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7. PERFORMING ORGANIZATION NAMES AND ADDRESSES
College of William and Mary
P.O. Box 8795

Williamsburg, VA 23187 -8795

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14. ABSTRACT

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19a. NAME OF RESPONSIBLE PERSON
Sarah Day

19b. TELEPHONE NUMBER
757-221-2013

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RPPR Final Report
as of 14-Aug-2023

Agency Code: 21XD

Proposal Number: 72624MI **Agreement Number: W911NF-18-1-0306**

INVESTIGATOR(S):

Name: Sarah Day
Email: sldayx@wm.edu
Phone Number: 7572212013
Principal: Y

Organization: **College of William and Mary**
Address: P.O. Box 8795, Williamsburg, VA 231878795
Country: USA
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Report Date: 08-Oct-2022 Date Received: 18-Apr-2023
Final Report for Period Beginning 13-Jul-2018 and Ending 08-Jul-2022
Title: A Combinatorial and Topological Framework for Deriving Dynamics from Data
Begin Performance Period: 13-Jul-2018 **End Performance Period:** 08-Jul-2022
Report Term: 0-Other
Submitted By: Sarah Day Email: sldayx@wm.edu
Phone: (757) 221-2013

Distribution Statement:

STEM Degrees: 3 **STEM Participants:** 4

Major Goals: From the proposal:

G1 Develop an algorithmic framework based on combinatorial models and approximations to merge delay reconstruction, regression analysis, and outer approximation for the extraction of dynamics from data.

G2 Utilize combinatorial models and combinatorial approximations on queried data sets (sets for which we may control sampling of predictors) for the purposes of dimension reduction for high-dimensional systems and the study of implicitly-known systems.

G3 Develop meaningful topological and geometric measurements of spatially-explicit systems and study the dynamics of the obtained data.

G4 Source and research a variety of different types of data sets to explore the advantages and limitations of the proposed methods.

Accomplishments: G1 Develop an algorithmic framework based on combinatorial models and approximations to merge delay reconstruction, regression analysis, and outer approximation for the extraction of dynamics from data.

This has been the biggest contribution of work on this grant. We established a general construction for combinatorial approximations for queried, measured, and delay reconstructed data. This allows for different surrogate models to serve as the backbone for the construction and the incorporation of explicit error bounds based on the chosen surrogate model and available data and error estimates. We also developed a method for using the combinatorial approximation to calculate measures of robustness for computed results. These methods are recovering accurate information for queried and time series data from the Kot-Schaffer model. They are also uncovering detailed and complicated dynamics in delay reconstructed data from a magneto elastic ribbon experiment. The overall framework is presented in Day and Kalies (preprint 2023), with foundational work appearing in Batko et al (submitted 2022) and Dowling et al (submitted 2022).

G2 Utilize combinatorial models and combinatorial approximations on queried data sets (sets for which we may control sampling of predictors) for the purposes of dimension reduction for high-dimensional systems and the study of implicitly-known systems.

This work has been carried out on a range of models including 1D, conceptual examples (Batko et al, submitted

2023) and the infinite-dimensional Kot-Schaffer model (Day and Kalies, preprint 2023). In the Kot-Schaffer model, we demonstrate that combinatorial approximation on even sparse, queried data yields results comparable to known rigorous results for the model. This represents a reduction from the infinite dimensional system to a two dimensional projected system.

G3 Develop meaningful topological and geometric measurements of spatially-explicit systems and study the dynamics of the obtained data.

In Storch and Day 2019, topological measurements in the form of Betti numbers are used to study dynamics and early warning signals of global extinction events in a spatially-explicit model ecological system. In Storch and Day 2022, these methods are extended and combined with geometric methods to characterize differing routes to extinction. In moving towards applying these methods to satellite and GIS data, the methods are currently being extended to incorporate morphological and other image processing techniques into a multiparameter persistence framework.

G4 Source and research a variety of different types of data sets to explore the advantages and limitations of the proposed methods.

In addition to the data sets listed above, we have also studied flour beetle population abundance data undergoing bifurcation in Berliner, Day, and Kalies (preprint 2023). This represents an important shift to studying nonstationary data and shows that an extension of the techniques may detect bifurcations, even in sparse and noisy data. The techniques allow for forecasting from measurements of sparsely measured, noisy, and non stationary systems.

Training Opportunities: Nothing to Report

Results Dissemination: In addition to talks listed in years 1-3,

Day presented work on this grant in a presentation on Topological Data Analysis in Population Pattern Quantification at the Workshop on Stability and Fluctuations in Complex Ecological Systems, Lorentz Center, Leiden, August 2022. This introduced the key of ideas of this work to a community of ecologists and physicists.

Kalies presented work on this grant in a virtual presentation on Order Theory and Dynamics at the DIMACS Workshop on Dynamics, Topology, and Robotic Control, New Brunswick, NJ, May 2021. This introduced the key of ideas of this work to a community of mathematicians, statisticians, and engineers.

Honors and Awards: Nothing to Report

Protocol Activity Status:

Technology Transfer: Nothing to Report

PARTICIPANTS:

Participant Type: PD/PI
Participant: Sarah Day
Person Months Worked: 10.00 **Funding Support:**
Project Contribution:
National Academy Member: N

Participant Type: Co PD/PI
Participant: William Kalies
Person Months Worked: 8.00 **Funding Support:**
Project Contribution:
National Academy Member: N

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)
Participant: Laura Storch
Person Months Worked: 4.00 **Funding Support:**
Project Contribution:
National Academy Member: N

Participant Type: Graduate Student (research assistant)
Participant: Sean Perry
Person Months Worked: 15.00 **Funding Support:**
Project Contribution:
National Academy Member: N

Participant Type: Undergraduate Student
Participant: Michael Getaneh
Person Months Worked: 2.00 **Funding Support:**
Project Contribution:
National Academy Member: N

Participant Type: Undergraduate Student
Participant: Krishna Tejo
Person Months Worked: 2.00 **Funding Support:**
Project Contribution:
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Participant Type: Undergraduate Student
Participant: Sage Stanish
Person Months Worked: 1.00 **Funding Support:**
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ARTICLES:

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Journal: AMS Contemporary Mathematics
Publication Identifier Type: DOI Publication Identifier: 10.1090/conm/736/14846
Volume: Issue: First Page #:
Date Submitted: 8/26/21 12:00AM Date Published:
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Article Title: Towards the prediction of critical transitions in spatially extended populations with cubical homology
Authors: Laura Storch, Sarah Day
Keywords: Biology and other natural sciences, topology
Abstract: The prediction of critical transitions, such as extinction events, is vitally important to preserving vulnerable populations in the face of a rapidly changing climate and continuously increasing human resource usage. Predicting such events in spatially distributed populations is challenging because of the high dimensionality of the system and the complexity of the system dynamics. Here, we reduce the dimensionality of the problem by quantifying spatial patterns via Betti numbers (?0 and ?1), which count particular topological features in a topological space. We illustrate how Betti numbers can be used to characterize spatial patterns by type, which in turn may be used to track spatiotemporal changes via Betti number time series and characterize asymptotic dynamics of the model parameter space. We hope these preliminary results will be used to aide in the prediction of critical transitions in spatially extended systems.
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Journal: Journal of Neuroscience Methods
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Publication Location:
Article Title: Topological portraits of multiscale coordination dynamics
Authors: Mengsen Zhang, William D. Kalies, J.A. Scott Kelso, Emmanuelle Tognoli
Keywords: Coordination Dynamics Metastability Oscillators Topological data analysis Complex systems
Abstract: Living systems exhibit complex yet organized behavior on multiple spatiotemporal scales. To investigate the nature of multiscale coordination in living systems, one needs a meaningful and systematic way to quantify the complex dynamics, a challenge in both theoretical and empirical realms. The present work shows how in- tegrating approaches from computational algebraic topology and dynamical systems may help us meet this challenge. In particular, we focus on the application of multiscale topological analysis to coordinated rhythmic processes. First, theoretical arguments are introduced as to why certain topological features and their scale-dependency are highly relevant to understanding complex collective dynamics. Second, we propose a method to capture such dynamically relevant topological information using persistent homology, which allows us to effectively construct a multiscale topological portrait of rhythmic coordination. Finally, the method is put to test in detecting transit
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Journal: Journal of Dynamics and Differential Equations
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Volume: Issue: First Page #:
Date Submitted: 4/2/23 12:00AM Date Published:
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Article Title: Lattice structures for attractors III
Authors: W.D. Kalies, K. Mischaikow, R. VanderVorst
Keywords: Booleanization, Conley form, Morse decomposition, distributive lattice, Birkhoff-Stone Representation Theorem.
Abstract: The theory of bounded, distributive lattices provides the appropriate language for describing directionality and asymptotics in dynamical systems. For bounded, distributive lattices the general notion of 'set-difference' taking values in a semilattice is introduced, and is called the Conley form. The Conley form is used to build concrete, set-theoretical models of spectral, or Priestley spaces, of bounded, distributive lattices and their finite coarsenings. Such representations build order-theoretic models of dynamical systems, which are used to develop tools for computing global characteristics of a dynamical system.
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Volume: 554 Issue: First Page #:
Date Submitted: 4/2/23 12:00AM Date Published: 9/15/22 4:00AM
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Article Title: Topological early warning signals: Quantifying varying routes to extinction in a spatially distributed population model
Authors: Laura Storch, Sarah Day
Keywords: Critical transitions, extinction event, early warning signals, pattern formation, spatial ecology, computational topology
Abstract: Understanding and predicting critical transitions in spatially explicit ecological systems is particularly challenging due to their complex spatial and temporal dynamics and high dimensionality. Here, we explore changes in population distribution patterns during a critical transition (an extinction event) using computational topology. Computational topology allows us to quantify certain features of a population distribution pattern, such as the level of fragmentation. We create population distribution patterns via a simple coupled patch model with Ricker map growth and nearest neighbors dispersal on a two dimensional lattice. We observe two dominant paths to extinction within the explored parameter space that depend critically on the dispersal rate d and the rate of parameter drift, ?. These paths to extinction are easily topologically distinguishable, so categorization can be automated. We use this population model as a theoretical proof-of-concept for the methodology, and argue that
Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
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Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 0-Other
Journal: preprint
Publication Identifier Type: Publication Identifier:
Volume: Issue: First Page #:
Date Submitted: 4/14/23 12:00AM Date Published:
Publication Location:
Article Title: Bifurcation from Data
Authors: Alexander Berliner, Sarah Day, William Kalies
Keywords: forecasting, bifurcation, critical transitions
Abstract: Computing nonlinear dynamics directly from data typically involves explicit model construction. Here, we propose a method for combinatorial approximation construction that combines statistical estimation techniques with combinatorial and topological tools in order to compute dynamics without needing to fit an explicit model. This framework allows for both denoising and incorporation of error estimates. We also provide means of calculating robustness of computed results. As illustration, we apply the methods to Kot-Schaffer projection data and measurement time series from a magnetoelastic ribbon experiment.
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Journal: preprint
Publication Identifier Type: Publication Identifier:
Volume: Issue: First Page #:
Date Submitted: 4/14/23 12:00AM Date Published:
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Article Title: Combinatorial Approximation: Dynamics from Data
Authors: Sarah Day, William Kalies
Keywords: nonlinear dynamics, Gaussian process, estimation, Conley index theory
Abstract: Computing nonlinear dynamics directly from data typically involves explicit model construction. Here, we propose a method for combinatorial approximation construction that combines statistical estimation techniques with combinatorial and topological tools in order to compute dynamics without needing to fit an explicit model. This framework allows for both denoising and incorporation of error estimates. We also provide means of calculating robustness of computed results. As illustration, we apply the methods to Kot-Schaffer projection data and measurement time series from a magnetoelastic ribbon experiment.
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Volume: Issue: First Page #:
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Article Title: Identifying Nonlinear Dynamics with High Confidence from Sparse Data
Authors: Bogdan Batko, Marcio Gameiro, Ying Hung, William Kalies, Konstantin Mischaikow, Ewerton Vieira
Keywords: Sparse data, Gaussian Process, Nonlinear Dynamics, Uncertainty Quantification
Abstract: We introduce a novel procedure that, given sparse data generated from a stationary deterministic nonlinear dynamical system, can characterize specific local and/or global dynamic behavior with rigorous probability guarantees. More precisely, the sparse data is used to construct a statistical surrogate model based on a Gaussian process (GP). The dynamics of the surrogate model is interrogated using combinatorial methods and characterized using algebraic topological invariants (Conley index).
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RPPR Final Report
as of 14-Aug-2023

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 5-Submitted
Journal: SIAM Dynamical Systems (submitted)
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Article Title: Identifying Nonlinear Dynamics with High Confidence from Sparse Data
Authors: Bogdan Batko, Marcio Gameiro, Ying Hung, William Kalies, Konstantin Mischaikow, Ewerton Vieira
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Abstract: We introduce a novel procedure that, given sparse data generated from a stationary deterministic nonlinear dynamical system, can characterize specific local and/or global dynamic behavior with rigorous probability guarantees. More precisely, the sparse data is used to construct a statistical surrogate model based on a Gaussian process (GP). The dynamics of the surrogate model is interrogated using combinatorial methods and characterized using algebraic topological invariants (Conley index).

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: **Y**

Partners

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I certify that the information in the report is complete and accurate:
Signature: Sarah Day
Signature Date: 4/18/23 11:18AM

Sarah Day
William & Mary
co-PI William Kalies (FAU)

A Combinatorial and Topological Framework for Deriving Dynamics from Data

Proposal 72624-MA
Contract #W911NF-18-1-0306

Objective

- Develop a flexible, model-free framework for extracting local and global dynamics from data

- Important applications include dimension reduction for high dimensional model systems as well as the study of dynamics in systems measured by noisy and/or sparse time series data

Approach

- Define and construct combinatorial representations (approximations and models) of dynamics that bypass the need for explicit model equations while allowing topological tools to draw conclusions about dynamics

- Test and adapt the techniques on data sets generated by models and given by physical/time series measurements of systems.

- Develop metrics for quantifying the stability and robustness of computed results.

Scientific Barriers

- We did not run into unexpected computational barriers but next phases will include testing the techniques on larger, more complicated data sets as well as noisy and sparse time series.

- As expected higher dimensional reconstructions lead to computational limitations.

Significance

- Both dimension reduction for high dimensional systems and understanding dynamics in systems measured only by sparse time series data represent important areas of research in modeling and simulation of dynamic systems.

- This project represents a more flexible, cohesive, and more broadly applicable approach than others previously taken in this field.

Accomplishments by year

Year 1

- We successfully integrated Gaussian process techniques into the construction of combinatorial approximations.

- Studies of dimension reduction in the Kot-Schaffer model and dynamics reconstruction for magnetoelastic ribbon data began showing success.

- We demonstrated the use of combinatorial model construction for obtaining a coarse, global understanding of dynamics in the case of sparse data

Year 2

- We demonstrated the utility of these methods in dimension reduction using known rigorous results for the Kot-Schaffer model as a benchmark.

- We incorporated a robustness parameter into combinatorial construction.

- We calculated and compared error estimates from test set data to robustness results for both the Kot-Schaffer model and the magnetoelastic ribbon time series data.

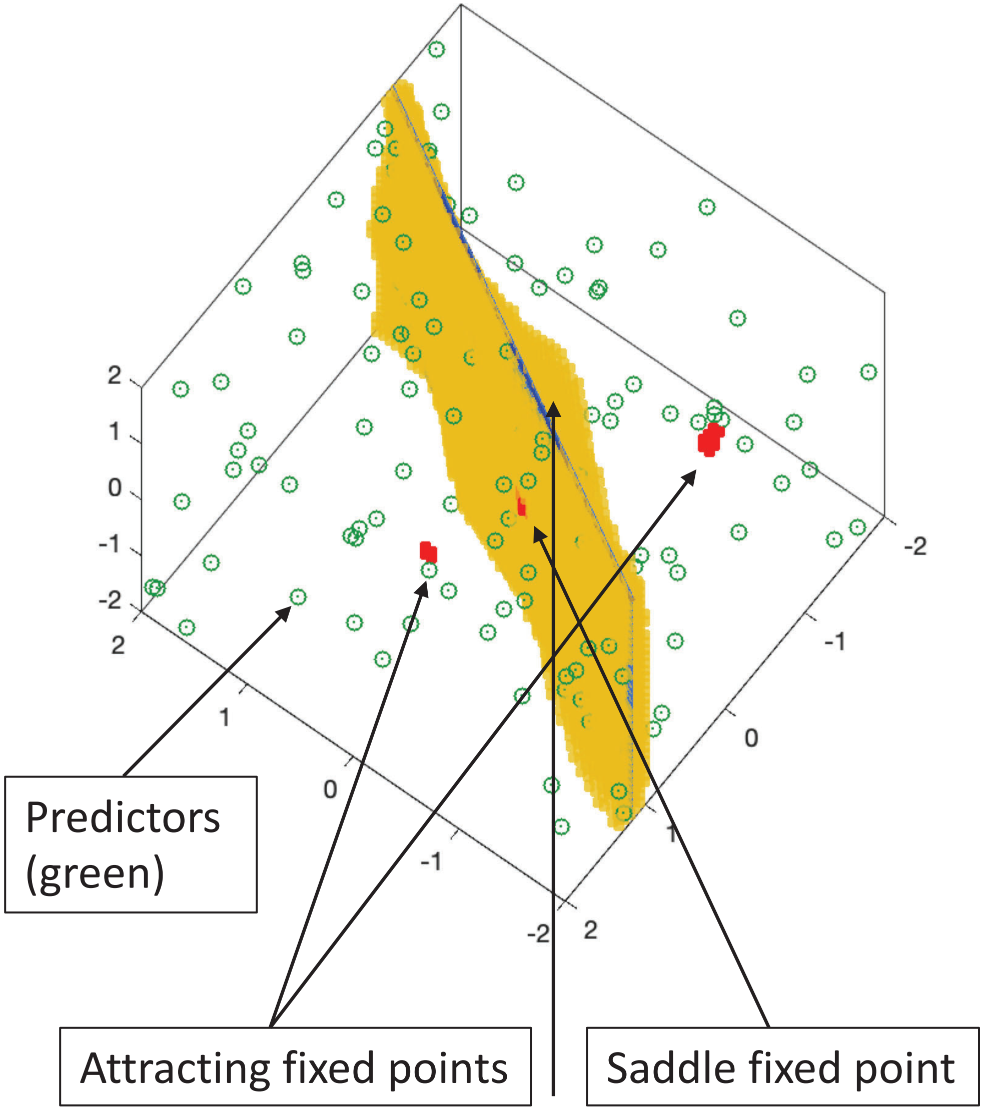
Year 3

- Using data from flour beetle experiments, we developed and tested methods for approximating bifurcation structure.

- We used topological measurements to characterize early warning signals in spatial population simulations and to detect differing routes to extinction.

- We developed code and general procedures for combinatorial approximation from models and data.

Combinatorial Model: Bistability and Basins of Attraction from Sparse Data



Predictors (green)

Attracting fixed points

Saddle fixed point

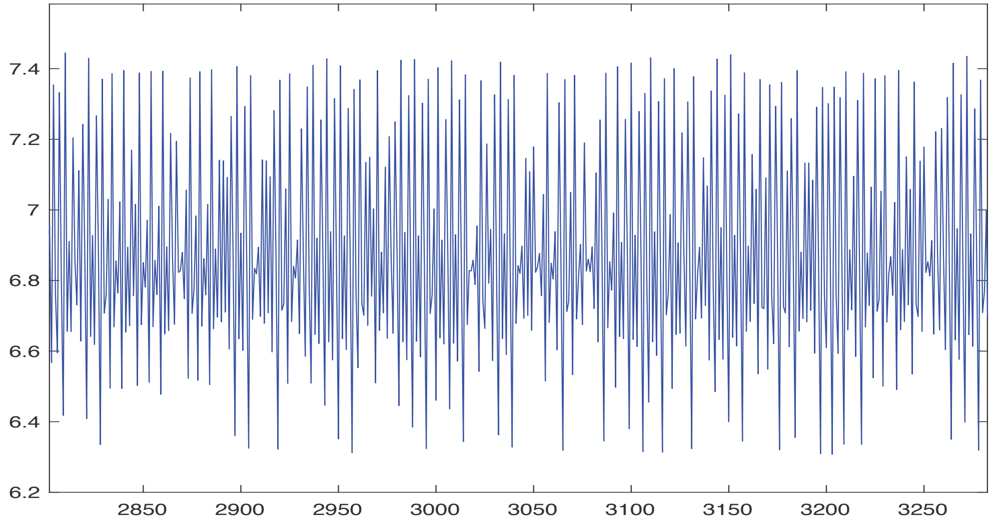
True separatrix is a hyperplane (blue), and an approximate covering of the separatrix computed by combinatorial model (yellow)

Global dynamical structure, consisting of two attractors, their basins of attraction, and a saddle, can be detected in 4-D with a sparse predictor set of approximately 150 points.

These simulated points (green, first three dimensions shown for visualization) from a 4-dimensional ODE exhibiting bistability are used to construct the combinatorial model.

The separatrix between domains of attraction can be approximated by adaptive sampling of predictors and refinement of the grid.

In what follows, we focus on results using *combinatorial approximations* (***abbreviated CA***), *robustness parameters*, and *error estimates* for the infinite dimensional **Kot-Schaffer Ricker (KSR) model** from ecology, and then the **Magnetoelastic Ribbon (MER) time series data** shown below.

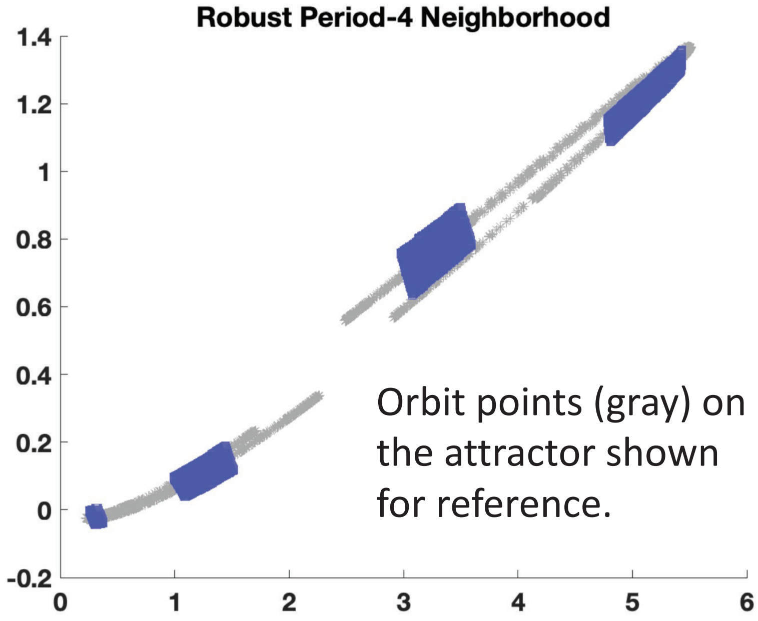


<u>MER Time Series</u>: Length 100,000 time series of voltage readings sampled at drive frequency 1.2 Hz measuring displacement of a magneto elastic ribbon under an oscillating magnetic field. Voltage is measured up to 0.001 volts.

Robust Periodic Orbit Validation in 2D projections of the Infinite-Dimensional KSR Model.

Robust neighborhood of a period-4 orbit (blue) obtained from a CA using 289 predictors on a 17x17 grid. For any map whose images are covered by this CA, the computed results hold. Using pointwise error estimates in this neighborhood, approximately 89.6% of the test points had x-images that were covered by this CA and 100% of the y-images were covered.

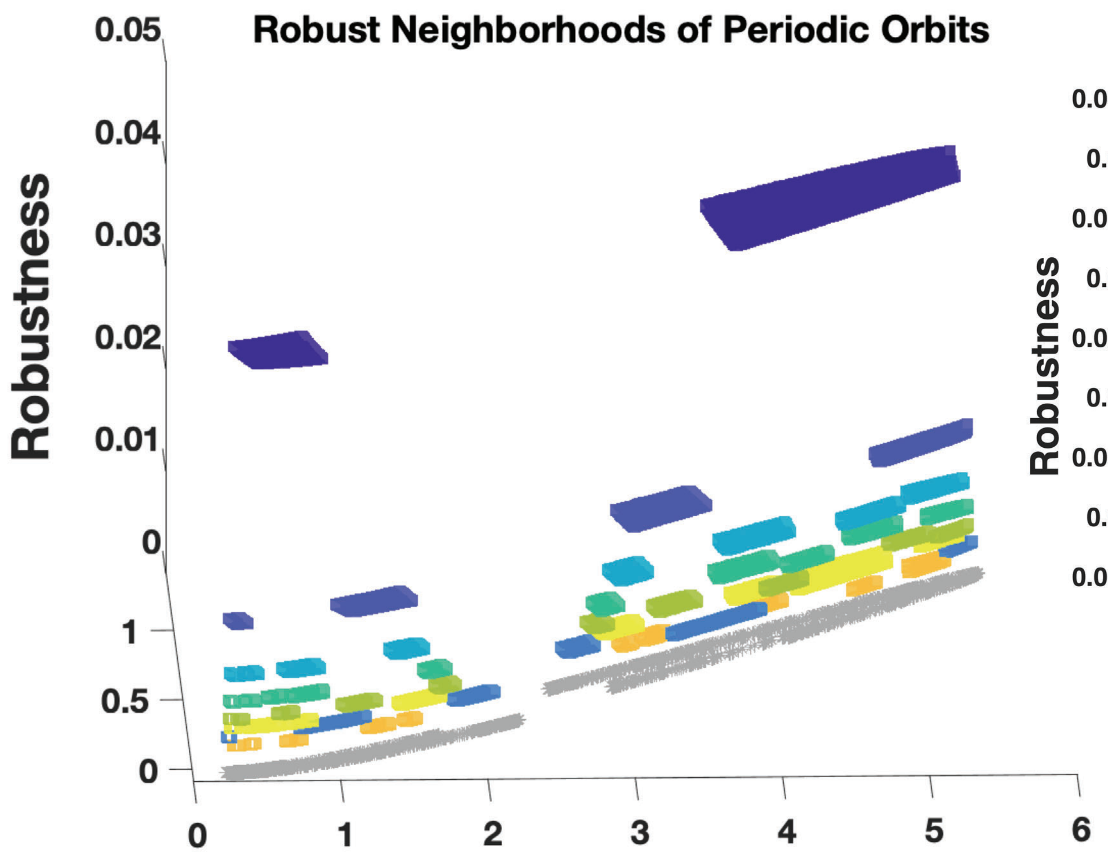
While the period-4 orbit cannot be validated by this CA for the exact 2D KSR projection, it is validated (using the Conley index) for a large class of dynamical systems. Indeed, repeating this computation with more predictors does produce a neighborhood that contains all test point images indicating high confidence that the period-4 orbit exists in the underlying 2D KSR projection.

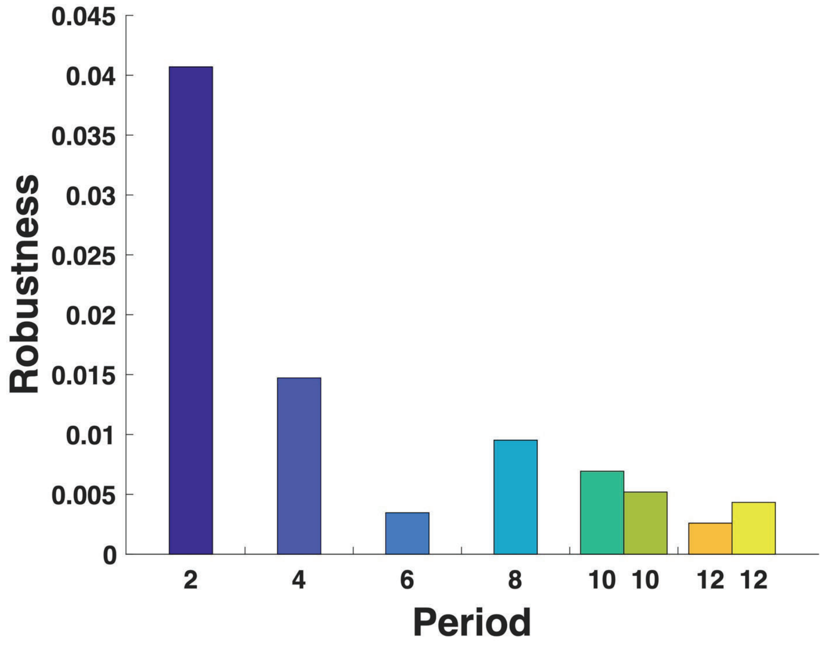


Orbit points (gray) on the attractor shown for reference.

A computed **robustness parameter** ρ, together with the CA, characterizes the (infinite) class of maps for which the results hold. Pointwise error estimates from test data allow us to estimate the robustness parameter needed for the system, given the data. A summary table of computed periodic orbits and their phase space visualization are given on the next slide.

Robust Periodic Orbits from 2D KSR Projection Data

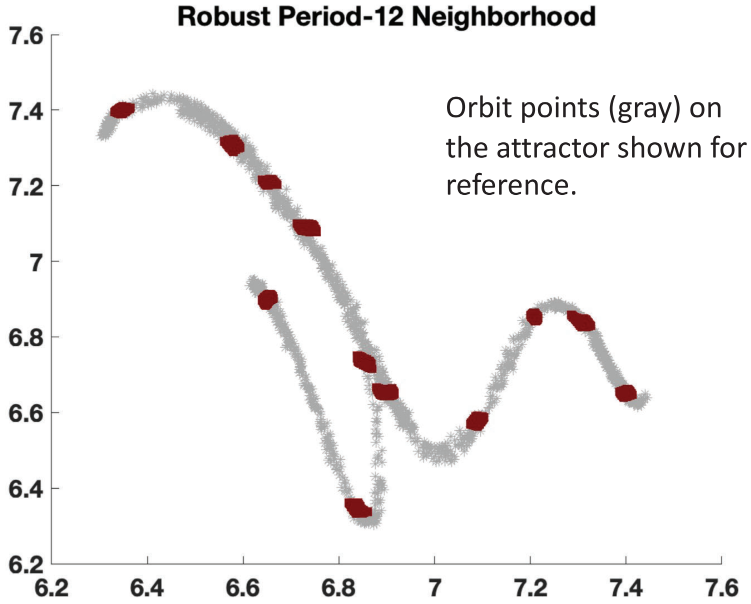




Period	ρ_x	ρ_y	Max_error_x	Max_error_y	% Covered in x	% Covered in y
2	0.0407	0.0543	0.0766	0.0313	91.1%	100%
4	0.0147	0.0196	0.0348	0.0159	89.6%	100%
6	0.0035	0.0046	0.0310	0.0772	68.4%	83.6%
8	0.0095	0.0127	0.0776	0.0313	85.0%	91.7%
10	0.0069	0.0092	0.0776	0.0313	85.5%	88.5%
10	0.0052	0.0069	0.0775	0.0312	82.3%	85.5%
12	0.0026	0.0035	0.0776	0.0313	80.0%	80.6%
12	0.0043	0.0058	0.0776	0.0313	83.1%	90.5%

Note: all of the periodic orbits proven to exist in previous work on KSR are found robustly using this much cheaper, data-driven approach.

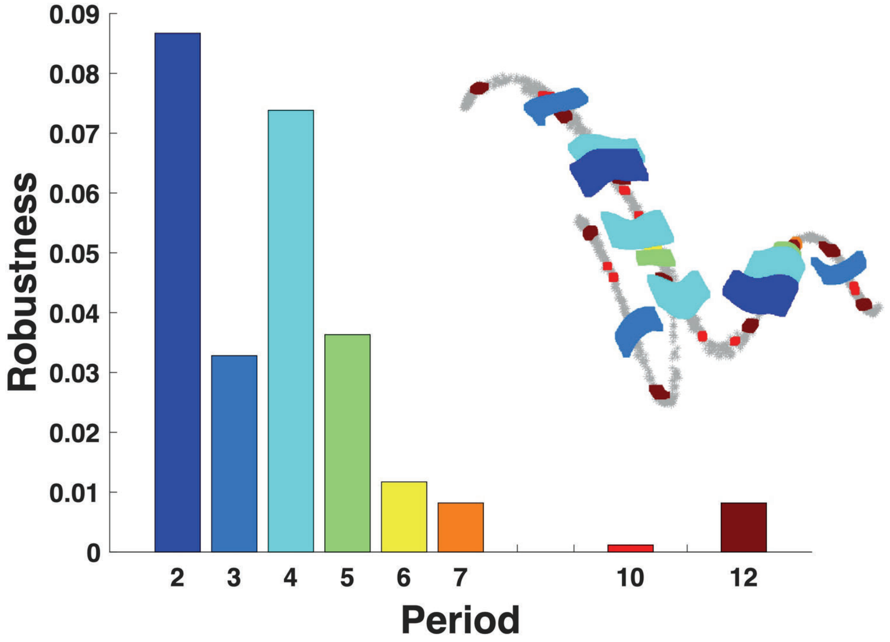
Robust Periodic Orbits from MER Time Series Data

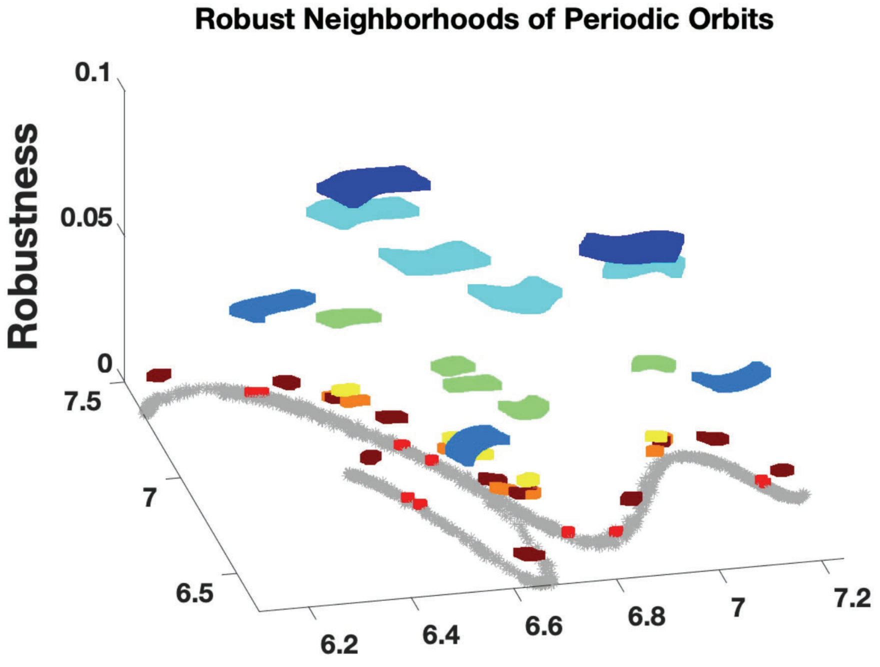


Orbit points (gray) on the attractor shown for reference.

Using a CA built from 400 predictors from a 2D delay reconstruction of the time series, we compute robust periodic orbits of various periods; eight are shown below, including a period-12 orbit shown also to the left.

These results further demonstrate that this method can extract dynamics from data, including when using delay reconstruction of physical data with measurement error.





Computational Dynamics and Systems Described by Data

Bill Kalies
Florida Atlantic University

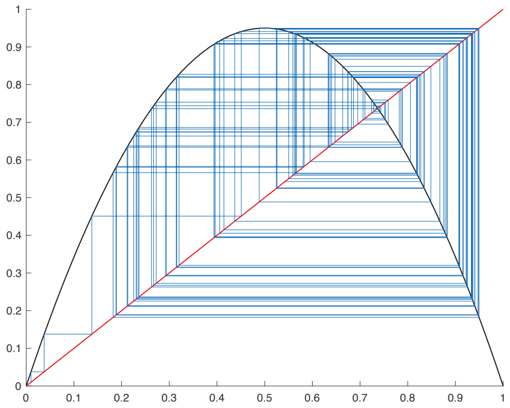
University of Toledo
February 27, 2023

Outline

- Overview of nonlinear dynamics
  - Algebraic structures
- Computational dynamics via Conley theory
  - Combinatorial outer approximation
- Nonparametric combinatorial models
- Robustness of combinatorial models from data
- Future work

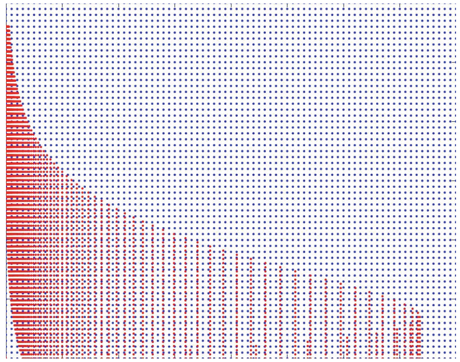
Dynamical Systems

$$\phi\colon X \times \mathbb{T} \to X \text{ such that } \phi(x,0) = x \text{ and } \phi(x, s+t) = \phi(\phi(t,x), s)$$



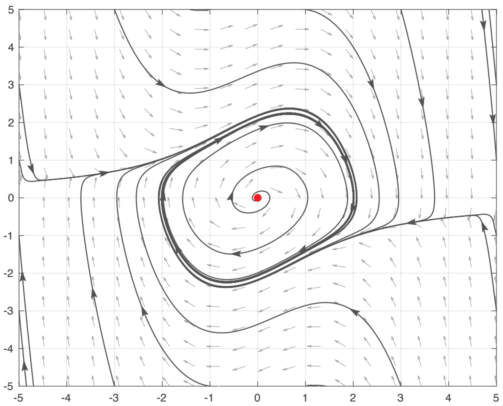
Discrete-Time

$$\mathbb{T} = \mathbb{Z}^+$$



Iterated Maps

$$\phi(x,1) = f(x)$$

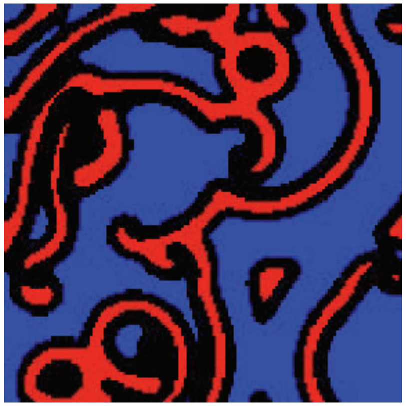


Continuous-Time

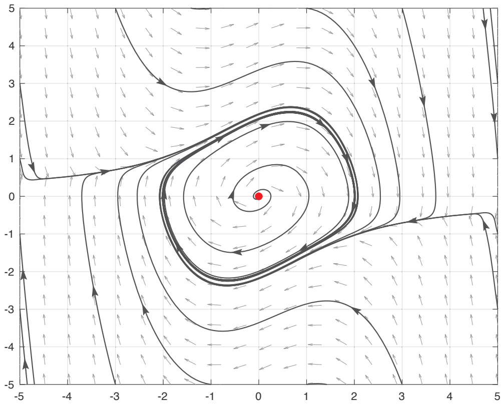
$$\mathbb{T} = \mathbb{R}^+$$

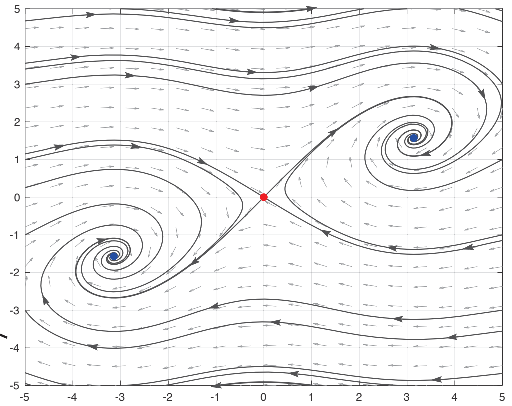
Differential Equations

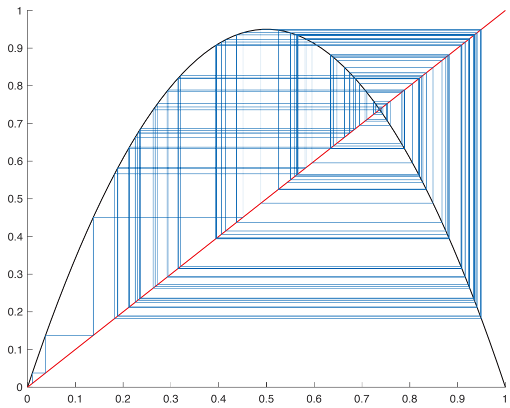
$$\phi_t(x,t) = F(\phi(x,t))$$



Asymptotic Dynamics





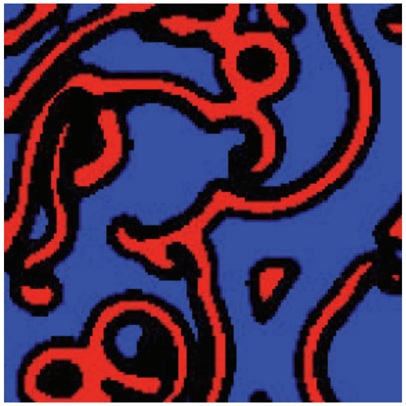


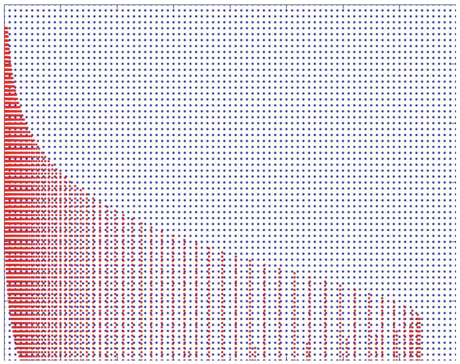
Fixed Point

Periodicity

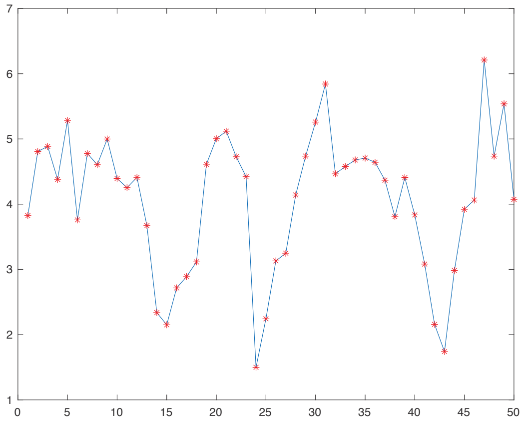
Multistability

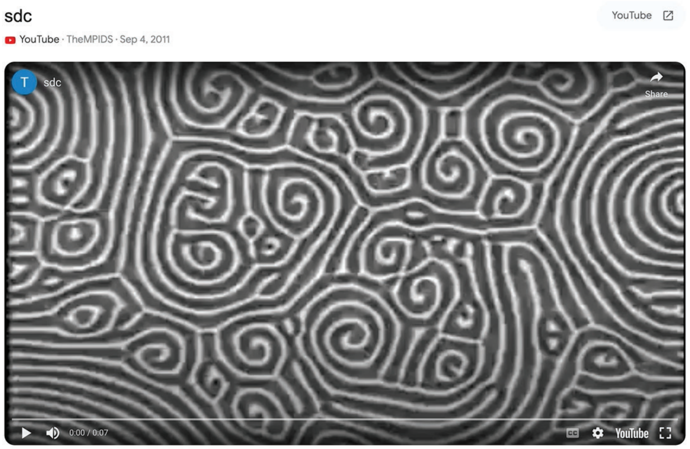
Chaos

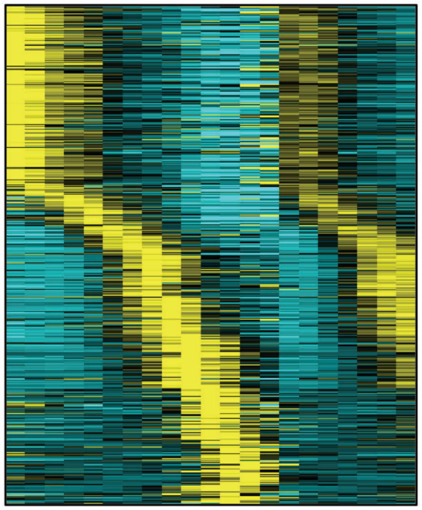


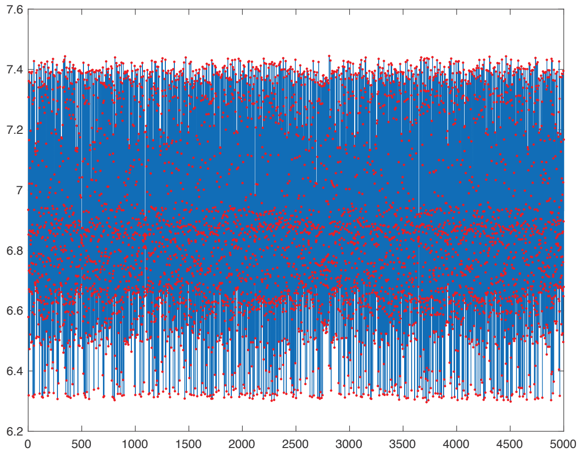


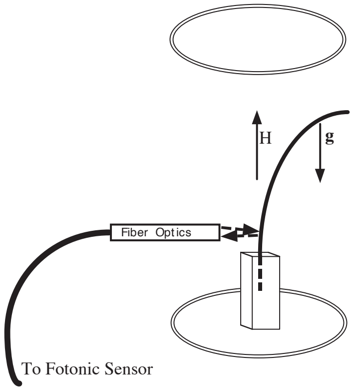
Dynamical Data





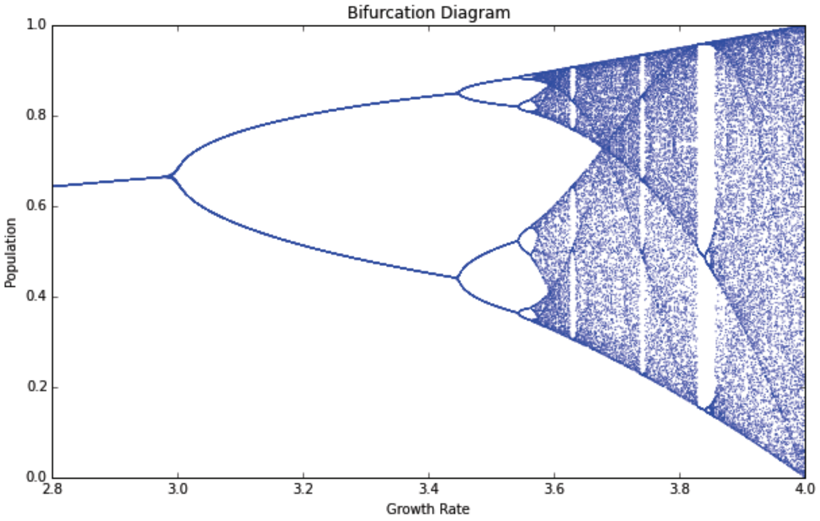






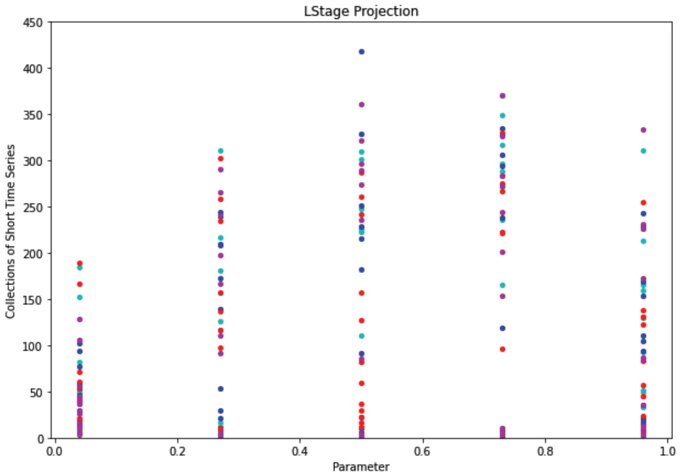
Dynamics can change with parameters

$$f(x) = rx(1-x)$$



Classifying up to topological conjugacy (all scales) is difficult.

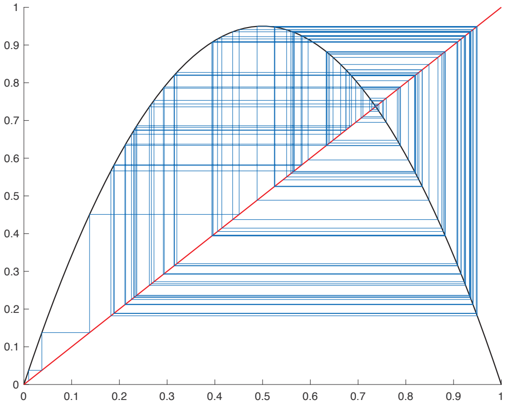
Bifurcations can occur on a Cantor set of positive measure.

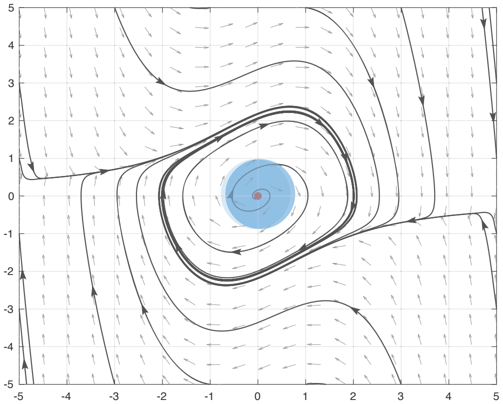


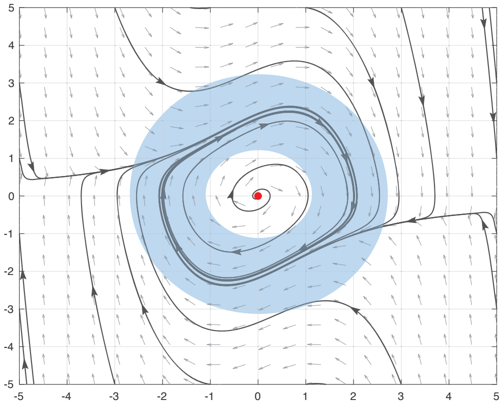
Invariant sets

$\gamma\colon \mathbb{T} \to X$ such that $\phi(\gamma(t), s) = \gamma(t+s)$ is an *orbit* through $x_0 = \gamma(0)$

$S \subset X$ is *invariant* if $\phi(S, t) = S$ for all $t \geq 0 \Leftrightarrow S$ is a union of orbits





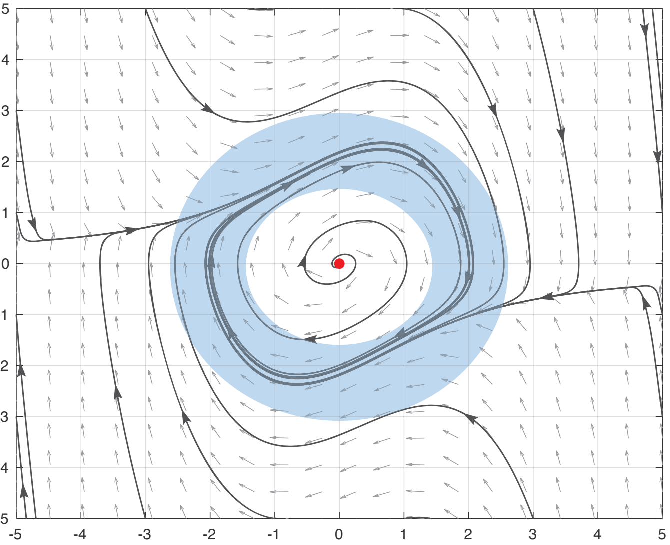


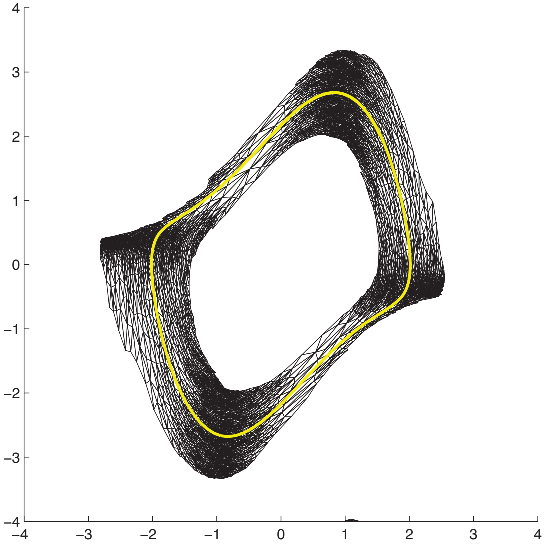
S is an *isolated invariant set* if there is a compact neighborhood N such that
S is the maximal invariant set in N, denoted by $\mathrm{Inv}(N)$, and $S \subset \mathrm{int}(N)$.
In this case, N is an *isolating neighborhood* for S.

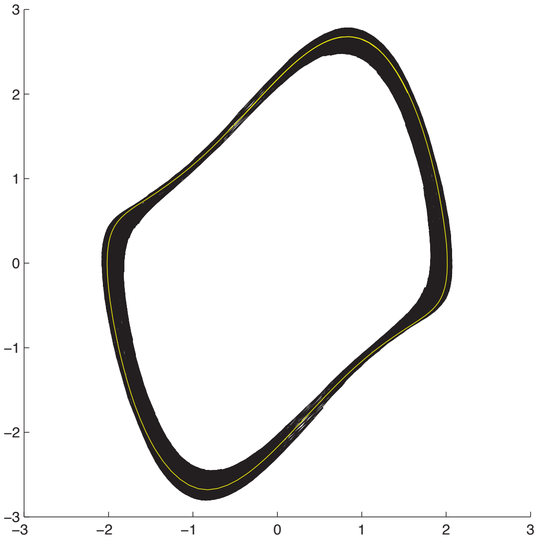
Attractors

$$\omega(U) = \bigcap_{t>0} \mathrm{cl}\left(\phi(U,[t,\infty))\right)$$

$U \subset X$ is an *attracting block (attracting neighborhood)* if $\phi((\mathrm{cl}(U)),t) \subset \mathrm{int}(U)$ for all $t > 0$ (for all $t > t_0$ for some $t_0 > 0$).

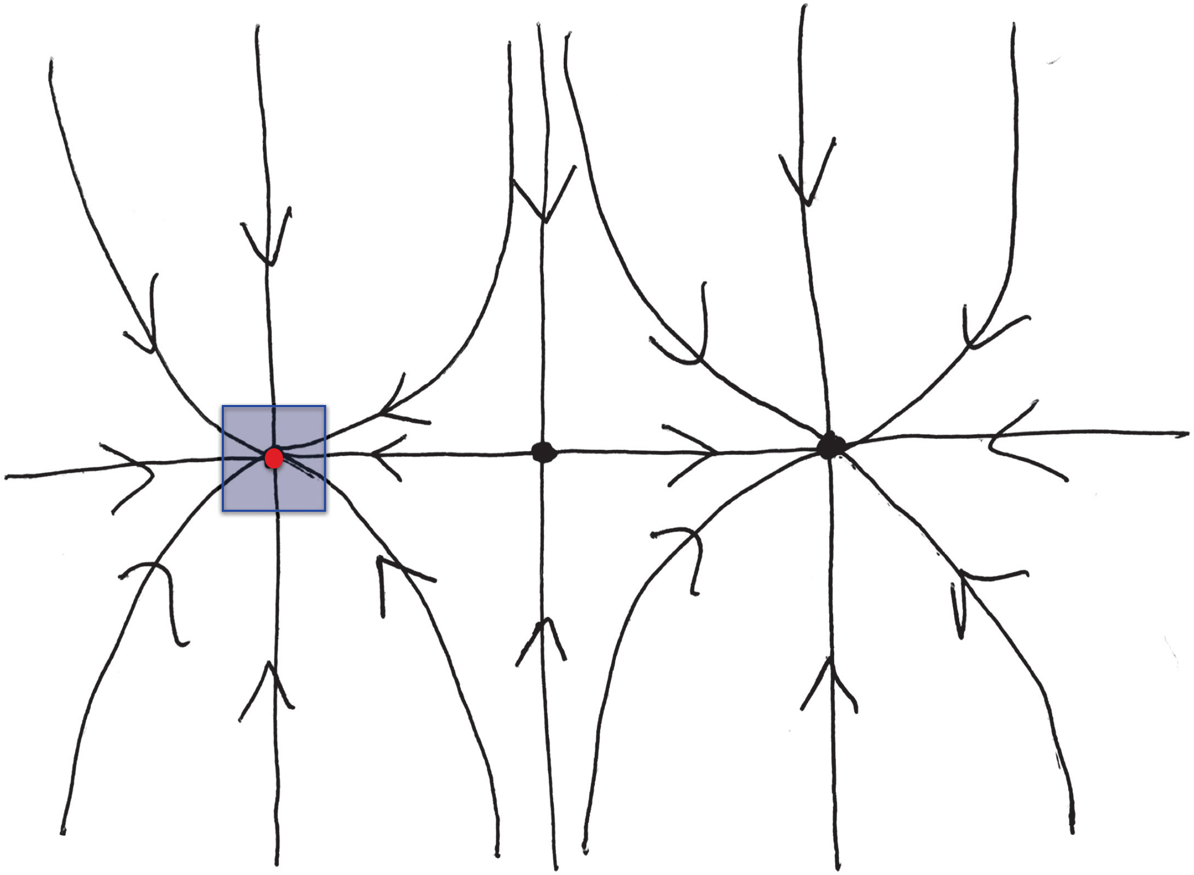


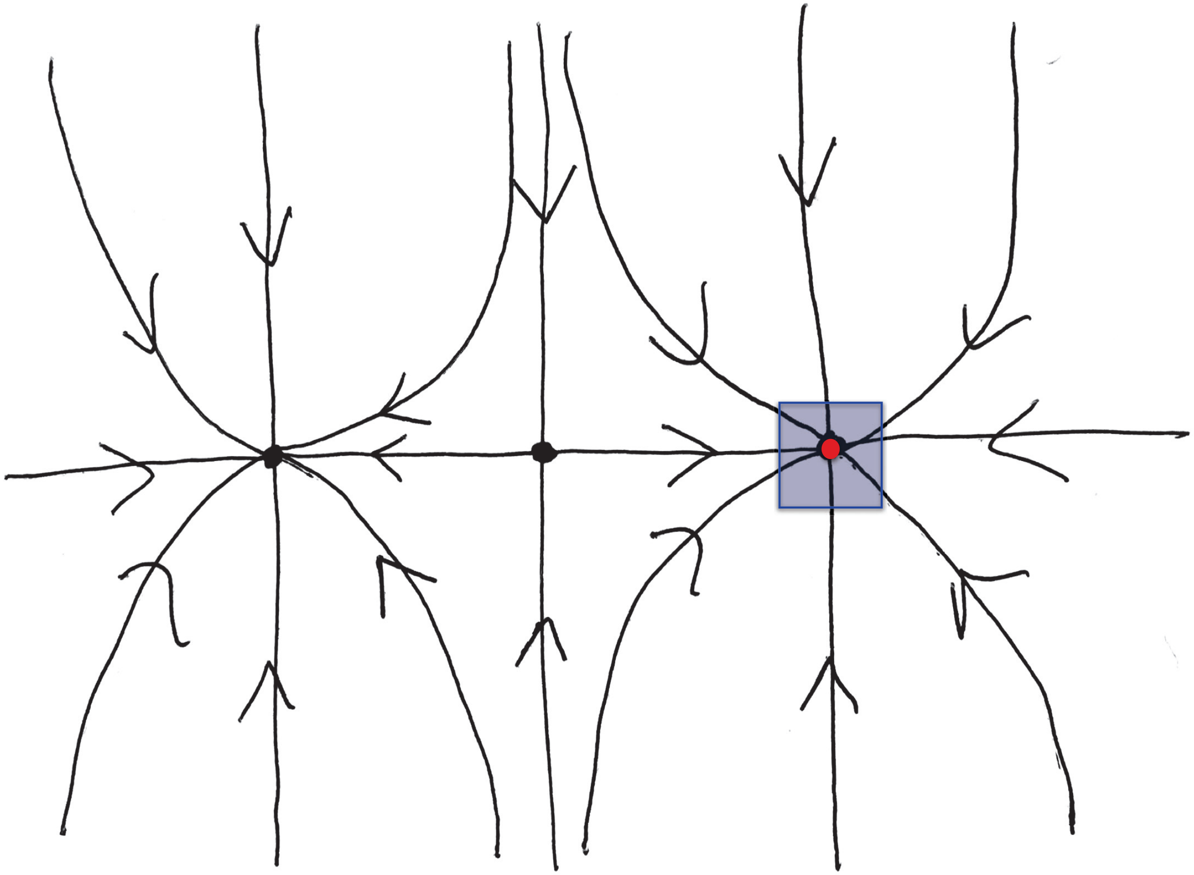


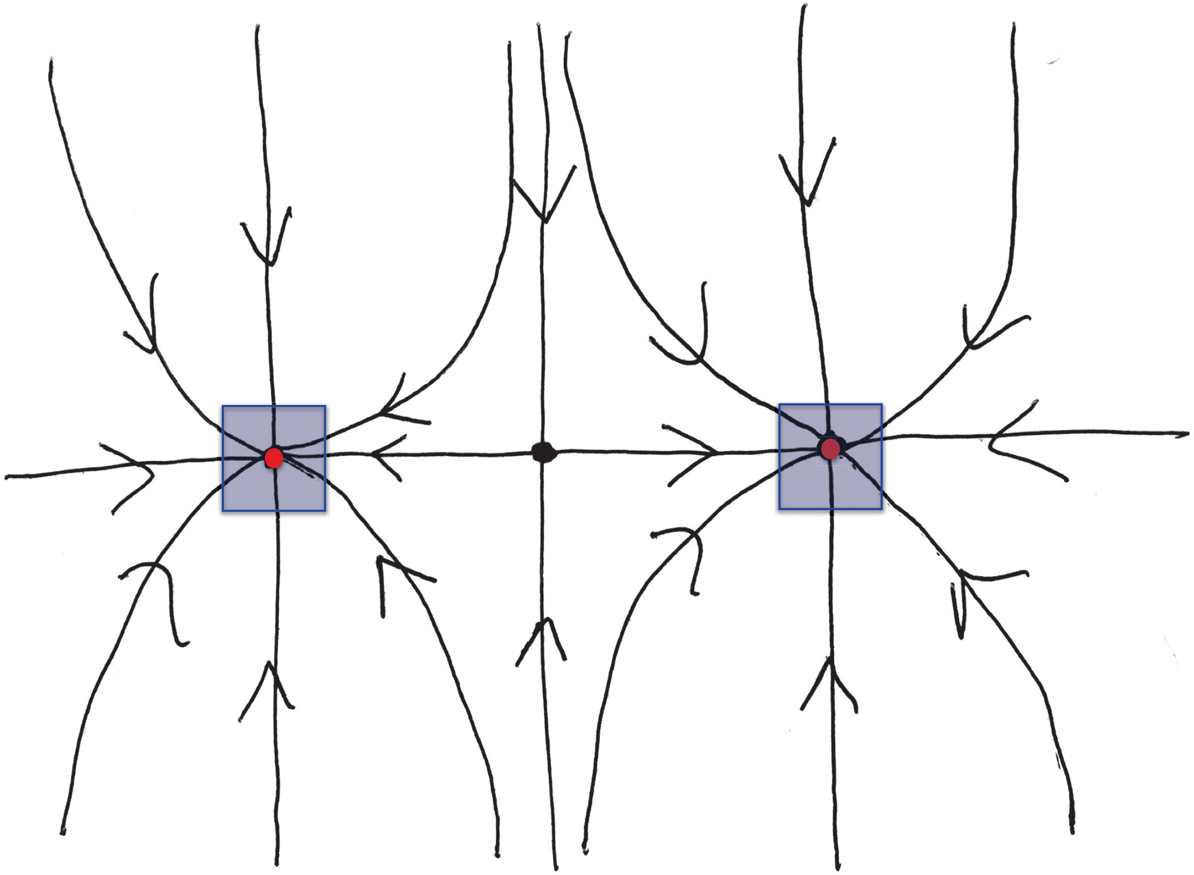


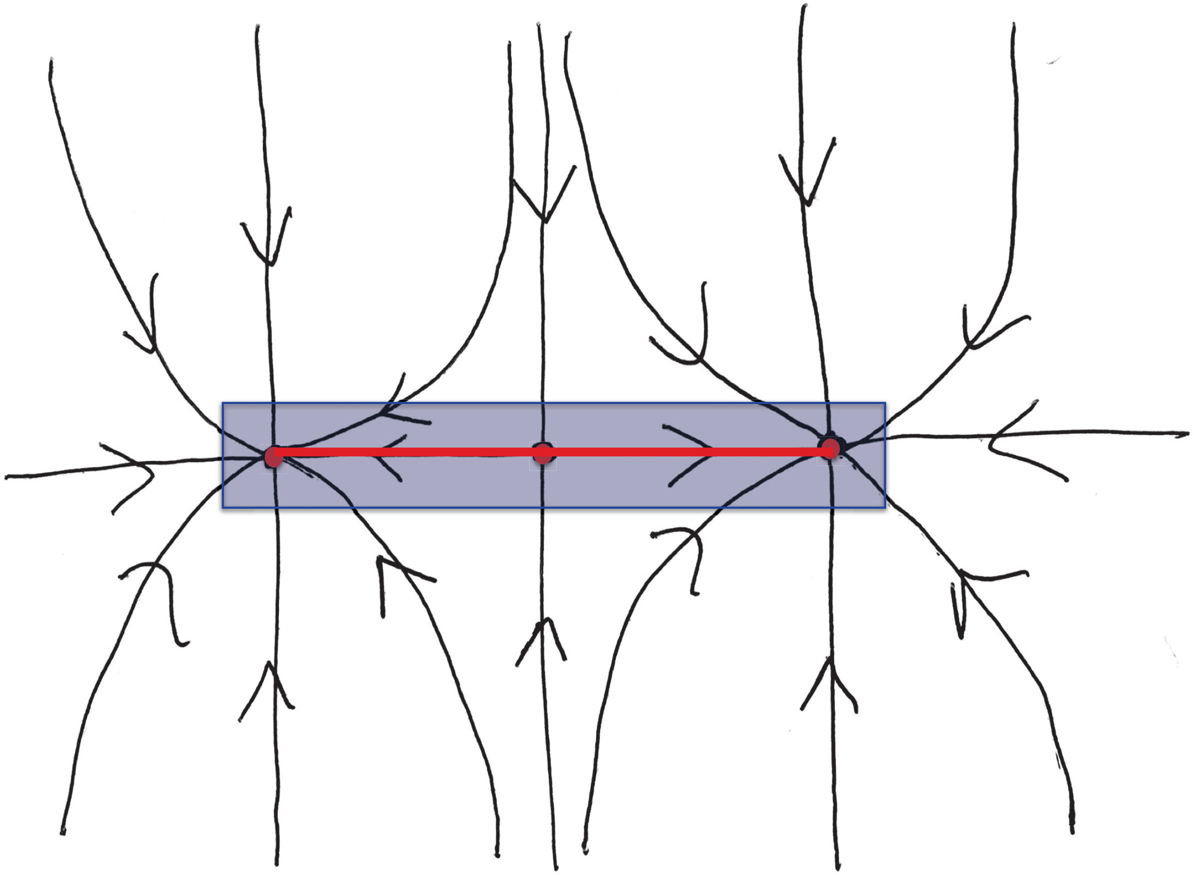
$A \subset X$ is an *attractor* if $A = \omega(U)$ for some attracting neighborhood U

An example: time-T map of $\dot{x} = x - x^3,\ \dot{y} = -y$









Algebraic structures in dynamics

The set of all attractors $\mathsf{Att}(\phi)$ is a bounded, distributive lattice: $A \vee A' = A \cup A'$ and $A \wedge A' = \omega(A \cap A')$.

There is a dual homomorphism ${}^*\colon \mathsf{Att}(\phi) \to \mathsf{Rep}(\phi)$ where $(\mathsf{Rep}, \cup, \cap)$ is the bounded, distributive lattice of repellers.

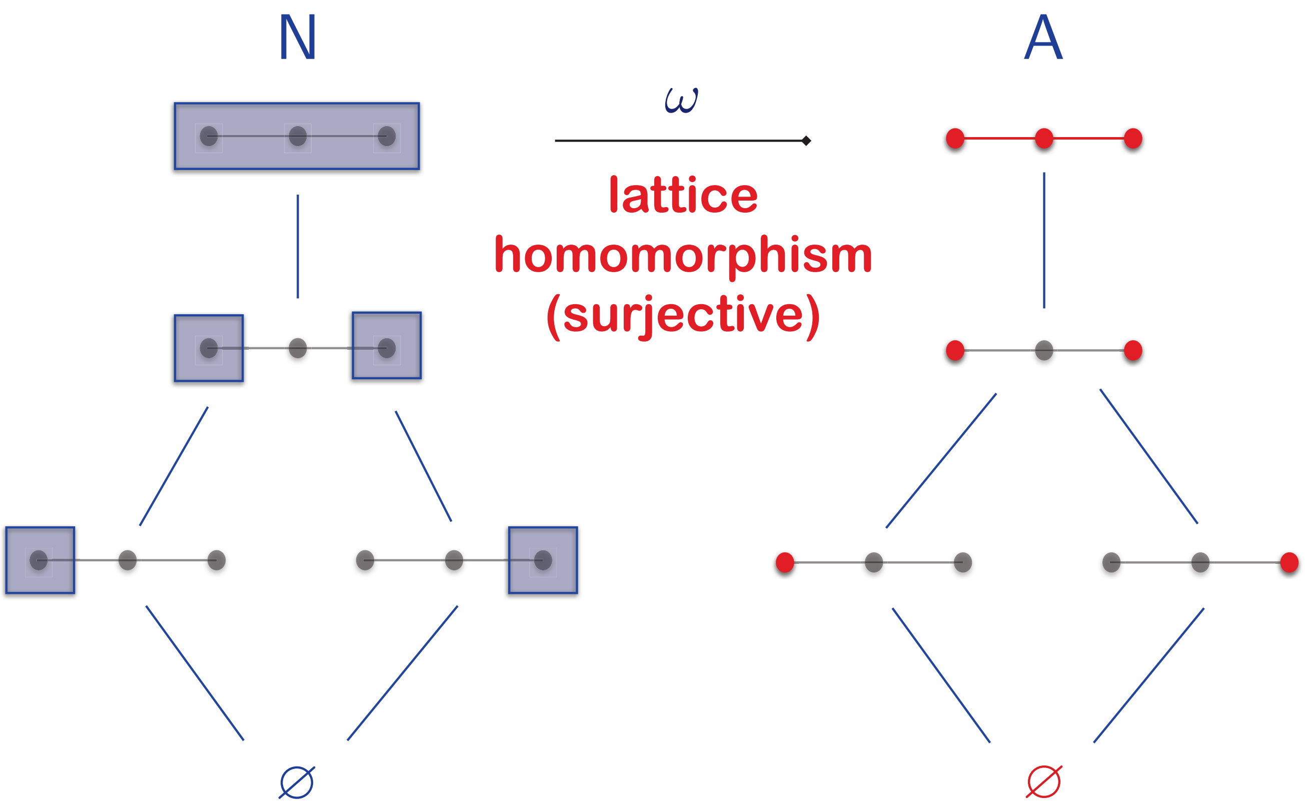
$\omega\colon (\mathsf{ANbhd}(\phi), \cup, \cap) \to (\mathsf{Att}(\phi), \vee, \wedge)$ is a surjective, homomorphism.

Birkhoff's Theorem: finite, distributive lattices are dually equivalent to finite posets. This equivalence can be specified concretely in the dynamics setting in the form of Morse decompositions.

Lattice structures of attractors I, II, III (Kalies, Mischaikow, and VanderVorst)

A computational approach to Conley's decomposition theorem (Kalies, Mischaikow, and VanderVorst)

An algorithmic approach to lattices and order in dynamics (Kalies, Kasti, and VanderVorst)



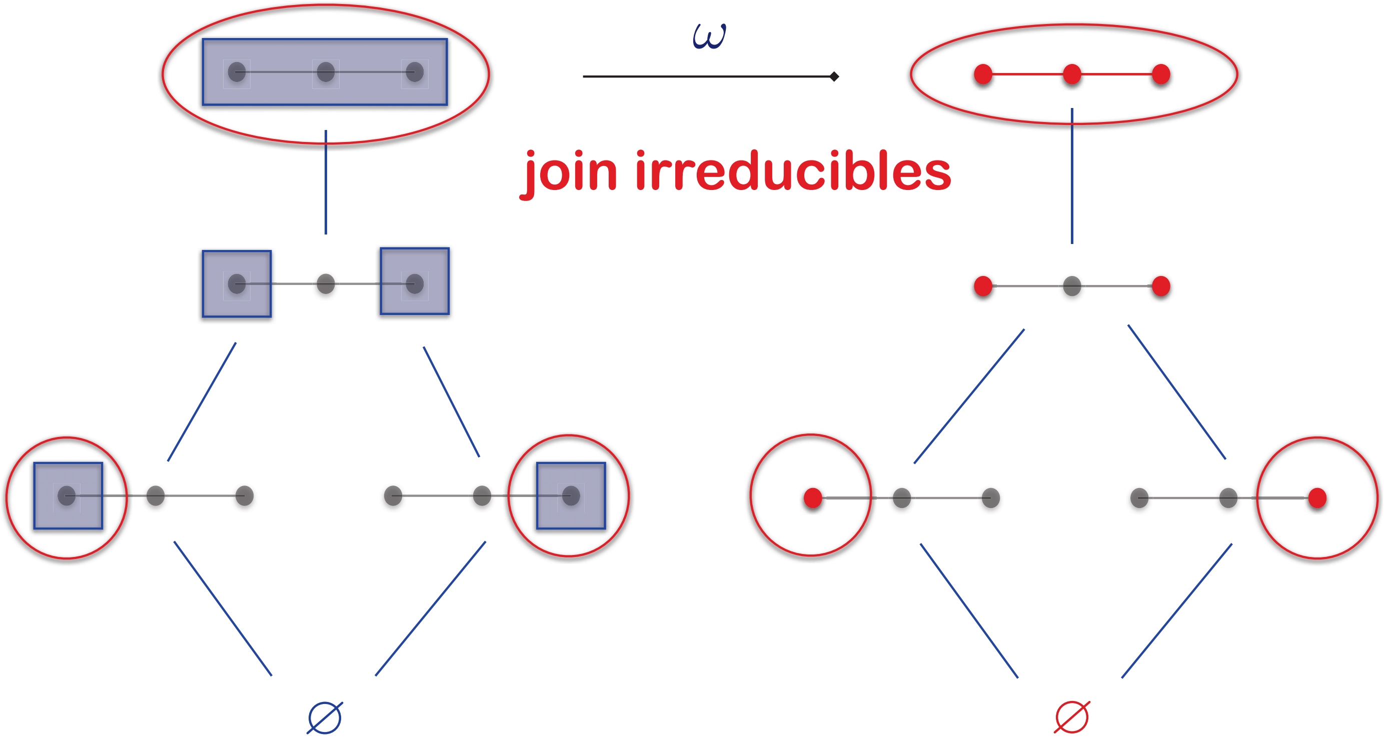
attracting block lattice

attractor lattice

$$\begin{tikzcd} \mathsf{N} \arrow[r, "\omega"] & \mathsf{A} \end{tikzcd}$$

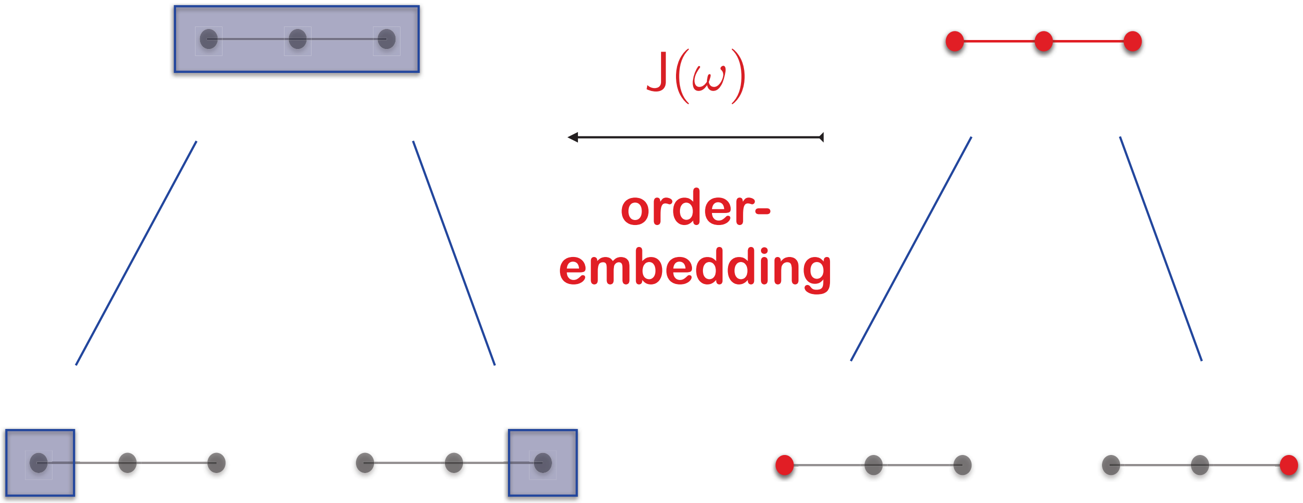
lattice homomorphism (surjective)

attracting block lattice **attractor lattice**



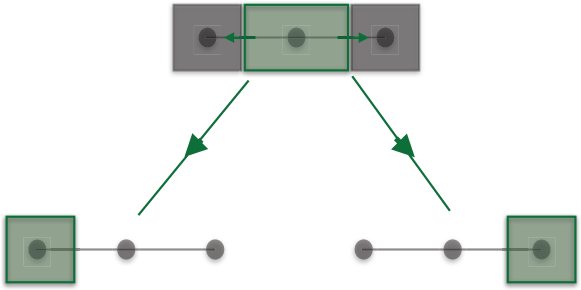
join irreducibles

J is a contravariant functor



Conley form

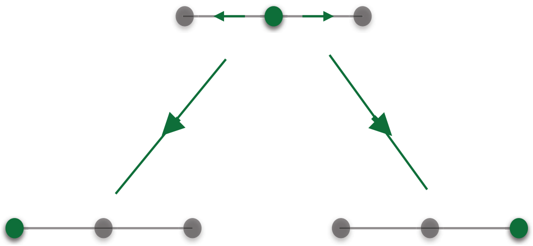
poset of isolating neighborhoods



$T(N)$

$$\xleftarrow{\pi}$$

poset of invariant sets

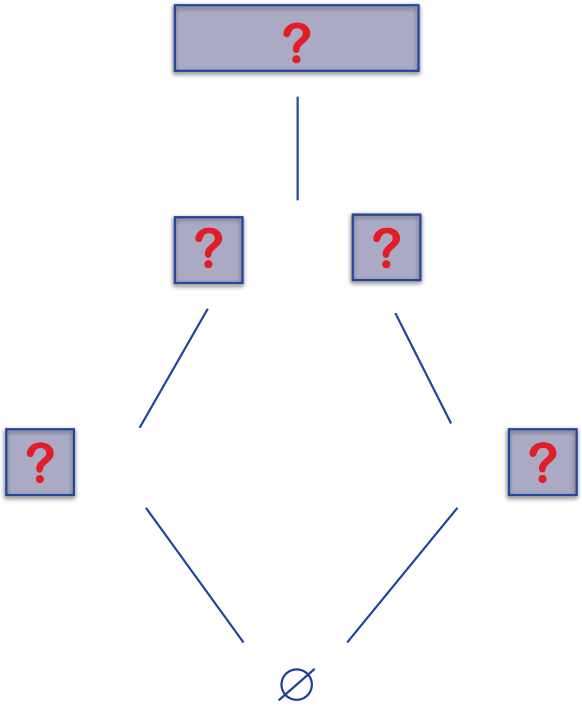


$M(A)$

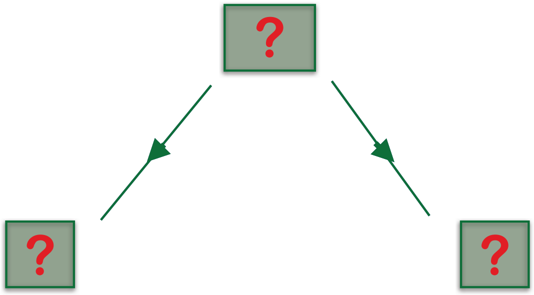
(tesselated) Morse decomposition

$$M(A) \hookrightarrow T(N)$$

attracting block lattice



poset of isolating neighborhoods



These structures are robust!

Combinatorial Dynamics

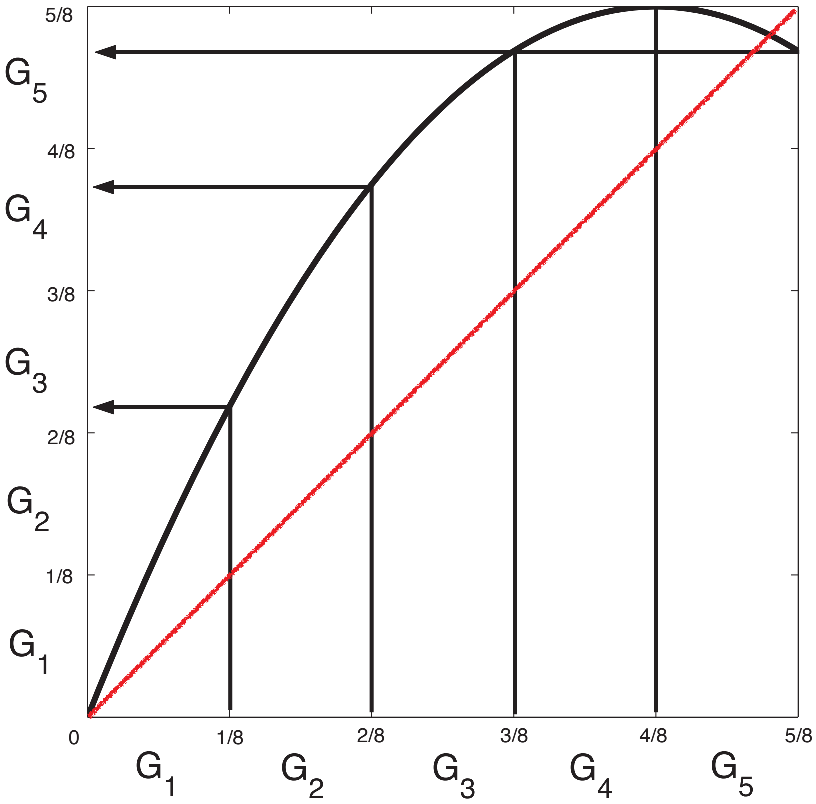
$X = [0, 5/8] \qquad f\colon X \to X$

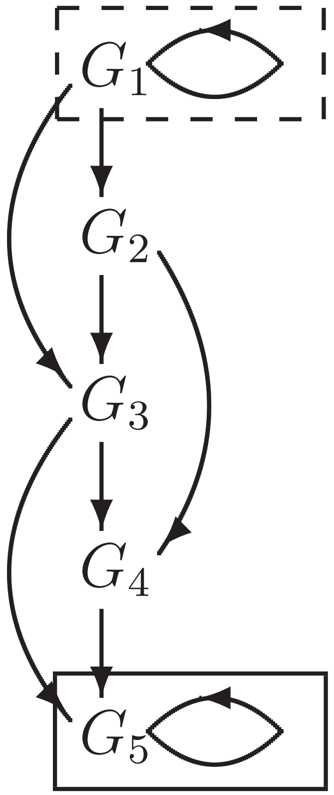
$f(x) = 2.5x(1-x)$

$\mathcal{X} = \{G_1, G_2, G_3, G_4, G_5\} \qquad \mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$

Multivalued Map ⇔ Directed Graph

⇔ Binary Relation





$|G_1| = [0, 1/8]$

$|\{G_1, G_2, G_3\}| = [0, 3/8]$

$f([0, 1/8]) = [0, 35/128]$

$\mathcal{F}(G_1) = \{G_1, G_2, G_3\}$

$f(|G_1|) \subset \operatorname{int}|\mathcal{F}(G_1)|$

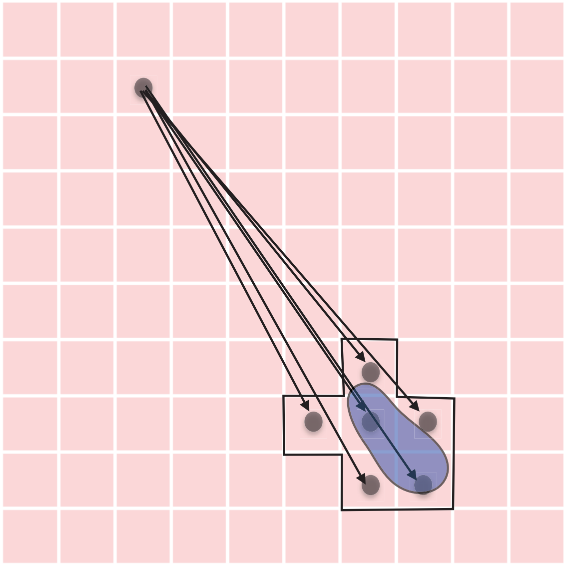
Outer Approximation

$\mathcal{X}$ is a labeling set for some finite *"grid"* on a compact metric space X

$\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is an *outer approximation* of $f\colon X \to X$ if

$f(|\mathcal{U}|) \subset \operatorname{int}|\mathcal{F}(\mathcal{U})|$ for all $\mathcal{U} \subset \mathcal{X}$

For explicit maps, OA's can be rigorously computed using interval arithmetic with outward rounding.



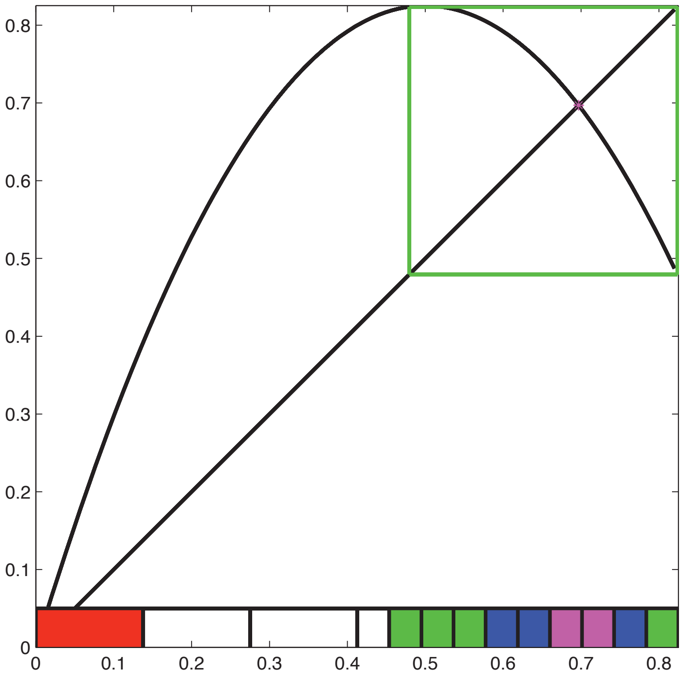
Proposition: If $\mathcal{F}$ is an outer approximation of f, and $\mathcal{U}$ is *forward invariant*, i.e. $\mathcal{F}(\mathcal{U}) \subset \mathcal{U}$, then $|\mathcal{U}|$ is an attracting block, which must contain a nonempty attractor for f.

Proposition: $\{f : \mathcal{F}$ is an outer approximation of $f\}$ is open in the space of dynamical systems on X.

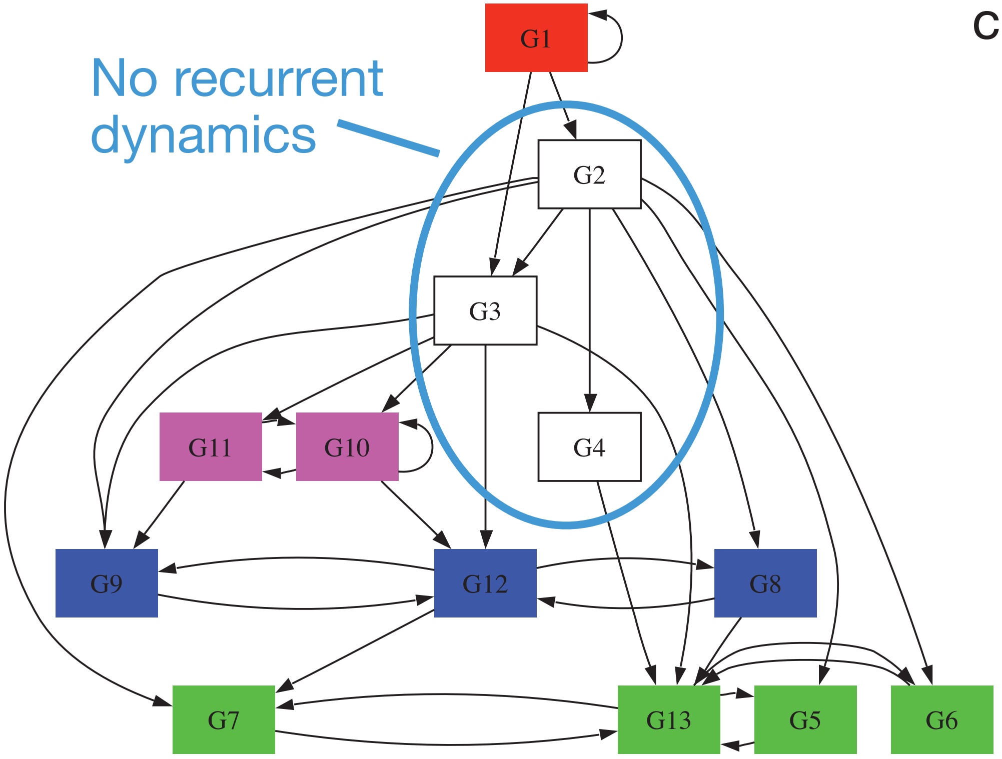
Proposition: If x_n for $n \in \mathbb{Z}$ is an orbit of f, i.e. $x_{n+1} = f(x_n)$, then any sequence $G_n \in \mathcal{X}$ with $x_n \in G_n$ is a walk through the graph of $\mathcal{F}$, i.e. $G_{n+1} \in \mathcal{F}(G_n)$.

Recurrence

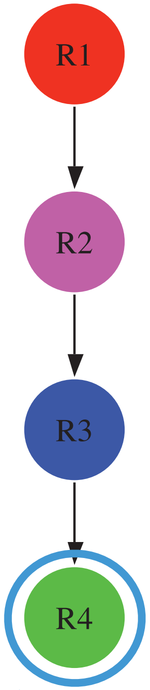
$$f(x) = 3.3x(1-x)$$



Strongly connected components $\hookleftarrow$ Morse graph of recurrent components



No recurrent dynamics

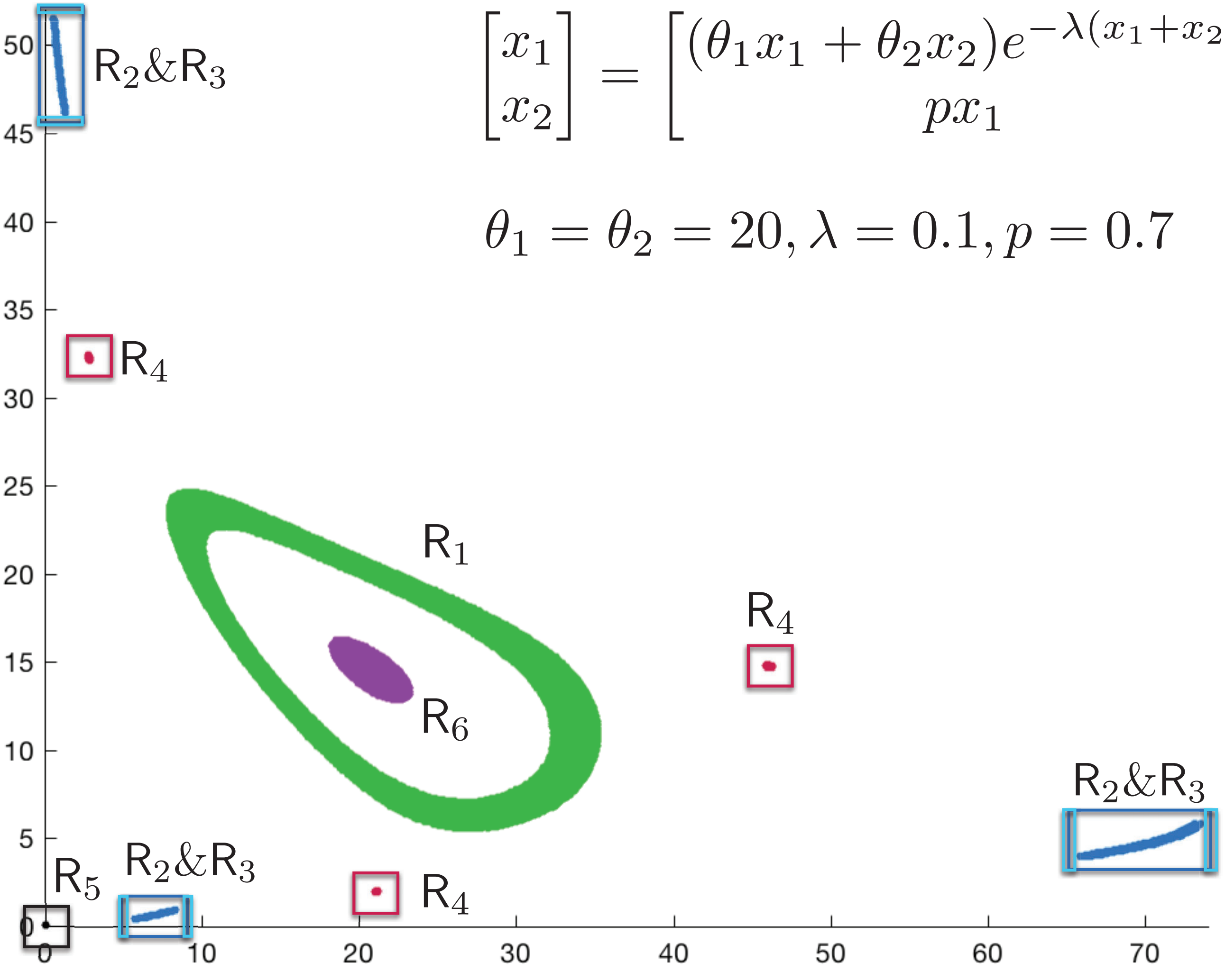


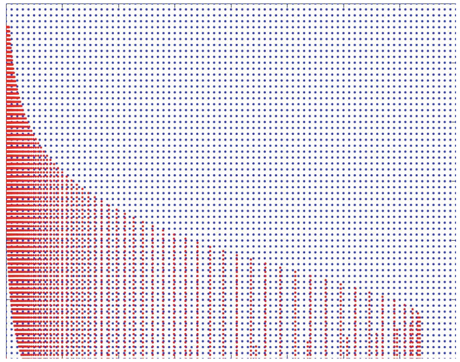
R4 contains a nonempty attractor

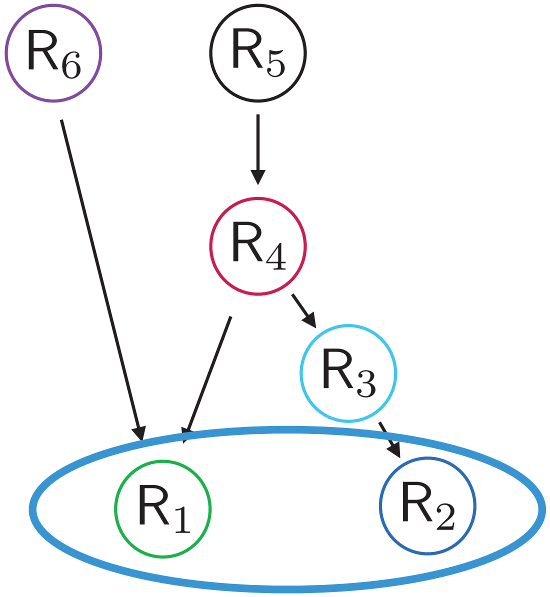
There exist fast graph algorithms to compute the Morse graph from the multivalued map.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2) e^{-\lambda(x_1+x_2)} \\ p x_1 \end{bmatrix}$$

$$\theta_1 = \theta_2 = 20, \lambda = 0.1, p = 0.7$$





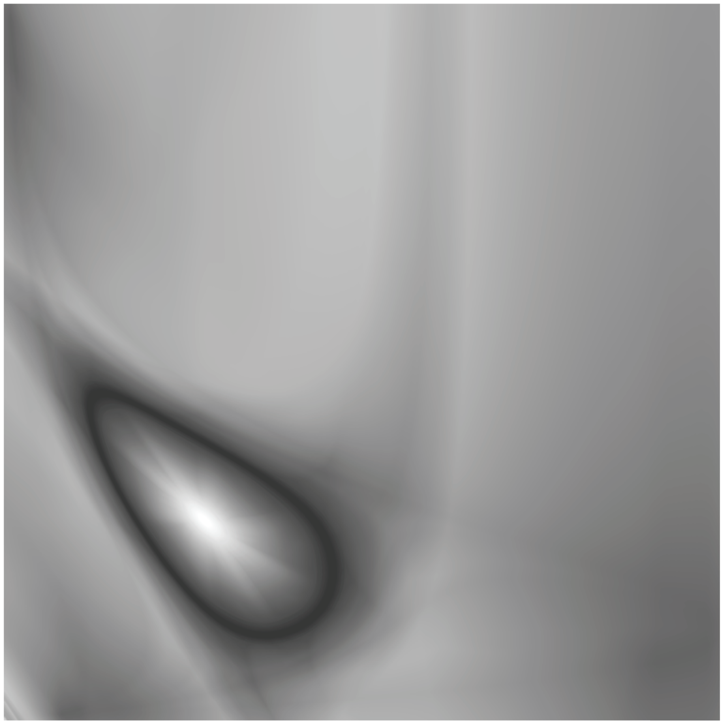


Multistabilty!

Conley's Decomposition Theorem

Fundamental Theorem of Dynamical Systems:

Let $\phi \colon X \times \mathbb{T} \to X$ be a dynamical system on a compact metric space X. Then there exists a set $\mathcal{R} \subset X$, the *(chain) recurrent set*, and a continuous function $V \colon X \to [0,1]$ such that if $x \notin \mathcal{R}$, then $V(x) > V(\phi(x,t))$ for all $t > 0$.



- $\mathcal{R}$ is the union of countably many *(chain) recurrent components*, $\mathcal{R}_i$.

- The *(complete) Lyapunov function* V can be chosen so that $\mathcal{R}_i \subset V^{-1}(\sigma_i)$ for distinct values $\sigma_i \in [0,1]$.

Efficient computation of Lyapunov functions for Morse decompositions (Goullet, Harker, Kasti, Kalies, Mischaikow)

Conley Index and Reconstruction of Dynamics

Algebraic topological invariant for isolated invariant sets.

Homological index is computable from the information contained in an outer approximation with nice images. There are fast algorithms in the cubical setting.

Theorems guarantee certain types of dynamics within an isolating neighborhood:

- Nontrivial Conley index implies nonempty invariant set.
- Lefschetz fixed point theorem for existence of fixed points.
- Extension of Lefschetz theorem to existence of periodic orbits.
- Extension of Lefschetz theorem to existence of chaotic dynamics.
- Connection matrix theory for existence of connecting orbits between Morse sets.

Conley Index

Given a pair of compact sets (N, L) with $L \subset N$ define the index map $f_{(N,L)} \colon \mathbb{N} \times (N/L, [L]) \to (N/L, [L])$ by

$$f_{(N,L)}([x]) = \begin{cases} [f(x)] & \text{if } x, f(x) \subset N \backslash L, \\ [L] & \text{otherwise.} \end{cases}$$

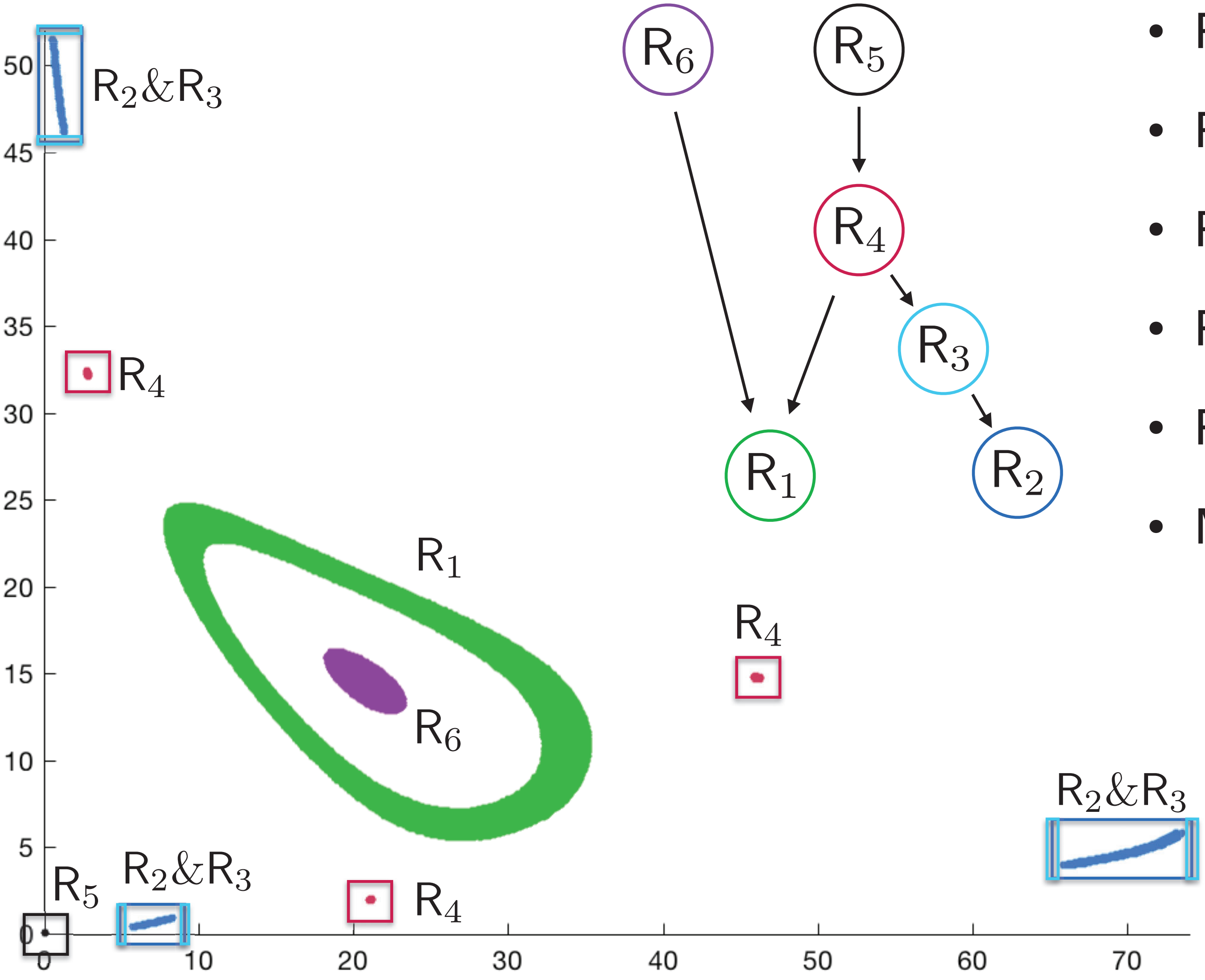
(N, L) is an *index pair* if $f_{(N,L)}$ is well-defined, continuous, and $\mathrm{cl}(N \backslash L)$ isolates $\mathrm{Inv}(N)$.

Definition: If (N, L) is an *index pair*, then the *(homological) Conley index* for $S = \mathrm{Inv}(N)$ is

$$\mathrm{Con}_*(S) = [H_*(N/L, [L]), (f_{(N.L)})_*]_s$$

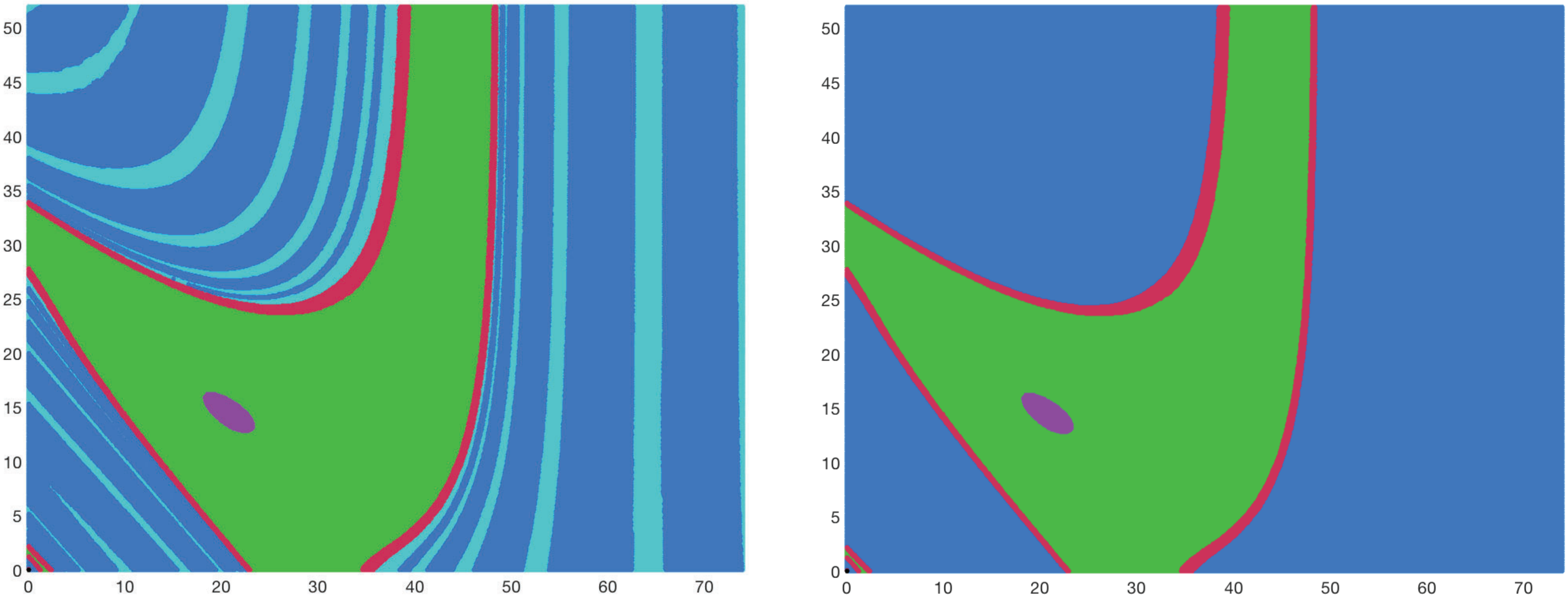
where $[\cdot, \cdot]_s$ denotes shift equivalence class.

Theorem: If $\mathrm{Con}_*(S)$ is nontrivial, i.e. $(f_{(N,L)})_*$ is not nilpotent, then $S \neq \varnothing$.



- R_6 contains a fixed point
- R_5 has trivial index*
- R_4 contains a period-3 orbit
- R_3 has trivial index
- R_2 contains chaotic dynamics
- Multistability

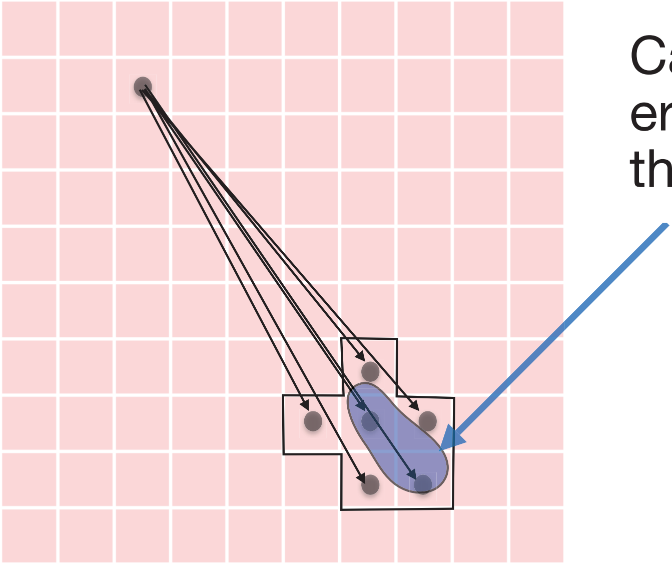
Basins of Attraction via order theory



An algorithmic approach to lattices and order in dynamics (Kalies, Kasti, and VanderVorst)

Computer-assisted proofs of dynamics

- Dynamics extracted from a combinatorial model are rigorously valid for an open class of systems, ie. all continuous selectors of the multivalued map.
- Methods do not require a close approximation / fine resolution; they are accurate but not necessarily precise, ie. the extracted dynamics is correct but may not be interesting if the model is very coarse.

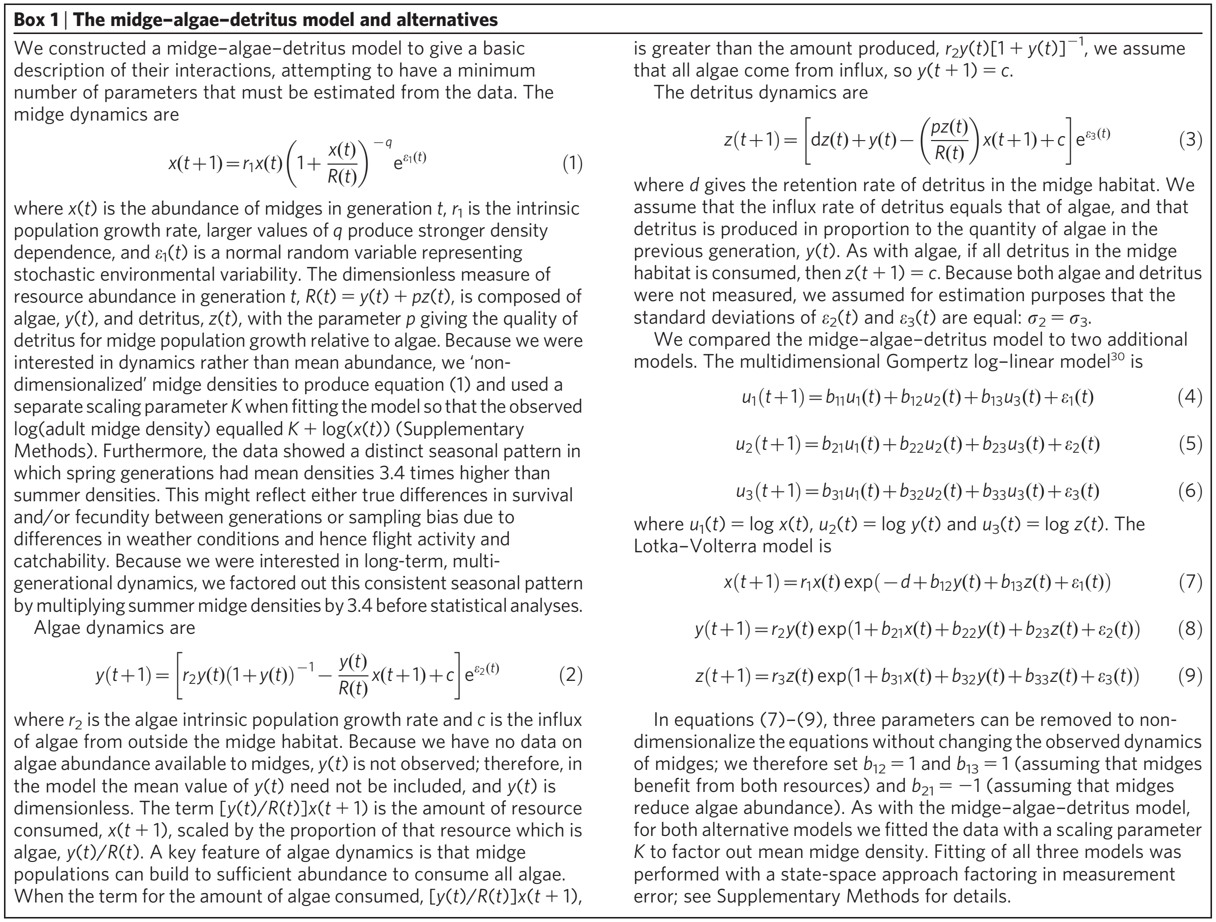


Can incorporate experimental error, modeling error, statistical measures of confidence into the images of the multivalued map.

Motivation for combinatorial models from data

- Biological phenomena do not have well-defined explicit models from first principles.
- Observable dynamics in biology should occur very robustly and parameters are often hard to measure precisely.
- DSGRN (Dynamical Signatures of Gene Regulatory Networks) — successfully exploited structure of model equations to use the combinatorial methods to identify classes of parameters where a network has a biologically relevant behavior. (7-dimensional phase space and 61 dimensional parameter space). (Mischaikow, et.al.)
- There has been rapid development of techniques to identify explicit model systems (e.g. SINDy, Koopman operators) — advantage of precise predictions, but combinatorial methods could provide validation or suggest a lack of understanding of essential features.

Example



Box 1 | The midge–algae–detritus model and alternatives

We constructed a midge–algae–detritus model to give a basic description of their interactions, attempting to have a minimum number of parameters that must be estimated from the data. The midge dynamics are

$$x(t+1) = r_1 x(t)\left(1 + \frac{x(t)}{R(t)}\right)^{-q} e^{\varepsilon_1(t)} \qquad (1)$$

where $x(t)$ is the abundance of midges in generation t, r_1 is the intrinsic population growth rate, larger values of q produce stronger density dependence, and $\varepsilon_1(t)$ is a normal random variable representing stochastic environmental variability. The dimensionless measure of resource abundance in generation t, $R(t) = y(t) + pz(t)$, is composed of algae, $y(t)$, and detritus, $z(t)$, with the parameter p giving the quality of detritus for midge population growth relative to algae. Because we were interested in dynamics rather than mean abundance, we 'non-dimensionalized' midge densities to produce equation (1) and used a separate scaling parameter K when fitting the model so that the observed log(adult midge density) equalled $K + \log(x(t))$ (Supplementary Methods). Furthermore, the data showed a distinct seasonal pattern in which spring generations had mean densities 3.4 times higher than summer densities. This might reflect either true differences in survival and/or fecundity between generations or sampling bias due to differences in weather conditions and hence flight activity and catchability. Because we were interested in long-term, multi-generational dynamics, we factored out this consistent seasonal pattern by multiplying summer midge densities by 3.4 before statistical analyses.

Algae dynamics are

$$y(t+1) = \left[r_2 y(t)(1+y(t))^{-1} - \frac{y(t)}{R(t)} x(t+1) + c\right] e^{\varepsilon_2(t)} \qquad (2)$$

where r_2 is the algae intrinsic population growth rate and c is the influx of algae from outside the midge habitat. Because we have no data on algae abundance available to midges, $y(t)$ is not observed; therefore, in the model the mean value of $y(t)$ need not be included, and $y(t)$ is dimensionless. The term $[y(t)/R(t)]x(t+1)$ is the amount of resource consumed, $x(t+1)$, scaled by the proportion of that resource which is algae, $y(t)/R(t)$. A key feature of algae dynamics is that midge populations can build to sufficient abundance to consume all algae. When the term for the amount of algae consumed, $[y(t)/R(t)]x(t+1)$, is greater than the amount produced, $r_2 y(t)[1 + y(t)]^{-1}$, we assume that all algae come from influx, so $y(t+1) = c$.

The detritus dynamics are

$$z(t+1) = \left[dz(t) + y(t) - \left(\frac{pz(t)}{R(t)}\right) x(t+1) + c\right] e^{\varepsilon_3(t)} \qquad (3)$$

where d gives the retention rate of detritus in the midge habitat. We assume that the influx rate of detritus equals that of algae, and that detritus is produced in proportion to the quantity of algae in the previous generation, $y(t)$. As with algae, if all detritus in the midge habitat is consumed, then $z(t+1) = c$. Because both algae and detritus were not measured, we assumed for estimation purposes that the standard deviations of $\varepsilon_2(t)$ and $\varepsilon_3(t)$ are equal: $\sigma_2 = \sigma_3$.

We compared the midge–algae–detritus model to two additional models. The multidimensional Gompertz log-linear model[30] is

$$u_1(t+1) = b_{11}u_1(t) + b_{12}u_2(t) + b_{13}u_3(t) + \varepsilon_1(t) \qquad (4)$$

$$u_2(t+1) = b_{21}u_1(t) + b_{22}u_2(t) + b_{23}u_3(t) + \varepsilon_2(t) \qquad (5)$$

$$u_3(t+1) = b_{31}u_1(t) + b_{32}u_2(t) + b_{33}u_3(t) + \varepsilon_3(t) \qquad (6)$$

where $u_1(t) = \log x(t)$, $u_2(t) = \log y(t)$ and $u_3(t) = \log z(t)$. The Lotka–Volterra model is

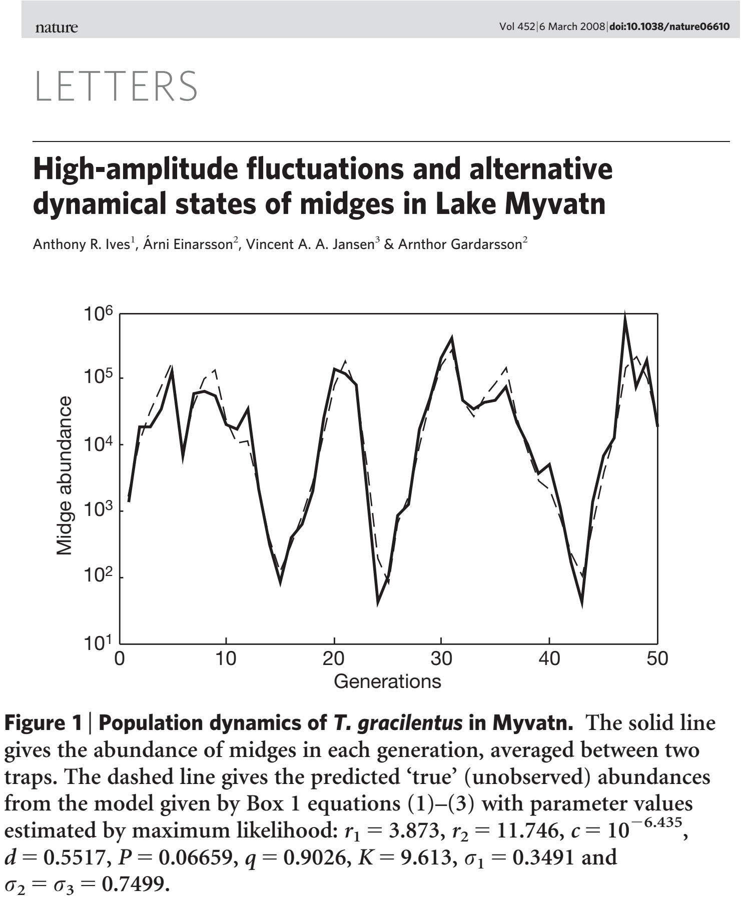
$$x(t+1) = r_1 x(t) \exp(-d + b_{12}y(t) + b_{13}z(t) + \varepsilon_1(t)) \qquad (7)$$

$$y(t+1) = r_2 y(t) \exp(1 + b_{21}x(t) + b_{22}y(t) + b_{23}z(t) + \varepsilon_2(t)) \qquad (8)$$

$$z(t+1) = r_3 z(t) \exp(1 + b_{31}x(t) + b_{32}y(t) + b_{33}z(t) + \varepsilon_3(t)) \qquad (9)$$

In equations (7)–(9), three parameters can be removed to non-dimensionalize the equations without changing the observed dynamics of midges; we therefore set $b_{12} = 1$ and $b_{13} = 1$ (assuming that midges benefit from both resources) and $b_{21} = -1$ (assuming that midges reduce algae abundance). As with the midge–algae–detritus model, for both alternative models we fitted the data with a scaling parameter K to factor out mean midge density. Fitting of all three models was performed with a state-space approach factoring in measurement error; see Supplementary Methods for details.

Ives, Einarsson, Jansen, Gardarsson



nature Vol 452 | 6 March 2008 | doi:10.1038/nature06610

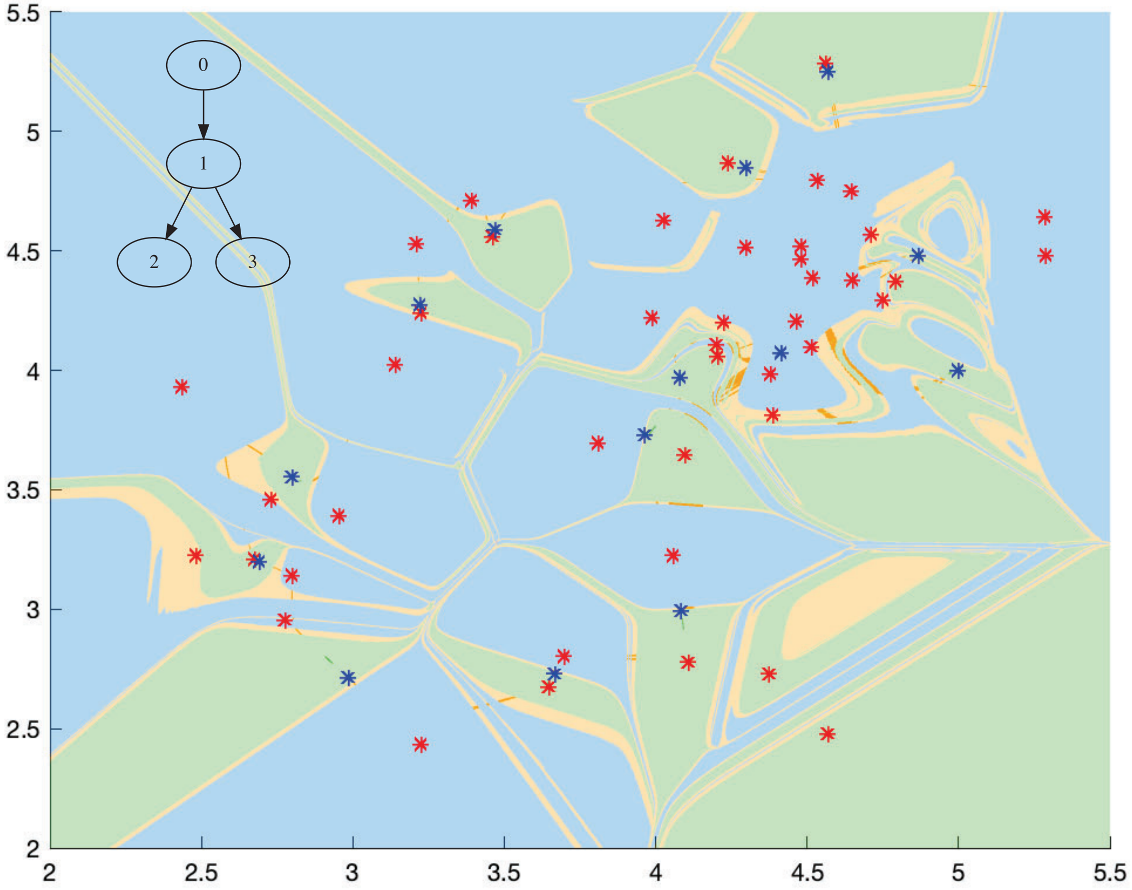
LETTERS

High-amplitude fluctuations and alternative dynamical states of midges in Lake Myvatn

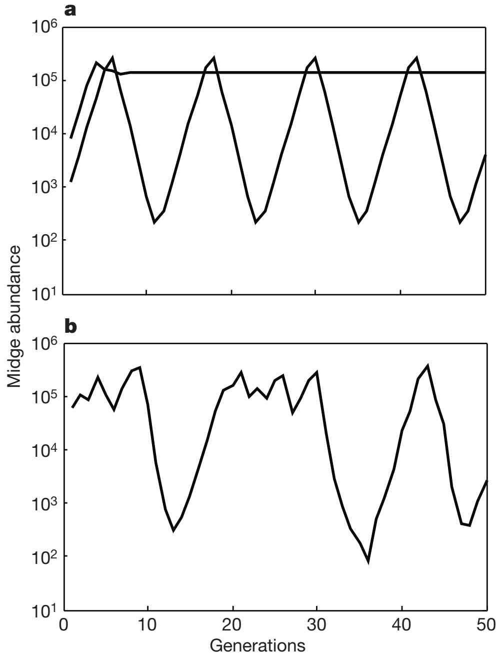
Anthony R. Ives[1], Árni Einarsson[2], Vincent A. A. Jansen[3] & Arnthor Gardarsson[2]

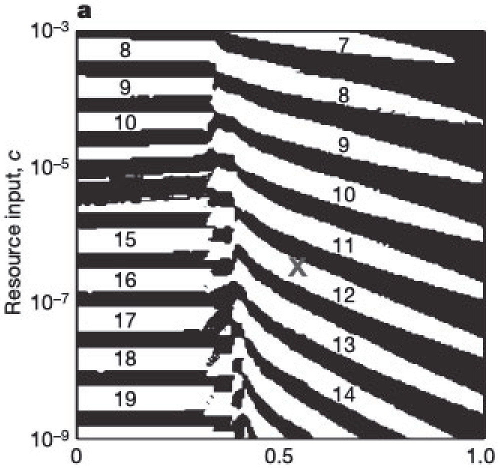
Figure 1 | Population dynamics of *T. gracilentus* in Myvatn. The solid line gives the abundance of midges in each generation, averaged between two traps. The dashed line gives the predicted 'true' (unobserved) abundances from the model given by Box 1 equations (1)–(3) with parameter values estimated by maximum likelihood: $r_1 = 3.873$, $r_2 = 11.746$, $c = 10^{-6.435}$, $d = 0.5517$, $P = 0.06659$, $q = 0.9026$, $K = 9.613$, $\sigma_1 = 0.3491$ and $\sigma_2 = \sigma_3 = 0.7499$.

Example



2-dimensional delay reconstruction of the data





Surrogate models

- Points in $X \subset \mathbb{R}^d$ and corresponding images in X .
- Partition into a set of predictors and a set of test points with corresponding observations and test images.
- Nonparametric regression to a surrogate model.
    - Gaussian process regression, kernel regression, machine learning …

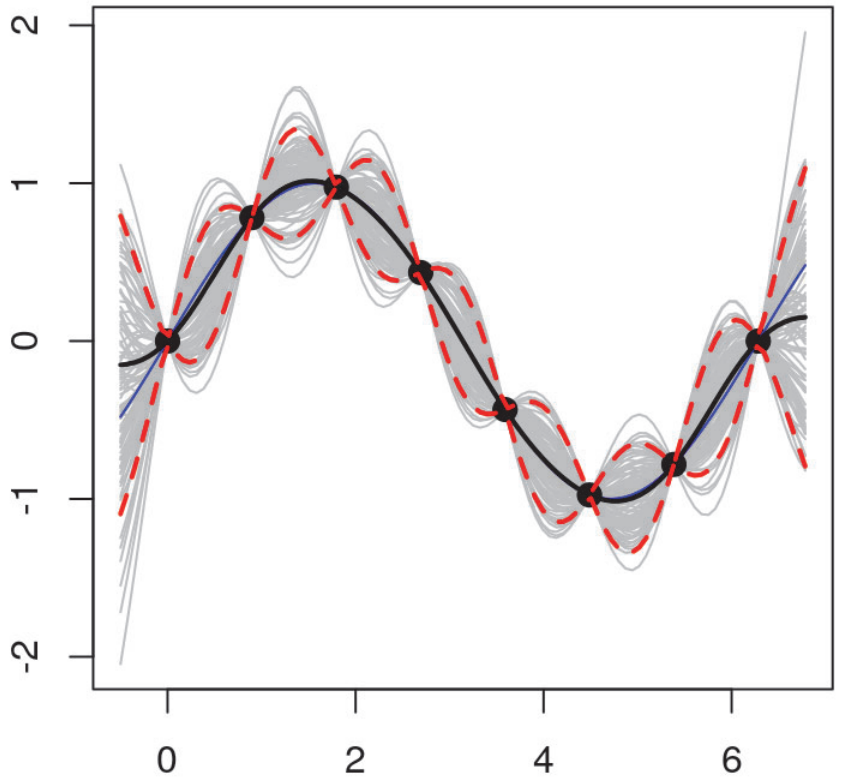
Gaussian process regression

A stochastic process where any finite collection of realizations are modeled by multivariate normal distributions and hence characterized by their mean and covariance.

Condition the process based on observed data and use the posterior mean and covariance as a tool for regression.

Blue curve: sampled function
Black curve: mean function
Gray curves: realizations
Red curves: 90% quantiles

Surrogates **by Grammarcy**
https://bookdown.org/rbg/surrogates



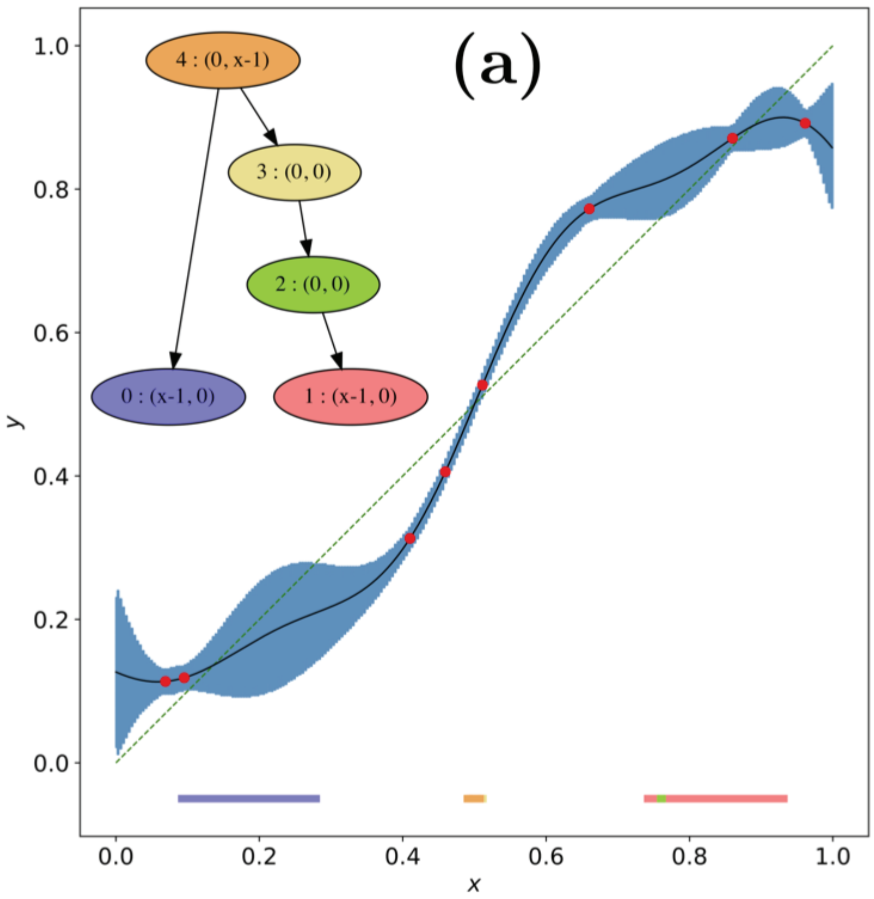
Idea

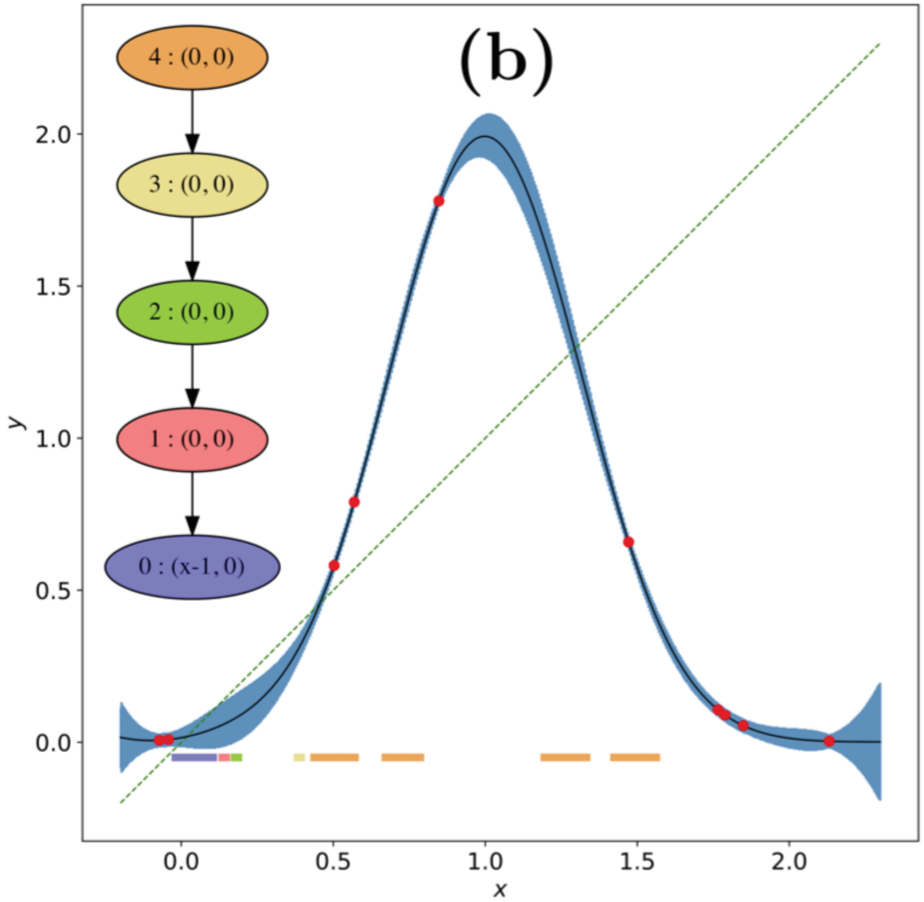
Use data to generate a surrogate model and characterize statistical confidence in the model predictions.

Use the surrogate model to construct a combinatorial approximation which is an outer approximation for the surrogate model incorporating levels of confidence into the combinatorial approximation.

Coarsen the combinatorial approximation to characterize the robustness of the computed dynamics; ie. the open class of systems

1d Examples


(a)
4 : (0, x-1)
3 : (0, 0)
2 : (0, 0)
0 : (x-1, 0)
1 : (x-1, 0)
1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.2
0.4
0.6
0.8
1.0
x


(b)
4 : (0, 0)
3 : (0, 0)
2 : (0, 0)
1 : (0, 0)
0 : (x-1, 0)
2.0
1.5
1.0
0.5
0.0
y
0.0
0.5
1.0
1.5
2.0
x

1d Procedure

A: $\mathcal{T} = \{(x_n, y_n) \colon y_n = f(x_n)\}$ where the unknown, continuous function f is a realization from a Gaussian process with a prespecified, semipositive kernel $k(\cdot,\cdot;\theta)$ where theta is a vector of unknown parameters associated to the kernel.

Step 1: Given the data set $\mathcal{T}$, estimate the unknown parameters and construct a GP surrogate model with predictive mean $\mu \colon X \to \mathbb{R}$ and predictive covariance function Σ.

Step 2: Choose a finite cell complex $\mathcal{X}$ whose geometric realization as a regular CW-complex is X. Construct a closed set $G \subset X \times X$ with the following property: G is the geometric realization of products of cells from $\mathcal{X}$ and each fiber $G_x := G \cap (\{x\} \times X)$ is nonempty and contractible. Use the combinatorial representation of G to identify potential dynamics and compute their associated Conley indexes.

1d Construction

- Use $k(x,x') = e^{-|x-x'|^2/\theta}$ where θ is estimated by maximal likelihood.

- Assume L provides a bound for the Lipschitz constant with confidence $(1-\delta)^{1/2}$, that is,

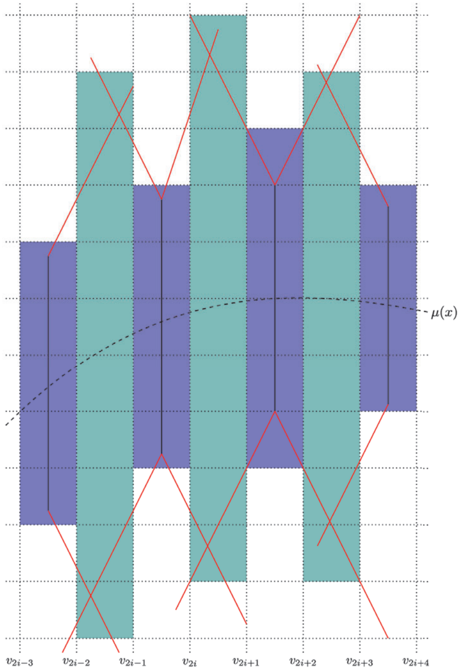
$$\mathbb{P}(\forall x_1, x_2 \in X |g(x_1) - g(x_2)| \leq L|x_1 - x_2|) \geq (1-\delta)^{1/2}.$$

- Select a finite set $S \subset X$ and corresponding confidence intervals such that

$$\mathbb{P}(g(v) \in E_{\Sigma(v)}(\mu(v), r(v)) \; \forall v \in S) \geq (1-\delta)^{1/2}.$$

- Construct a set G that satisfies $\mathbb{P}(\mathsf{G}(g) \subset G) > 1-\delta$ and the smallest combinatorial map $\mathcal{F}$ covering G.

- The dynamics extracted from G has a confidence level of at least $1-\delta$, i.e. the dynamics is valid for any continuous function h with $\mathsf{G}(h) \subset G$, and realizations of the GP are such selectors of G with probability at least $1-\delta$.



1d Theorem

Theorem: Let $\mathcal{T}$ be a data set that satisfies assumption **A** where $\{x_n : n = 1, \ldots, N\}$ are chosen i.i.d. from the uniform distribution, and assume the kernel k is smooth enough. Let $\alpha > 0$ and $\delta \in (0, 1)$. There exist $\epsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that the set G satisfies

$$\mathbb{P}\left(\sup_{x \in X} \operatorname{diam}(\tilde{G}_x) < \alpha\right) > 1 - \delta \quad \textit{and} \quad \mathbb{P}(\mathsf{G}(g) \subset \tilde{G}) > 1 - \delta$$

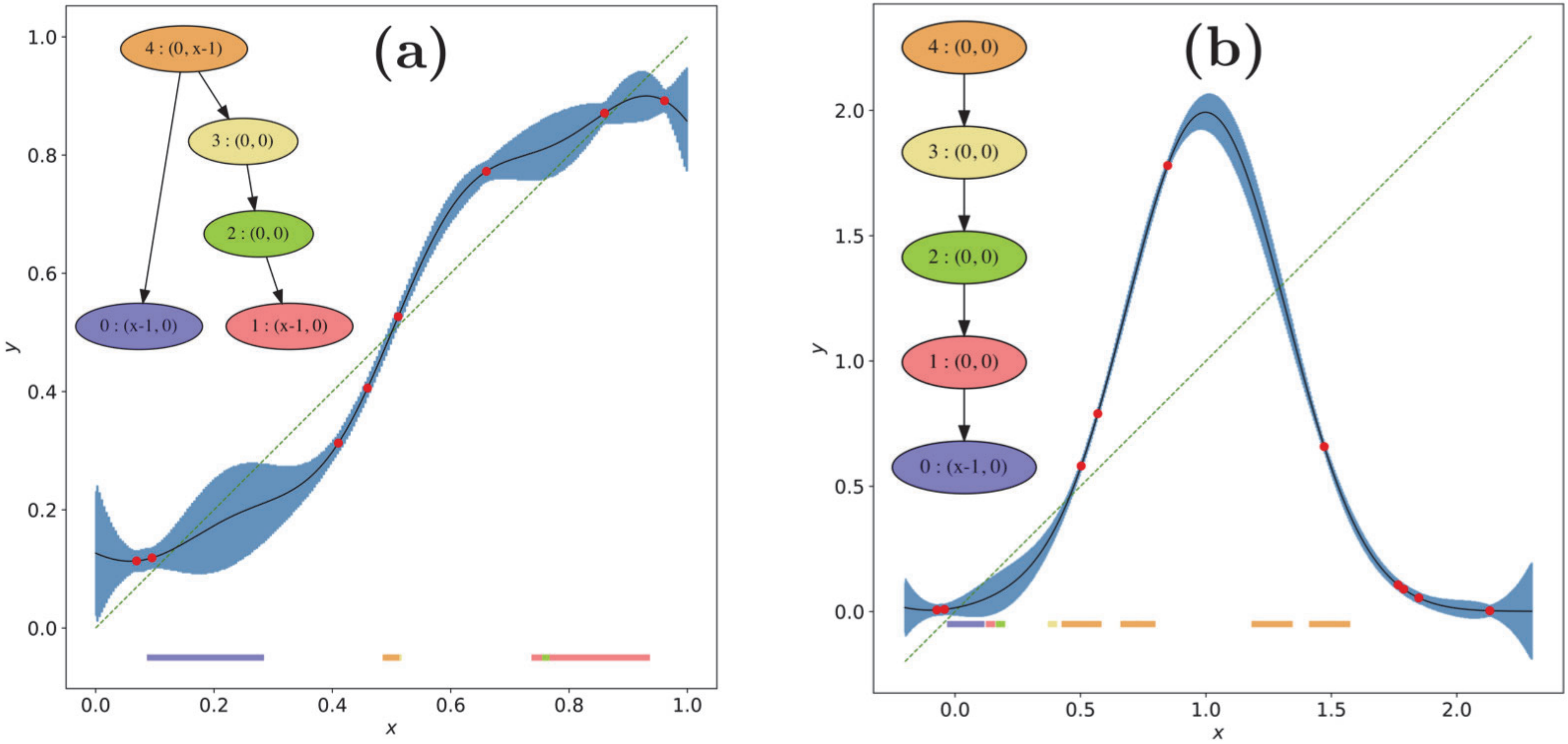
provided that $N > n_0$, g is a GP constructed as in Step 1, and as in Step 2, $X \subset \mathbb{R}^d$ is a compact regular CW-complex indexed by a cell complex $\mathcal{X}$ with $\operatorname{diam}(\mathcal{X}) < \epsilon_0$.

Smaller α allows more detailed dynamics to be extracted from the combinatorial map.

Smaller δ implies higher confidence in the extracted dynamics.

Identifying Nonlinear Dynamics with High Confidence from Sparse Data (Batko, Gameiro, Hung, Kalies, Mischaikow, Vieira)

Dynamics with 95% confidence level (Lipschitz constant double that of the sampled function).



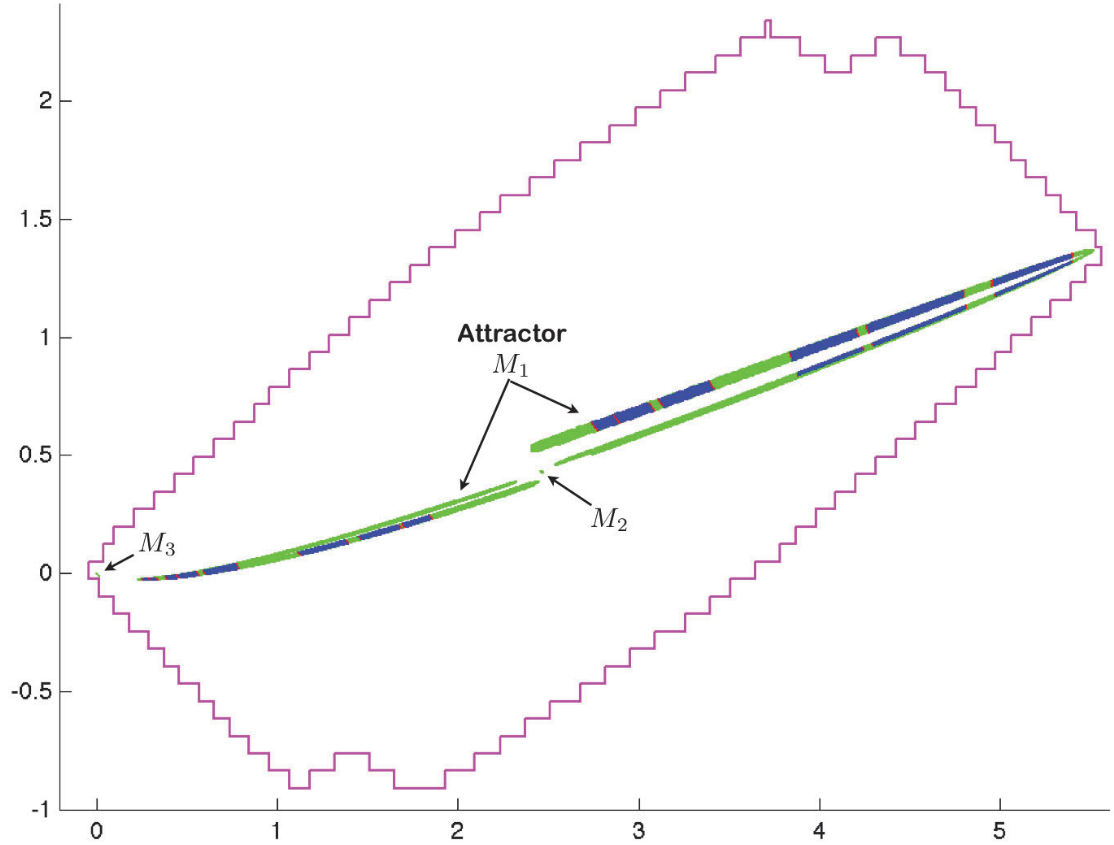
Bistability and fixed points

Chaos within an unstable invariant set

Example: Kot-Schafer-Ricker Integrodifference Equation

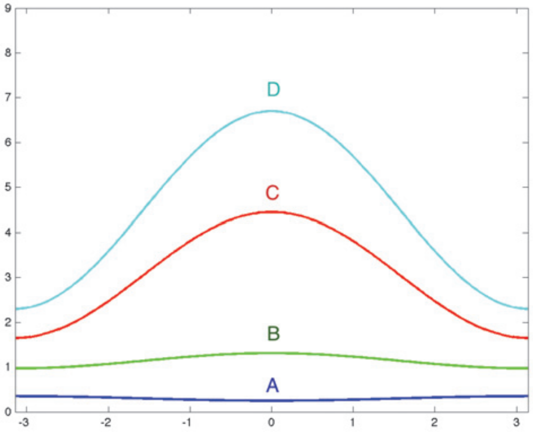
$$\Phi[a](y) := \frac{1}{2\pi}\int_{-\pi}^{\pi} b(x,y)\, G[a](x)dx$$

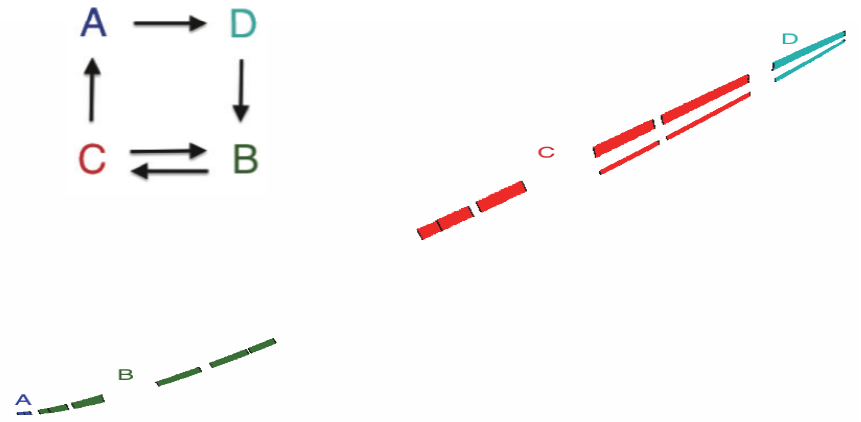
$$G[a](x) = \mu\, c(x)\, a(x)\, e^{-a(x)}$$

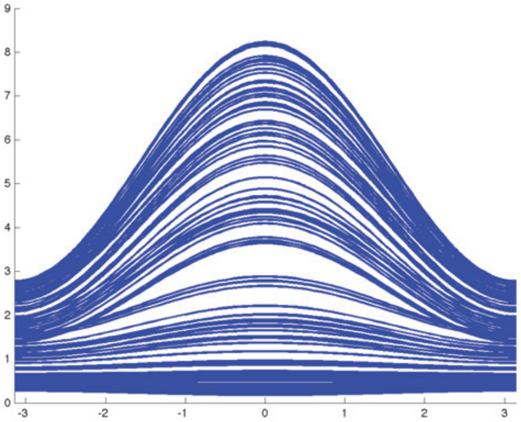


Rigorous computation of the global dynamics of integrodifference equations with smooth nonlinearities (Day, Kalies)

Rigorous KSR results





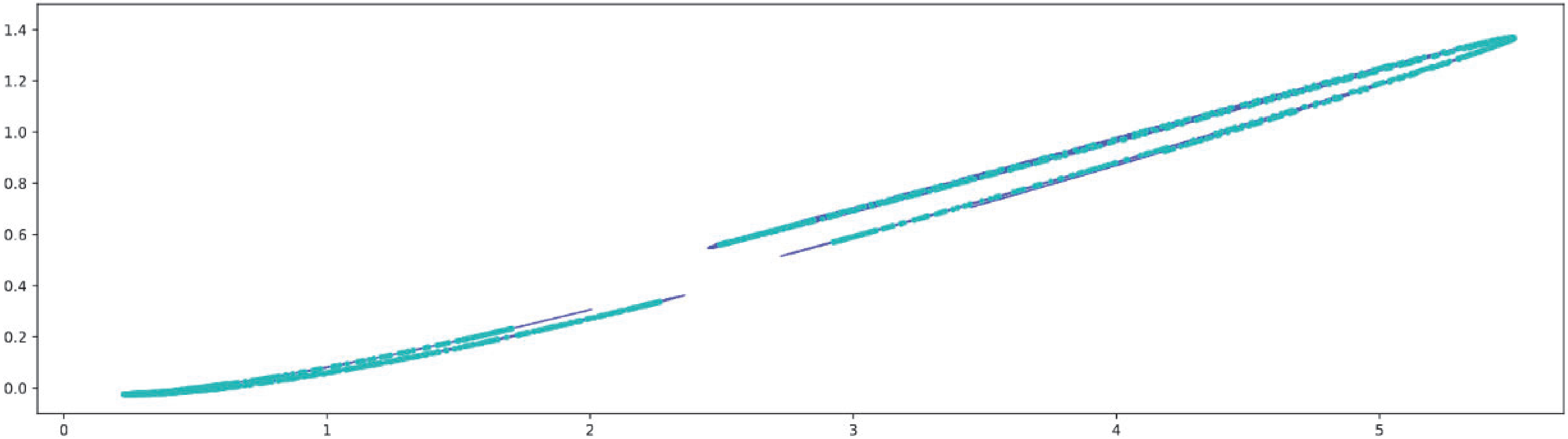


Rigorous proof:

- Morse decomposition
- Chaos
- Periodic orbits in M_1

1 Per-2, 1 Per-4, 2 Per-6,
1 Per-8, 2 Per-10, 2 Per-12

Example: KSR from data



Surrogate: Gaussian Process model using standard radial basis functions

Maximum error on test points:

$[2.45 \times 10^{-3}, 1.40 \times 10^{-3}]$

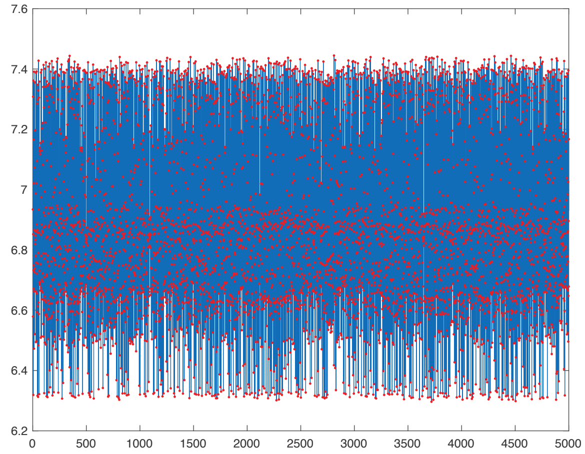
100 Predictor points simulated from projection onto the first two modes

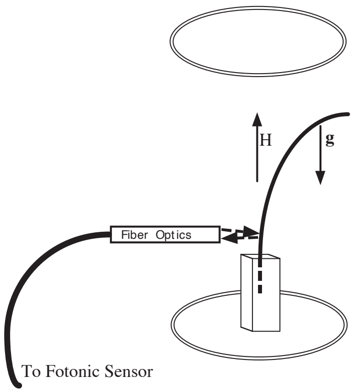
1500 Test points

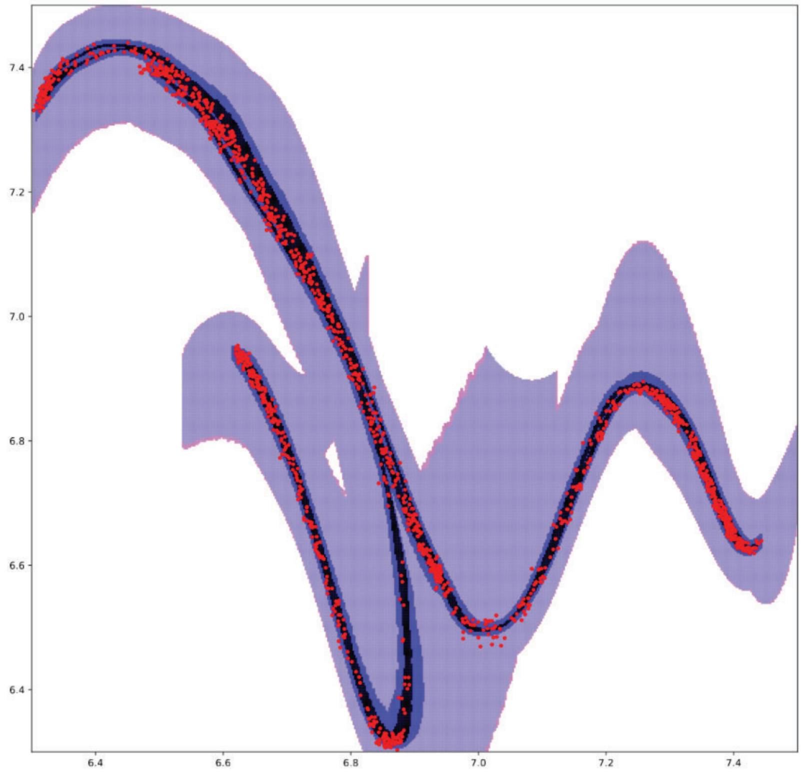
Able to isolate most of the periodic orbits in the attractor that were found in the rigorous computation.

Example: Magneto-elastic Ribbon Data

Time series of 30,000 data points which are measurements of the position of a magneto-elastic ribbon driven by an oscillating magnetic field.



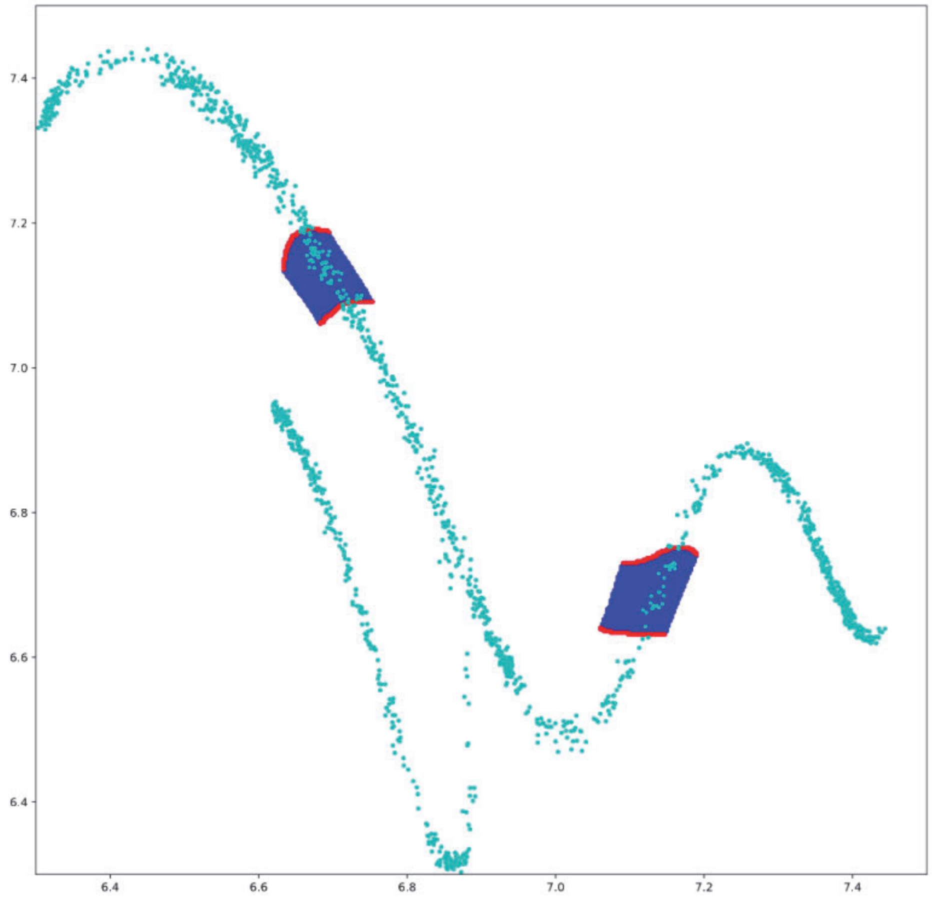




Joshua Reiss, PhD dissertation

Mischaikow, Mrozek, Reiss, Szymczak

Robust period-2 orbit in MER



Isolating neighborhood after the combinatorial multivalued map was expanded locally near a small initial isolating neighborhood near the period-2 orbit.

All of the images of data points in the isolating neighborhood are covered by the data.

Additional robustness of the Conley index computations.

Current work

- Explore the increased robustness from expanding the multivalued map locally and along the data to incorporate more confidence in results. (Day, Kalies)
- Incorporate parameter dependence
    - Surrogate models including the parameter — flour beetle data. (Day, Kalies)
    - Theoretical foundations for the computation of algebraic invariants for bifurcations, using sheaf cohomology and computational algorithms for cellular sheaves. (Dowling, Kalies, VanderVorst)
- Use machine learning models to build more efficient combinatorial models in higher dimensions by learning the attractor structure. (Gameiro, Gelb, Kalies, Kramar, Mischaikow, Tatasciore)

Thank you!

NSF

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1951

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AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

Topological early warning signals: A role for algebraic topology in ecological forecasting

Laura Storch
Oregon State University / William & Mary

Background

- Changing dynamics of populations and ecosystems
  - Global climate change
  - Habitat destruction and degradation
  - Increasing pressures on the population (e.g., harvesting)

- If we improve our understanding of ecosystem dynamics and change, can we better inform management and conservation efforts?

World Wildlife Fund

The Guardian

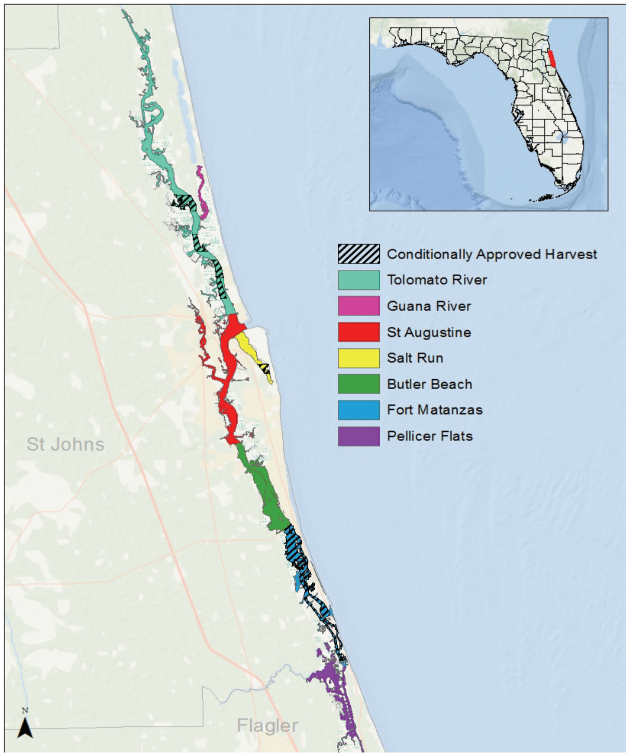
Background

1. Using field data to inform models of oyster population dynamics and directly communicate with stakeholders
2. Understanding the dynamics of nonlinear/chaotic populations (chaotic population dynamics modeling)
3. Understanding how model dynamics differ from the dynamics of population time series (nonparametric forecasting)
4. Using changing spatial patterns to inform/predict impending dynamical changes (topological data analysis)

Background

1. Using field data to inform models of oyster population dynamics and directly communicate with stakeholders

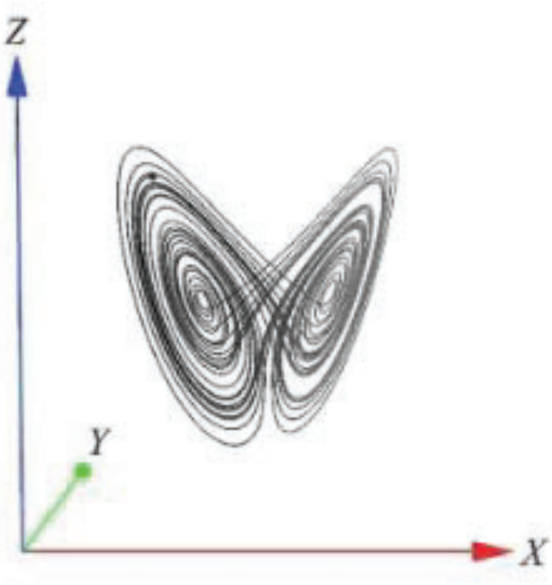

Photo: Adrienne Breef-Pilz



Storch et al. (in prep) *Marine Ecology Progress Series*

Background

2. Understanding the dynamics of nonlinear/chaotic populations (chaotic population dynamics modeling)



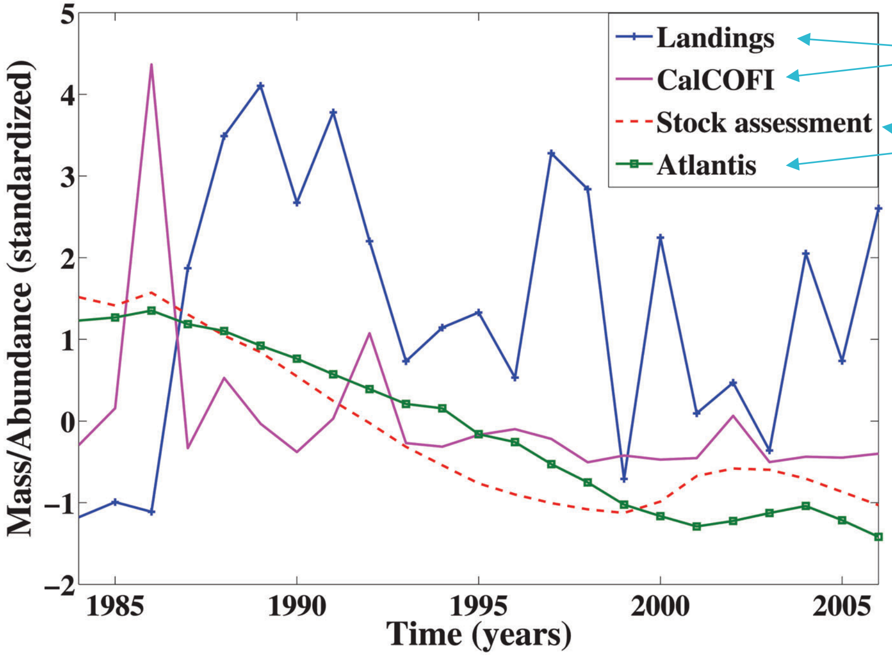
$$\frac{dX}{dt} = \sigma Y - \sigma X$$

$$\frac{dY}{dt} = -XZ + \rho X - Y$$

$$\frac{dZ}{dt} = XY - \beta Z$$

Background

3. Understanding how model dynamics differ from the dynamics of population time series (nonparametric forecasting)



data

model outputs

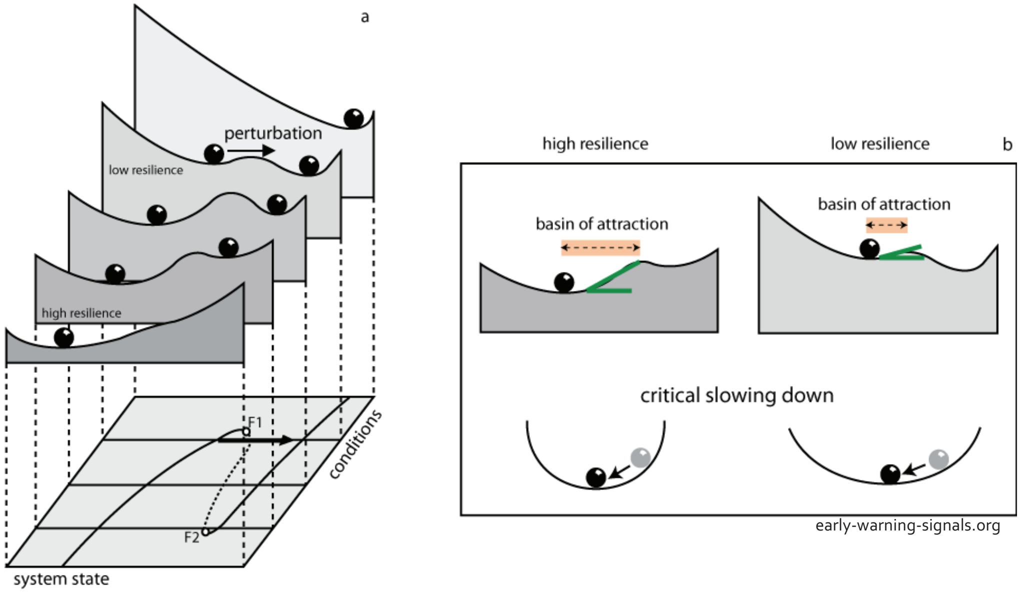
Storch et al. 2017

Background

1. Using field data to inform models of oyster population dynamics and directly communicate with stakeholders
2. Understanding the dynamics of nonlinear/chaotic populations (chaotic population dynamics modeling)
3. Understanding how model dynamics differ from the dynamics of population time series (nonparametric forecasting)
4. **Using changing spatial patterns to inform/predict impending dynamical changes (topological data analysis)**

Critical transitions

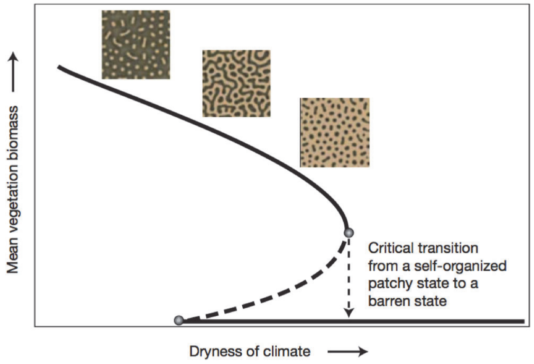
- Critical transition / regime shift : Abrupt, irreversible shift in the state of an ecosystem



early-warning-signals.org

Early warning signals of critical transition

- Critical slowing down (slowed recovery from perturbation) Scheffer et al. 2009
- Increased autocorrelation Scheffer et al. 2009
- Increased variance Scheffer et al. 2009
- Spatially explicit:
  - Increased spatial correlation Daikos et al. 2010, Kefi et al. 2014
  - Increased spatial variance with peaking of spatial skewness Guttal & Jayaprakash 2009, Kefi et al. 2014
  - Spectral reddening of DFT Carpenter & Brock 2010



Scheffer et al. 2009

Early warning signals of critical

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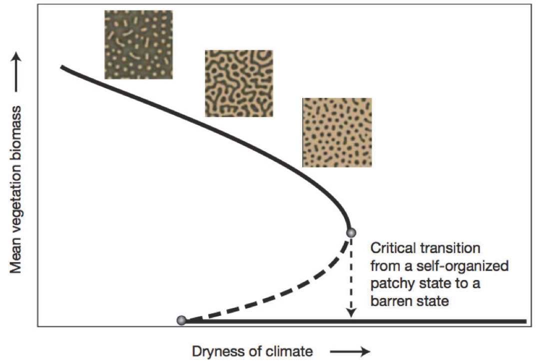
- Increased autocorrelation Scheffer et al. 2009
- Increased variance Scheffer et al. 2009
- Spatially explicit:
  - In et al. 2014
  - e with peaking of spatial skewness Guttal &
  - eddening of DFT Carpenter & Brock 2010

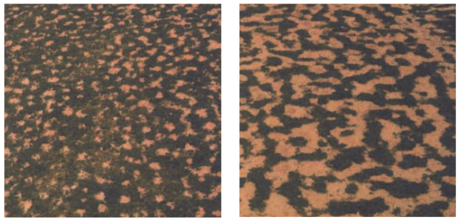
Statistical early warning signals will not be able to predict critical transitions in a chaotic system (Adamson et al. 2020)



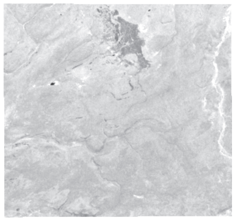
Scheffer et al. 2009

Spatially distributed populations

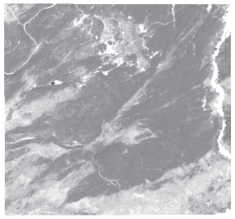
- Understanding and predicting changing dynamics of **spatially distributed** populations is an active area of current ecological research
  - High dimensional problem
  - Complex spatiotemporal dynamics
- Spatially distributed populations can form patterns



Rietkerk et al. 2004



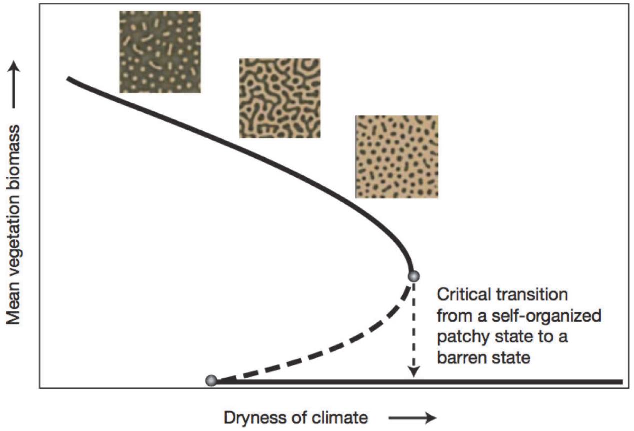
Pre-fire



Post-fire

Spatially distributed populations

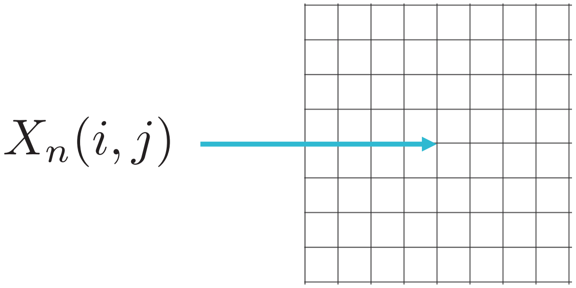
- Here, we ask: Can we correlate the dynamics of spatially distributed populations with the patterns they form?
- Can we use information about changing spatial patterns to predict critical dynamical transitions (e.g., extinction events)
- Can we reduce the dimensionality of the problem?



Scheffer et al. 2009

Population model

Growth phase: scaled Ricker map with an Allee effect Ricker 1954 :



Population model

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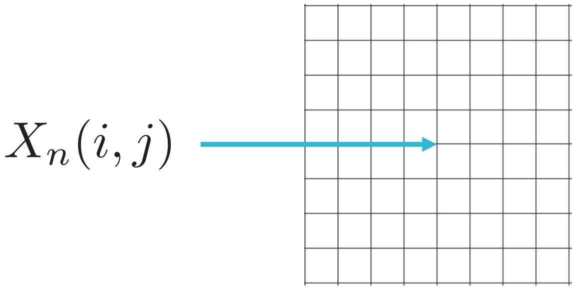
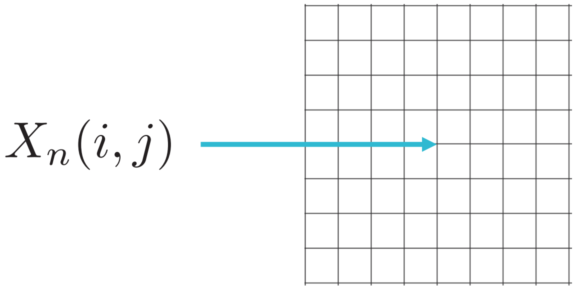


Population model

Growth phase: scaled Ricker map with an Allee effect Ricker 1954 :



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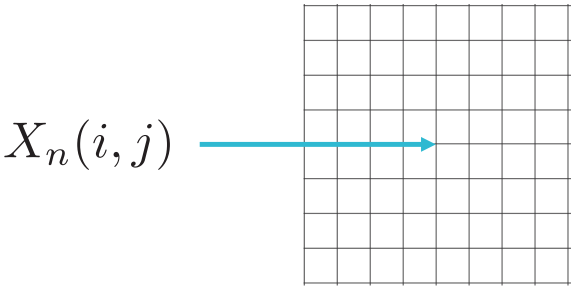
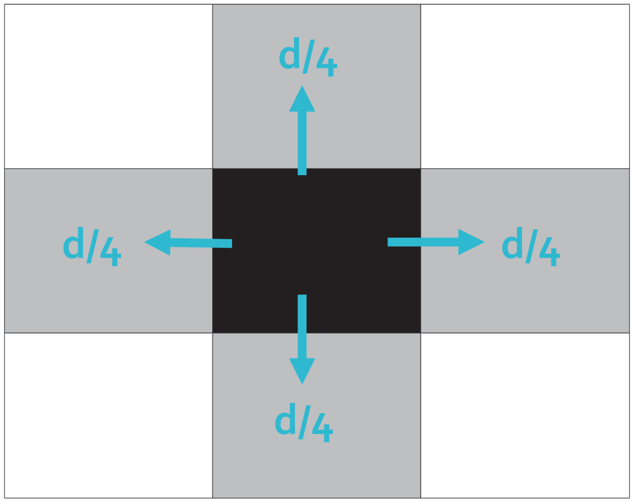


Population model

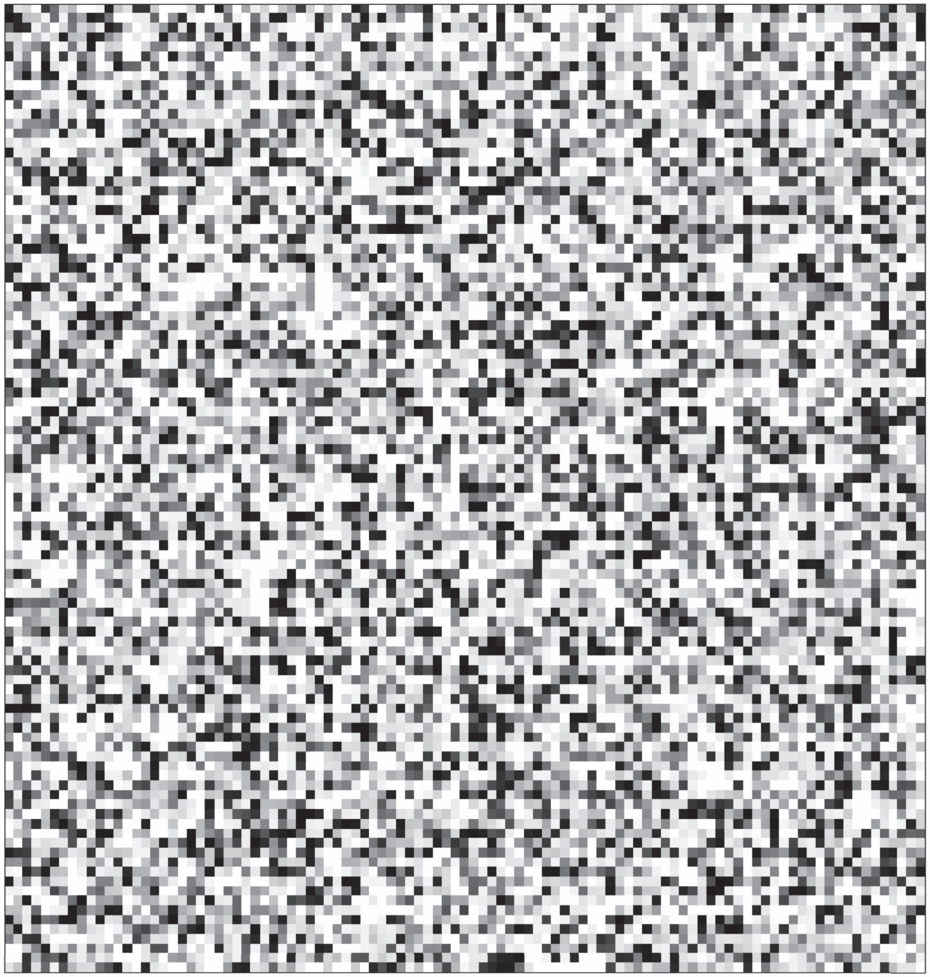
- **Dispersal phase**: coupled patch dispersal with absorbing boundaries

$$X_{n+1}(i,j) = (1-d)\bar{X}_n(i,j) + \frac{d}{4}\left[\bar{X}_n(i-1,j) + \bar{X}_n(i+1,j) + \bar{X}_n(i,j+1) + \bar{X}_n(i,j-1)\right]$$

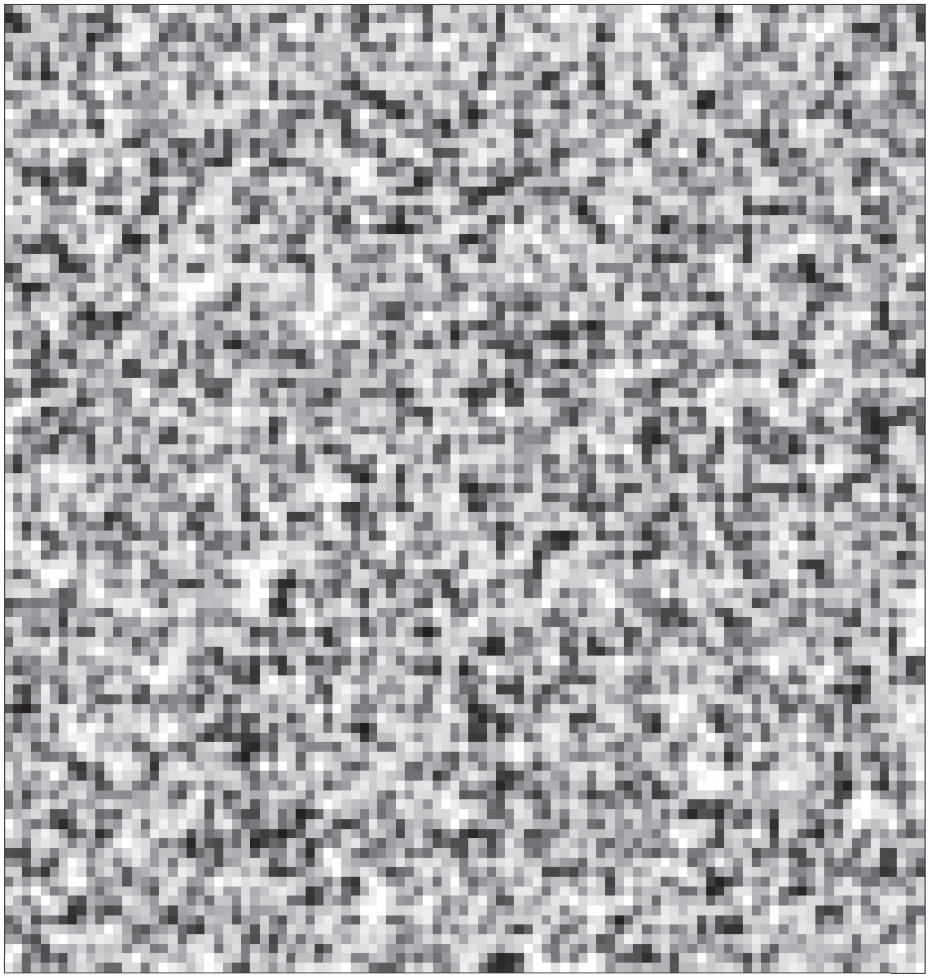
d = fraction of the population dispersing to neighboring patches



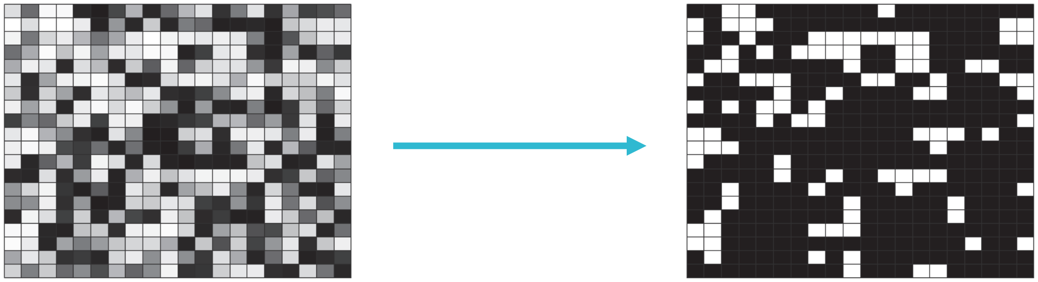
Population model – small dispersal



Population model – large dispersal

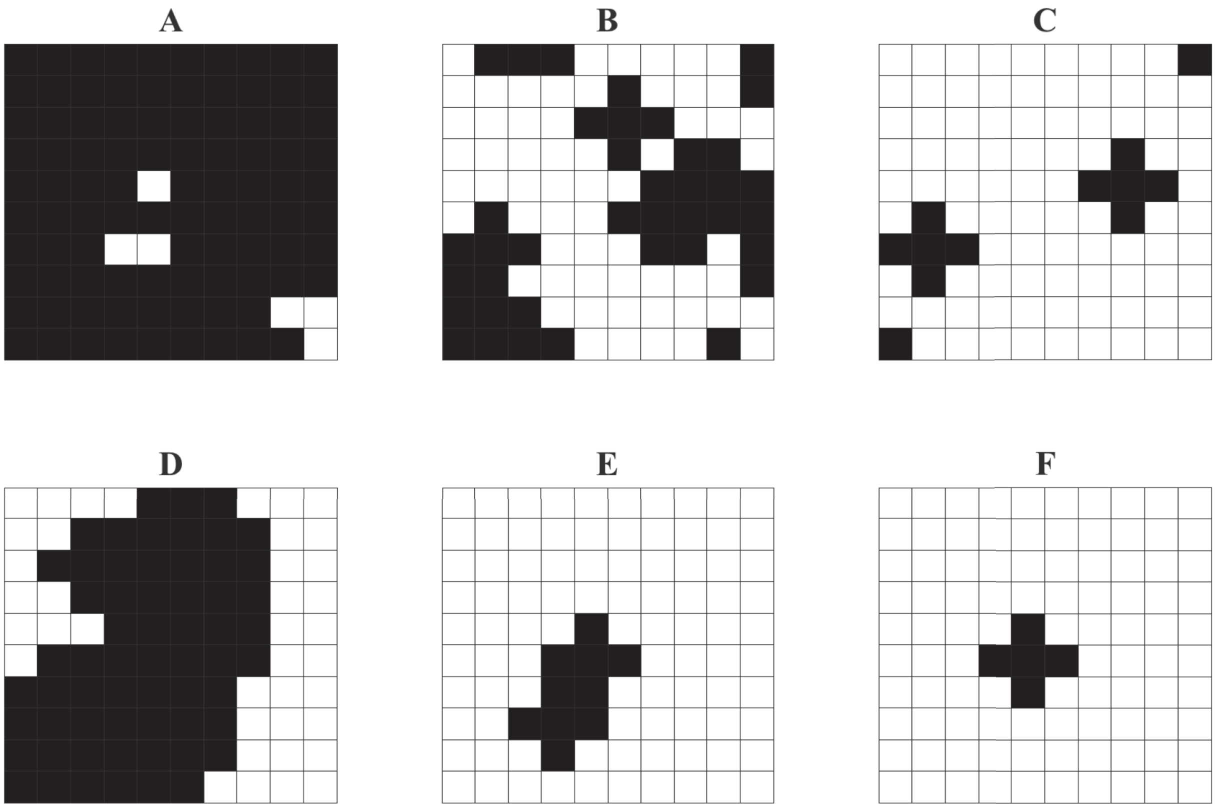


Global thresholding



We employ **cubical homology** to quantitatively classify spatial patterns

- We identify the first and second **Betti numbers** for spatial population time series ("connected components" and "holes")
- Also reducing dimensionality of problem



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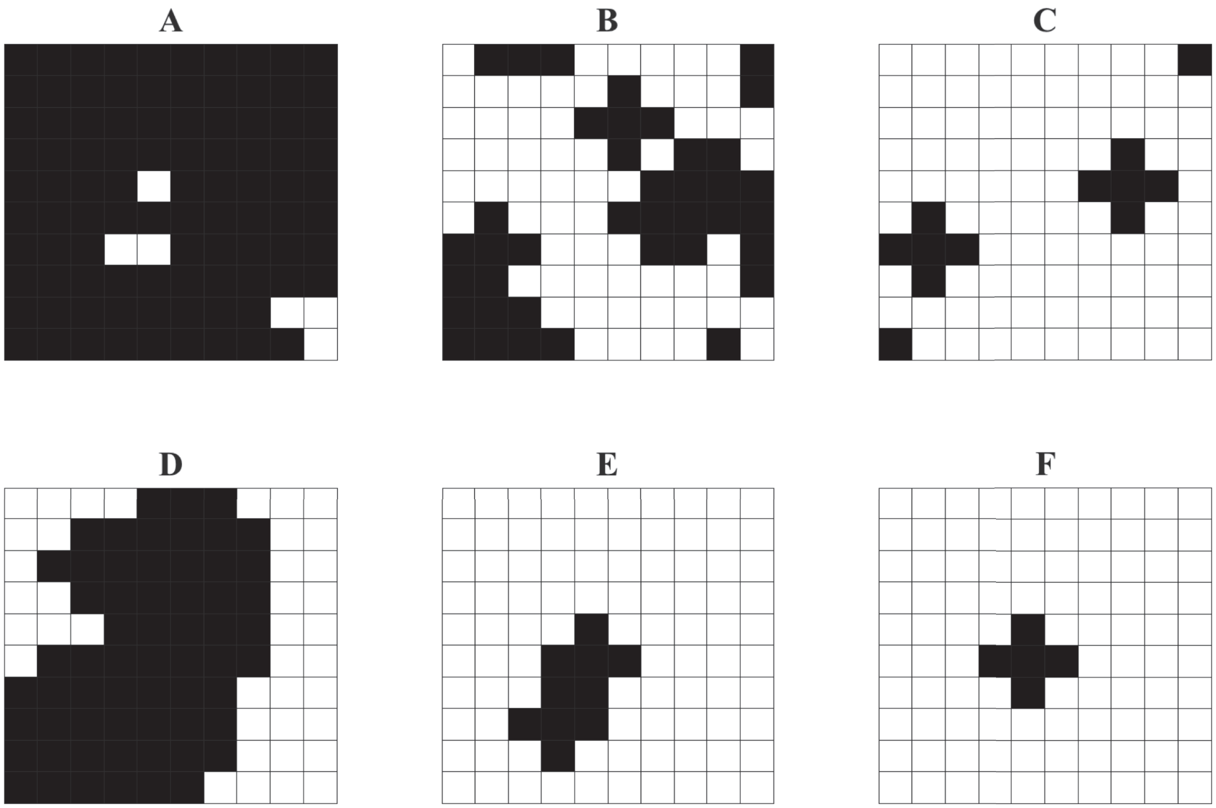
$\beta_0 = 1, \;\; \beta_1 = 2$



We employ **cubical homology** to quantitatively classify spatial patterns

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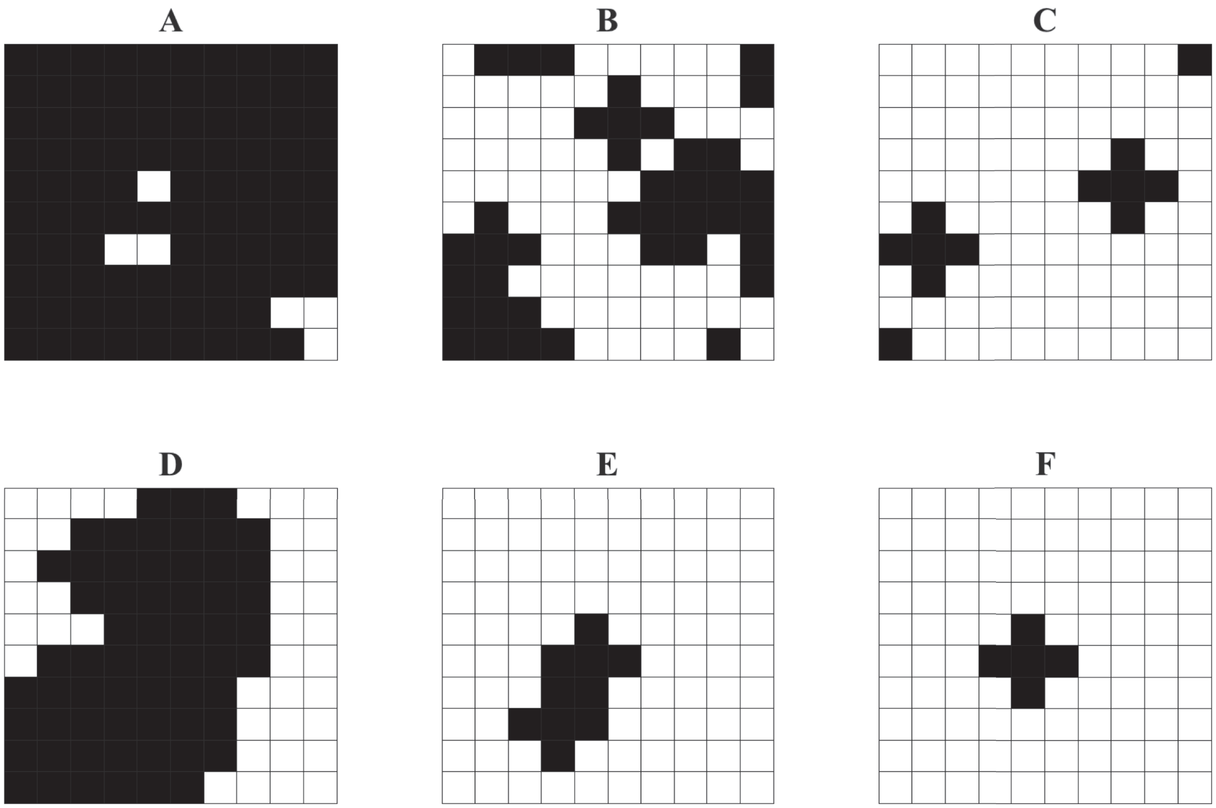
$\beta_0 = 5, \quad \beta_1 = 1$



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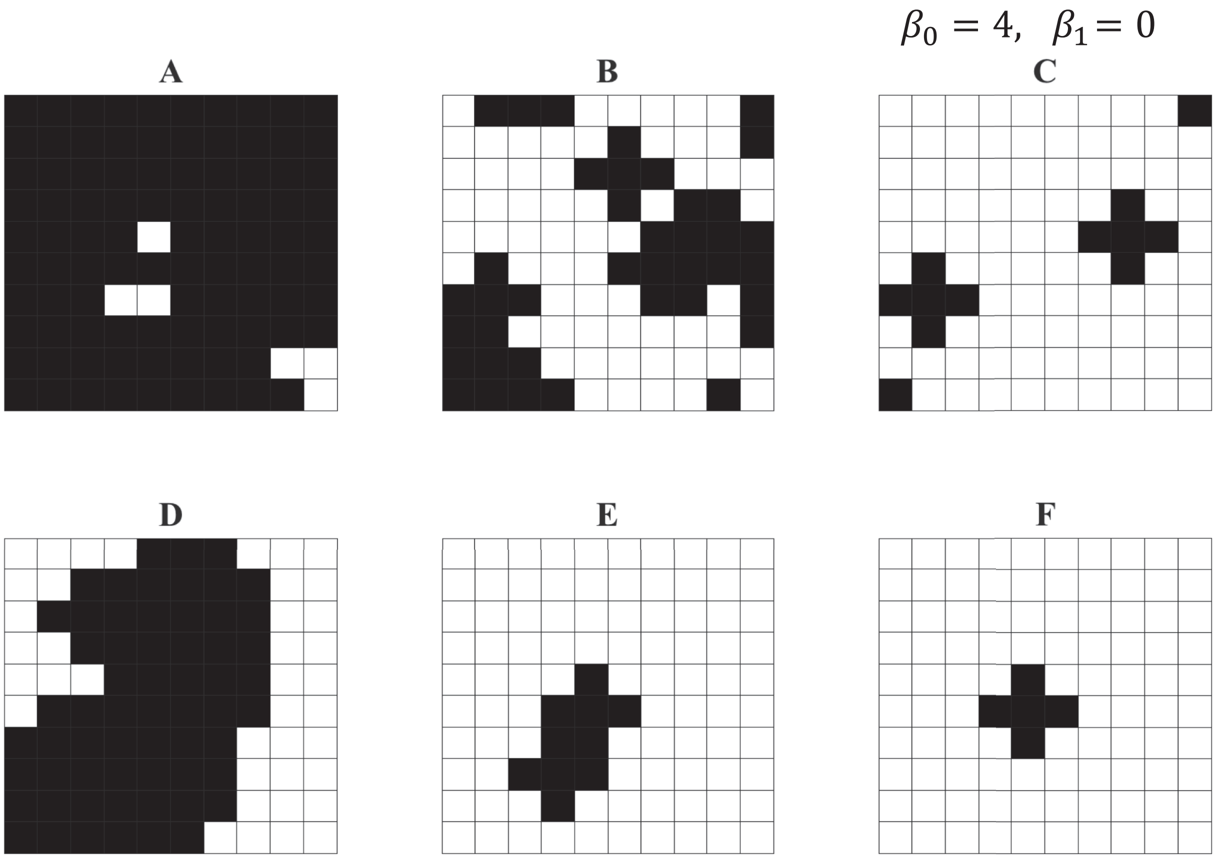

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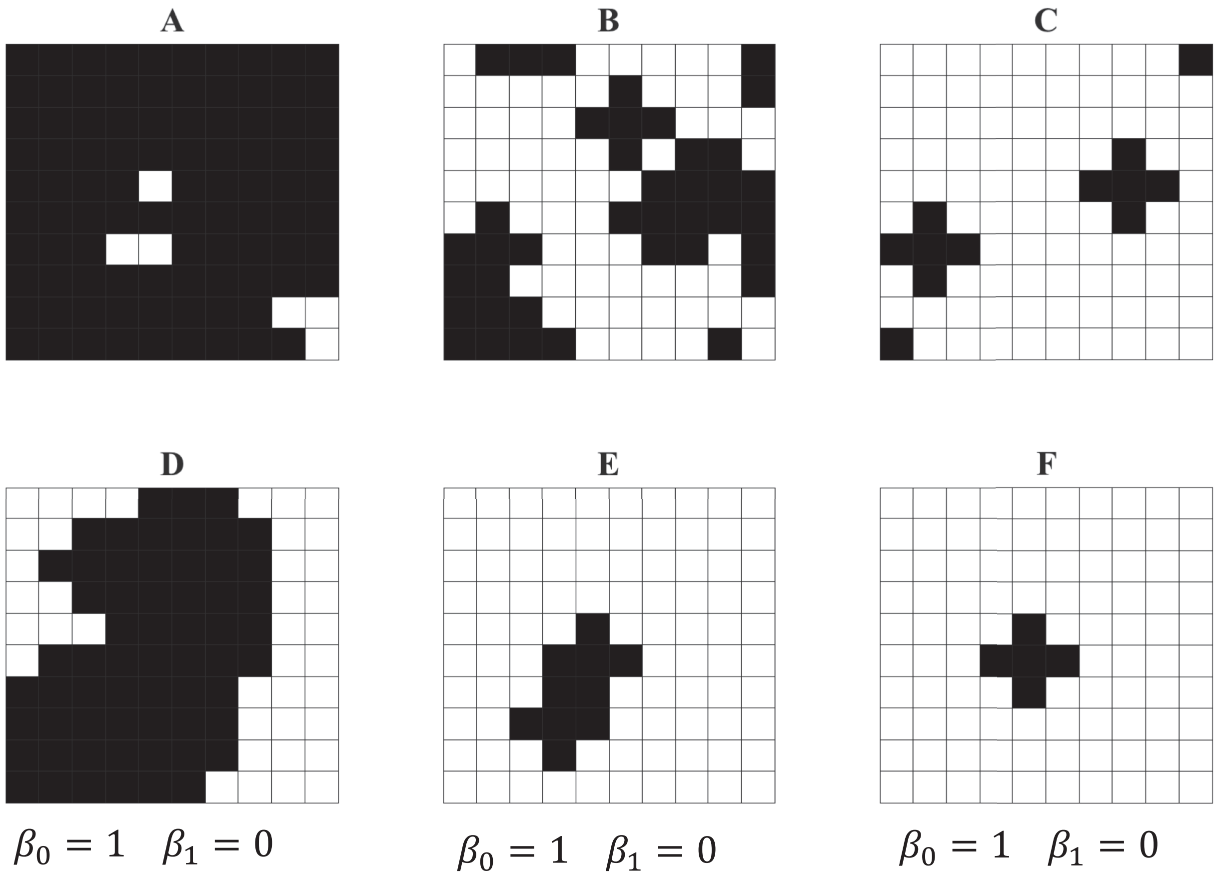
- We identify the first and second **Betti numbers** for spatial population time series ("connected components" and "holes")
- Also reducing dimensionality of problem

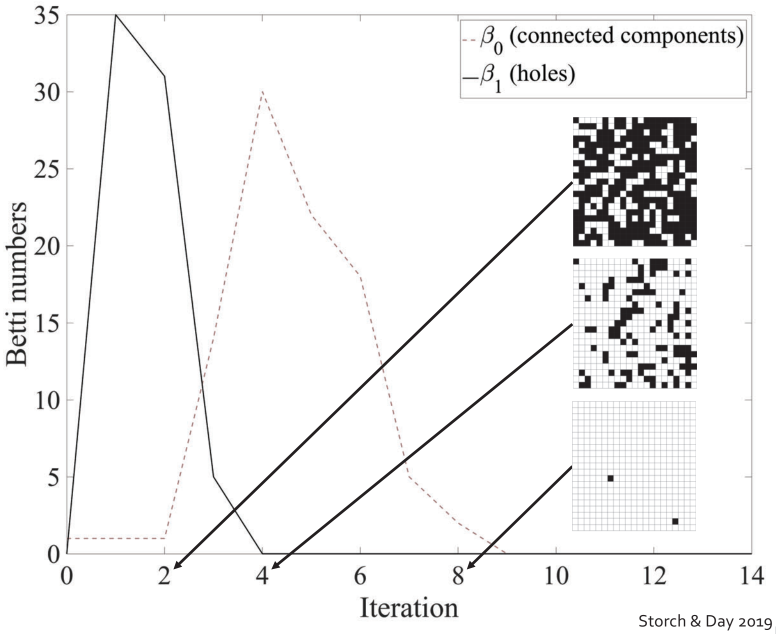


$\beta_0 = 1 \quad \beta_1 = 0$ $\quad$ $\beta_0 = 1 \quad \beta_1 = 0$ $\quad$ $\beta_0 = 1 \quad \beta_1 = 0$

Extinction event

As the population goes extinct:

- Patches become decoupled from neighboring patches (connected components increase while holes decrease)
- Isolated patches slowly go extinct (connected components decrease)

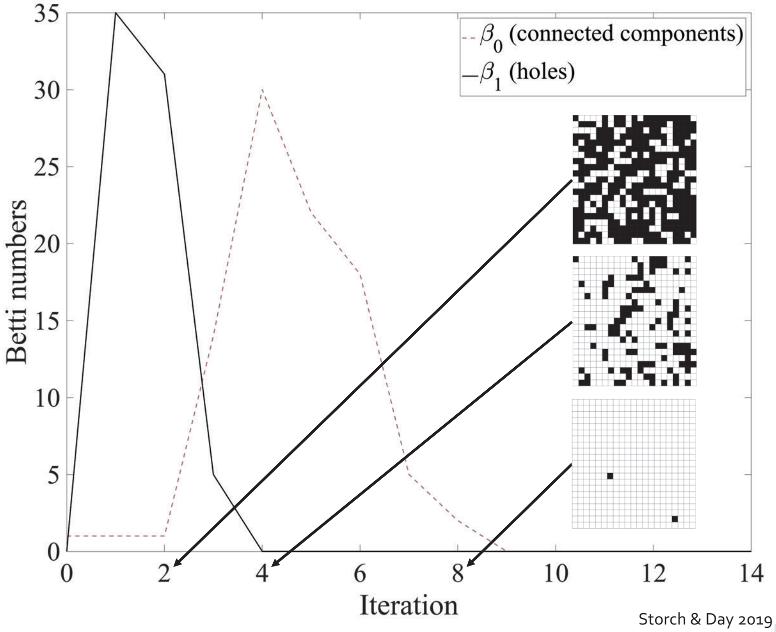


Storch & Day 2019

$N = 21$, $r = 8$, $d = 0.1$, $\epsilon = 0.07$

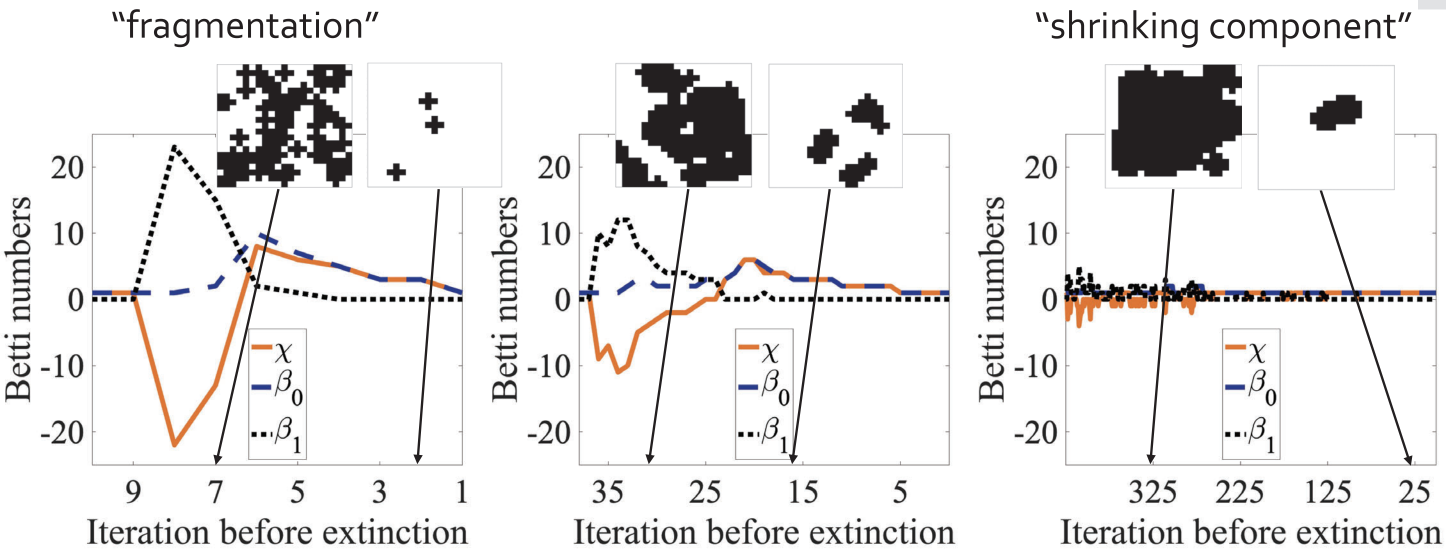
Extinction event

- Is this a universal "route to extinction" observed across the entire parameter space?



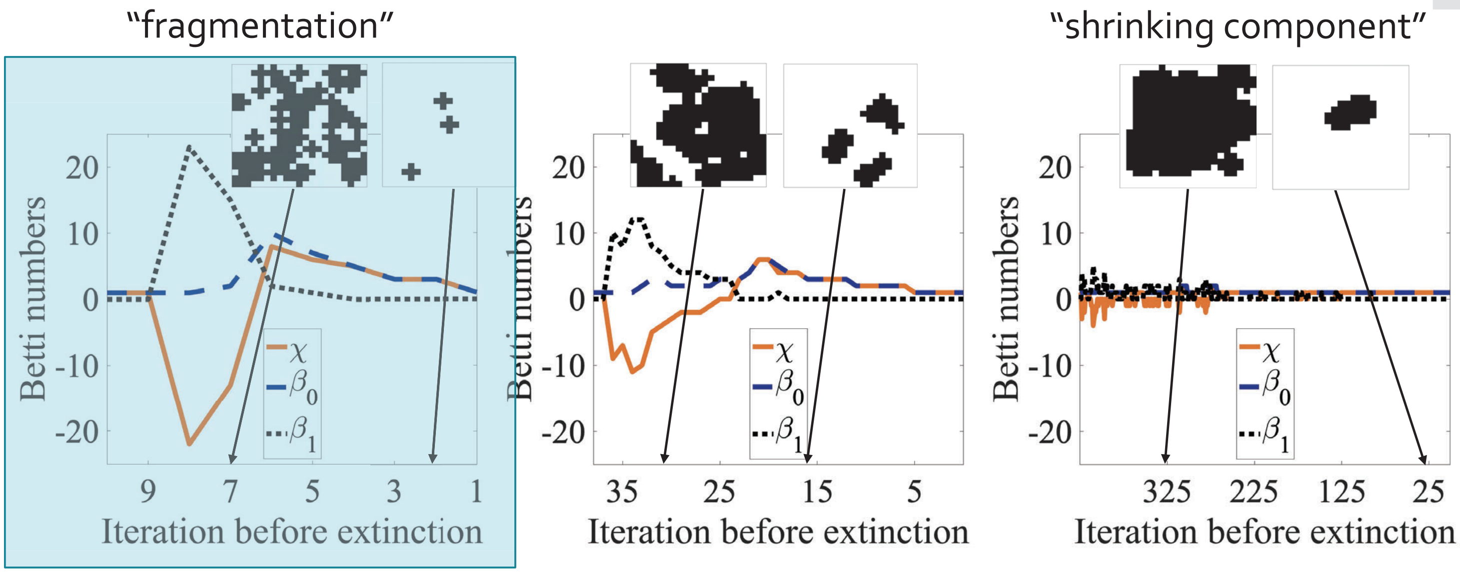
Storch & Day 2019

Extinction events



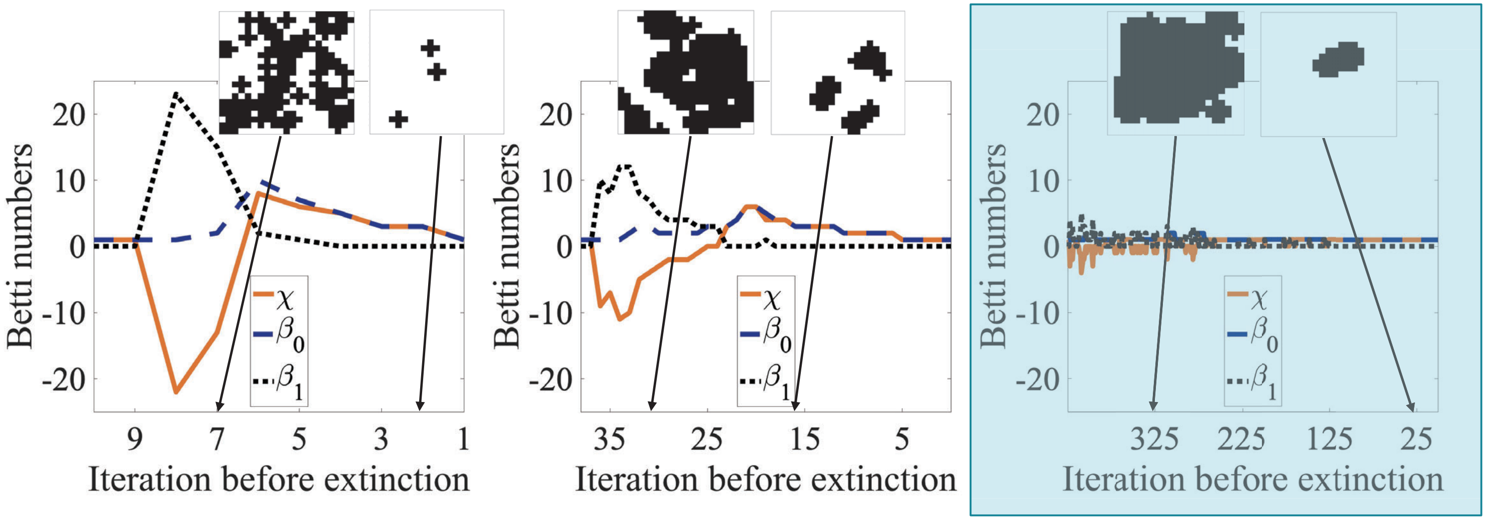
Storch & Day (submitted) *JTB*

Extinction events



Storch & Day (submitted) *JTB*

Extinction events

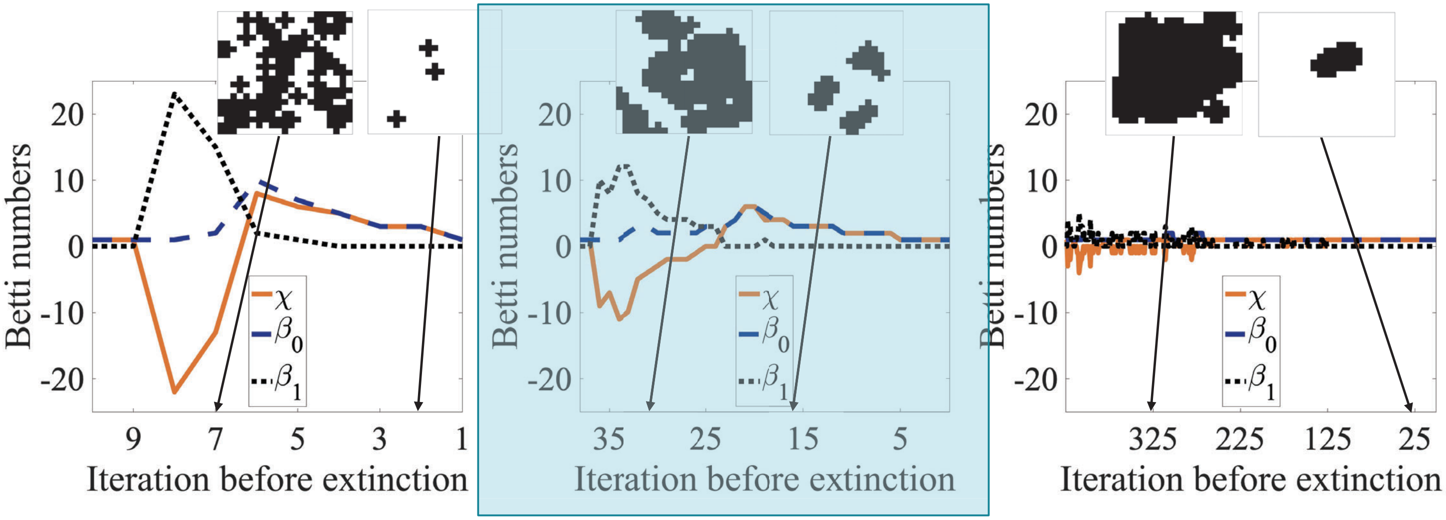


“fragmentation”

“shrinking component”

Storch & Day (submitted) *JTB*

Extinction events

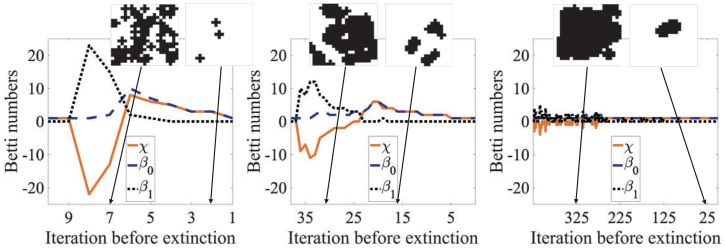


"fragmentation"

"shrinking component"

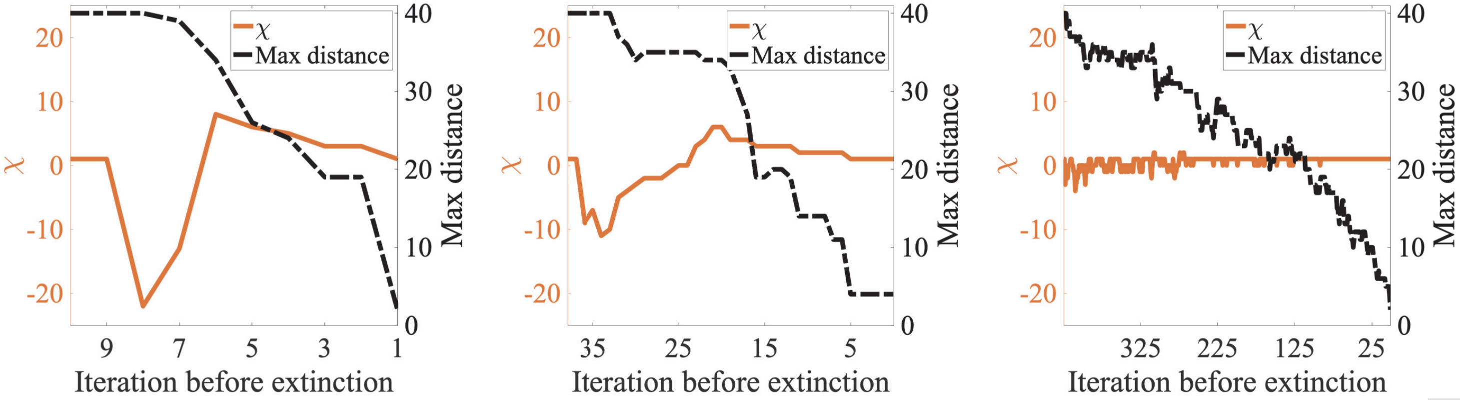
Storch & Day (submitted) *JTB*

Extinction events



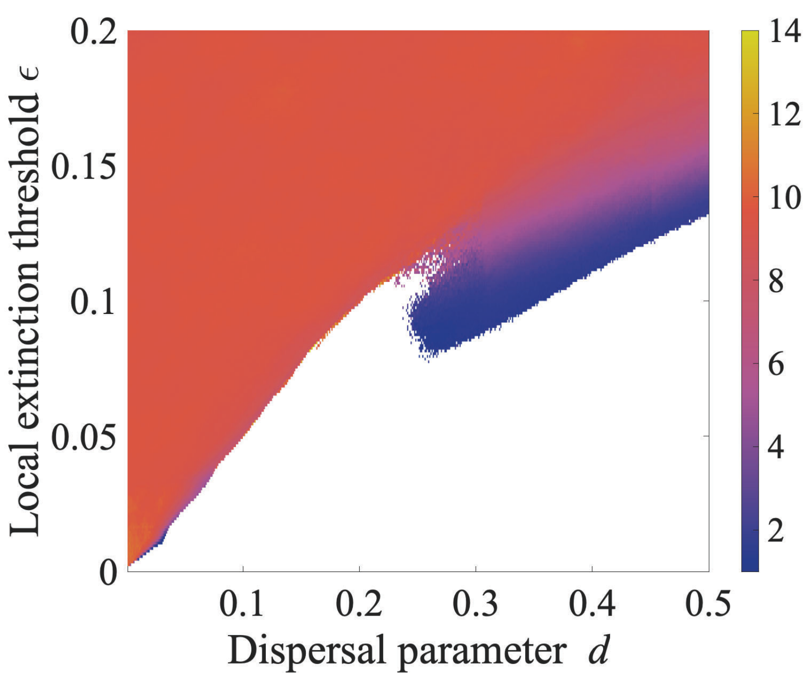
“fragmentation”

“shrinking component”

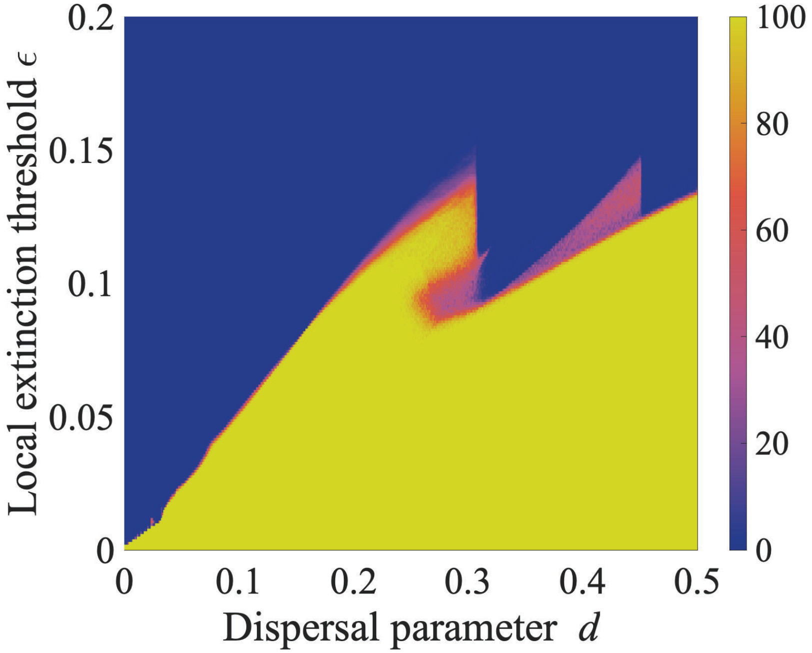


Storch & Day (submitted) *JTB*

Maximum β_0

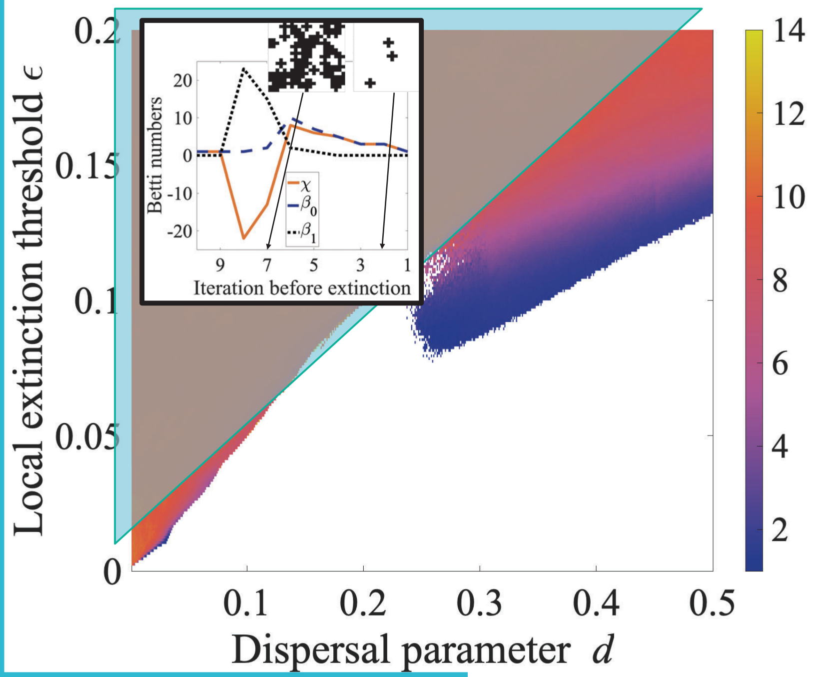


Probability of survival

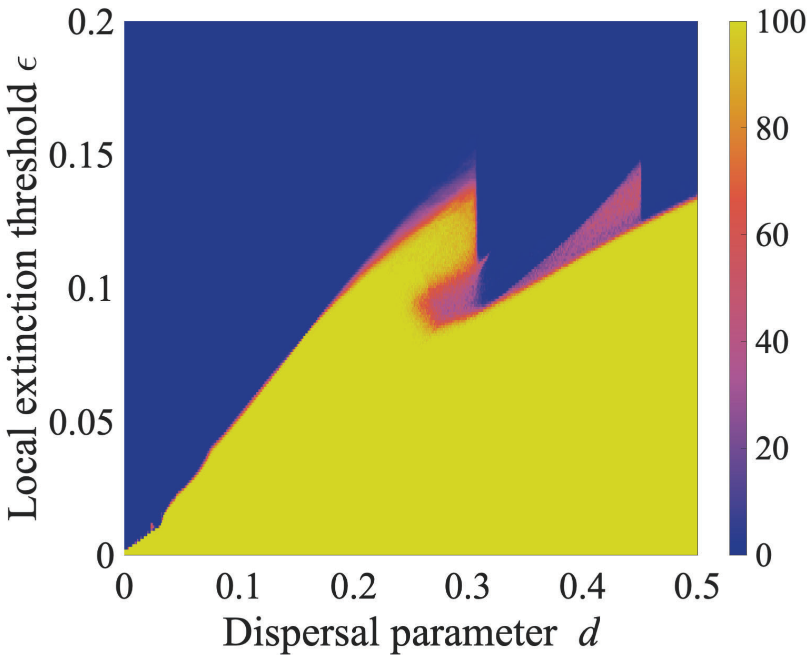


Storch & Day (submitted) *JTB*

Maximum β_0

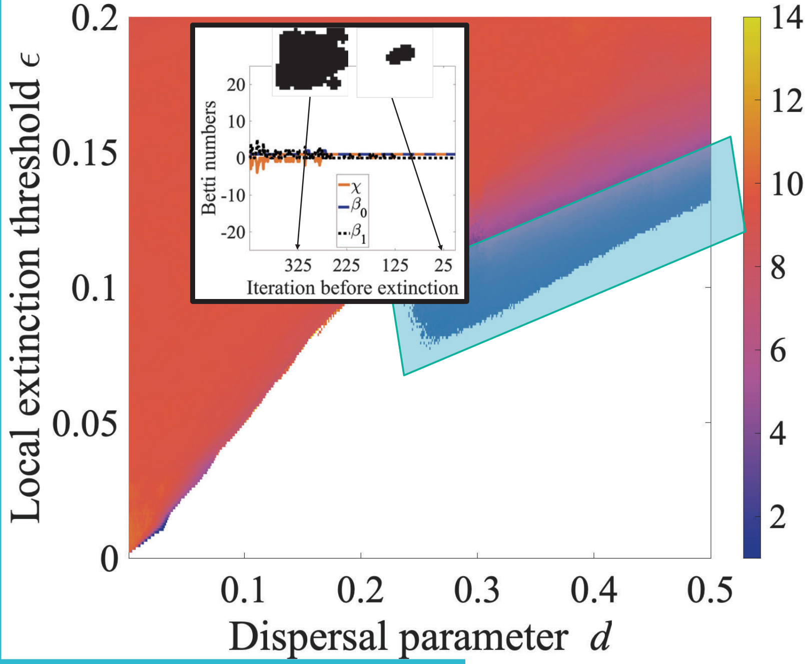


Probability of survival

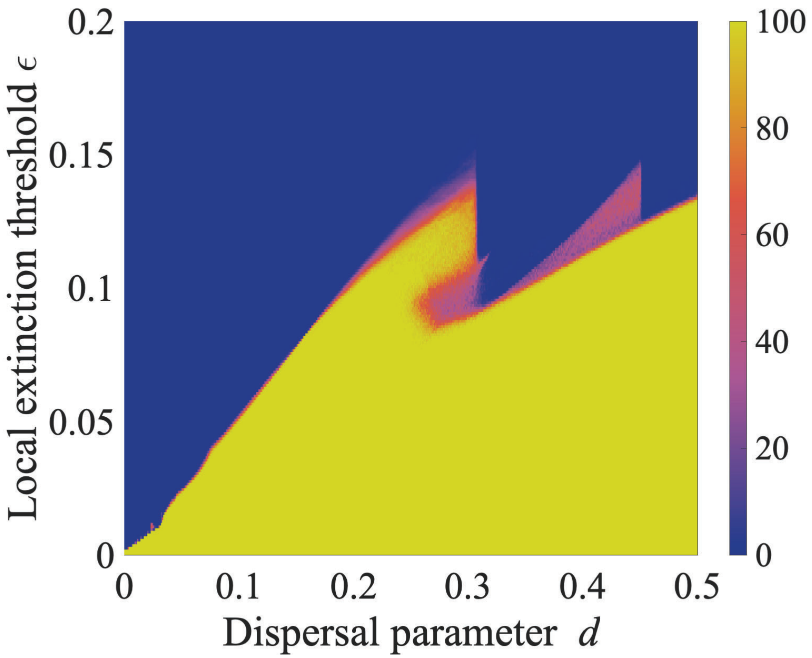


Storch & Day (submitted) *JTB*

Maximum β_0

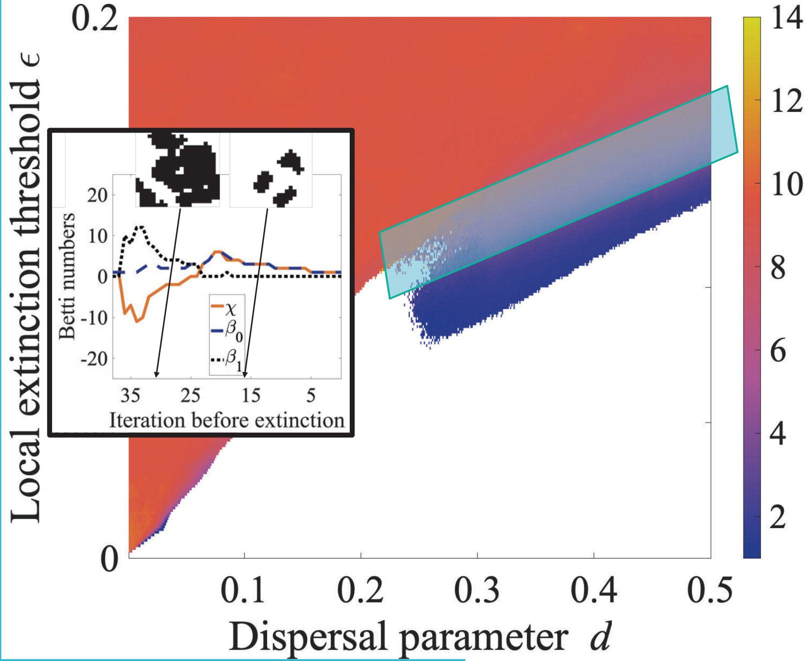


Probability of survival

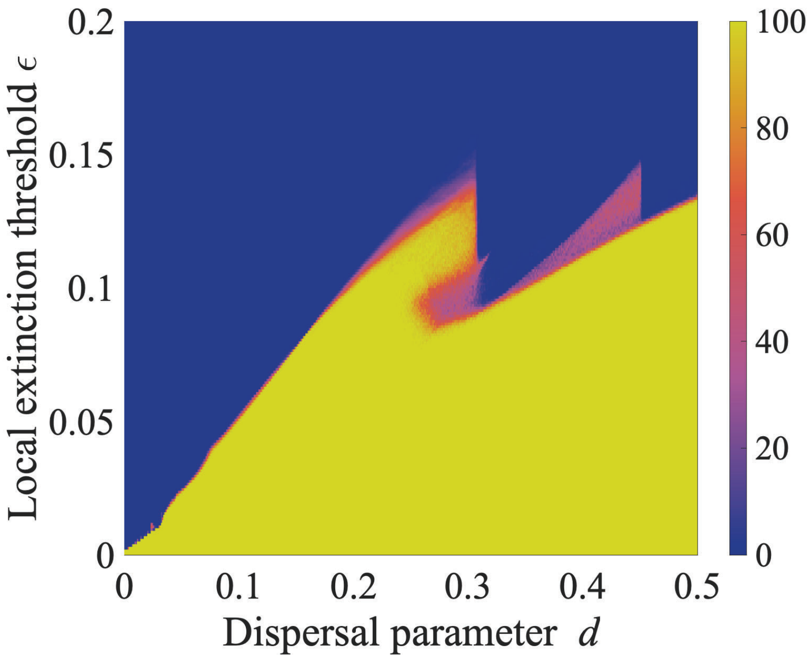


Storch & Day (submitted) *JTB*

Maximum β_0



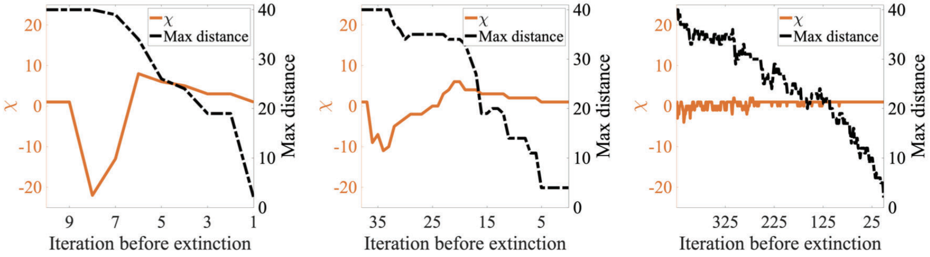
Probability of survival



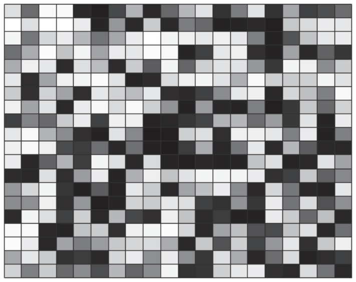
Storch & Day (submitted) *JTB*

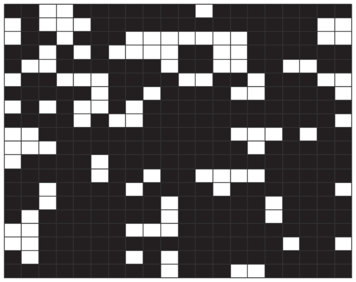
Summary

- We can characterize the different routes to extinction based on the changes in topological features over time
- There is no universal “route to extinction” (but two main possibilities)
  - Likelihood of a given extinction event depends on initial conditions
- Automate classification of the two routes
- Prediction tool for early warning signals in **chaotic systems**



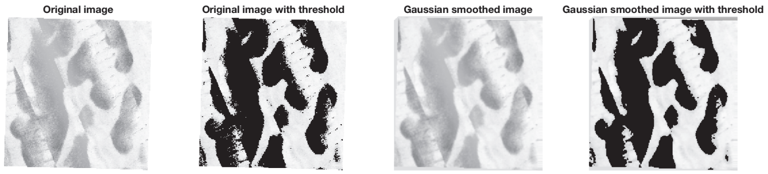
Applications to greyscale & higher dimensional data

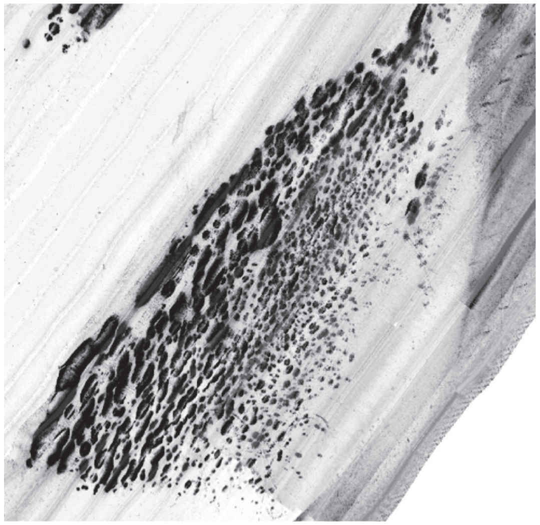




Applications to greyscale & higher dimensional data

- Persistent homology:
  - Predicting critical dynamical transitions
  - Image processing
  - Model validation



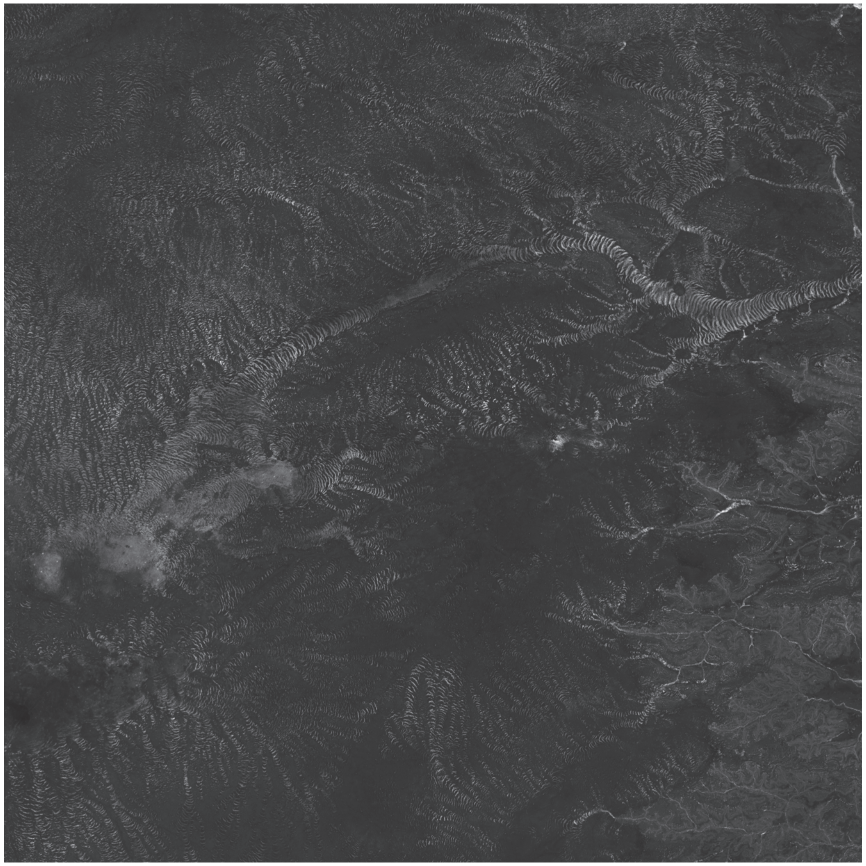


Sonar image of oyster reefs distributed across a riverbed

(obtained from David Bruce & Jay Lazar, NOAA Chesapeake Bay Office, NMFS)

Applications to greyscale & higher dimensional data

Arid grasslands pattern identification

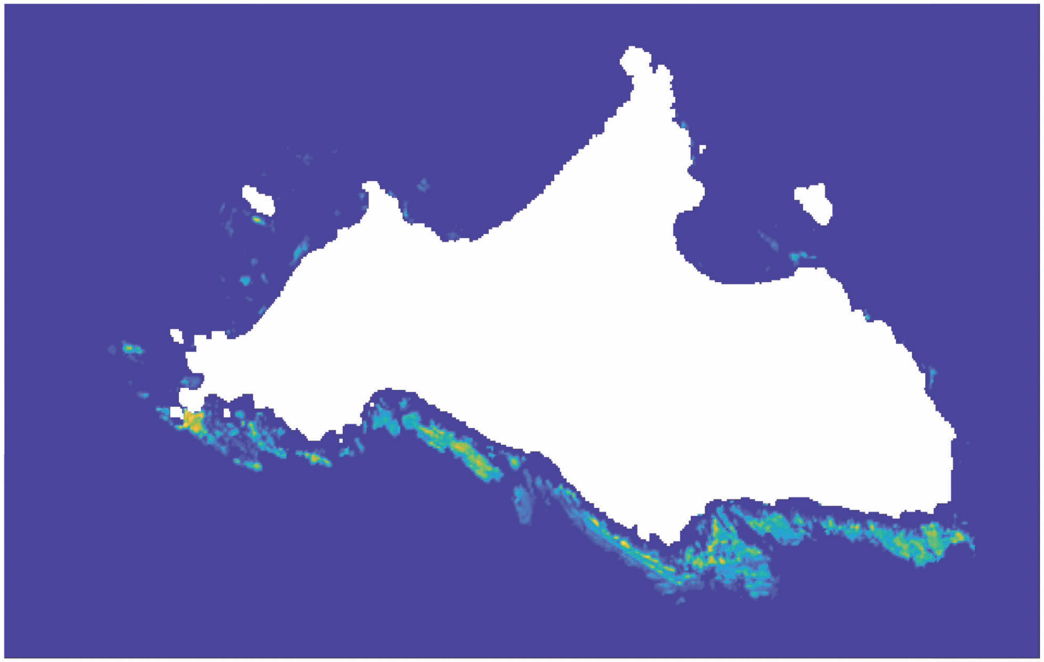


Data obtained from Chad Topaz at Williams College

Applications to greyscale & higher dimensional data

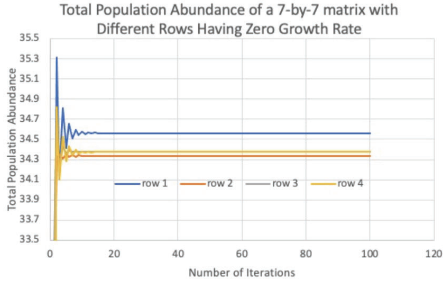

mbc.ucsd.edu

Channel Islands changing kelp cover

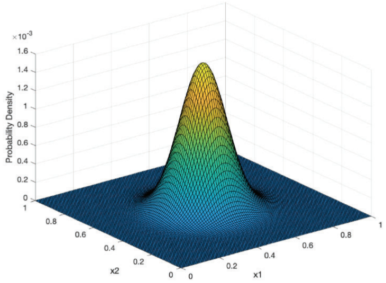

Data obtained from Tom Bell at Woods Hole Oceanographic Institute

Student research examples

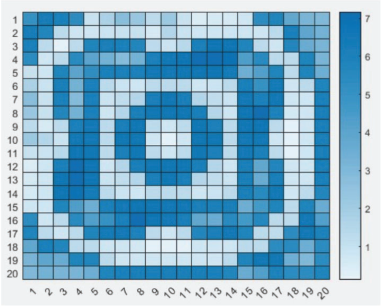
- Merielyn Sher: Studying the effects of habitat fragmentation on population growth



- Michael Getaneh: Altering the coupled patch dispersal model to a more ecologically realistic Gaussian kernel dispersal model



- Xinyao Wang: Population spread under differing dispersal scenarios



Student research examples

References

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- **Storch, Laura** & Day, Sarah (2019) Towards the prediction of critical transitions in spatially extended populations with cubical homology. Contemporary Mathematics 736: 31 – 48.
- **Storch, Laura** et al. (2017) Stock assessment and end-to-end ecosystem models alter dynamics of fisheries data. PLoS one 12(2): e0171644.
- **Storch, Laura** et al. (in prep) Fine-scale heterogeneity across environmental gradients exceeds differences among estuaries. Marine ecology progress series.
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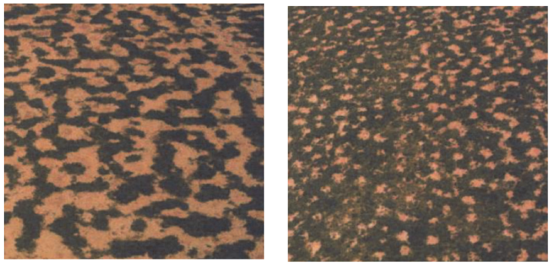
- Adamson, Matthew et al. (2020) Forecasting resilience profiles of the run-up to regime shifts in nearly one-dimensional systems. Journal of the Royal Society Interface 17: 20200566.
- Kefi, Sonia et al. (2014) Early warning signals of ecological transitions: Methods for spatial patterns. PLoS ONE 9: 1 – 13.
- Guttal, Vishwesha & Jayaprakash, Ciriyam (2009) Leading indicators of regime shifts in spatial ecological systems. Theoretical Ecology 2: 2– 13.
- Dakos, Vasilis et al. (2010) Spatial correlation as leading indicator of catastrophic shifts. Theoretical Ecology 3: 163 – 174.
- Ricker, Bill (1954) Stock and recruitment. Journal of the Fisheries Research Board of Canada 11: 559 – 623.
- Carpenter, Stephen & Brock, William (2010) Early warnings of regime shifts in spatial dynamics using the discrete fourier transform. Ecosphere 1(5): art10.

Thank you!

Questions?

- Contact: laura.storch@oregonstate.edu



Rietkerk et al. 2004