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RPPR Final Report

as of 25-May-2023

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Agreement Number: W911NF-19-1-0039

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Final Report for Period Beginning 07-Jan-2019 and Ending 06-Jan-2022

Title: Failure Probability and Safety Gain of Super-Strong Biomimetic Nacreous Materials

Begin Performance Period: 07-Jan-2019

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STEM Participants: 3

Major Goals: The basic objective is the mathematical modeling of the failure probability, P_f , with focus on the low probability tail up to one in a million (i.e. $<1.E-6$), and assessment of safety advantages of the nacreous, biomimetic and similar architected quasibrittle materials.

Prior to this grant, there existed only two probability models of material strength – 1) the weakest link chain, or series coupling (leading to Weibull statistics), and 2) the fiber bundle, or parallel coupling (leading to the normal, or Gaussian, distribution). These are nearly identical in the central probability range, but their extrapolations to probability $1.E-6$, the maximum tolerable in engineering design, can differ as 2 : 1.

The objective has been to model, for the first time, materials exhibiting regular alternations of both parallel and series couplings. This is important for nacreous biomimetic materials whose microstructure network connectivity resembles a fishnet pulled diagonally, and for other similar architected materials such as the printed octet material microstructures, which offers the highest strength-to-weight ratio for a given printing material. Special emphasis is placed on ensuring the failure probability, P_f , to be less than one in a million, which is a degree of reliability generally required of engineering structures, including most of those for Army applications. This objective requires overcoming four main challenges:

1) Determining the P_f of nacreous materials in the probability distribution tail up to one in a million for two basic cases: Brittle, i.e., suddenly failing links, and quasibrittle (progressively softening) links (the former was addressed in the initial phase of this project and the latter was completed this year).

2) Since the size effect is the major indicator of the statistical distribution of materials strength, develop a statistical size effect relation for these materials based on large-scale simulations, and extend it from tensile loading to shear loading.

3) Determine the size effect on the strength probability distribution when the materials specimen represented by the fishnet is scaled up geometrically.

4) Develop a finite element code for large-scale random Monte Carlo simulations that makes it possible to run million repetitions of failure simulations efficiently taking into account various softening behaviors of the internal links, in order to calibrate the strength distribution.

The third year of funding and the no-cost extension allowed tackling the third goal – clarification of failure probability scaling to large structure size.

Accomplishments: Safety experts generally agree that a safe engineering design requires the failure probability of structure to be less than one in a million ($<1.E-6$). Many engineers assume that it suffices to determine the mean and coefficient of variation of the failure load and then apply the normal (or Gaussian) distribution to extrapolate to the probability level of $1.E-6$. But this assumption is wrong.

Prior to this project, there existed only two models to reach $1.E-6$: 1) the series coupling (or weakest link) model,

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leading to the Weibull distribution (1939) and 2) the parallel coupling (or fiber bundle) model (Daniels 1945), leading to the Gaussian distribution. Although they almost coincide for failure probabilities down to 1 percent, they differ as much as 2 : 1 at the level of $1.E-6$. This extremely low probability cannot be directly verified by repeated histogram testing since >10 million test repetitions would be needed. What is required is a physically based probabilistic theory, which can be verified by size effect.

This project developed the "fishnet" statistics, which is only the third probabilistic model of structural failure that can be solved exactly. Combining a regular alternation of series and parallel couplings, it mimics the microstructure of nacre which behaves like a fishnet pulled diagonally because the micro-platelets are organized in a brick-and-mortar fashion. This architecture explains the astonishing tensile strength of nacre, two orders of magnitude higher than the bulk strength of its main constituent [CaCO_3 , aragonite]. The mean strength of nacre was explained deterministically, but at $1.E-6$ it almost doubles compared to the mean. So the fishnet architecture not only greatly enhances the mean material strength but it further greatly enhances the material strength.

It is now shown that, in terms of the failure with probability $1.E-6$ increases by a factor of almost 2 which is what matters for engineering design, civilian as well as military. One lesson from this is that material architectures that include combinations of series and parallel connections are much stronger in terms failure probability than those that adhere to the weakest link model. This is important for various printed materials where material architectures can be designed.

In the first two years of investigation the analytical modeling and the computer simulations of failure probability dealt with systems of no more than thousand links. In these, the failure of up to 3 links generally suffice to cause the structural failure, which simplifies calculations.

Simulations of larger systems indicate strong qualitative size effect. Massive super-computer simulations are required, but even limited studies already reveal that structures with fishnet architecture consisting of 100.000 series and parallel couplings fail as a whole only after about 20 links fail, and that the failed links develop in a row that resembles a crack.

Therefore, a continuum limit of the fishnet system failure probability has been developed and in turned out that it is represented, probabilistically, not by an elastic continuum but by the Laplace differential equation. It was found that a crack in a continuum represented by Laplace equation can be easily solved by means of complex variables. The solution revealed that a single crack in such a continuum body, of rectangular shape, suffices to cause a failure of the entire body. This is the hallmark of weakest-link behavior, or Weibull distribution, . It was also found that the Weibull modulus increases with the system size, which is an advantageous feature.

A new differential equation relating the slope of the probability distribution curve in Weibull coordinates to the applied stress and to the number of links in the system has been derived and computationally verified. This work is no complete and a paper on these new results will be submitted for journal review soon.

More complicated material architectures suitable for printing have also been studied. It was found that the octet material architecture, which is optimal for maximizing material strength-to-weight ratio, has similar failure probability properties to the fishnet architecture, and likewise offers further strength advantages at the $1.E-6$ failure probability level. This is also include in the aforementioned paper in preparation, joint with my student Houlin Xu, dealing with the scaling up the fishnet statistics to large systems containing about $1.E5$ links. This has appeared to be significant aspect not anticipated in the original proposal.

A plenary lecture summarizing the results of this project is planned to be given at the Annual Meeting of the Society of Engineering Science in Orlando on June 6, 2023.

Some important new problems cropped up, such as the effects of crack parallel stresses on fracture probability, as revealed by the "gap" test, recently conceived in the PI's lab, and also by fracture postpeak tests of composites enabled by PI's 2019 patent No. US 10.416.053 B2. But these are major issues that have to be relegated to the next research proposal.

THREE SPINOFFS:

1) The computational mechanics community recently became enamored with the so-called peridynamics, which is a material model of very different link connectivity – in a set of random material points, each material point is linked to all nearby points within a certain specified distance, called a horizon. This model makes sense for molecular dynamics but not for solids, because the particle-skipping interactions if implies do not exist on macroscale (except perhaps for felt fabrics for which peridynamics was not intended, and not used).

Peridynamics relies on an adaptation of the LAMMPS code, very efficient and realistic for molecular dynamics, was adapted to solids on a scale orders of magnitude higher. Large grants on this subject were awarded by AFOSR (MURI), ORNL, Sandia, NSF, etc., which I showed to have been wasted. The two reported papers in JAM and ES, and two international webinars with thousand viewers, demonstrated the error in the link connectivity concept of peridynamics and the inability to simulate various essential experiments hitherto ignored by the computational community.

The impact has already been major. In the last World Congress on Computational Mechanics (WCCM) in 2018,

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164 papers dealt with this subject (and with the Phase-Field models), while at the ECOMASS Congress last June in Oslo only 4! As a result, further waste of grant funds will be avoided.

2) The other significant spinoff, whose error was also revealed in the aforementioned two paper in JAM and ES, was the Phase Field model for fracture. This is again a computationally very efficient approach that can be easily made mesh-independent, which does the opposite -- it totally omits consideration of link connectivity on the microscale, and assumes failure to be totally controlled by the fracture energy at one point. This model can actually represent well in finite element computations the classical linear elastic fracture mechanics invented by Griffith in 1921 – but is this what we need? No, we need more,

The phase-field model fails to represent the failure of modern materials with complex link connectivity and large fracture process zones of interest today. Under significant funding, the Phase-Field model has recently flourished in the computational community, again because inattention to experiments -- selective experimental “verification”, ignoring numerous other relevant experimental data reported in the literature.

3) A new effect in fracture has been discovered in my group in 2022 (PNASm 117, 5). The fracture energy of materials changes significantly in presence of crack parallel normal stresses. Explanation: fracture in 2D cannot be a line, but must a fracture process zone with triaxial damage at its front. This has probabilistic implications for future study.

Training Opportunities: I trained one undergraduate (Kevin Chen), 2 graduate students and one postdoc

Results Dissemination: A description of dissemination during the reporting period.

1) With my two former students as co-authors, I published a new textbook at Oxford UP, U.K., which is receiving high level of attention

<http://www.civil.northwestern.edu/people/bazant/PDFs/Papers/B9-Quasibrittle%20Fracture%20Mech%20&%20Size%20Eff-Oxford%20UP.pdf>

2) My book <http://www.civil.northwestern.edu/people/bazant/PDFs/Papers/B7-Probabilistic%20Mechanics%20of%20Quasibrittle%20Structures.pdf>

also deals with the present subject of research

3) Note the 12 uploaded journal articles

4) I presented 2 international webinars dealing at least partly with our results under this grant, with about 1000 viewers each worldwide. They are:

- 12/05/22 Lepafrog-computational fracture mechanics: “Webex at Boeing website”

- TAM/SPREE: 2/9/22: Peridynamics Phase-field Critique: https://youtu.be/3aNiTC_igM4

- ARMA shale fracture Webinar 7/22/21: <https://www.youtube.com/watch?v=HvVOobdHUIw>

- To CanTAM/IUTAM Webinar 3/23/21: <https://www.youtube.com/watch?v=MONUIvCvoog>

- On May 19, 2023 I will present a webinar on Failure Risk at Oak Ridge National Lab

5) After Covid (2022), I presented 6 plenary lectures at 6 conferences dealing at least partly with the results of this grant, including

EURO-C Vienna, IABSE Prague, ECOMASS Oslo, ICONSON Sardinia, and Int. Conf. on Emerging Topics in Mechanics, Houston, SES Annual at Texas A&M, ASME/IMECE in Columbus, OH

Also one invited paper at ARMA Annual Meeting in Santa Fe, NM

6) My regular Research Group Meetings (open to outsiders), and my graduate PhD level course based on my new book (which also attracts auditors).

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Honors and Awards: Honors and Awards received since Jan. 6, 2022.

- 1) Selected to receive in May 2023 an Honorary Doctorate (my 8th) from the University of Minnesota, Minneapolis.
- 2) Elected as International Fellow of Canadian Academy of Engineering (CAE)
- 3) Elected as Foreign Member of the Academy of Turin (Italy, founded by Lagrange in mid 1700s)
- 4) Selected to receive in May 2023 the Lifetime Achievement Award from ICDM Board (Int. Con. Damage Mechanics).
- 5) ASCE Zdenek P Bazant Medal for Failure and Damage Prevention, created in 2015, was awarded to Huajian Gao, NAE/NAS, Nanyang University, Singapore
- 6) PI's Citations: H-index: 147(as of 3/2023), citations: 90,300, i10 index: 686(Google, incl. self-cit.).
- 7) Also note: In the 2019 Stanford University citation survey (published in PLoS, posted in Mendeley data), incl. over 200,000 engineering authors, the PI was ranked no. 1 in civil engineering and no. 2 in engineering of all fields, worldwide, among approx. 250K engineering authors (as published in PLoS, this ranking excluded self-citations, was weighted for first and last author, and number of co-authors, and filtered out reciprocal citations and citation "farms"); www.asce.org/templates/award-detail.aspx?id=11613. He maintained the same position in repeat rankings, incl. 2021.

Protocol Activity Status:

Technology Transfer: The PI promulgated his US Patent 10,416,053, 2019, on fracture testing of composites in lectures and at conferences.

Likewise he publicized his "gap test", a new test developed in 2020 partly under this project, which identifies unambiguously the effect of crack-parallel stress on material fracture.

PI's co-author and former postdoc, Dr. Marco Salviato, publicized the safety modeling partly revealed in this project at Boeing.

It turns out that the safety implications of this grant are of great interest to Boeing (Dr. Salviato, my collaborator on this grant, who left Northwestern for the University of Washington, Seattle, is sharing some of our results with Boeing).

PARTICIPANTS:

Participant Type: Graduate Student (research assistant)

Participant: Houlin Xu

Person Months Worked: 12.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: PD/PI

Participant: Zdenek Pavel Bazant PhD, Drhc

Person Months Worked: 8.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Graduate Student (research assistant)

Participant: Wen Luo

Person Months Worked: 2.00

Project Contribution:

National Academy Member: Y

Funding Support:

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Date Published: 3/1/20 6:00AM

Publication Location:

Article Title: General Fishnet Statistics of Strength: Nacreous, Biomimetic, Concrete, Octet-Truss, and Other Architected or Quasibrittle Materials

Authors: Wen Luo, Zdeněk P. Bažant

Keywords: computational mechanics, failure criteria, mechanical properties of materials, micromechanics, stress analysis, structures

Abstract: The fishnet probabilistic model was recently developed to characterize the strength distribution of nacre-like biomimetic materials. It reveals that the unique fishnet-like connectivity of the material microstructure brings about enormous safety gain at the extremely low failure probability level of one out of a million, desired for engineering structures. The gist of the theory is that the material microstructure plays a determining role in its failure probability tail. Therefore, a carefully designed connectivity for a material microstructure not only enhances its mean strength but also significantly reduces its marginal failure risk. Here, we first show that the initially introduced series expansion and the newer formulation based on order statistics are, in the fishnet model, essentially equivalent. From that we develop a neat general form of the fishnet statistics. Then, we extend our theoretical approach to the strength distributions of architected nanomaterials such as the print

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Article Title: Design of quasibrittle materials and structures to optimize strength and scaling at probability tail: an aperçu

Authors: Zdeněk P. Bažant

Keywords: size effect, scaling, quasibrittle fracture, material randomness, failure probability, strength distribution, interatomic bond failure, activation energy, Weibull distribution, weakest link chain

Abstract: The objective in materials or structure design has been to maximize the mean strength. However, as generally agreed, engineering structures, such as bridges, aircraft or microelectromechanical systems must be designed for tail probability of failure less than 10^{-6} per lifetime. But this objective is not the same. Indeed, a quasibrittle material or structure with a superior mean strength can have, for the same coefficient of variation, an inferior strength at the less than 10^{-6} tail. This tail is unreachable by histogram testing. So, one needs a rational theory, physically based and experimentally verified indirectly, which is feasible by size effect. Focusing on the results at the writer's home institution, this inaugural article (written three years ex post facto) reviews recent results towards this goal, concerned with quasibrittle materials such as concretes, rocks, tough ceramics, fibre composites, bone and most materials on the micrometer scale. The theory is anchored at the atomi

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Article Title: A Precis of Fishnet Statistics for Tail Probability of Failure of Materials with Alternating Series and Parallel Links

Authors: Z. P. Bažant

Keywords: probabilistic mechanics, material strength, fracture mechanics, strength scaling, size effect, nacreous materials, biomimetic materials, micromechanics of failure, nanoscale fracture, scale bridging, tolerable risk

Abstract: During the last dozen years it has been established that the Weibull statistical theory of structural failure and strength scaling does not apply to quasibrittle materials. These are heterogeneous materials with brittle constituents and a representative volume element that is not negligible compared to the structure dimensions. A new theory of quasibrittle strength statistics in which the strength distribution is a structure size dependent graft of Gaussian and Weibull distributions has been developed. The present article gives a precis of several recent studies, conducted chiefly at the writer's home institution, in which the quasibrittle statistics has been refined to capture the statistical effect of alternating series and parallel links, which is exemplified by the material architecture of staggered platelets seen on the submicrometer scale in nacre. This architecture, which resembles a fishnet pulled diagonally, intervenes in many quasibrittle materials. The fishnet architecture

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Article Title: Fishnet model with order statistics for tail probability of failure of nacreous biomimetic materials with softening interlaminar links

Authors: Wen Luo, Zdeněk P. Bažant

Keywords: Failure probability Fracture mechanics Structural strength Monte carlo simulations Lamellar structures Material architecture Structural safety Size effect Scaling Probability distribution function (pdf) Quasibrittle materials Brittleness

Abstract: The staggered (or imbricated) lamellar “brick-and-mortar” nanostructure of nacre endows nacre with strength and fracture toughness values exceeding by an order of magnitude those of the constituents, and inspires the advent of new robust biomimetic materials. While many deterministic studies clarified these advantageous features in the mean sense, a closed-form statistical model is indispensable for determining the tail probability of failure in the range of 1 in a million, which is what is demanded for most engineering applications. In the authors' preceding study, the so-called ‘fishnet’ statistics, exemplified by a diagonally pulled fishnet, was conceived to describe the probability distribution. The fishnet links, representing interlaminar bonds, were considered to be elastic perfectly-brittle. However, the links may often be quasibrittle or almost ductile, exhibiting gradual postpeak softening in their stress-strain relation. This paper extends the fishnet statistics to links with

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Article Title: Fishnet statistics for probabilistic strength and scaling of nacreous imbricated lamellar materials

Authors: Wen Luo, Zdeněk P. Bažant

Keywords: Structural safety Probabilistic mechanics Nacre Biomimetic materials Random strength Probability distribution tail Imbricated lamellar structures Staggered platelet structures Monte Carlo simulations Failure risk Fracture Brick and mortar structures

Abstract: Similar to nacre (or brick masonry), imbricated (or staggered) lamellar structures are widely found in nature and man-made materials, and are of interest for biomimetics. They can achieve high defect insensitivity and fracture toughness, as demonstrated in previous studies. But the probability distribution with a realistic far-left tail is apparently unknown. Here, strictly for statistical purposes, the microstructure of nacre is approximated by a diagonally pulled fishnet with quasibrittle links representing the shear bonds between parallel lamellae (or platelets). The probability distribution of fishnet strength is calculated as a sum of a rapidly convergent series of the failure probabilities after the rupture of one, two, three, etc., links. Each of them represents a combination of joint probabilities and of additive probabilities of disjoint events, modified near the zone of failed links by the stress redistributions caused by previously failed links. Based on previous nano- and m

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Publication Location:

Article Title: Fishnet model for failure probability tail of nacre-like imbricated lamellar materials

Authors: Wen Luo, Zdeněk P. Bažant

Keywords: failure probability, biomimetic materials, fracture mechanics, strength, size effect

Abstract: Nacre, the iridescent material of the shells of pearl oysters and abalone, consists mostly of aragonite (a form of CaCO_3), a brittle constituent of relatively low strength (~ 10 MPa). Yet it has astonishing mean tensile strength (~ 150 MPa) and fracture energy (~ 350 to $1,240$ J/m²). The reasons have recently become well understood: (i) the nanoscale thickness (~ 300 nm) of nacre's building blocks, the aragonite lamellae (or platelets), and (ii) the imbricated, or staggered, arrangement of these lamellae, bound by biopolymer layers only ~ 25 nm thick, occupying $<5\%$ of volume. These properties inspire manmade biomimetic materials. For engineering applications, however, the failure probability of $\sim 10^{-6}$ is generally required. To guarantee it, the type of probability density function (pdf) of strength, including its tail, must be determined. This objective, not pursued previously, is hardly achievable by experiments alone, since $>10^8$ tests of specimens would be needed. Here we outline a statistical

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Article Title: Fishnet Statistical Size Effect on Strength of Materials With Nacreous Microstructure

Authors: Wen Luo, Zdeněk P. Bažant

Keywords: size effect

Abstract: The statistical size effect has generally been explained by the weakest-link model, which is valid if the failure of one representative volume element (RVE) of material, corresponding to one link, suffices to cause failure of the whole structure under the controlled load. As shown by the recent formulation of fishnet statistics, this is not the case for some architected materials, such as nacre, for which one or several microstructural links must fail before reaching the maximum load or the structure strength limit. Such behavior was shown to bring about major safety advantages. Here, we show that it also alters the size effect on the median nominal strength of geometrically scaled rectangular specimens of a diagonally pulled fishnet. To derive the size effect relation, the geometric scaling of a rectangular fishnet is split into separate transverse and longitudinal scalings, for each of which a simple scaling rule for the median strength is established. Proportional combination of both

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Article Title: Coefficient of Variation of Shear Strength of RC Beams and Size Effect

Authors: Wen Luo, Jia-Liang Le, Mohammad Rasoolinejad, Zdenek P. Bažant

Keywords: random heterogeneous material, shear fracture, probability distribution, strength scaling

Abstract: In shear failure, the reinforced concrete (RC) beams always develop, in a stable manner, a finite length crack before the maximum load is reached. Thus the crack tip location cannot sample a large volume of the material with random strength since a small region in which the crack tip can lie is fixed by fracture mechanics. Consequently, the size effect on mean strength cannot be statistical. It must be predominantly energetic, or deterministic, and thus must follow the Type-2 size effect law. What has not yet been clarified is the size effect of the coefficient of variation (CoV) of beam strength, which is important for anchoring the shear probability distribution of strength and choosing the safety factor. In this study, we run thousands of simulations using Abaqus-Explicit with microplane model M7, each with a random input of material strength and Young's modulus for each finite element in the structure. The CoV of strength is found to decrease with the structure size when geometrical

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Publication Location: Washington, D.C., USA

Article Title: Size effect on branched sideways cracks in orthotropic fiber composites

Authors: Abdullah Dönmez, Zdeněk P. Bažant

Keywords: fiber composites, random heterogeneous material, fracture mechanics, strength scaling, crack directions, crack kinking,

Abstract: In edge-notched tension specimens of unidirectional fiber-polymer composites, an initial transverse crack or notch might not propagate forward. Instead, sideways cracks parallel to fibers may grow axially from the initial crack tip. When the fibers are inclined from the specimen axis, a sideways crack propagates along the inclined fibers in highly orthotropic cross-ply composites. Unless a potential sideways orthogonal cohesive crack is considered to exist a priori at the right location, the cohesive crack model cannot predict the sideways branching correctly. The crack band model can, because it involves a damage model with tensorial strength criterion. We obtain approximate analytical solutions by the method of stress relief zones bordered by lines of slope k calibrated by J-integral. Our analysis, verified by finite

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Article Title: Failure probability of concrete specimens of uncertain mean strength

Authors: Jia-Liang Le, Zdeněk P. Bažant

Keywords: Probability distribution, random material strength, Material uncertainty, strength scaling

Abstract: A general framework for analyzing the failure probability of concrete specimens with the mean strength sampled from a large database is presented. The framework takes explicitly into account the level of uncertainty of the existing strength database involving different concretes tested in different labs. This uncertainty strongly influences the probability distribution of the nominal strength of each structure. The energetic (nonstatistical) size effect in structural strength is also incorporated into the analysis. The proposed framework is demonstrated by analyzing the reliability of reinforced concrete beams against diagonal shear failure. The analysis considers four different levels of uncertainty of the mean strength database, with coefficients of variation 5%, 10%, 25% and 40%. It is shown that the overall failure probability depends strongly on the specimen or structure size, and that this size effect is also influenced by the uncertainty level of the database. In addition,

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Article Title: Critical comparison of phase-field, peridynamics, and crack band model M7 in light of gap test and classical fracture tests

Authors: Zdenek P. Bazant, Hoang T. Nguyen, A. Abdullah Donmez

Keywords: Fracture, quasibrittle materials, material testing, random material strength, strength scaling

Abstract: The recently conceived gap test and its simulation revealed that the fracture energy G_f (or K_{Ic} , J_{cr}) of concrete, plastic-hardening metals, composites, and probably most materials can change by $\pm 100\%$, depending on the crack-parallel stresses σ_{xx} , σ_{zz} , and their history. Therefore, one must consider not only a finite length but also a finite width of the fracture process zone, along with its tensorial damage behavior. The data from this test, along with ten other classical tests important for fracture problems (nine on concrete, one on sandstone), are optimally fitted to evaluate the performance of the state-of-art phase-field, peridynamic, and crack band models. Thanks to its realistic boundary and crack-face conditions as well as its tensorial nature, the crack band model, combined with the microplane damage constitutive law in its latest version M7, is found to fit all data well. On the contrary, the phase-field models perform poorly. Peridynamic models (both bond based and state bas

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Article Title: Precip of gap test results requiring reappraisal of line crack and phase-field models of fracture mechanics

Authors: Zdenek P. Bazant, A. Abdullah Donmez, Hoang T. Nguyen

Keywords: Fracture mechanics, Crack-parallel stresses, T-stress, Cohesive crack model, Crack band model, Nonlocal models, Material characteristic length, Fracture testing, Quasibrittle behavior, Fracture process zone, Phase-field models, Yielding zone

Abstract: This paper presents a brief review of the recent advances in fracture mechanics at Northwestern University and Istanbul Technical University, prompted by the recent discovery of the gap test—a test that makes it easy and unambiguous to determine the effects of crack-parallel stresses on the mode-I fracture energy and, in consequence, on the nominal strength of structures of different sizes (aka, the size effect). The standard fracture specimens cannot reveal these effects since they have zero or negligible crack-parallel stresses. In addition, these effects cannot be reproduced by the standard, widely used, fracture models including the linear elastic fracture mechanics (LEFM), the cohesive crack model (CCM), as well as the popular computational models such as the extended finite element (XFEM) and the phase-field models (PFM). Therefore, it will be necessary to adopt fracture models that can reflect the tensorial damage behavior in the fracture process zone (FPZ), which is governed by

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Article Title: Smooth Crack Band Model—A Computational Paragon Based on Unorthodox Continuum Homogenization

Authors: Yupeng Zhang, Zdeněk P. Bažant

Keywords: continuum mechanics, fracture mechanics, damage mechanics, quasibrittle materials, material heterogeneity, nonlocal continuum, gradient models, numerical simulation, finite element method, strain localization instability, localization limiters, computational mechanics, energy release rate, failure criteria, micromechanics

Abstract: The crack band model, which was shown to provide a superior computational representation of fracture of quasibrittle materials (in this journal, May 2022), still suffers from three limitations: (1) The material damage is forced to be uniform across a one-element wide band because of unrestricted strain localization instability; (2) the width of the fracture process zone is fixed as the width of a single element; and (3) cracks inclined to rectangular mesh lines are represented by a rough zig-zag damage band. Presented is a generalization that overcomes all three, by enforcing a variable multi-element width of the crack band front controlled by a material characteristic length l_0 . This is achieved by introducing a homogenized localization energy density that increases, after a certain threshold, as a function of an invariant of the third-order tensor of second gradient of the displacement vector, called the sprain tensor γ , representing (in isotropic materials) the magnitude of its Lapl

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Article Title: Scaling in size, time and risk—The problem of huge extrapolations and remedy by asymptotic matching

Authors: Zdeněk P. Bažant, Hoang T. Nguyen, A. Abdullah Dönmez

Keywords: Fracture mechanics Size effect on strength Creep and shrinkage Water diffusion Cement hydration Failure probability Experimental databases Concrete structures

Abstract: The scaling of structural response of concrete structures to large structure sizes, to long service lives and to tolerable failure probabilities is a problem of order-of-magnitude extrapolations, which are intractable by AI and machine learning and require significant theoretical advances. The present review, based on a lecture at Yonggang Huang's 60th birthday symposium in Houston, summarizes the recent advances, with a focus on those achieved at Northwestern University. Reliable extrapolation requires two-sided asymptotic matching. Most existing databases provide support on only one extreme of the range of size, time or failure probability, but theoretical support can be obtained for the asymptotic behaviors on both sides of the range. The advantage is that the asymptotics are much simpler than the behavior in the central, transitional, range. In closing it is explained that realistic extrapolations are required to mitigate the calamitous CO₂ emissions from cement and concrete industry.

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Book Title: Quasibrittle Fracture Mechanics and Size Effect:A First Course

Authors: Zdenek P Bazant, Jia-Liang Le, Marco Salviato

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Authors: Wen Luo

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I certify that the information in the report is complete and accurate:

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Signature Date: 4/3/23 8:09PM



ORIGINAL PAPER

Size effect on branched sideways cracks in orthotropic fiber composites

Abdullah Dönmez · Zdeněk P. Bažant

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Abstract In edge-notched tension specimens of unidirectional fiber-polymer composites, an initial transverse crack or notch might not propagate forward. Instead, sideways cracks parallel to fibers may grow axially from the initial crack tip. When the fibers are inclined from the specimen axis, a sideways crack propagates along the inclined fibers. Unless a potential sideways orthogonal cohesive crack is considered to exist a priori at the right location, the cohesive crack model cannot predict the sideways branching correctly. The crack band model can, because it involves a damage model with tensorial strength criterion, instead of scalar stress-separation law. We obtain approximate analytical solutions by the method of stress relief zones bordered by lines of slope k calibrated by J -integral. Our analysis, verified by finite element simulations, yields formulae for the structure size effect of orthogonal or inclined sideways cracks, taking into account the effect of the length of original transverse crack or notch, and the effect of the ratio of fracture energies Γ_f and Γ_s of the forward and sideways cracks, respectively. The sideways cracks develop only if the ratio Γ_s / Γ_f is small

enough, below a certain critical value (on-the-order of 10^{-1}) depending on material parameters. Measuring the size effect of sideways cracks yields Γ_s and the corresponding fracture process zone size. The theory is shown to match Nairn's tests with varying notch depths. The results can be applied to all kinds of orthotropic composites.

Keywords Crack branching · Quasibrittle fracture mechanics · Crack band model · Cohesive crack model · Strength scaling · Fracture energy · Fracture process zone · Unidirectional fiber composites · Laminates · Crack stability

1 Introduction

In edge-notched tensile strip specimens of highly orthotropic fiber composites, such as unidirectional prepregs or laminates of a two-directional orthotropic layup with weak transverse fibers, the crack often does not grow along the notch direction. Rather, a sideways crack branched in the direction of tension and normal to the notch direction may typically develop (Fig. 1). Similar behavior is known for wood.

Observations of the branched sideways cracks in experiments have often been invoked to claim that fracture mechanics is not applicable to fiber composites. It is, of course, true that the linear elastic fracture mechanics (LEFM) does not apply. It is also true that the cohesive crack model (Barenblatt 1962; Hillerborg et al.

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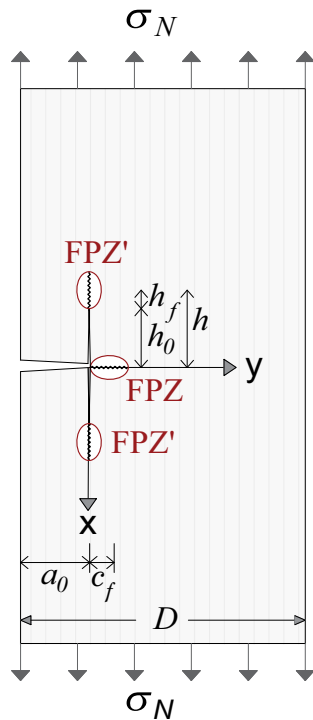


Fig. 1 General sideways crack propagation

1976), being a line crack model with a scalar relation between stress and crack opening, cannot predict sideways crack branching, unless a potential sideways cohesive crack zone is postulated to exist a priori at the right place where branching is suspected, as done by (Noselli et al. 2013; Tankasala et al. 2017, 2018). However, this is not a general prediction model.

Predicting the germination of a sideways crack from the fracture process zone (FPZ) requires quasibrittle fracture mechanics. Unlike the cohesive crack model, it takes into account the tensorial damage in the FPZ at the crack front. Its simplest form, to be used here, is the crack band model. Originated in the 1980s (Bažant 1982, 1984; Bažant and Oh 1983; Bažant and Planas 1997), it is distinct from, and more general than, the cohesive crack model. That model cannot capture the tensorial strength limits and damage in the FPZ. Neither can it capture the huge effect of crack-parallel compression, the axial splitting fracture under uniform compression, the transition from distributed cracking to a localized crack, and the tensorial behavior of FPZ under nonproportional or fatigue loading. Neither can these features be captured by the phase-field model (Ming-Yuan and Hutchinson 1989; Parmigiani and Thouless

2006; Martinez and Gupta 1994; Suo et al. 1991), as it now exists. Yet they can be by the crack band model.

The basic idea of the crack band model (Bažant and Oh 1983; Bažant and Planas 1997; Bažant and Le 2017) for strain-softening in tension as well as shear is to avoid spurious mesh-sensitivity by adjusting the material parameters that control the steepness of the softening post-peak stress–strain relation and describe smeared cracking. In this way, the energy dissipated by fracture per unit length of the crack band (and per unit out-of-plane thickness) is made approximately independent of the element size. The adjustment of material parameters depending on element size also circumvents the problems of unlimited strain localization and of ill-conditioning associated with partial differential equations for the strain-softening continuum. The crack band offers a simple way to capture the deterministic (or energetic) size effect (Bažant and Oh 1983; Bažant and Planas 1997).

The simplest way to use the crack band model is to use an element size, h , such that a sudden (or vertical) stress drop from the peak-stress point (or generally from the limit point at which the tangential stiffness matrix becomes singular) would dissipate an energy equal to the fracture energy of the material. If h is kept the same for various structure sizes, no adjustment of post-peak softening slope is needed, which is what is done here (Bažant and Oh 1983; Bažant and Planas 1997). Such a fixed element size may then be regarded as the material characteristic length, $h = l_0$ (akin to Irwin's length, l_0 , in cohesive fracture mechanics).

Early studies of sideways crack branching, aka kinking, used criteria based on the near-tip stress field, which cannot capture the energetic (non-statistical) size effect. The first such study apparently belongs to Cook et al. (1964), who assumed the existence of a weak plane orthogonal to the notch. This approach was later refined in (Goree and Gross 1980; Buczek and Herakovich 1985; Gupta et al. 1992). An explicit solution of the stress field near the crack tip in a unidirectional composite was given by Sih et al. (1965). They also studied energy-based criteria and proposed for fiber composites a criterion based on a critical strain energy density, which, however, was not generally accepted.

The power of the crack band model for predicting sideways cracks was recognized by (Pineda and Waas 2013; Xu et al. 2015; Xu and Waas 2016), who exploited the tensorial nature of the stress and strain within the band to predict orthogonal sideways cracks

in fiber polymer composites. Here we aim to use the crack band model to extend Waas et al.'s work to the problem of scaling and derive closed formulae for the size effect.

The size effect is a quintessential property of fracture mechanics. It is particularly important for quasibrittle fracture, and cannot be captured by strength criteria alone. The existence of size effect is one proof that the failure of composites cannot be predicted from stress or strain criteria alone, which predict no size effect, yet are still widely encountered in practice. It is still widely believed that the size effects observed in fiber composites are solely Weibullian. But this is erroneous and can be gravely misleading. The statistical aspect is only secondary and becomes significant only for very large structures; see (Bažant and Le 2017; Bažant 2019).

2 Orthotropic elastic constants of unidirectional fiber composites predicted from constituent properties

First, we need to formulate the elastic constitutive model. We consider orthotropic fiber laminates with unidirectional fiber reinforcement in the x -direction (Fig. 1), and, up to the material strength limit, we treat the laminate as orthotropic linearly elastic. The two-dimensional orthotropic elastic stress–strain relations are:

$$\begin{aligned}\sigma_{xx} &= E_{xx}\epsilon_{xx} + E_{xy}\epsilon_{yy} \\ \sigma_{yy} &= E_{xy}\epsilon_{xx} + E_{yy}\epsilon_{yy} \\ \tau_{xy} &= G_{xy}\gamma_{xy} \\ E_{xx} &= \frac{E_1}{1 - \nu_{12}\nu_{21}}; \quad E_{xy} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \\ E_{yy} &= \frac{E_2}{1 - \nu_{12}\nu_{21}}; \quad G_{xy} = G_{12}.\end{aligned}\quad (1)$$

where σ_{xx} , σ_{yy} = normal stresses, τ_{xy} = shear stress, ϵ_{xx} , ϵ_{yy} = normal strains and $\gamma_{xy} = 2\epsilon_{xy}$ = shear angle. Very good estimates are provided by the Hill–Hashin–Christensen–Lo formulae developed in (Hill 1965; Hashin 1965; Christensen and Lo 1979; Huang et al. 2011).

$$\begin{aligned}E_1 &= V_f E_f + V_m E_m + \frac{4(\nu_f - \nu_m)^2 V_f (1 - V_f)}{\frac{V_f}{k_m} \frac{1 - V_f}{k_f} \frac{1}{G_m}} \\ E_2 &= \frac{2}{\frac{0.5}{K_L} + \frac{0.5}{G_{23}} + \frac{2\nu_{12}^2}{E_1}} \\ G_{xy} &= G_m \frac{G_f + G_m + V_f (G_f - G_m)}{G_f + G_m - V_f (G_f - G_m)} \\ \nu_{12} &= V_f \nu_f + V_m \nu_m \\ &\quad + \frac{(\nu_f - \nu_m) V_f (1 - V_f)}{\frac{V_f}{k_m} + \frac{1 - V_f}{k_f} + \frac{1}{G_{12}}} \left(\frac{1}{k_m} - \frac{1}{k_f} \right) \\ K_L &= \frac{3k_m + G_m}{3} + \frac{V_f}{\frac{1}{k_f - k_m + \frac{G_f - G_m}{3}} + \frac{1 - V_f}{k_m + \frac{4G_m}{3}}} \\ G_{23} &= G_m \left(1 + \frac{V_f}{\frac{G_m}{G_f - G_m} + \frac{(k_m + 7G_m/3)(1 - V_f)}{2k_m + 8G_m/3}} \right) \\ k_f &= \frac{E_f}{3(1 - 2\nu_f)}; \quad k_m = \frac{E_m}{3(1 - 2\nu_m)}\end{aligned}\quad (2)$$

where V generally refers to volume fraction, E to modulus of elasticity, ν to Poisson's ratio and G to shear modulus, and subscripts f and m refer to the fibers and matrix.

The preliminary analysis used Bažant's 1968 simpler approximation (Bažant 1968; Bažant and Cedolin 2010) based on a mixture of parallel and series couplings. It gave similar results even though the estimate of E_{yy} has a significant error (almost 50% in some cases).

3 Simple approximate analysis based on idealized energy release zones

3.1 Method of "stress relief zones" bordered by "stress diffusion" lines of slope calibrated by J -integral

Consider an elongated rectangular strip with a transverse notch of length a_0 , as shown in Fig. 2a. Uniform uniaxial tension, σ_N , is applied at remote ends. Reinforcing fibers in the longitudinal direction, and nonexistent (or weak) in the transverse direction, cause the fracture energy Γ_f for forward (i.e., transverse) crack propagation to be much larger than the fracture energy Γ_s for the longitudinal crack propagation (orthogonal to the notch); i.e., $\Gamma_f \gg \Gamma_s$. Obviously,

Γ_s represents a shear fracture energy predominantly, because the vertical plane emanating from the notch tip (Fig. 2b) is subjected to shear stress, due to different longitudinal strains caused by longitudinal contraction of the unloaded material on the left of that crack.

We will now use the simple method of “stress relief” zones bordered by “stress diffusion” lines. In this method, which met with success in some other fracture problems (Bažant 1984; Bažant and Planas 1997, e.g.), we assume the stress σ in the cross-hatched triangular zone limited by a line of slope $1/k$, emanating from the center of the FPZ at the notch or crack tip (Fig. 2), to get reduced to zero, and the stress outside this zone to remain unaffected by the notch or crack. At first, this method may look too crude, because in reality the stress field varies gradually, and the stress concentration near the crack front is ignored.

It has nevertheless been shown that this method leads to the correct general form of the equation relating the load, fracture-caused displacement, and structure size D . To make not only the form but also the equation coefficients exact, the slope k must be determined either by matching the energy release rate based on k to the J -integral calculated by FE or by a comparison with other features of a full elasticity solution, if available.

The useful feature of this method is that, for geometrically scaled structures, (1) the optimal slope k is exactly the same for LEFM and almost the same for nonlinear fracture models, and that (2) the stress relief zones consist of two parts, one of which, representing the unloaded undamaged material, scales with D quadratically, and another, corresponding to the damaged FPZ, scales with D linearly; i.e., $\mathcal{G} \propto D^2$ or D . The combined scaling can thus give the correct form of quasibrittle size effect, which is transitional between ductility and brittleness. It was via the stress relief zones that Bažant’s size effect law (SEL) for concrete and other quasibrittle materials was originally derived in 1984 (Bažant 1984). The resulting equation form of SEL was later validated in several other ways, e.g., by asymptotic matching combined with equivalent LEFM and J -integral analysis (Bažant and Planas 1997; Bažant and Le 2017; Bažant and Kazemi 1990; Bažant 1997, 2005). Recently, the same method was successfully used for reinforced concrete shear walls (Rasoolinejad and Bažant 2019).

3.2 Forward crack propagation

For fracture analysis, we need the density of complementary energy of unidirectional composite subjected to uniform uniaxial stress $\sigma_{xx} = \sigma_N$ along the fibers. Inverting the stress–strain relations in Eqs. (1) with (2), we get the material compliance matrix and particularly the effective elastic modulus $\hat{E}$ for uniaxial stress along fibers. This yields the longitudinal strain $\epsilon_{xx} = \sigma_{xx}/\hat{E}$ and the complementary energy density

$$\bar{\Pi}^* = \sigma_N^2/2\hat{E}, \quad \hat{E} = E_{xx} - E_{xy}^2/E_{yy} \quad (3)$$

Denote now as c_f the half-length of the FPZ in the sense of equivalent linear elastic fracture mechanics (LEFM); c_f is approximately equal to $\frac{1}{3}$ to $\frac{1}{2}$ of the FPZ length, and is approximately a material constant [c_f can be precisely identified through the size effect analysis of geometrically scaled specimens; see Fig. 2 and (Bažant and Yu 2011; Cusatis and Schaufert 2009)].

Let $a = a_0 + c_f$ = total length of the equivalent LEFM crack. The complementary strain energy is assumed to be released from two symmetric triangular stress-relief zones bounded by certain slopes k (shaded in Fig. 2b);

$$\Pi^* = b \frac{\sigma_N^2}{2\hat{E}} 2 \frac{a \cdot ka}{2} \quad (4)$$

where b is the thickness of the laminate plate. So, the energy release rate of the forward crack (transverse to the fibers), which must be during propagation equal to the fracture energy, Γ_f , for transverse fracture, is

$$\mathcal{G} = \frac{1}{b} \frac{\partial \Pi^*}{\partial a} = \frac{\sigma_N^2}{2\hat{E}} 2 k(a_0 + c_f) = \Gamma_f \quad (5)$$

From this, for transverse crack propagation,

$$\sigma_N = \sqrt{\frac{\hat{E} \Gamma_f}{k(a_0 + c_f)}} = \frac{\sigma_0}{\sqrt{1 + D/D_0}} \quad (6)$$

$$\text{where } D_0 = c_f(D/a_0), \quad \sigma_0 = \sqrt{\hat{E} \Gamma_f / c_f k}. \quad (7)$$

where $D_0 = c_f(D/a_0)$ is a constant for geometrically scaled structures of different sizes, for which the notch lengths must also be similar, i.e., a_0/D is the constant. Note that Eq. (6) has the exact form of Bažant’s SEL (Bažant 1984; Bažant and Kazemi 1990), although parameters k (depending on structure shape) and material constant c_f must be determined separately. This law has been rigorously derived by asymptotic matching based on equivalent LEFM, or on J -integral (Bažant and Planas 1997; Bažant 1997, 2005).

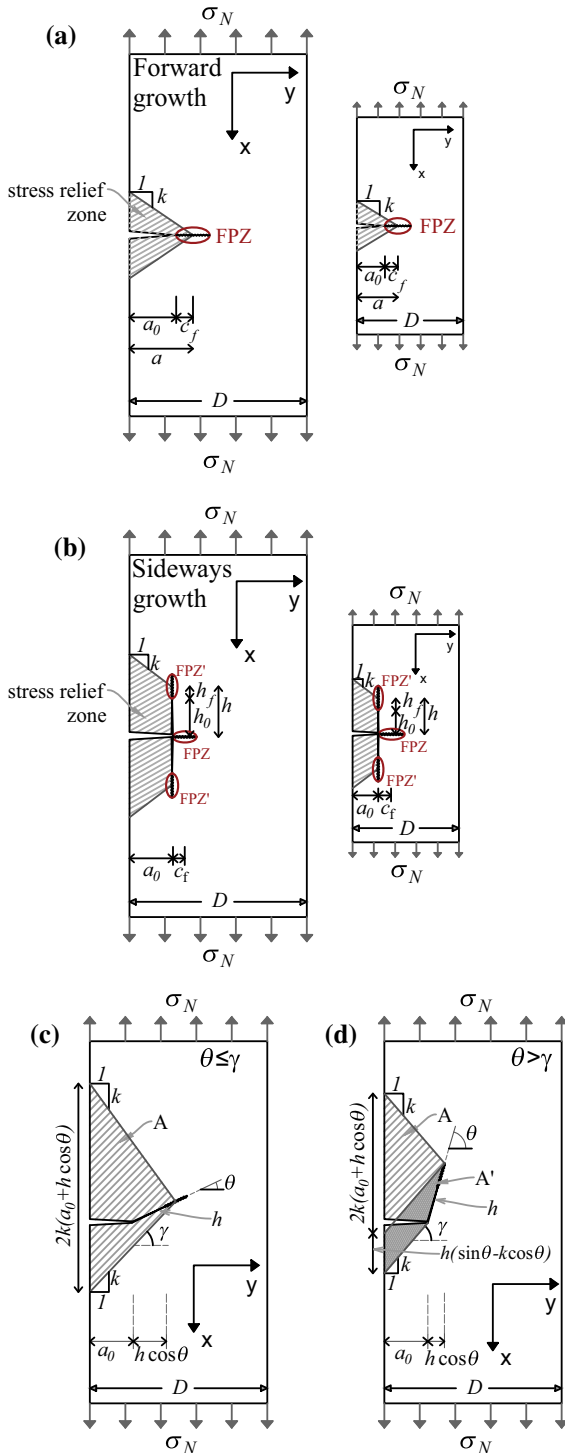


Fig. 2 Crack propagation in **a** forward, **b** sideways for scaled structures and **c**, **d** inclined directions

What is particularly useful in this simple approach is that, once the correct k -value for a given specimen geometry is identified and a realistic c_f is known, one can easily obtain the correct scaling, particularly the quasibrittle deterministic (or energetic) size effect, giving the transition from quasi-ductile failure to brittle failure. To get a general FE solution, one would have to supplement it by asymptotic matching, which would be much more tedious. Any accurate analytical solution of the boundary value problem would be possible only for $c_f = 0$, i.e., for LEFM, emulating the approach in Suo et al. (1991).

3.3 Propagation of orthogonal sideways cracks

Let us now analyze orthogonally branched sideways cracks propagating in the direction of axis x . The origin of coordinates (x, y) is placed at the tip of the notch or preexisting crack (Fig. 2b). The notch in y -direction (transverse to fibers) causes the material to unload in the vicinity of the notch on the left of axis x but not in the region on the right of that axis. So the material to the left of x contracts in x direction, while to the right x it does not. It is thus intuitively clear that shear stresses τ_{yx} must develop along axis x . A normal stress across that axis must develop as well. This is what intuitively explains, along with the material weakness along axis x , the propagation of sideways crack parallel to the applied tension.

The energy release rate at constant load σ_N does not increase as the sideways crack grows, as confirmed by what follows. Neither it decreases. This means that the propagation of branched sideways crack is at the limit of stability, and so these cracks can grow simultaneously in opposite directions (i.e., both up and down in Fig. 2b). Experiments (Nairn 1988a; Newaz 1985; Lee and Pharr 2019) confirm it.

We consider a longitudinal crack emanating from the transverse crack tip (Fig. 2b); h_0, h = its lengths up to the beginning of its FPZ and up to the approximate half-length of the FPZ (making h the length of an equivalent LEFM crack); $h_f = h - h_0$ is roughly the half-length of the FPZ, treated approximately as a constant.

The total release of the complementary strain energy from the cross-hatched stress-relief zone in Fig. 2b, consisting of one triangle and one rectangle, is

$$\Pi^* = b \frac{\sigma_N^2}{2E} 2 \left(a_0 h + \frac{a_0 \cdot a_0 k}{2} \right) \quad (8)$$

where $h = h_0 + h_f$. The rate of energy release from one longitudinal sideways crack, which must be equal to the fracture energy, Γ_s , for sideways crack propagation, must be equal to effective fracture energy Γ_s for the sideways cracks

$$\mathcal{G} = \frac{1}{b} \left[\frac{\partial \Pi^*}{\partial h} \right]_P = \frac{\sigma_{Ns}^2}{2\hat{E}} a_0 = \Gamma_s \quad (9)$$

Γ_s is fracture energy of sideways crack, which is here assumed to be approximately a constant. The crack is not of pure mode II, but is mixed with a mode I component because non-negligible normal stresses develop on the crack (this is captured automatically if the fracture front is described tensorially, as in the crack band model). The nominal strength of structure with a sideways crack then is

$$\sigma_{Ns} = \sqrt{\frac{2\hat{E}\Gamma_s}{a_0}} \quad (10)$$

Note that this result is independent of h and h_f , i.e., of the length of the sideways crack and of its FPZ length. This means that, after reaching the maximum load, there load evolution must exhibit a quasi-plastic plateau, instead of gradual softening which is endemic for forward propagation. However, as seen in Eq. (10), there is a size effect of initial relative cracks length, a_0 . For geometric similarity, i.e., for $a_0 \propto D^{-1/2}$, this size effect is of the LEFM type, $\sigma_{Ns} \propto (\text{size})^{-1/2}$, which is the strongest possible.

When the load is gradually increased, the sideways crack will start growing at the first opportunity it has. It will grow when

$$\sigma_{Ns} \leq \sigma_N \quad \text{or} \quad \frac{2\hat{E}\Gamma_s}{a_0} = \frac{\hat{E}\Gamma_f}{k(a_0 + c_f)} \quad (11)$$

$$\text{or} \quad \boxed{\frac{\Gamma_s}{\Gamma_f} = \frac{a_0}{2k(a_0 + c_f)}} \quad (12)$$

If, in Eq. (12), the left-hand side $>$ the right-hand side, the sideways fracture cannot grow, and if $<$, the energy released by growth exceeds the energy dissipated at the front of sideways fracture, the sideways crack will grow dynamically.

Since $c_f \geq 0$, the last equation lets us state a weak criterion that is necessary, though not sufficient, for sideways crack growth to happen:

$$\boxed{\Gamma_s < \Gamma_f/2k} \quad (13)$$

Solving Eq. (12) for a_0 further reveals the existence of a critical length a_{0cr} of transverse notch or crack below which the sideways crack cannot form:

$$a_{0cr} = \frac{c_f}{\Gamma_f/2k\Gamma_s - 1} \quad (14)$$

Rather, a short forward crack will grow first, and then a sideways crack will start once a_{0cr} is reached, provided that Γ_s is small enough, such that $\Gamma_s < \Gamma_f/2k$. In view of inequalities (11) and (12), we thus have the limits:

$$\lim_{\Gamma_s/\Gamma_f \rightarrow k/2} a_{0cr} = \infty \quad (15)$$

$$\lim_{\Gamma_s/\Gamma_f \rightarrow \infty} a_{0cr} = 0 \quad (16)$$

It is interesting to check the size effect on nominal strength when the specimen is scaled up geometrically, i.e., when a_0/D is kept constant, D being the specimen size. From Eq. (12),

$$\boxed{\sigma_{Ns} = \sqrt{C/D}} \quad \text{where} \quad C = 2\hat{E}\Gamma_s(D/a_0) \\ = \text{constant} \quad (17)$$

So the crack lengths h and h_0 are irrelevant and the sideways cracks exhibit the LEFM scaling even if the FPZ size, h_f , is finite. This simplifies the testing of Γ_s but prevents the use size effect testing to measure h_f for the orthogonal sideways cracks. Note also that the size effect on the sideways crack is stronger than it is on the forward crack, becoming identical only in the limit of infinite size.

To determine Γ_s , one may test the strength, or σ_N , of specimens of different sizes D . Keeping the same notch ratio a_0/D (and the same length-to-width ratio of the specimen) would make the calculations simpler. Because of scatter, test repetition is necessary. Unlike the testing of Γ_f and c_f (Bažant and Kazemi 1990; Bažant et al. 1996), two-dimensional regression is unnecessary. It is a simple single variable statistics because the FPZ size, h_f , need not be known. For each measured σ_{Ns} , one can use Eq. (17) to calculate the values of:

$$\Gamma_s = \frac{\sigma_{Ns}^2 D}{2\hat{E}(D/a_0)} \quad (18)$$

From the collection of measured Γ_s -values, one gets their mean and standard deviation. It is of course possible to test specimens with sufficiently different a_0 values and dissimilar sizes D . But, according to the experi-

ence with concrete (Bažant and Planas 1997), one must expect a more scattered collection of measured Γ_s values, especially because the present approach based on energy release zones becomes less accurate when the specimen shapes are not similar.

3.4 Non-orthogonal sideways crack propagation along inclined fibers

Consider now a single-edge-notched tension (SENT) specimen in which the fibers deviate from the longitudinal direction by either 45° or -45° . Let x be the fiber direction y normal to it. For a uniaxial tensile stress σ_N applied in the longitudinal direction, the Mohr circle readily gives, in coordinates x and y , the stresses are $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_N/2$. Inverting Eqs. (1) gives:

$$\begin{aligned}\epsilon_{xx} &= \frac{E_{yy} - E_{xy}}{E_{xx}E_{yy} - E_{xy}^2} \frac{\sigma_N}{2} \\ \epsilon_{yy} &= \frac{E_{xx} - E_{xy}}{E_{xx}E_{yy} - E_{xy}^2} \frac{\sigma_N}{2}\end{aligned}\quad (19)$$

The complementary strain energy density is then calculated as

$$\begin{aligned}\bar{\Pi}^* &= \frac{1}{2}(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{xy}\gamma_{xy}) \\ &= \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy} + \gamma_{xy}) \frac{\sigma_N}{2} \\ &= \frac{\sigma_N^2}{2E_{45}},\end{aligned}\quad (20)$$

$$\frac{1}{E_{45}} = \frac{1}{4} \left(\frac{E_{xx} + E_{yy} - 2E_{xy}}{E_{xx}E_{yy} - E_{xy}^2} + \frac{1}{G_{xy}} \right) \quad (21)$$

Next consider a bi-directional composite in which one half of the fibers run in direction x and one half in direction y . The strains, taken separately for each inclination, are equal, including γ_{xy} . The stresses, which are equal for both inclinations, are summed. But the stresses for both cases are equal, too.

Therefore, the foregoing equations for stresses and complementary energy remain valid. The only difference, which is irrelevant for the present analysis, is that the specimen cross sections remain normal to the longitudinal axis while, in the case of fibers inclined in only one direction, the shear strain causes the cross sections to rotate.

More generally, the unidirectional fibers in the notched specimen can run at any angle θ with the notch. We will not bother to set up the equations for the stresses, but it is obvious that $\hat{E}$ is a certain effective modulus depending on angle θ . When the fiber direction forms a skew angle θ with the notch or existing crack (Fig. 2c), the crack propagates in that sideways direction, too. Here one might think of two options: (1) Two inclined cracks propagating symmetrically on both sides of the notch, or (2) only one inclined propagating on one side of the notch. Calculations showed, however, that the former option is unstable under load control, because, unlike the orthogonal sideways cracks, the equilibrium propagation of inclined cracks would require a decreasing load. So we consider an inclined crack on one side only, as shown in Fig. 2c, d, and consider that the maximum load occurs as soon as the FPZ forms, i.e., for $h = h_f$, with $h_0 = 0$.

Denote $\gamma = \arctan k$ = angle of stress diffusion line with the notch, and θ = angle of the fibers and the sideways crack with the notch. For simplicity, we consider the stress relief zones on both sides of the notch to be bordered by the same slope k , as shown in Fig. 2c, d. Actually, on the side opposite to inclined crack one might expect a different slope k' , but there is no way to determine it in a simple way since, from the equivalence with J -integral, we can identify only one slope.

The energy release areas are A and A' , respectively, as marked and shaded in Fig. 2c, d. For both cases the total area, A_T , may be written as

$$\begin{aligned}A_T &= A + A', \quad A = k(a_0 + h \cos \theta)^2 \\ A' &= \langle \sin \theta - k \cos \theta \rangle \left(\frac{1}{2} h^2 \cos \theta + h a_0 \right)\end{aligned}\quad (22)$$

where $\langle x \rangle = \max(x, 0)$ = positive part of the argument = Macauley brackets (so A' vanishes when $\theta < \gamma$). The complementary energy density driving the fracture is assumed to be $\sigma_N^2/2\hat{E}$ where $\hat{E}$ is a certain effective elastic modulus (depending on fiber content and fiber orientation). For the case of $\theta = 45^\circ$, $\hat{E}$ is equal to E_{45} in Eq. (21). The estimate of complementary energy released by fracture is $\Pi^* = bA_T(\sigma_N^2/2\hat{E})$. Then, introducing dimensionless crack length parameters,

$$h = \eta D, \quad h_0 = \eta_0 D, \quad a_0 = \alpha_0 D \quad (23)$$

we obtain the energy release rate of the branched sideways crack in the form:

$$\begin{aligned} \mathcal{G} &= \frac{1}{b} \frac{\partial \Pi^*}{\partial h} = \frac{\sigma_N^2}{2\hat{E}} D \tilde{g}(\eta, \alpha_0) \tilde{g}(\eta, \alpha_0) \\ &= (\alpha_0 + \eta \cos \theta)(2k \cos \theta + (\sin \theta - k \cos \theta)) \end{aligned} \quad (24)$$

It may be checked that Eqs. (5) and (9) are the limiting cases for $\theta = 0$ and $\theta = \pi/2$. For the non-orthogonal fibers and crack, it is found that the energy release rate at constant load, or $\tilde{g}(\eta, \alpha_0)$, is increasing with crack length h (which is the case of positive geometry). This means that the maximum load is reached as soon as the sideways FPZ of length h_f is formed ($h = h_f$). After that moment, there is postpeak softening and the non-orthogonal sideways crack grows dynamically.

In Eq. (24), we may now approximate function $\tilde{g}(\eta, \alpha_0)$ by the first two terms of Taylor series expansion with respect to η at constant α_0 and θ :

$$\tilde{g}(\eta, \alpha_0) = \tilde{g}_0(\alpha_0) + \tilde{g}'_0(\alpha_0)h_f/D \quad (25)$$

where h_f , the half-size of the FPZ, is assumed to be approximately constant when size D is scaled. Next we may set $\mathcal{G} = \Gamma'_s$ and, considering scaled geometrically similar structures with similar cracks, for which η, α_0 are constant, we obtain

$$\sigma_N = \sqrt{\frac{\hat{E} \Gamma'_s}{D[\tilde{g}(\alpha_0) + \tilde{g}'(\alpha_0)h_f/D]}} = \frac{\sigma_0}{\sqrt{1 + D/D_0}} \quad (26)$$

$$\text{where } D_0 = h_f \frac{\tilde{g}'(\alpha_0)}{\tilde{g}(\alpha_0)}, \quad \sigma_0 = \sqrt{\frac{\hat{E} \Gamma'_s}{\tilde{g}'(\alpha_0)}}. \quad (27)$$

4 Material strength limits for FE analysis with crack band model

The cohesive crack model, which is a line crack model with a one-dimensional stress–strain law, requires one strength limit, the tensile strength. Generalized to mixed mode, it requires a potential depending on two scalar variables, the normal and tangential displacements. But for FE fracture analysis with the crack band model, we need a triaxial tensorial strength criterion for the material in the FPZ.

In the crack band model, infinitely many microcracks of infinitely small openings are imagined to be continuously distributed (or smeared) over the finite element. This can be conveniently modeled by reducing the material stiffness and strength in the direction

normal to the crack band after the peak strength of the material has been reached. Such changes of the stiffness matrix are relatively easy to implement in a FE code, and, hence, the appeal of smeared cracking. The evolution of the cracking process down to zero stress at full fracture implies strain softening, i.e., a postpeak gradual decline of stress at increasing strain. So, according to the crack band concept, using a post-peak softening law, or defining a characteristic length, or keeping the same element size in the damage or failure zones, are all equivalent.

Matzenmiller et al. (1995) used a continuum damage mechanics coupled with failure criteria for the anisotropic materials. This method is adopted here, and the damage tensor $\mathbf{Q}$ is used for the transition between the undamaged and damaged material. For plane stress conditions, this transition is addressed as follows:

$$\hat{\boldsymbol{\sigma}} = \mathbf{Q} \boldsymbol{\sigma} \quad (28a)$$

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{q_{11}} & 0 & 0 \\ 0 & \frac{1}{q_{22}} & 0 \\ 0 & 0 & \frac{1}{q_{12}} \end{bmatrix}; \quad \hat{\boldsymbol{\sigma}} = \begin{bmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\tau}_{xy} \end{bmatrix}; \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad (28b)$$

where $\hat{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$ correspond to the stress vectors for damaged and undamaged material respectively. The damage variables q_{11} , q_{22} and q_{12} are the functions of the strain values. The threshold values for the strains can be found by using the Hashin's criteria: (Hashin 1980)

$$\text{for } \sigma_x > 0: \quad \frac{\varepsilon_x}{e_{ft}} < 1 \quad (29a)$$

$$\text{for } \sigma_x < 0: \quad \left| \frac{\varepsilon_x}{e_{mc}} \right| < 1 \quad (29b)$$

$$\text{for } \sigma_y > 0: \quad \left(\frac{\gamma_{xy}}{\gamma_m} \right)^2 + \left(\frac{\varepsilon_y}{e_{mt}} \right)^2 < 1 \quad (29c)$$

$$\text{for } \sigma_y < 0: \quad \left(\frac{\gamma_{xy}}{\gamma_m} \right)^2 + \left(\frac{\varepsilon_y}{e_{mc}} \right)^2 < 1 \quad (29d)$$

where e_{ft} , e_{mc} , e_{mt} , γ_m refer to tensile strain threshold of the fibers and to the limit values of compressive, tensile and shear strains of the matrix, respectively. These thresholds correspond to f_{ft} , f_{mc} , f_{mt} , and τ_m in terms of stress quantities. Note that Eq. (29a) depends on only the fiber strength while Eqs. (29b, 29c, 29d) are independent of the fiber parameters.

The damage variables q_{11} , q_{22} and q_{12} can be calculated as in Eq. (30).

$$\text{for } \sigma_x > 0 : q_{11} = \exp \left(-c_f \left\langle \frac{\varepsilon_x}{e_{ft}} - 1 \right\rangle \right) \quad (30a)$$

$$\text{for } \sigma_x < 0 : q_{11} = \exp \left(-c_m \left\langle \left| \frac{\varepsilon_x}{e_{mc}} \right| - 1 \right\rangle \right) \quad (30b)$$

for $\sigma_y > 0 :$

$$q_{22} = \exp \left(-c_m \left\langle c_{sy} \left(\frac{\gamma_{xy}}{\gamma_m} \right)^2 + \left(\frac{\varepsilon_y}{e_{mt}} \right)^2 - 1 \right\rangle \right) \quad (30c)$$

$$q_{12} = \exp \left(-c_m \left\langle \left(\frac{\gamma_{xy}}{\gamma_m} \right)^2 + \left(\frac{\varepsilon_y}{e_{mt}} \right)^2 - 1 \right\rangle \right) \quad (30d)$$

for $\sigma_y < 0 :$

$$q_{22} = \exp \left(-c_m \left\langle c_{sy} \left(\frac{\gamma_{xy}}{\gamma_m} \right)^2 + \left(\frac{\varepsilon_y}{e_{mc}} \right)^2 - 1 \right\rangle \right) \quad (30e)$$

$$q_{12} = \exp \left(-c_m \left\langle \left(\frac{\gamma_{xy}}{\gamma_m} \right)^2 - c_{ys} \left(\frac{\varepsilon_y}{e_{mc}} \right)^2 - 1 \right\rangle \right) \quad (30f)$$

where c_f , c_m , c_{sy} and c_{ys} are the parameters associated with the softening behavior of fiber failure in tension, matrix compression failure, coupling of shear with matrix tensile failure, and coupling of matrix tension with shear failure, respectively. In constructing the damage variables in Eqs. (30), the following assumptions are adopted:

- In compression, the fiber kinking is ignored. Hence, only the matrix strength governs compression failure in the fiber direction.
- The failure in the y-direction (transverse to fiber, weak direction) is controlled only by the matrix. The coupling between the the shear and tension actions is characterized by parameter c_{sy} in Eq. (30c, 30e).
- For the shear damage variable, q_{12} , the positive effect of compressive stresses on the shear failure is taken into account by $-c_{ys}$.
- Since the rupture of fibers is much more brittle than the matrix failure, c_f is taken as 10-times greater than c_m . The coupling of shear to tension or compression failures of the composite is assumed to be 10-times weaker than the coupling of compression or tension failures of the matrix to the shear failure.

After each violation of the strength criterion in Eq. (29), the elastic constants, for that element only, are recalculated according to the Eq. (31);

$$\begin{aligned} E_{xx} &= \frac{q_{11} E_1}{1 - q_{11} q_{22} \nu_{12} \nu_{21}}; \quad E_{yy} = \frac{q_{22} E_2}{1 - q_{11} q_{22} \nu_{12} \nu_{21}} \\ E_{xy} &= \frac{q_{11} q_{22} \nu_{21} E_1}{1 - q_{11} q_{22} \nu_{12} \nu_{21}}; \quad G_{xy} = q_{12} G_{12} \end{aligned} \quad (31)$$

For a simplified analysis, we consider large numbers of c_m and c_f corresponding to a near-vertical drop in the stress–strain response of one finite element. Since the element size, l_0 , is fixed so that a vertical stress drop would represent fracture energy of the material, the foregoing strength limits imply the fracture energy for the opening mode of cracks in the forward direction y, i.e.,

$$\Gamma_f = l_0 \frac{f_{yt}^2}{2E_{xx}} \quad (32)$$

The fracture energy of the sideways crack cannot be defined precisely, as a constant. Although the shear failure of this crack is driven mainly by τ_{xy} , it is also influenced σ_y which is nonzero and varies along the crack. As a reasonable approximation, we assume:

$$\Gamma_s = l_0 \max \left(\frac{\tau_m^2}{2G_{xy}}, \frac{f_{mt}^2}{2E_{yy}} \right) \quad (33)$$

Vice versa, if the fracture energies are known (e.g., from size effect tests), the strength limits may be set as

$$f_{yt} = \sqrt{2E_{xx}\Gamma_f/l_0} \quad (34)$$

$$\tau_m = \sqrt{2G_{xy}\Gamma_s/l_0}, \quad f_{mt} = \sqrt{2E_{yy}\Gamma_s/l_0}. \quad (35)$$

5 Verification and calibration by finite element fracture analysis

The foregoing algorithm is implemented in user subroutine VUMAT in commercial FE software (ABAQUS Explicit). Spurious mesh sensitivity and ill-conditioning due to strain-softening are avoided by using the crack band model (Bažant 1984; Bažant and Planas 1997). For best accuracy, the element size is kept constant, the same for every structure size D . The FE mesh consists of linear quadrilateral plane stress elements.

The numerical model for sideways cracking is calibrated by Nairn's (1988a) double-edge notched (DEN) test specimens. Nairn tested 16 UD graphite-epoxy

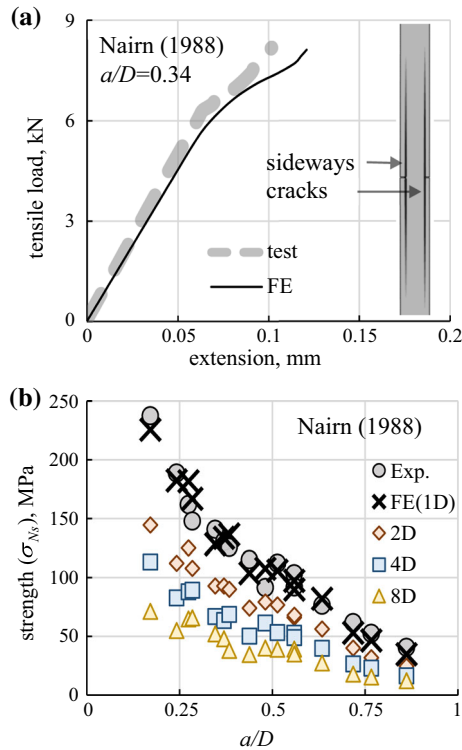


Fig. 3 Sideways crack **a** typical load vs extension curve of a UD fracture specimen, **b** calibration by different initial crack configurations Nairn (1988a)

composite specimens with fiber volume fraction 0.55, having different initial crack length configurations. The material properties are obtained using his test results and Eq. (2). The FE prediction of the orthogonal sideways cracking is shown in Fig. 3a, b.

Since, according to our simplified analysis, the energy release rate for orthogonal sideways crack propagation is constant, the structure is at the limit of stable crack growth. So the cracks could propagate in one or both sideways directions. But computations show a small equilibrium load decrease at the start of sideways cracks, which implies instability for a small crack advance, and causes the sideways crack to grow on only one side, as confirmed by computations. The sideways strength, which may also be called the splitting strength, is defined by the maximum stress at which the sideways crack starts.

Orthogonal sideways crack propagation is simulated for geometrically scaled specimens of four sizes $D = 6.35, 12.7, 25.4, 50.8$ mm, with relative values $D, 2D, 4D, 8D$. The calculated nominal strengths are

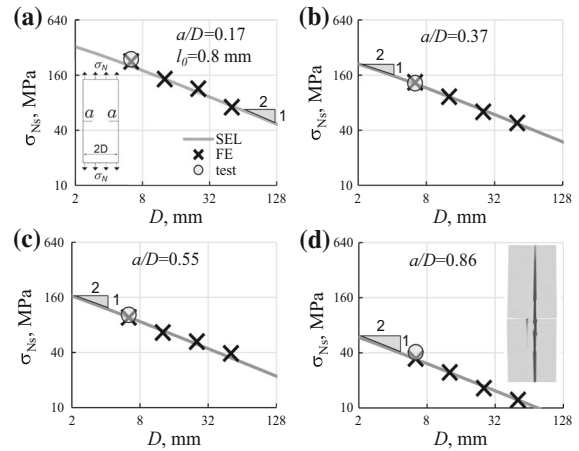


Fig. 4 Size effect predictions of sideways strength for different a/D values. The experimental data points are shown in circles Nairn (1988a)

plotted in Fig. 4a–d. The simulations are repeated for each relative transverse notch (or crack) lengths a/D . As seen in the Figs. 3 and 4, σ_{Ns} decreases as the initial (or forward) crack length increases.

FE calculation of the energy release rate $\mathcal{G}$ makes it possible to calibrate the value of the slope k of the energy release zone boundary for the present specimen geometry. Such calculations were made for both the isotropic ($V_f = 0$) and unidirectional transversely isotropic ($V_f = 0.5$) specimens. The $\mathcal{G}$ values were calculated by the J -integrals for forward crack propagation, using FE. The slopes k were then calculated from Eq. (4) and plotted in Fig. 5a. As seen, k is not constant but increases with the degree of anisotropy and with the relative initial crack length a/D .

The calculated k -values have then been used to determine the critical energy release rates $\mathcal{G}$ for the sideways propagation, Eq. (12). Their comparison is presented, for both isotropic and orthotropic specimens, in Fig. 5b, c. It is not surprising that the critical $\mathcal{G}$ depends on the relative initial crack length a/d (and in general surely also on the structure shape). An increase in a/D leads to a lower critical ratio Γ_s/Γ_f , which implies that the sideways cracks propagate more easily for larger relative initial crack lengths a/D .

It is noteworthy that the energy release rate of sideways propagation of initial crack, computed by FE, is independent of k . This confirms Eq. (9). Hence the k -values cannot be deduced through the fracture energies of sideways crack propagation. The energy release rates

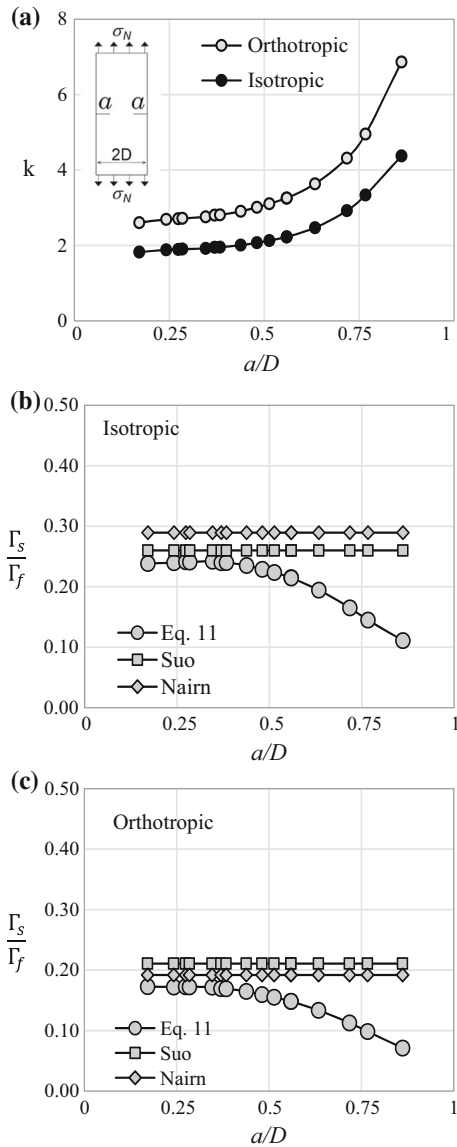


Fig. 5 The variations of **a** the slope of stress diffusion line (k) for both isotropic and orthotropic (UD) cases, and the critical ratio of fracture energies for **b** isotropic, **c** orthotropic cases with respect to dimensionless initial crack length (a/D), [(in comparison with approaches of Suo et al. (1991) and Nairn (1988b)]

of sideways propagation is also independent of the relative sideways crack length, h/a . To document it, see Fig. 6a, b, showing the J -integral for various values of a/D and h/D values. It is shown that the energy release rates are almost constant as h/a is increased. Moreover, the sideways fracture energy G_s is proportional to the length, a , of the initial forward crack.

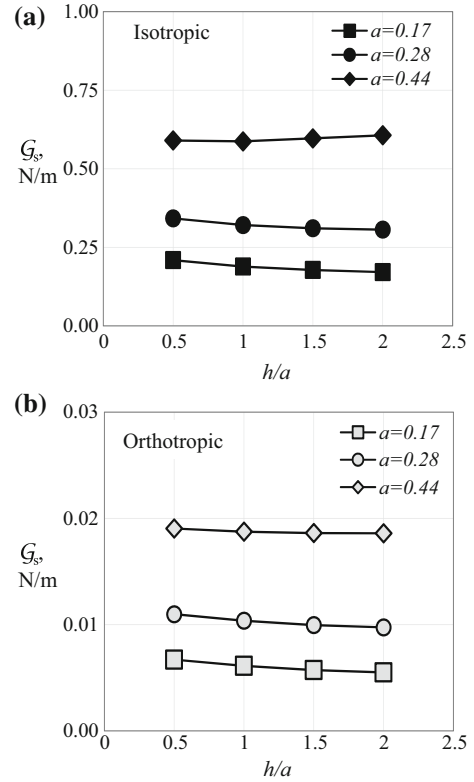


Fig. 6 The variation of energy release rates of sideways cracks propagations for **a** isotropic and **b** orthotropic cases with respect to ratio of h/a for three different initial forward crack length cases

The energy released by crack propagation into the tip of an oblique crack depends upon the initial crack length (a), the deflected crack length (h), the inclination angle (θ), and the k values. Keeping a and h constant, we find the energy release to vary with the inclination angle as seen in Fig. 7. Figure 7a shows that the energy release rate increases as the inclination angle becomes smaller. Eq. (24) can then be used to determine the corresponding k values. For the isotropic case, the smallest k is obtained at $\theta = 30^\circ$ ($k = 3.85$). Figure 7b shows the energy release rate variation for the case of cross-ply fibers oriented at 45° . For this special case, the slope of the stress 'diffusion' lines has its smallest value at $\theta = 45^\circ$ ($k = 1.79$). Note that the slope of stress diffusion lines (k) becomes smaller for the case of $\pm 45^\circ$ plies, which is opposite to the case of forward propagation in unidirectional composites (Fig. 5a).

As an example of the proposed FE model, we use the test data by Andersons et al. (2010). We calibrate the model using test data for different ply angles in a

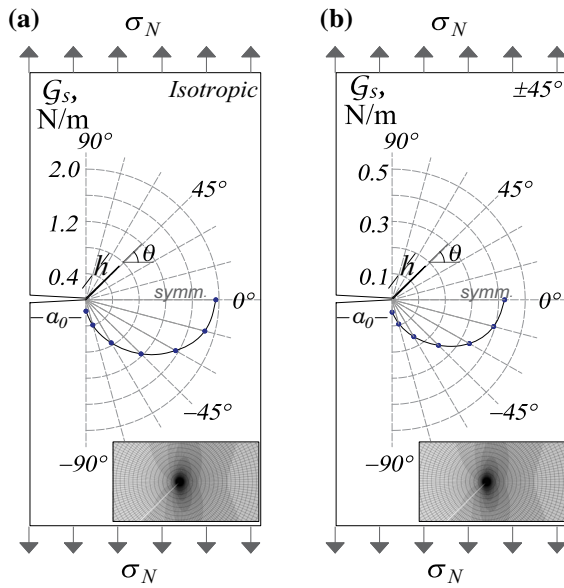


Fig. 7 The variation of energy release rates of inclined crack propagations for **a** isotropic and **b** angle-ply ($\pm 45^\circ$) cases with respect to inclination angles (θ)

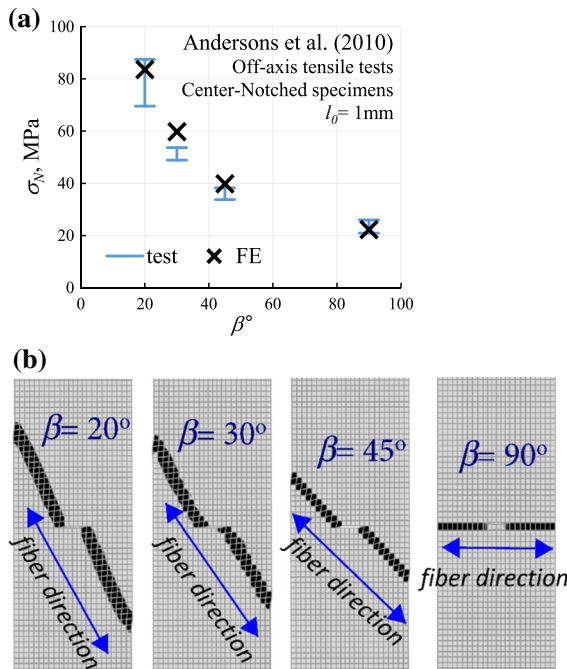


Fig. 8 **a** Off-axis tests of fracture specimen (center edge notched) and **b** corresponding crack propagation according to FE fit (Andersons et al. 2010)

uni-directional glass-epoxy composite under uniaxial tension. The strength values of the center-notched off-axis specimens for various fiber angles are shown in Fig. 8a. The fiber inclination has caused the fracture to emanate from the center notch tips (Fig. 8b).

6 Conclusions

1. While the sideways crack branching in fiber composites, whether orthogonal or inclined, cannot be predicted by the cohesive crack model, it can be by the crack band model—thanks to the damage in the FPZ being modeled as tensorial rather than scalar or vectorial.
2. In unidirectional (UD), as well as strongly orthotropic cross-ply (sCP) composites, the orthogonally branched sideways cracks exhibit a LEFM size effect, i.e., the structural strength is $\propto D^{-1/2}$.
3. The orthogonal sideways cracks (in UD, sCP) grow at constant load and constant energy release rate, and thus are, under load control, at the limit of stable equilibrium, i.e., grow in neutral equilibrium. They grow simultaneously and symmetrically to both sides of the initial forward crack.
4. The inclined sideways cracks grow at a decreasing load and thus are unstable under load control. They propagate dynamically as soon as the full FPZ forms at the tip of the initial transverse crack or notch. Two-sided propagation is unstable under load control, and thus they can propagate to only one side of the initial transverse crack.
5. Like the forward crack, the sideways cracks, whether inclined or orthogonal, follow Bažant's quasibrittle energetic size effect law. The size effect is stronger in sideways propagation than in the forward growth.
6. The orthogonal sideways cracks (in UD, sCP) can propagate from the initial notch or crack tip only if the sideways-to-forward ratio of fracture energies, Γ_s/Γ_f , is small enough, about 10^{-1} by order of magnitude for orthogonal fracture propagation. Ratio Γ_s/Γ_f increases with decreasing inclination of fibers (or sideways crack) from the initial crack.
7. The sideways fracture energy Γ_s and the corresponding fracture process zone (FPZ) size (or material characteristic length) can be identified from size effect tests with sideways crack growth, as an extension of the size effect method for test-

ing the fracture energy and FPZ size of concrete and other quasibrittle materials.

8. The load required to run the sideways crack decreases with the length of the initial transverse crack or notch. But that length cannot be shorter than the FPZ length.
9. The slopes k of the “stress diffusion lines” bounded by the approximate “stress relief zones” can be effectively calibrated by the J -integral computed by finite elements. Slope k depends on structure geometry and material properties.
10. The foregoing conclusions are verified by the finite element crack band model. Suppressing the spurious mesh sensitivity and ill-conditioning of strain softening, this model captures the effect of both the length and width of the fracture process zone.
11. After a recent demonstration of the existence of gradual postpeak softening in composites and of its agreement with fracture energy (Salviato et al. 2016), the present explanation of sideways crack propagation removes the last argument of some practitioners against the applicability of fracture mechanics and non-statistical size effect to fiber composites.

7 Review of broadly relevant literature

The problem of crack path deflection and branching appears not only for fiber composites but also more broadly for various other materials, whose studies are partly relevant to the present one. One question that has been studied is the stress-based criterion for crack deflection. The first sideways cracks study, based on the local crack-tip stress fields, belongs to Cook et al. (1964). They proposed a stress criterion for the crack deflection into a weak plane perpendicular to primary crack. A stress based criterion was later studied in Goree and Gross (1980), Buczek and Herakovich (1985) and Gupta et al. (1992). Explicit solutions of stress fields near a crack tip in unidirectional composite was given by Sih et al. (1965).

Energy-based criteria were also studied. Sih et al. (1975) proposed for fiber composites a criterion of critical strain energy density, in which they used both the homogeneous and discrete anisotropic elasticity models. Cotterell and Rice (1980) considered the stress intensity factors for different kinking crack angles and related them to the fracture energy release rates for

various possible crack directions emanating from the primary crack tip. Many authors followed their idea (Ming-Yuan and Hutchinson 1989; Martnez and Gupta 1994; Ming-Che and Erdogan 1983; Thouless et al. 1989).

Ming-Yuan and Hutchinson (1989), He and Hutchinson (1989), proposed for crack kinking out of a bi-material interface the criterion $\mathcal{G}_s/\Gamma'_s > \mathcal{G}_f/\Gamma'_f$. For the case of an interface between two identical materials they proposed reducing the kinked energy release rate in the criterion to 25% of the energy for forward propagation. He and Hutchinson's energetic criterion for crack kinking was extended to homogeneous anisotropic elastic solids by Suo et al. (1991). Suo et al. rescaled the elastic compliance properties of unidirectional composite for plane strain (or plane stress) using the nondimensional elastic parameters $\lambda = E_2/E_1$ and, $\rho = (E_1 E_2)^{0.5}/(2G_{12}) - (\nu_{12}\nu_{21})^{0.5}$, where subscripts 1 and 2 refer to the principal directions of material orthotropy. Considering a pure remote mode I loading and sideways and forward propagation options, Suo et al. proposed, approximately, $0.26\lambda^{-1/4}$ as the critical ratio of the energy release rates. On the other hand, Nairn (1988b), using the “shear-lag” model (Hedgepeth 1961), estimated this ratio as $0.5(G_{12}/E_1)^{1/2}$. A more recent and accurate work for the finite solid geometries was implemented by Xie et al. (2005).

Parmigiani and Thouless (2006) formulated a cohesive zone model combining the strength-based and energy-based criteria, in which cohesive zones of various kinking directions are assumed to co-exist. The kinked direction depended on both the fracture toughness and cohesive strength of various potential directions.

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CORRECTION

Correction to: Size effect on branched sideways cracks in orthotropic fiber composites

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1. The second line of Eq. 24 should have $\sigma_N^2/2\hat{E}$ in front.
2. Eq. 28b should read:

$$Q = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{12} \end{bmatrix}$$

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Fishnet Statistics for the Strength Distribution
of Nacre-like Biomimetic and Architected Quasibrittle Materials

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SUBMITTED TO THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Theoretical and Applied Mechanics

By

Wen Luo

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ABSTRACT

Fishnet Statistics for the Strength Distribution
of Nacre-like Biomimetic and Architected Quasibrittle Materials

Wen Luo

The staggered (or imbricated) lamellar “brick-and-mortar” nanostructure of nacre endows nacre with strength and fracture toughness values exceeding by an order of magnitude those of the constituents, and inspires the advent of new robust biomimetic materials. While many deterministic studies clarified these advantageous features in the mean sense, a closed-form statistical model is indispensable for determining the tail probability of failure in the range of 1 in a million, which is what is demanded for most engineering applications. So far, there is no undisputed theoretical basis for the probability distribution of strength of nacre. This study presents a numerical and theoretical study of the strength distribution and of the corresponding statistical size effect. After reasonable simplifications of the shear bonds, an axially loaded lamellar shell is statistically modeled as a square fishnet pulled diagonally. A finite element (FE) model is developed in Matlab (which has later been ported to C++) and used in Monte Carlo simulations of strength. An analytical model for failure probability of the fishnet is developed and

matched to the computed statistical histograms of strength for various sizes. It appears that, with increasing size, the pdf of strength slowly transitions from Gaussian to Weibull distribution.

Then, the fishnet statistical model is extended to capture the failure risk of nacre-like biomimetic materials with softening quasibrittle links. The probabilistic analysis is enabled by assuming the postpeak softening of a link to occur as a series of finite drops of stress and stiffness. The maximum load of the structure is approximated by the strength of the k th weakest link (order statistics), where k is in itself a random variable captured by the geometric-Poisson distribution. The analytically obtained probabilities are compared and verified by histograms of strength data obtained by millions of Monte Carlo simulations for each of many nacreous bodies with different link softening steepness and with various overall shapes. Finally, we show that the initially introduced series expansion and the newer formulation based on order statistics are, in the fishnet model, essentially equivalent. From that, we develop a neat general form of the fishnet statistics. Using this general form of strength distribution, we apply it to architected nanomaterials such as the printed octet-truss carbon nanolattices, as well as to quasibrittle particulate composites such as concrete. We show that the three-dimensional assembly of fishnets further enhances the tail strength at the 10^{-6} probability quantile, compared to two-dimensional (2D) fishnet statistics.

Part of this work has been presented in the following publications:

- (1) Luo, W., Bazant, Z.P., 2020. General Fishnet Statistics of Strength: Nacreous, Biomimetic, Concrete, Octet-truss, and Other Architected or Quasibrittle Materials. *Journal of Applied Mechanics* , 87(3).

- (2) Luo, W. and Bažant, Z.P., 2019. Fishnet statistical size effect on strength of materials with nacreous microstructure. *Journal of Applied Mechanics*, 86(8), p.081006.
- (3) Luo, W. and Bažant, Z.P., 2018. Fishnet model with order statistics for tail probability of failure of nacreous biomimetic materials with softening interlaminar links. *Journal of the Mechanics and Physics of Solids*, 121, pp.281-295
- (4) Luo, W. and Bažant, Z.P., 2017. Fishnet statistics for probabilistic strength and scaling of nacreous imbricated lamellar materials. *Journal of the Mechanics and Physics of Solids*, 109, pp.264-287.
- (5) Luo, W., Le, JL, and Bažant, Z.P., 2020. Coefficient of Variation of Shear Strength of RC Beams and Size Effect. *Journal of Engineering Mechanics*. (submitted)

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CHAPTER 1

Fishnet Statistics for the Strength Distribution of Brittle Nacreous Materials

1.1. Introduction

Nacre-like imbricated (staggered) structures (see Fig. 1.1) are commonly seen in both nature and man-made materials, and are of biomimetic interest. The mechanical merits of their characteristic 'brick-and-mortar' structure such as high fracture toughness and insensitivity of flaws have been well established. For example, Wang et al. [2] showed that the nano-asperities on the platelets of hydrated nacles are the main cause of high resistance during the sliding of platelets and the robust mechanical behavior on the macro-scale. Gao et al. [3] found that the stress concentrations at flaws and the damage in nacre are significantly mitigated on the nano-scale. Shao et al. [4] demonstrated that the fracture toughness of nacre is greatly enhanced by the crack bridging effects in the staggered microstructure.

On the other hand, studies on statistical aspects of the 'brick-and-mortar' structures are rare, especially in regard to the type and tail of the probability distribution of strength, and no undisputed theoretical basis has been established so far. For general quasi-brittle structures their failure probabilities are usually formulated by the weakest-link model and its well known limiting Weibull distribution is widely used. When the representative

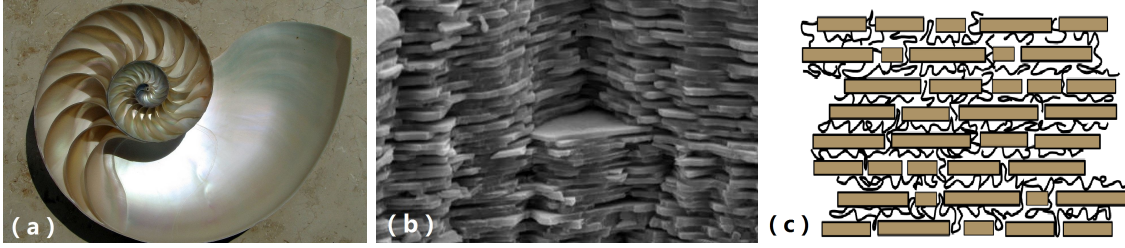


Figure 1.1. a) Picture of nacre inside a nautilus shell; b) Electron microscopy image of a fractured surface of nacre; c) Schematic of the microscopic structure of nacre; All images are from Wikipedia, <https://en.wikipedia.org/wiki/Nacre>.

volume elements (RVE) are not negligible compared to the size of the whole structure, the finite weakest-link model [5, 6, 7, 8, 9] is sometimes considered.

Meanwhile, limitations of classical Weibull-type approach have long been realized [9, 10, 11, 12], and applicability of the weakest-link chain model has been questioned, e.g., by Bertalan [13]. According to this model, the maximum load (which is, under load control, the failure load) occurs as soon as one representative volume of material (RVE) fails. This neglects the parallel couplings due to lateral interactions and the stress redistribution causing the release of stored energy due to crack growth. Additionally, Monte Carlo simulations show [13] the failure probability, P_f , to deviate from the Weibull distribution for small nominal stress. Yet, in structural design, the lower tail of P_f that is most important.

Unlike the weakest link model, fiber bundle model takes account of the effect of stress redistribution by considering various load sharing rules among fibers. The fiber bundle model with the rule of equal load sharing was studied rigorously by Daniels [14], who showed that its failure probability converges to the normal (or Gaussian) distribution as the number of fibers tends to infinity.

Later, a chain-of-bundles model was formulated Harlow and Phoenix [15, 16] to obtain the failure probability of fibrous composites under uniaxial tension. They assumed that cross sections of the composite with certain finite spacing along the loading direction are statistically independent. So these cross sections are related to each other through series coupling in a chain and the individual cross sections are treated as bundles with some hypothetical local load-sharing rule. A significant deviation of P_f from Weibull distribution is predicted. Due to overwhelming obstacles to large scale simulation at that time (1978), they analyzed only bundles with less than 10 fibers, and no verification through Monte Carlo simulations was given.

The new idea of this paper is to model the essence of failure behavior of the 'brick-and-mortar' structure of nacre under uniaxial tension by a brittle square fishnet pulled along one of the diagonals. Similar to the weakest link model, we obtain the failure probability of fishnet by calculating its counterpart—the survival probability, $1 - P_f$. We consider not only the probability that every link would survive, but also the probabilities that of the fishnet remaining safe after some links have failed. These additional probabilities are then identified to be the cause of deviation of P_f from the weakest-link model, producing Weibull distribution at the lower tail of P_f .

The results of Monte Carlo simulations offer a concrete verification to the analytical derivation fishnet statistics theory and provide fruitful insights. In previous studies of electrical circuits, Monte Carlo simulations have mostly followed the random fuse model (RFM) [17, 13], in which the elastic structure is simplified as a lattice of resistors with random burnout thresholds. The RFM simulates the gradual failure of resistor networks under increasing voltage, which is similar to the failure process of brittle elastic materials

under uniaxial loading. Here, however, the emphasis is on the finite element method (FEM) with random element (or link) strength because it reflects better the mechanical failure of fishnet. To obtain a sufficient number of samples ($> 10^5$) within a not too long run time, a simple finite element code is here developed, in Matlab. A unprecedented number, 10^6 , is obtained within a few days of computation. With such a large number (10^6) of samples, the resulting histograms become visually identical to the theoretical cumulative probability density function (cdf) of failure probability P_f .

In nature, the mechanical responses of hydrated and dry nacles are quite different: Hydrated nacles like those in the shell of pearl oysters and abalones exhibit strong non-linearity and a yielding plateau under tension, compression and shear [2]. However, dry nacles are mostly brittle and the ductility is vanishingly small under uniaxial tension as well as compression. In this chapter, only a dry (and brittle) nacre is considered. Its behavior up to brittle failure may be treated as linearly elastic.

1.2. Simplification of the Staggered Structure for Statistical Purposes

To simplify and capture the essence of load transmission in the staggered imbricated structure of nacreous materials, we ignore the relatively weak tensile bonds providing the longitudinal connections of the adjacent lamellae (or platelets) and assume that most of the strength and stiffness is provided by the shear bonds between parallel lamellae (Fig. 1.2(a)). Next, if we focus on the longitudinal strength of nacre under tensile load parallel to the lamellae, the bending stiffness of the thin polymer layers providing transverse connections of the lamellae do not play a significant role. So, for failure under longitudinal tension, we may imagine the lamellae to be replaced by their centroids, shown

as the black dots in Fig. Fig. 1.2 (a)), with the centroids, or nodes, connected by bars, or links (dashed lines in Fig. 1.2 (a)), that transmit only axial forces.

Thus we obtain a system of nodes and links shown in Fig. 1.2 (b), equivalent to a truss. Obviously, this truss looks like a fishnet. For example, in Fig.1.2(a) the lamellae (or platelet) A,B and C are mapped to nodes A,B and C in Fig. 1.2(b) and the shear bonds AB and AC in Fig.1.2(a) are replaced by truss elements AB and AC in Fig.1.2(b), respectively. This is equivalent to pulling an elastic fishnet pulled along the longitudinal diagonals of the parallelogram cells.

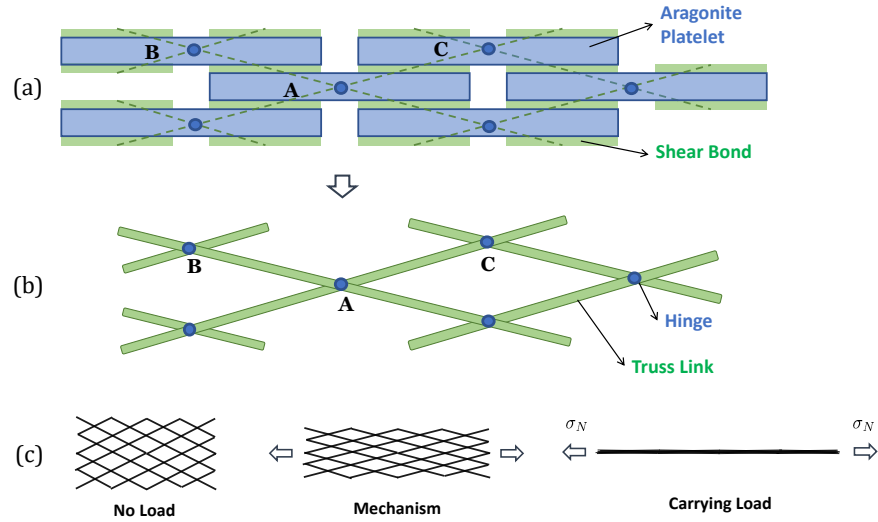


Figure 1.2. a) Schematic illustration of original nacre micro-structure; b) Equivalent fishnet structure; c) Deforming mechanism of fishnet.

The truss is not statically determinate. Each of the parallelograms in the fishnet forms a mechanism which will, under longitudinal tension, shrink immediately at the start of loading into a set of coinciding lines (or fibers) with staggered connections among them (Fig. 1.2(c)). Evidently, this is a simplification, which nevertheless retains the essence of

transverse interactions between the adjacent rows of lamellae. For the purpose of illustration, the stress field in the fishnet will always be displayed in the original (Lagrangian) coordinates, i.e., before the collapse of fishnet into coinciding lines.

The stress redistribution captured by the fishnet model plays a key role in the failure probability of fishnet. In this regard, the fishnet is different from each of the other two basic models for the statistics of failure, namely the weakest link chain model and the fiber bundle model.

1.3. Series Expansion of Survival Probability

Consider a general 2D fishnet structure consisting of N links whose failure probabilities σ_i are i.i.d. random variables. In contrast to a chain, more than one link may fail before the fishnet fails. Therefore, we define the random variable $X(\sigma)$ = number of links that have failed after stress σ is imposed at the boundary.

Now consider a fishnet with $N = m \times n$ links, i.e., with m rows and n columns, pulled in the direction of rows. Since events $\{X(\sigma) = k\}$ are mutually exclusive, the whole probability space can be partitioned by these exclusive events into $N + 1$ subsets: $\{X(\sigma) = k\}, k = 0, 1, \dots, N$. Different from a chain or a fiber bundle, the failure of fishnet, however, cannot be characterized as either $\{X(\sigma) = 1\}$, $\{X(\sigma) = m\}$ or $\{X(\sigma) = N\}$ because its failure also depends on the position of failed links, i.e. the whole structure fails when the n_f failed links form section cutting through all the rows (vertically in Fig. 1.2(c)). So the condition $n_f \geq m$ is necessary for failure but not sufficient. The failed links may be scattered in the fishnet discontinuously. Based on these considerations, the

probability of fishnet survival may be expressed as:

$$(1.1) \quad 1 - P_f(\sigma) = P_{S_0}(\sigma) + P_{S_1}(\sigma) + P_{S_2}(\sigma) + \dots$$

where $P_f(\sigma) = \mathbb{P}(\sigma_{max} \leq \sigma)$, σ_{max} = nominal strength of structure; and $P_{S_k}(\sigma) = \mathbb{P}(X(\sigma) = k)$, $k = 0, 1, 2, \dots, N$. Since the event $\{X(\sigma) = 0\}$ means that no link can fail under the load σ , we have

$$(1.2) \quad P_{S_0}(\sigma) = [1 - P_1(\sigma)]^n \text{ where } P_1(\sigma) = \mathbb{P}(\sigma_i \leq \sigma).$$

If we consider only the first term P_{S_0} on the right-hand side of Eq.(1.1), the model is equivalent to the classical weakest-link model,i.e.,

$$(1.3) \quad 1 - P_f(\sigma) = P_{S_0}(\sigma) = [1 - P_1(\sigma)]^n$$

Physically, this means rearranging all the links into a chain with only 1 row and N columns. Because the other terms in the sum of Eq. (1.1) are excluded, the weakest-link model gives a strict upper bound of failure probability of the fishnet.

1.4. Two-Term Fishnet Statistics

To get a better approximation, let us now truncate the sum in Eq. (1.1) after the second term P_{S_1} . Let σ_N be the nominal stress representing the average stress in the cross section of the fishnet, and σ_i be the stress at the i^{th} link when the nominal stress is $\sigma_N = \sigma$. Also denote $\eta_i^{(1)} = \sigma_i/\sigma_N$, which is the ratio of stress change in link i when some links failed. Based on our previous analysis of stress redistribution, we assume $\eta_j^{(1)}$ not to depend on the position of the failed link. For the two-term model, the survival

probability is

$$(1.4) \quad 1 - P_f(\sigma) = P_{S_0}(\sigma) + P_{S_1}(\sigma),$$

where P_{S_0} is given by Eq.(1.2). and $P_{S_1}(\sigma)$ is the probability that, under load $\sigma_N = \sigma$, there is one and only one link that has failed. The expression for P_{S_1} further reduces to

$$(1.5) \quad P_{S_1}(\sigma) = NP_1(\sigma) \cdot \prod_{i=1}^{N-1} [1 - P_1(\lambda_i \sigma)], \quad \lambda_i = \begin{cases} \eta_i, & \eta_i \geq 1 \\ 1, & \eta_i < 1 \end{cases}$$

This equation has been derived without using any knowledge of stress redistribution or any assumptions about load sharing rules. To get a simpler estimate of P_{S_1} , we can make two reasonable simplifications of stress redistribution:

1. The stress redistributes only in the vicinity of a failed link. So we assume that links farther away are undisturbed, with all $\eta_i^{(1)} = 1$, and assume the stress to change only a few links, whose number is denoted as ν_1 (not including the failed link);
2. To keep P_f either as an upper bound or an optimum prediction, we replace all the ν_1 redistributed stresses $\eta_i^{(1)}\sigma$ by the maximum $\eta_{max}^{(1)}\sigma$ (or by some weighted average $\eta_a^{(1)} \cdot \sigma$ among them).

Based on these two additional assumptions, P_{S_1} is further simplified to

$$(1.6) \quad \boxed{P_{S_1}(\sigma) = NP_1(\sigma)[1 - P_1(\sigma)]^{N-\nu_1-1}[1 - P_1(\eta_a^{(1)}\sigma)]^{\nu_1}}$$

As a reasonable estimate (to be checked by Monte Carlo simulations), we choose ν_1 such that all the $\eta_i^{(1)}$'s that are smaller than 1.1 would be replaced by 1, and then determine the value of $\eta_a^{(1)}$ by approximating Eq. (1.5) by Eq. (1.6), i.e.,

$$(1.7) \quad \prod_{i=1}^{N-1} [1 - P_1(\lambda_i \sigma)] \simeq [1 - P_1(\sigma)]^{N-\nu_1-1} [1 - P_1(\eta_a^{(1)} \sigma)]^{\nu_1}$$

Based on calculation, for the fishnet we are studying right now and various strength distribution $P_1(\sigma)$ of links, $\nu_1 = 4 \sim 8$ and $\eta_a^{(1)} = 1.34 \sim 1.36$. From this, we can see that $\eta_a^{(1)}$ is very close to $\eta_{max}^{(1)}$, meaning that using the maximum redistributed stress can actually give us a pretty good prediction of P_f .

To get the failure probability, we plug Eq. (1.6) into Eq. (1.4):

$$(1.8) \quad 1 - P_f(\sigma) = [1 - P_1(\sigma)]^N + NP_1(\sigma)[1 - P_1(\sigma)]^{N-\nu_1-1} [1 - P_1(\eta_a^{(1)} \sigma)]^{\nu_1}$$

$$(1.9) \quad = [1 - P_1(\sigma)]^N \left\{ 1 + \frac{NP_1(\sigma)}{1 - P_1(\sigma)} \left[\frac{1 - P_1(\eta_a^{(1)} \sigma)}{1 - P_1(\sigma)} \right]^{\nu_1} \right\}$$

Denoting

$$(1.10) \quad P_\Delta = \frac{1}{1 - P_1(\sigma)} \left[\frac{1 - P_1(\eta_a^{(1)} \sigma)}{1 - P_1(\sigma)} \right]^{\nu_1}$$

we finally get

$$(1.11) \quad \boxed{1 - P_f(\sigma) = [1 - P_1(\sigma)]^N \cdot \{1 + NP_1(\sigma)P_\Delta(\sigma, \eta_a^{(1)}, \nu_1)\}}$$

From Eq. (1.11), we can see that the main difference between weakest link model with the two-term fishnet model is the extra term:

$$(1.12) \quad NP_1(\sigma) \cdot P_\Delta$$

In the expression for P_Δ , $[(1 - P_1(\eta_a^{(1)}\sigma))/(1 - P_1(\sigma))]^{\nu_1}$ is the conditional probability of survival for the links in the region of stress redistribution with load $\eta_a^{(1)}\sigma$, given that they have been carrying load σ without failing. In other words,

$$(1.13) \quad 1 - \left[\frac{1 - P_1(\eta_a^{(1)}\sigma)}{1 - P_1(\sigma)} \right]^{\nu_1}$$

is a cumulative distribution function (cdf) of structural failure probability. Therefore, its complement has the following properties:

$$\lim_{\sigma \rightarrow 0} \left[\frac{1 - P_1(\eta_a^{(1)}\sigma)}{1 - P_1(\sigma)} \right]^{\nu_1} = 1 \text{ and } \lim_{\sigma \rightarrow \infty} \left[\frac{1 - P_1(\eta_a^{(1)}\sigma)}{1 - P_1(\sigma)} \right]^{\nu_1} = 0$$

As for the other term in P_Δ , i.e. $1/(1 - P_1(\sigma))$, it tends to 1 as σ tends to zero and blows up as σ approaches ∞ . Therefore as σ tends to 0, the term P_Δ tends to 1. Numerical study of P_Δ shows that for most cases with P_1 being Gaussian, Weibull or other typical grafted distributions, P_Δ always tends to 0 as σ approaches ∞ as long as $\eta_a^{(1)} > \eta_0$, where the constant η_0 depends on P_1 and is greater but very close to 1 (e.g $\eta_0 = 1.02$).

For structures such as bridges or aircraft, it is generally desired that the structural failure probability be less than 10^{-6} , and in that case we obviously need to pay attention only to the remote left tail of $P_f(\sigma)$, i.e., for $\sigma \rightarrow 1$. In that case we have

$$(1.14) \quad 1 - P_f(\sigma) \simeq [1 - P_1(\sigma)]^N [1 + NP_1(\sigma)]$$

To elucidate the difference from the classical weakest link model, it helps to transform the foregoing equation (Eq. (1.14)) to the Weibull scale, which has the coordinates:

$$(1.15) \quad X^* = \ln \sigma, \quad Y^* = \ln[-\ln(1 - P_f)]$$

For comparison, we plot in this scale also the weakest link model. For $\sigma \rightarrow 0$, we have $P_1 \rightarrow 0$, and so we think of using the approximation $\ln(1 + x) \simeq x$ for small x . However, with these approximations, the logarithm of Eq. (1.14) gives 0. So we must use the second order approximation $\ln(1 + x) \simeq x - x^2/2$. This leads to:

$$(1.16) \quad \ln[1 - P_f(\sigma)] \simeq N \ln[1 - P_1(\sigma)] + \ln[1 + NP_1(\sigma)]$$

$$(1.17) \quad \simeq N \left(-P_1 - \frac{P_1^2}{2} \right) + NP_1 - \frac{(NP_1)^2}{2}$$

$$(1.18) \quad = -N(N+1) \frac{P_1^2}{2}$$

The previous derivation shows that by adding the term P_{S_1} , the dominant term of $\ln[1 - P_f]$ changed from $P_1(\sigma)$ to $[P_1(\sigma)]^2$, which will be immediately seen to have a huge effect on the tail of P_f . To obtain the Weibull scale plot, we multiply both sides by -1 and take again the natural logarithm; this yields:

$$(1.19) \quad Y^* = \ln[-\ln[1 - P_f(\sigma)]] = \ln \frac{N(N+1)}{2} + 2 \ln P_1(\sigma)$$

For comparison, the Weibull plot for the weakest-link model is

$$(1.20) \quad Y^* = \ln[-\ln[1 - P_f(\sigma)]] = \ln N + \ln P_1$$

Since the independent variable of Weibull plot is $X^* = \ln \sigma$, and $\ln P_1 \simeq \ln(\sigma^{m_0}) = m_0 \ln \sigma$, the slope of the Weibull plot for the Weibull distribution is the Weibull modulus m_0 . Comparing the expressions of Eq. (1.19) and Eq. (1.20), one can find that the slope of the far left tail of Weibull plot in the two-term fishnet model is changed from m_0 into $2m_0$, which is twice as steep as in the weakest link model.

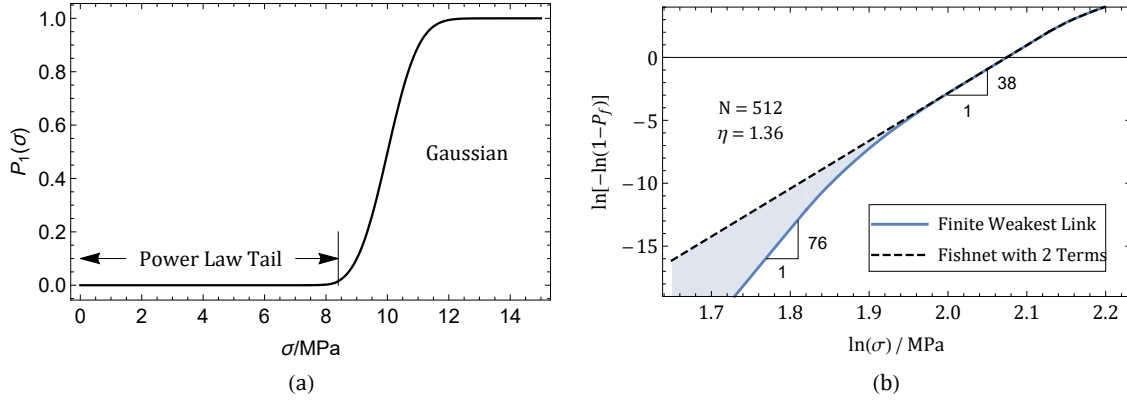


Figure 1.3. a) Cumulative distribution function (cdf) of failure for a single truss element; b) Comparison between finite weakest link model and fishnet model with 2 terms in expansion (Weibull scale).

On the other hand, in the region where P_Δ is close to 0, i.e., for large enough σ , Eq. (1.11) reduces to the weakest link model. Thus the addition of the term P_{S_1} affects only the far left tail of the cdf of structural strength and has no effect on the right tail. Physically, this is because when σ_N is very small (far below the mean), the links surrounding the failed link are generally strong enough to bear the redistributed load without breaking. But the weakest-link model characterizes this case as a failure of the whole structure, and this is why it overshoots $P_f(\sigma)$ when the load is small. However, when σ_N is large (far above the mean), the survival probability of the links surrounding

the failed one is very small, regardless of the failed link. So, on the far right tail, the two-term fishnet model is essentially the same as the weakest-link model.

Fig. 1.3(b) shows the difference between the finite weakest-link model and the two-term fishnet model for the case in which $N = 512$, $\eta_a^{(1)} = 1.36$, $\nu_1 = 6$, and the cdf $P_1(\sigma)$ of structure strength is given in Fig. 1.3(a). Note that $P_1(\sigma)$ is the Gaussian (or normal) distribution with a power law tail grafted at cumulative probability $P_1(\sigma) = 0.015$. From Fig. 1.3(b), one should note that the two-term fishnet model shows, in the Weibull plot, a smooth transition of slope from $m_0 = 38$ to $2m_0 = 76$. Note that, since the two-term fishnet statistics does not consider all the terms in Eq. (1.1), it is still an upper bound for the true $P_f(\sigma)$.

The difference between the weakest-link model and the two-term fishnet model is illustrated by the diagram in Fig.1.3(b), in which $\sigma = 6.05$ MPa. The probability calculated from the two-term fishnet model is $1.19 \cdot 10^{-6}$, whereas the value calculated from the weakest-link model is 2.95×10^{-5} . This is 24.8 times higher than the value obtained by the two-term fishnet model. This unexpected observation shows that when σ is small, the weakest-link model gives a much too high prediction of failure probability, and the two-term fishnet model yields much higher estimates of failure loads, provided that the fishnet action indeed takes place.

1.4.0.1. Location of the Slope Transition for the 2-Term Fishnet Statistics.

It is not enough to know that the left terminal slope in the Weibull plot is doubled. One needs to know also at which σ value the slope transition is centered. According to Eq. 1.1, this value is controlled by two parameters, $\eta_a^{(1)}$ and ν_1 . Here $\eta_a^{(1)}$ is the ratio of the average redistributed stress to the initial uniform stress, and ν_1 is the number of links

in the redistribution zone. To study the effects of these parameters, we fix one of them plot P_Δ for various values of the other. We consider $N = 512$ and $\nu_1 = 6$. The calculated plots of P_Δ and P_f for $\eta_a^{(1)} = 1.1, 1.3$ and 1.6 are given in Fig. 1.4.

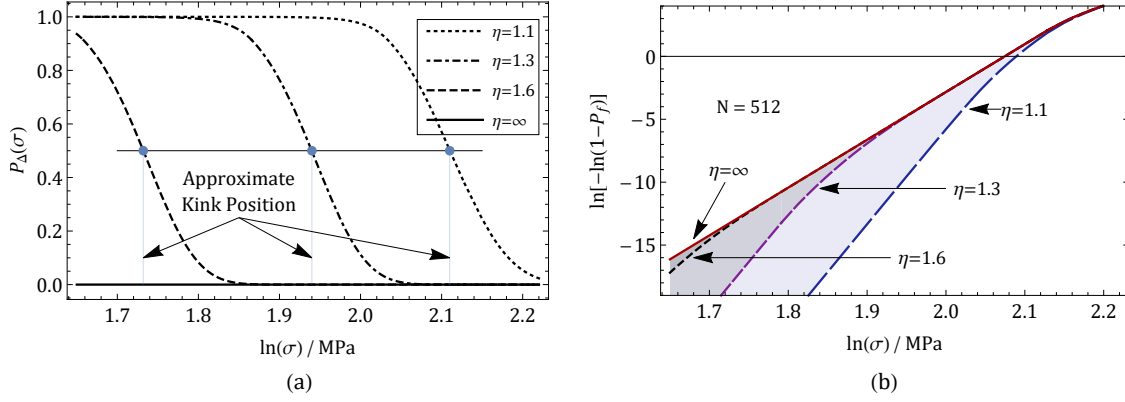


Figure 1.4. a) Approximate position of transition for $\eta_a^{(1)} = 1.1, 1.3$ and 1.6 ; b) Cumulative distribution function (cdf) of $P_f(\sigma)$ in Weibull scale with different $5\eta_a^{(1)}$.

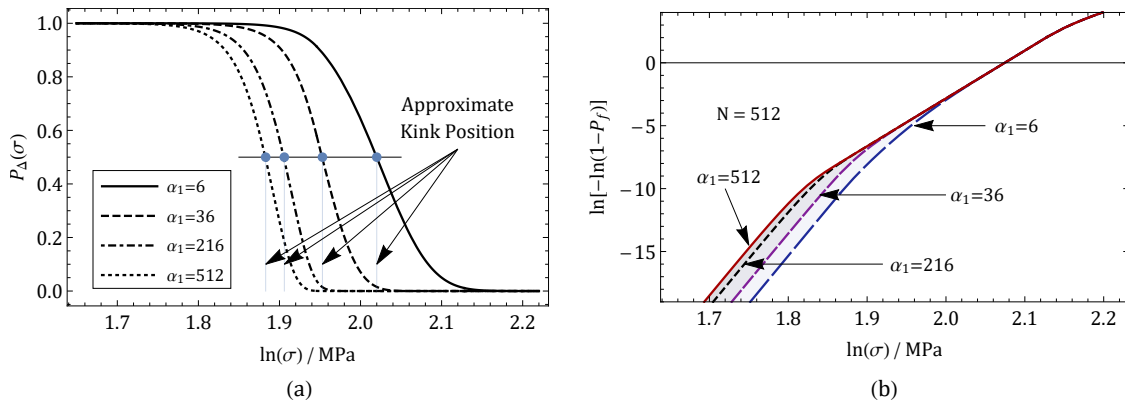


Figure 1.5. a) Approximate position of transition for $\nu_1 = 6, 36, 216$ and 512 ; b) Cumulative distribution function (cdf) of $P_f(\sigma)$ in Weibull scale with different ν_1 .

As the value of P_Δ varies from 0 to 1, the slope of cdf in Weibull scale gradually transitions from m_0 to $2m_0$. Thus the center of the transition is the value σ_T of σ at which $P_\Delta(\sigma_T) = 0.5$. So we set $P_\Delta = 0.5$ for various values of $\eta_a^{(1)}$, and compare the σ_T values. Shown in Fig. 1.4(a), they are seen match well with the kink locations in Fig. 1.4(b). Also note that the kink location shifts dramatically to the left when $\eta_a^{(1)}$ is increased from 1.1 to 1.6. Finally, as $\eta_a^{(1)} \rightarrow \infty$, the kink position approaches $\sigma_T = 0$, which means that the fishnet degenerates into the weakest-link model—as long as 1 link fails, the stress in its neighborhood becomes infinitely large, then the structure must fail right after a single link fails.

Next, we fix $\eta_a^{(1)} = 1.2$ and let $\nu_1 = 6, 36, 216$ and 512. Note that, since $N = 512$, the largest ν_1 value cannot exceed 512. Similarly, as ν_1 increases, the kink location shifts to the left, but the shift is much smaller than the change of $\eta_a^{(1)}$. Thus we conclude that the stress ratio $\eta_a^{(1)}$ has a big effect on the location of slope change from m_0 to $2m_0$, while ν_1 has a much smaller effect.

1.4.0.2. Three-Term Fishnet Statistics. After a link fails, the next one to fail will be either a neighboring link or a link located farther away. Since these two events are mutually exclusive, we may write:

$$(1.21) \quad P_{S_2} = P_{S_{21}} + P_{S_{22}},$$

where $P_{S_{21}}$ (or $P_{S_{22}}$) is the probability of $\{X = 2\}$ when the failed link is close to (or far away from) the previously failed link. Furthermore, the probability density function (pdf) characterizing $P_1(\sigma)$ is denoted as $\psi(\sigma) = dP_1(\sigma)/d\sigma$. If the second link failure after the original link failure happens in the neighborhood of the first one, we let ν_2 be the number

of links that will endure the redistributed stresses after 2 links (after the original one) fail and let the new stress ratio be $\eta^{(2)}$ (note that $\eta^{(2)} > \eta_a^{(1)} > 0$). In both foregoing cases, since there are only 2 failed links (after the original one), we still assume that the stress in the links far away from the failed ones remains undisturbed and equals the remotely applied stress σ (this ceases to be valid if number of failed links keeps increasing further). Thus we may write:

(1.22)

$$P_{S_{21}} = \binom{N}{1} \binom{\nu_1}{1} \int_0^\sigma \int_{x_1}^{\eta_b^{(1)}\sigma} \psi(x_1)\psi(x_2)dx_2dx_1 [1 - P_1(\sigma)]^{N-\nu_2-2} [1 - P_1(\eta^{(2)}\sigma)]^{\nu_2}$$

(1.23)

$$P_{S_{22}} = \binom{N}{1} \binom{N - \nu_1 - 1}{1} \times \int_0^\sigma \int_{x_1}^\sigma \psi(x_1)\psi(x_2)dx_2dx_1 [1 - P_1(\sigma)]^{N-2\nu_1-2} [1 - P_1(\eta^{(2)}\sigma)]^{2\nu_1}$$

where x_1 is and $\eta_b^{(1)}$ is a parameter calculated from $\eta_i^{(1)}$. This parameter is used by another approximate relation which is similar but different from that for $\eta_a^{(1)}$:

$$(1.25) \quad \nu_1 \int_0^\sigma \int_{x_1}^{\eta_b^{(1)}\sigma} \psi(x_1)\psi(x_2)dx_2dx_1 \simeq \sum_{j=1}^{\nu_1} \int_0^\sigma \int_{x_1}^{\eta_j^{(1)}\sigma} \psi(x_1)\psi(x_2)dx_2dx_1$$

Further note that

$$(1.26) \quad \int_0^\sigma \int_{x_1}^\sigma \psi(x_1)\psi(x_2)dx_2dx_1 = \frac{1}{2}[P_1(\sigma)]^2$$

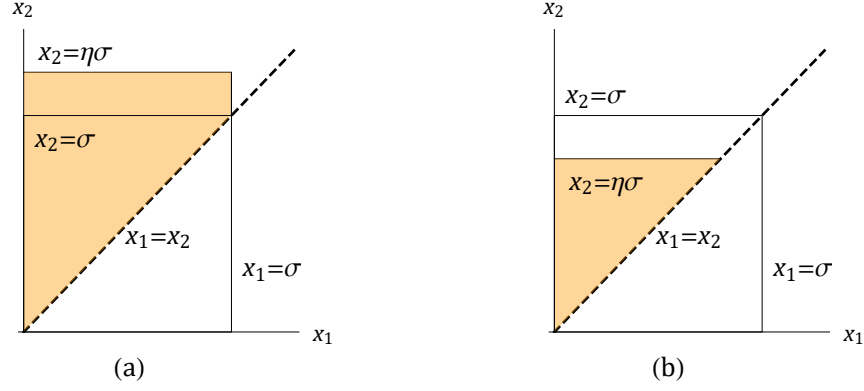


Figure 1.6. a) Schematic showing the domain of integration when $\eta \geq 1$; b) domain of integration when $\eta < 1$.

and then, according to Fig. 1.6(a),

(1.27)

$$\int_0^\sigma \int_{x_1}^{\eta_b^{(1)}\sigma} \psi(x_1)\psi(x_2)dx_2dx_1 = \int_0^\sigma \int_{x_1}^\sigma \psi(x_1)\psi(x_2)dx_2dx_1 + \int_0^\sigma \int_{x_1}^{\eta_b^{(1)}\sigma} \psi(x_1)\psi(x_2)dx_2dx_1$$

(1.28)

$$= \frac{1}{2}[P_1(\sigma)]^2 + P_1(\sigma)[P_1(\eta_b^{(1)}\sigma) - P_1(\sigma)]$$

(1.29)

$$= P_1(\sigma)P_1(\eta_b^{(1)}\sigma) - \frac{1}{2}[P_1(\sigma)]^2$$

Finally, we obtain

$$(1.30) \quad P_{S_{21}} = N\nu_1 \left\{ P_1(\sigma)P_1(\eta_b^{(1)}\sigma) - \frac{1}{2}[P_1(\sigma)]^2 \right\} \cdot [1 - P_1(\sigma)]^{N-\nu_2-2} \cdot [1 - P_1(\eta^{(2)}\sigma)]^{\nu_2}$$

$$(1.31) \quad P_{S_{22}} = \frac{1}{2}N(N - \nu_1 - 1)[P_1(\sigma)]^2 \cdot [1 - P_1(\sigma)]^{N-2\nu_1-2} \cdot [1 - P_1(\eta^{(2)}\sigma)]^{2\nu_1}$$

So, considering three terms in Eq. 1.1, we find the failure probability of fishnet to be

(1.32)

$$\begin{aligned}
 1 - P_f(\sigma) &\simeq [1 - P_1(\sigma)]^N \cdot \{1 + P_{S_1}/[1 - P_1(\sigma)]^N + P_{S_2}/[1 - P_1(\sigma)]^N\} \\
 (1.33) \quad &= [1 - P_1(\sigma)]^N \cdot \left\{ 1 + NP_1(\sigma)P_\Delta + N\nu_2 \left[P_1(\sigma)P_1(\eta_b^{(1)}\sigma) - \frac{1}{2}[P_1(\sigma)]^2 \right] P_{\Delta_{21}} \right. \\
 (1.34) \quad &\left. + \frac{1}{2}N(N - \nu_1 - 1)[P_1(\sigma)]^2 P_{\Delta_{22}} \right\},
 \end{aligned}$$

where

$$(1.35) \quad P_\Delta = \frac{[1 - P_1(\eta_a^{(1)}\sigma)]^{\nu_1}}{[1 - P_1(\sigma)]^{\nu_1+1}}, \quad P_{\Delta_{21}} = \frac{[1 - P_1(\eta^{(2)}\sigma)]^{\nu_2}}{[1 - P_1(\sigma)]^{\nu_2+2}}, \quad P_{\Delta_{22}} = \frac{[1 - P_1(\eta_a^{(1)}\sigma)]^{2\nu_1}}{[1 - P_1(\sigma)]^{2\nu_1+2}}$$

Similar to P_Δ , $\lim_{\sigma \rightarrow 0} P_{\Delta_{21}} = 1$ and $\lim_{\sigma \rightarrow 0} P_{\Delta_{22}} = 1$.

Note that the foregoing expression for P_{S_2} is not exact expression only an approximate estimate because we simplified, in two steps, the stress redistribution as a uniform scaling of stress by factors $\eta(b)^{(1)}$ and $\eta^{(2)}$. They are both greater than 1. In reality, however, some regions near the group of failed links get unloaded. These are regions in which the stress is smaller than the remotely applied stress σ . The failure probability of these regions is the integral of joint density over the domain shown in Fig. 1.6(b). This integral makes a more accurate solution very complicated, since stronger links could fail prior to the weaker ones if unloading happens in the weak links. Nevertheless, our estimation can still be quite accurate if $\eta_b^{(1)}$ is chosen properly, based on Eq. (1.25).

Since the probability P_{S_2} consists of two terms, namely $P_{S_{21}}$ and $P_{S_{22}}$, another interesting question arises: which one of these two terms is larger? Eq. (1.22) and Eq. (1.23) tell us that $P_{S_{21}}$ is proportional to $N[P_1(\sigma)]^2$, while $P_{S_{22}}$ is proportional to $N(N - \nu_1 - 1)[P_1(\sigma)]^2$,

which approximately equals $N^2[P_1(\sigma)]^2$. Therefore, $P_{S_{22}}$ is much larger than $P_{S_{21}}$. This means that the second link failure will most likely appear far away from the first one if the left tail of $P_1(\sigma)$ is relatively heavy but such that the effect of $[P_1(\sigma)]^2$ is still recognizable. This tendency to scattered damage will be discussed later.

1.4.0.3. Ramifications to Higher-Order Terms in Failure Probability. Intuitively, P_{S_k} is proportional to $[P_1(\sigma)]^k$ when σ is close to 0, and so Eq. (1.1) is similar to a Taylor series expansion of some real function in terms of $P_1(\sigma)$. The higher the order, the smaller the contribution to the total sum. If the tail of $P_1(\sigma)$ is very light, $[P_1(\sigma)]^k$ for $k \geq 3$ are usually too small to be seen on the cdf directly. So, in order to see the effect of the higher order terms, we need, once again, to study them in the Weibull scale. This scale exponentially magnifies tiny differences in the tail of a distribution.

What will happen if we include more and more terms on the right-hand side of Eq.1.1? Will they cause further slope increases in the Weibull plot? The answer is affirmative: The higher-order terms of survival probability would lead to further slope increase of cdf in the Weibull scale. From the derivation of Eq. (1.16), we know that the very reason for slope increase by the factor of 2 is that the term $P_1(\sigma)$ canceled out by Taylor expansion, allowing $[P_1(\sigma)]^2$ to dominate. So, if there is another slope increase from $2m_0$ to $3m_0$, the square terms $[P_1(\sigma)]^2$ in Eq. (1.16) must vanish after adding more terms of P_{S_k} in Eq.(1.1).

In order to show that the slope of cdf could further increase due to the presence of more terms in the expansion of survival probability we must use the exact law of stress redistribution, otherwise it could lead to false conclusions. However, the exact law of stress redistribution is almost impossible to obtain analytically. Therefore, instead of working

on a general fishnet, it is much easier if we consider a special form of it i.e. fiber bundle: by changing the aspect ratio $m : n$ gradually to $N : 1$, a fishnet with a fixed number of links reduces to a fiber bundle. Then we can use the law of stress redistribution of a fiber bundle in derivation and study the slope change of cdf and the case for general fishnet is studied numerically by Monte Carlo simulations.

Now, consider a fiber bundle consisting of N fibers and we truncate Eq.(1.1) at P_{S_2} . Consequently P_{S_0} , P_{S_1} and P_{S_2} can be expressed as: Similarly,

$$(1.36) \quad \int_0^\sigma \int_0^{N\sigma/(N-1)} \psi(x_1)\psi(x_2)dx_2dx_1 = P_1(\sigma)P_1[N\sigma/(N-1)] - \frac{1}{2}[P_1(\sigma)]^2$$

Therefore,

$$(1.37) \quad P_{S_2} = N(N-1) \left\{ P_1(\sigma)P_1[N\sigma/(N-1)] - \frac{1}{2}[P_1(\sigma)]^2 \right\} \left[1 - P_1\left(\frac{N\sigma}{N-2}\right) \right]^{N-2}$$

The probability of survival of the whole fishnet can then be expressed as:

$$(1.38) \quad 1 - P_f(\sigma) \simeq P_{S_0} + P_{S_1} + P_{S_2}$$

Then we factor out $[1 - P_1(\sigma)]^N$ and let σ tends to 0:

$$(1.39) \quad 1 - P_f(\sigma) \simeq [1 - P_1(\sigma)]^N \cdot \left\{ 1 + NP_1(\sigma) + N(N-1)P_1(\sigma)P_1[N\sigma/(N-1)] - \frac{N(N-1)}{2}[P_1(\sigma)]^2 \right\}$$

By the same procedure, we take natural log on both sides and then use Taylor expansion of $\log(1+x)$ and collect all the terms of order smaller than $[P_1(\sigma)]^3$: We can see that considering the term P_{S_2} in Eq. (1.1) could make all the first and second order terms of

$\log[1 - P_f(\sigma)]$ vanish, which means $[P_1(\sigma)]^3$ dominates when σ is small enough and the slope of cdf of fishnet failure in Weibull scale can increase by a factor of 3.

1.4.0.4. Upper and Lower Bounds of Failure Probability of Fishnet. As is already seen in previous sections, the fishnet possesses two geometrical limits i.e. chain (weakest link) and fiber bundle: the structure reduces to a chain if we set the aspect ratio to $m : n = 1 : N$ and to a fiber bundle if we set the ratio to $m : n = N : 1$ (Loads are applied horizontally). Intuitively, given the same amount of links, a fiber bundle would be much more reliable than a single chain: failure of one link will not necessarily lead to rupture of the whole bundle. Therefore, we conjecture that the upper and low bounds of fishnet's failure probability should be that of a weakest link and fiber bundle consisting of the same number (N) of links, respectively (see Fig. 1.7). Furthermore, by assuming that $P_f(\sigma)$ changes continuously as the fishnet gradually transforms from a chain to a bundle, the failure probability decreases when the aspect ratio m/n increases at a fixed value of the product $m \times n = N$.

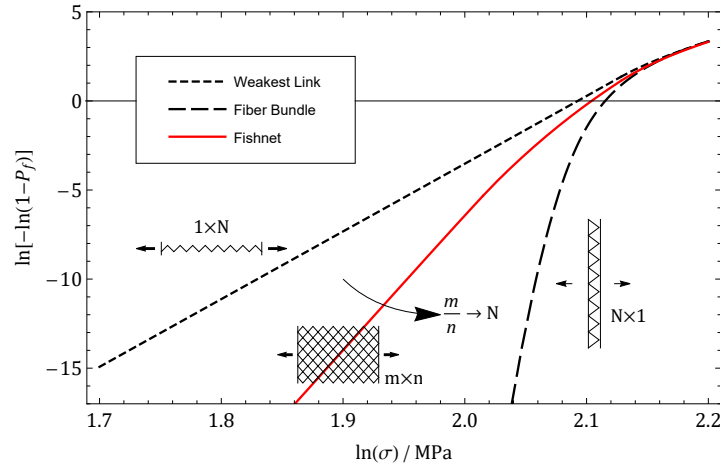


Figure 1.7. Schematic showing the change of failure probability of fishnet by changing its aspect ratio m/n gradually from $1 : N$ to $N : 1$.

1.5. Monte Carlo Simulation of Fishnet Failure

1.5.1. Setup of Finite Element Model

To verify the theory, numerical Monte Carlo experiments are conducted by finite element simulations (in Matlab). In the finite element model (FEM) model, the size of rectangular fishnet is defined by its number of rows and columns $m \times n$. Each link is treated as an elastic truss element. All the elements (or links) have the same cross section area and Young's modulus, and differ only in their strength, s_i , which is a random variable that follows the probability distribution $P_1(\sigma)$. The nodes at the left fishnet boundary are fixed in x direction but allowed to slide freely in the transverse y direction. The fishnet is loaded under displacement control. The tensile load is applied by prescribing increments of a uniform displacement u_0 at the right boundary nodes while allowing transverse free sliding.

1.5.2. Random Variable Generation

Random variables s_i are generated by the inverse-transform method. First, we use the random number generator in Matlab to generate random variables x_i in the interval $[0, 1]$, and then apply the inverse of P_1 to each x_i , i.e. $s_i = P_1^{-1}(x_i)$. Based on previous research[18], $P_1(s)$ is defined by grafting a power-law left tail onto a Gaussian distribution.

The stiffnesses of links (EA/l_e) are not treated here as random variables. This is not a major simplification. To explain why not, consider an $m \times n$ fishnet with independent identically distributed (i.i.d.) random stiffness $K_{i,j}$ for each link. Now, if the x -displacement at left boundary is zero and on the right boundary is nu_0 , then the elongation of each link is u_0 . So, if we pick any cross-section, the total force and nominal

stress are:

$$(1.40) \quad P = u_0 \sum_{r=1}^m K_r, \quad \sigma_N = \frac{u_0}{A} \left(\frac{1}{m} \sum_{r=1}^m K_r \right)$$

where K_r are the element stiffnesses along the r -th cross-section of fishnet, and A is the cross-section area of each element. This means that the random variable σ_N is proportional to the sample mean of K_r . According to the weak law of large numbers the sample mean converges to true mean with probability 1 as $m \rightarrow \infty$. Intuitively, this means that given any form of the distribution of $K_{i,j}$, the nominal stress σ_N always follows the law of a degenerate Gaussian distribution, i.e., the Dirac Delta function, centered at $\sigma_N = Ku_0/A$ if the cross-section contains a sufficiently large number of elements, where K is the mean of K_r . Therefore, no matter what the distribution is for the stiffness of a single element, its influence on the randomness of nominal strength negligible. This is also true if the fishnet is slightly damaged, because we can always find a cross-section that is sufficiently far away from the zone of failed links. Indeed, the elongation of these elements on the chosen cross-section is still approximately u_0 , which means that Eq. (1.40) still holds. Therefore, it suffices to randomize strength of each link while considering the element length, elastic modulus and cross-section area as deterministic.

1.5.3. Element Deletion and Propagation of Cracks

The time steps are indexed by the total number (k) of failed links. At the beginning of time step k , we set $u_0 = 1$ and calculate the stresses of each element $\sigma_j^{(k)}$. Then the only element, indexed by i , that is going to fail in this new time step is the one whose stress

satisfies the condition:

$$(1.41) \quad \lambda^{(k)} = s_i / \sigma_i^{(k)} = \min_j \{s_j / \sigma_j^{(k)}\}$$

where s_j is the strength of the j^{th} element. Since the constitutive law is linear elastic, $\lambda^{(k)}$ is the load multiplier such that if we reset $u_0 = \lambda^{(k)}$ and recalculate the stress field, we will have $\sigma_i^{(k)} = s_i$ and $\sigma_j^{(k)} < s_j$ for $i \neq j$. So the final stress state of each element at the current time step is calculated by $\sigma_{j_{new}}^{(k)} = \lambda \sigma_i^{(k)}$. After updating the stresses, the critical i^{th} element is deleted, and its element stiffness matrix $\mathbf{K}_{e_i}^{(k)}$ will not be assembled into the global stiffness matrix $\mathbf{K}^{(k)}$ for future loading steps. This process keeps going until the global stiffness matrix becomes singular, which means that a crack has already gone through the cross-section and the fishnet has failed.

Indexing the time steps by the number of failed links allows us to obtain not only the peak load but also the complete load-displacement curve without worrying about stability issues. In fact, the load-displacement curve of elastic fishnet shows a strong snap-back instability, as discussed paper.

1.5.4. Results of FE Simulation

Fig. 1.8(a) shows the load-displacement curve of a typical result of FE simulation on 16×32 fishnets with random strength. The distribution of random strength is a Gaussian distribution $N \sim (10, 0.4^2)$ grafted with a power law tail at $P_1(\sigma) = 0.015$. Often, peaking of the load and a strong snap-back instability is observed right already before the rupture of the first or second link. This is no surprise because, if there is no randomness in

strength, the fishnet will reach the maximum load (and fail) as soon as any one of the links succumbs.

So it is the strength randomness that makes it possible for the whole structure to survive after failure of a few links. On the other hand, in real nacreous materials, in which the shear connections between the lamellae do not act locally as pin nodes and resist bending moments, a few links (or shear connections) will probably have to break before the maximum load is reached. This would be similar type I failure in brittle heterogenous materials, in which several microcracks between grains must form in a representative volume element (RVE) before a macrocrack propagation from the RVE is initiated. Thus it might be necessary to generalize the present theory by stipulating that a certain small number of links forming an RVE must break before the maximum load is reached. This means that the true nacreous behavior would be a transition from Type I fracture to a fishnet fracture. Such a generalization, however, must be relegated to further work.

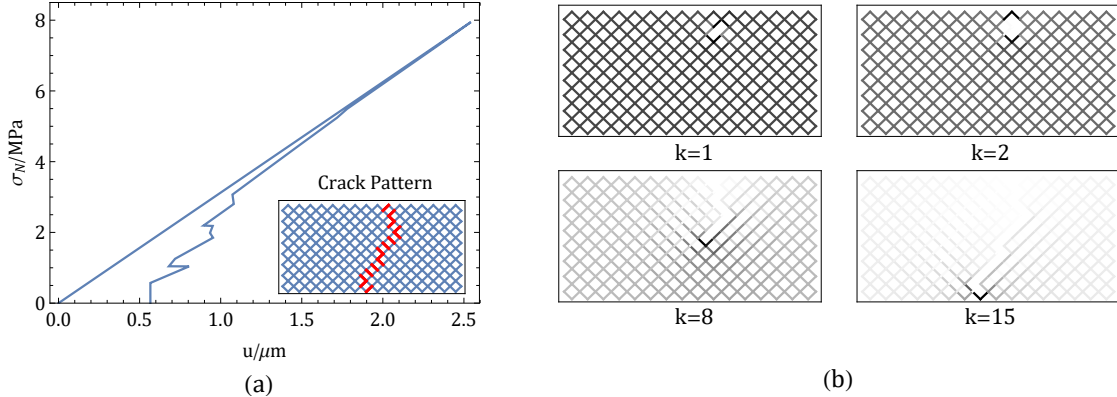


Figure 1.8. a) Load-displacement curve of a 16×32 fishnet with random strength together with its crack pattern; (b) Stress evolution as crack propagates through the fishnet (the darker the color, the larger the normalized stress σ/σ_{max}), where k is the number of failed links.

Fig. 1.8(b) illustrates the stress field of 4 typical fishnet frames ($k = 1, 2, 8, 15$) in the process of crack propagation. In this particular case, subsequent failures localize immediately after the crack initialization, and no scattered damage is observed. One can see that the crack path in this case has a strong tendency to move vertically and form a cross-section, even though the crack path is random. Another observation is that as crack propagates, sharper contrast of darkness in the figures are seen, meaning that the difference between maximum stress σ_{max} and nominal stress σ_N becomes larger and larger, making the whole structure even weaker.

1.5.5. Verification of Fishnet Theory

Millions of finite element simulations are run to verify the fishnet statistics. The criterion of convergence is set as follows: First we consider the curve of the estimated distribution obtained from many runs of fishnet sample of size n_1 in Weibull scale. Then we increase the sample size n_1 by one and consider the curve of the newly estimated distribution. The part of the curve that does not change much upon doubling the sample size is considered as converged.

To obtain an accurate estimate of the distribution, especially in its left tail, at least on million (10^6) samples have been computed for each case. Based on this scenario, the distribution estimated from the histogram of the results of all computer runs converges very well to the exact distribution in the region $Y = \ln[-\ln(1 - P_f)] > -10$ in Weibull scale. This corresponds to the interval $4.54 \times 10^{-5} \leq P_f < 1$.

1.5.5.1. Two-Term Fishnet Model. When $P_1(\sigma)$ is a Gaussian $N(10, 0.8^2)$ grafted with a power law tail at $P_f = 0.015$, its tail is relatively light and 2-term fishnet model

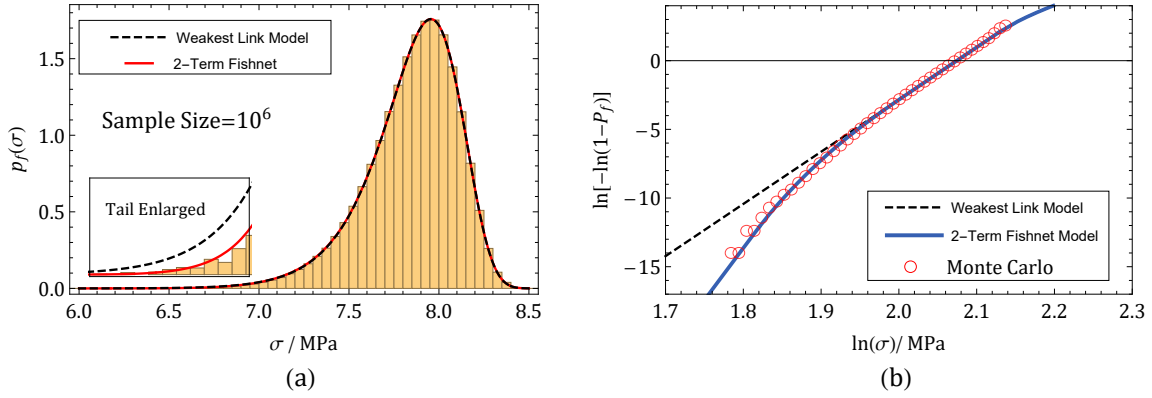


Figure 1.9. (a) Histogram of Monte Carlo Experiment of 10^6 samples (frequency is normalized to probability density) on 16×32 fishnets compared with the density functions of weakest link model and 2-term fishnet model; (b) Same data converted into cumulative probability and plotted in Weibull scale.

works very well. The expression of $P_1(\sigma)$ is chosen as

$$(1.42) \quad P_1(\sigma) = \begin{cases} 0.11338 \cdot (\sigma/10)^{38}, & \sigma \leq 8.4 \text{ MPa} \\ 0.015 + \{0.481 - 0.504 \cdot \text{Erf}[0.884(10 - x)]\}, & \sigma > 8.4 \text{ MPa} \end{cases}$$

where

$$(1.43) \quad \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

Fig. 1.9(a) shows the estimated probability density function (pdf), $p_f(\sigma) = dP_f(\sigma)/d\sigma$, obtained from the histogram of Monte Carlo simulations, and compares this pdf with the prediction of the weakest-link model (black dashed line) and of the two-term fishnet model (red continuous line). These two predictions are almost indistinguishable in the range $\sigma > 6.7 \text{ MPa}$ and match very well the Monte Carlo experiments. However the difference shows up in the tail ($\sigma \leq 6.7 \text{ MPa}$). Indeed, the two-term fishnet model fits the

Monte Carlo data much better than the classical weakest-link model. The difference is more obvious when it is plotted in Weibull scale (Fig 1.9(b)): the results of Monte Carlo simulations match the two-term fishnet model everywhere in the figure, while the weakest link model works well only in the range $\ln \sigma > 1.9$.

1.5.5.2. Three-Term Fishnet Model. The effect of P_{S_2} in Eq. (1.1) becomes more significant if the tail of $P_1(\sigma)$ gets heavier. Now consider the case where $P_1(\sigma)$ is a grafted distribution of Gaussian $N(10, 0.8^2)$ and a power law tail with the grafting point $P_f(\sigma) = 8.955\%$:

$$(1.44) \quad P_1(\sigma) = \begin{cases} 2.551 \cdot [1 - e^{-(\sigma/12)^{10}}], & \sigma \leq 8.6 \text{ MPa} \\ 0.08955 + \{0.436 - 0.474 \cdot \text{Erf}[0.884(10 - x)]\}, & \sigma > 8.6 \text{ MPa} \end{cases}$$

To see the effect on the left tail “thickness”, we calculate that, for $\sigma = 6.7 \text{ MPa}$, a change of the grafting point from $P_1(\sigma) = 0.1\%$ to $P_1(\sigma) = 9\%$ increases P_1 from 2.79×10^{-6} to 7.5×10^{-3} (note that these grafting point values are much higher than what was identified for particulate composites [9]). This increase makes the tail much “thicker” than in the previous case. Once again, for this particular strength distribution $P_1(\sigma)$ of a single element, and for the 16×32 fishnet, one million samples of Monte Carlo simulations have been obtained.

Fig. 1.10 shows the estimated density function of P_f with $P_1(\sigma)$ as mentioned above and its comparison with the predictions of the weakest link model, 2-term (Eq. (1.11)) and 3-term (Eq. (1.30)) fishnet model. We can see that the weakest link model gives a poor prediction of failure probability, while the fishnet models are much better. Note that, as more and more terms of P_{S_k} are considered in the fishnet model, the skewness

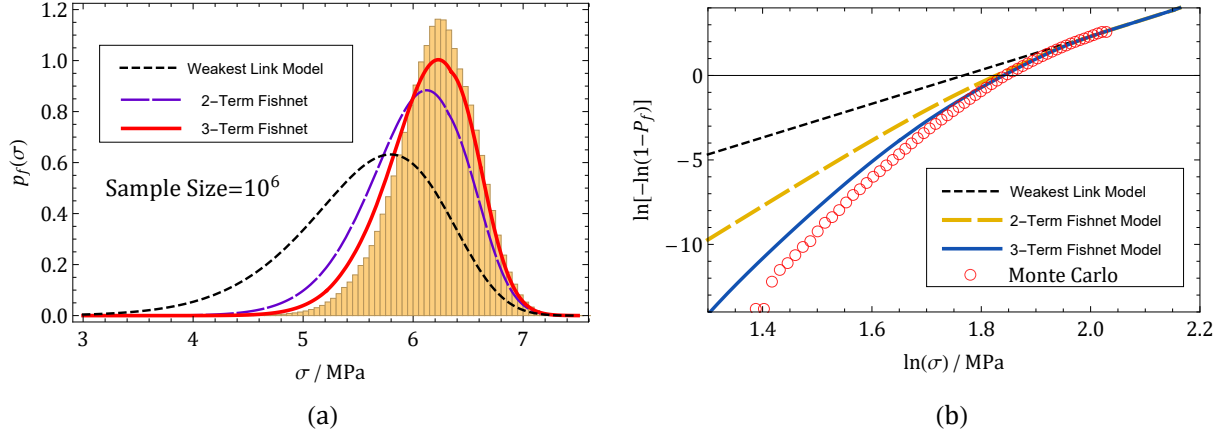


Figure 1.10. (a) Histogram of Monte Carlo Experiment of 10^6 samples (frequency is normalized to probability density) compared with the probability density functions of weakest link model, 2-term and 3-term fishnet model; (b) Same data converted into cumulative probability and plotted in Weibull scale.

of the corresponding distribution becomes smaller and smaller. From Fig. 1.10(b), all of the three predictions are upper bounds of the true failure probability. As σ decreases, the Monte Carlo results begin to deviate from the two-term fishnet model and a further slope increase is observed. As we see, the three-term fishnet model gives the best prediction. To get a still more accurate prediction, we would need to consider the next term, P_{S_3} , or even more terms, in the expansion of Eq. (1.1).

1.5.5.3. Scattered Damage vs. Localized Damage. The previous Monte Carlo simulations show that changing the tail of strength distribution of a single link could have a huge effect on the failure probability of the whole structure. To explain it intuitively, we run finite element simulations for a 32×32 square fishnet for two different strength distributions of a single link $P_1(\sigma)$ and $P_1^*(\sigma)$, where $P_1(\sigma)$ is the Gaussian distribution

grafted with a light power law tail (Eq. (1.42)) and $P_1^*(\sigma)$ is the one with a heavy Weibull tail (Eq. (1.44)).

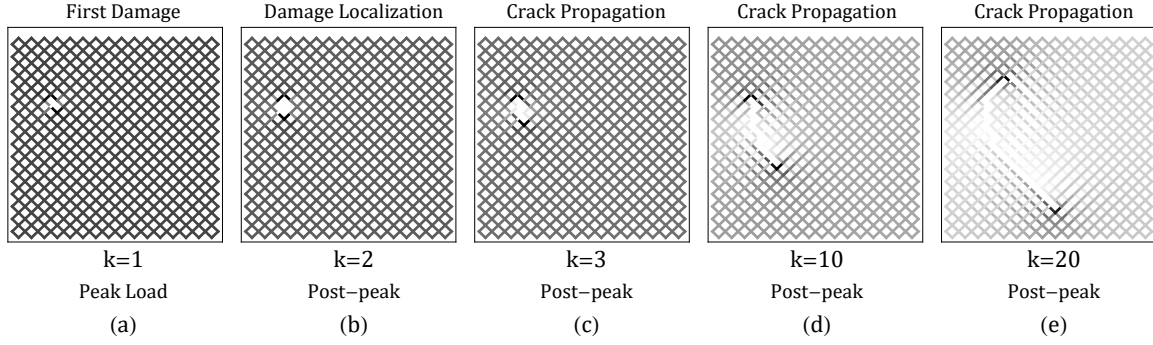


Figure 1.11. Stress evolution of a 32×32 square fishnet whose strength distribution of a single link is P_1 , which has a thin power law tail. In each figure, the stress is normalized by the maximum σ_{max}^k of that moment. Pure black corresponds to σ_{max}^k and pure white corresponds to zero stress.

First, consider the case of $P_1(\sigma)$. A typical simulation result is shown in Fig. 1.11(a) ~ (e). Due to the fact that $P_1(\sigma)$ has a very thin power law tail, the probability that there exists an extremely weak link is vanishingly small. Therefore, in most cases, the peak load is reached right before the first rupture. Then the successive failures localize and form a single crack. This means that, with a very high probability, the failure of structure is the result of rupture of a single link. Then the weakest link model predicts the P_f quite accurately, as long as σ is not too small.

The scenario gets more complicated as the tail of P_1 becomes thicker. Very likely, a few fishnet links have a very low strength and their failures will affect the fishnet strength very little. Moreover, these very weak links are scattered through the whole fishnet. As the loading of fishnet begins, these weakest links will fail successively at very small nominal stresses, while the rest of the “normal” links are still sound and safe. As long as the first

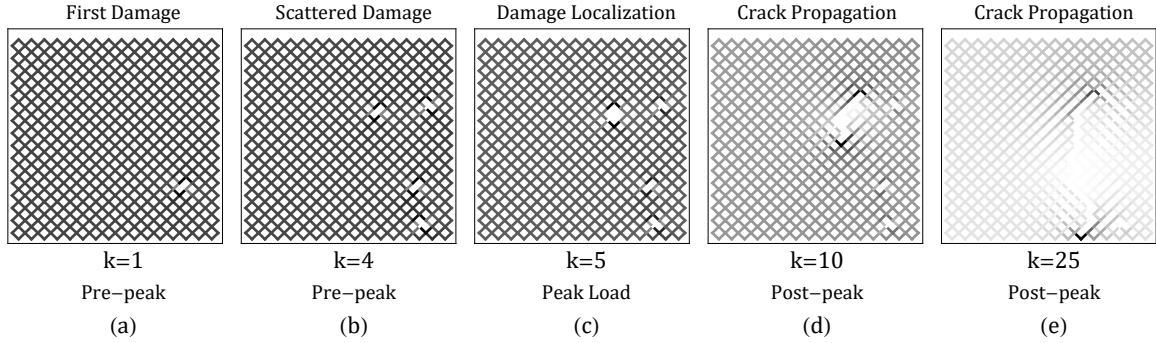


Figure 1.12. Stress evolution of a horizontally pulled 32×32 square fishnet in which the strength distribution, P_1^* , of a single link has a thick Weibull tail. In each figure, the stress is normalized by the maximum σ_{max}^k reached at that moment. Perfect black corresponds to σ_{max}^k and perfect white corresponds to zero stress.

few damages are scattered, the stress field will keep to be almost uniform except near these failed links. In this case, the failure of one or two links need not lead to the failure of the whole structure, and the weakest-link model two-term fishnet models cannot give good estimates of P_f . This is exactly why we need to consider more higher-order terms in the formula (Eq. (1.1)) of survival probability.

Fig. 1.12 (a)–(e), shows the typical pattern of stress evolution when the strength distribution of links, $P_1^*(\sigma)$, has a thicker Weibull tail. In this particular case, the damage zone does not localize until the fifth rupture, occurring just after the peak load is reached. Once the damages start to localize, the stress concentration at the crack front will only get higher and higher, making the rest of the links bear smaller and smaller stresses and next failures more likely to localize. Consequently, the nominal stress keeps decreasing. In this process, damage localization offers a positive feedback to the system: the current localization leads to a higher stress concentration, making the next failure more likely to localize. Such a process keeps going until the structure fails. Eventually the stability

of fishnet (under load control) is lost and the peak load occurs right at the moment of damage localization, which depends strongly on the “heaviness” of the tail of $P_1(\sigma)$.

1.5.5.4. Transition from Chain to Bundle (Shape Effect). Based on the fact that the peak load is most likely reached at the onset of damage localization helps to decide how many terms of P_{S_k} are needed to get an accurate enough prediction of P_f : For a given $P_1(\sigma)$ of strength distribution of links and a given size and shape of fishnet, the approximate number of terms we need to add in the fishnet model should be greater than the average number of scattered failures before localization, and the more terms get added, the more accurate the estimation will be in the tail of P_f .

It thus becomes clear that the fiber bundle must give the lower bound on the strength of all fishnets consisting of the same number of links. The fiber bundle has the largest cross section, and no stress concentration. So the stability limit of a bundle is reached the latest and each term P_{S_k} is the largest among all fishnets. Therefore we have to consider most terms P_{S_k} in the expansion of survival probability Eq. (1.1) so as to get an accurate estimation of P_f . Hence, the survival probability, $1 - P_f$, of a bundle is the upper bound for those of all fishnets and, accordingly, the failure probability, P_f , of a bundle is the lower bound on all the failure probabilities of all fishnets.

Fig. 1.13 shows the transition of failure probability P_f of fishnets of various aspect ratios, consisting of the same number, N , of links. The circles with different colors corresponds to results of Monte Carlo simulations. Since we want to verify only the qualitative effect of aspect ratio on P_f , we reduce the sample size from 10^6 to 2×10^5 . Accordingly, we focus only on the range $\ln[-\ln(1 - P_f)] \geq -6$, in which the histogram has converged quite close to the true distribution.

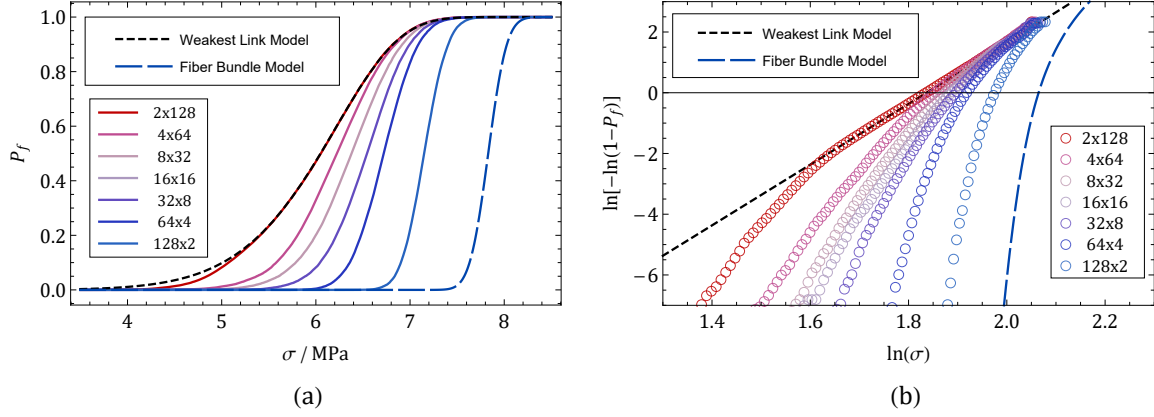


Figure 1.13. a) Monte Carlo simulations showing the transition of P_f as aspect ratio of fishnet is changed from $1 \times N$ to $N \times 1$; b) Same data plotted in Weibull scale.

As shown in the figure, the weakest link model (black dashed line) is the strict upper bound of all fishnets with the same number of links, and the fiber bundle (dark blue dashed line) is the strict lower bound. When the fishnet is long and thin, for example $m \times n = 2 \times 128$ (dark red circles), the corresponding P_f can be described quite well by the two-term Fishnet model, which shows a gradual slope increase by a factor of 2 as $\sigma_N = \sigma$ tends to 0.

This last case might look similar to the real nacre shell, which is much longer than wide. However, in nacre shells and similar structures, the highly stressed zone is often highly localized, as in bending, and then the shell may rather behave as our fishnet with a similar length and width (the statistics of fishnets under flexure, or transverse stress gradient, as well as the statistics of three-dimensional fishnet generalization, would probably be similar, but its study must be relegated to a subsequent article).

Returning to the uniformly stressed long and narrow fishnet, it cannot be described adequately by the two-term fishnet, because this model applies if only one additional link

fails before the maximum load. The higher-order terms in the expansion of P_f matter. Although they are hard to calculate analytically, insight can be gained by Monte Carlo simulations. We run simulations for fishnets of size 64×16 and 16×64 , and then count the average number r_p of scattered link failures before the peak load is reached (the distribution of r_p should in fact lead to the Poisson distribution with rate parameter μ_p when total number of links $N \rightarrow \infty$). For fishnets of size 64×16 , $\mu_p \simeq 5.2$, while $\mu_p \simeq 4.4$ for fishnets of size 16×64 . This shows that more terms of P_{S_k} are needed for short and wide fishnets than for the long and narrow ones. Due to the fact that the laws of stress redistribution around scattered failures are almost identical, the common terms of P_{S_k} , $k \leq k_0$ in P_f for both fishnets are almost the same. The only difference is that the next few terms P_{S_k} , $k > k_0$ for the short and wide fishnets are much larger than those for the long and narrow ones. The cause is the delay of damage localization.

1.5.6. Comparison with the Chain of Bundles

Attempts have previously been made [15, 16, 19, e.g.] to use a chain of fiber bundles to model the failure probability of fiber composites under uniaxial tension. A long specimen under uniaxial tension is subdivided, by imagined cross sections of an assumed spacing, into fictitious segments, each of which is modeled as a fiber bundle. This is an approach that has physical basis in the microstructure of a fiber composite but not of nacre-like materials.

If a fiber bundle with equal load sharing is represented mechanically by loading through rigid platens, each parallel coupling reduces the reach of a power law tail by about one order of magnitude [5, 9]. For 10 parallel fibers, the distribution becomes Gaussian except

for a power law (or Weibull) tail reaching to the probability of only about 10^{-20} . The parallel coupling of fibers extends the reach of the Weibull tail [5, 9], but about 10^{22} bundles in the chain would be needed to approach the Weibull distribution expected for a very long structure. Obviously, such mechanics based equal load-sharing hypothesis gives clearly unreasonable predictions.

The problem with power law (or Weibull) tail shortening is avoided by some convenient, non-mechanical, purely intuitive, load-sharing rules for the transfer of load from a failed fiber to its neighbors. A convenient load-sharing rule can give a very different probability distribution G_n for each bundle, with a realistic reach of power law tail. Like the fishnet, a chain of bundles, each with a suitable G_n , can then predict, for low probabilities, a reasonable slope increase in the low-probability range of Weibull plot, as shown by Harlow and Phoenix [15, 16].

In the fishnet model, by contrast, this slope decrease is due to the addition of non-zero probability P_{S_k} of structure survival after a failure of k links. The slope of P_f will increase, at least by a factor of 2, due to adding the first term, P_{S_1} . As mentioned before, adding more terms P_{S_k} increases the slope further. Importantly, no separate hypothesis about transverse load sharing or redistribution is needed in the fishnet model to get the above result.

1.6. Conclusions

- (1) Based on similar mechanical responses, nacre-like imbricated laminar materials under uniaxial tension can be modeled as fishnets with square holes pulled in the diagonal direction.

- (2) Severe stress redistribution around isolated damages of fishnets are confined in their close neighborhood, but greatly affect the failure probability.
- (3) Assuming the link strength values in a $m \times n$ fishnet to be i.i.d. random variables characterized by failure probability $P_1(\sigma)$, one can express the fishnet failure probability as a finite series $P_f(\sigma) = 1 - \sum_{k=0}^{m(n-1)} P_{S_k}(\sigma)$ in which $P_{S_k}(\sigma)$ is the probability of the event that total of k links have failed while the structure is still safe, while P_{S_0} corresponds to the classical survival probability of a chain with $m \cdot n$ links. Apart from the first term in the series, the remaining terms tend to 0 as σ approaches ∞ and tend to $c_k[P_1(\sigma)]^k$ as σ approaches to 0. Consequently, in the upper tail P_f is essentially the same as the tail of weakest link model while in the lower tail, P_f exhibits a major deviation from the weakest link model and curves downward in the Weibull scale.
- (4) As σ decreases, the slope of P_f in the Weibull plot gets steeper by a factor of 2 due to the effect of P_{S_1} . The center of transition to increased slope depends on the level of stress concentration near the failed link. A higher stress concentration causes the transition to happen at smaller σ . The further term in the expansion cause smaller and smaller increases of slope of P_f in Weibull plot, and the subsequent slope transitions are centered at smaller σ values, due increasing stress concentration.
- (5) The fishnet probability P_f exhibits strong shape effects. By changing the aspect ratio m/n from $1/N$ to $N/1$, a fishnet gradually transforms from a chain to a fiber bundle, and the corresponding failure probability transits smoothly from the Weibull distribution to the Gaussian (or normal) distribution. When the shape $m \cdot n$ is fixed, a larger value of m/n corresponds to a strictly smaller P_f

for all σ . So, when $m \cdot n$ is fixed, the weakest-link model and fiber bundle model give, respectively, the strict upper and lower bounds of the failure probability, P_f .

- (6) There is a strong size effect, similar to, though different from, the Type 1 quasi-brittle size effect characterizing particulate or granular materials and fiber composites.

CHAPTER 2

Strength Distribution for Fishnets of Postpeak-softening Links

2.1. Introduction

As already described in chapter 1, the fishnet model with series expansion can predict the probability of failure for a nacreous material with perfectly brittle links. In this chapter, we extend the fishnet statistics to nacreous materials with quasibrittle links. The softening may give a more realistic characterization of the interlaminar bond failures in some nacreous structures.

The strength and fracture energy of nacre, the shell of pearl oyster or abalone, exceeds by an order-of-magnitude the strength of its constituents (95% CaCO_3). This remarkable property has been shown to originate from the imbricated (or staggered) ‘brick-and-mortar’ arrangement of nanoscale aragonite platelets bonded by a bio-polymer [2, 3, 4, 19]. Thus, the nacre’s nanostructure is of great interest for developing new ultra-strong and ultra-tough biomimetic materials.

Most studies have so far been deterministic. However, it is generally agreed that most engineering structures (bridges, airframes, electronic components, etc.) must be designed for failure probability not exceeding 10^{-6} per lifetime (which is negligible compared to the risk of death in car accident, 10^{-2} , and is similar to the risk of death by a falling tree, by lightning, etc.). To design for failure risk $< 10^{-6}$, a theoretically based analytical closed-form probability distribution is indispensable. A direct experimental verification

of the distribution, by histograms testing, is impossible because $> 10^8$ test repetitions would be required to verify the distribution tail at 10^{-6} , which is obviously beyond reach. Therefore, the experimental verification must rely on other predictions, and the predicted size effect is most useful.

A previous study [20, 21] presented a new statistical model, the fishnet statistics, that can predict the probability tail of a nacreous material in which the fishnet links are perfectly brittle, i.e., their stress drops suddenly to zero as soon as the strength limit of the link is reached. The strength probability distribution of a fishnet was shown to lie between those of a fiber bundle [14, 22] and of a finite (or infinite) chain [9, 23, 5, 6, 7, 8, 11, 24]. The distances from the mean to the point of 10^{-6} failure probability differ, for these two limiting cases, by about 2:1, and the fishnet distribution can provide a continuous transition between these two limiting distributions.

In this study, we extend the fishnet statistics to links that are quasibrittle, exhibiting progressive postpeak softening of various steepness. The softening may give a more realistic characterization of the interlaminar bond failures in some nacreous structures.

Before reaching the maximum load (which indicates the stability limit and failure if the load is controlled), the fishnet may already contain various numbers, 0, 1, 2,..., of failed links. Based on this observation, the fishnet statistical model splits the fishnet survival event into a union of disjoint events corresponding to different numbers of failed links, which implies a summation of survival probabilities:

$$(2.1) \quad 1 - P_f(\sigma) = P_{S_0}(\sigma) + P_{S_1}(\sigma) + P_{S_2}(\sigma) + \cdots$$

where P_f is the failure probability and $P_{S_k}(\sigma)$ ($k = 1, 2, 3, \dots$) are the probabilities of the whole fishnet surviving under load (or nominal stress) σ while there are exactly k failed links under load σ . This formulation works quite well for fishnets with brittle links but it also poses two difficulties.

First, in obtaining the second and third term (P_{S_1} and P_{S_2}) in the foregoing expansion, an equivalent uniform redistributed stress needs to be used for regions near the failed link, based on the stress field from finite element simulation. The error of doing so is negligible when we truncate the expression at the second or third terms (which was shown to give, for brittle links, sufficient accuracy). But it becomes considerable as more higher-order terms are added. This is because, at the lower tail, P_{S_k} is of the same magnitude as $P_1(\sigma)^k$, and so the higher-order terms can be easily ruined by the errors from the previous terms.

Second, as the links become less brittle, more widely scattered damages tend to occur before the peak load, and so more higher-order terms need to be included to predict the failure probability accurately. It is, unfortunately, far more tedious to calculate them. In the previous study, P_{S_2} had to be separated into two parts, to distinguish the cases of two failed links which are either close to, or far away, from each other.

Proceeding similarly, one would have to partition the higher-order terms based on the relative positions of the k failed (or damaged) links and track the stress history for each single case. As k increases, the formula would become too complicated. So the existing fishnet model is suitable only when the links are brittle or almost brittle, in which case it suffices to consider only a few terms in the fishnet expansion (Eq. (2.1)).

For fishnets with softening links, a modified approach is needed to calculate the failure probability. Although the brittle fishnet model is not applicable to a softening fishnet, its concept has inspired two key ideas to tackle the softening fishnet:

1) Instead of a sudden drop of link stiffness to zero stiffness, the progressive continuous postpeak softening is decomposed into a series of sudden stiffness drops, each of them from one link stiffness to the next lower stiffness (Fig.2.1.a).

2) The probability distribution of maximum load (or strength) of the softening fishnet is approximated by the order statistics. In other words, for softening links it is no longer the weakest link but the k -th weakest link that matters for the maximum load of the whole fishnet. . The order statistics allows us to bypass the treatment of complicated stress redistribution in softening fishnets. The order k equals the extent, or number N_c , of the link damages right before the maximum load, with N_c being in itself a discrete random variable.

The probability mass function (pmf) of N_c is approximated by the geometric Poisson distribution, which represents a distribution appropriate for the random cluster nature of the damages, typical of nacreous systems. This distribution (also called Pólya-Aeppli distribution) was originally formulated for the insurance industry, to estimate the total cost of claims in a period of time, and later it was used to model the defects in softwares and the process of random word substitutions in a DNA molecule [25, 26, 27, 28].

One common feature of these applications is that they can be described by a collection of clusters in which the number of clusters and the number of objects in each cluster are both random. This feature suggest using 1) the geometric Poisson distribution to model

the process of scattered damage accumulation, and 2) the total number of damaged links at the peak load of a softening fishnet.

Combining the distributions of the extent of damage, N_c , with the corresponding order statistics, we develop here a probabilistic model for the strength of a fishnet system. To verify the theory, we pick 4 typical softening slopes of links (ranging from almost brittle to almost ductile) and run Monte Carlo simulations 10^6 -times for each case of softening slope.

Finally, it must be stressed that material scientists and engineers developing new materials or structures should strive to maximize not only the mean strength but also the tail strength at probability level 10^{-6} . It can happen that a material or structure of a lower mean strength (and the same coefficient of variation) would have a higher tail strength at 10^{-6} , and vice versa. Cognizant of this fact, we always run at least a million Monte Carlo simulations for each case.

2.2. Stochastic Failure: Qualitative Study

Before embarking on the analytical formulations of failure probability, we begin by presenting some background information and qualitative results, particularly numerical simulations of the stochastic load-displacement curves and damage patterns. This facilitates understanding of the problem to be solved.

2.2.1. Numerical Treatment of Softening and Model Configuration

The numerical method to simulate softening fishnets under uniaxial tension is similar to that in [20, 21] where only brittle links are considered. The two dimensional fishnets

are here still treated, in Matlab, as one-dimensional structures, thanks to considering a collapsed configuration in which the degree of freedom in the transverse direction is omitted and replaced by point-wise connections.

Only the strengths of the links are treated as random variables and, as before [5, 20], they obey the grafted Gauss-Weibull distribution;

$$(2.2) \quad P_1(x) = \begin{cases} 2.55 \left(1 - e^{-(x/12)^{10}}\right), & x \leq 8.6 \text{ MPa} \\ 0.526 - 0.474 \operatorname{erf}[0.884(10 - x)], & x > 8.6 \text{ MPa} \end{cases}$$

Since each shear bond in the nacreous system is treated as a single link, we assume the strengths of the links to be statistically independent, i.e., uncorrelated. This implies the autocorrelation length of the random strength field to be equal to the link length. For the downward drops of link stiffness in the post-peak phase, we assume the softening strength after each drop to be fully correlated to the initial strength of the same link.

The main idea, and the main difference compared to the previous study, is the numerical realization of softening behavior, which is here achieved by replacing continuous softenings with a sequence of small stress drops, or “discrete softenings”. This avoids dealing with a tangential stiffness matrix and allows us to use sequentially a linear finite element solver for what is a nonlinear problem.

The idealized constitutive behavior of discrete softening of the links is depicted in Fig. 2.1.a. Each softening link is initially treated as a perfectly elastic-brittle truss element that does not fail immediately after reaching its maximum load (F_0) but degenerates discontinuously into, or “jumps to”, a weaker linear elastic truss element possessing a slightly lower stiffness and strength. This corresponds to the change of constitutive behavior from

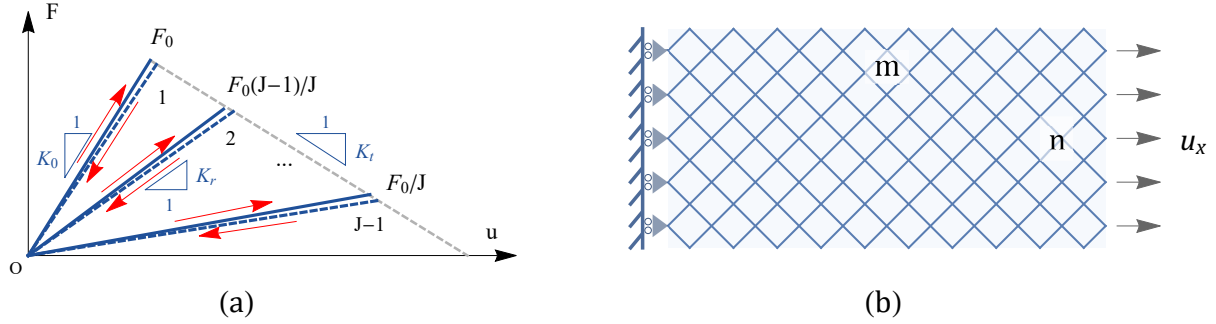


Figure 2.1. a) Numerical treatment of softening behavior of each single link;
b) Schematic showing model configuration and loading conditions.

$\overline{01}$ to $\overline{02}$. Once the stress in the weakened link has reached its new strength $F_0(J-1)/J$ (where J denotes the total number of uniform jumps), the link jumps again to the next weaker one. This series of jumps continues until the link strength is finally reduced to 0, in which case the link has fully failed. Clearly, upon increasing J , the behavior of the link gradually approaches continuous softening. Here we set J to a relatively large number, such as $J = 20$, to reflect a realistic softening while also keeping the computational cost acceptable.

The reduction of link strength after each jump is F_0/J . The residual link stiffness follows from the condition that the point (u_{max}, F_{max}) must lie on the continuous softening curve. Here we consider only linear softening, for which the residual stiffness K_r is a simple function of initial and softening stiffnesses K_0 and K_t ($K_t < 0$) and of the damaged softening state $(J-i)/J$, where i is the number of discrete jumps that has already occurred.

The dimension of fishnet is here considered as $N = m \times n = 16 \times 32 = 512$ (see Fig. 2.1.b), length of links $L = 0.01$ mm, cross-section area $A = 1$ mm², and Young's

modulus $E = 1$ MPa. These properties are chosen purely for simplicity since the system is linear elastic. Changing these constants does not change the system's behavior.

As for boundary conditions, the left end of the fishnet is fixed in the x-direction (roller support) while displacement u_x of all the nodes at the other end is prescribed to be equal (see Fig. 2.1.b). Thanks to linear elasticity of the links, a simple linear finite element solver could be used to calculate the critical elongation u_x such that, in the current step, one and only one link would be about to soften. So, the variable used to control the numerical process is neither the load nor the displacement but the number of softening jumps, i.e., adding one more discrete softening jump in the structure. After each discrete step, the whole structure is treated as a brand new one and is loaded from the original stress-free state up to the attainment of the strength limit in the the next link, which can be different from the last softened link. A similar procedure was used in the previous simulations of brittle fishnets [20] (and also in previous deterministic simulations of masonry [29]). Thus a detailed algorithm is here omitted.

2.2.2. Numerical Results

To study the behavior of fishnets with various softening tangential stiffnesses, we introduce a fixed random strength field for all simulations. We consider link softening with three subsequent tangential stiffnesses: $K_t = -0.5K_0$, $K_t = -0.1K_0$ and $K_t = -0.01K_0$ and with two different numbers of softening jumps of the links, $J = 20$ and $J = 500$, while keeping all the other conditions the same.

2.2.2.1. Load-Deflection Curves and Stress Evolution. Fig. 2.2 shows the load-displacement curves as well as the stress field evolution for various cases. By comparing

Fig.2.2a-c, one realizes that the general shape of load-displacement curves (mean behavior) follows the trend shown by the crack band theory [30] for band fronts of various widths. When $K_t = -0.5K_0$, the links behavior is close to brittle (i.e., to a vertical stress drop) and the whole fishnet exhibits strong snap-back instability in the post-peak behavior (Fig. 2.2.a) (as already seen in the previous study [20] for brittle links). As the softening slope gets flatter ($K_t = -0.1K_0$), the snap-back instability is mitigated (Fig. 2.2.b). With an even flatter softening slope for each link ($K_t = -0.01K_0$), the snap-back instability no longer exists and a post-peak softening curve is observed (Fig. 2.2.c).

Apart from the shape of load-displacement curve, the softening slopes K_t of links are found to have a huge impact on the location of maximum load. As the softening slope of links, K_t , increases gradually from $-\infty$ to 0, there is a larger number of softening jumps before the maximum load is reached. Thus the history of stress redistribution gets quite complicated when K_t approaches 0, as can be seen from the load-displacement curve in Fig. 2.2.c where the maximum load is reached after a long period of wavy stress redistribution. In other words, the problem becomes strongly history dependent when the links are not brittle.

The total number of discrete jumps allowed for each link, J , affects the stochastic failure of fishnets as well. Intuitively a larger J would allow a smoother softening process and result in a more realistic stress-strain curve. When $J = 20$ (Fig. 2.2.c), a plateau is observed on the load-displacement curve (between point A and B), and it disappears as J is increased to 500 (Fig. 2.2.d). This change of behavior gets more noticeable as the softening slope gets flatter. Intuitively, the reason is that the next discrete softening is more likely to be pushed elsewhere rather than localize when softening slope is flat.

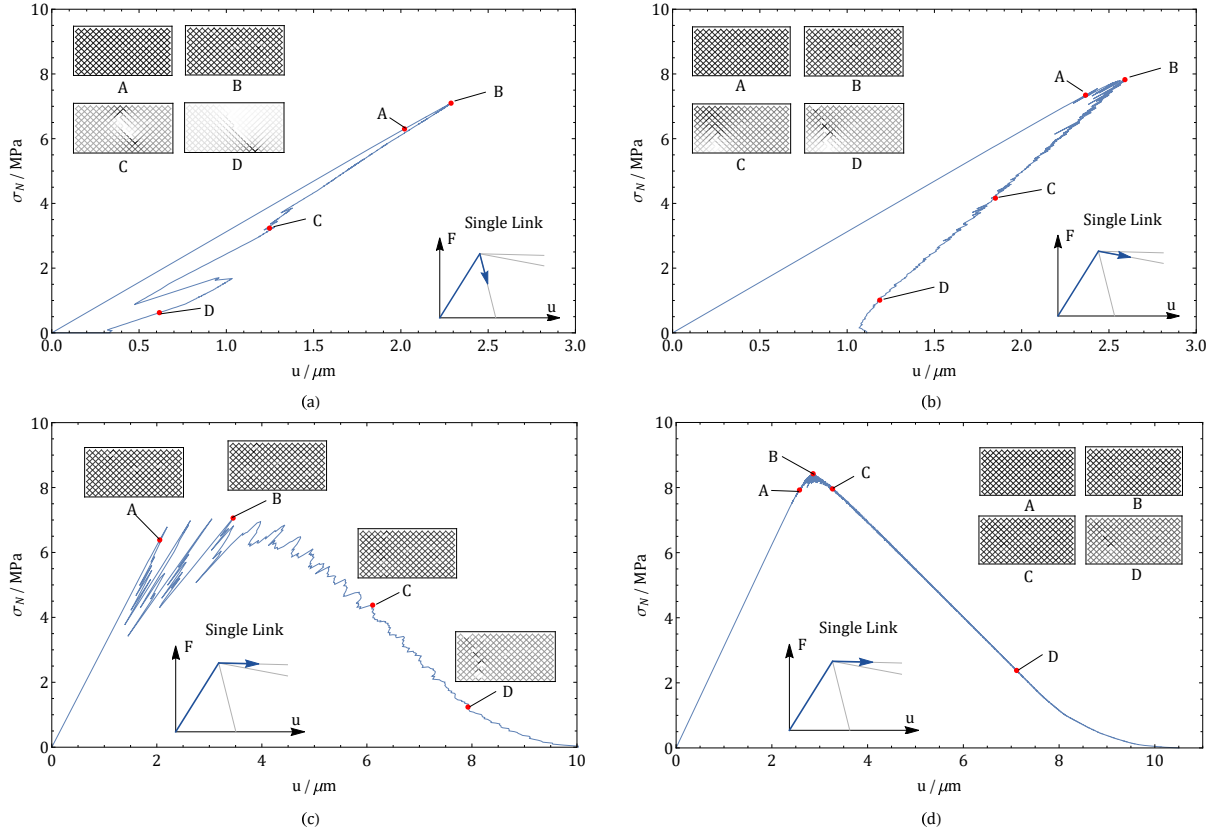


Figure 2.2. Load-displacement curves for fishnets with the same random link strengths and different tangential softening moduli (and stress fields σ_i/σ_{max} at 4 typical stages) : a) $K_t = -0.5K_0$, $J = 20$; b) $K_t = -0.1K_0$, $J = 20$; c) $K_t = -0.01K_0$, $J = 20$; d) $K_t = -0.01K_0$, $J = 500$. Magnitude of stress (normalized by the maximum of that frame) is indicated by the darkness.

Thus more jumps with smaller stress drops are required to scatter the damaged links. A smaller J imposes larger stress drops at each jump (F_0/J) and thus skips many possible ways that could have stopped the damage localization. For steep softening, though, most damages tend to keep localizing for many steps. Therefore, combining a few consecutive small jumps into a big one does not affect the outcome of softening process, and having a relatively small J is enough to capture the realistic behavior of the nearly brittle fishnet system.

One notable phenomenon in flat softening of the links is that stress field becomes quite smooth and nearly uniform during the whole loading process (Fig. 2.2c and d). This serves as the basis of the mathematical modeling of the failure probability.

2.2.2.2. Damage Pattern. To track the damage evolution during the whole process, we introduce a new variable $\mathcal{D} = j/J$, where j is the number of discrete softening jumps that a link has already undergone; $\mathcal{D}$ indicates the extent of damage for a single link, and $\mathcal{D} = 1$ means that the link has failed completely. Fig. 2.3 shows the evolution of $\mathcal{D}$ for the cases discussed in the preceding section, in which 4 plots within each row correspond to the damage field at 4 typical stages (A, B, C and D) marked in Fig. 2.2.

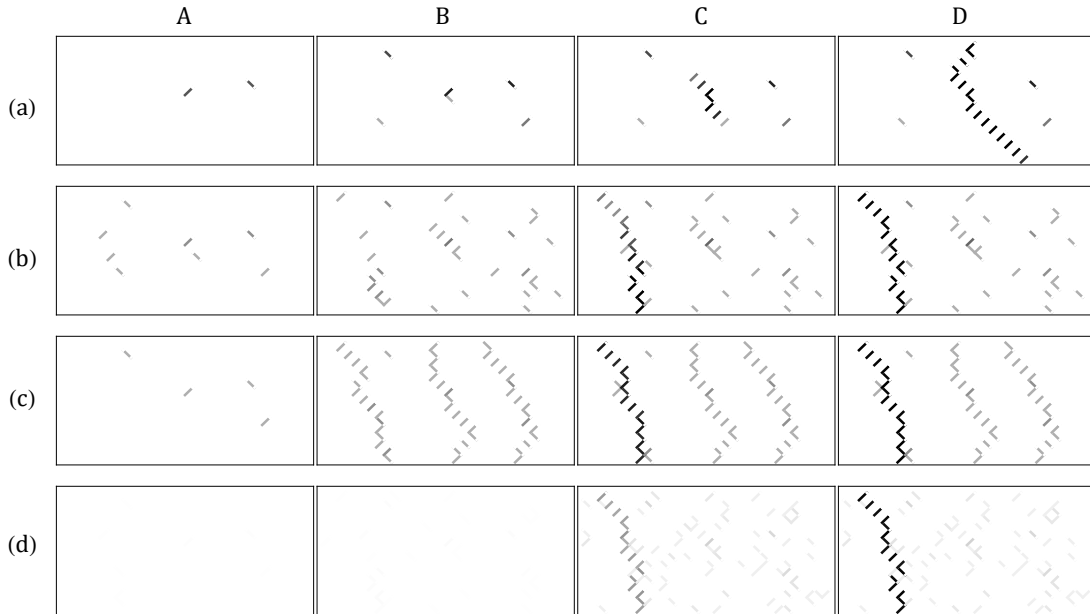


Figure 2.3. Damage (softening) evolution of the fishnets shown in Fig. 2.2. Rows: (a) $K_t = -0.5K_0$, $J = 20$; (b) $K_t = -0.1K_0$, $J = 20$; (c) $K_t = -0.01K_0$, $J = 20$; (d) $K_t = -0.01K_0$, $J = 500$; Columns: A - prepeak, B - peak, C and D - postpeak. The level of damage $\mathcal{D}$ is indicated by the darkness and pure black corresponds to $\mathcal{D} = 1$.

Row (a) in Fig. 2.3 shows the damage evolution of fishnets with links that are almost brittle (steep softening slope). Before the peak load, only a few scattered damages show up and, at each site of damage, the damage level has a relatively high value ($\mathcal{D} \simeq 0.5$). As the softening slope becomes flatter, more scattered damages show up (row (b), column B), each with a lower value of $\mathcal{D}$ compared with case (a) (row (a), column B). In addition, the final crack pattern of case (b) is completely different from case (a) due to the change in the softening behavior of links. For an even flatter softening slope (row (d)-B), the damages at the peak load are too weak to be seen via bare eyes. Note that we do not take case (c) into consideration because, in this case, $J = 20$ is not large enough to reflect the realistic softening of links with slope $K_t = -0.01K_0$, and thus the three softening bands at the peak load (row(c), column B) are not realistic.

2.3. Failure Probability

In this section, we begin formulating the failure probability of fishnets with softening links. When the softening slope K_t is steep, the behavior of the whole fishnet is almost elastic-brittle. Therefore its failure probability can be very well estimated by the previous fishnet model [20, 21]. However, when K_t is very close to zero, which might be more similar to the real nacre, the previous model may not be analytically tractable. More specifically, if we want an accurate estimation of P_f we have to consider all possible cases of stress evolution up to the maximum load, which is too tedious to be done for softening fishnet due to its complicated random stress history shown in Fig 2.2.b, c and d. To this end, a different analytical model is needed to accurately predict the failure probability when the fishnet links are gradually softening, and especially when K_t is nearly 0.

To get the failure probability, we first define a new discrete random variable N_c as the number of previously damaged links upon reaching the peak-load, based on which we then partition the event of structure failure as follows:

$$(2.3) \quad P_f(x) = \mathbb{P}(\sigma_{max} \leq x) = \sum_{k=0}^N \mathbb{P}(N_c = k) \mathbb{P}(\sigma_{max} \leq x \mid N_c = k)$$

Then we try to approximate the conditional probability in Eq. (2.3) by the distribution of the k -th smallest minimum, $s_{(k)}$, of the strengths of the links:

$$(2.4) \quad \mathbb{P}(\sigma_{max} \leq x \mid N_c = k) \simeq \mathbb{P}[s_{(k)} \leq x]$$

Finally we study the distribution of the random variable N_c .

In a nutshell, the general idea is to approximate P_f using linear combination of the distribution of order statistics (a set of bases) whose combination coefficients are given by the probability $\mathbb{P}(N_c = k)$.

2.3.1. Bounding Nominal Stress from Above by Order Statistics

The key part of our model is that we use the k -th smallest minimum (i.e., an order statistic) $s_{(k)}$ of the link strengths to bound from above the nominal stress σ_N when $N_c = k$. To see why this is possible, we consider the history of stress and residual strength field in time. Note that under uniaxial loading, the nominal stress σ_N is defined to be the total load divided by the original cross-section area, which equals the mean value of the redistributed stress field σ_i . Each fishnet link is assigned an index based on its location (from left to right and top to bottom), so as to identify those that undergo damage.

Fig. 2.4 shows the stress field σ_i and residual strength field s_i^R plotted against the link index for four typical stages of a random realization based on the damage extent (measured by the number of damaged links). Note that the 2-D stress and strength fields are plotted as 1-D vectors. In this way, we put aside the geometry temporarily and focus only on the magnitudes of stress and strength for the whole structure. Recall that at each step we use the criterion $s_i^R = \sigma_i$ to find the currently softening link, and so the residual strength curve touches the actual stress-strain curve at one and only one place in each frame (marked by the circle) and $s_j^R > \sigma_j$, for all $j \neq i$ is strictly satisfied for the rest of the links, i.e., the rest of links will not undergo further damage under the load in the current step.

Fig. 2.4.a shows the very first step in the numerical simulation, in which the stress field right before the first softening is recorded. The stress field (thin straight line) is strictly constant and equals the strict minimum strength of links $s_{(1)}$, and the weakest link is the first one to undergo softening. Therefore, the assumption that the nominal stress (dashed line) at the k -th step equals the strength of the k -th weakest link ($\sigma_N = s_{(1)}$) holds for the first step ($k = 1$).

Fig. 2.4.b shows the second step in the simulation. The residual strength field is almost unchanged ($s_{(1)}$ reduced by $s_{(1)}/500$), while some small disturbances in the stress field are observed for a few links due to the decrease of stiffness for the previously softened link. But still the redistributed stress field is almost uniform and most link stresses are almost equal to the nominal stress σ_N of that step. Most importantly, the stress curve touches the residual strength curve at the second weakest link ($\sigma_N \simeq \sigma_i = s_{(2)}$). So, in the second

step, it is no longer the weakest link but the second weakest link that determines the nominal stress and softening process.

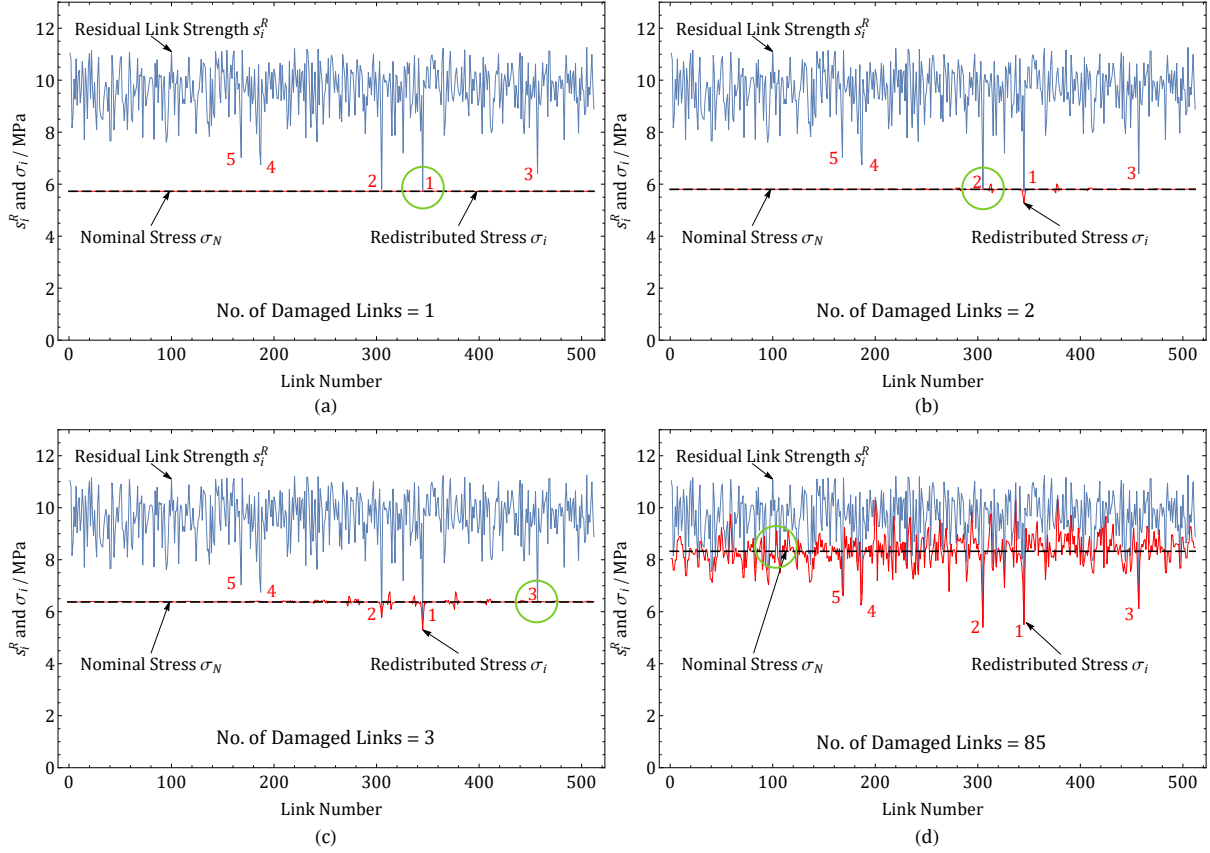


Figure 2.4. Plot of residual strength field s_i^R and redistributed stress field σ_i against link index ($K_t = -0.01K_0$, $J = 500$) for a typical random realization. The 5 weakest links are marked with index 1,2,3,4,5 and the circle indicates the link that is to fail at the current step ($s_i^R = \sigma_i$).

Note that, in the second step, the current damage occurred at a different place from the first softening link. If the second damage occurred at the same place as the first one did, or, in other words, if the damage localizes, the stress at the damaged link will be way below the nominal stress σ_N and the equilibrium criterion will be $\sigma_i = 499s_{(1)}/500 \simeq s_{(1)}$. So, the condition $\sigma_N \simeq s_{(2)}$ will not be satisfied.

Fortunately, damage localization decreases the nominal stress in general and it will increase to a higher level when the localization stops. So the steps where damage localization happens contribute little to the maximum load. We can simply ignore them and only care about the steps in which the damage does not localize. Each time when the damage moves to a new place, the total number of links that are damaged in the fishnet will increase by 1. Therefore we let k denote the total number of damaged links in a fishnet, so as to keep track of the steps where damages did not localize.

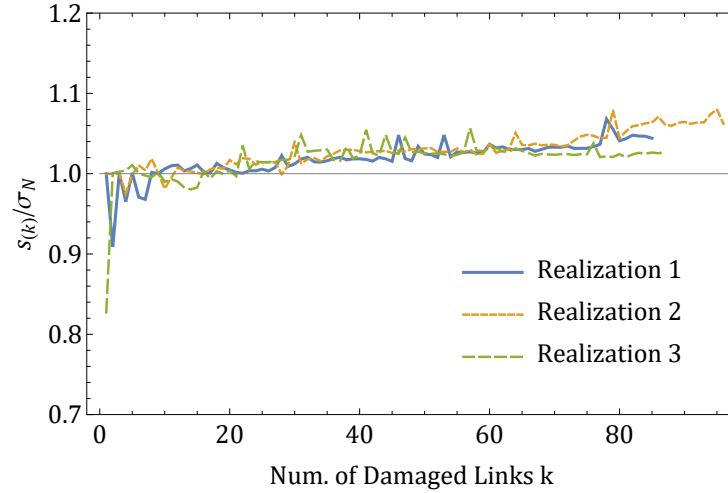


Figure 2.5. Plot of the quantity s_k/σ_N against k , number of damaged links, until the maximum load is reached ($K_t = -0.01K_0$, $J = 500$). If there are multiple values of σ_N for one k , the maximum is used.

Fig. 2.4.c shows the fourth step, in which $k = 3$ (i.e., the damage localized in the third step and spread out again in the fourth step). Once again we can clearly see that $\sigma_N \simeq s_{(3)}$ holds. As more damages occur, the stress field is no longer nearly uniform when the maximum load is reached (see Fig.2.4.d). To see whether the assumption $\sigma_N \simeq s_{(k)}$ still holds at the maximum load, we plot $s_{(k)}/\sigma_N$ against k up to the step in which the peak load is reached, as shown in Fig. 2.5.

Note that if the damages localize in a few steps, k will not increase. Then we choose the largest σ_N for various steps with the same k . The thick line corresponds to the same realization as used in Fig. 2.4, while the other two dashed lines correspond to two new realizations. One can easily see that, despite the fact that N_c varies a lot for various realizations, the ratio $s_{(k)}/\sigma_N$ varies only little and remains slightly above 1 throughout the process. This justifies our method of using the k -th smallest minimum to approximate the nominal stress σ_N when number of damaged links equals k .

Note that we have so far defined two variables k and N_c , which seem to be similar. However, they are not; N_c is a random variable that depends on the maximum load and takes different values for different realizations while k is a deterministic variable that characterizes the number of damaged links at any given time.

2.3.2. Relation between nominal stress and order statistics: γ_k

The nominal stresses are bounded from above in our formulation ($\sigma_N < s_{(k)}$). If we directly use the distribution of $s_{(k)}$ to replace that of nominal stress σ_N , we will get a lower bound on the failure probability, which could lead to considerable deviation of our estimate from the true P_f . This deviation could be significant when N_c is large, as is the case of flat softening, because the ratio s_k/σ_N becomes strictly larger than 1 (though only by a small amount) when k is large (see Fig. 2.5).

It is therefore necessary to incorporate the qualitative behavior of s_k/σ_N into our model and approximate σ_N by $s_{(k)}/\gamma_k$ instead of $s_{(k)}$ alone, while γ_k , strictly greater than 1, is taken as the mean curve of $s_{(k)}/\sigma_N$ of a few random realizations up to the maximum

load in Fig. 2.5. In this way, the distribution of $s_{(k)}/\gamma_k$, compared to that of $s_{(k)}$, is much closer to the distribution of σ_N .

In fact, if we would not introduce γ_k , it would lead to a logical contradiction. Suppose that the nominal stress, σ_N , at the k -th step strictly equals $s_{(k)}$. This would give an increasing sequence, and thus the nominal stress would never decrease, contradicting the fact that the maximum load can be reached before the end of displacement control. This observation tells us that γ must be considered and that it will always increase to ∞ at the end of displacement control ($\sigma_N \simeq s_{(k)}/\gamma_k \rightarrow 0$), even though it is very close to 1 up to the maximum load. Intuitively, γ_k indicates the extent of damage localization and stress concentration.

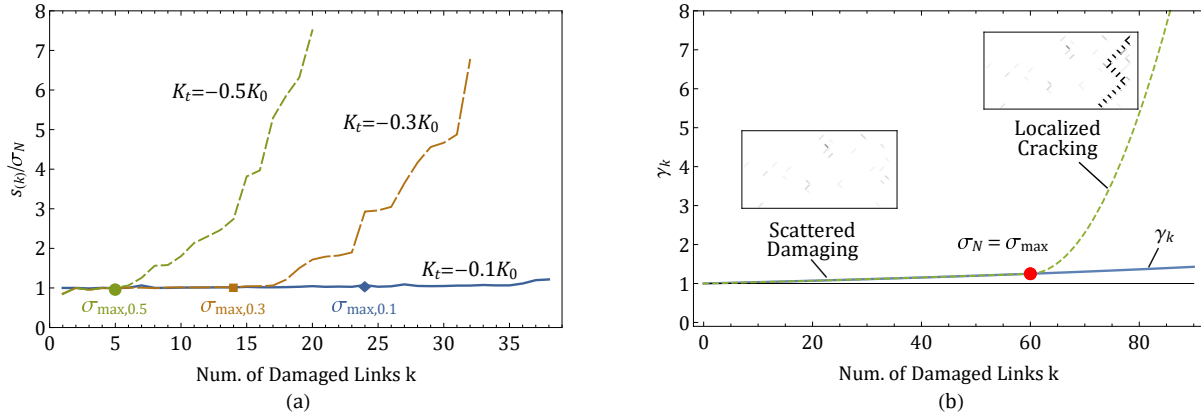


Figure 2.6. (a) Evolution of the ratio $s_{(k)}/\sigma_N$ against k for various softening slopes K_t ($J = 20$); (b) Schematic showing the qualitative behavior of γ_k .

To see the qualitative behavior of γ_k , we plot in Fig. 2.6.a the evolution of the ratio $s_{(k)}/\sigma_N$ for fishnets with 3 different softening slopes ($K_t/K_0 = -0.1, -0.3$ and -0.5). In all three cases, the same random strength field is used so that the softening slope would be the only variable. It is observed that the curves stay very close to 1 at the beginning. Then, after a critical point, suddenly a sharp transition happens and the curve blows up.

The difference, for the flat softening case ($K_t = -0.1K_0$), is that $s_{(k)}/\sigma_N$ starts to blow up long after the maximum load while $s_{(k)}/\sigma_N$ tends to blow up much earlier and right after the maximum load, as the softening slope becomes steeper.

This observation matches our conclusion in the previous study [20] of brittle fishnet, namely that maximum load is reached right before damage localization. It is clear that, in all three cases, the maximum load (see the dark disk, square and diamond markers in the figure) is reached before the sharp transition, and so it is guaranteed that our order statistics approximation works.

Note that in Fig. 2.6.a, the three curves almost coincide before they blow up and the softening slope K_t affects only the position of the sharp transition. Specifically, the smaller the $|K_t|$ is, the later the sharp transition and the maximum load will materialize. So, γ_k , defined as the mean curve of $s_{(k)}/\sigma_N$ only up to the maximum load, is independent of the softening slope K_t although it depends on the fishnet shape and size, and on the number J of the discrete jumps.

Different damage patterns before and after the sudden transition of $s_{(k)}/\sigma_N$ indicate two distinct system behaviors during the whole process—the damages are: 1) small in extent and large in amplitude at the beginning, but 2) large in amplitude and small in extent after the transition (see Fig. 2.6.b). Initially, the damages tend to be very weak and uniformly scattered as if they are not too correlated to each other. After the curve bends sharply upward, the damages begin to accumulate and localize in a small region which later forms a contiguous crack. Interestingly, this phenomenon has been noted by Krajcinovic and Rinaldi [31], who describe the stochastic damage process of quasi-brittle

materials as “ergodic” before the peak load and as of “avalanche class” after the peak load.

Fig. 2.7 shows the individual behavior of $\gamma_k \sigma_N$ and $s_{(k)}$ instead of their ratio for a typical random realization ($K_t = -0.2K_0$). It can be seen that when scattered damages continue showing up, the scaled nominal stress follows closely with the k -th smallest minimums until the maximum load is reached, after which damages localize and the nominal stress drastically drops to 0. As a result, the ratio $s_{(k)}/\sigma_N$ blows up after the structure strength gets reduced to zero. It is interesting to note that the general shape of the plot of order statistics $s_{(k)}$ looks similar to $P_1^{-1}(x)$, which can be explained by the inverse transform sampling theory used in pseudo-random number generation.

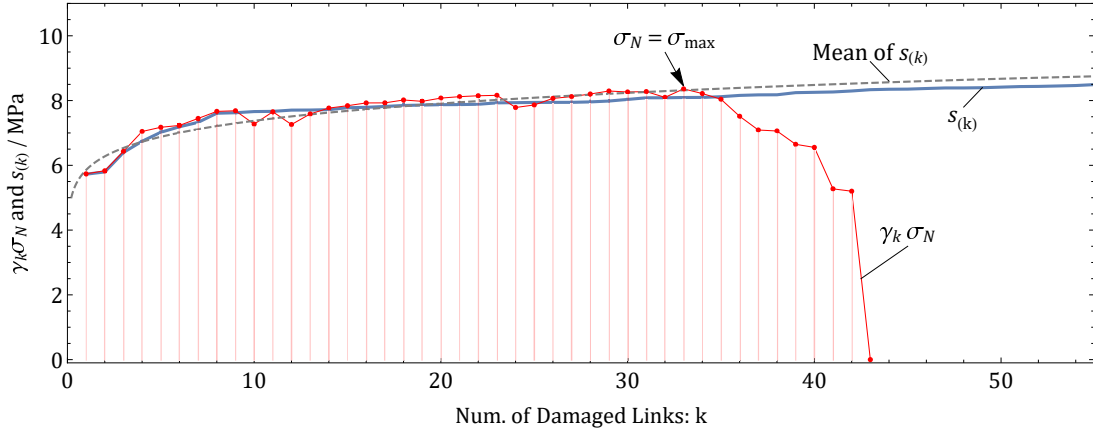


Figure 2.7. Evolution of scaled nominal stress $\gamma_k \sigma_N$ and order statistics $s_{(k)}$ against k

2.3.3. k -th Smallest Minima

According to the foregoing analysis, we approximate the conditional probability in Eq. (2.3) by using the distribution of order statistics (i.e., the k -th Smallest Minima):

$$(2.5) \quad \mathbb{P}(\sigma_{max} \leq x \mid N_c = k) \simeq \mathbb{P}[s_{(k)}/\gamma_k \leq x] = \mathbb{P}[s_{(k)} \leq \gamma_k x] = W_k(\gamma_k x)$$

where $W_k(x)$ is the cdf of $s_{(k)}$. The detailed derivation of $W_k(x)$ can be found in [32], and we only give a brief review of order statistics in this section.

The k -th smallest minimum is related to the k -th largest maximum since

$$(2.6) \quad \min\{X_1, X_2, \dots, X_n\} = -\max\{-X_1, -X_2, \dots, -X_n\}$$

So it suffices to study the distribution $G_k(x)$ of the k -th largest maximum and then the distribution of $s_{(k)}$ can be obtained via the relation:

$$(2.7) \quad W_k(x) = 1 - G_k(-x)$$

Consider now the exceedances of levels u_n by identical identically distributed (i.i.d.) random variables $X_1, X_2, \dots, X_n$ having distribution P_1 . Here u_n is the normalized threshold ($u_n = a_n x + b_n$) used in the extreme value statistics, and S_n denotes the number of exceedances of a level u_n by $X_1, X_2, \dots, X_n$. Clearly, the mean of S_n is $n(1 - P_1(u_n))$, i.e., the total number of random variables times the probability of a random variable being greater than u_n . The probability that there are exactly k random variables greater than

u_n follows the binomial distribution:

$$(2.8) \quad \mathbb{P}\{S_n \leq k\} = \sum_{s=0}^k \binom{n}{s} [1 - P_1(u_n)]^k P_1(u_n)^{n-k}$$

Now, $n[1 - P_1(u_n)] \rightarrow \tau < \infty$ as $n \rightarrow \infty$ represents a condition for P_1 to be in the domain of attraction of 1 of the 3 possible types of limiting distributions, So, from the classical Poisson limit for the binomial distribution it follows that

$$(2.9) \quad \mathbb{P}\{S_n \leq k\} \rightarrow e^{-\tau} \sum_{s=0}^k \frac{\tau^s}{s!}, \text{ as } n \rightarrow \infty$$

where $e^{-\tau} = G_0(x)$ is one of the three types of limiting distributions (Gumbel, Fréchet and Weibull). Therefore,

$$(2.10) \quad \mathbb{P}\{M_n^{(k)} \leq u_n\} = \mathbb{P}\{S_n < k\} = G_0(x) \sum_{s=0}^k \frac{[-\ln(G_0(x))]^s}{s!} = G_k(x),$$

where $M_n^{(k)}$ is the k -th largest maximum of $\{X_1, X_2, \dots, X_n\}$. Typically, when $k = 0$, $M_n^{(0)}$ is the strict maximum and its distribution follows the classical limit distribution $G_0(x)$. From the relation between $G_k(x)$ and $W_k(x)$, Eq. (2.7), it follows that

$$(2.11) \quad W_k(x) = 1 - [1 - W_0(x)] \sum_{s=0}^k \frac{\{-\ln[1 - W_0(x)]\}^s}{s!},$$

where W_0 is the limiting Weibull distribution.

Alternatively, we could replace $1 - W_0(x)$ by $[1 - P_1(x)]^N$ if N is large. Note that here we do not replace x by $-x$ because x in Eq. (2.10) is negative, representing $-X_i$, while x is positive in Eq. (2.11). This means that we have already converted x into its opposite number, and so x represents X_i .

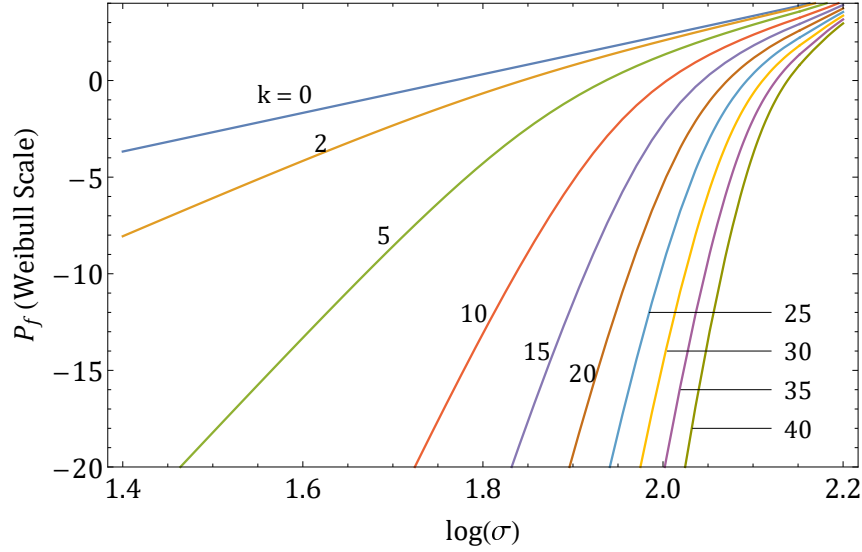


Figure 2.8. Plot (Weibull scale) of samples from the family of cdf's of the k -th smallest minimum in a collection of 512 i.i.d. random variables that follows the distribution P_1 .

Fig. 2.8 shows the curves of $W_k(x)$ in Weibull scale for various k values. As one can see, the spectrum of curves spans the region in which the true failure probability P_f could possibly lie and, therefore, it can serve as the basis to represent P_f .

2.3.4. Critical Number of Softening Links N_c

The failure probability P_f in Eq. (2.3) depends not only on the order statistics but also on the discrete random variable N_c , which is the number of damaged links at the peak load. It is again too tedious to get the exact probability mass function (pmf) of the distribution due to its history dependence. We therefore seek an approximation based on the physical nature of N_c —the damaged links appear in clusters. To be specific, a cluster keeps growing until the next damage appears far away from the current one, forming a

new cluster. Thus, the random variable N_c can be expressed as the following sum,

$$(2.12) \quad N_c = \sum_{s=1}^{N_{\text{cluster}}} Y_s,$$

where not only the number of damaged links in each cluster Y_s but also the number of clusters N_{cluster} at maximum load are random variables. We assume that N_{cluster} follows Poisson distribution with parameter λ and Y_s follows geometric distribution with parameter θ . Therefore, the sum N_c follows the geometric Poisson (Pólya-Aeppli) distribution [25], whose pmf is

$$(2.13) \quad p_k = \mathbb{P}(N_c = k) = \begin{cases} \sum_{s=1}^k e^{-\lambda} \frac{\lambda^s}{s!} \binom{k-1}{s-1} \theta^s (1-\theta)^{k-s}, & k = 1, 2, 3, \dots \\ e^{-\lambda}, & k = 0 \end{cases}$$

and the mean and variance are

$$(2.14) \quad \mathbb{E}N_c = \frac{\lambda}{\theta}, \quad \text{Var}(N_c) = \mathbb{E}[(N_c - \mathbb{E}N_c)^2] = \frac{\lambda(2-\theta)}{\theta^2},$$

where λ is the average number of clusters at maximum load and θ is the probability of success on each “trial”, i.e., that the size of current cluster increases by 1.

In our formulation, we assumed that N_{cluster} follows the Poisson distribution and Y_s follows the geometric distribution. This assumption is true only in the approximate sense, and is the main source of error for the value of p_k . More specifically, Poisson distribution of N_{cluster} requires that the clusters would not interact with each other, which is not strictly satisfied because the clusters affect each other through the redistributed stress

field. Fortunately, such interactions are very weak, making the assumption justifiable in the approximate sense.

In addition, the geometric distribution of Y_s requires that the probability of success on each “trial” be a constant, which is not strictly satisfied as well. For softening fishnets, the success of a trial can be interpreted as the current softening link lying in the neighborhood of the previous one, so that the size of the current cluster could increase by 1. Due to stress redistribution, the links near the damaged links will have higher stresses than the average, and so the future damages will more likely appear near the damaged links. So, strictly speaking, the probability of success on each trial is not constant.

Nevertheless, in softening fishnets with a rather flat softening slope. the stress concentration is very weak, and then the geometric distribution assumption is applicable in the approximate sense. On the other hand, for relatively brittle fishnets, characterized by steep softening slopes, the geometric Poisson distribution could give noticeable underestimation of p_k for small k . To overcome this problem, we will adjust the formulation of P_f by introducing two parameters.

2.3.5. Formulation of P_f

Now that we have the distributions of N_c and $s_{(k)}$, the failure probability P_f of softening fishnet in Eq. (2.3) reduces to

$$(2.15) \quad P_f(x) = \sum_{k=0}^N p_k W_k(\gamma_k x),$$

where p_k is given in Eq. (2.13) and $W_k(x)$ by Eq. (2.11). In this formulation, the collection of functions W_k will not change if P_1 and N are fixed. So the effect of fishnet shape m/n ,

softening slope K_t , and number of allowed jumps for each link J on P_f , is embedded in γ_k and p_k . And since K_t is independent of γ_k , the softening slope affects the failure probability by influencing the distribution of N_c , i.e., the number of damaged links at the peak load.

Note that from Fig. 2.5, γ_k tends to be, in the first few steps, considerably smaller than 1. So, the first few terms in the sum Eq. (2.15) could give significant overestimation at the lower tail of P_f when the softening links are not very brittle. Apart from γ_k , the errors in the estimation of the coefficients p_k by the geometric-Poisson distribution carry over to the final failure probability. To get a more accurate result, we introduce into our model two additional parameters k_0 and δk :

$$(2.16) \quad P_f(x) = \sum_{k=k_0}^N p_k W_{k'}(\gamma_{k'} x), \quad k' = k - \delta k$$

where $k_0 \geq 1$ and $\delta k \geq 1$ represent, respectively, the truncation and shifting of the terms of order statistics. These two parameters allow us to compensate for a part of the error in the estimation of γ_k and p_k . Intuitively, since the difference between the scaled maximum load ($\gamma_k \sigma_{max}$) and order statistics ($s_{(k)}$) is relatively large when the maximum load is reached right after damage initiation, we simply rule out the possibility that the structure could fail when $N_c < k_0$. Hence, k_0 stands for a threshold of truncation for the order statistics. Note that directly omitting the first terms, k_0 , in Eq. (2.15) would be incorrect since it would break the partition of unity for coefficients p_k , i.e., $\sum_{k=0}^N p_k = 1$.

So, after the truncation, we always renormalize p_k to ensure satisfying the partition of unity exactly. Apart from the truncation, replacing k by $k - \delta k$ for $W_k(\gamma_k)$ shifts the order statistics to the left in the sequence, which thickens the lower tail of P_f . As shown

in the following, this adjustment is useful especially when the links are relatively brittle, in which case the geometric Poisson estimation of p_k for small k is not very accurate. On the other hand, when K_t is very flat (i.e, the postpeak almost plastic), δk is not needed and can be set to 0. At the same time, since in our model the nominal stresses are not bounded from below, the upper bound on P_f is not guaranteed. Choosing a proper value for δk converts the lower bound on δk into an upper bound on the lower tail of P_f .

2.4. Numerical Verification

2.4.1. Verification of N_c

To predict the failure probability accurately, the key is to get an accurate estimation of the coefficients $p_k = \mathbb{P}(N_c = k)$, which are assumed to come from the Geometric Poisson distribution. For various softening slopes, 10^4 Monte Carlo simulations have been run for each case. The histograms as well as the optimum fits of the probability mass functions (pmf) are shown in Fig. 2.9. Note that the sample size (10^4) chosen here is quite large and would be difficult to achieve through histogram testings in the lab. However, a much smaller sample size, sufficing only for the sample mean and variance, can be used to estimate the two parameters, λ and θ , needed for the geometric Poisson distribution (Eq. (2.14)).

The histograms from Fig. 2.9 can be closely fitted, except for some small discrepancies, by the geometric Poisson distribution. As the softening slope becomes steeper, the underestimation by geometric-Poisson distribution becomes larger (Fig. 2.9.d) and this could lead to underestimation of P_f at its lower tail. As already mentioned, a shift by δk was introduced to partially overcome this problem. Keep in mind, though, that for the

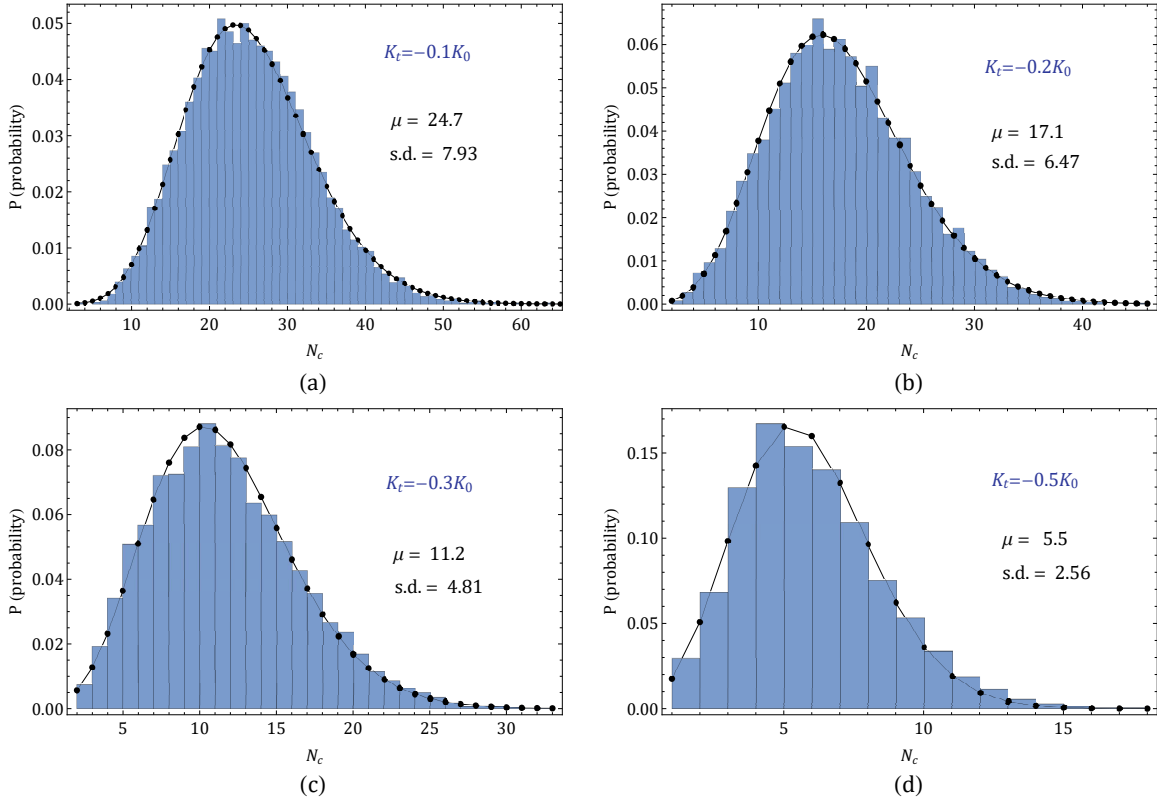


Figure 2.9. Optimum fit of histograms of N_c with geometric Poisson distributions for fishnets of various softening slopes and same number of discrete link softening ($J = 20$). The mean (μ) and standard deviation ($s.d.$) are shown in the figure.

limit of steep softening, which is a vertical stress drop, the original fishnet model [20, 21] gives an accurate prediction.

Fig. 2.10 shows the estimations of the distribution of N_c for various softening slopes, all plotted in one figure. It is clear that, as the softening slope, K_t , becomes closer to 0 (quasi-plastic), the mean and variance of N_c increases. This verifies our previous observation that N_c increases as the fishnet links become more plastic (or ductile), while fishnets with brittle links tend to fail at damage initiation. In other words, flatter softening slope delays the occurrence of maximum load.

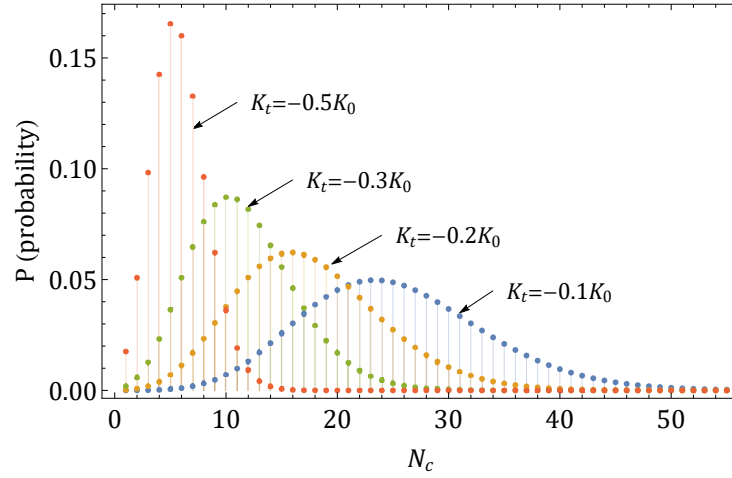


Figure 2.10. Geometric Poisson distribution of N_c for fishnets of various softening slopes plotted in the same figure.

2.4.2. Verification of P_f

To describe P_f completely, we must choose an accurate expression for γ_k . For the current configuration of the numerical model (size, shape and material properties), we let

$$(2.17) \quad \gamma_k = \frac{N}{N - k} = \frac{512}{512 - k}$$

This expression is chosen to be the mean curve of $s_{(k)}/\sigma_N$ of a few random realizations before they blow up (see Fig. 2.11). Coincidentally, γ_k in this particular case is the same as the deterministic expression of $s_{(k)}/\sigma_N$ for a brittle fiber bundle with N fibers. Alternatively, a linear fit of γ_k is also possible since the curve is very flat and smooth. Once the curve is chosen, γ_k remains fixed for various softening slopes K_t since it only affects the location at which the ratio $s_{(k)}/\sigma_N$ blows up.

Fig. 2.12 shows the comparison of the failure probability P_f obtained by analytical model and histogram data for various softening slopes. Also included are the histogram

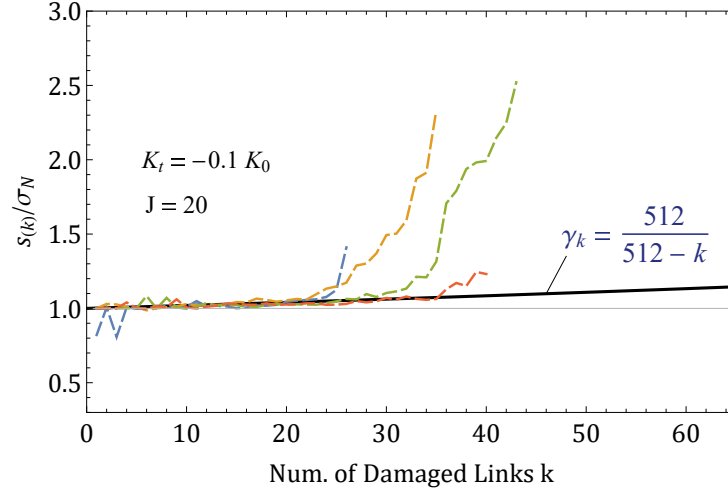


Figure 2.11. Optimum fit of γ_k for the fishnet with $K_t = -0.1K_0$ and $J = 20$. and 3-term fishnet model for fishnets with brittle links ($|K_t/K_0| = \infty$) taken from previous study [20].

Table 2.1. parameters k_0 and δk for cases of various K_t

$ K_t/K_0 $	0.1	0.2	0.3	0.5
k_0	5	5	5	5
δk	0	2	3	3

Table. 2.1 shows the truncation and shifting parameters (k_0 and δk) used in the analytical model. $k_0 = 5$ is the same for all softening slopes for simplicity and the shifting parameter δk gradually increases for increasing magnitude of softening slope K_t . It is used to overcome part of the underestimation of p_k and P_f for their lower tails. Note that $\delta k \leq k_0$ should always be satisfied. As for fishnets with very ductile links ($|K_t| \leq 0.1|K_0|$), we set $\delta k = 0$ because most fishnets would fail after the size of damage become very large, where the order statistics s_k (and their distribution) are close to each other. So, shifting the order of $W_k(\gamma_k x)$ makes very little difference and the model without shifting works just fine.

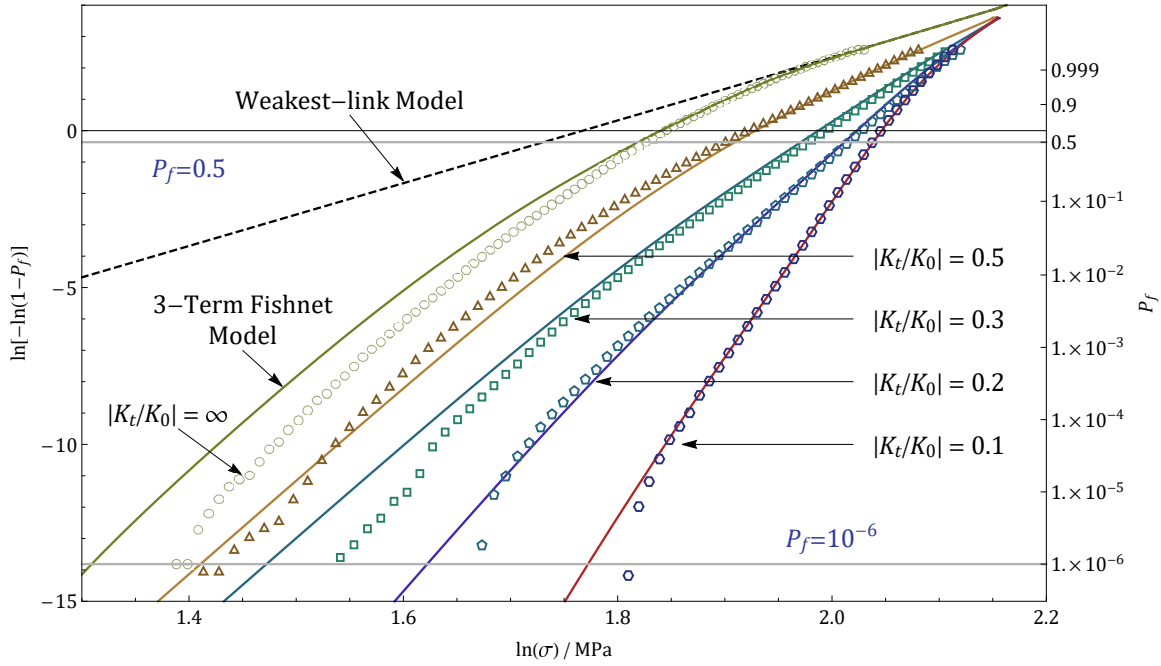


Figure 2.12. Histogram of nominal strengths (discrete markers) in Weibull scale for fishnets with different softening slopes and results by using order statistics and the original 3-term fishnet model (solid lines). The sample size is 10^6 for each case.

Fig. 2.12 shows that the values given by Eq. (2.16) match quite well the histogram data obtained by Monte Carlo simulations. The discrepancies between the model and the data become larger when the softening slope K_t increases. The most accurate estimation is obtained for the quasi-plastic case ($K_t = -0.1K_0$). This is expected because, for quasi-plastic cases, the geometric Poisson distribution of N_c produces very little error. So the model works very well without introducing k_0 and δ_k . These parameters are introduced mainly for the fishnets with links that are quasibrittle to nearly brittle ($|K_t| \gg 0.1K_0$).

Since approximating the distribution of N_c introduces error, one may wonder why not use directly the histogram of N_c for the coefficients p_k ? Indeed, in this way we could get even better estimations. But the sample size of histogram of N_c in our case is 10^4 , which

is still too large for any histogram test to achieve. So, in application, using the sample mean and variance to infer the parameters of geometric Poisson distribution remains to be a much more practical and reliable way.

Clearly, the decrease of magnitude of the softening slope K_t makes the whole structure much safer, especially at the $P_f = 10^{-6}$ level. When $K_t = -0.5K_0$, the strength at which $P_f = 10^{-6}$ is about 4.06 MPa, while, when $K_t = -0.1K_0$, this strength increases to about 6.05 MPa—a strength increase of almost 50%! For comparison, the strength enhancement at the median level ($P_f = 0.5$) is about 22% (6 MPa to 7.77 MPa). Though the numbers will change for different model configurations, the considerable strength increase at the lower tail of failure probability is found to be a common feature when the softening slope changes from steep to flat.

2.5. Conclusions

- In the early stage of uniaxial loading, the damages (or partial stress drops) spawn within the fishnet in a scattered fashion. The nominal stress σ_N keeps staying very close to the strength of the k -th weakest link $s_{(k)}$, where k is the number of damaged (i.e., softened) links in the fishnet.
- After a certain critical moment of loading, new damages cease to be scattered and begin to localize. In the process, the nominal stress σ_N begins to deviate from $s_{(k)}$ and quickly drops to 0.
- The softening slope, K_t , of links controls the stochastic “time” (N_c) of damage localization. A flatter softening slope delays the occurrence of damage localization by increasing the mean of N_c .

- Based on the fact that the maximum loads are reached before the damage localization, the structure strength is approximated by the strength of the N_c -th weakest link at maximum load s_{N_c} , multiplied by a constant, γ_{N_c} , which is slightly greater than 1. The probability of failure for the whole fishnet can be formulated on the basis of this approximation.
- Both Monte Carlo simulations and analytical results show that fishnets with links of flatter softening slopes, which may represent the real nacre more closely, have a much higher strength at the extremely low failure probability level, 10^{-6} , compared with fishnets with links having steep softening slopes.

CHAPTER 3

Statistical Size Effect of Strength Distribution

3.1. Introduction

In chapter 1 and 2, we have developed the probabilistic model for the strength distribution of nacre-like biomimetic materials. In the current chapter we first explore the statistical size effect of quasibrittle fishnets. Then we establish a connection between the size effect relation and the lower tail strength distributions.

The amazing mechanical robustness of bioinspired hierarchical materials, such as nacre [2, 19, 33], the shell of a conch [34, 35] or the exoskeleton of a figeater beetle [36], has been extensively studied over the past two decades. Understanding of the deterministic toughening mechanism and of the critical role of the hierarchical fine-scale structure in enhancing material toughness has been achieved and fostered the advent of novel bioinspired materials [37, 38, 39]. However, the stochastic aspects of these hierarchical materials and their effects on the mechanical behavior have received relatively little attention. Nukaka [40] et al. simulated the process of random fracturing of nacre by the random fuse model (RFM). Recently, Najmul [41] et al. modeled the fracturing process using discrete element method (DEM) with randomized material properties. The effect of randomized parameters on strength has been discussed only qualitatively.

To clarify the effect of nacreous material architecture on the failure probability distribution up to the tail probability of 10^{-6} , the connectivity of the lamellae in nacre

under longitudinal tension has been simplified as a diagonally pulled fishnet with brittle links. In addition to the classical weakest-link and fiber-bundle model [42, 43, 14], the fishnet turned out to be the third failure model whose probability distribution is analytically tractable [20, 21]. The fishnet links were assumed to be brittle, with a steep drop after reaching the link strength. The fishnet strength distributions have been compared with those of fiber bundle [14, 22] and of the finite weakest-link model [44, 24, 9, 18, 6, 7, 45, 46]. Subsequently, based on order statistics, the fishnet model was extended [47] to take into account possible gradual postpeak softening of the individual links. In that case, to determine the failure probability distribution at the peak load, it was found necessary to consider the number of damaged links at the peak load as a random variable required as an input.

A big challenge for most engineering structures, such as aircraft, bridges or micro-electronics, is that they are supposed to be designed for failure probability less than 10^{-6} per lifetime. The tolerable load that satisfies this probability is next to impossible to determine experimentally, since about 10^8 test repetitions would be needed. Therefore, a theory is needed. It must, of course, be experimentally verified by other means. The best and virtually the only means available is the size effect on the nominal strength of the structure. As already shown for calibration of the Gauss-Weibull distribution of quasibrittle materials such as concrete, rocks, tough ceramics, fiber polymer composites, etc. [9], it suffices, for geometrically scaled specimens, to obtain experimental data for the structure size effect on the mean strength. Here, for the sake of simplicity, instead of the sample mean strength, we consider the structure size effect on the sample median strength even though its variance is higher.

It should be noted that there are two types of size effect [9]. Type 2 [48, 49] is typical for reinforced concrete. It occurs when geometrically similar large cracks develop in a stable manner prior to the reaching the peak load, and is also obtained in specimens with large scaled notches. Type 2 size effect is caused by the size dependence of the release of stored strain energy. Its mean is deterministic because the crack tip location at maximum load is determined by mechanics.

Type 1 [50], which is the type considered here, occurs to structures that become unstable as soon as a macrocrack initiates from the zone of distributed damage. The macrocrack initiation can occur at many places within the random strength field, which is why the mean size effect originates from the randomness of material strength, except for the effect of stress redistribution due to finite size of the damaged zone, captured, e.g., by the finiteness of the weakest-link chain model [9] (for a broad review quasibrittle and fishnet failure probability; see [51]).

In the fishnet model, similar to all Type 1 failures, the peak load is decided by the failure of only a few links among many. Thus, in a scaled-up large specimen, the few links controlling the peak load can occur in very many locations in the structure and, for a large enough specimen, can thus sample a broad range of failure probability distribution. This is the intuitive explanation of the fishnet size effect. Note also that, except for very large structures, the classical Weibull statistical size effect cannot be applied to nacreous and other hierarchical or architected materials. The reason is that the one-to-one correspondence between the size effect and the strength distribution is no longer valid for materials and structures that do not behave like a chain, such as a fishnet.

Here we first study the statistical size effect for rectangular fishnets with softening links. Instead of the exact expression, we formulate a tight upper bound on the lower tail distribution of the structural strength. We achieve it by splitting the geometrically similar scaling into longitudinal and transverse scalings, and then analyze them separately. As expected, longitudinal scaling gradually transforms the structure into a chain, and transverse scaling into a fiber bundle. Proportional combination of the two types of scaling finally gives us a size effect transitioning between a chain and a fiber bundle. Apart from the scaling relations of the mean and median, we also study those of the variance and coefficient of variation (CoV).

3.2. Median versus Mean Strength

First we try to formulate the median size effect, which describes the dependence of median structure strength on its size. It is similar to the mean size effect but is determined more easily. In the Weibull-scale plot of the strength distribution, i.e., in the plot with ordinate $\ln[-\ln(1 - P_f)]$ (where P_f = failure probability), the median is the point of ordinate $\ln(\ln 2)$ (for which $P_f = 0.5$).

On the other hand, even though the mean of strength could be quite hard to calculate analytically from the distribution, the mean would have practical advantages: The sample mean would converge faster than the sample median, in the light of the law of large numbers. Therefore, one could get an accurate estimation of the true mean by a rather small number of tests. As a result, the sample mean is always preferred in testing.

To keep the advantages of both the mean and median, we derive the size effect curve of the median, because of simplicity. But, to infer the failure probability, we treat the

sample means from the tests as the medians. The effect of replacing the median strength with the mean strength in the calculations is discussed later.

3.3. Correspondence between P_f and Its Median Size Effect

As mentioned in the introduction, the main purpose of obtaining the expression of statistical size effect is to infer the failure probability, i.e. the strength distribution P_f . Of particular interest is the lower tail of P_f , which is almost impossible to reach with the traditional histogram testing. It is clear that for each individual strength distribution $P_f(\sigma)$, there is one and only one median strength, $\sigma_{0.5}$, that, by definition, satisfies the condition that $P_f(\sigma_{0.5}) = 0.5$.

The reverse, however, is not necessarily true. Given a mean or median size effect curve of a structure, there may or may not be only one strength distribution that corresponds to the size effect curve. An example of multiple correspondence is the Gaussian (normal) distribution. Its variance could vary while keeping the mean fixed, therefore, knowing the mean value is not enough to fully determine the probability distribution. On the other hand, the Weibull distribution, in which the mean and variance are fully coupled [44], is an example of one-to-one correspondence:

$$(3.1) \quad s^2 = \mu^2 \left(\frac{\Gamma \left(1 + \frac{2}{m} \right)}{\Gamma^2 \left(1 + \frac{1}{m} \right)} - 1 \right),$$

where μ and s^2 are the mean and variance of the Weibull distribution, respectively, and m is the Weibull modulus. Eq. 3.1 indicates that, given the constant m , the variance is completely fixed if the mean is known (the variance-mean coupling disappears for the

three-parameter Weibull distribution, but its non-zero threshold is incorrect for strength statistics [9]).

As shown previously [20], the strength distribution of the fishnet is in the middle of the transition from Weibull to Gaussian distribution. Thus it must be concluded that the correspondence between the median (mean) size effect curve and the strength distribution P_f is not one-to-one. Therefore, it is impossible to get the exact distribution function only from the size effect curve. Instead, to make sure the prediction would be safe for the design purposes, we seek an upper bound on P_f from the mean size effect test data.

3.4. Formulation of Median Statistical Size Effect

Fig. 3.1 shows the geometries of the four fishnets used in the current study. The fishnet with length 128 and width 64 is the largest one, due partially to space limitations. For all the sizes, the aspect ratios are kept constant, as 1:2. We first study the evolution of strength distribution under longitudinal and transverse scaling, respectively. Then the scaling of geometrically similar structures is treated as the combination of the two.

Same as previous studies [20, 47], the structure of a nacreous material is simplified into a rectangular fishnet consisting of r rows and s columns of links (the rows run in the direction of tension). The mechanical behavior is similar to the so-called pantographic structures [52, 53], except that we consider the truss links to be pint-jointed, with no rotational stiffness. Based on the arguments in [6, 9, 45], the link strengths are treated

as random variables that obey the grafted Gauss-Weibull distribution:

$$(3.2) \quad P_1(\sigma) = \begin{cases} 2.55 \left(1 - e^{-(\sigma/12)^{10}}\right), & \sigma \leq 8.6 \text{ MPa} \\ 0.53 - 0.47 \operatorname{erf}[0.88(10 - \sigma)], & \sigma > 8.6 \text{ MPa} \end{cases}$$

The link strength distribution is here considered to be directly given, instead of being derived from a homogenized continuous random field. In other words, the autocorrelation length is set to be the same as the link length, reflecting the discrete nature of the fine-scale structure of nacreous materials. Also, by doing this, we have direct control over the link strength distribution $P_1(\sigma)$ and its power-law tail is strictly preserved.

The fishnet links are quasibrittle and obey a linear softening law. The brittleness of a link is characterized by the ratio $|K_t/K_0|$, where K_0 is the initial elastic stiffness of the undamaged fishnet links, and K_t (< 0) is the tangential softening stiffness. The steeper the softening slope, the more brittle the link. The fishnets are pulled uniaxially under displacement control. Accordingly, the nominal strength, σ , is measured from the boundaries: the total reaction force at peak load divided by the cross-section area. Our main interest lies in the strength distribution, $P_f(\sigma)$, of the whole fishnet and the size effect on its median. Fig. 3.2 shows some of the typical numerical results, such as the stress-displacement curves and damage evolution. Since the mechanical response for each size is similar, we show the results for only one size: $r = 32$ and $s = 64$.

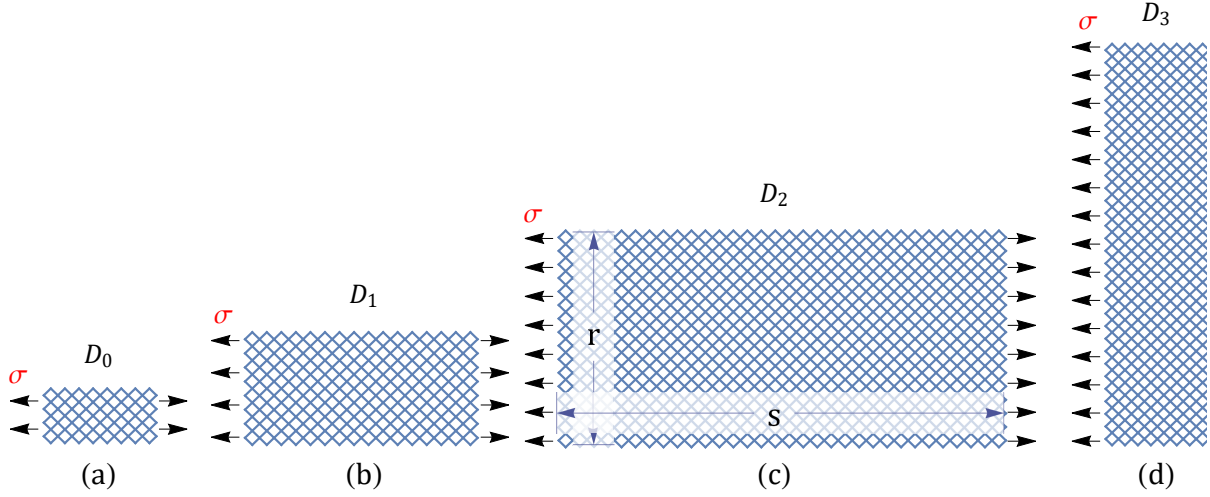


Figure 3.1. Schematic showing 4 geometrically similar fishnets under uniaxial tension: (a) 8×16 ; (b) 16×32 ; (c) 32×64 and (d) 64×128 (partially shown). r and s denote the number of fishnet rows and columns respectively.

3.4.1. Longitudinal Scaling

3.4.1.1. Weakest-Link Behavior. To study the effect of longitudinal scaling on the strength distribution, we fix the number of rows of the fishnet at 16 ($r = 16$) and let the number of columns increase geometrically from 8 to 64 ($s = 8, 16, 32$ and 64).

The structure begins behaving more like a chain as its length increases while fixing its width. Intuitively, this implies that the limiting strength distribution of the structure is Weibullian and that long enough segments of the chain have almost independent strengths. For a long chain of n segments, each of which has a survival (or strength) probability $P_s(\sigma)$, the failure probability (or strength) of the chain, P_f , satisfies the equation:

$$(3.3) \quad 1 - P_f(\sigma) = [1 - P_s(\sigma)]^n$$

In other words, the survival probability of the whole chain is the joint probability of survival of all its segments (or, in other words, no segment can fail if the whole chain is

safe). Taking a logarithm on both sides of Eq. 3.3 twice yields:

$$(3.4) \quad \ln\{-\ln[1 - P_f(\sigma)]\} = \ln[-\ln(1 - P_s(\sigma))] + \ln n$$

Since the ordinate and coordinate used in the Weibull-scale plot (or Weibull paper) is $\ln \sigma$ and $\ln[-\ln(1 - F(\sigma))]$, respectively, the graph of P_f for the whole chain in Weibull scale is equivalent to shifting the graph of the segment, P_s , upward by distance $\ln n$. This applies, of course, only for long enough fishnets, and so deviations must be expected for short fishnets.

Fig. 3.3 shows the strength histograms obtained by Monte Carlo simulations of quasibrittle fishnets of 4 different lengths (each point in these histograms represents the mean of several hundreds of fishnet FEM simulations). It is clear that for fishnets whose lengths are greater than their width ($s > r$), the strength distributions are nearly straight lines in Weibull scale. In addition, the vertical distance between neighboring histograms is a constant, $\ln 2$, which is the logarithm of the ratio of the neighboring fishnet lengths, $\ln s_2/s_1$.

These observations verify the weakest-link assumption of long fishnets. It is also observed that for the shortest fishnet with length 8, the upper and lower tail of its histogram deviate significantly from the straight line. This makes sense because this fishnet, due to its short length, is more like a fiber bundle than a chain. Similar to the chain-of-bundles model [54], this shows that the weakest-link model is a good approximation only when long enough longitudinal sections are considered as the longitudinal representative volume elements (RVE). The model would be invalid if either short sections or single links were treated as the RVE.

It is interesting, though not surprising, that the convergence of histogram to Weibull distribution is fast in the central region close to the mean, while both the upper and lower tails converge much slower. The Weibull modulus, which represents the slope of the distribution in the Weibull scale, does not converge to the biggest value, $m = 80$, at its extreme lower tail (Fig. 3.3) but to an intermediate value $m = 50$. Given that the Weibull modulus of the link strengths $P_1(\sigma)$ is $m = 10$. The converged Weibull modulus is 5-times as large as that of a single link, which is due to the transverse coupling of links along the transverse direction. In other words, there are, on average, 5 damaged links for each shortest independent section prior to the maximum load. This interesting phenomenon could be related to the post-peak softening of individual links and deserves further study, which is beyond the scope of the current paper.

3.4.1.2. Dependence of the Median Strength on Fishnet Length. As already discussed, we try to establish a relation between the size effect curve and one corresponding upper bound of P_f . Relying on the fact that the histogram is always concave (i.e., no center of curvature lies above the histogram curve), we choose the tangent line through the median strength as the upper bound of P_f . Typically, we care about the slope near the mean or median strength, because this slope is easy to obtain from a limited number of tests. The straight lines in Fig. 3.3 are the tangent lines through the median strengths of each histogram. It is clear that they are the upper bounds for each corresponding distribution. Moreover, these upper bounds are Weibull distributions, which satisfy the weakest-link assumption, that is, the vertical distance between two adjacent lines is determined by the logarithm of the length ratio of the two fishnets, $\ln s_2/s_1$. Suppose the strength distribution in the Weibull scale is $Y = mX + C$ for the fishnet of reference

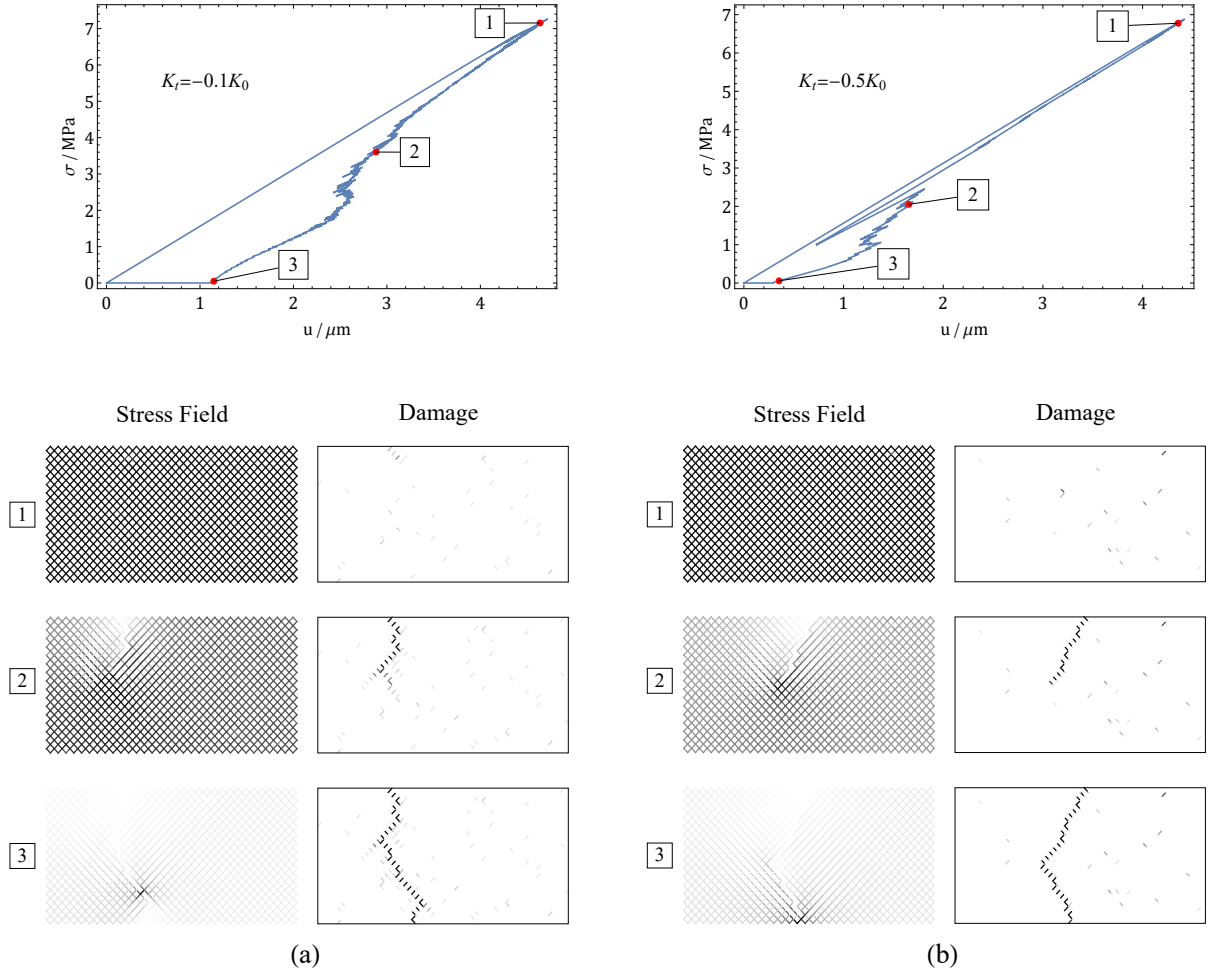


Figure 3.2. typical stress-displacement curves for fishnets of size $r \times s = 32 \times 64$ and its evolution of stress field and damage: a) $K_t = -0.1K_0$; b) $K_t = -0.5K_0$ (more brittle). For each case, the number of discrete softening jumps to reach zero stress in each link $J = 20$.

length, s_0 , where $X = \ln \sigma$, and $Y = \ln[-\ln(1 - P_f)]$. Then, from Eq. 3.4, the median strength, $\sigma_{0.5}$, of the fishnet with length s satisfies the following relation in Weibull scale:

$$(3.5) \quad Y = mX + C + \ln(s/s_0), \text{ where } X = \ln \sigma_{0.5}$$

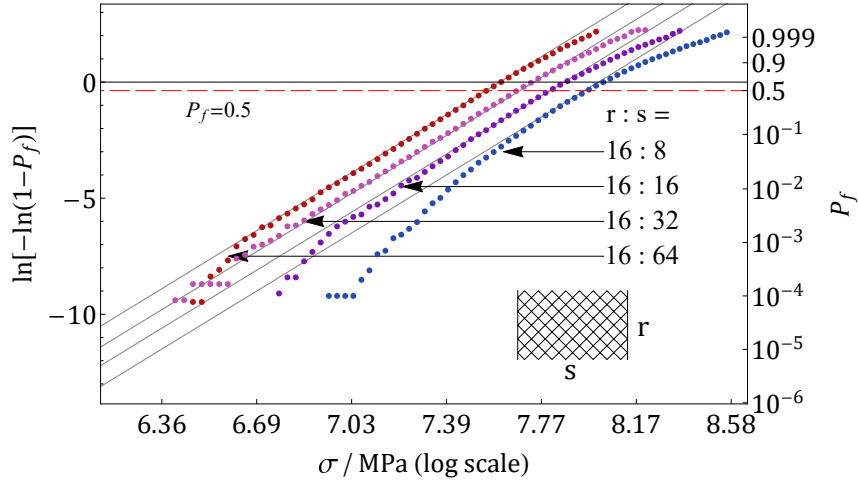


Figure 3.3. Strength histogram (in Weibull scale) of various fishnets under **longitudinal** scaling. The widths of all fishnets, r , are fixed at 16 rows while their lengths, s , vary geometrically from 8 to 64 columns. For each case, the sample size is 10^4 , softening slope $K_t = -0.1K_0$, and number of discrete softening jumps for each link $J = 20$.

Solving for X yields:

$$(3.6) \quad \ln \sigma_{0.5} = X = -\frac{1}{m} \ln(s/s_0) + \frac{1}{m}(Y - C)$$

This represents the classical Weibull statistical size effect[44], where the logarithm of median (or mean) strength, $\ln \sigma_{0.5}$, depends linearly on the logarithm of normalized structure size (or length), $\ln(s/s_0)$, and the slope of the line is $-1/m$.

3.4.2. Transverse Scaling

3.4.2.1. Evolution of Histogram under Transverse Scaling. To study the effect of transverse scaling on the strength distribution of the whole structure, we fix the number of columns of the fishnet at 16 links ($s = 16$) and let its number of rows increase in geometric sequence from 8 to 64 ($r = 8, 16, 32$ and 64). Fig. 3.4 shows the strength

histograms obtained by Monte Carlo simulations of quasibrittle fishnets of the 4 different widths.

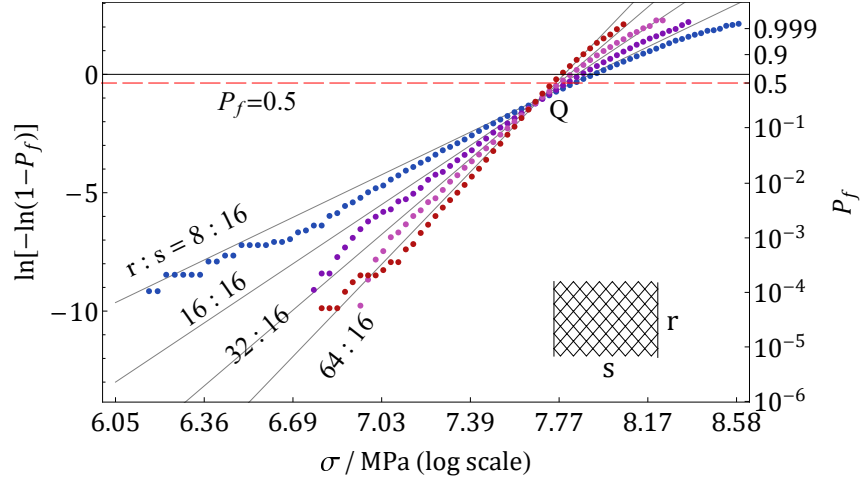


Figure 3.4. Strength histogram (in Weibull scale) of various fishnets under **transverse** scaling. Lengths of all fishnets, s , are fixed at 16 while their widths, r , vary in geometric sequence from 8 to 64. For each case, the sample size is 10^4 , the softening slope $K_t = -0.1K_0$, and the number of discrete softening jumps to reach zero stress in each link $J = 20$.

As can be seen, the pattern of strength distributions under transverse scaling is significantly different from longitudinal scaling shown in Fig. 3.3. As the fishnet becomes wider in the vertical direction (transverse to the directions of tension), the variance of the resulting strength histogram becomes smaller and the apparent Weibull modulus becomes larger. Because more links could fail or soften before reaching the maximum load of the whole structure. As a consequence, more survival probabilities need to be considered to formulate the strength distribution, leading to a larger slope at the lower tail.

It is striking that all the histograms intersect at almost a single point, Q , on the curve (see Fig. 3.4). This is explained by two observations:

- (1) The upper tail region (e.g. $P_f \geq 0.9999$ or $\ln[-\ln(1 - P_f)] \geq 2.22$) of the histogram always follows the weakest-link rule. Because the additional terms from the fishnet statistics do not affect the upper tail of the histogram in that the applied load is much larger than the average link strength, the neighborhood of any failed link is unlikely to survive under the redistributed stresses. The result is that the failure of any single link leads to the failure of the whole structure, same as in the weakest-link chain.
- (2) Except for the upper tail, the remaining histograms are almost straight lines in the Weibull scale. And the slope (Weibull modulus), m , increases as the fishnet width is scaled up in geometric sequence (see the straight line fit in Fig. 3.4). In addition, m grows almost linearly with $\ln r$.

With these two observations, it is now clear that, as the right end of histogram shifts up under transverse scaling, the slope also increases. Therefore, the histograms for fishnets of any two widths must intersect at a point. The reason why they intersect at a single point, Q , is that both the upward shift and the slope increase are nearly proportional to $\ln r$.

Note that the location of point Q depends on the rate of slope increase of the histogram under transverse scaling of the fishnet. The greater the rate, the higher the location of the intersection point. Meanwhile, the rate of histogram slope increase is related to the brittleness (or softening slope K_t) of the fishnet links. Therefore, the location of point Q can be seen as an indicator of the material brittleness. As the links become more brittle, the rate of slope increase gets smaller, causing the position of Q to shift further away to the lower left of the graph.

On the other hand, one can check that, as the links become more ductile, Q will be closer to the median of the distributions. The reason is that, in the limiting case of elastic-perfectly plastic links ($K_t \rightarrow 0$), the fishnets of various widths follow (in the light of the central limit theorem) the Gaussian distributions of the same mean (or median) but different variances. In that case, point Q lies exactly on the median line. Thus, the coordinate of point Q is bounded from above by the median line: $\ln[-\ln(1-P_f)] = \ln \ln 2$.

In the limiting case of $r : s \rightarrow \infty$, the strength distribution approaches that of a fiber bundle consisting of softening fibers. Again, the brittleness of the links governs the softening slope of the limiting constituent fiber. The resulting load-displacement curve is the mean curve of the responses of each single fiber. Since the number of fibers in the limiting bundle is infinite, the overall load-displacement curve is almost deterministic (converges to the mean load-displacement curve) in light of law of large numbers, though strengths of each individual fibers are random variables.

Consequently, the bundle strength is the maximum load from the limiting deterministic load-displacement curve and the corresponding failure probability is a degenerate distribution (Heaviside step function) with finite mean and zero variance. But in this case, the limiting bundle strength is strictly smaller than the mean fiber strengths due to softening of fibers. To be more specific, we could calculate analytically the mean load-extension curves for quasibrittle bundles. Without loss of generality, we ignore the dimensions of stress and strain (e.g. normalized by some constant stress and strain), and consider the fiber stiffness to be equal to 1 and fiber peak load to follow the distribution $G(\sigma)$, whose density function is $g(\sigma) = G'(\sigma)$. Then the average load-extension curve for

the whole bundle is:

$$(3.7) \quad f(\sigma) = \int_{\sigma}^{\infty} \sigma g(\tau) d\tau + \int_{\eta\sigma}^{\sigma} \left(\tau + (\sigma - \tau) \frac{K_t}{K_0} \right) g(\tau) d\tau$$

where

$$(3.8) \quad \eta = \frac{K_t}{K_t - K_0}$$

The first integral in Eq. 3.7 reduces to $\sigma[1-G(\sigma)]$, which is the mean load-displacement curve for brittle bundles, as shown by Daniels [14] in 1945. It only accounts for the situation where fibers survive under load σ (as brittle fibers directly fail after strength is reached). The second integral in Eq. 3.7 considers the case where fibers are softened but still not completely failed, and thus contributes to the bundle strength. The lower bound of the second integral is $\eta\sigma$ rather than 0, because the residual strength of the softened fiber is $\tau + (\sigma - \tau)K_t/K_0$, which is greater than zero when $\tau > \eta\sigma$. Otherwise, the fiber has failed completely, contributing no load. Note that if $K_t \rightarrow -\infty$ (perfectly brittle fibers), the value of η approaches 1, making the second integral in Eq. 3.7 vanish. Then the result reduces to the brittle case studied by Daniels.

Fig. 3.5 shows the comparison of average load-extension curves of bundles with three different brittlenesses. It is clear that as constituent fibers become more brittle, the peak load decreases.

As the other extreme case, in which the links are perfectly elastic-plastic, i.e. a ductile bundle, the law of large numbers directly applies to the bundle strength, while in the previous case it only applied to the entire load-displacement curve. Therefore, the bundle strength is not only a constant, it is exactly the same as the mean fiber strength.

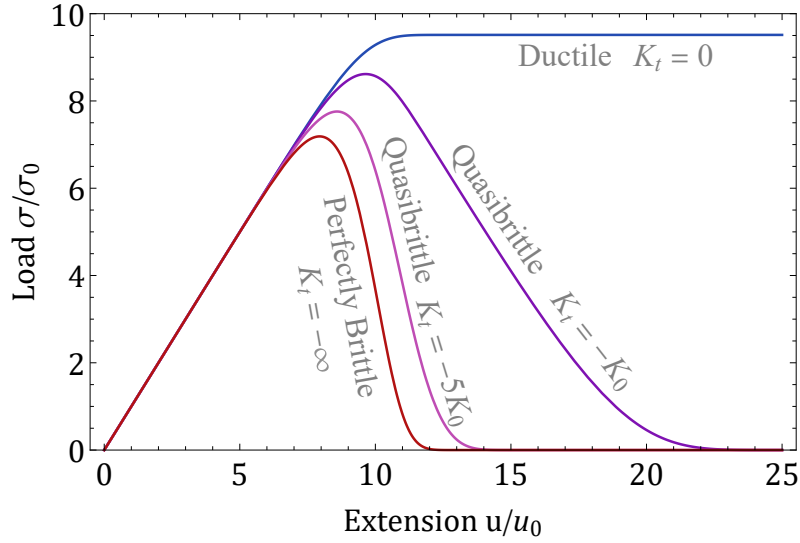


Figure 3.5. Comparison plot of analytical results of mean load-extension curves of bundles with different fiber brittleness. Fiber strengths are i.i.d. random variables and follow the same distribution $G(\sigma)$ for all 3 cases.

To conclude, for extremely wide fishnets, for which $r : s \rightarrow \infty$, their points of rotation Q are all on the median line $y_0 = \ln \ln 2$, but their horizontal positions x_0 vary due to different post-peak softening slopes of the fibers, and are bounded by the mean fiber (or link) strength $\bar{\sigma}$, i.e., $x_0 \leq \ln \bar{\sigma}$, which is an equality that holds for perfectly ductile bundles.

3.4.2.2. Dependence of the Median Strength on Fishnet Width. Now that we have clarified how the histograms evolve under transverse scaling of the fishnet width, we can obtain the relation between the median strength and the fishnet width. Let the coordinates of the intersection point Q be (x_0, y_0) . Then the family of Weibull approximations of the true strength distributions under transverse scaling can be expressed as:

$$(3.9) \quad Y - y_0 = m(r)(X - x_0),$$

where $m(r)$ is the apparent Weibull modulus that, for simplicity, is assumed to depend linearly on $\ln r$:

$$(3.10) \quad m(r) = m_0[1 + c \ln(r/r_0)]$$

Here, m_0 is the Weibull modulus for the reference fishnet width, and c is the rate of slope (or Weibull modulus) increase. Combining Eq. 3.9 and Eq. 3.10, and plugging in $Y = \ln(\ln 2)$, one can solve for the median strength, $X = \sigma_{0.5}$:

$$(3.11) \quad \ln \sigma_{0.5} = \frac{\ln(\ln 2) - y_0}{m_0[1 + c \ln(r/r_0)]} + x_0$$

3.4.3. Statistical Median Size Effect

As discussed earlier, the scaling of geometrically similar structures is treated as the combination of longitudinal and transverse scaling in a proportional fashion. To make sure that the scaling is geometrically similar and the aspect ratio is a constant, the scaling of fishnet length and width must be done at the same rate:

$$(3.12) \quad s = Ds_0, \text{ and } r = Dr_0,$$

where $D > 1$ is the dimensionless size. Combining the effect of longitudinal (Eq. 3.6) and transverse (Eq. 3.11) scaling on the strength distribution gives:

$$(3.13) \quad Y - y_0 = m_0[1 + c \ln(r/r_0)](X - x_0) + \ln(s/s_0),$$

Here the first term on the right-hand side of the equation is the same as the expression for transverse scaling (Eq.3.9). The second term takes into account the longitudinal (chain)

scaling (Eq.3.5). Finally, we replace r/r_0 and s/s_0 by the dimensionless size D and set $P_f = 0.5$, or equivalently $Y = \ln \ln 2$. Solving for X yields:

$$(3.14) \quad \ln \sigma_{0.5} = X = \frac{\ln \ln 2 - y_0 - \ln D}{m_0(1 + c \ln D)} + x_0$$

Eq. 3.14 is the median size effect relation that describes the relation between the logarithms of the median strength X and the dimensionless size, D . There are four parameters in this relation; they are: x_0 , y_0 , c and m_0 , where (x_0, y_0) is the coordinate of the point of rotation, Q , under transverse scaling; c is the rate of slope increase for histograms under transverse scaling; m_0 is the apparent Weibull modulus for the reference size fishnet. Note that m_0 is generally not the same as the Weibull modulus of the link strength distribution, and depends on the shape of the chosen reference size fishnet.

3.5. Inference of P_f from Size Effect

The ultimate goal of obtaining the size effect curve (Eq. 3.14) is to infer the upper bound on the failure probability distribution (or the strength distribution) of the whole structure for various sizes. This is achieved by substituting into Eq. 3.13 the parameters obtained from the size effect curve:

$$(3.15) \quad Y = \ln[-\ln(1 - P_f)] = m_0(1 + c \ln D)(X - x_0) + y_0 + \ln D$$

Solving for $P_f(\sigma)$ and using the fact that $X = \ln \sigma$ yields the upper bound (i.e., Weibull) estimation of the strength distribution:

$$(3.16) \quad P_f(\sigma) = 1 - \exp \left\{ -n \left(\frac{\sigma}{\sigma_0} \right)^m \right\},$$

where $n = De^{y_0}$, $\sigma_0 = e^{x_0}$ and $m = m_0(1 + c \ln D)$.

Fig. 3.6 and Fig. 3.7 show the histograms of fishnet strength for various sizes. It is a common feature that both the mean and the variance of strength decrease as the fishnet size increases. In addition, the variances for the more ductile case ($K_t = -0.1K_0$) are much smaller than those for the more brittle case ($K_t = -0.5K_0$);

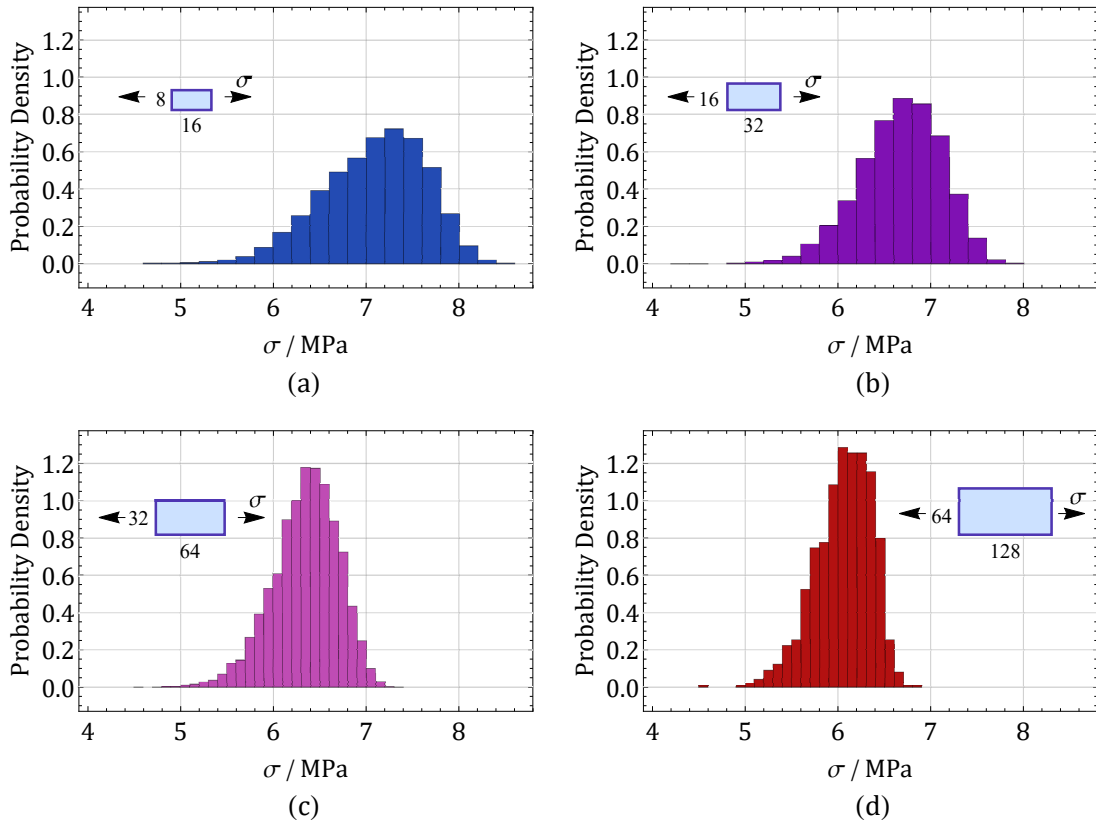


Figure 3.6. Strength histograms of fishnets ($K_t = -0.5K_0$) of 4 typical sizes under uniaxial tension: a) 8×16 ; b) 16×32 ; c) 32×64 ; d) 64×128 . The counts have been normalized to probability density.

To illustrate the mean or median size effect, Fig. 3.8 shows the median size effect curves using the median strengths of the samples obtained from the histograms of the Monte Carlo simulations in Figs. 3.6–3.7. The optimum fit of the data by Eq. 3.14 is also

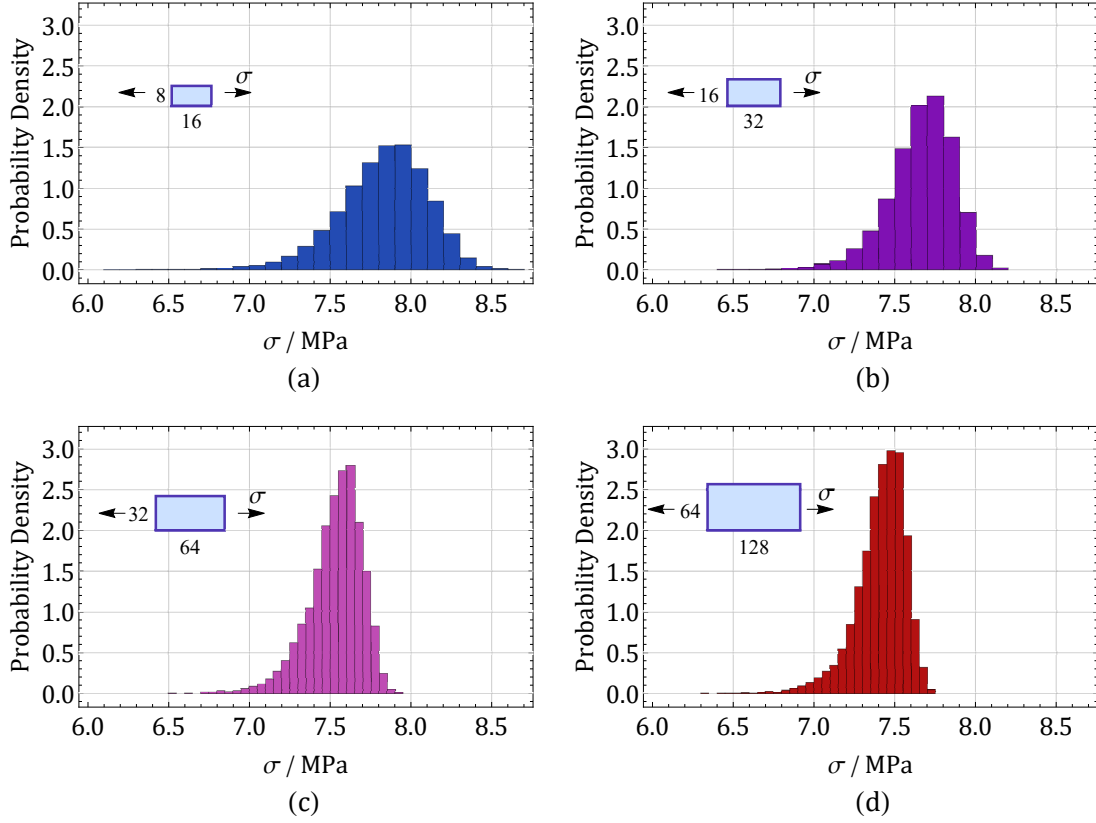


Figure 3.7. Strength histograms of fishnets ($K_t = -0.1K_0$) of 4 typical sizes under uniaxial tension: a) 8×16 ; b) 16×32 ; c) 32×64 ; d) 64×128 . The counts have been normalized to probability density.

shown for comparison. It is clear that the size effect relation is monotonically decreasing and has a convex curvature, which is a general feature of the type-1 size effect [24, 9]. As discussed earlier, parameters c and y_0 indicate the brittleness of the fishnet links. By comparing the parameters of the optimum fit in Fig. 3.8a,b, we see that the rate of slope increase under transverse scaling, c , increased from 0.25 to 0.27 as the links became less brittle. Meanwhile, the distance between Q and the median line, $y_0 - \ln \ln 2$, decreased from 2.63 to 0.93: the more ductile the material, the closer it is to the median line for Q .

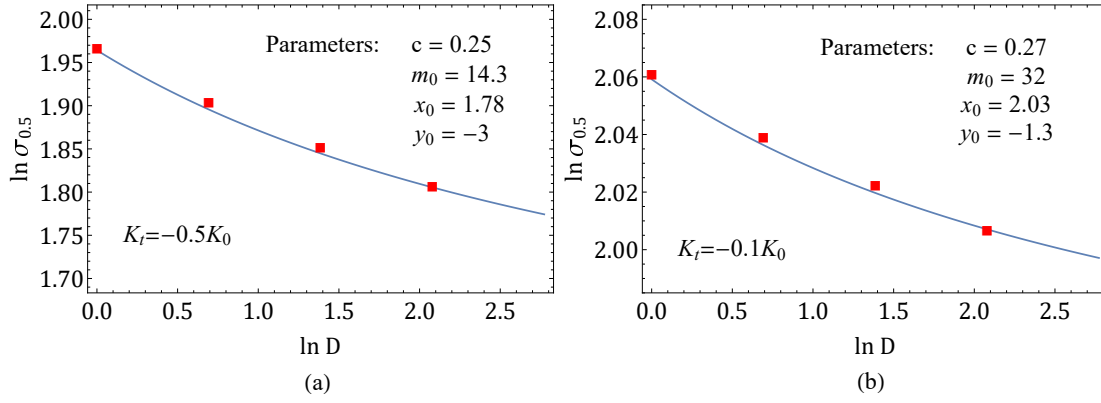


Figure 3.8. Optimum fit of the size effect relation ($\ln \sigma_{0.5}$ vs $\ln D$) using the sample median strength obtained from Monte Carlo simulations. Two different brittleness levels given by $K_t = -0.5K_0$ and $-0.1K_0$ are considered here. The data points are the sample median strengths for the 4 typical sizes and the curve is the optimum fit.

Using the optimal parameters, one obtains the predictions of the upper bound failure probability for the fishnet. The predictions and actual histograms are shown, for comparison, in the Weibull scale in Fig. 3.9. In general, the predictions match the histograms pretty well in the regions near and above the mean strength. Since the curve of the actual failure probability keeps bending downward as the slope keeps increasing while moving toward the lower tail, it is no surprise that the histograms starts to deviate from the Weibull upper bound, which is a straight line.

Note that all the predictions intersect at a single point. An interesting consequence of this is that, below the point of intersection, the structure strength might increase as the size increases, which is exactly the opposite of the trend at the mean level. Whether this reverse trend would actually happen in the probability range that we are interested in ($10^{-7} \leq P_f \leq 10^{-5}$) depends on the brittleness of the material. This indicates that, even though the structure becomes weaker as it becomes larger, in the sense that the mean

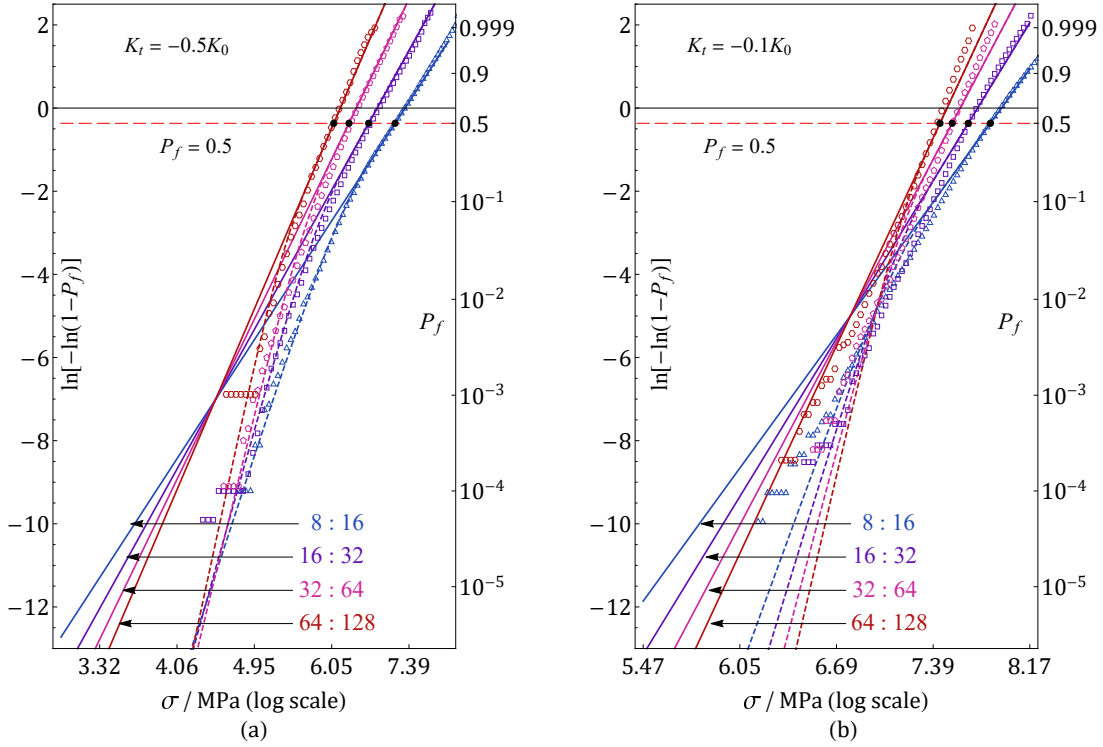


Figure 3.9. Comparison of predicted (thick lines) upper bound strength distributions of the fishnet with the actual histograms (discrete markers). Optimum fit by homogenized 2-term fishnet model are shown by dashed curves.

strength decreases, it could become stronger at the level of $P_f = 10^{-6}$ in the lower tail of failure probability. In other words, a reverse size effect can occur at the tail of failure probability level.

Finally, due to a much faster convergence rate, one could also use the mean strengths from the test data to replace the median strengths used in the current model. This will lead to a safer (less tight) upper bound compared to using the median. The reason is that, for Weibull distribution of modulus $m > 3.44$, the median is slightly greater than the mean (precisely, the mean $= \sigma_0 \Gamma(1 + 1/m)$, and the median $= \sigma_0 (\ln 2)^{1/m}$). On the other hand, for symmetric distributions such as the Gaussian, the mean is exactly equal to

the median. Since general strength distributions of fishnets lie between those of Weibull and Gaussian distributions, their medians are slightly larger than their mean values. This fact, which is the consequence of negative skewness of the Weibull and related fishnet distributions, means that the substitution of a mean for the median predicts the safety to be slightly higher than it actually is.

3.6. Homogenization of Fishnet Statistics

Noting that the actual strength histogram deviates downward from our upper bound prediction and bends down more in the lower tail, We may naturally ask whether there exists a general model that could capture this phenomenon. For this purpose, we resort to the two-term fishnet model with a homogenized (equivalent) link strength distribution $\tilde{P}_1(\sigma)$.

First, we start with the upper bound prediction Eq. 3.16, which is the Weibull distribution obtained in the previous section and is denoted as $P_{f_0}(\sigma)$. The predicted Weibull modulus m in $P_{f_0}(\sigma)$ is larger than the value for a single link m_0 . The ratio m/m_0 indicates the size of the equivalent coarse-grained “link”. Therefore, the total number of equivalent “links” is $N_{\text{eq}} = rs/(m/m_0)$. Then, $\tilde{P}_1(\sigma)$ is calculated by reverse scaling from $P_{f_0}(\sigma)$ for the equivalent chain back to a single link:

$$(3.17) \quad \tilde{P}_1(\sigma) = 1 - [1 - P_{f_0}(\sigma)]^{1/N_{\text{eq}}}$$

Finally, we apply the two-term fishnet model to the coarse-grained system:

$$(3.18) \quad P_f(\sigma) = P_{f_0}(\sigma) - N_{\text{eq}} \tilde{P}_1(\sigma) [1 - \tilde{P}_1(\sigma)]^{N_{\text{eq}} - \nu - 1} [1 - \tilde{P}_1(\eta\sigma)]^\nu,$$

where ν is the equivalent size of the stress redistribution region, and η is the equivalent stress redistribution factor.

The dashed curves in Fig. 3.9 show the optimum fit of the histogram data given by Eq. 3.18. The homogenized two-term fishnet model shows the ability to capture the lower tail behavior of the strength histograms of fishnets with various sizes. However, this improvement does not come without a price. The model requires more information apart from the mean (or median) size effect. More specifically, one must know the value of m_0 in $P_1(\sigma)$, ν and η a priori, while the upper bound prediction (Eq. 3.16) only requires the size effect relation, which is much easier to obtain by tests.

3.7. Scaling Relations for Variance and CoV of Strength

Apart from the scaling relations for the mean and median values of the strength distributions, the size effect on the variance and coefficient of variation (CoV) are also critical to characterize the reliability of a structure. One would expect that the scaling relations for variance and CoV (the standard deviation divided by the mean) should depend heavily on the brittleness of the material characterized by K_t : if the material is ductile (very low $|K_t|$, the scaling law of variance is dictated by the central limit theorem, as for the ductile fiber bundle); On the other hand, if the material is brittle (very high $|K_t|$), the variance should have a different scaling relation and is mainly governed by the weakest-link rule. Therefore, we first examine the foregoing two limiting cases. Then the cases where the fishnet links exhibit gradual post-peak softening (medium $|K_t|$) should lie between the two extreme cases.

For the weakest-link model consisting of n links, the strength distribution is

$$(3.19) \quad P_f(\sigma) = 1 - e^{-n(\sigma/\sigma_0)^m} = 1 - \exp\left(\frac{\sigma}{n^{-1/m}\sigma_0}\right)^m$$

Based on this cdf, we could calculate its variance:

$$(3.20) \quad \text{Var} = (n^{-1/m}\sigma_0)^2 \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right) \right] \propto n^{-2/m}$$

Since, for two-dimensional scaling, $n = \alpha D^2$, we have

$$(3.21) \quad \text{Var} \propto D^{-4/m}$$

From Eq. 3.1, the CoV of Weibull distributions is a constant, that is, independent of n .

For the ductile case, the structural limit is a ductile bundle of n fibers. From the central limit theorem, the variance, Var , scales as n^{-1} . Since $n = \alpha D^2$, we have

$$(3.22) \quad \text{Var} \propto D^{-2}$$

In addition, the sample mean does not change with n ; therefore, the CoV scales as $\sqrt{\text{Var}}$, that is, $\text{CoV} \propto D^{-1}$.

To sum up, we show the scaling relations of variance and CoV for the brittle and ductile limits in the following table:

	Var	CoV
Brittle	$D^{-4/m}$	constant
Ductile	D^{-2}	D^{-1}

Fig. 3.10 shows the measured scaling relations of variance and CoV as functions of the dimensionless size $D = r/r_0 = s/s_0$ in log-scale. Here, the size of the smallest fishnet ($r_0 = 8, s_0 = 16$) is chosen as the reference size. The brittle and ductile limit of the scaling relations of variance and CoV match very well with the analytical expressions (Eq. 3.21 and 3.22). The slope measured for the variance of the weakest-link model is -0.38, which is slightly greater than the value predicted by the theory ($-4/m = -0.4$). This small discrepancy is due to the fact that $P_1(\sigma)$ is not completely a Weibull distribution but only that it has a Weibull lower tail.

From both figures, it is clear that the variance and CoV of the fishnet strength distribution lie in the middle of the two limits, and as fishnet links become more brittle, the variance and the CoV tend to those of the chain. As the links become more ductile, the behavior of variance and CoV is more similar to the ductile bundle. As expected, the scaling relations of variance and CoV of the strength distributions for general fishnets depend on the brittleness of the links. They transit continuously from that of a brittle chain to that of a ductile bundle. Interestingly, the CoV of the more ductile fishnet ($K_t = -0.1K_0$) gradually tends to a constant very fast (see Fig. 3.10.b). This is because, under longitudinal scaling, the overall response of the fishnet becomes more brittle, making its strength distribution converge to a Weibull distribution. As a result, the evolution of CoV tends to a constant for the large size limit. Similar behavior is also seen for the more brittle fishnet, but if the fishnet is already more brittle and has a relatively small CoV, this phenomenon is not as significant.

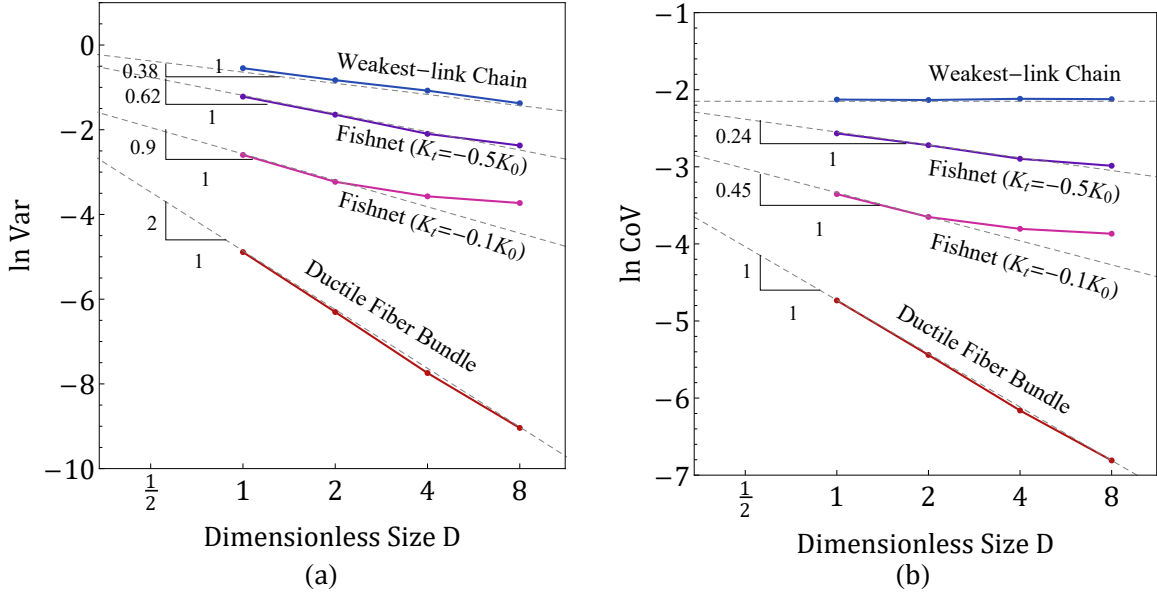


Figure 3.10. Two-dimensional geometrically similar scaling relations of a) variance; b) CoV for the weakest-link chain (brittle limit), fishnets and the ductile fiber bundle (ductile limit). Weibull modulus for the strength distribution, $P_1(\sigma)$, of links is $m = 10$.

3.8. Conclusions

- (1) The statistical median size effect relation for nacreous biomimetic materials is not difficult to derive. But an analytical mean size effect relation appears to be much more difficult to obtain.
- (2) An effective way to obtain the size effect relation is to analyze first the scalings for longitudinal and transverse (or lateral) scaling separately, and then superpose the results to obtain the two-dimensional scaling.
- (3) In view of the negative skewness of all the distributions, replacing the mean by the median yields a safe bound on the failure probability at any given stress, i.e., give a conservative result.

- (4) The longitudinal scaling leads to a vertical upward shift of the distribution in the Weibull scale, while the transverse scaling leads to a counter-clockwise rotation of the distribution about a fixed point located below the median.
- (5) Because of the rotation of the failure probability distribution at transverse scaling, a reverse size effect at two-dimensional scaling may take place, i.e., a trend opposite to the mean size effect, may occur: The strength at which $P_f = 10^{-6}$ may increase as the structure size increases. This is, of course, advantageous for design safety.
- (6) An upper bound, i.e., safe, estimate for the tail of the failure probability of nacreous materials of any size and shape can be inferred from the calibrated parameters of the size effect relation.
- (7) Unlike histogram testing, the size effect method of estimating a tight upper bound on the failure probability distribution requires only about six tests for each of several scaled fishnet sizes.
- (8) Millions of Monte Carlo simulations confirm the analytical results.
- (9) The inference method developed here is applicable to nacreous materials whose links have any postpeak softening properties, i.e., are brittle, quasi-brittle or almost elastic-perfectly plastic.

CHAPTER 4

General Form of Strength Distribution in Fishnet Statistics

4.1. Introduction

As has been discussed in Chapter 1 and 2, we have developed two models for the tail strength distribution of nacre-like biomimetic materials. In this chapter, we show that the series expansion of the survival probability discussed in Chapter 1 and the order statistics formulation in Chapter 2 are essentially equivalent. Based on this observation, we propose a general expression for the fishnet statistics and then apply it to other novel quasibrittle architected materials.

Material scientists and engineers developing new engineering structures and materials have been focusing mainly on enhancing the mean strength, while the extreme lower tail of the strength distribution $P_f(\sigma)$ has long been ignored. However, the consensus of safety experts [55, 56] is that engineering structures must be designed with failure probability not exceeding 10^{-6} , so that engineers do not add appreciably to the risks people inevitably face (the risk of dying in a car accident is in the U.S. about 1 in a hundred per lifetime, while 1 in a million is the risk of death by lightning, or by a falling tree, or by a fall in the bathroom). Maximizing the mean strength is not enough. It can even happen that that a material or structure of a lower mean strength would have, for the same coefficient of variation, a higher tail strength at 10^{-6} , and vice versa (see Fig. 4.4). Therefore, a realistic probabilistic model for structure strength is necessary.

The amazing strength of bioinspired hierarchical materials such as nacre [2, 3] and shell of conch [34], which is about ten-times greater than the strength of its main constituent, have been extensively researched over the past two decades. Studies of the toughening mechanism engendered by the hierarchical structure have fostered the development of novel bioinspired materials. However, the stochastic aspects have received little attention. Here we first review our recently proposed fishnet statistical model for nacreous biomimetic materials [57, 20, 47, 58] (as presented at the Century Fracture Mechanics Summit in Singapore on April 8th, 2019). Then we extend this model to architected nanomaterials, and show that it may also capture some failure probability features of quasibrittle materials in general, including concrete or fiber composites.

Traditionally, only two major analytically tractable models for strength statistics have been available—the weakest-link chain (or the series coupling model, due to Weibull [59], 1939) and the fiber bundle (or the parallel coupling model, due to Daniels [14], 1945). Based on the power-law form the failure probability tail, which is justified by the extreme value theorem [43] and, according to [60, 18], also by the statistics of interatomic bond failures, the weakest-link model leads to the Weibull distribution, while the strength of a fiber bundle follows the Gaussian (or normal) distribution [14].

Further it has been shown that, to capture quasibrittleness and the plastic-brittle transition of size effect, the number of links in the weakest-link model must be finite and depend on the structure size [60, 9, 24, 61, 18, 7, 45]. The theory has recently been extended for the strength distribution of MEMS devices [62]. The lower tail of the strength distribution of fiber bundles with a general fiber softening constitutive law has been clarified by Salviato et al. [22].

Both the weakest-link and fiber bundle models are idealizations of the internal material failure mechanism, representing simplified internal connectivity and force transmission. They enable the mathematical formulations of their strength probability distribution. Recently [57, 20, 47], the force transmission in nacreous materials has been idealized as a fishnet with brittle links pulled diagonally, which allowed a mathematical formulation of the fishnet statistics. The fishnet represents the third probabilistic model of strength distribution that is tractable analytically.

Unlike a single chain, whose failure and maximum load is triggered by the rupture of a single link, the weakest one, a fishnet possesses multiple structural redundancies. Before reaching the maximum load, the fishnet may already contain various several failed links, whose number generally increases with the coefficient of variation on link strength scatter. To capture it, the fishnet statistical model splits the fishnet survival event into a union of disjoint events corresponding to different numbers of failed links at the prescribed load σ . Thus the failure probability of fishnet gets reduced to a series of powers of the distribution, $P_1(\sigma)$, of link strength.

This formulation works well for brittle nacreous materials, because the failed links at the peak load are not that many. A few terms in the series expansion suffice for a good approximation of the distribution. In this sense, the fishnet model series expansion can be seen as a perturbation from the weakest-link model. On the other hand, it is relatively hard to extend the expansion method to quasibrittle materials, whose links exhibiting progressive postpeak softening of various steepness. As the links become less brittle, more widely scattered damages occur before the peak load. So more higher-order terms have to be considered. It is, however, very tedious to calculate them and their expressions

vary for different geometry and loading conditions. Therefore, another approach is needed for fishnets with softening links.

In [47], it was shown that the strength distribution of a fishnet with gradually softening links can be well approximated by order statistics. It is no longer necessary to consider the weakest of the remaining links. Rather it suffices to consider the k -th weakest link that determines the peak load of the whole structure. The order k reflects the extent of damages right before the maximum load, with k being in itself a discrete random variable. The probability mass function (pmf) of the order is approximated using the geometric Poisson distribution (also called Pólya-Aeppli distribution), which accounts for the random cluster nature of the damages.

Here we show that the series expansion and the theory based on order statistics are actually equivalent—both of them are power series of the link strength distribution $P_1(\sigma)$. Then, we propose a general expression for the fishnet probabilistic model based on this observation. The fishnet model demonstrates the potential of significantly reducing the failure risk of a material by carefully designing its microstructure such as the distinctive “brick-and-mortar” structure of nacre.

Recently, it has been shown that a thin film of cellulose nanocrystal (CNC) has a statistical size effect to similar to fishnet’s [63], implying that the fishnet model may apply to this type of material. Thanks to the general formulation of the fishnet probabilistic model, we can apply at the low probability tail to quasibrittle particulate materials such as concrete, to fiber composites, and to architected nanomaterials such as the octet-truss nanolattice.

First introduced by Fuller [64], deterministic modeling of the octet-truss lattice system has been pursued intensively due to its high stiffness-to-mass ratio and excellent energy absorption capability [65, 66]. Therefore, it has long been perceived as an alternative to replace foams and honeycombs in lightweight structures. With the advent of 3D printing, the manufacturing of octet-truss nanolattice became easier.

In a deterministic manner, the octet-truss properties have been studied both numerically and analytically [67, 68, 69]. However, the reliability and probability distribution of the nanostructure strength, which have not been studied yet, will become increasingly important. More nano-architected materials will doubtless appear in engineering applications.

The Weibull statistics is the large-size asymptotics of the Type 1 energetic-statistical size effect [9, 24]. The energetic release part, dominant for small sizes, has been explained by stress redistributions due to a large fracture process zone (FPZ), and can be regarded as the limit case of the energetic size effect, Type 2, for vanishing initial stress-free crack. The fishnet statistics, on the other hand, can be regarded as a modification, or generalization, of the Type 1 size effect law, in which the energy release release aspect is reflected in the interaction of multiple link breaks prior to reaching the maximum load.

4.2. Comparison Between Series Expansion of Survival Probability and Order Statistics

Up to now, we have considered two different approaches to obtain P_f in the fishnet model. In this section, we show that the two approaches are actually consistent. In other words, the fishnet model with order statistics can be written in the form of power series

expansion in terms of $P_1(\sigma)$. $P_1(\sigma) \sim (\sigma/\sigma_0)^m$ at the lower tail. Based on Eq. (2.11), the expression of P_f using order statistics is

$$(4.1) \quad P_f(\sigma) = \sum_{k=0}^N c_k W_k(\gamma_k \sigma) = 1 - \sum_{k=0}^N c_k (1 - P_1(\gamma_k \sigma))^N \sum_{s=0}^k \frac{(-N \ln(1 - P_1(\gamma_k \sigma)))^s}{s!}$$

Let N be sufficiently large. Then $(1 - P_1(\sigma))^N$ tends to the Weibull distribution, i.e., to $e^{-N(\sigma/\sigma_0)^m}$, where $P_1 \sim (\sigma/\sigma_0)^m$. For simplicity, let $x = \sigma/\sigma_0$. Substitution into the foregoing expression yields:

$$(4.2)$$

$$1 - P_f(x) = c_0 e^{-N(\gamma_0 x)^m} \cdot 1$$

$$(4.3) \quad + c_1 e^{-N(\gamma_1 x)^m} \cdot [1 + \gamma_1^m N x^m]$$

$$(4.4) \quad + c_2 e^{-N(\gamma_2 x)^m} \cdot \left[1 + \gamma_2^m N x^m + \gamma_2^{2m} \frac{(N x^m)^2}{2!} \right]$$

$$(4.5) \quad + c_3 e^{-N(\gamma_3 x)^m} \cdot \left[1 + \gamma_3^m N x^m + \gamma_3^{2m} \frac{(N x^m)^2}{2!} + \gamma_3^{3m} \frac{(N x^m)^3}{3!} \right]$$

$$(4.6) \quad + c_4 e^{-N(\gamma_4 x)^m} \cdot \left[1 + \gamma_4^m N x^m + \gamma_4^{2m} \frac{(N x^m)^2}{2!} + \gamma_4^{3m} \frac{(N x^m)^3}{3!} + \gamma_4^{4m} \frac{(N x^m)^4}{4!} \right]$$

$$(4.7) \quad + \dots$$

After the Taylor series expansion of $e^{-N(\gamma_k x)^m}$ in the foregoing expression, Eq. (4.1) can be rewritten in the form of Eq. (1.1). A unique correspondence between the coefficients a_k , γ_k and c_k can then be obtained by equating the two expressions.

Further it can be shown, via Taylor series expansion, that the distribution of the k th order statistics has the leading term of $(\sigma/\sigma_0)^{km}$. Therefore, its lower tail in the Weibull-scale plot has slope km . In other words, the slope of P_f at its lower tail is fully controlled by the first non-zero coefficient c_k .

Noting that γ_k is very close to 1 in general (e.g., 1.04), we could, for simplicity, replace γ_k for various k values by their average, γ . Then, the survival probability reduces to a neat expression:

$$(4.8) \quad 1 - P_f(\sigma) = e^{-N(\gamma x)^m} \cdot \left[f(0) + f'(0)N\gamma^m x^m + f''(0)\frac{(N\gamma^m x^m)^2}{2!} \right. \\ (4.9) \quad \left. + f'''(0)\frac{(N\gamma^m x^m)^3}{3!} + f^{(4)}(0)\frac{(N\gamma^m x^m)^4}{4!} + \dots \right]$$

or

$$(4.10) \quad \boxed{P_f(\sigma) = 1 - e^{-N\gamma^m x^m} \cdot f(N\gamma^m x^m)}$$

where

$$(4.11) \quad f(0) = 1$$

$$(4.12) \quad f'(0) = 1 - c_0$$

$$(4.13) \quad f''(0) = 1 - (c_0 + c_1)$$

$$(4.14) \quad f'''(0) = 1 - (c_0 + c_1 + c_2)$$

$$(4.15) \quad \dots$$

$$(4.16) \quad f^{(k)}(0) = 1 - \sum_{i=0}^{k-1} c_i$$

The expression in the brackets of Eq. (4.9) can be seen as a Taylor series expansion of some function $f(y)$ about $y = 0$, where $y = N\gamma^m x^m$. The sequence of derivatives $\{f^{(k)}(0)\}$, strictly decreasing from 1 to 0, is the complementary cumulative distribution function of the damage extent N_c . In this sense, f is determined purely by the stress redistribution process, which in turn depends on the mechanical properties of the material.

Thus the determination of P_f gets reduced to the characterization of the function f or, equivalently, of its derivatives at 0. In the special case of the weakest-link model, function f is simply a constant equal to 1, and P_f then reduces to the Weibull distribution.

4.3. Application of Fishnet Statistics to Quasibrittle Materials in General

4.3.1. Tensile Strength Distribution of Octet-Truss Nanolattice

A careful look at the octet-truss lattice structure reveals that it is actually a union of three systems of parallel fishnets aligned in three orthogonal directions. From Fig. 4.1, it

is clear that the three families of parallel fishnets lie in xy , yx and xz planes. Because the lattice consists of fishnets in three directions, the connectivity (or nodal coordination number) at each node is 3-times larger than that of the fishnet. This greatly enhances the redundancy of the microstructure, positioning the octet-truss tail farther away from the brittle weakest-link tail and closer to the ductile fiber-bundle tail. Therefore, compared to a single 2D fishnet, we expect a huge safety gain of the octet-truss structure at the lower tail of the strength distribution.

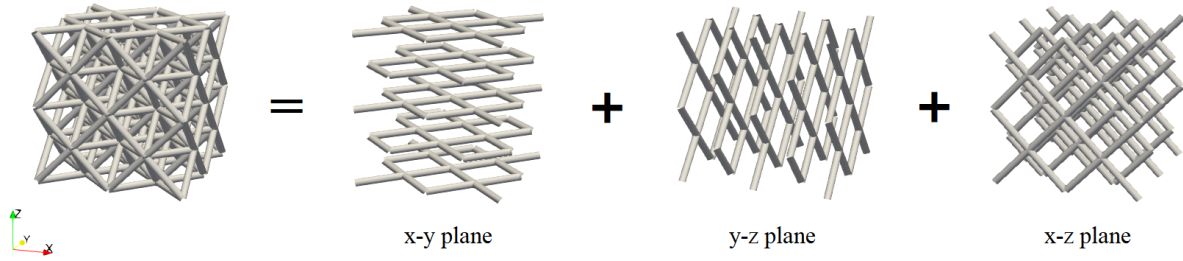


Figure 4.1. octet-truss nanolattice can be seen as the union of fishnets aligned in three orthogonal directions. Each fishnet node on its inside has connectivity 4, while each node inside of an octet-truss nanolattice has connectivity 12.

Here we model the strength distribution of an $8 \times 4 \times 4$ octet-truss nanolattice, pictured in Fig. 4.2. Many materials have been used to fabricate the lattice structure on the meso- or nano-scale, e.g., pyrolytic carbon[67] and copper[68, 69]. With no loss of generality, we take the properties of the nanolattice truss links to be of the same magnitudes as reported in the literature (see table 4.1). To model the gradual failure process, a finite element (FE) code with an infinitesimal strain measure is developed in C++ using Eigen 3.3.7 [70] linear algebra library. The visualization of the results is conducted in Paraview 5.7 [71]. Instead of using the encastre boundary conditions on one end, the nodes on the outer faces of the yz , xz and xy plane with outward normals $(-1, 0, 0)$, $(0, -1, 0)$ and $(0, 0, -1)$

are subjected to boundary conditions $u_x = 0$, $u_y = 0$ and $u_z = 0$, respectively. This is to guarantee that every middle cross-section undergoes the same contraction. Then uniaxial tension is applied by prescribing the horizontal displacements u_x of the nodes on the outer face with outward normal $(1, 0, 0)$. The remaining two boundaries are considered as free surfaces with no external load.

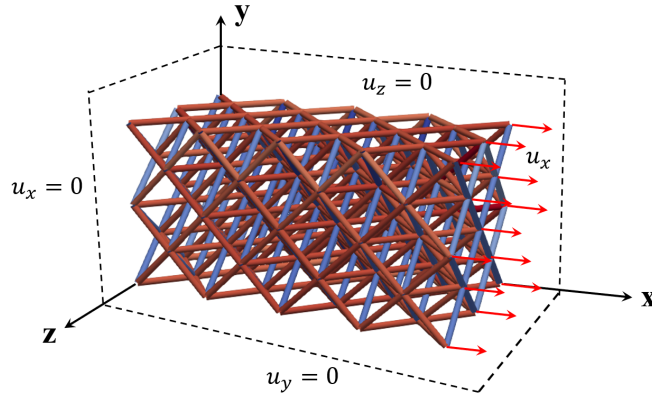


Figure 4.2. Loading and boundary conditions for the $8 \times 4 \times 4$ nanolattice in finite element simulation

Table 4.1. Truss Properties

Young's Modulus	Mean Strength	Length	Cross-section Diameter
10 GPa	$\simeq 100$ MPa	$2 \mu\text{m}$	$0.5 \mu\text{m}$

The strengths of the truss elements (or links) are randomized based on a grafted Gauss-Weibull distribution [60], which has a Weibull lower tail $P_1(\sigma) = 1 - \exp[-(\sigma/100)^{10}]$, and on a Gaussian distribution, $\mathbf{N}(100, 16^2)$ for the rest. The current settings of mean strength and element stiffness guarantees that the failure strains of the links are very small, and so our assumption of a linearized infinitesimal strain measure holds. The grafting point is set at $P_f = 10^{-3}$ (as in [47]). The truss links are here considered brittle; i.e, once the strength limit of a link (or truss element) is reached, its stress immediately drops to 0.

In each simulation, the structure is first loaded by a trial displacement u_0 , then the ratio of strength over stress is calculated for each link. The minimum ratio is the smallest load multiplier that will cause one and only one link failure in the whole structure. Next, all the variables are updated by the smallest multiplier, according to the linear elasticity of the structure. Then the critical element is deleted by removing its contribution from the global stiffness matrix. Finally, the damaged structure is treated as a new one, and is loaded again from origin by the trial displacement u_0 . This process is repeated until complete failure. In other words, the driving parameter for the simulation is the number of failed links in the truss system, while displacement control is used within each step. In these simulations, we suppress compression failures of the truss links by assigning to them very large strength. The links under compression are those in the yz plane. They are under compression from the beginning, and are excluded from our probabilistic modeling.

One million Monte Carlo simulations have been conducted using a FE code. Fig. 4.3 shows the load-displacement curves of 1000 samples. The peak loads are marked by solid points. Since the truss elements are treated as brittle, the load-displacement curves exhibit snap-back instability in the post-peak regime. If the maximum load is reached right before the failure of the weakest link, the peak load point will be on the line of initial stiffness (or slope) of the load-displacement curve. Otherwise, the peak load point will lie underneath the line of initial stiffness. As is clear from Fig. 4.3, most of the peak loads lie slightly off the initial stiffness line, which indicates that they happen after a few links have failed. Only a small portion of peak loads are triggered by the first failure of links. This is beneficial for safety and occurs thanks to the redundancies in the octet-truss connectivity.

Before modeling the strength distribution of the octet-truss nanolattice, we must define the nominal stress, σ . We do so as follows:

$$(4.17) \quad \sigma = c_n \frac{P}{A},$$

where P = total reaction force from the cross-section, A = cross-section area of the octet-truss lattice, and c_n = a normalizing constant to convert the average stress in a cross-section to the stress in a tensile member. For the current octet-truss nanolattice, its cross-section area is $A = 13 \times \pi d^2/4 = 2.553 \mu m^2$, and $c_n = 0.4784$.

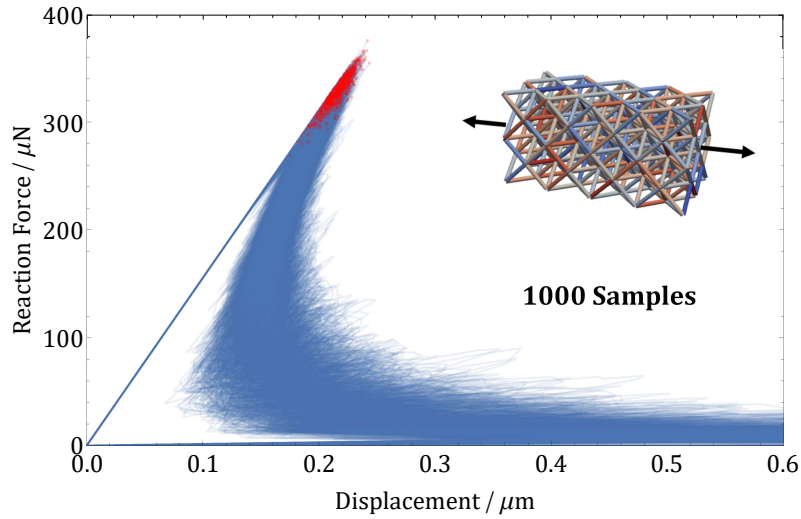


Figure 4.3. Load-displacement curves for 1000 Monte Carlo simulations. Maximum loads are marked by solid points.

For the analytical strength distribution, we evaluate and compare several existing models: 1) weakest-link model, 2) 2-term fishnet model, 3) 3-term fishnet model, 4) the general fishnet distribution (Eq. 4.10), and 5) the fiber-bundle model. Evaluating the

weakest-link model is straightforward:

$$(4.18) \quad P_f(\sigma) = 1 - [1 - P_1(\sigma)]^N$$

where $N = 320$ = number of members in tension under the initial loading. The elements in the yz plane are under compression, and are not counted. Then, the 2-term fishnet model can be evaluated on the basis of the weakest-link model by adding the probability that the whole fishnet remains safe under σ with one and only one failed link:

$$(4.19) \quad P_f(\sigma) = 1 - [1 - P_1(\sigma)]^N - NP_1(\sigma)[1 - P_1(\sigma)]^{N-\nu-1}[1 - P_1(\eta\sigma)]^\nu,$$

where $\eta = 1.15$; $\nu = 5$ gives the best fit. Again, the 3-term fishnet model can be formulated on the basis of the 2-term fishnet model; [20]:

$$(4.20)$$

$$(4.21) \quad P_f(\sigma) = 1 - [1 - P_1(\sigma)]^N - NP_1(\sigma)[1 - P_1(\sigma)]^{N-\nu-1}[1 - P_1(\eta\sigma)]^\nu - N\nu_1[1 - P_1(\sigma)]^{N-\nu_2-2}[1 - P_1(\eta^{(2)}\sigma)]^{\nu_2} \int_0^\sigma \int_{x_1}^{\eta_b^{(1)}\sigma} \psi(x_1)\psi(x_2)dx_2dx_1$$

$$(4.22) \quad - N(N - \nu_1 - 1)[1 - P_1(\sigma)]^{N-2\nu_1-2}[1 - P_1(\eta^{(2)}\sigma)]^{2\nu_1} \int_0^\sigma \int_{x_1}^\sigma \psi(x_1)\psi(x_2)dx_2dx_1,$$

where $\psi(\sigma) = P_1'(\sigma)$ = probability density for the link strength, $\nu_1 = 8$, $\nu_2 = 18$, $\eta^{(2)} = 1.5$ and $\eta_b^{(1)} = 1.115$.

As can be seen from the previous formulations, the expression of the failure probability becomes increasingly complicated as we keep adding more survival probabilities. It will

be much easier to use the general expression of the fishnet model (Eq. 4.10) if many higher order terms should be considered.

The key step is to characterize the unknown decreasing function f (or its derivatives at 0) in the expression Eq. 4.10. The optimum fit using this expression can be achieved by setting $\gamma = 1.075$ and $f(0) = f'(0) = f''(0) = \dots = f^{(4)}(0) = 1$. The remaining higher-order derivatives remain 0. In other words, it is simply the distribution of the 5th-order statistic of link strengths.

Finally, for the fiber bundle model [14], the limiting distribution is a Gaussian (normal) distribution with mean $= \tau[1 - P_1(\tau)]$ and standard deviation $= \tau\sqrt{P_1(\tau)[1 - P_1(\tau)]/n}$, where τ is the value of x such that the average stress-strain curve, $xP_1(x)$, takes its maximum value.

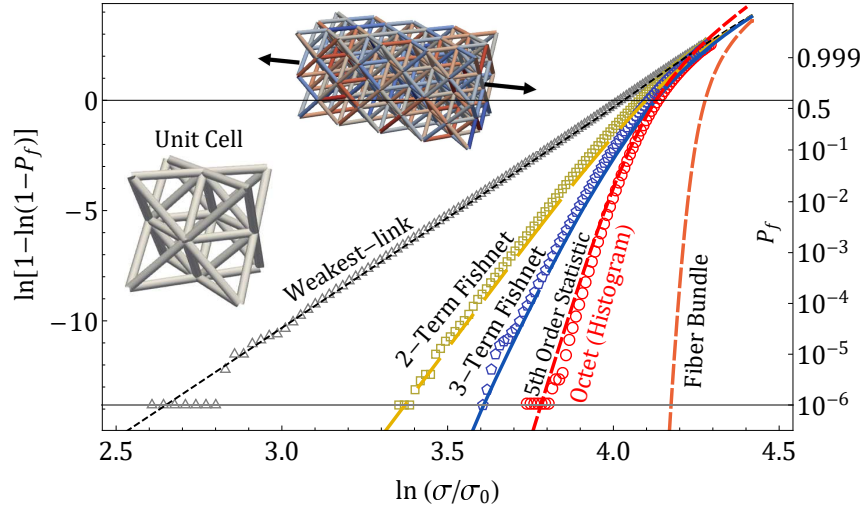


Figure 4.4. Strength histogram (shown in circles) of the strength of octet-truss nanolattices under Weibull scale compared with different analytical models. The sample size is 10^6 .

Fig. 4.4 shows the strength histogram (shown by circle points) of the octet-truss nanolattice together with various analytical probabilistic models of material strength,

as mentioned above. Also shown in the figure are the computed histograms of: 1) the load at the first step (first link failure) (triangles); 2) the maximum load of the first two steps (squares); and 3) the maximum load of the first three steps (pentagons). Since the sample size of our Monte Carlo simulations is 10^6 , the histograms have reached $P_f = 10^{-6}$ probability level and have converged at about $P_f = 10^{-5}$.

It is clear that the general expression of the fishnet model, Eq. (4.10) gives, for the distribution of 5th-order statistic), the closest match of the computed histogram, except at the upper tail. This is because we used a single γ to replace the different values of γ_k for various k values. The weakest-link model, the 2-term and 3-term fishnet models, are upper bounds on the actual strength distribution. They match very well the histogram of the maximum loads of the first two and three steps respectively. It is clear that, as we add more terms in Eq. 1.1 for the survival probability, the distribution gradually approaches the histogram. Also note that the Gaussian distribution for the fiber bundle gives a lower bound.

At the extremely low probability level of $P_f = 10^{-6}$, the 2-term fishnet model gives a nominal strength that is 92% higher than the weakest-link model does. Comparing the 2-term and 3-term fishnet models, the 3-term fishnet model predicts a further 27% strength increase. The actual histogram and the distribution of the 5th-order statistic indicates an additional 18.5% nominal strength increase relative to that of the 3-term fishnet model at the $P_f = 10^{-6}$ tail. However, at the median level (i.e., for $P_f = 0.5$), the strength values given by these models are very close. The median strength increase is merely 12% between the weakest-link model to the distribution of the 5th order statistic. But the 10^{-6} quantile has a 209% increase, which is enormous.

It thus becomes clear that paying attention only to the mean strength of a material does not suffice to optimize material or structure design. It does not guarantee robustness and reliability. The connectivity of the microstructure does play a determining role in the tail strength of a material. A smart design of such connectivity, as in the case of octet-truss nanolattice, makes a huge difference for optimizing a printed material microstructure.

Within the range of brittle materials, the brittle fiber bundle gives the lowest possible failure probability for each prescribed load and the highest possible strength at all probability levels. It is the theoretical limit of architected material strength. But there is a serious limitation for the fiber bundle—it can carry load in only one direction.

The 3D lattices, on the other hand, can carry multiple loads in different directions. In this sense, although the octet-truss nanolattice is not as reliable as the fiber bundle under loads of a single orientation, in terms of overall reliability, for loads of different orientations it surpasses the fiber bundle.

To sum up, there are always trade-offs in material performance under loads of different orientations.

4.3.2. Tensile Strength Distribution of Concrete and Composites in General

A homogeneous specimen (or structure) of a brittle material with a negligible material strength variability and an FPZ negligible compared to specimen dimensions fails as soon as a crack begins to grow, provided that the structure geometry is positive (which means a geometry for which the derivative of energy release rate with respect to the crack length at constant load is positive). Obviously, the strength must then follow the weakest-link

model exactly. However, for quasibrittle materials, there are deviations. The Gauss-Weibull distribution was shown to capture them [60, 18, 9], but not necessarily for the extreme lower tail in the 10^{-6} range.

Recent simulations of the tensile force transmission chains in a concrete prism loaded in tension [72] reveal a combination of axial and inclined interparticle forces. The inclined ones represent lateral interactions and suggest a partial fishnet action to be present. This is certainly a possibility since no existing histogram data give information below $P_f = 0.01$. The same is probably true of all quasibrittle materials, including coarse-grain ceramics, granular rocks, sea ice, wood and fibre composites, especially textile composites.

4.3.2.1. Correction of Weibull’s histogram plots of test data for concrete. Let us now consider concrete. The earliest, and still the most extensive, test data on the distribution of the tensile strength of concrete can be found in Weibull’s paper in 1939 [59]. He tested three batches of specimens, cured in water for 2, 7 and 28 days. The statistical sample sizes were 680, 1082 and 1106 respectively. The whole histogram testing took about one year to finish. The original unit used for stresses and strengths in Weibull’s paper was kg/cm^2 (which equals 0.098067 MPa). We keep here this unit in discussing his tests. We now explore the possibility of using our fishnet model to fit these original test data and discuss the strength predicted at 10^{-6} probability level.

For each data set for various materials presented in Weibull’s paper, he or his assistant fitted the measured histograms using either 2-parameter or 3-parameter Weibull distribution, whichever fitted the best. For the tensile strengths of concrete, they chose the 3-parameter Weibull distribution because the histogram was not a straight line in the 2-parameter Weibull scale. But in 2007 it was shown [60], based on the probability of

interatomic bond breaks, that a finite threshold used in the 3-parameter Weibull distribution is, for material strength distribution, physically unjustifiable (since it conflicts with Kramer's rule of transition rate theory for statistics of interatomic bond breaks, whose validity is beyond doubt). Therefore, we convert Weibull's test data from the 3-parameter Weibull scale to the 2-parameter Weibull scale.

It was during this data conversion process that we uncovered an inconsistency (apparently enduring since 1939) in the types of logarithm used for the vertical and horizontal axes, as presented in Weibull's original paper [59]. For the horizontal (log-strength) axis, his assistant who did the data fitting must have used the common (or decadic, Briggs') logarithm (of base 10), or else the model parameters presented in Weibull's paper would not match the data, while for the vertical axis, he or she must have used the natural logarithm (base e), because assuming the common logarithm would lead to wholly unreasonable tail probabilities for the test data. Therefore, we converted all Weibull's data to the 2-parameter Weibull scale using the natural logarithm.

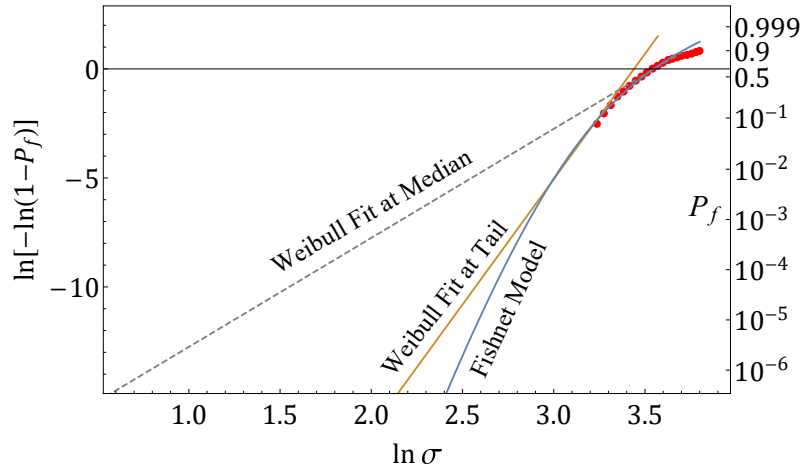


Figure 4.5. Fitting of the Weibull's 1939 histogram test data on the tensile strength of concrete (7 days in water) by 2-parameter Weibull distribution and fishnet model.

4.3.2.2. Analysis of corrected plots of Weibull’s histograms. The solid points in Fig. 4.5 show the histogram data of Weibull’s original tests on concrete specimens converted into 2-parameter Weibull scale: $X = \ln \sigma$, and $Y = \ln(-\ln(1 - P_f(\sigma)))$. Although his sample size for concrete was 1082, the data points in Weibull’s histogram plots reach only $P_f = 0.1$ and $P_f = 0.9$ for the lower and upper tails, respectively. This is, because before plotting the data, Weibull first grouped the data into bins, each aggregating about 1 to 10 points, which greatly decreased the histogram scatter but reduced the sample size (i.e., the number of data points). In contrast to the 3-parameter Weibull plot, the test data under 2-parameter Weibull scale can no longer be fitted well by a straight line.

Also shown in Fig. 4.5 are the optimum fits of the test data by various statistical models. The optimal number of links for the fishnet statistics is $n = 100$. It is clear that the tangent (Weibull distribution) through the median section gives a much more conservative prediction than the tangent through lower tail, and even more conservative compared to the fishnet model.

The fishnet model prediction indicates a further slope increase of the curve at its lower tail, leading to a 5.13% tail strength increase compared to the Weibull distribution fit at data tail. This extra safety gain comes from the redundancy of the force chain for concrete under uniaxial loading. The pattern of the force chain was simulated by Nitka and Teichman [72] using the discrete element method (DEM). Though not the same, the force chain pattern is similar to that of a fishnet. A histogram test with a much larger sample size would help clarify which model of the two gives a more realistic prediction at the extreme lower tail. A better check could result from the analysis of size effect.

4.4. Conclusions

- (1) The fishnet model with a series expansion of survival probabilities for brittle nacreous material is equivalent to the recent formulation with order statistics for fishnets with links of arbitrary softening slopes. Both of them can be written in the form of a power series in terms of the link strength distribution.
- (2) A unified general formula for the fishnet probabilistic model is possible and its mathematical form has been proposed.
- (3) The fishnet model is applicable to the strength distribution of octet-truss nanolattices. Their nodal connectivity, which is three times than that of the 2D fishnet, results into more extensive and more uniform stress redistribution after a link fails. This leads to a major safety gain at the risk level of one out of a million.
- (4) Millions of Monte Carlo simulations of the failure of the octet-truss nanolattice confirm the analytical results.
- (5) The fishnet model has been used to fit the original Weibull's test data on the strength distribution of concrete. The prediction of strength at 10^{-6} probability level shows a small increase compared to that given by the Weibull distribution fit of the histogram lower tail. It indicates that the force chain connectivity in concrete could also provide some structural redundancy, though not as much as in nacre.
- (6) Unnoticed since 1939, Weibull's original plot of his test data utilized different and mutually incompatible types of logarithms for the horizontal and vertical axes.

CHAPTER 5

Coefficient of Variation of Shear Strength of RC Beams and Size Effect

5.1. Introduction

In the previous chapters, the failure probability are developed for structures and materials that fail without the stable crack propagation prior to the peak load. In this chapter, we focus on cases where stable crack growth are observed before the load capacity is reached. Due to this difference in failure mode, the failure risk formulation changes accordingly. In this study, we are interested in the strength distribution of the reinforced concrete (RC) beams with an special emphasis on its coefficient of variation (CoV).

The reliability of reinforced concrete structures is a major concern in modern structural design. When assessing the strength and reliability of RC beams, one crucial aspect is the size effect. How does structural strength scale with structure is already known reasonably well [73]. But how does the strength scatter scale?

Big majority (in fact, 87%) [74] of all the available laboratory test data is based on relatively small beams, with a depth smaller than 20 in. (508 mm) while, in practice, much larger beams are critical and are often designed. Due to this discrepancy between laboratory and practice, it is inevitable to rely on a theoretical but realistic size effect law, to extrapolate to large beams based from smaller ones tested in the lab.

When studying the deterministic failure of quasibrittle materials, there are two types of size effect on structural strength. Type 1 size effect [73, 74, 1] occurs in structures that fail (under load control) catastrophically as soon as a macrocrack initiates at some place within the structure.

Particularly relevant to the current paper is the type 2 size effect law [49, 48], which is typical for RC structures and has recently been incorporated into the ACI design code (ACI Standard 318-2019), for beam shear, punching and strut-and-tie design. The Type 2 size effect occurs when geometrically similar stable cracks develop prior to the peak load, which is common for RC structures due to the existence of reinforcement [75, 76, 77, 78, 79, 80, 81, 82]. It also applies to specimens with pre-existing large notches. The type 2 size effect is caused by the size dependence of the strain energy release due to critical crack advancement.

Apart from the deterministic size effect laws, important is also the statistical size effect which arises from the random heterogeneity inherent to the material, i.e. statistical size effect [8, 24]. It involves the scaling relation for the mean strength and its coefficient of variation (CoV). Knowing such scaling is necessary to evaluate the reliability of engineering structures at the one-out-of-one-million (10^{-6}) failure risk level.

For concrete, its heterogeneity and the irregular geometry of grains lead to considerable variability of the overall structure response. As for typical quasibrittle materials, the strength of plain concrete follows the classical Gauss-Weibull statistical size effect. Its mean strength decreases as structure size grows, with a limiting slope $-n/m$ in plot of log-mean strength versus $\log D$. Here, $n = 1, 2$ or 3 = number of spatial dimensions in which the structure is scaled. For not too wide beams, the proper choice is $n = 2$ since

the shear fracture must advance simultaneously through the beam thickness; m is the Weibull modulus. The CoV of material strength is a constant for the large size limit, due to the Weibull tail of the strength distribution.

The Weibull statistical size effect is not directly applicable to reinforced concrete structures. Because of the effect of reinforcement, the fundamental hypothesis of Weibull statistics [83, 59, 84] is invalid—the local failure of any representative volume element (RVE) can no longer trigger a catastrophic failure of the whole structure. Because the reinforcement stabilizes the shear crack propagation, the peak load is reached only long after the growth of a long crack. Consequently, the strength variability and its statistical distribution do not follow Weibull statistics and need to be studied.

As the continuation of a previous work in ICOSAR proceedings [85], this paper studies the scaling of failure probability of RC beams without stirrups and analyzes the strength CoV. We first introduce the concept of sample p -quantiles [86, 87, 88, 89, 32] of strength statistics to model the failure probability of RC beams and then combine it with the analysis of Monte Carlo numerical simulations. Finally, we analyze the large size limit of the strength CoV that helps evaluate the reliability of engineering reinforced concrete structures.

For the numerical simulations of the stochastic damage and crack evolution in quasi-brittle materials, the lattice models [90, 91, 92] have been favored because they can directly represent with a lattice system the randomly irregular intergranular connectivity on the mesoscale. Some valuable qualitative findings on the random behavior can be obtained from such simulations. For instance, a large amount of micro cracks are found at the peak load in simulations of plain concrete [93]. The randomness in notched beams

affects only the variability of their strength, but not the mean behavior [94]. Replication of the mesoscale structure of concrete in these models helps obtaining realistic results. This comes, however, at the price of high computational cost of the simulations of large structures.

To reduce the computational cost and enable running, fast enough, numerous Monte Carlo experiments, we opt for the microplane model M7 [95, 96] embedded in the framework of finite element analysis. Instead of modeling the random intergranular connectivity on the mesoscale, we use regular finite elements with random material properties to account for the material heterogeneity of concrete. The random properties are sampled from a Gauss-Weibull distribution with m -value (Weibull modulus) being less than the macroscale value 24, according to the hierarchical model[8].

For mechanicians, the term ‘beam shear’ calls for explanation. The shear crack actually grows as a Mode I (opening) crack. It is called a shear crack because it is created by shear force V in the beam.

5.2. Model Configuration

Since the size effect is a quintessential property of concrete fracture, we consider beams of three total depths: $H_1 = 200\text{mm}$, $H_2 = 400\text{mm}$ and $H_3 = 800\text{mm}$ (see Fig. 5.1). The corresponding effective beam depths (from top face to steel centroid, excluding the concrete cover) are $D_1 = 160\text{mm}$, $D_2 = 360\text{mm}$, and $D_3 = 740\text{mm}$. The beams of all sizes are geometrically similar in two dimensions (2D), i.e., only the lengths and depths are scaled, while the thickness b is kept constant at 20 mm. This avoids the wall effect, minor but non-negligible, which would complicate evaluation. The choice of dimensions is based

on important previous test data for shear failure of reinforced concrete (RC) beams [97]. Since the concrete cover, $H - D$, does not contribute to the beam strength appreciably, D rather than H is considered as the characteristic dimension. The relative shear span $a/D = 3$ is kept constant, to ensure geometric scaling.

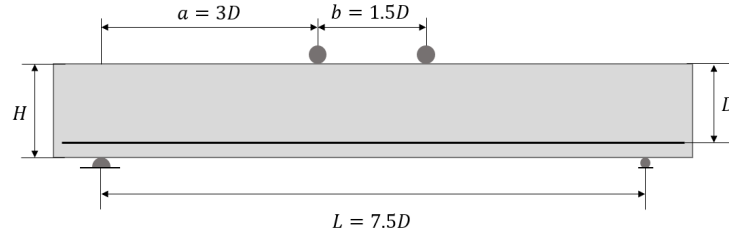


Figure 5.1. Geometry of the RC beam.

The RC beams are subjected to four-point loading under displacement control. For simplicity, the rebar is modeled by truss elements with Young's modulus 200 GPa. The truss elements nodes rigidly attached to the concrete elements nodes. To reduce the computational cost, the reinforcing bar is attached to only one surface of the specimen on which the boundary condition of symmetry is applied, and so the equivalent specimen thickness is 40 mm. The reinforcement ratio ρ is set to 1.2% for all three beam sizes, and is chosen the same as in the experiments in [97]. This ratio prevents the steel from yielding. As a consequence, almost all specimens fail right after the peak load if load control is used. The simulation results shown later in the current paper confirm this behavior.

In simulations with the crack band model [30, 44], which we adopt to avoid spurious mesh sensitivity, the element size is dictated by the fracture energy, G_f . Based on the aggregate size as well as experience with typical concrete we use cubical finite elements of dimensions $20\text{mm} \times 20\text{mm} \times 20\text{mm}$. The element type is C3D8R, i.e., cubic element with reduced integration. The damage constitutive law of concrete is the microplane model

M7, which is implemented in ABAQUS as VUMAT subroutine for damage and failure of concrete [95, 96].

The Young's modulus, E , and radial scaling parameter k_1 of microplane model, which jointly control material compression strength, are here randomized on the basis of the grafted Gauss-Weibull distribution [8, 6]. Even though both Young's modulus and strength are random, they are strongly correlated. For simplicity, we consider them to be perfectly correlated, related by the approximate ACI formulas: $E = 4735\sqrt{f'_c}$ (f'_c, E in psi). This means, for instance, that an element of high strength will also have a high Young's modulus. Furthermore, based on experience, we assume the autocorrelation length of the random field to be equal the fracture process zone size, which is roughly equal to the element size. This means that the autocorrelation need not be considered in the finite element simulations, as long as the element size does not change, which applies here.

At the beginning of each run, random numbers are first generated for each element (and also for each Gauss point, due to reduced integration). They are based on a grafted distribution with Gaussian core of mean $\mu = 9$ and standard deviation $s = 1.843$, and with a remote lower Weibull tail.

Since the magnitude of the M7 model parameter k_1 in deterministic simulations is on the order of 10^{-4} , the random value of k_1 for each element is evaluated by dividing the generated random numbers by 10^5 . Young's moduli are generated in a similar fashion. To guarantee that each element has a unique pair of randomized strength and modulus, the random properties are assigned according to the element index that has passed into the VUMAT subroutine.

Fig. 5.2 shows one random realization of the material properties of the beam. To visualize the result, the dimensionless random number is stored in the state variable SDV190 in Abaqus. The darker the color, the stronger (and also stiffer) the element. Apart from the randomized model parameter k_1 , other major M7 parameters are given in Table. 5.1

Table 5.1. Microplane M7 model parameters for concrete.

$E/(\text{MPa})$	ν	k_1	k_2	k_3	k_4
$\sim 2.5 \times 10^4$	0.18	$\sim 1 \times 10^{-4}$	110	30	100

For each size of the beam, one thousand Monte Carlo finite element simulations of shear failure with random parameters have been run. The maximum shear forces V are obtained from the recorded load-displacement curves, after which they are normalized to nominal strength of the beam according to $\sigma_N = V/bD$ (where b = beam thickness).

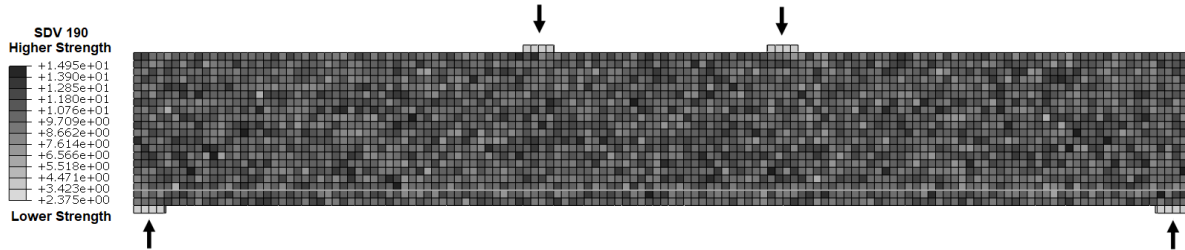


Figure 5.2. One realization of random strength and modulus of the median size RC beam ($H = 400$ mm). The darker the color, the stronger the element (in terms of strength and stiffness).

5.3. Size Effect on Mean and CoV of Nominal Shear Strength

5.3.1. Is Crack growth a Random Process?

The diagonal shear crack initiates at the longitudinal reinforcement at the bottom of the beam, and then it propagates upwards in a stable manner under increasing load. At

peak load, the crack front lies roughly at distance $0.2D$ from the top face of beam. The location of crack initiation at bottom is random over a broad range, as seen in Fig. 5.3 where the crack paths from tests at various beams sizes, scaled to one size, are collected. In perfunctory discussions it was sometimes thought that the crack should be treated like a Markov process, or some other random process, leading to an even greater spread of the random tip locations at peak load.

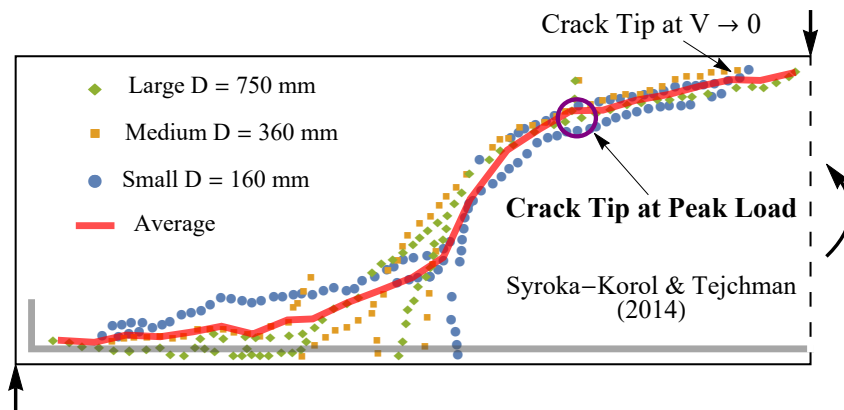


Figure 5.3. Crack patterns documented in scaled beams of different sizes

The opposite it true, however, as experimentally best demonstrated by Syroka-Korol and Teichman, Fig. 5.3. If no random scatter is considered, fracture mechanics forces the crack tip to lie, at peak load, at a certain precise location. If the random scatter is considered, fracture mechanics forces the tip to lie in a relatively small region, much smaller than the region of crack initiation points at bottom. Thus the cracks, widely scattered at bottom, tend to converge to this region. Therefore, the random spread of crack tip location at peak load is much smaller than the random spread of the crack initiation point. So, random process modeling of shear crack growth would not be realistic

unless subordinated to fracture mechanics. This would be difficult, and there is no need for it.

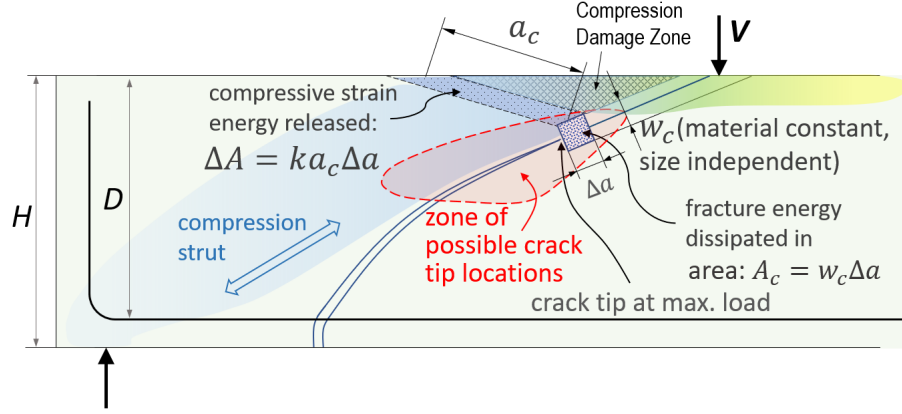


Figure 5.4. Idealized intuitive fracture increment at peak load.

5.3.2. Mean Size Effect on Strength

Despite the randomness considered in the model, we must consider the mean behavior of the RC beam to be dominated by the quasibrittle fracture mechanics ([49, 44]) and analyze it deterministically. Fig. 5.4 shows the idealized diagonal shear crack at the peak load. As argued in [98] and demonstrated in [74, 1], the tensile stress across the crack (called aggregate interlock stress), as well as the mode I fracture energy, is at peak load reduced virtually to zero, and the failure at peak load is caused by crack-parallel compression. This is documented by the field of vectors of principal compressive stress at peak load, computed in [1] and displayed in Fig. 5.5. The shear-compression crushing of element Δa in Fig. 5.4 causes a slip of the cross-hatched wedge-shaped zone and diminishes the compression stress and strain energy in that wedge.

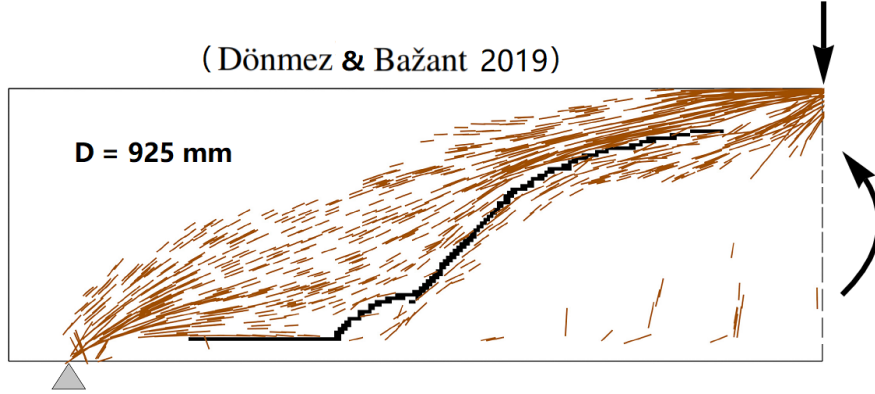


Figure 5.5. Minimum principal compressive stresses in RC beam [1].

Imagine the shear crack to extend by the distance Δa . Like for all quasibrittle materials, the width, w_c , of the fracture process zone (FPZ) of the crack, simulated as a propagating crack band, is approximately a material constant and is size independent. To satisfy the energy balance of the fracture process, the elastic energy released in the whole structure during incremental crack advance Δa must equal (per unit beam thickness) the work of fracture dissipated by Δa :

$$(5.1) \quad -\bar{U}(\Delta A_c + \Delta A) = G_f w_c \Delta a,$$

where $\bar{U}$ = strain energy increment at peak load, A = area of major strain energy release, ΔA_c = area of increment crack band and G_f = average fracture energy of concrete (mixture of mode-I and mode-II). The region of significant energy release is a band above the crack increment (see shaded area in Fig. 5.4). We then express the average energy density change as $C\sigma_N^2/2E$, where C is a constant, due to geometric similarity, to take account of the spacial variation of strain energy. Let the length of the energy relief band

be a_c , Eq. 5.1 then reduces to:

$$(5.2) \quad C \frac{\sigma_N^2}{2E} (ka_c \Delta a + w_c \Delta a) = G_f w_c \Delta a,$$

where k is the direction cosine that projects Δa to the width direction of the compressive strain energy release band. Rearranging Eq. 5.2 and leave σ_N on the left gives:

$$(5.3) \quad \sigma_N = \frac{\sigma_0}{\sqrt{1 + D/D_0}},$$

where $\sigma_0 = \sqrt{2EG_f/C}$, $D_0 = w_c/(k\alpha_c)$ and $\alpha_c = a_c/D$. This is the classical type-2 deterministic size effect [48, 49, 24]. Note that the Young's modulus E and fracture energy G_f are actually random variables for each element, so the quantities in the above mean size effect relation (Eq. 5.3) are their expected values.

5.3.3. Size Effect of CoV of Strength

5.3.3.1. Sample p -quantiles. Although the size effect on the mean shear strength of RC beams is dictated by fracture mechanics, the scatter of nominal strength, in terms of the variance and coefficient of variation, is rooted in the randomness of the material. We start from the mean size effect relation, Eq. 5.3, and allow its parameters to take random values. For instance, we let $\sigma_0 = \sqrt{2EG_f/C}$ to be random, where E and G_f no longer take the mean values. Because, the parameters in the size effect relation do not refer to an arbitrary element in the structure but to the critical shear crack increment when the peak load is reached, we cannot directly use in our model the distribution of E and G_f defined for each single element previously.

The critical element that triggers the peak load only shows up in a relatively small region along the major diagonal shear crack close to the load point in top. To get the size effect of strength CoV, we first need a rough estimate of the probability distributions of the local material parameters critical for σ_0 and D_0 in the size effect relation. As the shear crack propagates, its local crack tip trajectory tends to grow in the direction of relatively weak elements in the randomly heterogeneous material, making the crack trajectory deviate from the direction of maximum tensile stress. On the other hand, the overall crack trajectory must still roughly follow the deterministic solution, which is governed by fracture mechanics.

As a consequence, the properties of the extremely strong elements do not directly affect the peak load. The reason is that the critical element is very unlikely to be among those extremely strong ones, unless the elements surrounding the crack tip are all strong, in which case it simply means that the mean strength is on the high side. On the other hand, some extremely weak element is also unlikely to be the critical one since the elements neighboring the weak one are likely to be much stronger. So the overall load, as well as the nominal stress, will continue to increase as the crack keeps growing after the failure of an extremely weak element. Intuitively, it is thus neither the extremely strong nor the extremely weak elements that determine the peak load.

Based on this reasoning, we adopt the p -th sample quantile, $\sigma_0(p)$ (where $0 < p < 1$), of n i.i.d. (identical identically distributed) random variables, to approximate the ductile limit of Type 2 size effect, σ_0 , in Eq. 5.3. The value of p should lie somewhere between the quantile limits, 0 and 1, standing for the weakest and strongest element in the structure. The sample size n represents the number of elements potentially critical for the peak load.

The distributions of sample quantiles are closely related to those of the order statistics. Both of them are random variables arising from an ordered list of random variables. The sample quantile is characterized by the relative order percentage (p) in the list of all variables, rather than the absolute order considered in the order statistics.

To compare the two random variables further, consider a list of i.i.d. random variables arranged in ascending order $X_1, X_2, \dots, X_n$, where X_k is the k th order statistic. The sample p -quantile corresponds to the (np) th order statistic, X_{np} . Fig. 5.6 illustrates the relation of the limiting distributions of sample quantiles and order statistics. If we fix the order k and let $n \rightarrow \infty$, we obtain the limiting distribution of order statistics; while if we fix the ratio $k/n = p$ and let the order k increase with n , we obtain the limiting distribution of the sample p -quantile, which is known to be the normal (or Gaussian) distribution regardless of the distribution of X_i (see [99]). In the special case of $p = 0.5$, the corresponding sample quantile is simply the median of n i.i.d. random variables.

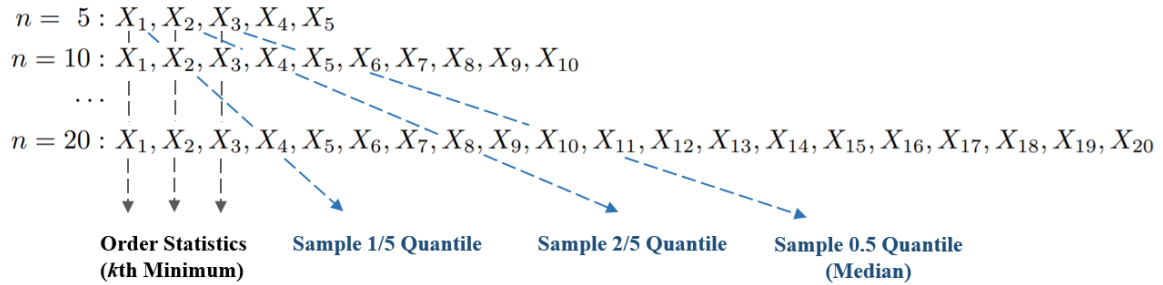


Figure 5.6. Relation between sample quantiles and order statistics

The strength of the critical element, σ_0 , is approximated by the (np) th weakest element in the whole critical region. As the RC beam becomes larger, the size of the possible region for the critical crack tip, n , also grows. For increasing sample size n , the distribution of the p -th sample quantile converges to the normal distribution with mean $x_p = F^{-1}(p)$ and

variance $p(1-p)/(nf(x_p)^2)$, where $f(x_p) = F'(x_p)$ is the value of the probability density function (p.d.f.) of material strength evaluated at the p -th theoretical quantile.

In cases where the variation of D_0 is negligible compared to that of σ_0 , we simply treat D_0 as a deterministic constant. Then the nominal strength of the RC beam would follow asymptotically normal distribution:

$$(5.4) \quad \sigma_N \sim N \left(\frac{F^{-1}(p)}{\sqrt{1 + D/D_0}}, \frac{p(1-p)}{nf(x_p)^2(1 + D/D_0)} \right)$$

where $F(x)$ is the cumulative distribution function of the element strength, and N is the unit normal distribution function.

For the cases in which the randomness of D_0 is not negligible, its effect on the scatter of σ_N is discussed later. For the derivation of the distribution of sample quantiles of i.i.d. random variables, see [99].

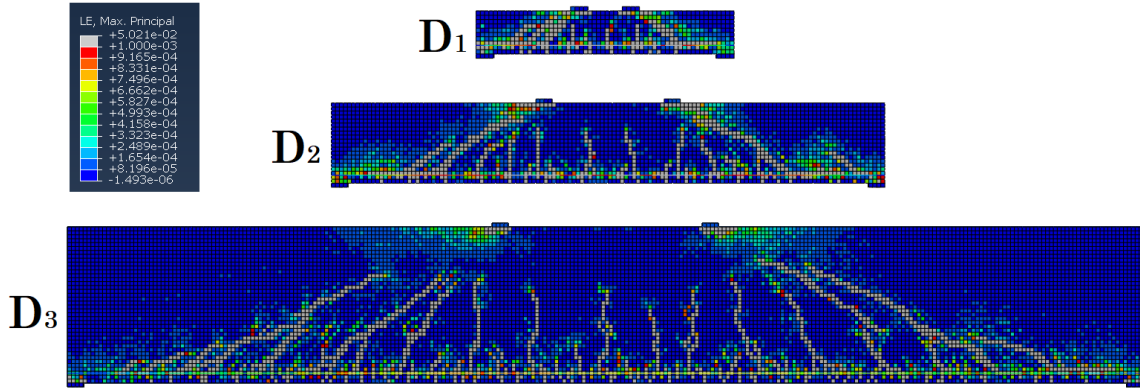


Figure 5.7. Crack pattern of one random realization for each beam size.

5.3.3.2. Critical Region of Possible Crack Tip Locations. Since we are interested in the relation between the CoV of the nominal shear strength and the structure size D , the natural question is: How does the size of the critical region n of all possible crack tip

locations scale with the structure size D ? There are three possibilities:

1) The critical region size, n , does not depend on structure size D and remains constant, in which case the critical region becomes negligibly small compared to the whole structure as D tends to infinity;

2) n is proportional to D^2 , i.e., the area of the critical region grows proportionally with the specimen area;

3) n increases with the growth of D but the rate is slower than D^2 .

To figure out which case is correct, we conduct FE Monte Carlo simulations of the RC beams and try to obtain the histogram of the vertical coordinates of the critical crack tips. Since the maximum principal log-strain (LE-MaxPrincipal in Abaqus) is normally used to indicate the location of a crack, we superimpose for each beam size the logarithmic strain field at peak load for all 1000 random simulations, and then we take their average. This averaged superposition of strain fields reflects the relative likelihood (or probability density) for cracks/damages to appear at each location in the beam given that elastic deformations are negligible compared to those due to crack opening. To focus on the vertical location distribution of the critical crack tip, we add up the strain values for each finite element in the same row after removing elements that remain purely elastic, as indicated by setting a threshold strain. Then, we obtain an average strain field that only depends on the vertical coordinate along the beam depth direction.

Note that there are many parallel cracks that appear close to the dominant shear crack (see Fig. 5.7), and that the crack density gradually decreases as widely spread parallel cracks at the bottom the beam gradually localize into a single major diagonal shear crack

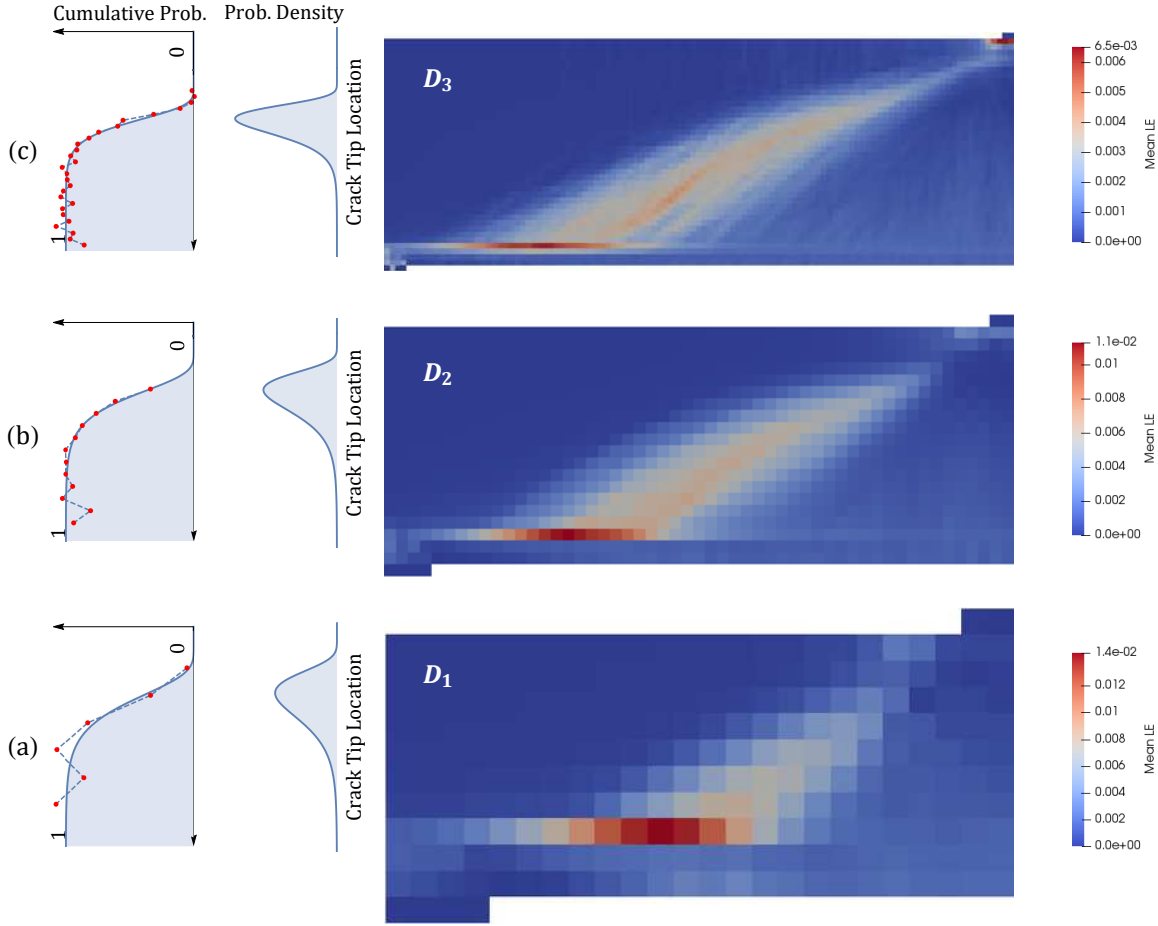


Figure 5.8. Average log strain field (LE) at peak load of 1000 samples for each of the 3 beam sizes obtained by superposition. Shown on the left are the probability density and cumulative distribution of the vertical location of the critical crack tip. The beams of three sizes are rescaled to the same size, for comparison. The mesh sizes are kept the same for all specimens, although they look different in this figure, due to rescaling.

while progressing toward the top. As a result, the summed strain keeps increasing from top to bottom of the beam. It even grows below the reinforcement bar without plateauing.

Interpreting such a strain field as the cumulative probability for the location of the critical crack tip would lead to the wrong conclusion that the structural failure could be

triggered by elements in the concrete cover at bottom. To obtain the vertical location distribution purely for the critical diagonal crack tip, we must remove the strain contribution from the densely populated parallel cracks at the concrete cover.

To do so, we subtract from the summed strains a linearly increasing function of the distance measured from the top of the beam. The parameters of this linear function are chosen so that the strain values at the bottom of the beam show a plateau while the whole function is still monotonically increasing. Finally, we normalize the summed strains for each row to the range of 0 to 1, such that it becomes a cumulative probability measure. By fitting the normalized strain field, we obtain a cumulative probability distribution function (c.d.f.) for the vertical location of the critical crack tip. The probability density can then be easily calculated by taking the derivative of the c.d.f.

The visualization of the averaged strain field is conducted using Paraview 5.5.2 [71]. Fig. 5.8 shows the mean strain fields for specimens of three sizes and the fitted probability distribution of vertical location of the critical crack tip. Now that we are interested only in the scatter (or standard deviation) of the crack tip location, it is not necessary to figure out exactly what type of distribution it follows. Due to a slight positive skewness of the data (measured from top to bottom), we choose the lognormal distribution to fit it.

From this figure, one can observe that, as the structure size increases, the critical crack tip tends to concentrate on smaller regions after the structure is rescaled to the chosen reference size. On the other hand, if we make a direct comparison to the distributions before rescaling, the standard deviation increases as structure size grows. More specifically, the standard deviations for the scatter of crack tip locations are 19.5mm, 32.0mm, and 44.4mm for D_1 , D_2 and D_3 respectively.

These observations show that the size of the critical crack tip region n grows as D increases, but the rate of growth is slower than D^2 . Therefore we replace n by $(kD)^c$, where $0 \leq c \leq 2$ is the exponent introduced to capture the growth rate of n as D increases, and k is the ratio of the critical region height to D for the smallest beam. The typical value for k is around 1/4. Then, substituting this into the variance of the nominal strength, we obtain its coefficient of variation (CoV) by dividing the standard deviation by the mean:

$$(5.5) \quad \text{CoV}(\sigma_N) = \frac{\sqrt{p(1-p)}}{x_p f(x_p)} (kD)^{-c/2}, \text{ where } x_p = F^{-1}(p)$$

For the sake of simplicity, this formulation is based on one single sample p -quantile of the material strengths in the critical region. In reality, it could be a linear combination of several sample quantiles near the theoretical p -quantile. Then the calculation of CoV must rely on the limiting joint distribution of different sample quantiles, which are multivariate normal distributions, but it does not alter the critical scaling exponent $-c/2$ in the expression for the CoV of strength. To see this, consider the limiting joint distribution of two different sample quantiles $X_{[np]}$ and $X_{[nq]}$:

$$(5.6) \quad \begin{bmatrix} X_{[np]} \\ X_{[nq]} \end{bmatrix} \xrightarrow{d} N \left(\begin{bmatrix} x_p \\ x_q \end{bmatrix}, \frac{1}{n} \begin{bmatrix} \frac{p(1-p)}{f(x_p)^2} & \frac{p(1-q)}{f(x_p)f(x_q)} \\ \frac{p(1-q)}{f(x_p)f(x_q)} & \frac{q(1-q)}{f(x_q)^2} \end{bmatrix} \right)$$

where the first vector is the mean, and the second matrix is the covariance matrix. The distribution of the linear combination of $X_{[np]}$ and $X_{[nq]}$ is essentially a double integral of the multivariate normal density function, in which the factor $1/n$ is a constant that carries over to the variance. Since $n \sim D^{-c/2}$, the scaling exponent $-c/2$ is preserved even if σ_0 is the linear combination of several sample quantiles.

The current formulation bears some resemblance to that of the strength distribution of quasibrittle fishnets [47, 58]. Indeed, by counting the damages at the peak load for either fishnets or fiber bundles under transverse scaling, it is clear that the average critical order k that determines the peak load increases linearly with the number of links n . In other words, their ratio $p = k/n$, which is the sample p -quantile of link strengths, tends to a constant.

As a result, the large size limiting strength distributions of both fishnets and fiber bundles are normal distributions. For the same reason, the strength distribution of RC beams in the present study also follows the normal distribution.

On the other hand, the two cases are not exactly the same. For RC beams, the peak load will be triggered by material failure only in a small zone in the RC beam, whereas, for fishnets under uniaxial tension, the failure at any place in the structure could potentially lead to the peak load.

5.4. Strength Histogram

One thousand simulations have been run for each of the three beam sizes. The normalized load-displacement curves σ_N versus u/u_0 are shown in Fig. 5.9 ($\sigma_N = V/(bD)$, $V = \text{shear force}$). The normalizing constant u_0 is chosen for each size such that all the curves have the same initial stiffness. It is clear from the figure that both the critical load-point displacement and the peak load are random and that there is size effect, i.e., the nominal strength decreases as structure size increases. All the curves can be divided into four major sections according to the decrease of slope, depicting four distinct stages of failure process: 1) initial linear elastic response; 2) development of tensile cracks at

the bottom of the beam; 3) development of the major diagonal shear crack (peak load); 4) catastrophic failure of the whole RC beam, if the load is controlled (postpeak). Also it can be noticed in the figure is that the scatter of the nominal strength decreases as structure size increases.

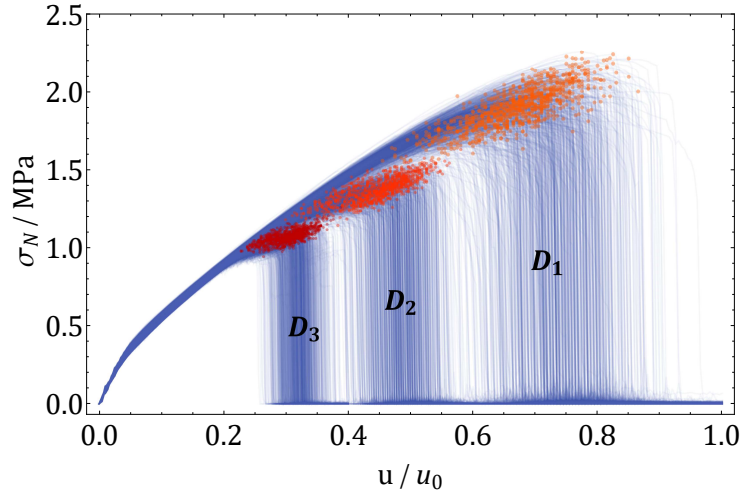


Figure 5.9. Normalized stress-displacement curves of 1000 samples of structure of size D_2 . Dark points indicate peak load locations.

Before analyzing the scatter of nominal strength, we first verify the type-2 size effect law [48, 49] (Eq. 5.3) for the mean strength of RC beams. Eq. 5.3 can be rearranged to the linear regression plot [44]:

$$(5.7) \quad y = Ax + C$$

where $x = D$ and $y = 1/\sigma_N^2$. The parameters σ_0 and D_0 in the size effect relation can be expressed in terms of A and C : $\sigma_0 = 1/\sqrt{C}$ and $D_0 = C/A$. The optimal values for A and C by fitting the simulated data of $1/\sigma_N^2$ are $A = 0.001006$ and $C = 0.143$, from which we obtain the values of the size effect parameters: $D_0 = 142\text{mm}$ and $\sigma_0 = 2.65\text{MPa}$. Fig. 5.10

shows the mean size effect curve of the scaling of RC beams together with all the simulated beam strengths. Despite the randomness of the nominal strength, the mean value follows the type-2 size effect as expected, confirming that such behavior is dominated by the deterministic energy release during the diagonal crack propagation.

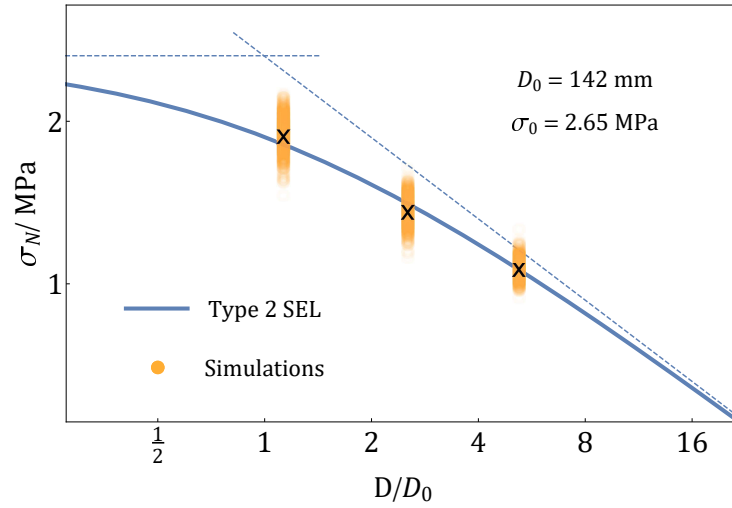


Figure 5.10. Size effect plot for the 2D scaling of the RC beam. The solid curve shows the type-2 size effect law (SEL), and the overlapping circles are simulated nominal strengths (1000 points for each size).

Fig. 5.11 shows the histograms normalized to probability density of the nominal shear strength for the 3 beam sizes. Shown in the subplots on the right are the same histogram data in normal probability papers. The data form nearly straight lines for all 3 sizes, indicating the histograms follow normal distribution (although nothing can be inferred for the tails). The fitted normal distributions are plotted as solid curves on top of the histograms. From the histograms, it is clear that the variance of nominal shear strength decreases as structure size increase.

Since the mean strength also decreases as structure size increases, we are interested in the change of the coefficient of variation (CoV), which is the standard deviation normalized

by the mean. Therefore, we plot the CoV against D in log-log scale (see Fig. 5.12). Now, it is clear that the CoV also decreases as the structure size increases, which confirms the analysis based on the scatter of critical crack tip location. The average slope in the log-log plot is -0.285 .

To compare it with the classical statistical strength models, the weakest-link chain model with Gauss-Weibull link strength distribution has a constant CoV under size scaling, and the fiber bundle model has a CoV that scales as $1/\sqrt{D}$. The slope of -0.285 found here for the RC beams lies between the slopes for these two models.

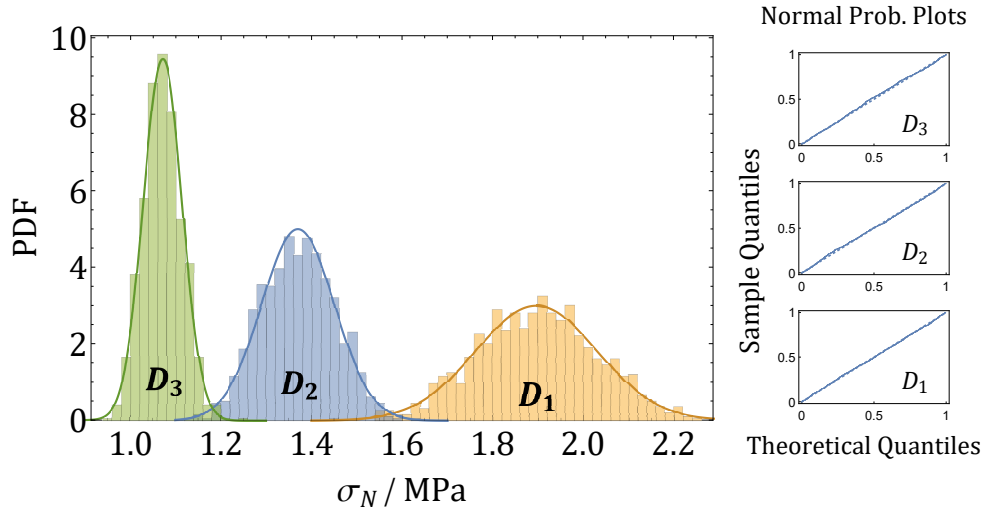


Figure 5.11. Histograms of nominal strengths for the beams of 3 sizes (1000 samples for each size) shown on the left and their normal probability plots shown on the right.

It is worth noting that the slope of strength CoV for the RC beam is smaller in magnitude than the slope for the ductile fiber bundle. This is mainly because the size of the critical crack-tip region, n , does not grow linearly with the structure size, D , but rather $n \propto D^{0.57}$. Aside from the fact that the histograms follow normal distributions (Fig. 5.11), it is clear that there does not exist any “weakest-link” type behavior in the

RC beams under shear loading. Otherwise, the tiny Weibull tail of the distribution of the inputs (k_1 and Young's modulus E) would dominate under the action of the weakest-link, leading to Weibull distributions for the nominal strength, which clearly is not the case. Despite the quasibrittleness of concrete, the reinforcement prevents a sudden chain-like failure of the whole structure and ensures a stable propagation of the diagonal shear crack through about 80% of cross section depth, up to the peak load. This is the major reason for the structure strength to have a Gaussian, rather than Weibull, strength distribution.

Now that we know the scaling relation of CoV, we could use this information to roughly infer the quantile p that governs the strength of the beam using Eq. 5.5. The slope in the log-log plot, -0.285 , corresponds to the exponent $-c/2$. Thus, $c = 0.57$. In other words, the number n of the potential critical crack tip locations scales as $D^{0.57}$.

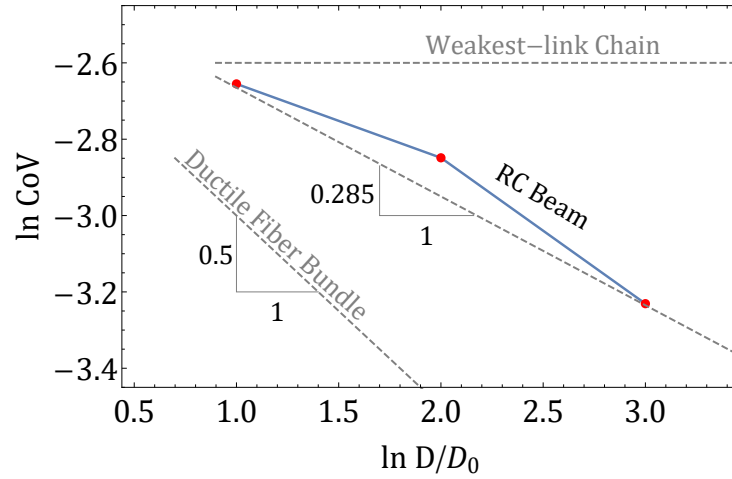


Figure 5.12. Coefficient of variation (CoV) of shear strength of RC beams for 3 different beam sizes, and comparison with the weakest-link model and the fiber bundle model.

5.5. Large Size Limit of Strength CoV

Up to now, we have studied the CoV of shear strength under the assumption that only σ_0 in Eq. 5.3 is random. This assumption is valid when D/D_0 is not too large compared with 1. Because of the square root and constant 1 in the denominator of the mean size effect relation, the effect of randomness of D_0 is not as large as that of σ_0 . In other words, the scatter of nominal shear strength σ_N is mainly affected by the randomness of σ_0 for small and medium size RC beams. But when the size becomes larger, the effect of random D_0 could become non-negligible.

Therefore, we need to study the large-size limit behavior of the RC beam by considering also the randomness of D_0 . Since thousands of Monte Carlo simulations for extremely large RC beams would take impracticably long time to finish, we need a mathematical estimation of the scaling of strength CoV for extremely large sizes. With the notation $D_0 = w_c/(k\alpha_c)$, Eq. 5.3 for $D \rightarrow \infty$ reduces to

$$(5.8) \quad \sigma_N \simeq \frac{\sigma_0 \sqrt{D_0}}{\sqrt{D}}$$

In the expression of D_0 , both the normalized length of the energy release band, α_c , and the characteristic length, w_c , could be considered as random variables. But because the individual distributions of w_c and α_c are not known any better than that of D_0 , it is appropriate to assign a distribution directly to D_0 . Then we analyze how the variance and CoV will change if D_0 in Eq. 5.8 is random. Since D_0 , as the transitional size, refers to the whole structure, and D/D_0 is an indicator of the brittleness of the whole structure, the distribution of D_0 should not scale with the effective structure size D .

Denote the mean and variance of $\mu = \sqrt{D_0}$ by $\mu_{\sqrt{D_0}}$ and $s_{\sqrt{D_0}}^2$, and the mean and variance of σ_0 by μ_{σ_0} and $s_{\sigma_0}^2$. To estimate them, we at first consider D_0 and σ_0 to be independent, and then the variance of the product $\sigma_0\sqrt{D_0}$ can be calculated:

$$(5.9) \quad \text{Var}(\sigma_0\sqrt{D_0}) = \mathbb{E}[(\sigma_0\sqrt{D_0})^2] - \mathbb{E}[\sigma_0\sqrt{D_0}]^2$$

$$(5.10) \quad = \mathbb{E}[\sigma_0^2]\mathbb{E}[D_0] - \mathbb{E}[\sigma_0]^2\mathbb{E}[\sqrt{D_0}]^2$$

$$(5.11) \quad = (s_{\sigma_0}^2 + \mu_{\sigma_0}^2)(s_{\sqrt{D_0}}^2 + \mu_{\sqrt{D_0}}^2) - \mu_{\sigma_0}^2\mu_{\sqrt{D_0}}^2$$

Since we assumed that the distribution of D_0 does not scale with size D , we let $\alpha = \mu_{\sqrt{D_0}}^2$ and $\beta = s_{\sqrt{D_0}}^2$. The variance of the product reduces to

$$(5.12) \quad \text{Var}(\sigma_0\sqrt{D_0}) = (\alpha + \beta)(s_{\sigma_0}^2 + \mu_{\sigma_0}^2) - \alpha\mu_{\sigma_0}^2 = (\alpha + \beta)s_{\sigma_0}^2 + \beta\mu_{\sigma_0}^2$$

Following from $\mu_{\sigma_0} = x_p = F^{-1}(p)$ and $s_{\sigma_0}^2 = p(1-p)/(nf(x_p)^2)$, the above expression can be rewritten as

$$(5.13) \quad \text{Var}(\sigma_0\sqrt{D_0}) = (\alpha + \beta)p(1-p)/(nf(x_p)^2) + \beta x_p^2$$

With this, the standard deviation of σ_N can be obtained by multiplying the square root of variance $\text{Var}(\sigma_0\sqrt{D_0})$ with the factor $1/\sqrt{D}$:

$$(5.14) \quad s_{\sigma_N} = \frac{\sqrt{(\alpha + \beta)p(1-p)/(nf(x_p)^2) + \beta x_p^2}}{\sqrt{D}}$$

Meanwhile, the mean value of σ_N follows the large size limit of the Type 2 size effect, which is essentially the size effect of linear elastic fracture mechanics (LEFM):

$$(5.15) \quad \mu_{\sigma_N} = \mathbb{E}[\sigma_0 \sqrt{D_0}/\sqrt{D}] = \frac{\sqrt{\alpha} x_p}{\sqrt{D}} \propto D^{-1/2}$$

Normalizing the standard deviation of σ_N by its mean, we obtain the large-size scaling relation for the strength CoV:

$$(5.16) \quad \text{CoV}(\sigma_N) = \frac{s_{\sigma_N}}{\mu_{\sigma_N}} = \sqrt{\frac{(1 + \beta/\alpha)p(1-p)}{nx_p^2 f(x_p)^2} + \beta/\alpha}$$

where $\sqrt{\beta/\alpha} = \text{CoV}(\sqrt{D_0})$, which is a constant. As $D \rightarrow \infty$, the region size n of the possible diagonal crack tip also tends to infinity. As a result, the first quotient under the square root vanishes. Therefore, for extremely large beams, $\text{CoV}(\sigma_N)$ tends to the constant $\sqrt{\beta/\alpha}$, which is the coefficient of variation for $\sqrt{D_0}$.

Two remarks should be made on this result. First, we do not assume any specific type of distribution for D_0 or $\sqrt{D_0}$ in the foregoing derivation. The result is valid as long as $\sqrt{D_0}$ has a finite mean and variance. Second, this result relies on the assumption that σ_0 and D_0 are statistically independent. In reality, D_0 should have some degree of correlation with σ_0 . Therefore, the large size limit of the strength CoV might not be exactly $\sqrt{\beta/\alpha}$. Since that D_0 does not scale with D , D_0 and σ_0 cannot be fully coupled ($D_0 \propto \sigma_0$). According to this argument, the constant mean and standard deviation of D_0 will show up in the expression of $\text{CoV}(\sigma_N)$ in a similar way as in the case of independence, which will lead to a non-zero strength CoV for $D \rightarrow \infty$.

With this large size constant CoV, we propose an approximate size effect equation for the nominal strength CoV, ω , for RC beams:

$$(5.17) \quad \omega = \omega_0 \left[1 + \left(\frac{D_b}{D} \right)^r \right]^{\frac{1}{2r}}$$

where D_b = characteristic length and r = positive empirical constant. This formula can be seen as the asymptotic matching of two limiting cases: 1) ductile limit for small sizes ($D \rightarrow 0$), characterized by the ductile fiber bundle model with its strength CoV scaling as $D^{-1/2}$, and 2) constant CoV for large sizes ($D \rightarrow \infty$) with $\omega_0 = \sqrt{\beta/\alpha} = \text{CoV}(\sqrt{D_0})$. The three typical beam sizes that we used in the simulations fall in the intermediate range of the size effect relation. Combining this with the Freudenthal reliability integral [100, 101], the current size effect relation (Eq. 5.17) for strength CoV could be further used in the structural reliability analysis presented in [102] for the safety factor in RC beam design.

5.6. Conclusions

- (1) The simulations confirm that the ultimate failure of the RC beam is triggered by the compression-shear failure of a small region at the top of the beam. This is similar to the failure of the compression strut in the strut-and-tie model, which was shown to occur near the top of the beam and close to the tip of the diagonal crack [74, 1, 44].
- (2) Despite the fact that concrete is quasibrittle, the reinforcement enables a stable propagation of the diagonal shear crack before the peak load is reached. Therefore, the weakest-link model for plain concrete does not apply.

- (3) The mean nominal strength of the RC beams follows the Type-2 size effect, which is deterministic, energetic.
- (4) Based on Monte Carlo simulations, the zone of the possible critical shear cracks becomes larger as structure size increases, and its growth rate is smaller than D^2 . Equivalently, as we rescale structures strictly to the same size, the critical zone for the largest structure reduces to a point.
- (5) The basic idea advanced here is that the shear strength follows the distribution of the sample p -quantile of strengths in a small critical region, leading to normal distribution for the large size limit.
- (6) The justification for this idea is that the number of critical material elements of random strength that can trigger failure grows with the size of the near-tip critical region. This region, in turn, is not constant but grows with the structures size, which causes that what matters is not the number but the percentage, or quantile, of the elements of random strength in the structure.
- (7) There is a significant size effect on the coefficient of variation (CoV) of strength. The CoV of strength decreases as the structure size increases. But the CoV decrease weakens at large sizes, and the large size limit of the CoV of strength is a constant.

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Vita

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- (1) Luo, W., Bažant, Z.P., 2020. General Fishnet Statistics of Strength: Nacreous, Biomimetic, Concrete, Octet-truss, and Other Architected or Quasibrittle Materials. *Journal of Applied Mechanics* , 87(3).
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His conference proceedings include:

- (1) Bažant, Z.P., Rasoolinejad, M., Dönmez, A. and Luo, W., 2019, August. Dependence of fracture size effect and projectile penetration on fiber content of FRC. In IOP Conference Series: Materials Science and Engineering (Vol. 596, No. 1, p. 012001). IOP Publishing.
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