1-MS\mu_d)\textstyle\sum_{i=1}^{K} I_i(t)\lambda_i] = \left[1-2\left(\frac{a_1}{a_1+b_1}\right)\left(\frac{a}{a+b}\right)\left[\frac{{a_1}^2(a_1+b_1+1)+a_1b_1}{(a_1+b_1)^2(a_1+b_1+1)}\right]\left[\frac{a^2(a+b+1)+ab}{(a+b)^2(a+b+1)}\right]\right]\lambda_I\beta\left(1-\frac{1}{(1+\beta t)^2}\right). \tag{28}$$

For the second term in Eq. 19, the variance can be expressed as

$$Var\left[\sum_{i=1}^{K}[1-I_i(t)]\lambda_i\right]=\sum_{i=1}^{K}[E[([1-I_i(t)]\lambda_i)^2]-E([1-I_i(t)]\lambda_i)^2]. \tag{29}$$

Using the same techniques as the previous derivations yields a result for the finite form given by

$$Var\left[\sum_{i=1}^{K}[1-I_i(t)]\lambda_i\right]=\frac{K\alpha\beta^2(\alpha+1)}{(1+\beta t)^{(\alpha+2)}}-K\left[\frac{\alpha\beta}{(1+\beta t)^{(\alpha+1)}}\right]^2, \tag{30}$$

and taking the limit results in

$$\lim_{K\to\infty} Var\left[\sum_{i=1}^{K}[1-I_i(t)]\lambda_i\right]=\frac{\lambda_I\beta}{(1+\beta t)^2}. \tag{31}$$

Combining Eqs. 28 and 31 to get the total variance for the system failure rate yields

$$\begin{aligned}\lim_{K\to\infty} Var[\lambda_s] = \Bigg[1-2\left(\frac{a_1}{a_1+b_1}\right)\left(\frac{a}{a+b}\right)\\ +\left[\frac{{a_1}^2(a_1+b_1+1)+a_1b_1}{(a_1+b_1)^2(a_1+b_1+1)}\right]\left[\frac{a^2(a+b+1)+ab}{(a+b)^2(a+b+1)}\right]\Bigg]\lambda_I\beta\left(1-\frac{1}{(1+\beta t)^2}\right)\\ +\frac{\lambda_I\beta}{(1+\beta t)^2}.\end{aligned} \tag{32}$$

2.3 System Failure Rate Distribution

The assumption that the individual failure mode rates follow a Gamma distribution implies that the initial system failure rate is also Gamma distributed. Results in Wayne and Modarres[2] show the system failure rate after corrective actions can also be assumed to be approximately Gamma, and the mean and variance can be used to find the Gamma parameters by equating the moments of the distribution. If we define the mean in Eq. 18 as μ and the variance in Eq. 32 as σ^2, the moment-based estimators are

$$\alpha=\frac{\mu^2}{\sigma^2} \tag{33}$$

$$\tau=\frac{\mu}{\sigma^2} \tag{34}$$

Simulation can be used to examine how well the assumed Gamma distribution approximates the actual system failure rate distribution. Figure 1 contains a histogram of the failure rate distribution from 100,000 replications of a simulated reliability growth

test. There were 500 failure modes assumed to be in the system, and 1,000 h of reliability growth testing was assumed. The initial Mean Time Between Failure (MTBF) was 100 h, which is also the expected value of the initial Gamma distribution for the mode failure rates. Setting β = 0.001 provides the remaining parameter to define the distribution. For the MS and FEF, the mean of each Beta distribution was set to 0.7 and the coefficient of variation was set to 0.01. The coefficient of variation is the ratio of the standard deviation to the mean, which provides two equations that can be solved to yield the specific Beta distribution parameters. The blue curve is the Gamma distribution determined from the results in Sections 2.1 and 2.2 and Eqs. 33 and 34. The approximate Gamma matches the true distribution very closely.

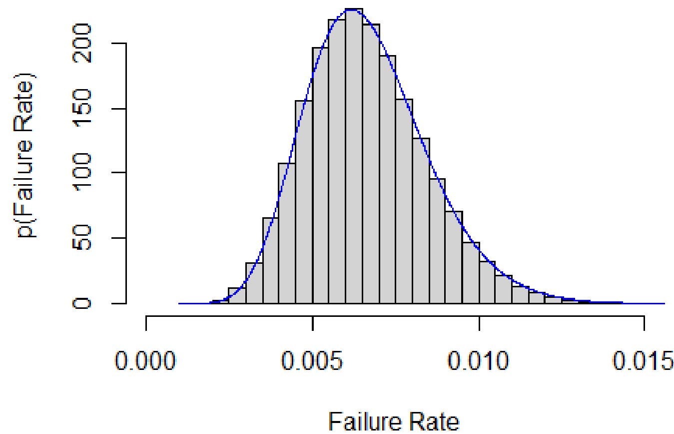

Figure 1. Failure rate distribution comparison with 1,000 h of reliability growth testing

Figure 2 shows the results using the same input assumptions but increasing the amount of reliability growth test time to 2,500 h. The distribution has shifted to the left when compared to Figure 1, which is due to the increase in test hours and additional failure modes that were observed and had corrective actions applied. The Gamma distribution from Eqs. 33 and 34 still matches the true distribution very closely.

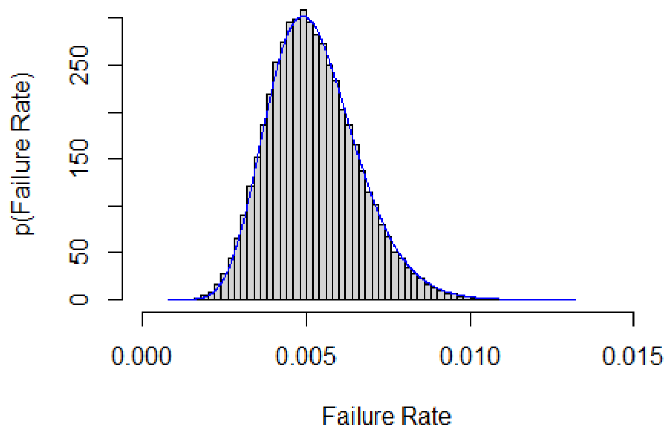

Figure 2. Failure rate distribution comparison with 2,500 h of reliability growth testing

The distribution on the system failure rate can be further understood by examining the properties of the Gamma distribution and the initial failure rate for the system. Setting t = 0 in the variance defined by Eq. 32 results in the variance of the initial failure rate being defined by $\lambda_I \beta$. This means that systems with lower initial failure rates and higher initial MTBF values will have lower variance in the failure rate prior to reliability growth testing. Figure 3 shows a comparison of distributions after 3,000 h of reliability growth testing, where the initial MTBF is the only difference between the two cases. The distribution using an initial MTBF of 100 h is not only shifted to the right. It is also significantly wider than the distribution resulting from an initial MTBF of 500 h because of the differences in the initial variance values.

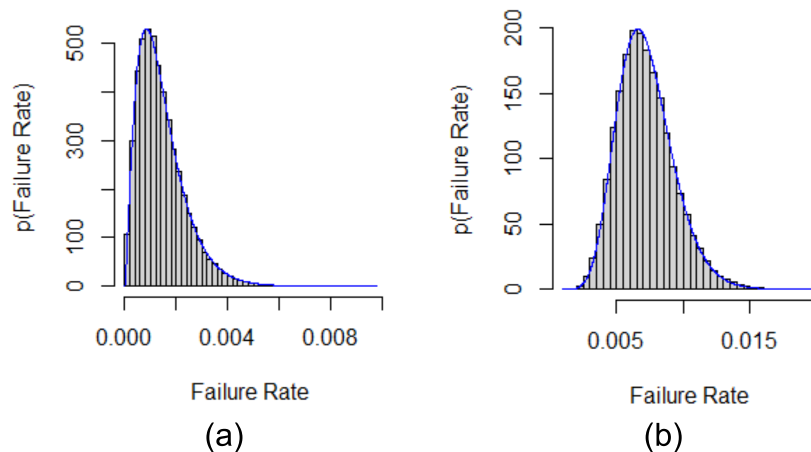

Figure 3. Failure rate distribution with initial MTBF values of (a) 500 h (b) 100 h

This is an important property to remember when constructing planning curves, because the initial variance in the failure rate impacts the variance throughout the growth testing. It will generally provide more of an impact on the resulting uncertainty in the failure rate distribution than the distributions on the MS and FEF. This is because planning curves usually represent the observation and correction of a relatively small fraction of the total number of failure modes in the system, which can reduce the impact of the distributions on the MS and FEF.

3. DEFINING THE PLANNING CURVE

The reliability growth planning curve is based on the desired initial and goal failure rates or MTBFs. It is also dependent on the method used for assessing reliability at the end of the reliability growth program. Two methods for determining the end point are reliability assurance testing and reliability demonstration testing. The specifics of each method will determine the number of allowable failures in the assessment, which in turn helps to determine the start and end points of the planning curve.

3.1 Reliability Assurance Testing

A Bayesian assurance test combines the last phase of Developmental Testing (DT) with an additional assurance test to provide information on whether the system is meeting its reliability requirement. The DT results are used as prior information and combined with the results from the assurance test to give an overall reliability assessment. Posterior consumer and producer risks are calculated, along with the overall posterior probability acceptance for the test. More discussion on the assurance testing approach can be found in Wayne,[3] which describes how a range of DT results can be examined to develop an assurance test plan.

After the assurance test plan is developed, the first step in the process is to define the probability of seeing the desired results in the DT (i.e., the number of observed failures is less than or equal to the allowable failures). For a DT of length T, the probability of seeing at most n allowable failures is given as

$$p(x \leq n) = \sum_{i=0}^{n} \frac{T^i \tau^\alpha \Gamma(\alpha + i)}{i!\,\Gamma(\alpha)(\tau + T)^{(\alpha+i)}} \,. \tag{35}$$

The α and τ are both unknown Gamma parameters in Eq. 35, but they can be expressed completely in terms of common reliability growth planning parameters using the expected value and variance results from Section 2. First, define the expected value in Eq. 18 in terms of the Gamma parameters in Eqs. 33 and 34. Simplifying the right-hand side yields

$$\frac{\alpha}{\tau} = \lambda_I \left[1 - \frac{\beta t}{(1+\beta t)} \frac{a_1 a}{(a_1 + b_1)(a + b)}\right]. \tag{36}$$

Using the definition of the variance for the Gamma along with Eq. 36 then yields

$$Var[\lambda_s] = \frac{\alpha}{\tau^2} = \frac{\lambda_I \left[1 - \frac{\beta t}{(1+\beta t)} \frac{a_1 a}{(a_1 + b_1)(a + b)}\right]}{\tau}. \tag{37}$$

Rearranging Eq. 37 and substituting Eq. 32 for the variance on the left-hand side gives the expression for τ in terms of reliability growth parameters in Eq. 38.

$$\tau = \frac{\left[1 - \frac{\beta t}{(1+\beta t)} \frac{a_1 a}{(a_1+b_1)(a+b)}\right]}{\left[1 - 2\left(\frac{a_1}{a_1+b_1}\right)\left(\frac{a}{a+b}\right) + \left[\frac{a_1{}^2(a_1+b_1+1)+a_1 b_1}{(a_1+b_1)^2(a_1+b_1+1)}\right]\left[\frac{a^2(a+b+1)+ab}{(a+b)^2(a+b+1)}\right]\right]\beta\left(1 - \frac{1}{(1+\beta t)^2}\right) + \frac{\beta}{(1+\beta t)^2}} \tag{38}$$

Equation 38 shows that τ is a function of the reliability growth test time, the distributional parameters for the FEF and MS, and the β parameter associated with the distribution of the mode failure rates. The test length and FEF and MS distributions will be assumed by the user. The β parameter is the only remaining unknown, and it can also be expressed in terms of common reliability growth planning inputs.

To find the β parameter, first define the growth potential failure rate, λ_{GP}, as the theoretical lower bound on the failure rate that can be achieved after all modes are observed and corrected with an assumed level of fix effectiveness. The expected value of the growth potential failure rate can be expressed as the limit of the expected failure rate in Eq. 18.

$$E[\lambda_{GP}] = \lim_{t\to\infty} \lim_{K\to\infty} E[\lambda_s] = \left[\frac{ab_1 + a_1 b + b_1 b}{(a_1+b_1)(a+b)}\right]\lambda_I. \tag{39}$$

The reliability growth potential is often used to manage risk when planning a test program. It is not generally achievable in the limited testing associated with a reliability growth program, but the growth potential failure rate needs to be lower than the desired goal failure rate for the growth program. The ratio of the goal MTBF to the growth potential MTBF is a common metric used in reliability growth planning, with typical values ranging from 0.6 to 0.8. Combining Eq. 39 with the expected goal failure rate in Eq. 18 gives the ratio defined by

$$R = \frac{\left[\frac{ab_1+a_1 b+b_1 b}{(a_1+b_1)(a+b)}\right]}{\left[\frac{ab_1+a_1 b+b_1 b}{(a_1+b_1)(a+b)}\right]\frac{\beta t}{(1+\beta t)} + \frac{1}{(1+\beta t)}}\ . \tag{40}$$

Rearranging Eq. 40 then provides the solution for β as

$$\beta = \frac{(ab_1 + a_1 b + b_1 b)(1-R) - a_1 a R}{t(R-1)(ab_1 + a_1 b + b_1 b)}\ . \tag{41}$$

The value of τ in Eq. 38 can then be expressed entirely in terms of common reliability growth planning parameters. Choosing a desired probability of seeing at most n allowable failures for the left-hand side of Eq. 35 then provides a solution for the α

parameter. The remaining unknown in Eqs. 18 and 32 is the λ_I, but rearranging Eq. 36 provides a solution in terms of α, τ, and the other planning parameters.

$$\lambda_I = \frac{\alpha}{\tau\left[1 - \frac{\beta t a_1 a}{(1+\beta t)(a_1+b_1)(a+b)}\right]} \qquad (42)$$

These variables completely define the expected value and variance forms in Eqs. 18 and 32, which then yields the desired Gamma distribution with parameters defined in Eqs. 33 and 34.

3.2 Reliability Demonstration Testing

If a traditional reliability demonstration test is to be used instead of an assurance test, the allowable number of failures in the demonstration test can be used in the same manner as for the DT in the assurance test application. The number of allowable failures is found by bounding the probability of passing the test at the desired significance level (i.e., 1 – confidence) when the true MTBF of the system is equal to the MTBF requirement. The cumulative Poisson distribution is used, and n is the largest integer value of k satisfying the inequality in Eq. 43:

$$\sum_{i=0}^{k} \frac{\left(\frac{T}{M_R}\right)^i}{i!} exp\left(-\frac{T}{M_R}\right) \leq 1 - \tilde{\alpha}. \qquad (43)$$

For the inequality in Eq. 3, $\tilde{\alpha}$ is the significance, T is the length of demonstration test, and M_R is the MTBF requirement.

The probability of passing the demonstration test is then just the probability of observing at most the allowable number of failures, which is identical to the form shown in Eq. 35. The results in Eqs. 36–42 will hold, and the planning curve can be determined using the same methods.

4. EXAMPLE APPLICATION

To demonstrate the methodology developed in Sections 2 and 3, this example assumes that the end of the reliability growth testing includes a demonstration test of 1,000 h with three allowable failures. This demonstration test could precede an assurance test, where the allowable failures are determined by considering the posterior risks with a Bayesian model, or it could be a traditional demonstration test where the allowable failures are determined from the desired confidence level and the MTBF requirement. The desired probability of seeing the allowable failures defined in Eq. 35 is set to 0.70. The mean MS and FEF are also assumed to be 0.95 and 0.75, respectively; the coefficients of variation are assumed to be 0.03 and 0.20, respectively. This yields the Beta distributions for each parameter shown in Figures 4 and 5. The distributions can easily be adjusted to reflect historical data from similar systems or other relevant information by adjusting the mean and/or coefficient of variation. The desired ratio of the goal MTBF to the growth potential MTBF is set to 0.70, and the amount of reliability growth testing is assumed to be 5,000 h.

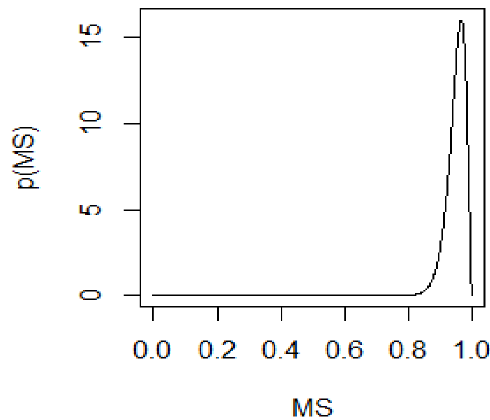

Figure 4. Distribution on MS for example application

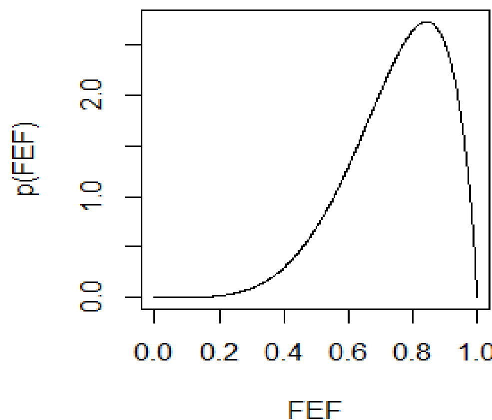

Figure 5. Distribution on FEF for example application

Figure 6 shows the resulting reliability growth planning curve resulting from the process outlined in Section 3.1. After determining the appropriate parameters using Eqs. 35–42, the solid curve shows the expected value calculated using Eq. 18. The dashed curves provide a 90% probability interval for the Gamma distribution resulting from Eqs. 33 and 34. To simplify the plot and demonstrate the methodology, there are no corrective action periods assumed for this example. They can be applied here using the same approach as PM2,[1] and they are strongly recommended for applications on real programs. Other metrics associated with the reliability growth plan are also available, and they are identical to those defined by PM2.[1] For example, there are approximately 12 failure modes that will be observed in the planning curve depicted in Figure 6.

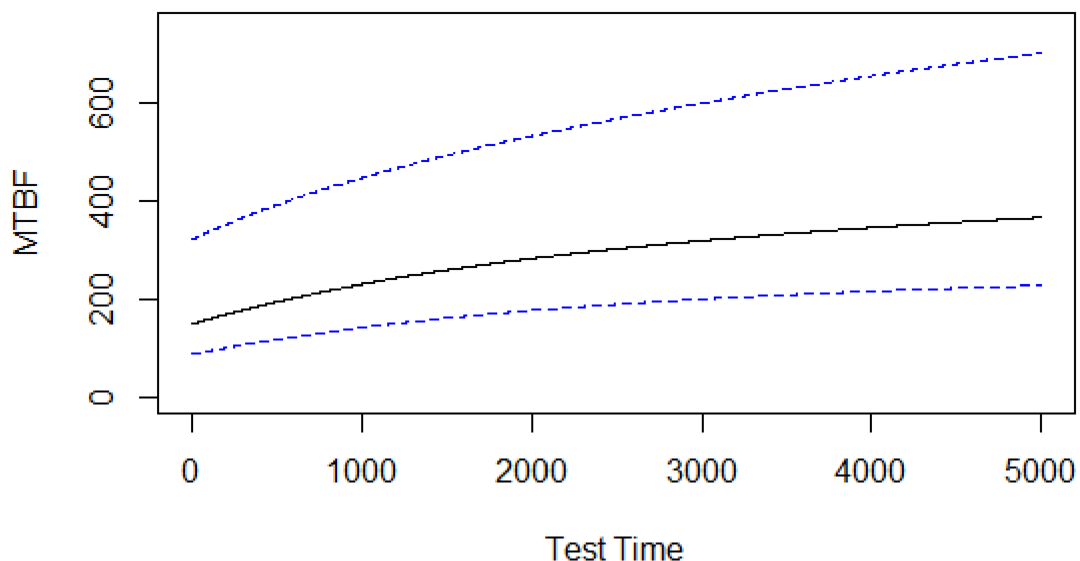

Figure 6. Reliability growth planning curve for example application

As shown in Figure 6, the MTBF grows from the initial value of 150 h to the goal value of 365 h, which provides a 0.70 probability of seeing no more than three failures in 1,000 h of additional testing. The probability interval on the curve also shows that the system MTBF is likely to range from 89 to 320 h at the start of testing, and 228 to 699 h at the end of growth testing.

The dashed curves that define the probability interval can help indicate whether a system is progressing reasonably throughout the growth testing. MTBF estimates that are higher than the interval may be sufficient evidence for truncating the reliability growth testing early, especially if the early estimates are above interval associated with the end of the test period. Conversely, early estimates below the interval would be cause for concern, and decision-makers should evaluate all available evidence before proceeding with additional reliability growth testing. Early estimates that are significantly below the interval on the planning curve would be strong evidence that the reliability

growth potential of the system is too low. This implies that the system is at risk of not being able to achieve its reliability requirement without additional reliability engineering activities outside of the reliability growth testing depicted by the planning curve.

5. CONCLUSIONS

As demonstrated, the inclusion of uncertainty on all reliability growth parameters provides a more complete treatment of the uncertainty associated with the reliability growth program. The approach serves as a natural extension of current reliability growth planning models used in the defense industry while explicitly modeling the additional uncertainty that exists in the problem. Considering the uncertainty in the planning allows for more informed decision-making regarding reliability progress while potentially reducing the risks associated with various decisions.

6. REFERENCES

1. Ellner, P.M., & Hall, J.B. (2006). *An approach to reliability growth planning based on failure mode discovery and correction using AMSAA projection methodology.* Proceedings of the Annual Reliability and Maintainability Symposium.

2. Wayne, M., & Modarres, M. (2015). A Bayesian model for complex system reliability growth under arbitrary corrective actions. *IEEE Transactions on Reliability*, 64–1.

3. Wayne, M. (2023). *Reliability growth planning with reliability assurance testing* (DEVCOM DAC-TR-2023-18). DEVCOM Analysis Center.

LIST OF ACRONYMS

DAC	DEVCOM Analysis Center
DEVCOM	U.S. Army Combat Capabilities Development Command
DT	Developmental Testing
FEF	Fix Effectiveness Factor
MS	Management Strategy
MTBF	Mean Time Between Failure
PM2	Planning Model Based on Projection Methodology

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