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RATE MEASUREMENT OF MARINE CHRONOMETERS, GIMBAL MOUNTED CHRONOMETER WATCHES, AND NON-GIMBAL NAVIGATING WATCHES UNDER CONTROLLED CLIMATIC CONDITIONS

PART V - THE INTERVAL-RECORD AND CHRONOGRAPH METHODS OF RATE MEASUREMENT

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CONTENTS

ABSTRACT

Previous reports in this series have been concerned with the visual, photo, and photo-record methods of daily rate measurement and the results obtained using the latter method. In this final report two methods of greater accuracy (an error reduction over previous methods by more than 10 times), the interval-record and the chronograph methods, are described and discussed; the results obtained using the first of these two methods are compared with previous results. The sensitivities and over-all measuring errors for both of these methods are estimated and compared with values for the three other methods. While the three methods previously considered yielded only values of daily rate, both methods considered in this report permit use of the data to measure, in addition, changes in or constancy of rate for intervals of less than a day.

It is shown that the interval-record and chronograph methods have comparable sensitivities and accuracies. From an analysis of the recorded data it is shown that a correct interpretation requires that trend lines be drawn. These trend lines, at least for the results presented, are adequately represented by two parallel straight lines, the spacing between which depends upon the measuring errors involved.

Certain limitations in the interpretation of the records are pointed out. These limitations serve as useful criteria in the design of future experiments using these methods of measurement to provide records which can be interpreted unambiguously. The interpretation of the recorded data is discussed.

The design of an experiment to investigate the temperature and pressure characteristics of timepieces is indicated. Using some of the design considerations the rate-temperature gradient ($\partial R/\partial \theta$) is calculated from the experimental data (for the marine chronometer) together with an estimate of the error.

PROBLEM STATUS

This is a final report. No further work on this phase of the problem is contemplated; work on other phases is continuing. The nine winding-inverting mechanisms, the winding sequence-control circuits, and the chronograph have been shipped to the Naval Instrument Repair Facility; the remainder of the instrumentation has been dismantled.

AUTHORIZATION

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RATE MEASUREMENT OF MARINE CHRONOMETERS, GIMBAL MOUNTED CHRONOMETER WATCHES, AND NON-GIMBAL NAVIGATING WATCHES UNDER CONTROLLED CLIMATIC CONDITIONS

PART V - THE INTERVAL-RECORD AND CHRONOGRAPH METHODS OF RATE MEASUREMENT

INTRODUCTION

Previous reports of this series have been concerned with describing the instrumentation,[1] defining the meaning of the term "rate,"[2] determining the over-all errors involved[3] in the use of three rate measuring methods,* and the presentation of results obtained using the photo and photo-record methods.[4] In this, the last of the series, two methods of greater accuracy are described and the available results are presented, discussed, and compared with the results of the previously used methods. These two methods are the interval-record and chronograph methods; the instrumentation involved has already been described in Part I.

INTERVAL-RECORD AND CHRONOGRAPH RATE MEASURING METHODS

The interval-record and chronograph methods depend for their increased accuracy on an extension of the photo-record method in which the record is obtained at more frequent than daily intervals. While the previously described methods are dependent upon two readings made at the start and end of the measuring interval, both of these methods use a greater number of readings (recordings of the timepiece timing marks), assuring an effective sampling of the sequence of errors:

$$E_N = e_1 + e_2 + \ldots + E_N = \sum_1^N e_j . \tag{1}$$

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*The visual method, the photo method, and the photo-record method, these titles being descriptive of the method of reading.

[1] Suski, H. M., "Rate Measurement of Marine Chronometers, Gimbal Mounted Chronometer Watches, and Non-Gimbal Navigating Watches Under Controlled Climatic Conditions, Part I - Instrumentation," NRL Report 4524, July 29, 1955

[2] Suski, H. M., "Rate Measurement of Marine Chronometers, Gimbal Mounted Chronometer Watches, and Non-Gimbal Navigating Watches Under Controlled Climatic Conditions, Part II - Definition of the Term 'Rate'," NRL Report 4568, August 4, 1955

[3] Suski, H. M., "Rate Measurement of Marine Chronometers, Gimbal Mounted Chronometer Watches, and Non-Gimbal Navigating Watches Under Controlled Climatic Conditions, Part III - The Estimated Accuracy of Three Rate Measuring Methods," NRL Report 4645, December 14, 1955

[4] Suski, H. M., "Rate Measurement of Marine Chronometers, Gimbal Mounted Chronometer Watches, and Non-Gimbal Navigating Watches Under Controlled Climatic Conditions, Part IV - Rate-Pressure and Rate-Temperature Trends," NRL Report 4716, April 24, 1956

The record of this sequence, when interpreted, gives the rate with a measuring error, E_R or

$$R = E_N' \pm E_R. \tag{2}$$

The quantity E_N' is the measured value of E_N, and E_R is the over-all measuring error of the method. It is shown in Appendix A that the improvement of the interval-record or chronograph method over the three previously described methods (Part III) results from the fact that E_N' approximates E_N more closely and that E_R is smaller than the correspon ing over-all errors E_v, E_p, and E_p', for the other methods.

The interval-record method is discussed in greater detail in Appendix A. Some of the features of the chronograph as intended for use in these experiments are described in Appendix B; a comparison of the interval-record and the chronograph methods shows that equivalent results can be expected with either method.* The essential differences are:

1. A 20-pen recorder was used to obtain a linear type† of record (interval-record) rather than the drum type† of record (chronograph).

2. Results obtained from the 20-pen recorder had to be replotted for ease of interpretation.

It was convenient in replotting the results to present them in the form of the drum-type of record (Figs. 1-5). The chronograph method can make use of either continuous or sampled recordings of the timepiece timing marks. The merits of using sampled recordings are discussed in Appendix C.

In the chronograph method samples are taken of the timepiece timing marks at regul (stylus-driving or x-axis) intervals; the positions of the timing marks (y-axis) on a recoı with a specified scale factor Y are recorded in each sample. The daily record may be reduced to a trend line‡ which, for the purpose of this report at least, can be taken to be a straight line, and which can be used as an aid in interpreting the recorded data.** The mathematical development (Appendix A) reveals that there are two approaches to the rate measuring problem. On the one hand, if the maximum rate which is expected to be meas ured is known by some means in advance, it is possible to select the marking interval or the frequency of recording timepiece timing marks and the record scale factor in such a way that unambiguous results are obtainable. On the other hand if a very wide range of

*For convenience, the methods which are the subject of this report will be referred to as the chronograph method, except where it is necessary to consider the two methods individually.

†See Appendix A

‡See Appendixes A and D; note that this tends to indicate the possibility that Case 2 of Part II describes the manner in which the period of oscillation changes (i.e., the period tends to remain constant until some climatic change induces a change in the period of oscillation).

**In Appendix D it is shown that two lines are required to show the trend, the separation between these lines being determined by the measuring errors, see Eqs. (D12) and (D13) of Appendix D.

rates is to be encountered, the maximum value of which is not known in advance, a correct interpretation of the record requires the value of the rate be determined approximately by another means (e.g., the visual method; see Part III) so that the true rate value can be obtained with greater accuracy from the record.

In the first case, where the maximum expected value of the rate R is known in advance it is possible to select* the marking interval to be

$$\frac{T}{C_\ell} < \frac{TY}{2|R|}, \tag{3}$$

where T is the measuring interval (86,400 seconds for daily rate measurements), C_ℓ is the limiting number of samples or intervals for recording timing marks, Y is the scale factor in seconds (the greatest value along the y-axis of the record), and R is the daily rate in seconds per day (spd).†

The choice of the scale factor Y may be based upon the standard timing mark interval or upon the interval between timepiece timing marks; the former was the case for the experimental results being presented in this report. For example, if it had been known in advance that $|R| = 40$ spd was the maximum value of rate to be expected (from Part IV), then with a value of $Y = 1$ second, $TY/2|R| = 1080$ seconds or 18 minutes. Hence, a marking interval of 15 minutes should have been chosen, to provide some safety factor. As it happened, a marking interval of 30 minutes was used, indicating that for $|R| > 24$ spd an ambiguous record is to be expected.

In the second case, perhaps the more general one, in which there is no advance knowledge of the maximum expected value of R, several interesting things have been found in Appendix A concerning the record. So long as the value of R can be estimated by another method‡ the resulting record can be interpreted correctly. First, it must be noted that once the scale factor Y and the marking interval from which the number of samples C in the interval T have been selected, the rate value R_s which will be measured from the record has the limiting values

$$0 \le |R_s| \le \frac{CY}{2}, \tag{4}$$

where again all quantities have the units of time.** The quantity R_s is measured directly from the record; R, the actual rate value, is obtained by interpreting the record. There are two cases to be considered in obtaining the correct interpretation of the measured value; the possible true values of R must be divided into intervals dependent upon the limits of the measured value R_s given by Eq. (4), or in general, by

$$\frac{sCY}{2} \le |R| \le (s+1)\frac{CY}{2}, \tag{5}$$

where $s = 0, 1, 2, 3, \ldots$

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*See Appendix A.

†Note that all terms have the units of time; the interval T is implicit in the definition of R. See Appendixes A and G.

‡See Table B1, Appendix B, for accuracy requirements.

**See Appendix G.

The two distinct cases to be considered are

$$(1)\ R = \frac{sCY}{2} + R_s, \tag{6}$$

which applies for even values of s (0,2,4,...), and

$$(2)\ R = \frac{(s+1)CY}{2} - R_s, \tag{7}$$

which applies for odd values of s (1,3,5,...).

To obtain R from R_s, the approximate value of R obtained from the auxiliary method is used in Eq. (5) to determine the integral value of s and whether it is even or odd, after which Eq. (6) or (7) can be used. Use will be made of these three equations below in interpreting the actual results.

It is of interest to point out a measuring limitation. The lower limit of rate which can be detected, as shown in Appendix A, is

$$|R|_{min} \equiv E_R; \tag{8}$$

the values for the chronograph method are given in Table 4. Therefore, the limits give by Eq. (4) should be rewritten as

$$E_R \leq |R_s| \leq C\frac{Y}{2}. \tag{4a}$$

For values of R_s or R near the lower limit it may be difficult to determine the algebraic sign of the rate from the record.

It is interesting to note also that points recorded on a drum-type (chronograph) reco are plotted in either of two ways: (1) positive x and y values, or (2) negative x and posit y values depending on the direction of travel of the styli. The replotted points of the line type (i. e., from the 20-pen recorder) have signs as given in (1).

THE SENSITIVITY OF THE CHRONOGRAPH METHOD OF RATE MEASUREMENT

It may be recalled from Part III that the sensitivity S of a rate measuring method is defined as the smallest value of rate which can be detected in the absence of measurir errors and mistakes, and depends upon limitations inherent in that method.

The chronograph method provides a measure of elapsed time $\bar{y}$ between a standard timing mark and the particular timepiece timing mark. The difference between two valı of $\bar{y}$ is a measure of rate. In general,*

$$\Delta\bar{y} = \bar{y}_b - \bar{y}_a, \tag{9}$$

*See Appendix D.

which is equivalent to the measurements made from the record in the photo-record method (Part III).

The quantities from which the value of S is determined, using the nomenclature of Part III, are: (1) the actual variation σ_w in timing marks, attributed to individual timepiece mechanisms, and (2) the error σ_t(= 1.3 msec*) introduced by recording the standard timing marks. Thus, for a single measurement of the quantity $\bar{y}$, the expected distribution is defined by the standard deviation,

$$\sigma_s' = \sqrt{\sigma_w^2 + \sigma_t^2} \quad . \tag{10}$$

For the two measurements required to obtain a value of $\Delta\bar{y}$, actually representing a measurement of either R or r, the sensitivity is

$$S = \sigma_s' \sqrt{2} \; . \tag{11}$$

Since both the photo-record and interval-record methods employed exactly the same instrumentation, the same value of S given by Eq. (11) is expected for each method.

Because no experimental data are available for the chronograph method, a different value of σ_t, and hence of S, might be expected since different relays and somewhat different marking conditions existed; only the value of σ_t is affected. It is possible that such a difference might have been compensated for by adjusting the relays; therefore the sensitivity of the chronograph method should be considered to be of the same order of magnitude as that of the interval-record method.

In the chronograph method, as will be shown later, the over-all measuring error approaches the value of S more closely than do the values for the three methods described in Part III. Because of this approach of the actual measurement to the sensitivity it is worth considering other factors which affect the sensitivity, but which were not considered in the analysis of the previous methods.

The increased accuracy obtainable permits use of the data for other purposes than just the measurement of R. It may be desired, for example, to investigate the constancy of rate from the available timing mark samples. In such cases the distribution of values of r_i about $\bar{r}$ might be considered (see Appendix A); the values of r are the cumulative errors obtained during the sampling subintervals T/C. The average value $\bar{r}$ can be determined either from

$$\bar{r} = \frac{R}{C} \tag{12}$$

or from

$$\bar{r} = \frac{1}{C}\sum_{1}^{C} r_i, \tag{13}$$

in which C is the number of sampling subintervals. Now, the value of $\bar{r}$ as determined by Eq. (12) may be calculated with a sensitivity given by Eq. (11), but that determined by Eq. (13) is calculated with a sensitivity

$$S' = \sigma_s' \sqrt{C} \; , \tag{14}$$

which can be seen to be considerably poorer than that given by Eq. (11), or $S' > S$, particularly for a large value of C such as that actually used, i.e., C = 48.

*From Part III.

It is also possible to use the trend lines to obtain specific values of r_i. Such use requires a knowledge of the smallest detectable slope variation, which is effectively a sensitivity, denoted by S_m. To determine a value for S_m, it is necessary to consider som details of the actual recording of the timing marks.

In the sampled type of record the sample consists of a small number of timing marks. As shown in Appendixes A and C, the cumulative error e causes a spread of timing marks which may overlap in a given sample. In the absence of mistakes and measuring errors, this spread can be represented by the quantity E_h.* The effect of E_h (or $E_{\Delta T}$) must be found. Each sample in the measurement is represented by the mean value $\bar{y}$ of the sample, and hence involves only half of E_h. Because a small sample is used, the error $E_{\bar{y}}$ in y is not negligible and the total error is therefore the combination of the independent components, i.e.,

$$E_s = \sigma_s' \pm \frac{1}{2}E_h \pm E_{\bar{y}} \tag{15}$$

or, using the results of Appendix D ($E_{\bar{y}} = \sigma_s'/\sqrt{h}$),

$$E_s = \sqrt{\sigma_s'^2 + \frac{E_h^2}{4} + \frac{\sigma_s'^2}{h}}. \tag{16}$$

For the case of the experimental data being considered in this report, wherein $h = 4$,

$$E_s = \sqrt{1.25\,\sigma_s'^2 + \frac{E_h^2}{4}}. \tag{17}$$

In Table C1 of Appendix C, it is shown that for extreme values of R (see Part IV) E_h is negligible compared with σ_s'; therefore

$$E_s = \sigma_s' \sqrt{1.25} = 1.12\,\sigma_s'. \tag{18}$$

The value E_s must be used in determining the sensitivity involved in measurements of r or R, for which

$$S_r = E_s \sqrt{2} = 1.58\,\sigma_s' \tag{19}$$

Now in the event that it is desired to use the record to detect changes in R, as determine from changes in slope ($m = (C/T)\ r_i$, from Appendix A), each value of slope requires at least the effective measurement of two points, hence Δm or Δr has a sensitivity $S_m = S_r \surd$ or

$$S_m = 2E_s = 2.23\,\sigma_s'. \tag{20}$$

Pertinent values of σ_s' and of S obtained from Part III are given in Table 1. Included in Table 1 are the values of sensitivity which have been found applicable for the chronograph method. The tabulated values apply directly for the interval-record method and are considered to represent the order of magnitude to be expected of the chronograph method.

*For a continuous recording of timing marks, where the minimum detectable x-axis movement is ΔT, the spread is represented by $E_{\Delta T} > E_h$, as shown in Appendix C.

TABLE 1
Sensitivity Values of Interval-Record Method
(Approximate Sensitivity Values of the Chronograph Method)

Timepiece		σ_s' * (msec)	$S = \sigma_s' \sqrt{2}$ * (spd x 10^{-3})	$S' = \sigma_s' \sqrt{C}$† (spd x 10^{-3})	$S_r = E_s \sqrt{2}$‡ (spd x 10^{-3})	$S_m = 2E_S$** (spd x 10^{-3})
Type	Ser. No.					
MC	4746	<1.8	<2.6	18	2.8	4.0
NW††	3753	13.4	19.0	132	21.2	29.5
NW††	6033	4.1	5.9	41	6.5	9.2
NW††	6979	48.6	68.8	477	76.8	108.3

*See Eq. (10) and (11); values from Part III
†from Eq. (14), C = 48
‡from Eq. (19)
**from Eq. (20)
††contain break-circuit contacts operating at 1-sec intervals, except 59th second every minute

THE OVER-ALL ERRORS OF MEASUREMENT OF THE CHRONOGRAPH METHOD OF RATE MEASUREMENT

Because no measurements were obtained with the chronograph, the following analysis of errors will hold for the interval-record and chronograph methods, but numerical values which apply for the former method are only approximate for the latter. While two values of $\bar{y}$ are required to obtain the daily rate, the errors associated with a single measurement of $\bar{y}$ will be considered first. It is found that in determining the over-all error it is appropriate to consider separately the errors associated with recording the timing marks (y-axis errors) E_y and those of the driving or x-axis, E_x.

The y-axis errors E_y include those errors previously considered in Part III and errors associated with the sampling process. The errors previously considered are: (1) the variation σ_w attributable to the timepiece itself (this quantity has a different value for each timepiece),* (2) the error introduced in recording the timing marks of each timepiece (this quantity is taken to be the same for each timepiece and is considered to have the magnitude of the variation σ_t obtained in recording the timing marks of the standard, WWV), (3) the variation σ_t obtained in recording the standard timing marks which are used as a reference for the measurement of the quantity $\bar{y}$,† and (4) the error introduced in the actual measurement of the quantity $\bar{y}$; as in Part III the symbol σ_m will be assigned.

These four errors represent those which were considered in analyzing the photo-record method (Part III), and which still apply to the chronograph method. The sampling

*The reasons for this are beyond the scope of this report.

†Unlike the photo-record method described in Part III, in which it was possible to measure $\bar{y}$ in either direction with respect to the reference (standard timing marks), the chronograph method makes use of a measurement of $\bar{y}$ in only one direction, as determined by the timepiece timing mark which occurs after the standard timing mark.

process introduces two additional errors which must be considered, E_h or $E_{\Delta T}$ (Appendix C) and $E_{\bar{y}}$ (Appendix D). The error E_h for sampled data is fixed by the number of marks involved in the recording process, which from Appendixes A or C is

$$E_h = \frac{1}{2}\sum_{1}^{h} e_j = \frac{1}{2} h\bar{e}, \tag{21}$$

where h is the number of marks in the sample ($h = 4$ for the experimental data being considered). The error E_h is independent of and does not include the effect of instrumentatio or measuring errors. The error $E_{\bar{y}}$ is encountered because it is expected that the averag ordinate on the record $\bar{y}$ will be used to represent the sample; $E_{\bar{y}}$ depends on the expected distribution of recorded timing marks (represented by σ) and the number h of marks used in the sample, or, from Appendix D

$$E_{\bar{y}} = \frac{\sigma}{\sqrt{h}}.$$

Thus, the uses of the record being considered in this report will involve these six errors. The first four errors, which are independent, may be combined statistically to give (as in Part III), for a single measurement of $\bar{y}$,

$$\sigma_{er}' = \sqrt{\sigma_w^2 + 2\sigma_t^2 + \sigma_m^2}. \tag{22}$$

The standard deviation σ_{er}' defines the combined distribution of errors which are involved in the determination of $E_{\bar{y}}$, or

$$E_{\bar{y}} = \frac{\sigma_{er}'}{\sqrt{h}}. \tag{23}$$

While the magnitude of $E_{\bar{y}}$ is determined by the distribution defined by σ_{er}', any particula sample will have specific values of σ_{er}' and $E_{\bar{y}}$ which are independent of each other. Hen

$$E_y' = \sqrt{\sigma_{er}'^2 + E_{\bar{y}}^2},$$

or

$$E_y' = \sigma_{er}'\sqrt{1 + \frac{1}{h}}. \tag{24}$$

The total y-axis error for a single measurement of $\bar{y}$ must include the fixed error E_h and the random error E_y', or

$$E_y'' = E_h \pm E_y', \tag{25}$$

for which a decision must be made concerning the range of the random error to be includ it may be desirable to increase the factor E_y' two or three times, to include a greater pe centage of the errors. Thus, Eq. (25) might be more appropriately written as

$$E_{yt} = E_h \pm 2E_y' \tag{26a}$$

or

$$E_{yt}' = E_h \pm 3E_y'. \tag{26b}$$

The use of the recorded data, as indicated in discussing sensitivity, affects the over-all error; there are two such uses which will be considered separately: (1) the measurement of R from the record, and (2) the measurement of changes ΔR in R.

For measuring R, two values of $\bar{y}$ are involved, or

$$\Delta\bar{y} = \bar{y}_b - \bar{y}_a. \tag{27}$$

Each of the values of $\bar{y}$ has associated with it a value of E_y'', Eq. (25), or E_{yt}, Eq. (26), consisting of the fixed error E_h and the random error E_y'. Since the difference between two values of $\bar{y}$ is obtained, each value of which has nearly the same value E_h and probably different values of E_y', the two values of E_h tend to cancel each other, or the difference between values of E_h can be considered to be negligible, since the difference comes from the values of e involved (see Table C1, Appendix C). Thus, for rate measurements alone,

$$E_y = E_y' \sqrt{2}. \tag{28}$$

For the experimental data being considered later, from Eq. (24),

$$E_y = 1.58\,\sigma_{er}'. \tag{29}$$

The terms of Eq. (29), obtained from Part III, are given in Table 2.

TABLE 2
The y-Axis Measuring Errors E_y for the Interval-Record Method
(Approximate Over-All Measuring Errors of the Chronograph Method)

Timepiece		σ_w	σ_{er}'*	E_y†
Type	Ser. No.	(μsec)	(μsec)	(spd x 10^{-3})
MC	4746	1.3	2.3	3.6
NW‡	3753	13.3	13.4	21.2
NW‡	6033	3.9	4.3	6.8
NW‡	6979	48.6	48.6	76.8

*from Eq. (22), same as Part III; σ_t = 1.3 msec and σ_m = 0.5 msec from Part III

†from Eq. (29)

‡contain break-circuit contacts operating at 1-sec intervals, except 59th second every minute

If reasonable care is taken in using the chronograph method, mistakes are not likely to occur;* this presupposes that approximate values of R will be obtained by other suitable means. Therefore, Eq. (28) gives the over-all y-axis measuring error for the chronograph method.

*See the discussion in Part III in which it is established that all measured values of spacings already include some possible instrumental mistakes which may have been made. Hence, the value σ_w already contains some effect of possible mistakes; therefore E_R combines all known factors involved in rate measurements.

The values of the terms of Eq. (28) are obtained largely from Part III, insofar as they apply. For the interval-record method, values of σ_w and σ_t from Part III apply directly. The value of σ_m is somewhat indeterminate because of the process used in transferring the data from the 20-pen records to the form used herein (Figs. 1-5). It is estimated that the value of σ_m is of the same order of magnitude as that used in Part III (σ_m = 0.5 msec), hence, this value will be used. For the chronograph method, somewhat different values of σ_t and σ_m may be expected; reasons for the difference in the value of σ_t have been given above.† It is expected that, in practice, the graph paper which was chosen for use with the chronograph would have been used in the measurement of $\bar{y}$; the errors thus introduced would determine the value of σ_m. However, it is possible that either or both of these methods could utilize the measuring system described in Part III, and hence that value of σ_m could be used. Thus it is likely that both methods would have values of E_R, which woul be of the same order of magnitude.

Let us obtain the over-all measuring error E_R which includes both the y- and x-axis errors; E_y, for rate measurements, is obtained from Eq. (28); E_x, from Appendix E. Since these errors are independent, their combination gives

$$E_R = \sqrt{E_y^2 + E_x^2}\ . \qquad (30)$$

From Eq. (28) and Appendix E

$$E_R = \sqrt{2E_y'^2 + \left(\frac{\sigma_f}{f} R\right)^2} \qquad (31)$$

which for the experimental data and the source of power actually used becomes

$$\left.\begin{aligned} E_R &= \sqrt{2.5\ \sigma_{er'}^2 + (4.17 \times 10^{-3}\ R)^2} \\ &= \sqrt{2.5\ \sigma_{er'}^2 + (17 \times 10^{-6})R^2} \end{aligned}\right\}. \qquad (32)$$

While the value of E_R depends on the value of R being measured, it can be approximated in two stages: (1) find the average value of R from the record, and (2) take the approximate value of

$$R_1 = \bar{R} \pm E_y\ , \qquad (33)$$

using the largest numerical value for calculating E_x (=4.17 x $10^{-3}|R_1|$), thus obtaining, a pessimistic value of E_R.

The second use of the data, in the measurement of changes ΔR in the rate, involves the measurement of at least two rate values and the use of the trend lines to aid in detecting the difference. The trend lines serve also to indicate the constancy of rate in any interval greater than the sampling interval. The difference ΔR can be obtained either from actual measured values of R or from measurement of the slope of the trend lines, each leading to the same result. In general, when comparing two rate values, the values

*This applies to the measurements made with the vernier calipers.

†See p. 5.

will be derived from unequal numbers (C') of sampling intervals, hence, it can be expected that each rate value has a different value of E_R. The over-all error in measuring ΔR is, therefore,

$$E_{\Delta R} = \sqrt{E_{R_1}^2 + E_{R_2}^2}\ , \tag{34}$$

or

$$E_{\Delta R} = \sqrt{2E_y^2 + E_{x_1}^2 + E_{x_2}^2}\ , \tag{35}$$

where

$$E_{R_1} = \sqrt{E_y^2 + E_{x_1}^2}$$

and

$$E_{R_2} = \sqrt{E_y^2 + E_{x_2}^2}\ .$$

In this case an approximation similar to Eq. (33) must be made first of the value of R (i.e., R_1); after which an approximation of the value of E_x can be obtained. Since trend lines or segments are being considered which do not in themselves give daily rate values, it is appropriate to use, from Appendix E, the quantity E_x' rather than E_x, or

$$E_x' = \frac{\sigma_f}{\overline{f}} R' = \frac{\sigma_f}{\overline{f}} \frac{C'}{C} R, \tag{36}$$

where $\sigma_f/\overline{f}$ is the relative frequency error of the power source, C' is the number of sampling intervals (T/C) used in drawing a particular line or segment, C is the total number of sampling intervals in the interval T, and R' is the total error obtained in the interval being considered and will be approximated by the value R_1.

Since Eq. (36) is intended to permit comparing rate values obtained from the record for intervals less than 24 hours (or, in general, less than T) but greater than T/C, it is useful to express R' in terms of the standard interval T/C:

$$\overline{r} = \frac{R'}{C'}\ . \tag{37}$$

Because $C\overline{r} = \overline{R}$,* it can be shown that

$$\Delta\overline{r} = \frac{\Delta\overline{R}}{C}\ . \tag{38}$$

Hence, comparisons can be made of rate values obtained for different values of C', and can be expressed readily in terms of $\overline{R}$ for ease in grasping directly the effect of specific rate changes in terms of daily rates.

*See Appendix A.

It must be kept in mind that the specific rate values thus obtained are subject to the error E'_x given by Eq. (36) which must be used in Eq. (35) to obtain appropriate values of E_{R_1}, E_{R_2}, and $E_{\Delta R}$.

EXPERIMENTAL RESULTS

The significant results which were obtained using the interval-record method are presented in a form similar to that which would have been obtained had the chronograph been available in its final form. These results were obtained during a continuous interval of 42-1/2 hours in which samples of timepiece and standard timing marks were recorded every half hour. Four complete, successive, actual records from the 20-pen recorder are shown in Fig. A2 of Appendix A. The variations in the length of record paper indicate the variation in starting and stopping the chart drive motor. In interpreting these records measurements were confined to the last three or four seconds of each sample. By eliminating the first second, allowance is made for the 20-pen record paper driving speed to stabilize, thereby reducing the chance of excessive deviations.*

Because the records shown in Fig. A2 of Appendix A are unwieldy if used directly, the timing marks for each timepiece were transcribed individually into the form shown in Figs. 1-5. The transcription of data required alignment of reference times for replotting individual timing marks; this is equivalent to measuring the quantity y (i.e., y_1, y_2, y_3, and y_4) from each second of the sample.

The records thus derived for each timepiece are used to obtain the following information: (1) the value of R measured in accordance with the method developed in Appendix A; (2) an indication, with the aid of the trend lines, of the constancy of rate throughout the recording interval; and (3) the measurement of a change in rate, in particular that caused by a change in climatic conditions during the interval of measurement. The climatic conditions under which the data were obtained were controlled during the 42-1/2 hours use of this rate measuring method. A change in temperature occurred during the measuring interval, and its effect on the rate will be shown.

It is helpful in interpreting the records to draw trend lines of the recorded timing marks. It is found in Appendix C that two lines† are required so that limits can be established within which a prescribed percentage of the actual recorded timing marks for complete trend lines or segments can be expected to occur. These two lines, then, represent upper and lower limits which encompass the majority of intermediate recorded marks. Thus, if approximately 95 percent of the timing marks are confined within the limits set by the trend lines, the rate of the timepiece can be said to be constant within the limits of error determined by drawing the lines. Because the over-all error involves the rate magnitude being measured, it is necessary to draw trend lines in two stages. In the first stage, that in which the approximate value R_1 is obtained using Eq. (33), particularly where $D > 1$, it will be found useful to draw the approximate trend lines using only the y-axis variations. In the second stage the value R_1, E_x or E'_x may be calculated, giving approximately the total variation E_R or $E_{\Delta R}$.

The approximate trend lines, from Appendix D, can be drawn between the following points:

$$\text{upper limit,} \quad x_a,\ \bar{y}_a + E_{yt};\quad x_b,\ \bar{y}_b + E_{yt} \qquad (39)$$

*Readings were rejected on the basis of a lack of constancy in the spacings of either or b recorded standard timing marks.

†In Appendixes A and C, the use of linear trend lines has been assumed. Under constan climatic conditions it is expected that the rate should remain constant; therefore, linea trend lines are to be expected.

and

$$\text{lower limit,} \quad x_a,\ \bar{y}_a - E_{yt};\quad x_b,\ \bar{y}_b - E_{yt}, \tag{40}$$

where

$$\bar{y} = \frac{1}{4}(y_1 + y_2 + y_3 + y_4) \tag{41}$$

which is the average of the four (or less) marks used to represent the entire sample obtained in any particular marking subinterval (T/C), and E_{yt} includes twice the standard deviation, given by Eq. (26),* thus, establishing the percentage of recorded marks which can be expected to fall between the trend lines (approximately 95 percent) in the absence of x-axis errors.

After the points through which the approximate trend lines are to be drawn have been obtained, an interpretation of the record is possible if the method of Appendix A is used. Thus, the average daily rate $\bar{R}$ (see Figs. 1a-5a) and an estimated value of R_1 can be obtained by using Eq. (33). Then it is possible to obtain, by including the x-axis errors, the estimated over-all error of $\bar{R}$.

The experimental data for four timepieces obtained by replotting the actual recorded data of the interval-record method, are given in Figs. 1-5 inclusive. Five figures are necessary because the data for one of the timepieces (NW 6979), unless interpreted properly, gives an ambiguous result when the scale factor $Y = 1$ sec (Fig. 4b). The data for NW 6979 serves to illustrate a case of possible ambiguous rate value interpretations as treated in Appendix A; for this particular timepiece two scale factors are used, $Y = 1$ sec (Figs. 4a and 4b) and $Y = 2$ sec (Figs. 5a and 5b), to illustrate the difference between the two possible interpretations. Figures 1b-5b show the data, for the four timepieces, on which the approximate trend lines have been drawn; for drawing the actual trend lines, points on the various line segments were used. The actual construction procedure of these approximate trend lines was: (1) to establish the point $\bar{y}$, from Eq. (41), (2) to calculate the limit E_{yt} using Eq. (26a), (3) to plot the limits given by Eqs. (39) and (40), and (4) to connect the points.

The specific values of E_{yt} obtained for the timepieces used are given in Table 3. These values aid in obtaining $\bar{R}$ for each of the timepieces. The value of $\bar{R}$ given in Figs. 1b-5b and also in Table 4, and the value of E_y' (from Table 3), permit $|R_1|$ to be calculated using Eq. (33). Next, values of E_x may be calculated by using the results of Appendix E, where it has been shown that for obtaining errors affecting daily rates,

$$E_x = 4.2 \times 10^{-3}\ R_1\ . \tag{42}$$

Finally, estimated values of E_R (the over-all error in R) are obtained from Eq. (32). The results of calculations are given in Table 4.

The values of E_R having been obtained for each of the timepieces, it is appropriate to see how close an estimate has been made by using the approximate trend lines. It will be recalled that these approximate lines were drawn in order to determine a value of E_x and that only the y-axis errors were used in their construction. Thus, the x-axis errors must be included—by considering that each point of the line has associated with it the fixed error E_h and the random errors E_y' and E_x'. The error E_x' is used, rather than E_x as pointed out in Appendix E, because line segments having C' instead of C ($C' < C$) sampling intervals are involved. The total variation becomes

$$E_R'' = E_h \pm \sqrt{E_y'^2 + E_x'^2}\ , \tag{43}$$

*It is necessary to use Eq. (26a) because each of the points through which the approximate lines are to pass must be considered separately and Eq. (26a) gives the y-axis errors for this case.

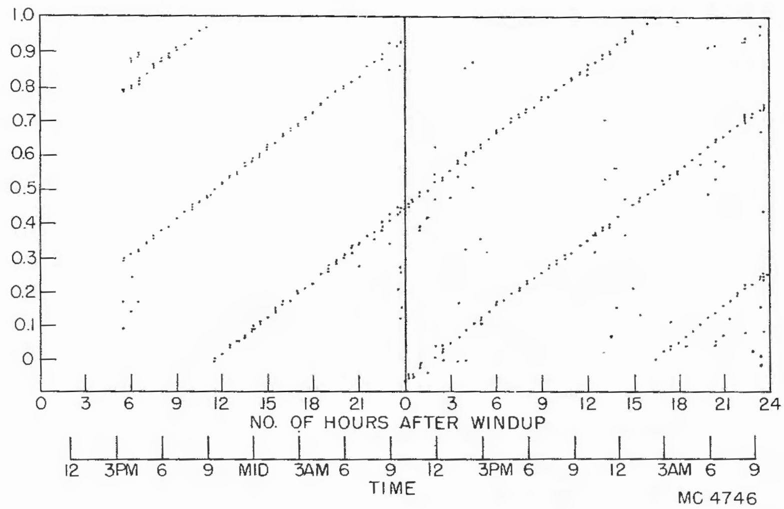

Fig. 1a - Replotted interval record data for MC 4746. (This shows the appearance of records as if obtained directly from a chronograph for $Y = 1$ second. Note that the record is for two days and that both ticks are recorded, hence the double set of data. The timing marks were obtained by using a condenser microphone and preamplifier.)

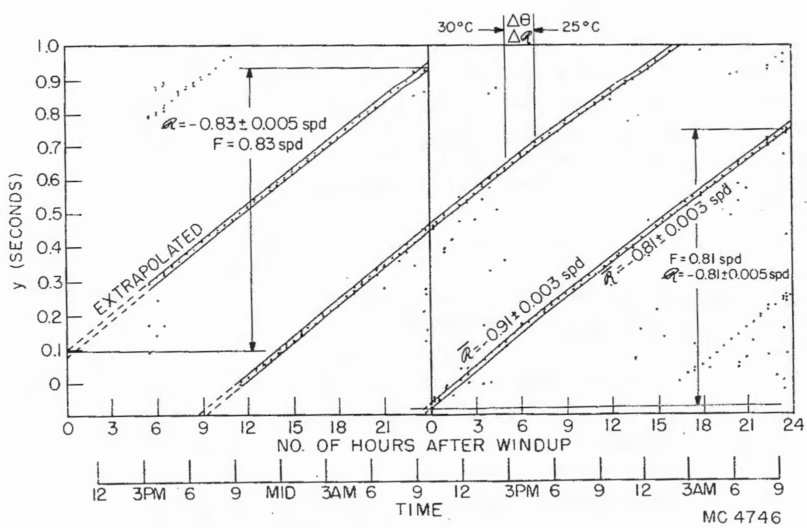

Fig. 1b - Interpreted record of the chronograph method of rate measurement for MC 4746 (values of E_{yt} from Table 3)

TABLE 3
Values of Estimated Errors Required for the Drawing of Approximate Trend Lines for Chronograph Records

Timepiece		E_h*	E'_y†	$2E'_y$	E_{yt}‡		Fig. No.
Type	Ser. No.	(sec)	(sec)	(sec)	(sec)	(in.)**	
MC	4746	0.0002	0.0026	0.0052	0.005	0.022	1
NW	3753	0.0004	0.0150	0.0300	0.030	0.134	2
NW	6033	0.0004	0.0048	0.0096	0.010	0.045	3
NW	6979	0.0004	0.0542	0.1084	0.110	0.493	4, 5

*from Table C1, Appendix C; h = 4 }

†from Eq. (24) and data of Table 2 } $\sqrt{1 + \frac{1}{h}} = 1.12$, for h = 4

‡from Eq. (26a)

**obtained by use of linear temporal equivalent of Part III, 1 sec = 4.484 in, for use in plotting points on records

TABLE 4
Estimated Over-All Errors for Daily Rate Measurements with the Chronograph Method

Timepiece		$\lvert\bar{R}\rvert$*	$\lvert R_1\rvert$†	E_x‡	E_R**	Remarks	
Type	Ser. No.	(spd)	(spd)	(spd)	(spd)		
MC	4746	0.83 0.81	0.834 0.814	0.0035 0.0034	0.005	1st day 2nd day	Fig. 1b
NW	3753	0.27 2.38	0.291 2.401	0.0012 0.0100	0.021 0.023	1st day 2nd day	Fig. 2b
NW	6033	3.18 3.10	3.187 3.107	0.0134 0.0129	0.015	1st day 2nd day	Fig. 3b
NW	6979	39.16 39.13	39.237 39.207	0.165 0.165	0.182		Fig. 4b Fig. 5b

*from Figs. 1b - 5b

†from Eq. (33), values of E'_y from Table 2

$\lvert R_1\rvert = \lvert\bar{R}\rvert + \lvert E_y\rvert$, i.e., $\lvert\bar{R} + E_y\rvert$ for positive $\bar{R}$

and $\lvert\bar{R} - E_y\rvert$ for negative $\bar{R}$.

‡from Eq. (40) [for daily rate values]

**from Eq. (32)

and, for constructing the trend lines so as to include approximately 95 percent of the timing marks and the maximum magnitude, the quantity to be used is

$$E_{Rt} = E_h \pm 2\sqrt{E_y'^2 + E_x'^2}\,. \tag{44}$$

Therefore, the trend lines should have been drawn between the following points:

$$\text{upper limit,}\quad x_a,\ \bar{y}_a + E_{Rt};\quad x_b,\ \bar{y}_b + E_{Rt} \tag{45}$$

$$\text{lower limit,}\quad x_a,\ \bar{y}_a - E_{Rt};\quad x_b,\ \bar{y}_b - E_{Rt}. \tag{46}$$

Values of E_{Rt} have been calculated and are compared in Table 5 with values of E_{yt} used in constructing the approximate trend lines. The values of E_{Rt} have been obtained from measurements taken from Figs. 1b, 2b, 3b, and 4b; these measurements and other data contained in Table 6 serve to illustrate the method of calculation. It can be seen from Table 6 that the approximate lines are not noticeably affected by disregarding the x-axis errors. It should be noted from Table 6 that Fig. 4b rather than Fig. 5b was used to obtain data for NW 6979; it was felt that greater numbers of points would be available for at least some of the lines.

TABLE 5
Comparison of Error Values Used in Construction of Actual and Approximate Trend Lines

Timepiece		E_x'*	E_y'†	$2\sqrt{E_x'^2 + E_y'^2}$*	E_h†	E_{Rt}‡	$2E_y'$†
Type	Ser. No.	(sec)	(sec)	(sec)	(sec)	(sec)	(sec)
MC	4746	0.00017 0.00169	0.0025	0.0050 0.0060	0.0002	0.0052 0.0062	0.005
NW	3753	0.00004 0.00170	0.0150	0.0300 0.0302	0.0004	0.0304 0.0306	0.030
NW	6033	0.00026 0.00121	0.0048	0.0096 0.0100		0.0100 0.0104	0.010
NW	6979	0.00008 0.00051	0.0542	0.1084		0.1088	0.110

*greatest and least values, from Table 6

†from Table 3

‡$E_{Rt} = E_h + 2\sqrt{E_x'^2 + E_y'^2}$

It seems appropriate to compare average daily rate values from the chronograph method with results given for the corresponding data groups in Part IV. This provides at the same time an opportunity to compare the over-all errors associated with the two rate measuring methods; see Table 7.

TABLE 6
Measured and Calculated Quantities Required in Obtaining Values of Over-All Errors for the Construction of Trend Lines

Timepiece	$\|R_1\|$* (sec per TC'/C)	C' †	$\bar{r}$ ‡ (sec per T/C)	$\bar{R}'$ ** (spd)	E'_x †† (sec)	E'_y ‡‡ (sec)	$2\sqrt{E_x'^2 + E_y'^2}$ (sec)	E_h ‡‡ (sec)	E_{Rt} (sec)
MC 4746 Fig. 1b	0.193	10	0.019	0.91	0.00017	0.0025	0.0050	0.0002	0.0052
	0.573	34	0.017	0.81	0.00169		0.0060		0.0062
NW 3753 Fig. 2b	0.405	6	0.065	3.12	0.00021	0.0150		0.0004	
	0.175	3	0.053	2.54	0.00004		0.0300		0.0304
	0.595	14	0.042	1.99	0.00072				
	1.090	18	0.055	2.62	0.00170		0.0302		0.0306
	0.155	3	0.047	2.24	0.00004		0.0300		0.0304
NW 6033 Fig. 3b	0.645	10	0.064	3.07	0.00056	0.0048	0.0096		0.0100
	0.800	11	0.072	3.48	0.00076		0.0098		0.0102
	1.000	14	0.068	3.28	0.00121		0.0100		0.0104
	0.425	7	0.060	2.88	0.00026		0.0096		0.0100
NW 6979 Fig. 4b	0.984	6	0.155	7.44	0.00051	0.0542	0.1084		0.1088
	1.034	5	0.196	9.40	0.00045				
	0.434	2	0.190	9.11	0.00008				
	0.904	3	0.284	13.60	0.00024				
	0.504	3	0.150	7.19	0.00013				
	0.864	5	0.162	7.77	0.00038				
	0.899	5	0.169	8.10	0.00039				
	0.934	5	0.176	8.44	0.00041				

*$|R_1| = |R'| + E'_y$ ‡‡ $\bar{r} = |R'|/C'$ †† from Eq. (36)

†measured from Figs. 1b, 2b, 3b, or 4b ** $\bar{R}' = \bar{r}C$ ‡‡ from Table 3

TABLE 7
Comparison of Chronograph and Photo-Record Methods of Rate Measurements

Data* Group No.	Temp.† (°C)	Average Daily Rate (spd) MC 4746	NW 3753	NW 6033	NW 6979	Rate Measuring Method
19	30±1	0±0.33	+0.8±0.81	+2.6±0.81	-39.2±0.81	photo-record
-		-0.83±0.005	+0.27±0.021	+3.18±0.015	+39.16±0.182‡ +39.13±0.182	interval-record**
20	25±1	+0.5±0.33	+2.2±0.81	+3.2±0.81	-39.6±0.81	photo-record
-		-0.81±0.005	+2.38±0.023	+3.10±0.015	‡	interval-record**

*data obtained from Part IV ($\bar{R}_{pr} \pm E'_p$)
†temperature change occurred as indicated in Figs. 1b, 2b, 3b; pressure, 750 ±0.5 mm Hg
‡because of incomplete recording, direct daily comparison is not possible; instead the values of $\bar{R}$ have been obtained from Figs. 4b and 5b for the 24-hour interval shown.
**data from Table 4 ($\bar{R}_{ir} \pm E_R$)

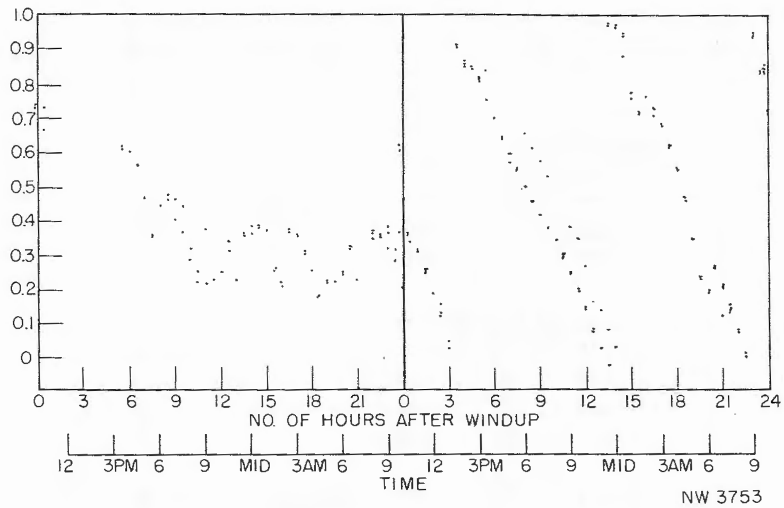

Fig. 2a - Replotted interval record data for NW 3753. (Note that the record in which Y = 1 second is for two days. The initial set of marks were obtained from daily record of the photo-record method. The timing marks were obtained from the break-circuit contacts of the timepieces.)

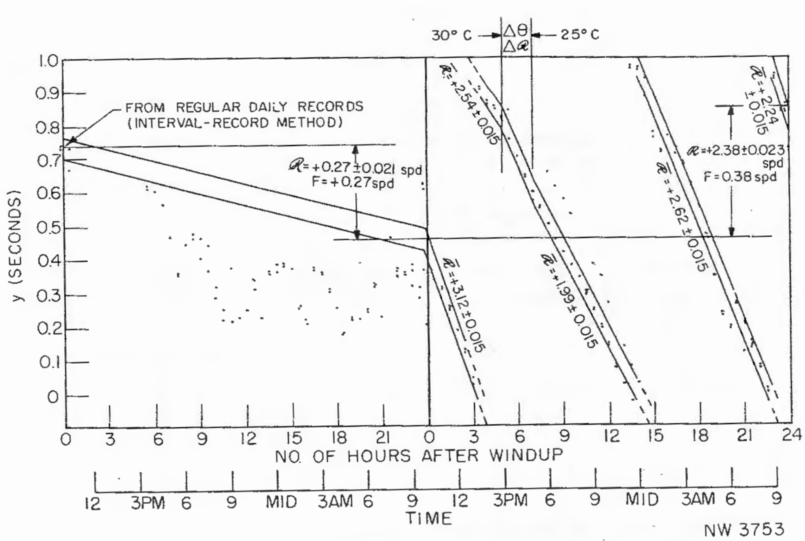

Fig. 2b - Interpreted record of the chronograph method of rate measurement for NW 3753 (values of E_{yt} from Table 3)

A valid comparison between the two sets of data given in Table 7 requires that the rate value criterion of Part IV be used. When adapted for this comparison, this criterion is

$$|\bar{R}_{pr} - \bar{R}_{ir}| \leq 6\sqrt{E_p'^2 + E_R^2}\,, \tag{47}$$

where $\bar{R}_{pr}$ and $\bar{R}_{ir}$ are the rate values obtained using the photo-record and interval-record methods, respectively, and E_p' and E_R are the over-all measuring errors associated with these methods. All values given in Table 7 satisfy this criterion, thereby establishing the equivalence of the results obtained. The difference between values of $\bar{R}_{pr}$ and $\bar{R}_{ir}$ for MC 4746 should be noted. This is striking evidence of the effect of an improved measuring method on the results obtained in detecting constancy of rate by methods having large measuring errors.

The constancy of rate, the second use to which the data may be put, is indicated from the records by the number of timing marks which are not within the limits set by the trend lines. Thus, in general, for the rate to be considered constant for a given time interval, as suggested by Case 2 of Part II, all timing marks should be plotted between the two trend lines during that interval, and this establishes the limit within which constancy can be detected. For example, Figs. 1b-5b show the timing marks which do not fall between the two trend lines. It can be seen first from Figs. 1b, 4b, and 5b that for these two timepieces the timing marks generally fall between the trend lines. For the marine chronometer (Fig. 1b, a single-line record*), there are some timing marks which fall outside the limits set by the trend lines; but these may be attributed, it is felt, to the use of the microphone and its associated sensitive preamplifier as a tick transducer. For the marine chronometer, because only a single trend line is involved, it appears safe to conclude without measuring individual values of r_i that within the limits of measuring error, the rate remained constant, up to the time when the temperature change occurred.

Examination of Fig. 3b shows that NW 6033 departs slightly from constancy and exhibits what might be called erratic behavior. It should be noted that the departures from the trend lines appear to be as great or greater than the error represented by the trend lines, indicating that a significant change in rate occurs during the daily interval.

In the case of NW 3753 (Fig. 2b), it is significant to note that there is a wide departure from constancy of rate during the first of the two days. Any attempt to give an explanation requires the use of a far greater amount of data† than is presently available. However, it is significant to note how readily the chronograph method permits the measurement of departures from constancy. It is interesting to point out, furthermore, that on the second day a radically different record (for NW 6979) was obtained: there is a tendency toward constancy, but erratic behavior is still evident. By way of comparison, it may be recalled that in Part IV the data for NW 3753 were not used in presenting the rate-temperature characteristics because it was found that both the rate-value‡ and recovery** criteria were not fulfilled. Figure 2b is perhaps further justification of the value of the employment of such criteria.

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*There are more timing marks than should occur for a single-line record because both ticks were recorded each second.

†Sufficient data to compare the performance of CW and NW to indicate any effect introduced by the break-circuit contacts would be required.

‡See Part IV.

**See Part IV.

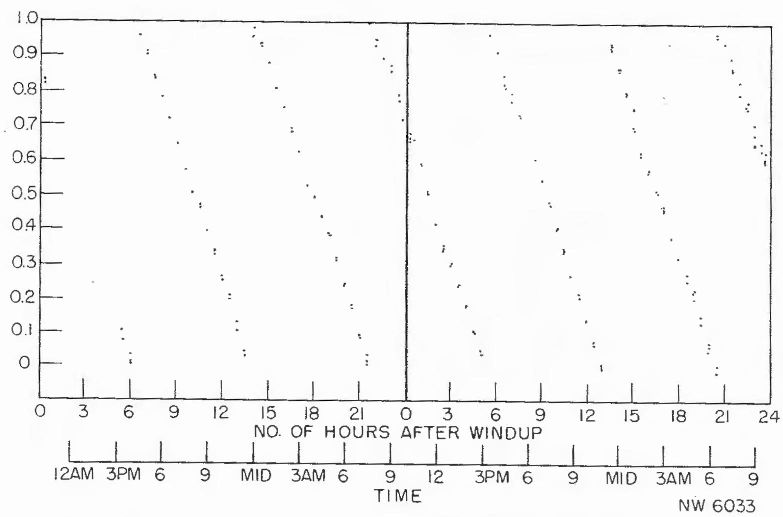


Fig. 3a - Replotted interval-record data for NW 6033. (Note that the record in which y = 1 second is for two days. The initial set of marks were obtained from daily record of the photo-record method. The timing marks were obtained from the break-circuit contacts of the timepieces.)

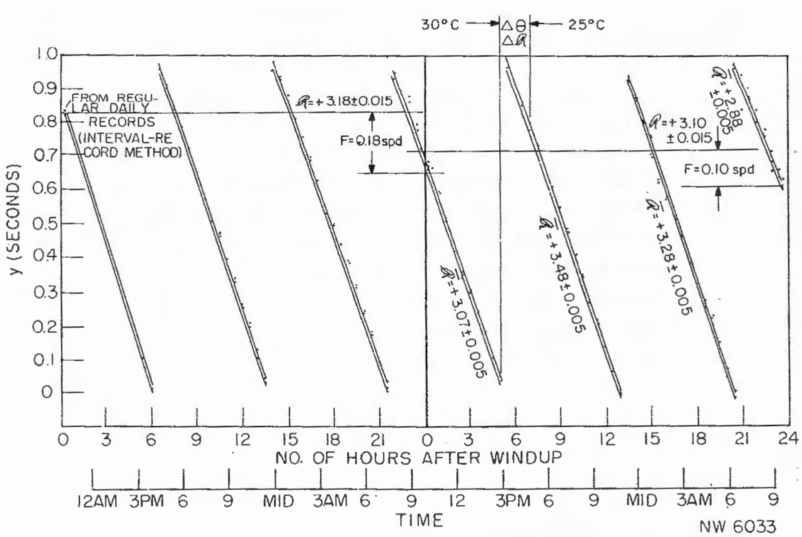


Fig. 3b - Interpreted record of the chronograph method of rate measurement for NW 6033 (values of E_{yt} from Table 3)

It must be pointed out that many of the timing marks of NW 6979 (Figs. 4b and 5b) were not recorded; the cause of this failure was not determined. However, from the available data, the recorded points do fall within the limits set by the trend lines. Since more than a single line segment is involved, a rate constancy criterion is required to indicate lack of constancy.

In the cases where $D > 1$ and the record has more than one trend line segment, the individual segments provide indication of constancy of the rate within that interval. It is necessary, however, that the segments be intercompared; for example, the slope may be used. Thus, for the case where $D > 1$ two conditions must be satisfied: (1) the timing marks must occur between the trend lines, and (2) the slopes of the individual segments must be constant; and these conditions are included in the approximate criterion,

$$|r_i - \bar{r}| \leq 6 \sqrt{E_{r_i}^2 + E_{\bar{r}}^2}. \tag{48}$$

which, as expected, is similar to the rate value criterion given by Eq. (47). Thus, regardless of the number of line segments, tests of constancy may be made by measuring specific values of r_i. This constancy criterion may be expressed in terms of the measured value of the average daily rate $\bar{R}$, and an equivalent daily rate $\bar{R}'$ obtained by measuring any specific segment or portion of a line, as, approximately,

$$|\bar{R}' - \bar{R}| \leq 6 \sqrt{E_{r_i}^2 + E_{\bar{r}}^2}. \tag{49}$$

In Eq. (49)

$$\bar{R}' = C \frac{R'}{C'}, \tag{50}$$

where R' is the measured value of rate corresponding to C' sampling intervals, and $C' < C$. The fraction R'/C' gives r_i, which may be extrapolated to an equivalent daily rate value when multiplied by C to permit the comparison, Eq. (49), to be made in proper units. Furthermore, the errors associated with the values $\bar{R}'$ and $\bar{R}$ are, respectively

$$E_{r_i} = E_h \pm \sqrt{E_y'^2 + E_x'^2} \tag{51}$$

and

$$E_{\bar{r}} = E_R. \tag{52}$$

Hence, the rate constancy criterion can be written as

$$|\bar{R}' - \bar{R}| \leq E_h + 6 \sqrt{E_y'^2 + E_x'^2 + E_R^2}. \tag{53}$$

Consider the application of this criterion, which may be used in comparing the equivalent daily rate values represented by any trend line segment. Since it is advisable to keep the limits of this criterion as small as possible,* only two values need be considered. And during any daily interval only the cases involving the maximum and minimum values of $\bar{R}'$ need be considered. Note that no such comparison is necessary for timepieces having single line records, e.g., MC 4746 on both days, and NW 3753 on the first day. Also, the criterion does not have to be applied if it can be determined by visual

*More values tend to add error terms.

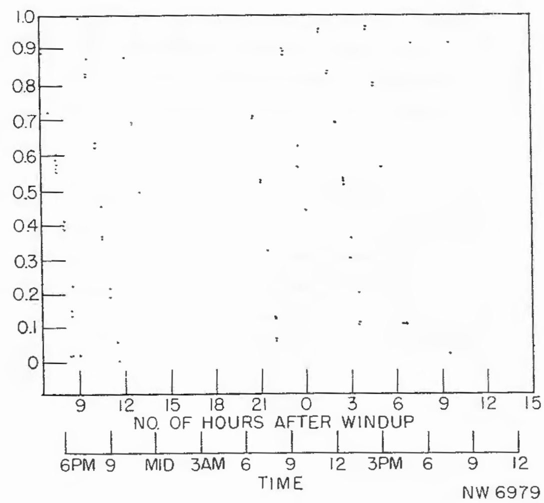


Fig. 4a - Replotted interval-record data for NW 6979. (Note that the record in which Y = 1 second is for portions of two days. The timing marks were obtained from the break-circuit contacts of the timepieces.)

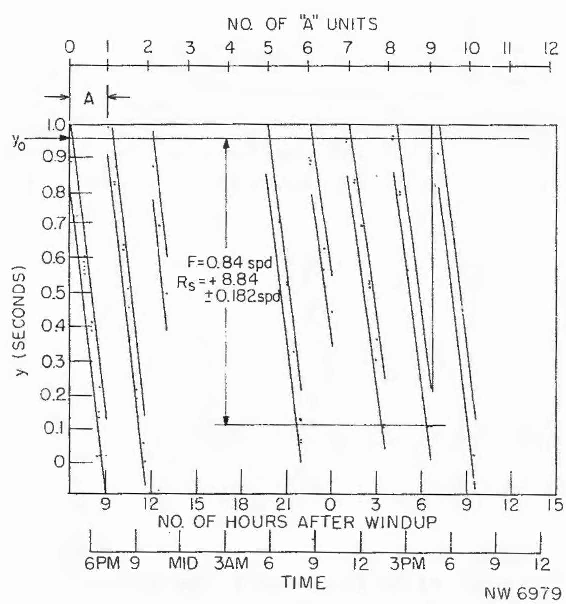
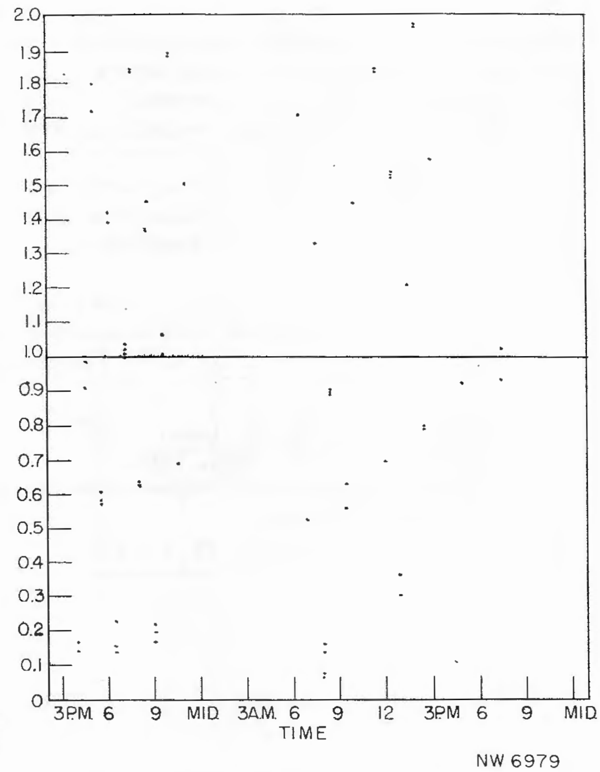


Fig. 4b - Interpreted record of the chronograph method of rate measurement for NW 6979. (This is an example of an ambiguous record: R_s is read, rather than R, if not properly interpreted. The values of E_{yt} are taken from Table 3.)

Fig. 5a - Replotted interval-record method data for NW 6979. (Same data as in Fig. 4a, but scale factor Y changed to 2 seconds.)



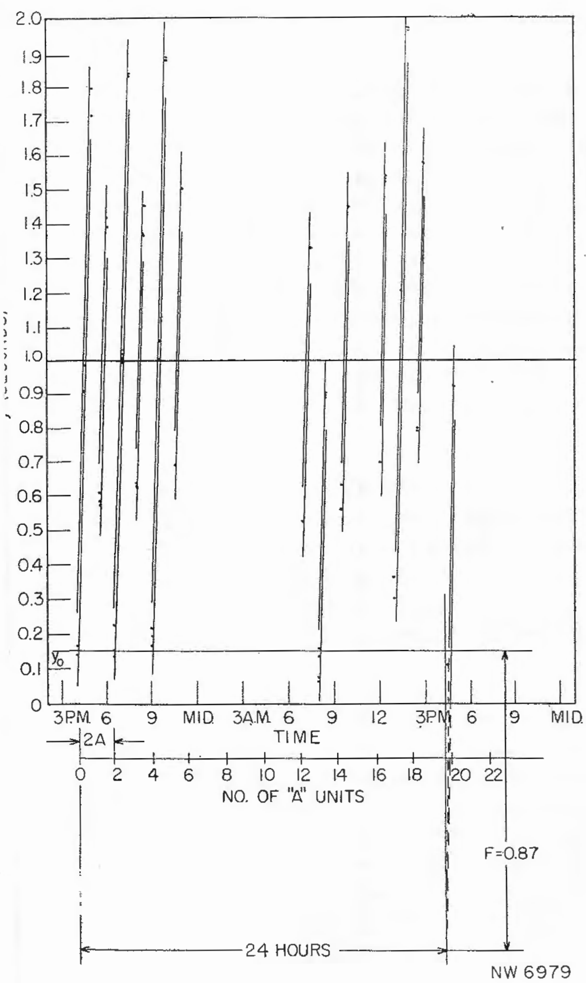


Fig. 5b - Interpreted record of the chronograph method of rate measurement for NW 6979. (Fig. 4b replotted Y = 2 seconds, to obtain unambiguous record. The record has been extended in order to measure R. The values of E_{yt} are taken from Table 3.)

examination of the record that the rate is not constant, e.g., NW 3753 or NW 6033. Furthermore, because of the temperature change which occurred on the second day, only the data for the first day are presented in Table 8. Since it has been determined already that NW 3753 and NW 6033 show erratic behavior, and MC 4746 does have constant rate, NW 6979 is the only timepiece for which, on the first day, the rate constancy criterion need be applied; the first five line segments were recorded before the temperature change occurred. The procedure of calculation is indicated in Table 8. It can be concluded that the rate of the timepiece was not constant.

TABLE 8
Application of Rate Constancy Criterion with Data of NW 6979 Taken from Fig. 4b

$\bar{R}'$* (spd)		$\bar{R}$† (spd)	$\lvert\bar{R}' - \bar{R}\rvert$ (spd)		$\sqrt{E_y'^2 + E_x'^2 + E_R^2}$ ‡ (spd)	$E_h + 6\sqrt{E_y'^2 + E_x'^2 + E_R^2}$ (spd)
Max	Min		Max	Min		
(13.60) -34.40	(7.19) -40.81	(8.84) -39.16	4.76	1.65	0.190	1.14

*data from Table 6

†data from Table 7

‡based on data from Tables 6 and 7

Consider now the third use of the data, that of measuring changes ΔR in rate which occur within a daily interval. In this case, while Eq. (30) still applies for calculating E_R from E_y' and E_x, Eq. (36) rather than Eq. (42) must be used to obtain E_x; hence, another value of E_R different from that given in Table 4 must be calculated, from which the value of $E_{\Delta R}$, using Eq. (34) can be obtained. To illustrate the procedure the over-all error associated with observed rate differences, e.g., that caused by a temperature change which occurred during the tests, will be calculated. Figures 1b, 2b, and 3b show the time interval being considered. For each of the timepieces represented by the three figures, in the record for the second day, there are three regions of the trend to be considered. In the first region, the initial temperature, +30 °C, exists. The second region shows the interval during which both the chamber and timepiece are adapting to the new temperature, a period of stabilization. The third period represents the performance at the second temperature, +25° C.

The determination of $E_{\Delta R}$ involves calculation of the individual values, E_{R_1} and E_{R_2} which are associated with the rate values whose difference is desired; values of E_R are given in Table 9, for each line segment of significance shown in Figs. 1b, 2b, and 3b (using the record for the second day only). In Fig. 1b both parts of the middle line were considered, leaving out the portion included within the indicated stabilization interval. In Figs. 2b and 3b, the line segments were considered individually, again leaving out the portion within the stabilization interval.

Because rate differences are of interest, the over-all error of the differences is obtained from the statistical combination of the values of E_R associated with the specific rate values involved. Of immediate concern is the rate difference obtained as a result of the temperature change indicated in Figs. 1b, 2b, and 3b. The record shown in Fig. 1b is a simple example because the record has a single trend line. The rate difference of interest is represented by the change in slope of the line segment just prior to the temperature change and that after stabilization has occurred; these two values are the only two obtained (see Fig. 1b and Table 6) for MC 4746. The value of $E_{\Delta R}$, obtained by using Eq. (34), is given in Table 10.

TABLE 9
The Individual Over-All Errors Involved in Determining Rate Values from Segments of Trend Lines

Timepiece	$E_R = \sqrt{E_y'^2 + E_x'^2}$* (sec)
MC 4746 Fig. 1b	0.0025 0.0030
NW 3753 Fig. 2b	0.0150† 0.0151 0.0150
NW 6033 Fig. 3b	0.0048 0.0049 0.0050 0.0048

*data from Table 6

†applies for the first 3 lines shown in Fig. 2b

TABLE 10
The Over-All Measuring Errors Associated with Rate Difference Values Obtained with the Chronograph Method

Timepiece		$E_{\Delta R}$ (spd)
Type	Ser. No.	
MC	4746	0.004
NW	3753	0.021
NW	6033	0.007

TABLE 11
Experimental Rate Differences Resulting from a Temperature Change

Timepiece		$\lvert \bar{R}_1 \rvert$ (+30°C) (spd)	$\lvert \bar{R}_2 \rvert$ (+25°C) (spd)	$\Delta R \pm E_{\Delta R}$ (spd)
Type	Ser. No.			
MC	4746	0.91	0.81	-0.10 ±0.004
NW	3753*	3.12	2.62	-0.50 ±0.021
NW	6033*	3.07	3.28	-0.21 ±0.007

*ignores lack of constancy of rate

In the case of the records shown in Figs. 2b and 3b, there are several line segments to be considered in obtaining the value of the rate difference. Certainly a pessimistic result, insofar as the over-all error is concerned, is obtained if the two greatest values of E_R (given in Table 9) are used. These results are given in Table 10.

The rate difference values obtained from the experimental records can be written

$$\Delta R = \bar{R}_2 - \bar{R}_1 \pm E_{\Delta R}. \qquad (54)$$

It should be remembered that $E_{\Delta R}$ is the standard deviation of a distribution of possible error values. This difference is obtained easily from Fig. 1b, since there are only two values. For the data represented by Figs. 2b and 3b, it is considered appropriate to use the two values associated with the longest line segments for the two particular climatic conditions, or the equivalent rate values obtained for the largest values of C′ (see Table 6). The results obtained by this procedure are given in Table 11.

It is significant to obtain rate differences as indicated particularly by the record shown in Fig. 1b because it permits the possibility of rapid investigation of rate variations, e.g., as a function of pressure and temperature. Some of the problems associated with such an investigation are considered in Appendix F where it is shown that the combined error associated with the temperature rate of change of rate (the partial derivative $\partial r/\partial\theta$) is estimated to be

$$E' = \frac{r_{\theta_2} - r_{\theta_1}}{\theta_2 - \theta_1} \sqrt{\frac{E_{\Delta R}^2}{\left(r_{\theta_2} - r_{\theta_1}\right)^2} + \frac{E_\theta^2}{\left(\theta_2 - \theta_1\right)^2}} \qquad (55)$$

where $\theta_2 - \theta_1$ (°C) is the temperature difference; $r_{\theta_2} - r_{\theta_1}$ (spd) the difference in rate values obtained at those temperatures; $E_{\Delta R}$ (spd) the value given in Table 11; and E_θ the variation associated with the control of temperature.

The rate of change of rate with respect to temperature may be written in incremental form as

$$\frac{\partial R}{\partial \theta} = \frac{\bar{R}_2 - \bar{R}_1}{\theta_2 - \theta_1} \pm E' = \frac{\Delta R}{\Delta \theta} \pm E'. \tag{56}$$

Using the value of $E_{\Delta R}$ given in Table 11 and from Part I: $3E_\theta = 1°C$, $E_\theta = 0.33°C$. The two temperature values are +30°C and +25°C for the experimental data being considered; and the corresponding rate differences are given in Table 11. Thus, average values of the derivative and estimated values of E' can be calculated and are found to be as follows:

Timepiece	$\frac{\Delta R}{\Delta \theta}$ (spd/°C)	E' (spd/°C)
MC 4746	0.02	0.002

In connection with the above tabulated data it should be noted that the calculated results for the two watches (NW) are not included as additional examples. Up to this point data for these watches have been included as a means of illustrating the procedures and differences to be expected for the several kinds of records obtained. But it will be recalled that these timepieces were shown above not to have a constant rate. Hence, it cannot be considered significant to use this data for the purpose of calculating rate derivatives.

It should be pointed out that the values of $E_{\Delta R}$ represent the error to be expected in obtaining the differences between any two rate values. It can be used to compare the rate values obtained by measuring the various segments in a multi-line record (such as those of Figs. 2b, 3b, 4b, or 5b). The comparison made above in Table 8 is an example.

SENSITIVITY AND OVER-ALL MEASURING ERROR OF FOUR RATE MEASURING METHODS

In conclusion, it is of interest to compare the sensitivities and measuring errors of all four methods of rate measurement considered in this series of reports. These results have been included (Table 12) with the previous results of Part III and it should be noted that sensitivities for certain uses of the chronograph method can be greater than those of the three previously considered methods. This is so because there are no comparable data for these additional uses for the previous methods. It is significant also to note the greater accuracy obtainable by the chronograph method for measuring either R (see E_R) or ΔR (see $E_R\sqrt{2}$).

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The suggestions made by S. Paull and C. V. Parker of the Security Systems Branch in their reading of the manuscripts of this series of reports are gratefully acknowledged. Early comments by T. I. Humphreys, formerly of the Security Systems Branch, also were helpful. Credit is due Miss G. C. Batchelder for her diligence in typing all the manuscripts.

TABLE 12
Sensitivities and Over-All Measuring Errors of Four Rate Measuring Methods

Method	Sensitivity (spd)								Over-All Measuring Error (spd)		
	$\sigma_s' \sqrt{2}$		$\sigma_s' \sqrt{C}$ *		$E_s \sqrt{2}$		$2E_s$				
	MC	CW	MC	CW	MC	CW	MC	CW	MC	CW	Symbol
Visual	0.50	0.20							0.35	0.36	E_v
Photo	0.50	0.20							0.11 to 0.33	0.79	E_p
Photo-Record	0.003	0.006 to 0.070							0.11 to 0.33	0.79 to 0.82	$E_p' = \sqrt{E_p^2 + E_R^2}$
Interval-Record or Chronograph	0.003	0.006 to 0.070	0.018	0.040 to 0.480	0.0028	0.006 to 0.080	0.004	0.009 to 0.108	0.005	0.015 to 0.182	E_R †
									0.004	0.007 to 0.021	$E_{\Delta R}$ ‡

*C = 48

†from Eq. (30), for measuring rate values (R), see Table 4 (for one timepiece)

‡from Eq. (34), for measuring differences in rate (ΔR), see Table 10 (for two timepieces)

* * *

APPENDIX A
Mathematical Description of the Interval-Record and Chronograph Methods of Rate Measurement

Measuring Methods and Equations of Rate

The three rate measuring methods already described and analyzed in Part III involve two determinations of the timepiece indications (i.e., readings), at the start and end of a 24-hour interval. The rate values $\mathcal{R}$ obtained by using these three methods can be represented, using the nomenclature of Part II, by the general equation

$$\mathcal{R} = E_N \pm e_r' , \tag{A1}$$

where E_N is the total error accumulated in the interval T, and the term e_r' arises from the "end-effects" which occur because the measured interval T cannot be determined exactly from actual timepiece indications. The term e_r' represents a theoretical limitation imposed by the timepiece period of oscillation and is independent of any measuring errors which may be present in these three methods of rate measurement. But the quantity e_r' may be shown to have a negligible effect on $\mathcal{R}$.* Thus, $\mathcal{R}$ is the quantity which it is desired to measure and E_N is its value in the absence of measuring errors or mistakes.

The magnitudes of the over-all measuring errors, including probable mistakes, were estimated for each of the three measuring methods. The results obtained and the distinction between the methods can be illustrated by writing the individual equations that hold. Let I denote a visual timepiece reading and P a photographic reading. Then

$$\mathcal{R} \equiv (I_f - I_i) - T \equiv \Delta I - T \tag{A2}$$

for the visual method,

$$\mathcal{R} \equiv (P_f - P_i) - T \equiv \Delta P - T \tag{A3}$$

for the photo method, and

$$\mathcal{R} \equiv (P_f + y_f) - (P_i + y_i) - T \equiv \Delta P + \Delta y - T \tag{A4}$$

for the photo-record method. The subscripts f and i are used to indicate final and initial readings in each case; Δ is used to indicate the difference obtained in readings. It should be noted that in each of these equations, the quantity T is not actually involved in the determination of daily rate† because the indications of timepieces are not totalized beyond 12 or 24 hours; the indicating system effectively having the capability of automatic zero-reset by virtue of the circular dials used.

In each of the foregoing equations $\mathcal{R}$ is measured approximately as E_N', with an accuracy E as estimated in Part III (a subscript is used to indicate the associated method); E_N' combined with E describes a distribution of measured values which has for its mean

*At least for the values encountered in this series of reports.

†$\mathcal{R}$ is the daily rate when T = 86,400 seconds.

value E_N'. Thus, E_N' is allowed to differ, because of inherent features of the measuring process, from E_N, the quantity which is being estimated by the measurement. Hence the equations may be rewritten as follows:

(visual method) $$E_N \equiv E_N' \pm E_v; \tag{A2a}$$

(photo method) $$E_N \equiv E_N' \pm E_p; \tag{A3a}$$

(photo-record method) $$E_N \equiv E_N' \pm E_p \pm E_R \tag{A4a}$$

or

$$E_N \equiv E_N' \pm E_p'; \tag{A4b}$$

where

$$E_p' = \sqrt{E_p^2 + E_R^2}. \tag{A5}$$

Actual values for the quantities E_v, E_p, E_R, and E_p' are tabulated in Table 12 in the main body of this report. It should be noted that E_p and E_p' are almost the same values, hence E_R is quite a small quantity ($E_R << E_p$).

These equations show that the difference to be expected in using each of the three methods arises from the difference in the values of E which actually exist when compared with the theoretically exact value of R expected. These equations are useful, in discussing the interval-record or chronograph method, to show the improvements to be expected.

Equations of the Interval-Record and the Chronograph Method*

An improvement in the accuracy of rate measurement is obtained by a reduction in the value of the over-all measuring error E. A further improvement can be expected if the end-effects can be eliminated from the measurement. It will be shown that the chronograph method permits this twofold improvement over the other three methods to be achieved because: (1) the interval T can be measured more exactly, and (2) the over-all measuring error is dependent upon E_R, rather than E_v, E_p, or E_p'.

To show how the improvement is possible it is necessary to point out the essential measuring differences existing between the chronograph method and the three previous methods. To make this comparison let us use the results of Part II, in which it is found that

$$\text{R} = Rt_s - T$$

or (A6)

$$\text{R} = Rt_s - N't_s,$$

where R (not an integer) represents the total number of timepiece periods (partial or complete) in the interval T, and N' (an integer) represents the number of nominal or standard periods in the interval T.

*The interval-record and chronograph methods are compared in Appendix B and their equivalence is established. The methods will be referred to herein simply as the chronograph method.

For the first three methods, the measuring process may be considered to be one in which two lengths (Rt_s and $N't_s$) are measured and compared, the unit of length being t_s. An uncertainty exists because the start and end of the interval T cannot be defined exactly in terms of events occurring within the timepiece; this uncertainty is included in the value $\mathcal{R}$.*

The improvements attained by using the chronograph method are as follows: (1) a more accurate measure of T in terms of timepiece events which permits an effective shift of the start of this interval to be coincident with the initial timepiece timing mark;* (2) a measure of the accumulation, which, from Part II, is

$$E_N = e_1 + e_2 + \ldots + e_N = \sum_1^N e_j, \tag{A7}$$

and (3) the rate value only depends on measurement of y on the record which only involves the error E_R , where

$$E_R << E_v, E_p, E_p'. \tag{A8}$$

It should be noted that because the chronograph method samples the entire sequence of Eq. (A7), it furnishes more information than the other three methods from which the behavior of particular timepieces is observable. The corresponding rate equation for the chronograph method is

$$\mathcal{R} = E_N \equiv E_N' \pm E_R. \tag{A9}$$

The Algebraic Sign of $\mathcal{R}$ and its Meaning

It is found from Eqs. (A7) and (A9) that the algebraic sign of $\mathcal{R}$ (and of r, as will be shown later) is dependent upon the sign of E_N , which is dependent upon the sign of $\bar{e}$ (the average error per period) or e_j (the actual error per period).† Because of the manner of defining‡ e_j and $\bar{e}$, both $\mathcal{R}$ and r represent the cumulative effect of comparing the actual ($\bar{t}$) with the nominal (t_s) timepiece periods of oscillation. This comparison gives the following result:

1. If $t > t_s$, $\bar{e}$ is negative and the timepiece runs slow ($\mathcal{R}$ and r are negative)

2. If $t < t_s$, $\bar{e}$ is positive and the timepiece runs fast ($\mathcal{R}$ and r are positive)

While it is true that smaller over-all errors permit measuring a smaller rate magnitude, it is impossible to measure zero rate** because of end-effects.

*See Eq. (30A) Part II

†See Part II

‡See Appendix F

**Zero rate means that $\bar{t} = t_s$, or $\bar{e} = \frac{1}{N}\sum_1^N e_j = \frac{E_N}{N} = 0$.

A further indication of the meaning of the algebraic sign of R is obtained by considering Eq. (A2), which, by transposing terms and neglecting T, is a statement of the fact that the final indication may be obtained from the algebraic sum of the initial indication and the rate (or total error during the elapsed interval). Therefore, if the timepiece runs slow or R is negative, $I_f < I_i$; and if the timepiece runs fast or R is positive, $I_f > I_i$. This reasoning applies of course for Eqs. (A3) and (A4).

The above convention of algebraic sign of R or r will be used in discussing the chronograph method. In developing the mathematics both positive and negative values of R and r are anticipated; in certain cases separate expressions are required, and in such cases both expressions will be given, otherwise the use of either sign is implied.

Considerations of the Record

Since the chronograph method centers about the recording of samples of the sequence given by Eq. (A7) let us first consider the record which is expected. The timepiece timing marks might be recorded continuously or samples might be taken periodically. The record which may be either of a linear type or of a drum type can be made to accommodate continuous or sampled recordings of the timing marks (of both timepieces and standards). The two types of records for the same arbitrary situation are shown in Figs. A1 and A2. To provide a reasonable measuring accuracy for such a record, the spacing between timing marks might be made to equal the linear temporal equivalent of Part III (4.484 inches ≡ 1 second); but an unreasonable length of record paper would be required for a continuous record. Furthermore, unless the entire record were examined, there would be difficulty in obtaining directly the value of the rate. A continuous record of the drum or chronograph type might be used to give a much more compact record than the linear type. It must be kept in mind that Fig. A2, for a continuous type of recording, represents successive timing marks on an exaggerated horizontal scale. In practice, the lead screw of a chronograph determines the x-displacement between timing marks;* for all reasonable lengths of lead screw a number of timing marks would overlap. The overlapping of timing marks is exaggerated by the effects of the errors associated with the rate measuring process. In Appendix C it is concluded that the drum type of record is easier to interpret, because the effect of overlapping marks is less pronounced if the timing marks are sampled, and if the sample consists of a small number of marks. Figure A3 illustrates a sampled record of the drum type.

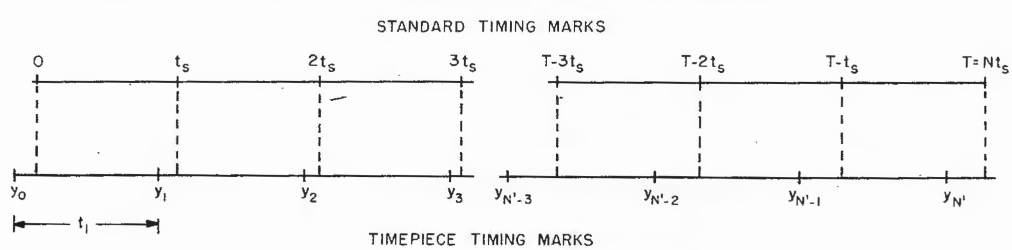


Fig. A1 - Linear type of record. (t_s is the nominal period of oscillation and t is the actual period of oscillation.)

*This matter is discussed in Appendix B.

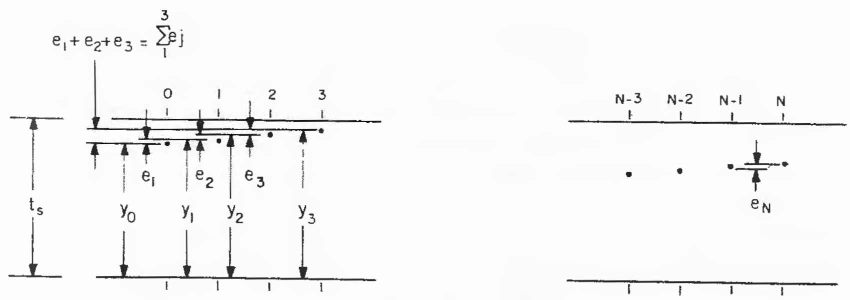


Fig. A2 - Drum or chronograph type of record. (A possible method of recording timing marks, either continuously or in samples. The situation shown is taken from Fig. A1, using a record scale factor of $Y = t_s$.)

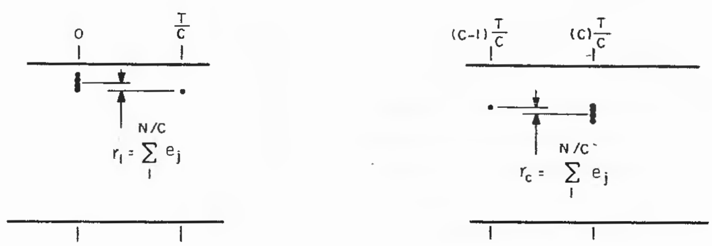


Fig. A3 - A sampled drum or chronograph type of record. (The rate r_i for a marking interval T/C is shown. The first and last marking sample ordinates, taken from Fig. A2, have been exaggerated to show the use of the average ordinate in determining r_i .)

The process of sampling may be described as follows. Let the interval T be divided into C subintervals. During each subinterval (of duration T/C sec), the error accumulated is

$$r_i = \sum_1^{N/C} e_j, \tag{A10}$$

where N is the total number of nominal periods in T (see Part II). The total error accumulated in the interval T is

$$E_N = \sum_1^{N} e_j = \sum_1^{C} r_i = r_1 + r_2 + \dots + r_C. \tag{A11}$$

By sampling, a recording of timing marks is required only once during each subinterval. Figure A2 shows the sampled portion obtained in the first subinterval, ignoring the effect of overlap. The first timepiece timing mark is shown by locating the point a arbitrarily between standard timing marks. Figure A3 shows the situation of Fig. A2, for the first subinterval, in the manner the timing marks actually might be recorded. The effect of overlap is shown, but the random effect in which the measuring errors might occur is not shown. From Fig. A3, the recorded sampled sequence will be as follows;

$$\left.\begin{aligned} t_1 &= t_s - y_0 + y_1, \\ t_2 &= t_s - y_1 + y_2, \\ t_3 &= t_s - y_2 + y_3, \\ &\vdots \\ t_h &= t_s - y_{h-1} + y_h, \end{aligned}\right\} \tag{A12}$$

where $t_1, t_2, t_3, \ldots t_h$ represent the actual timepiece periods of oscillation, t_s represents the nominal period of oscillation, and the y-terms are used to designate the actual time the successive timepiece timing marks are recorded. These equations are shown in Fig. A2 and may be written in terms of e, the error per period, as follows:

$$\left.\begin{aligned} y_0 - y_1 &= t_s - t_1 = e_1 \\ y_1 - y_2 &= t_s - t_2 = e_2 \\ &\vdots \\ y_{h-1} - y_h &= t_s - t_h = e_h \end{aligned}\right\} \tag{A13}$$

(If this sample, of which only a few, h, terms are to be used,* is continued through the entire subinterval, the sum of the individual terms must lead to the value

$$r_i = \sum_1^{N/C} e_j.)$$

In using the sampling technique the average position of the timing marks should be used to represent the entire sample. From Eq. (A12)

$$\bar{y} = \frac{1}{h}(y_1 + y_2 + \ldots + y_h). \tag{A14}$$

Associated with $\bar{y}$, ignoring the rate measuring errors, will be half the error accumulated in taking the sample, or from Eq. (A13)

$$e_{\bar{y}} = \frac{1}{2h} \sum_1^{h} e_j, \tag{A15}$$

but for all practical purposes, especially for small samples, the value of $e_{\bar{y}}$ is negligible and can be ignored.† Thus, a single timing mark $\bar{y}$ can theoretically represent the entire sample; it can be expected with certainty that $e_{\bar{y}} \ll E_R$. Of course, the effect of E_R cannot be ignored; the timing marks occur at random depending upon the actual magnitude of E_R.‡

The complete record is to consist of C samples obtained at intervals T/C, each sample being represented by a mark $\bar{y}$. Each of the samples is a term of the sequence expressed by Eq. (A11). Figure A3 shows two of the samples of the type of record expected.

*To provide redundancy of information to make the recording process more effective.

†See Appendix B.

‡The extent of the scatter is dependent upon both E_R and the record parameters yet to be discussed.

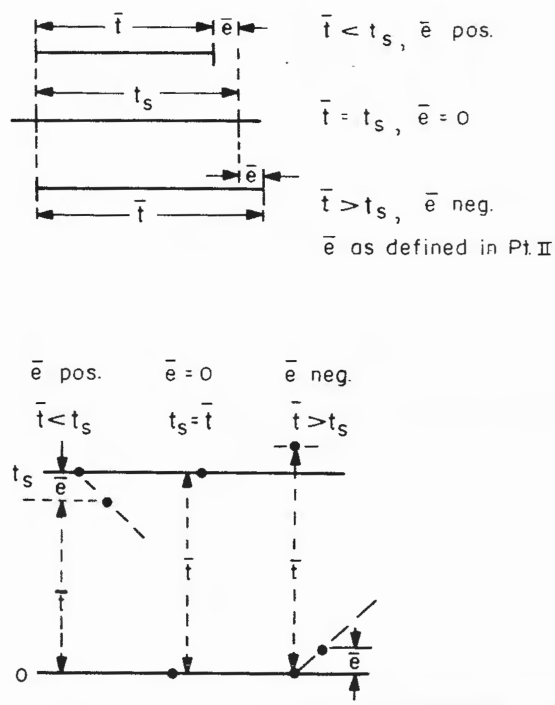


Fig. A4 - Effect of the algebraic sign of $\bar{e}$ on the record. (The type of record to be expected for positive, negative, and zero values of $\bar{e}$, and hence of both r and R, is shown for the types of record in Figs. A1 and A2; this applies also to the record shown in Fig. A3. The values of $\bar{e}$ have been exaggerated.)

The effect on the record of the algebraic sign of $\bar{e}$, and hence of r and R, is illustrated in Fig. A4. The two parts of Fig. A4 show in an exaggerated way how $\bar{e}$ has been defined (see Part II) and the expected appearance of positive, negative, and zero values of $\bar{e}$ on both the linear and drum types of records. These results, of course, apply to the sampled type of record shown in Fig. A3. Thus, the slope of the line joining recorded points plays an important part in the interpretation of the records obtained with the chronograph method. The slopes shown in Fig. A4 can be used as a guide in the interpretation of actual records.

Approximate Straight-Line Equation to Represent the Record (Trend Lines)

A straight line can be drawn between any two points of the record (to indicate the trend for purposes of interpretation) and the general equation for a straight line, in its slope (m)—intercept(b) form is

$$y = mx + b. \tag{A16}$$

The slope

$$m = \frac{\Delta y}{\Delta x}, \tag{A17}$$

where the y-increment is

$$\Delta y = -r_i \tag{A18}$$

(the minus sign is necessary to account for the algebraic sign inherent in the definition of e given in Part II and illustrated in Fig. A4), and the x-increment is

$$\Delta x = \frac{T}{C}, \tag{A19}$$

the duration of the subintervals. Hence,

$$m = -\frac{Cr_i}{T}. \tag{A20}$$

Note that m has a sign which is opposite to that of r_i.

The y-intercept is the initial point of the record (at $x = 0$), y_0, or

$$b = y_0, \tag{A21}$$

where the limits of y_0 will be considered later. Therefore

$$y = \left(-\frac{C}{T} r_i\right) x + y_0, \tag{A22}$$

where x will have integral values given in terms of T/C if the sampled record is to be reproduced. If

$$x = \frac{T}{C} x', \tag{A23}$$

then Eq. (A22) becomes,

$$y = -r_i x' + y_0, \tag{A24}$$

and

$$0 \leq x' \leq C \tag{A25}$$

for the complete record for the interval T. Equation (A22) or (A24) describes the line, to be called a trend line, and describes the actual record obtained if r_i is constant.

Equation (A24) can be rewritten in terms of the cumulative error or $\Delta y = E_N$, $\Delta x = T$, from which $m = E_N/T$, and from Eq. (A23) it follows that

$$y = -\frac{E_N}{C} x' + y_0, \tag{A26}$$

for which Eq. (A25) must apply. It is interesting to note that

$$\frac{E_N}{C} = \frac{1}{C} \sum_{1}^{C} r_i = \bar{r} \tag{A27}$$

and that

$$y = -\bar{r} x' + y_0. \tag{A28}$$

The limits of y_0 can now be given as

$$0 \leq y_0 \leq r_i. \tag{A29}$$

It is of further interest to find that by using the upper limit of Eq. (A25), or $x' = C$, in Eq. (A26) there is obtained the maximum traverse of recording marks across the record during the interval T, or

$$y_{max} = -E_N + y_0, \tag{A30}$$

which, incidentally, proves that the sampling technique can be used to measure E_N directly. This proof having been obtained in general terms, the result can be obtained for the daily rate R which represents the special value of E_N for which $T = 86,400$ seconds. Hence, Eqs. (A26) and (A27) may be rewritten as,

$$y = -\frac{R}{C}x' + y_0 \tag{A31}$$

where $x' = C$

and

$$\frac{R}{C} = \bar{r}\ , \tag{A32}$$

and Eq. (A28) may be used as a general straight-line equation using record coordinates.

The result expressed by Eq. (A31) suggests that at least the average rate may be obtained by measuring the y-displacement between the initial and final marks on the record In addition, the line represented by Eq. (A28) shows the trend of the daily timing mark pattern for a 24-hour interval, and is called a trend line because in practice random variations occur in timepiece timing marks and in the recording process; see Appendix C.

By using the straight-line equations given in Eqs. (A22), (A28), or (A31), further factors regarding the interpretation of the recorded pattern are to be considered.

Considerations for Selecting the Scale Factor Y

To obtain a practical record for interpreting the rate data it is necessary to consider what is involved in selecting the scales of the y- and x-axes, or the displacement driving axes. As an illustration of some of the problems encountered, the complete records of several actual subintervals have been joined, as shown in Fig. A5.

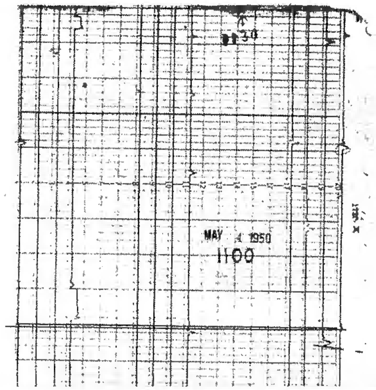
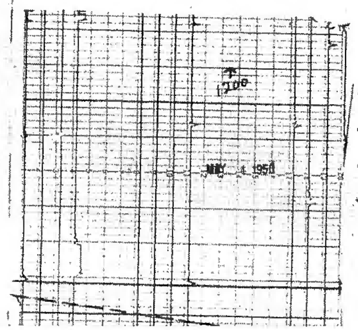
Using only the first four subintervals, the generation of a straight-line type of record is shown in Fig. A5, thereby illustrating the application of Eq. (A26) or (A31). In Fig. A5 the timing marks of two timepieces in each of the four subinterval records are joined by a straight line, one full and the other dotted. It should be noted that a complete daily record formed in this way, would require 48 subintervals to be placed side-by-side; this is an unwieldy means of interpreting the recorded data. An additional objection to the use of this type of record is in the possible intersection of the lines of the various timepieces. A more compact record which can be interpreted without ambiguity is required. Compactness and ambiguity are to be considered by examining the factors which affect selection of the scales.

The choice of a scale factor Y depends on the expected minimum and maximum values of $|R|$.* If Y is substituted for y_{max} in Eq. (A30), Y can be chosen to be large enough to detect a sufficiently small value of $|R|$, but small enough to obtain a simple pattern of timing marks; or

$$|R|_{min} \leq Y \leq |R|_{max} \tag{A33}$$

for a single line record.

*Use of $|R|$, the magnitude of R, indicates that the choice of Y is independent of the algebra sign of R; the record is to be used for plotting R with either algebraic sign.


MAY 4 1950
1100

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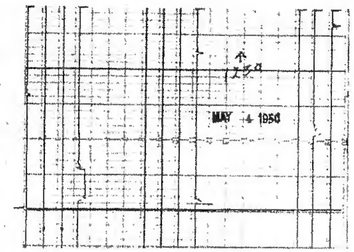
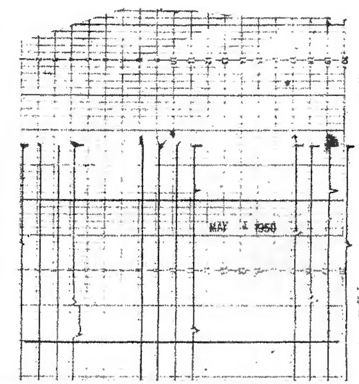

MAY 4 1950

MAY 4 1950

From a practical point of view the smallest value of rate to be measured $|R|_{min}$ should be consistent with the measuring capabilities of the rate measuring method employed (i.e., E_R, for this case). Hence,

$$|R|_{min} \equiv E_R. \qquad \text{(A34)}$$

A lower limit different from zero indicates that any practical method of rate measurement will only be capable of measuring small values of rate to within limits imposed by the measuring errors which are present. This limitation must be kept in mind when interpreting records obtained with the chronograph method.

Practical considerations are also useful in determining the value of $|R|_{max}$. In general, a knowledge of the value of $|R|_{max}$ will be unknown unless an auxiliary method is used to obtain it. The experimental results given in Part IV can be used; approximate values are:

chronometers	10 spd,
watches	50 spd.

The choice of a value of $Y = |R|_{max} = 50$ sec* would have assured obtaining a single-line (simple) record for all timepieces. In general, for a single-line record,

$$Y \geq |R|_{max}. \qquad \text{(A35)}$$

Let us examine the practicability of attempting to obtain single-line recordings for the case in which the measuring limits of R are known in advance. To set the value of Y at 50 seconds would require, preferably, that only a single timepiece (and standard) timing mark be made (in 50 seconds) to provide an unambiguous record.† This effect may be seen in Fig. 1a of the main body of this report, where both ticks of the chronometer are recorded on the record. It is required furthermore that the interval for isolating the single timing mark to be recorded be established with sufficient precision that there will be no confusion in interpreting the record to decide which of successive timepiece timing marks should be used. The confusion results from the fact that in 50 seconds the entire sample of h timing marks given by Eq. (A12) would undoubtedly be recorded. To avoid such confusion precise timing of the start of the sample is required, and the precision required is related to the nominal period of oscillation of a particular timepiece; or the marking interval must be $Y \pm 3\sigma_m$, for a normal distribution of the actual intervals, where

$$6\sigma_m \leq t_s, \qquad \text{(A36)}$$

which represents the allowable timing tolerance for the desired measurement. It will be recalled that for watches $t_s = 0.2$ second. Hence, $\sigma_m \leq 0.03$. Therefore, the marking interval of 50 seconds must be established within ± 0.1 second, representing an accuracy of 99.8 percent. In practice it is considered easier to use the timepiece and standard timing marks in the form in which they are made available without the introduction of

*For a discussion of the units of the terms and the definition of R see Appendix G.

†This suggests the need for gating and/or counting circuits since timepiece timing marks are obtained generally as electrical impulses at the same frequency as generated, except for the navigating watches, which are fitted with break-circuit contacts.

gating or count-down circuits. Hence, multiple-line records or segments of the trend line expressed by Eqs. (A22), (A24), or (A31) are to be expected. The scale factor Y is then determined from the standard timing mark interval or the timepiece timing mark interval.

A Criterion for Selecting the x-Interval T/C

The choice of a scale for the x or stylus-driving axis, in units of T/C for a sampled record, depends on the possible use to which the record is to be put. If only the average value of the daily rate $\bar{R}$ is desired, only the initial and final recorded timing marks are of importance. However, by noting deviations from the line connecting the initial and final recorded marks in each segment, departures from the average rate during the 24-hour interval become obvious. Therefore, if it is desired to investigate timepiece variations within any 24-hour interval, then the x-interval should be made large enough to permit the desired accuracy in locating a particular subinterval.

While the choice of the x-interval is relatively unimportant, some simplification of the interpretation of the record may be obtained if a value is chosen for the x or sampling interval, T/C. If the limiting value of $\bar{r}$ is r_ℓ and from Eq. (A32) $C\bar{r} = R$, the corresponding limiting value of C is

$$C_\ell = \frac{R}{\bar{r}_\ell} = \frac{2R}{Y}. \tag{A37}$$

Therefore, if the sampling interval T/C is chosen by using Eq. (A37),

$$\frac{T}{C_\ell} < \frac{TY}{2R}. \tag{A38}$$

Of course, Eq. (A38) is useful only if the maximum expected value of R is known, e.g., from previous results.

The Unambiguous Interpretation of the Record

The formation of multiple-line records is shown in Fig. A6 in which, as is to be expected, $Y < |R|$. The representation of Fig. A6 suggests the following:

$$|R| = DY + F, \tag{A39}$$

where D is an integer (0, 1, 2,...), and

$$0 \leq F \leq Y. \tag{A40}$$

Equation (A39) results from the division of the magnitude of $|R|$ by Y; i.e., Y is the divisor, D the integral quotient, and F the remainder. Note that if $D = 0$, a single-line record is obtained, and D gives the number of lines which the record will contain.

Now we must consider Eq. (A39) from the viewpoint of possible ambiguities. The slope of each line will be the same as that given by Eq. (A20). Of course, Eq. (A28) gives the expression for the complete line, but it may be used to obtain any segment of the line. It will be recalled that x' in Eq. (A28) is the number of units of time, T/C seconds, required for the record to reach a particular value of y. This fact may be used to find the x interval for a line segment to make a complete traverse across the record, and the y interval is Y. Thus this x interval, for the first expected complete line,* is obtained from Eq. (A28) as follows:

$$x'_{y=2Y} - x'_{y=Y} = -\left[\frac{2Y - y_0}{\bar{r}} \; \frac{Y - y_0}{\bar{r}}\right] = -\frac{Y}{\bar{r}}, \tag{A41}$$

and represents a possible limiting value for the x interval. But more will be said of this below.

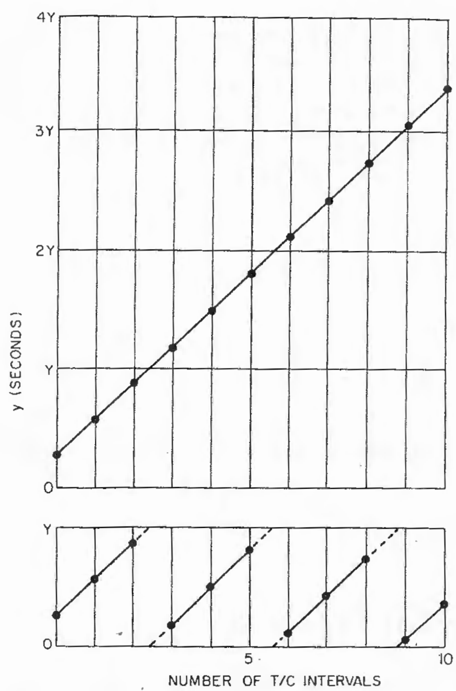


Fig. A6 - Formation of segments or multiple lines on a drum type of record ($|R| > Y$ or $|R| = DY + F$). (A partial record for an arbitrary value of R is shown.)

If the initial point of the record is ignored, as it was in obtaining Eq. (A41), a limiting value of $\bar{r}$ may be obtained for an unambiguous record. Two points are necessary in order to draw a line between successive points. For a limit case, in which $\bar{r} < Y$, the plotted situation will appear on the record as shown in Fig. A7. To interpret such a record, let $\Delta y = -r_s$, where $r_s \leq \bar{r} < Y$. For the situation shown in Fig. A7, there are two ways of writing the difference:

$$y_b - y_a = \Delta y = -r_s, \tag{A42}$$

or

$$Y - y_a + y_b = \bar{r}. \tag{A43}$$

*The first segment, in general, will be incomplete because of the arbitrary value which y_0 can assume.

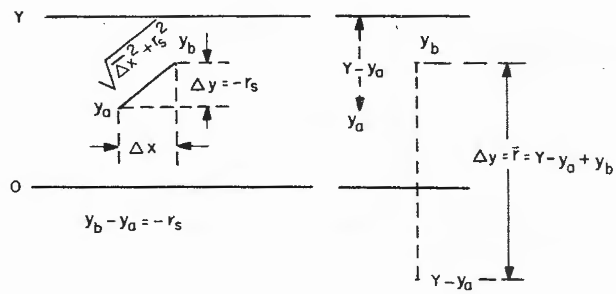


Fig. A7 - The problem of interpreting the record. (Either $\Delta y = -r_s$, or $\Delta y = \bar{r}$; $r_s = Y - \bar{r}$, for $r_s \leq \bar{r}$ within limits given in the test of Appendix A; drawing the minimum length of the line between g_a and g_b will in general give the distance $\sqrt{\Delta x^2 - \bar{r}_s^2}$.)

From the latter,

$$y_b - y_a = \bar{r} - Y;$$

hence,

$$r_s = Y - \bar{r}. \tag{A44}$$

It is to be expected that the length of line drawn directly between y_a and y_b will be

$$\sqrt{\Delta x^2 + \Delta y^2}$$

or, in general

$$\sqrt{\Delta x^2 + r_s^2} \leq \sqrt{\Delta x^2 + \bar{r}^2}; \tag{A45}$$

from which by use of Eq. (A44), a limiting value of $\bar{r}$ is obtained as

$$|\bar{r}_\ell| = \frac{Y}{2} \tag{A46}$$

and this value will assure obtaining an unambiguous record; the subscript ℓ is used to denote the limiting value. This result may be used to obtain a limit value for R; from Eq. (A32),

$$|R_\ell| = \frac{CY}{2}. \tag{A47}$$

And the corresponding limit number of lines for an unambiguous interpretation is, from Eqs. (A39) and (A43),

$$D_\ell = \frac{C}{2} - \frac{F}{Y}. \tag{A48}$$

Thus, $|R_\ell|$ represents the maximum daily rate measurable without ambiguity, D_ℓ is the number of lines to be expected for that value of $|R_\ell|$. In general, the expected valu of $|R_\ell|$ will not be known, hence, unambiguous records cannot be certain and other rate measuring means should be used to obtain a rough indication of the magnitude and sign of R; e.g., the visual method might be used.

The limit given by Eq. (A46) may be written as

$$0 \leq |\bar{r}| \leq \frac{Y}{2}, \tag{A49}$$

where $r_s = \bar{r}$, to represent the limit of unambiguous results obtainable for positive, negative, or zero values of $\bar{r}$. Furthermore, it can be written that

$$\frac{Y}{2} \leq |\bar{r}| \leq Y, \tag{A50}$$

where from Eq. (A44), $r_s = Y - \bar{r}$.

To verify the difference to be expected in the record between the limits given by Eqs. (A49) and (A50), let us set down the record as generated for each case.

Case 1: For values of $\bar{r}$ within the limits given by Eq. (A49), with exact values of r for purposes of generality, i.e., $0 \leq |r| \leq Y/2$:

$$\begin{array}{ll} \text{1st mark} & y_0 \\ \text{2nd mark} & y_0 + r_1 \\ \text{3rd mark} & y_0 + r_1 + r_2 = y_0 + \sum_{1}^{2} r_i \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot & \vdots\ \vdots\ \vdots\ \cdot\ \vdots \\ L_1\text{th mark} & y_0 + r_1 + r_2 + \ldots + r_{L_1} = y_0 + \sum_{1}^{L_1} r_i . \end{array} \tag{A51}$$

The L_1th mark represents the last mark made before the end of the record, whose scale factor is Y, has been reached. Or,

$$y_0 + \sum_{1}^{L_1} r_i \equiv Y, \tag{A52a}$$

where, $y_0 < Y$, and

$$y_0 + \sum_{1}^{L_1} r_i + y_1 = Y, \tag{A52b}$$

where $y_0 < Y$ and $y_1 < r_i$. In Eqs. (A51) and (A52) y_0, the arbitrary initial start of the record (see Fig. A1), is positive, which means that it occurs after the initial standard timing mark. In both Eq. (A51) and Eq. (A52) the sign of r_i is retained and determines the pattern obtained as shown in Fig. A4. These equations apply for all values r_i within the limits given by Eq. (A49). In Eq. (A52), the quantity y_1 is introduced to represent a fraction of the quantity r_i, i.e., $y_1 < r_i$.

Equation (A52) expresses the formation of the first line of the record. It should be pointed out that if

$$y_0 + \sum_{1}^{L_1=C} r_i \leq Y \tag{A53}$$

then the complete record is a single line. However, if such is not the case then the daily record may have several lines, the first complete line of which may be expressed as

$$(r_{L_1}+1) - y_1 + \sum_{L_1+1}^{L_2} r_i + y_2 = Y, \tag{A54}$$

where

$$y_1 < r_{L_1} + 1,\ y_2 < r_{L_2} + 1,\ L_2 < C.$$

The slope of either of the lines given by Eqs. (A52) and (A54) can be found to agree with Eq. (A20) or $m = -C\, r_i/T$, which verifies that for Case 1 and its limits, an unambiguou result for R can be obtained. It should be pointed out that a daily record would contain the number of lines D given in Eq. (A39). Of course, the value of y_0 may alter the value of D; hence the value of F would be altered correspondingly. By combining Eqs. (A39) and (A30), it is found that E_N or R is obtained from the record in terms of $y_{max} - y_0$. Therefore in practice, by using for measuring purposes a horizontal line through y_0, Eq. (A39) will be found to hold and D is the number of segments which cross the horizontal line through y_0. Otherwise, Eq. (A39) is modified as follows:

$$y_{max} = DY + F + y_0 \tag{A55}$$

or

$$y_{max} = DY + F', \tag{A56}$$

where

$$F' = F + y_0. \tag{A57}$$

Because of the limiting values of F and y_0,

$$0 \le F' \le 2Y. \tag{A58}$$

Thus,

$$DY + F'' \le y_{max} \le (D+1)Y + F'' \tag{A59}$$

where

$$F'' \le Y. \tag{A60}$$

Of course, for the lower limit in Eq. (A59) $F' = F'' \le Y$.

Now, consider Case 2 where $Y/2 \le |\bar{r}| \le Y$.

Case 2: The generation of the record may be investigated in the manner used for Case 1:

$$\begin{array}{ll}
\text{1st mark} & y_0 \\
\text{2nd mark} & y_0 + |r_1| - y_1 = Y \\
\text{3rd mark} & y_1 + |r_2| - y_2 = Y \\
\vdots & \vdots \\
\text{Kth mark} & y_K + |r_{K+1}| + |r_{K+2}| - y_{K+2} = Y \\
\vdots & \vdots \\
\text{Cth mark} & y_{C-1} + |r_C| = F',
\end{array} \tag{A61}$$

where $y_0 < Y$, and $y_1, y_2 \ldots < r_i$. In Eq. (A61) it can be seen that the markings occur at the points $y_0, y_1 \ldots y_K, y_{K+1} \ldots y_{C-1}$, and $y_{C-1} + r_C$ and that no two adjacent marks contain the sum $r_i + r_{i+1} < Y$. It is to be expected that subsequent points will be joined to form an apparent record, since, for any record the natural tendency will be to join these points with the shortest line. The slope of this apparent record as shown in Fig. A4 is

$$m_s = \frac{y_2 - y_1}{T/C}.$$

A value for m_s can be obtained from Eq. (A61), for the 3rd mark,

$$m_s = \frac{C}{T}(Y - r_2) \equiv \frac{C}{T}(Y - \bar{r}) \tag{A62}$$

or

$$m_s = \frac{C}{T} r_s, \tag{A63}$$

where

$$r_s = (Y - \bar{r}). \tag{A64}$$

In Eq. (A63) it should be noted that there is a reversal of sign; compare this result with Eq. (A20). This knowledge is necessary in interpreting the records; the sign of r_s determines the sign to be given r. Using a process similar to that used to obtain Eq. (A32),

$$R_s = Cr_s. \tag{A65}$$

From Eq. (A62) $R_s = C(Y - \bar{r}) = CY - C\bar{r}$, which, with Eq. (A32) reduces to

$$|R| = CY - |R_s|. \tag{A66}$$

Thus, Eq. (A66) provides a means for converting apparent values of R_s to true values, R, for Case 2.*

Since Eqs. (A42) and (A43) give the maximum values which can be obtained from the record unambiguously, the results obtained from considering the above two cases can be used to express the values of $\bar{r}$ and R in general terms and thereby permit the true values to be obtained from an interpretation of the record, with the aid, of course, of another rate measuring method of lesser accuracy. The limiting values of $\bar{r}$ for the two cases can be written in the general form

$$s\frac{Y}{2} \leq |\bar{r}| \leq (s+1)\frac{Y}{2} \quad (s = 0, 1, 2, \ldots), \tag{A67}$$

where Case 1 occurs for $s = 0$, and Case 2 for $s = 1$. A general expression for r may be written in two ways,

$$\bar{r} = s\frac{Y}{2} + r_s$$

or

$$\bar{r} = -\left[(s+1)\frac{Y}{2} - r_s\right],$$

where the minus sign signifies that $\bar{r}$ takes the sign opposite to r_s,

$$0 \leq |r_s| \leq \frac{Y}{2}, \tag{A68}$$

*Note that Eq. (A66) only gives the magnitude of the rate; the sign is determined from the sign of r_s, and hence of R_s, Eq. (A65).

and $\bar{r}$ is expressed in terms of either the lower or upper limits. The statement Eq. (A68) is merely another way of expressing Eq. (A46). From Eq. (A68) and Eq. (A65) it follows that

$$0 \leq |R_s| \leq C\frac{Y}{2} \quad \text{(A69)}$$

which is the maximum value of daily rate which can be measured from the record; this is a restatement of Eq. (A47).

If Case 1 is reconsidered with the generalized results $(s = 0)$, and the first way just given for expressing $\bar{r}$, then $\bar{r} = r_s$ and $C\bar{r} = Cr_s = R = R_s$, as expected. Now, if the value Y is added to r_s, then $\bar{r} = Y + r_s$ or $\bar{r} = 2Y + r_s$, etc., would all show the same y increment r_s. Therefore,

$$\bar{r} = s\frac{Y}{2} + r_s \qquad \text{(even values of } s\text{).} \quad \text{(A70)}$$

Hence

$$R = \frac{sCY}{2} + R_s \qquad \text{(even values of } s\text{)} \quad \text{(A71)}$$

gives the true value R from the measured value R_s, for even values of s in Eqs. (A71) and A67).

In reconsidering Case 2 $(s = 1)$, the second way of expressing F is found to apply, if odd values of s are used or,

$$\bar{r} = -\left[(s+1)\frac{Y}{2} - r_s\right] \qquad \text{(odd values of } s\text{),} \quad \text{(A72)}$$

where the minus sign signifies that $\bar{r}$ takes the sign opposite to r_s. If $s = 1$, it is seen that the results agree with Eq. (A64), as expected. For $s = 3$, $\bar{r} = 2Y - r_s$ and for $s = 5$, $\bar{r} = 3Y - r_s$; the y increment r_s of the record is the same and the same record will be generated. The true rate R is

$$R = -\left[(s+1)\frac{CY}{2} - R_s\right] \qquad \text{(odd values of } s\text{),} \quad \text{(A73)}$$

from which a proper interpretation of the record can be obtained.

The results given by Eqs. (A71) and (A73) permit, along with the use of Eq. (A67), a general means of interpreting records of the type which have been discussed. It is possible also to make use of the results in selecting values of Y to be used in specific tests.

* * *

APPENDIX B
The Chronograph

Description

An instrument for recording data directly on the drum type of record (see Appendix A) was built* for use with the instrumentation described in Part I. This chronograph had four relay-operated styli to record the timing marks of three timepieces and a standard for a complete 24-hour interval; the standard timing marks were to serve as a check on the timing calibration of the record. Each stylus was provided with a separate ribbon of a distinctive color to avoid confusion in obtaining the results from the record. It was necessary to stop the instrument to change the record.

It was possible to record timing marks with all four styli continuously. However, it was felt that there would be less confusion in interpreting the record if the timing marks were sampled at regular subintervals during the 24-hour interval. The program device, described in Part I, thus was expected to be used with the chronograph. This device was set to provide a sampling subinterval T/C of 30 minutes, or $C = 48$ subintervals in 24 hours (using the nomenclature of Appendix A). Once during each subinterval, as determined by cam settings, each of the four styli was permitted to record sequentially the timing marks of the three timepieces for five seconds, while recording the standard for 15 seconds.

Some flexibility in the use of the chronograph was made available by providing 7 sets of gears which permitted as many rotational speeds of the drum containing the record paper to be obtained. Hence, there was available a choice of the scale factor Y; the values of Y were 0.5, 0.2, 0.1, 0.05, 0.033, 0.025, and 0.020 seconds. Thus, measurements of R for the timepieces under test should have given, generally, multiple-line records, because in the nomenclature of Appendix A, $Y < |R|_{max}$.

For any 24-hour interval the scale factor Y would have a fixed value as determined by the rotational speed of the drum, thus establishing the units of the y axis of the record. The x interval was established by the program device as well as the length and pitch of the lead screw, a complete traverse of which took 24 hours. Since the length of the lead screw between stops was 10 inches, the fixed x interval between timing mark samples was 10 in/48 = 0.21 inches.

Since a complete traverse of the lead screw carrying the styli required 24 hours, in the design of the chronograph it was considered expedient to reverse its direction when it was restarted each day. This design feature, which required that the marking direction be correctly established in order that the resulting record be interpreted properly when removed from the drum,† introduced a possible record ambiguity not covered in Appendix A, and necessitated the use of an additional rate measuring method for establishing the approximate value of rate so that a correct interpretation of the value of the rate could be made.

*Only the constant-frequency power source was not completed.

†Otherwise there are two possible slopes which can be used to obtain the measured value of rate, R_s, from the record, thereby increasing the complexity of interpretation.

The chronograph record can be interpreted in accordance with the factors considered in Appendix A. The ambiguities to be expected are those considered in Appendix A which depend upon values of C and Y. The value of C which was to have been used has already been given as 48 for T = 86,400 seconds (24 hours). Limit values of Y available for the chronograph are tabulated with the corresponding values of R which could be measured; this table may be used as an aid in interpretation of records, once the value of s has been determined from an approximate rate measurement.

Included in Table B1 are the corresponding values for the interval-record method, for purposes of comparison. It can be readily seen that smaller values of rate can be measured by using the chronograph, because of the values of Y it is possible to obtain. It is interesting to note that the theory of Appendix A applies to both the interval-record and the chronograph methods. Included in Table B1 are the values of $|R_\ell|$ which, incidentally, define the limit of the approximate rate measuring method to be used, and of D_ℓ, the maximum number of lines expected.

TABLE B1
Limiting Values of Rate Measurable with the Chronograph and Interval-Record Methods

s	|R| (spd) $s\frac{CY}{2} \le \|R\| \le (s+1)\frac{CY}{2}$					
	Interval-Record Method		Chronograph Method			
	Y = 1 Second C = 48 $\|R_\ell\|$ = 24 Spd* D_ℓ = 24 Lines†		Y_{max} = 0.5 Second C = 48 $\|R_\ell\|$ = 12 Spd* D_ℓ = 12 Lines†		Y_{max} = 0.02 Second C = 48 $\|R_\ell\|$ = 0.48 Spd* D_ℓ = 1 Line†	
	24s	24(s+1)	12s	12(s+1)	0.48s	0.48(s+1)
0	0	24	0	12	0	0.48
1	24	48	12	24	0.48	0.96
2	48	72	24	36	0.96	1.44
3	72	96	36	48	1.44	1.92
4	96	110	48	60	1.92	2.40
5	110	144	60	72	2.40	2.88
6	144	168	72	84	2.88	3.36
7	168	192	84	96	3.36	3.84
8	192	216	96	108	3.84	4.32
9	216	240	108	120	4.32	4.80
10	240	264	120	132	4.80	5.28

*accuracy of approximate rate measuring method to aid interpretation

†$|R| = DY + F$; $F = 0$, for integral values of $|R|$

Comparison of Interval-Record and Chronograph Methods

In the foregoing description of the chronograph method of rate measurement it may be noted that in many respects it is similar to the interval-record method. Because of this similarity a comparison of the two methods is considered appropriate. To make this comparison, the features of both methods are to be listed side-by-side, as follows:

A. Interval-Record	B. Chronograph
1. 20-Pen Recorder	1. Chronograph Drum
2. Recording instrumentation (relays actuated for record)	2. Same circuits as in A (relays actuated for record)
3. Continuous-type record	3. Drum-type record
4. Record capacity - up to 20 sets of timing marks	4. Record capacity - up to 4 sets of timing marks
5. Interpreted after replotting; see Table B1 (separate record for each timepiece and more convenient)	5. Interpreted direction from record (see Table B1)
6. Long, bulky record	6. Compact record
7. Timepiece timing marks always associated with same column on record	7. Timepiece timing marks distinguished by use of four colors
8. Equations of Appendix A apply in interpretation together with associated ambiguities which are possible	8. Same as A
9. Additional ambiguity possible if direction of marking not noted on record	9. Same as A
10. Approximate value of rate by less accurate method necessary to obtain correct interpretation	10. Same as A
11. Scale factor Y chosen on basis of standard timing mark interval (sec) and driving speed of record (4.484 in/sec); it is possible to use scale factors which are multiple values of Y	11. Limits of scale factors given in Table B1
12. Sensitivity = S; over-all measuring error E_R - see Table 3	12. Approximately the same as A

The twelve points of comparison of the two methods of rate measurement reveal their similarity, and indicate the reasons for treating them as one.

* * *

APPENDIX C
The Case for Using a Sampled Type of Record

The drum type of record, which represents the most compact record obtainable, is particularly well suited for use with the chronograph; this type of record was to have been used with the instrument designed for this experiment. As indicated in Appendix A the timing marks may either be recorded on such a record continuously or they may be sampled.

If the timing marks are recorded continuously, a certain number of marks can be expected to overlap. The number which overlap depends upon the motion of the styli across the record. If $\mathcal{L}$ is the length of the lead screw, the styli travel at the rate of $\mathcal{L}/86,400$ (in/sec). If $\Delta\mathcal{L}$ represents the minimum movement of the styli which can be detected upon examining the record, then the time ΔT required for this movement is

$$\Delta T = \left(\frac{\Delta\mathcal{L}}{\mathcal{L}}\right)(86,400) \text{ sec.} \tag{C1}$$

By using Eq. (C1) the number of timing marks which will be recorded during ΔT, or which will overlap, can be determined for the three cases of interest: (1) for watches (i.e., CW with 5 ticks per second), $5\Delta T$; (2) for chronometers (MC with 2 ticks per second), $2\Delta T$; and (3) for timepieces with only a single tick per second, ΔT. The third case is that represented by timepieces such as NW which are provided with break-circuit contacts.*

During any time interval an accumulation of errors per period, $E_{\Delta T}$, can be expected. For the above cases, the accumulated error using the nomenclature of Part II is

for CW

$$E_{\Delta T} = \sum_{1}^{5\Delta T} e_j = 5\Delta T\bar{e}, \tag{C2}$$

for MC

$$E_{\Delta T} = \sum_{1}^{2\Delta T} e_j = 2\Delta T\bar{e}, \tag{C3}$$

for NW

$$E_{\Delta T} = \sum_{1}^{\Delta T} e_j = \Delta T\bar{e}. \tag{C4}$$

The quantity $E_{\Delta T}$ indicates that the record, in the absence of measuring errors, can be expected to show a smear; the magnitude of $E_{\Delta T}$ is a measure of how far the smear will extend on the record. Thus, $E_{\Delta T}$ may affect the sensitivity for the recording situation being considered and hence must be added to the over-all measuring error E_R (Appendix A). Since substantially all measuring errors are included within the range $6E_R$ ($\pm 3E_R$), the added effect of overlap in a continuous record is given by

$$\frac{1}{2}E_{\Delta T} \pm 3E_R.$$

---

*Such contacts provide an electrical impulse once each second, except for the 59th second during each minute.

For the sampled type record, however, Appendix A shows that the individual marks are spread over the range

$$E_h = \sum_{1}^{h} e_j = h\bar{e} \ , \qquad \text{(C5)}$$

where $h << 5\Delta T$. Therefore,

$$E_h << E_{\Delta T} < 5E_{\Delta T} \ . \qquad \text{(C6)}$$

In fact, for small samples, E_h may be considered to be zero.* Extreme magnitudes of measuring error are less probable for small samples than for large ones, hence it may be inferred that on the average

$$\frac{1}{2}E_{\Delta T} \pm 3E_R < \pm 3E_R' \ , \qquad \text{(C7)}$$

where $3E_{R'}$ is used as a measure of the maximum expected spread of the sampled timing marks, or

$$3E_R' < 3E_R \ . \qquad \text{(C8)}$$

Of course, as the number of timing marks in a sample is increased, $E_{R'}$ approaches the value E_R .

It is of interest to calculate the magnitudes of some of the quantities just considered. The length $\mathcal{L}$ of the lead screw, for the chronograph which was built, was 10 inches. If it is assumed that $\Delta\mathcal{L} = 0.01$ in, then $\Delta T = 86.4$ seconds, and for the three cases considered the number of timing marks recorded in a continuous record will be:

CW, $5\Delta T = 432$ timing marks,

MC, $2\Delta T = 172$ timing marks, and

NW, $\Delta T = 86$ timing marks.

Values of $E_{\Delta T}$ may be calculated by using values of $\bar{e}$ obtained from Appendix A, Part III; for MC, take $\bar{e} = 58 \times 10^{-6}$ (for $E_N = 10$ spd), and for CW or NW, take $\bar{e} = 93 \times 10^{-6}$ (for $E_N = 40$ spd). Hence, for

CW, $E_{\Delta T} = 0.04$ sec,

MC, $E_{\Delta T} = 0.01$ sec, and

NW, $E_{\Delta T} \cong 0.01$ sec.

The corresponding value, for the sampled record, E_h , may also be calculated for the purpose of comparison; for a sample of 4 timing marks, and with the foregoing values of $\bar{e}$,

CW or NW, $E_h = 0.0004$ sec, and

MC, $E_h = 0.0002$ sec.

For the scale factor used in the interval-record method ($\Upsilon = 1$ sec), the effect of either $E_{\Delta T}$ or E_h is negligible when compared with E_R . The effect, using different

*This may be verified by considering the magnitudes of e_j given in Appendix A of Part III.

scale factors, of the spread of timing marks may be seen in the data given in Table C1. In this tabulation, both continuous and sampled recordings are compared for Y values of both the interval-record and the chronograph methods; for the latter method only the larges and smallest available values have been used to show the range of values expected. The smaller values of Y for the chronograph method can produce records which may be either difficult or impossible to interpret.

TABLE C1
Comparison of Continuous and Sampled Recordings for the Interval-Record and Chronograph Methods of Rate Measurement

Method	Y (sec)	$E_{\Delta T}$ (sec)	$\frac{E_{\Delta T}}{Y} \times 100$ (% of scale factor)	E_h (sec)	$\frac{E_h}{Y} \times 100$ (% of scale factor)	$2E_R$* (sec)	$\frac{2E_R}{Y} \times 100$ (% of scale factor)
		Continuous Recording		Sampled Recording		Continuous or Sampled Recording	
Interval-Record	1.00	0.01	1 (1, 3)	0.0002	0.02 (1)	0.006	0.6 (1)
		0.04	4 (2)	0.0004	0.04 (2, 3)	0.140	14.0 (2, 3)
Chronograph	0.50	0.01	2 (1, 3)	0.0002	0.04 (1)	0.006	1.2 (1)
		0.04	8 (2)	0.0004	0.08 (2, 3)	0.140	28.0 (2, 3)
	0.02	0.01	50 (1, 3)	0.0002	1.00 (1)	0.006	30.0 (1)
		0.04	200 (2, 4)	0.0004	2.00 (2, 3)	0.140	700.0 (2, 3, 4)

*from Table III, where trend lines are employed
(1) applies to MC
(2) applies to CW
(3) applies to NW
(4) not interpretable

* * *

APPENDIX D
Trend Lines

Interpretation of the record can be facilitated by drawing trend lines which are straight lines whose slope is $\bar{R}$.* In Appendix A it has been established that a sample of timing marks only need be recorded once during each subinterval T/C . As the program device was set up and used there was a maximum of four timing marks in each marking subinterval T/C , each sample consisting of the values y_1, y_2, y_3, and y_4 or fewer.† Each sample has a mean ordinate

$$\bar{y} = 1/4(y_1 + y_2 + y_3 + y_4). \qquad \text{(D1)}$$

A straight line between any two values of $\bar{y}$ will determine $\bar{R}$. Since the values of y_i for a given sample are statistically distributed about $\bar{y}$, two parallel trend lines can be drawn to include any given percentage of recorded marks. The separation between these two lines is determined by the standard deviation E_{tot} of all values of $y - y_i$ which occur between the two samples used in constructing them. This standard deviation has two components, one resulting from the measurement and the other resulting from the use of only 4 values of y_i to determine $\bar{y}$.

The first of these components, for a single sample, is

$$\sigma_{er'} = \frac{E}{\sqrt{2}} \qquad \text{(D2)}$$

where $\sigma_{er'}$ is the standard deviation of the universe of values of $\bar{y} - y_i$ (obtained in Part III) and E is the over-all y-axis measuring error. The second component, for a single sample, can be represented by

$$E_{\bar{y}} = \frac{er'}{\sqrt{h}}, \qquad \text{(D3)}$$

where h is the number of marks used in the sample. The two independent components add statistically to give

$$E_{tot} = \sqrt{E_{\bar{y}}^2 + \sigma_{er'}^2}, \qquad \text{(D4)}$$

or

$$E_{tot} = \frac{E}{\sqrt{2}} \sqrt{1 + \frac{1}{h}}. \qquad \text{(D5)}$$

Thus, a measurement of $\bar{y}$ can be written as

$$\bar{y} \pm \frac{E}{\sqrt{2}} \sqrt{1 + \frac{1}{h}}. \qquad \text{(D6)}$$

---

*In Appendix A the equation of a straight line was used to approximate the trend. This implies that the rate is constant, and for purposes of presenting the experimental results this appears to be reasonable.

†While there are both y- and x-axis errors to be considered, this discussion will consider only the former. It is shown in the body of this report that the x-axis errors can be included later.

The interpretation of the patterns and the ultimate determination of the values of R involves the measurement of $\Delta\bar{y}$ as indicated in Appendix A, Eq. (A6); e.g., the quantity

$$\Delta\bar{y} = \bar{y}_f - \bar{y}_i, \tag{D7}$$

in terms of the final and initial values or, in general,

$$\Delta\bar{y} = \bar{y}_a - \bar{y}_b \tag{D8}$$

for use in drawing any trend line or segment. Since by Eq. (A29)

$$R \propto \Delta\bar{y} \ ,$$

the variation in rate as determined from two independent measurements of $\dot{\bar{y}}$ may be represented in general as

$$R \propto (\bar{y}_a - \bar{y}_b) + \left(\frac{E}{\sqrt{2}} \sqrt{1 + \frac{1}{h}}\right) \sqrt{2}$$

or

$$R \propto (\bar{y}_a - \bar{y}_b) + E \sqrt{1 + \frac{1}{h}} \ . \tag{D9}$$

The actual method of constructing the trend lines must be considered. Equation (D5) gives the error associated with a given point $\bar{y}$. It will be recalled from Part III that the over-all measuring errors E_R were given as the standard deviations of nearly normal distributions. The quantity used in Eq. (D5) is also the same statistical quantity. An entire line or segment may be drawn by using these limits. Two limit lines can be drawn, indicating the trend of the recorded marks; for the upper limit, the connecting points are

$$x_a,\ \bar{y}_a + E_{tot} \text{ and } x_b,\ \bar{y}_b + E_{tot}, \tag{D10}$$

and for the lower limit, the connecting points are

$$x_a,\ \bar{y}_a - E_{tot} \text{ and } x_b,\ \bar{y}_b - E_{tot}. \tag{D11}$$

The trend, represented by the two lines (D10) and (D11) for a timepiece whose rate is constant, should include within the line limits 68 percent of the recorded marks, since normal distributions are expected.

It may be desired, however, to obtain a pattern in which more of the marks will be included within the limit lines. To do so it is only necessary to increase the spread from $\pm E_{tot}$ to $\pm 2E_{tot}$. Therefore, to include within the limit trend lines 95 percent of the mark lines should be drawn as follows: for the upper limit, between the points

$$x_a,\ \bar{y}_a + 2E_{tot} \text{ and } x_b,\ \bar{y}_b + 2E_{tot}, \tag{D12}$$

and for the lower limit, between the points

$$x_a,\ \bar{y}_a - 2E_{tot} \text{ and } x_b,\ \bar{y}_b - 2E_{tot}. \tag{D13}$$

The process might be continued further, if desired. The limit lines expressed by (D12) and (D13) are those chosen for presenting the data of this report.

There may be some question concerning the number of timing marks which should be used in each sample. It appears that a balance must be struck between some minimum number and the number required to provide the necessary redundancy to obtain a reasonably correct indication of the location of $\bar{y}$. A reasonable number of marks should be used if it is expected that the recording process is to be made automatic and unattended. Sufficient data is not available from the use of the methods being considered in this report on which to base a realistic redundancy requirement. However, in the experiment, the redundancy requirement was almost eliminated by using the secondary standard. In fact, the secondary standard proved to be capable of being used with greater reliance than WWV, because there were no propagation effects involved in its use.

Having adopted a convention for constructing the trend lines, (D12) and (D13), it is possible either to determine the accuracy obtained when using a specific value of h, or to select a suitable number of marks (h) to obtain a desired accuracy or separation of the trend lines. In either case, the separation of the trend lines is obtained by calculating the value of E_{tot} from Eq. (D5), which is a function of h. To aid in making this calculation and to show the effect of increasing h, Fig. D1 has been drawn; the ratio E_{tot}/E_R is drawn as a function of h.

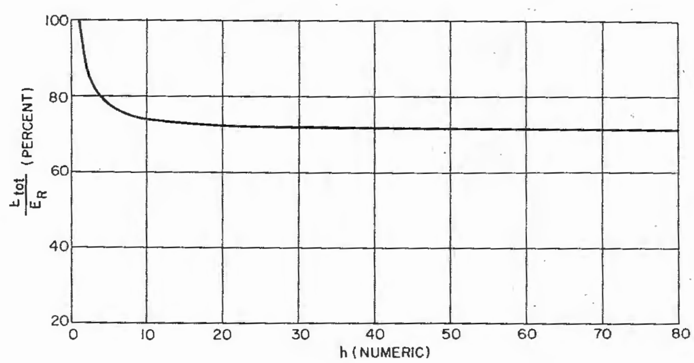


Fig. D1 - The ratio E_{tot}/E_R as a function of the number h of timing marks

By considering Eq. (D5), it is found that the ratio E_{tot}/E_R has the limits,

$$\frac{1}{\sqrt{2}} \leq \frac{E_{tot}}{E_R} \leq 1, \qquad \text{(D14)}$$

which also may be expressed as

$$0.707 \leq \frac{E_{tot}}{E_R} \leq 1.00 . \qquad \text{(D15)}$$

For the values of h shown in Fig. D1, it is seen that the lower limit is an asymptote. Figure D1 shows a rapid reduction for values of h up to 10 and for values greater than 25 an almost negligible reduction is obtained. Thus, it would be of questionable value to use more than 10 marks in a sample and probably it would be uneconomical to use more than 25 marks. It is interesting to point out that for the interval-record method as described in this report,

$$\frac{E_{tot}}{E_R} = 0.80,$$

for $h = 4$. The number 4 was chosen arbitrarily by setting the program device to record for 5 seconds; the first second was not used. For the chronograph method, recordings would have been obtained during the entire 5 seconds, all timing marks being recorded. Hence, for the three cases considered in Appendix C, for an interval of 5 seconds, the values of h and E_{tot}/E_R are given in Table D1.

TABLE D1

Values of E_{tot}/E_R Used in Evaluating Chronograph Record Data

Timepiece	h	$\frac{E_{tot}}{E_R}$
CW	26	0.72
MC	10	0.74
NW	5	0.78

* * *

APPENDIX E
The Error Introduced by Variations in the Sampling Interval T/C (X-Axis Error)

The program mechanism described in Part I as used in the experiment established the sampling interval T/C. The switching times were set by means of continuously rotating cams, driven through reducing gears by a synchronous motor from a 60-cps commercially supplied power source. It is considered that the greatest variation in the sampling interval results from frequency variations in the power source. The resulting error in T or T/C will be determined in terms of the causal variation, Δf that is to be expected. Thence, it is possible to relate the variation ΔT or $\Delta T/C$ to the effect expected on measured rate values, or ΔR.

Variations in the frequency of the power source may be represented as follows:

$$\bar{f} = f + \Delta f, \qquad \text{(E1)}$$

where f is the actual frequency, $\bar{f}$, the nominal frequency, and Δf the difference between the two frequencies; the units are cycles per second. The relative frequency error is $\Delta f/\bar{f}$.

If the frequency f is constant for an interval T seconds, it can be shown by using Eq. (E1) that the frequency error causes an error in time of

$$\Delta T = \frac{\Delta f}{\bar{f}} T. \qquad \text{(E2)}$$

The total error ΔT represents the error with which the final sample is marked, along the x or driving axis. For $T = 86{,}400$ sec and $\bar{f} = 60$ cps, the values of most direct concern,

$$\Delta T = 1440\, \Delta f. \qquad \text{(E3)}$$

The occurrence of this error affects the record and its proper interpretation in the following way. The samples are obtained either later or earlier than desired, or the x-axis shift previously mentioned is obtained. Consider the case of linear trend lines; a shift along the x-axis introduces a shift along the y axis with a consequent effect ΔR on the rate; this effect, from Eq. (E2), is

$$\frac{\Delta T}{T} = \frac{\Delta f}{\bar{f}} = \frac{\Delta R}{R}. \qquad \text{(E4)}$$

The effect on the rate may be obtained by using Eq. (E4); the relative frequency error introduces an equivalent error in the measured rate value, or

$$\Delta R = \frac{\Delta f}{\bar{f}} R, \qquad \text{(E5)}$$

and values of ΔR may be calculated for a given value of $\bar{f}$ if Δf and R are known.

Since the error given by Eq. (E5) is to be combined statistically with other errors, let

$$\Delta f = 3\sigma_f , \tag{E6}$$

where σ_f is the standard deviation of the distribution of frequency variations. Hence,

$$\Delta R = \frac{3\sigma_f}{\bar{f}} R . \tag{E7}$$

Equation (E7) may be used in the design of a chronograph to specify acceptable limits of error in measuring rate and thereby establish the frequency accuracy of the power source required to drive the lead screw. For such an application, it will be desirable to have the range

$$2\Delta R << Y \tag{E9}$$

or

$$2\Delta R \le |R|_{min}, \tag{E10}$$

or it follows that

$$\Delta R \le \frac{|R|_{min}}{2} . \tag{E11}$$

By using Eq. (E7) it can be shown that

$$\frac{\sigma_f}{\bar{f}} R \le \frac{|R|_{min}}{6} ,$$

or

$$\frac{\sigma_f}{\bar{f}} \le \frac{|R|_{min}}{6|R|_{max}} . \tag{E12}$$

If Eq. (8) in the main body of this report, is substituted in Eq. (E12),

$$\frac{\sigma_f}{\bar{f}} \le \frac{E_R}{6|R|_{max}} , \tag{E13}$$

which represents a useful criterion for specifying the allowable frequency error of the driving source, for a given over-all error E_R and a maximum value of rate which it is desired to measure.

While Eq. (E7) gives the desired result for the daily rate, it is useful to express that result for the case where trend lines are drawn. Thus, R' is used instead of R, where

$$R' = \frac{C'}{C} R, \quad (C' \leq C), \tag{E14}$$

in which C' is the number of sampling intervals T/C used in drawing the particular line or segment. Hence, letting $E_x' = \frac{\Delta R'}{3}$ *

$$E_x' = \frac{\sigma_f}{\bar{f}} R', \tag{E15}$$

or

$$E_x' = \frac{\sigma_f}{\bar{f}} \frac{C'}{C} R. \tag{E16}$$

It is of interest to show the effect on measured rate values of the actual use of a commercial source of power for determining the sampling interval. Through conversation with the local power company, it has been found that the frequency stability may be described as follows: $\sigma_f \leq$ 0.25 cps, the standard deviation for normal 24-hour intervals, and $|\Delta f|_{max} \leq$ 3 cps, the greatest deviation in any hourly interval. From Eq. (E7), the value normally to be expected will be*

$$E_x = 4.17 \times 10^{-3} R. \tag{E17}$$

Equation (E17) applies in all cases where the record is a single line. For all other cases Eq. (E16) applies. For the 60-cps commercial power used in the experiments, Eq. (E16) becomes*

$$E_x' = 8.68 \times 10^{-5} C' R. \tag{E18}$$

* E_x' is associated with R'; E_x with R.

* * *

APPENDIX F

Design of an Experiment for the Application of the Chronograph Method to the Investigation of Rate-Pressure and Rate-Temperature Characteristics

Consider the application of the chronograph method to the investigation of specific rate characteristics, i.e., rate-pressure and rate-temperature. Preliminary measurements will show whether r_i is constant, within the criterion given by Eq. (51) in the main body of this report, when the climatic conditions are maintained constant, and also indicate the difference in rate for the range of pressure and temperature being investigated. The following readings may be expected:

(0) r_{ref}

(1) $\left[r_{p_{max}} - r_{p_{min}}\right]$ θ_{max}

(2) r_{ref}

(3) $\left[r_{p_{max}} - r_{p_{min}}\right]$ θ_{min}

(4) r_{ref}

(5) $\left[r_{\theta_{max}} - r_{\theta_{min}}\right]$ p_{max}

(6) r_{ref}

(7) $\left[r_{\theta_{max}} - r_{\theta_{min}}\right]$ p_{min}

(8) r_{ref}

where p_{max} and p_{min} refer to the limiting pressure values, θ_{max} and θ_{min}, the limiting temperature values for the conditions being investigated, and the subscript "ref" is used to denote the conditions of p and θ selected as reference conditions. For the foregoing readings, the symbol outside of the brackets denotes the parameter being held constant.

While using the chronograph in these preliminary tests an auxiliary rate measuring method (e.g., one of those described in Part III) will give an average value of rate $\overline{R}$ equa to the sum of the individual rate values obtained in each step. The rapidity with which these preliminary readings are obtained depends upon (1) the time required for stabilizing both the chamber and timepiece ambient climatic conditions,* and (2) the time required for obtaining each of the required preliminary rate values. A reasonable number of samples is recommended both during the transitional interval when climatic conditions are stabilizing and during the measuring interval when these conditions can be considered to be constant. During the latter interval there should result a reasonable indication of possible erratic performance (lack of constancy of r) and a good measurement of the value of r.

A possible sequence of events for a preliminary test is:

1. Set selected reference conditions and stabilize
2. Maintain constant reference conditions (step 1) for a sufficient length of time for rate measurement
3. Change climatic conditions and stabilize (maximum pressure at maximum temperature)
4. Maintain conditions of step 3 constant for rate measurement
5. Change climatic conditions and stabilize (minimum pressure at maximum temperature)
6. Maintain conditions of step 5 constant for rate measurement
7. Same as step 1
8. Same as step 2
9. Change climatic conditions and stabilize (maximum pressure at minimum temperature)
10. Maintain conditions of step 9 constant for rate measurement
11. Change climatic conditions and stabilize (minimum pressure at minimum temperature)
12. Maintain conditions of step 11 constant for rate measurement
13. Same as step 1
14. Same as step 2
15. Change climatic conditions and stabilize (maximum temperature at maximum pressure)
16. Maintain conditions of step 15 constant for rate measurement
17. Change climatic conditions and stabilize (minimum temperature at maximum pressure)
18. Maintain conditions of step 17 constant for rate measurement

*Tests similar to those described in Part I, the results of which are included in the appendixes of that report, may be required to provide such information.

19. Same as step 1
20. Same as step 2
21. Change climatic conditions and stabilize (maximum temperature at minimum pressure)
22. Maintain conditions of step 21 constant for rate measurement
23. Change climatic conditions and stabilize (minimum temperature at minimum pressure)
24. Maintain conditions of step 23 constant for rate measurement
25. Same as step 1
26. Same as step 2

If each event is assigned a duration which can be represented by the number of sampling intervals required, C', the total duration of the preliminary test is

$$(C'_{tot}) \frac{T}{C} = (C'_1 + C'_2 + \ldots + C'_{26}) \frac{T}{C} = \frac{T}{C} \sum_{1}^{26} C'_i . \qquad \text{(F1)}$$

for the eight rate values corresponding to the four rate differences desired, the five reference values, and the 13 stabilization intervals.

An estimate of C'_{tot} is required to interpret rate values obtained from the records and may be made by attempting to select values for each C'. By considering the stabilization time required from tests similar to that shown in Fig. 1b of the main body of this report or to those whose result are given in the Appendixes of Part I, it might be assumed that a number C'_s might be selected with assurance that an adequate interval will be allowed for each stabilization; if another number, C'_m, be chosen to determine the duration of the measuring interval, such that

$$C'_m = n\, C'_s , \qquad \text{(F2)}$$

where n is a positive integer,

$$(C'_{tot}) \frac{T}{C} = \left[13C'_s + 13(n\, C'_s) \right] \frac{T}{C} = 13(n+1) \frac{T}{C} C'_s . \qquad \text{(F3)}$$

Furthermore, if it is assumed that C'_s (from Fig. 1b in the main body of this report) is about 5, and if $n = 3$, $C'_{tot} = 52$. This would indicate that, provided the timepiece rate is constant, the preliminary test (depending on the actual duration represented by C') for the specific values of T/C used in the interval-record method might be accomplished in about 26 hours, or the entire test might conceivably be recorded in a single traverse of a chronograph lead screw, i.e., 24 hours.

The five rate measurements at the reference conditions should give not only sufficient data for use of the recovery criterion derived in Part IV, but the actual rate values may be used also (together with allowable tolerances) in an application of the rate-value criterion, also derived in Part IV. These two criteria may be used for a measure of unacceptable or acceptable performance. As a result, in the former case, detailed investigation of faulty timepieces is avoided; and for those timepieces which satisfy both criteria, testing may continue.

Following the preliminary measurements, the detailed investigation can be continued to obtain the two families of characteristics: (1) rate-pressure at constant temperature, and (2) rate-temperature at constant pressure.

The rate-pressure family is obtained by measuring the rate at values of pressure differing by increments Δp for each of the temperatures

$$\theta_{min} \leq \theta_1 < \theta_2 < \theta_3 \ldots \leq \theta_{max}.$$

The values θ_1, θ_2, θ_3 ... can be selected as close together as desired, allowing the family to be obtained in as much detail as is deemed necessary for the temperature range being considered.

A value for Δp can be determined by considering the expected change in rate indicated by the preliminary measurements; i.e.,

$$\Delta p_1 = \frac{p_{max}-p_{min}}{\left| r_{p_{max}} - r_{p_{min}} \right| \theta_{max}} |\Delta r_1| \quad \text{(F4)}$$

or

$$\Delta p_2 = \frac{p_{max}-p_{min}}{\left| r_{p_{max}} - r_{p_{min}} \right| \theta_{min}} |\Delta r_2| \quad \text{(F5)}$$

and the greater of these two values should be used. It is seen that the value Δp depends upon the value $|\Delta r|$, for which a limiting value exists ($\pm E_{\Delta R}$ which determines the over-all separation between the trend lines); thus,

$$|\Delta r_1| \;,\; |\Delta r_2| \geq E \quad \text{(F6)}$$

for the rate difference to be detected. The error E represents the statistical combination of the over-all rate measuring error and the error associated with the degree of climatic control actually achieved. This composite error must be evaluated for specific instances. This procedure for estimating the minimum Δp to be used for investigating the rate-pressure characteristics, of course, is an approximation based on a linear characteristic.

The composition of the error E and its derivation may become evident if Eq. (F4) or (F5) is rewritten in incremental form as a partial derivative

$$\frac{\partial r}{\partial p} = \frac{\left| r_{p_{max}} - r_{p_{min}} \pm E_{\Delta R} \right|}{p_{max} \pm E_{pres} - p_{min} \pm E_{pres}} ,$$

where the component errors are: E_{pres} , resulting from the degree of pressure control available, and $E_{\Delta R}$, the over-all error for obtaining rate differences, assuming the temperature is constant. Upon statistical combination of the errors in the denominator,

$$\frac{\partial r}{\partial p} = \frac{r_{p_{max}} - r_{p_{min}} \pm E'_{\Delta R}}{p_{max} - p_{min} \pm E_{pres}\sqrt{2}} .$$

The errors of numerator and denominator may be combined by the method of Worthing and Geffner,* to give, for this case

$$E = \left(\frac{r_{p_{max}} - r_{p_{min}}}{p_{max} - p_{min}} \right) \sqrt{\frac{E_{\Delta R}}{\left(r_{p_{max}} - r_{p_{min}} \right)^2} + \frac{\left(2E_{pres}{}^2 \right)}{\left(p_{max} - p_{min} \right)^2}} . \qquad \text{(F7)}$$

In the same way, an approximation of the minimum value of $\Delta\theta$ can be obtained for investigating the rate-temperature characteristics within the pressure range

$$p_{min} \leq p_1 < p_2 < p_3 < \dots \leq p_{max} ,$$

the values $p_1, p_2, p_3, \dots$ being selected as desired to obtain as much of the family as may be required. As before, either

$$\Delta\theta_1 = \frac{\theta_{max} - \theta_{min}}{\left| r_{\theta_{max}} - r_{\theta_{min}} \right| p_{max}} \left| \Delta r_3 \right| \qquad \text{(F8)}$$

or

$$\Delta\theta_2 = \frac{\theta_{max} - \theta_{min}}{\left| r_{\theta_{max}} - r_{\theta_{min}} \right| p_{min}} \left| \Delta r_4 \right| , \qquad \text{(F9)}$$

*A. G. Worthing and J. Geffner, "Treatment of Experimental Data," p. 207, New York: Wiley, 1946 where

$$|\Delta r_3| \; , \; |\Delta r_4| \geq E' \; , \qquad \text{(F10)}$$

where E' is the composite error similar to and derived in the same way as E in Eq. (F7); or

$$E' = \left(\frac{r_{\theta_{max}} - r_{\theta_{min}}}{\theta_{max} - \theta_{min}}\right) \sqrt{\frac{E_{\Delta R}^2}{\left(r_{\theta_{max}} - r_{\theta_{min}}\right)^2} + \frac{E_\theta^2}{\left(\theta_{max} - \theta_{min}\right)^2}} \; . \qquad \text{(F11)}$$

In evaluating the test results of this report, an estimate of the values of E_{pres} and E_θ may be obtained using the results given in Part IV. Using the rate-pressure and rate-temperature characteristics together with the results obtained for the degree of climatic control obtained, the approximate variations are:

$$E_{pres} = (\text{pressure tolerance}) \times \frac{\Delta R}{\Delta p} \qquad \text{(F12)}$$

and

$$E_\theta = (\text{temperature tolerance}) \times \frac{\Delta R}{\Delta \theta} \qquad \text{(F13)}$$

where, for a conservative estimate, maximum values of $\partial R/\partial p$ and $\partial R/\partial \theta$ should be obtained from the experimental characteristics given in Part IV. In general, the standard deviation of the variation represented by the pressure or temperature tolerance may be taken as 1/3 of the tolerance (i.e., the half-range), or

$$E_\theta = \frac{1}{6}(\text{range of temperature control}) \qquad \text{(F14)}$$

and

$$E_{pres} = \frac{1}{6}(\text{range of pressure control}). \qquad \text{(F15)}$$

The two families of rate characteristics can be investigated within the above limits. Of course, these minimum values serve only as a guide to indicate how closely pressure or temperature points can be obtained. Sufficient information undoubtedly will be obtained from the first of each family to serve as a test for obtaining the remainder of the family.

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APPENDIX G
Glossary of Abbreviations, Definitions, and Symbols

Many of the following terms are either errors per timepiece period of oscillation, sums of such errors, or the rate for various intervals. Such terms can be recognized from the context and their units are those of time, usually seconds. The terms e and t, in particular, may be used individually or collectively. When used individually the units are understandably those of time. However, when they are used collectively to obtain such a term as E_R, the term rate becomes involved. When the term rate is used, there is involved both an accumulated error and an interval (e.g., T) during which the accumulation has taken place. As long as the interval is understood, the accumulated error can be expressed unambiguously in units of time; this of course is partly taken care of by the assignment of particular symbols. Thus, where the term rate is used the units are those of time (representing the error accumulated) and the interval of accumulation is either expressed or to be understood from the context.

A specific problem of units of measurement arises in the case of readings or indicati of timepieces. Any particular reading of a timepiece represents an observation of the pos tions occupied by the hands or the indicating system at a particular instant of time. Henc each reading must have the units of time. Where a time difference or the determination o an indicated elapsed time interval is concerned, the idea of rate becomes involved. As before, the difference between the standard and the indicated intervals, is expressed in convenient units of time; the term rate may be applied in units of time as long as the interval for which the time difference applies either is known or is understood.

$a, b, c, \ldots k, l$ - time intervals (sec) measured with respect to standard or reference timing marks; $a, b, c, \ldots k, l < t_s$ or Y.

C - number of marking subintervals in the interval T; for the interval-record method, $C = 48$, when $T = 86{,}400$ sec.

$C_\ell = \frac{2R}{Y}$ (numeric) - the limiting value of C which should be used in case the maximum expected value of R is known. Note: This limit value will assure a record which can be interpreted unambiguously without requiring tests to determine the value of s.

D - an integer (numeric) used to represent the expected number of lines in a recorded pattern, $D = 0, 1, 2, 3 \ldots$

$D_\ell = \frac{C}{2} - \frac{F}{Y}$ - the maximum possible number of lines expected in any recorded patterr when the scale factor is Y and the number of marking subintervals is C

E_N - the total error accumulated after N periods of oscillation $\bar{t}$, i.e., seconds accumulated during the indicated time interval;

$$E_N = \sum_{1}^{N} e_j$$

(see introductory note regarding units).

E_N' - the value of E_N obtained in the measurement of rate by a particular method, E_N being the true value; see introductory note regarding units

E_p - the over-all measuring error which results from the use of the photo-record method of rate measurement; see introductory note regarding units.

E_p' - the over-all measuring error which results from the use of the photo method of rate measurement;

$$E_p' \sqrt{E_p^2 + E_R^2} \cong E_p;$$

see introductory note regarding units.

E_R - the over-all error associated with the measurement of the quantity t_r, from records; associated with the photo-record, interval-record, and chronograph methods of rate measurement; see introductory note regarding units.

E_v - the over-all measuring error which results from the use of the visual method of rate measurement; see introductory note regarding units.

e - the error per period (sec) of a timekeeping instrument ($e = t_s - t$); subscripts denote specific values as indicated in context; specific values of e will be associated with corresponding values of t; $\bar{e}$, mean value which is associated with $\bar{t}$.

e_r' - the combined errors (sec) introduced by the initial and final periods; for Case 1,

$$e_r' = 2ce + \sum_1^m (e_i + e_f),$$

and for Case 2, $e_r' = 2ce - m(e_i + e_f)$ (see Part II for details).

$e_{\bar{y}} = \frac{1}{2h} \sum_1^h e_j$ - a specific average value of e.

F - a fraction of Y, $0 \leq F \leq Y$, (sec); see R; also $F' = F + Y_0$, where $0 \leq F' \leq 2Y$; and $0 \leq F'' \leq Y$ when $y_{max} = |R| + y_0 = (D+1)Y + F''$.

I - the time (sec) read or indicated visually by a timepiece at a particular instant; subscripts i and f are used to specify the initial and final indicated readings which may be obtained at the start and end of an interval, such as T. $\Delta I = I_f - I_i$ sec per measuring interval T (spd when $T = 86{,}400$ sec) is the indicated elapsed time interval between initial and final visual timepiece readings or indications.

L - an integer (numeric) representing the number of timing marks in a segment of a complete trend line; used especially where $L < C$, and the pattern includes more than a single traverse across the record (see Figs. 1, 2, 3, 4 in the main body of this report).

Linear temporal equivalent - 4,484 in $\cong$ 1 sec (interval-record and photo-record methods).

m - slope of a straight line, see y.

$m_s = \frac{C}{T}(Y - \bar{r})$ - apparent slope (numeric) for Case 2 of Appendix A, where $\frac{Y}{2} \leq r_i \leq Y$.

n - an integral number of timekeeping instrument periods of oscillation $\bar{t}$, comprising the least-count of the instrument when $t_c > \bar{t}$.

N - an integer representing the number of complete periods of oscillation $\bar{t}$ of a timekeeping instrument registered during the time interval T $(N \equiv T/\bar{t})$.

$\frac{N}{C}$ - number of nominal timepiece periods in the subinterval determined by the value C.

P - the time (sec) read or indicated photographically by a timepiece at a particular instant; subscripts i and f are used to specify the initial and final indicated readings obtained at the start and end of an interval, such as T. $\Delta P = P_f - P_i$ sec per measuring interval T (spd, where $T = 86{,}400$ sec) is the indicated elapsed time interval between initial and final photograph timepiece readings or indications.

p - absolute pressure (mm Hg). Δp is used to denote pressure differences; various subscripts are used to denote specific values as taken from the context.

r - the error accumulated during a subinterval (sec/subinterval T/C), used in the interval-record and chronograph methods; subscripts are used to denote specific values as indicated in the context, e.g.,

$$r_i = \sum_{1}^{N/C} e_j ,$$

the error accumulated during any subinterval.

$\bar{r} = \frac{R}{C} = \frac{1}{C} \sum_{1}^{C} r_i = \bar{r}_i = \frac{E_N}{C}$ - the average value of r (sec/subinterval T/C).

r_s - the apparent value of r obtained by measurement from the record;

$$\bar{r} = \frac{SY}{2} + r_s ,$$

for even values of s;

$$\bar{r} = (S+1)\frac{Y}{2} - r_s ,$$

for odd values of s;

$$0 \leq r_s \leq \frac{Y}{2} ,$$

the actual limits of values it is possible to measure from records.

$\mathcal{R}$ - the true value of the rate (spd); equals

$$\sum_{1}^{N} e_j \pm \left[2ce + \sum_{1}^{m} (e_i + e_f)\right]$$

for Case 1 and equals

$$\sum_{1}^{K} n_k e_k \pm [2ce + m(e_i + e_f)]$$

for Case 2 $[|\mathcal{R}| = DY + F]$.

$\mathcal{R}_\ell$ - $\frac{CY}{2}$ sec/interval T (spd when $T = 86{,}400$ sec), the maximum value of rate which can be measured from a record with scale factor Y and having C marking subintervals during T.

$\mathcal{R}_s$ - the apparent or measured value of the rate as obtained from a record;

$$\mathcal{R} = \frac{sCY}{2} + \mathcal{R}_s$$

for even values of s;

$$\mathcal{R} = \frac{(s+1)CY}{2} - \mathcal{R}_s$$

for odd values of s; $\mathcal{R} = \mathcal{R}_s$ for $s = 0$.

R - the effective number of both complete and incomplete periods of oscillation in the measuring interval T; $R = N \pm 2(m+c)$.

s - an integer (numeric), $s = 0, 1, 2, 3 \ldots$; its value is obtained once $\mathcal{R}$ is approximately known from the limits

$$\frac{sCY}{2} \leq \mathcal{R} \leq \frac{(s+1)CY}{2}.$$

S - sensitivity of a rate measuring method (sec/interval T, or spd when $T = 86{,}400$ sec); S may be considered to be the ultimate accuracy or the lower limit of total error, hence the smallest time difference which can be detected by a daily rate measuring method; $S \leq \mathcal{R}$ spd.

spd - seconds per day, a unit of $\mathcal{R}$, the daily rate.

t - the period of oscillation of any timekeeping instrument, i.e., the time (sec) between successive ticks; $\bar{t}$ is the average period.

t_h - specific values of t, for $h = 1, 2, 3 \ldots$

t_s - nominal period (sec) of oscillation of a timekeeping instrument chosen so that $1/\overline{t}$ is an integer which is constant for a given timepiece.

T - total measured time interval (sec).

$\frac{T}{C}$ - the sampling or marking interval (sec) used in the interval-record or chronograph methods.

$\frac{T}{C_\ell} < \frac{TY}{2R}$ - the critical marking interval to be used, when the maximum expected value of R is known, for obtaining an unambiguously interpretable record.

$x = \frac{T}{C}x'$ - introduced to simplify the equation of a linear trend line (for y); x' is given in units of C/T, $0 \le x' \le C$.

$y = mx + b$ - the slope-intercept equation of a straight line, a trend line, see m ; b is the y intercept.

y_0 - the position on a record of the initial timing mark, measured from the standard or reference timing mark; $y_0 < Y$ sec.

y_{max} - the greatest value of y (includes y_0) obtained for the trend line when $x' = C$.

Y - the record scale factor (sec), or the maximum possible value of y whi(can be measured unambiguously.

σ - standard deviation; the context and subscript indicates the quantity ass(ciated with this statistic; e.g., σ_w , σ_t , and σ_s' have the same meanir as used in Part III.

θ - temperature (°C); $\Delta\theta$ is used to denote a temperature difference; vari subscripts are used to denote specific values as taken from the context

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