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JSM 2023: Comparing Normal and Binary D-Optimal Design of Experiments by Statistical Power

Rebecca M. Medlin, Project Leader

Addison D. Adams

August 2023

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Executive Summary

In many applications binary response variables are unavoidable. Many have considered D-optimal design of experiments for generalized linear models. However, little consideration has been given to assessing how these new designs perform in terms of statistical power for a given hypothesis test. Monte Carlo simulations and exact power calculations suggest that D-optimal designs generally yield higher power than binary D-optimal designs, despite using logistic regression in the analysis after data have been collected. Results from using statistical power to compare designs contradict traditional design of experiments comparisons, which employ D-efficiency ratios and fractional design space plots. Power calculations suggest that practitioners that are primarily interested in the resulting statistical power of a design should use normal D-optimal designs over binary D-optimal designs when logistic regression is to be used in the data analysis after data collection.

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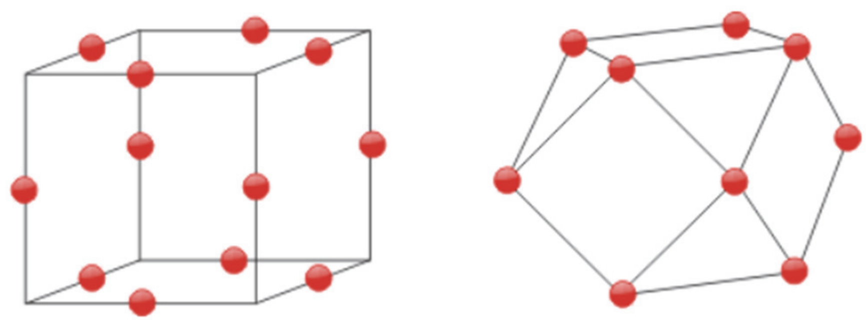

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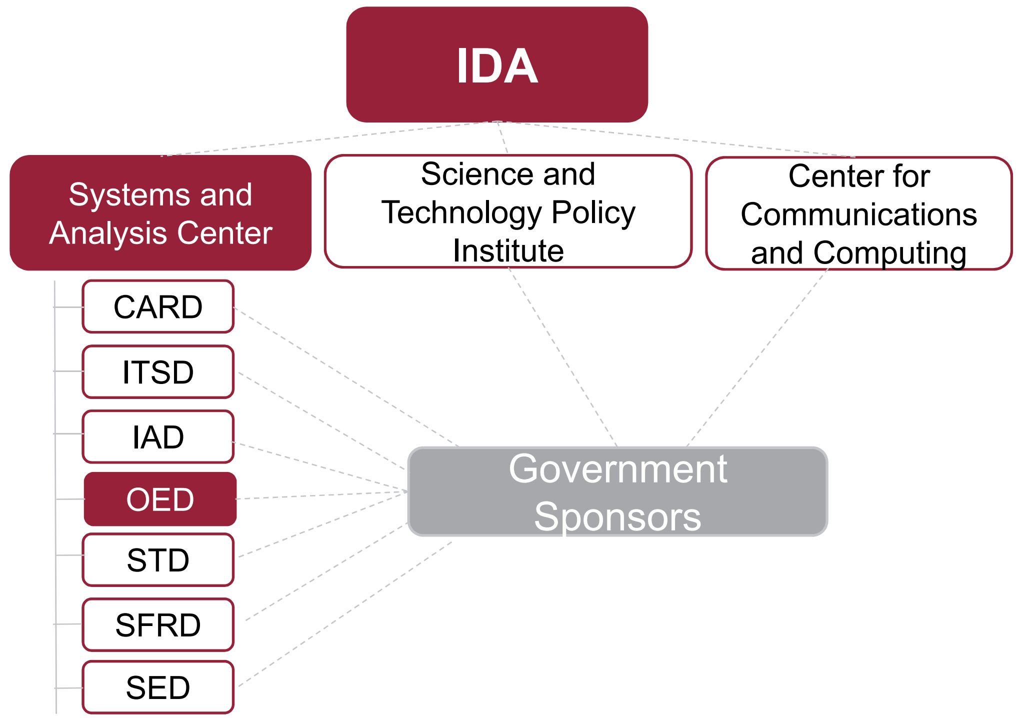
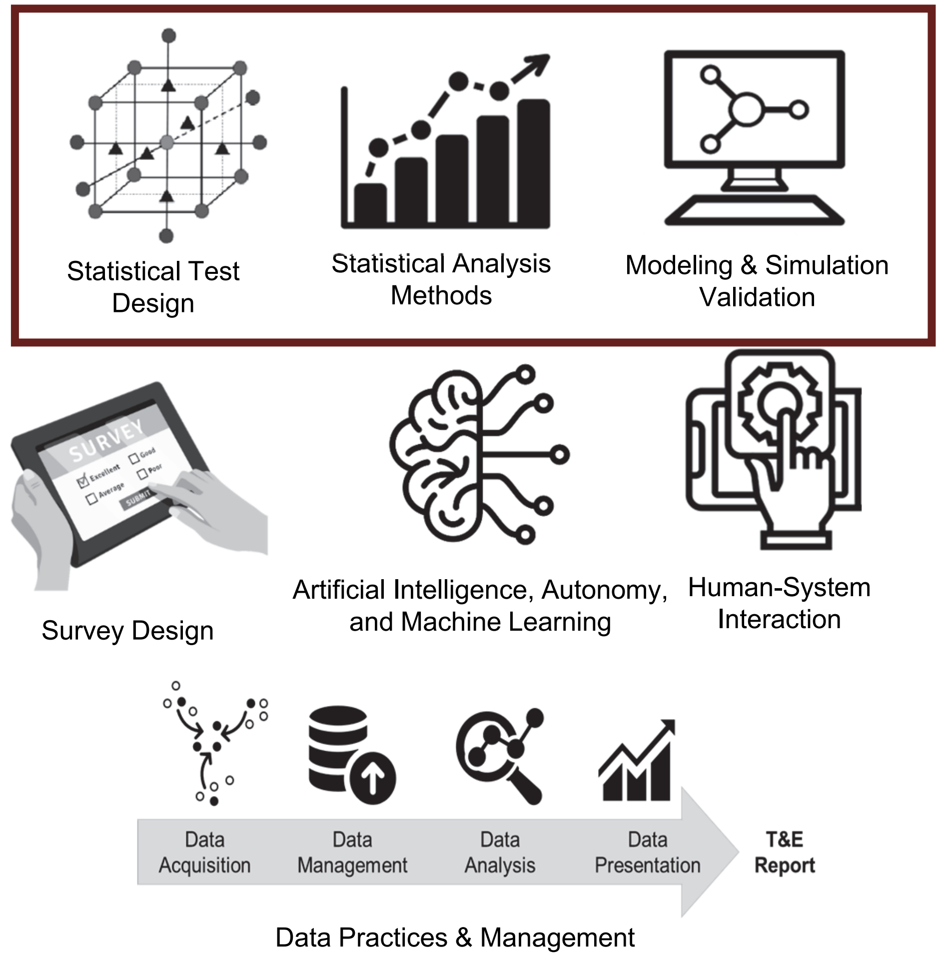


CARD – Cost Analysis and Research Division; ITSD – Information Technology and Systems Division; IAD – Intelligence Analyses Division; OED – Operational Evaluation Division; STD – Science and Technology Division; SFRD – Strategy, Forces and Resources Division; SED – System Evaluation Division

The Test Science Team provides expertise to all warfare areas in OED

We develop, apply, and disseminate statistical, psychological, and data science methodologies

Core Areas of Expertise



TestScience
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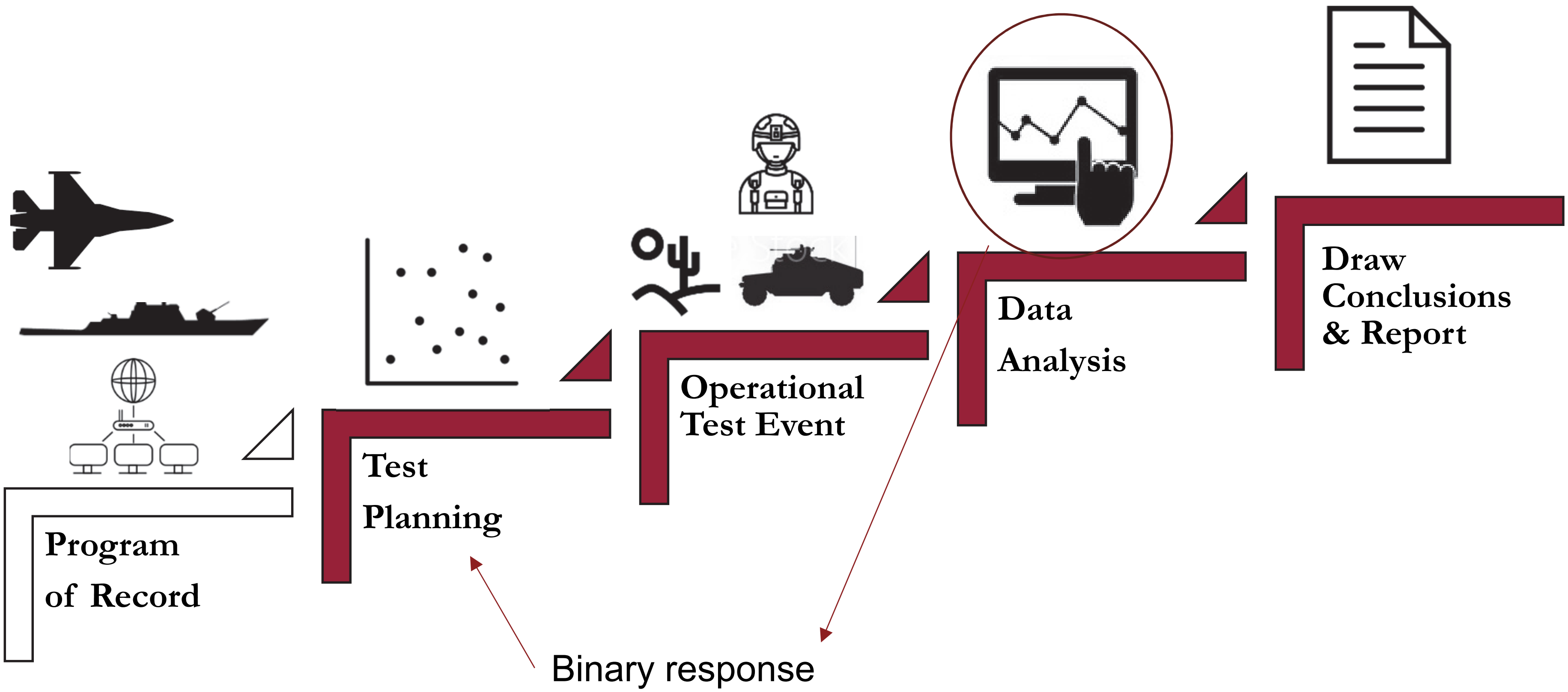
OED: Operational Evaluation Division; T&E: Test and Evaluation

Central Question: How should an operational test event be planned when the response of interest is a success or failure?



- Compare two designs
  - Normal D-optimal design
  - Binary D-optimal design: special case of DOE for GLM

DOE: Design of Experiments; GLM: Generalized Linear Models

Key findings

- We advocate to compare design by statistical power.
- Standard design comparisons favor a binary design.
- Generally, a normal D-optimal design results in higher statistical power than a binary D-optimal design.

DOT&E: Director, Operational Test & Evaluation; SECDEF: Secretary of Defense

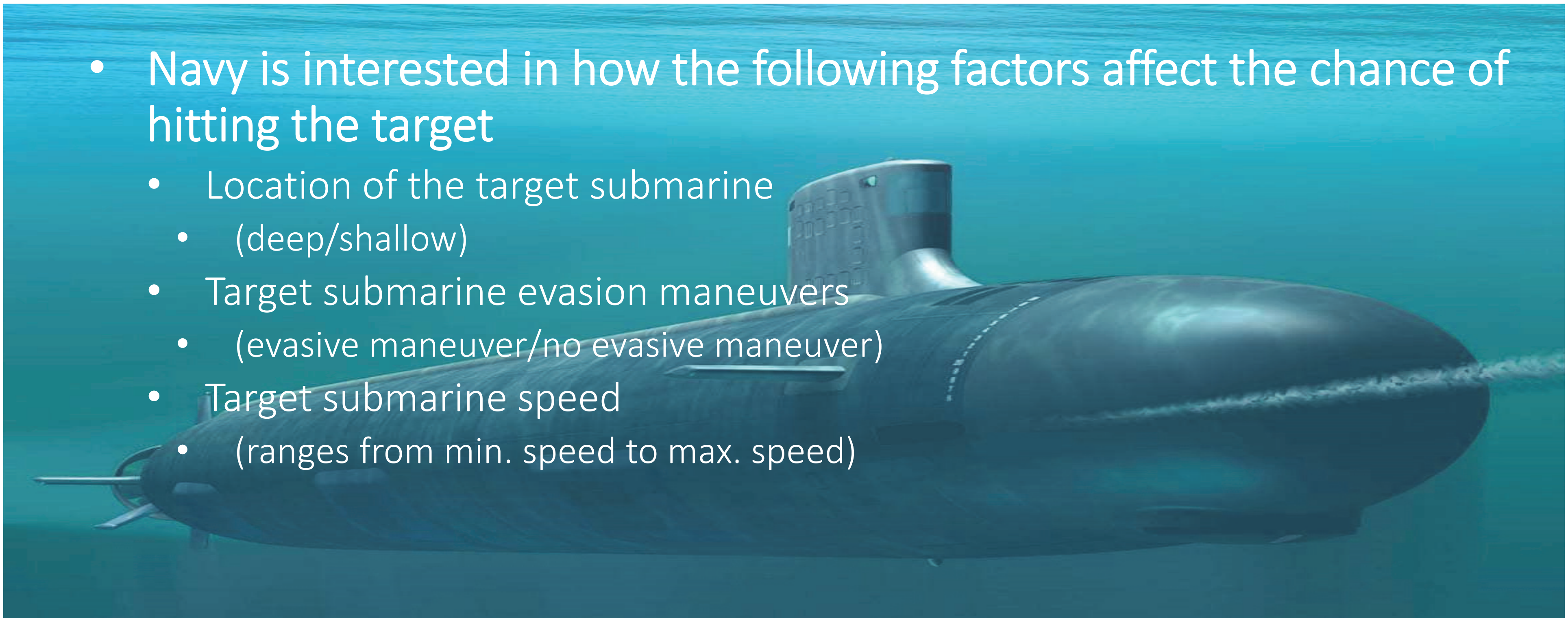
Experimental design for torpedo hit probabilities

- **D-optimal:** A design which minimizes the generalized variance of the parameter estimates

- **Hypothetical:** An operational test event is to be planned to explore condition effects on torpedo hit probability against an adversary submarine
- **Response:** Did the torpedo hit or miss the target boat?

Experimental design for torpedo hit probabilities

- Hypothetical Factors:
  - 2 categorical
  - 1 continuous



- Navy is interested in how the following factors affect the chance of hitting the target
  - Location of the target submarine
    - (deep/shallow)
  - Target submarine evasion maneuvers
    - (evasive maneuver/no evasive maneuver)
  - Target submarine speed
    - (ranges from min. speed to max. speed)

Experimental design for torpedo hit probabilities

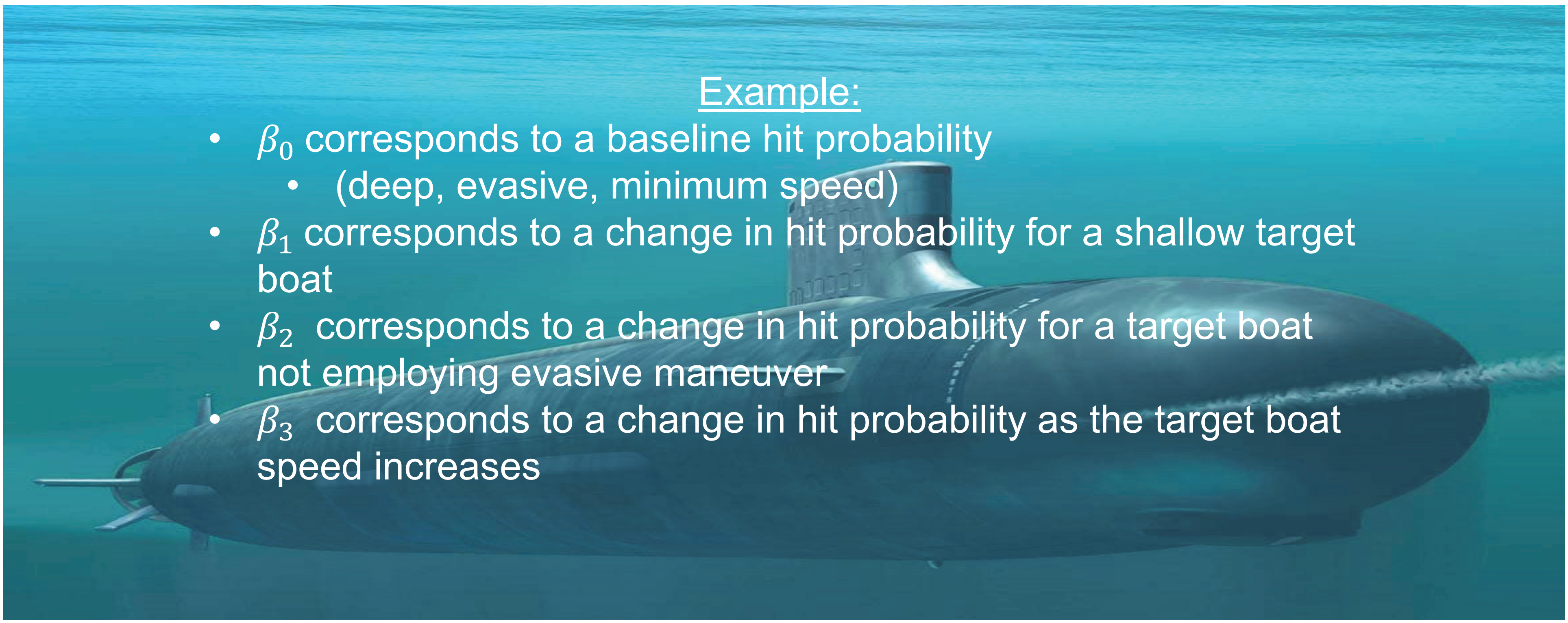
Analysis Model:

$$y_i \sim Bernoulli(\pi_i)$$

$$\text{where } \pi_i = \frac{\exp(x_i^T \boldsymbol{\beta})}{1 + \exp\left(x_i^T \boldsymbol{\beta}\right)}$$

Hypothesis Test:

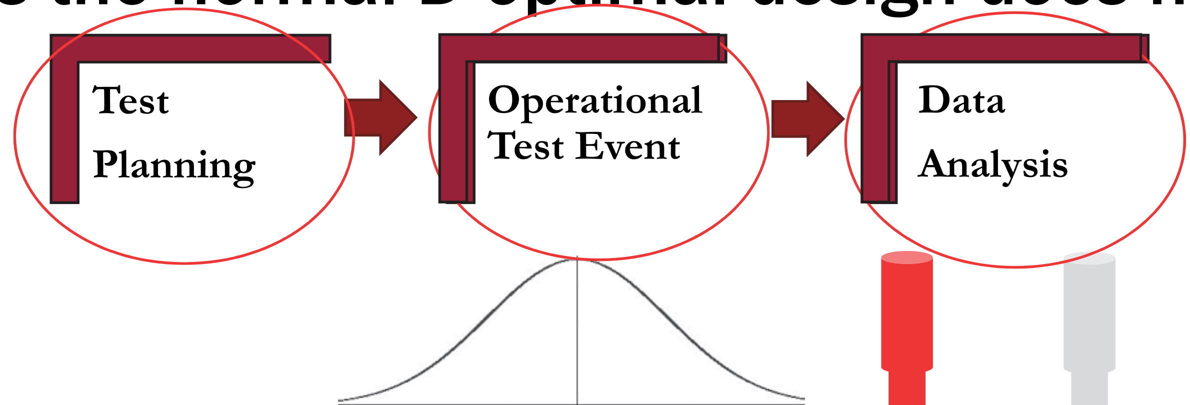
$H_0: \beta_1 = \beta_2 = \beta_3 = 0$ vs H_A: *At least one* $\beta_i \neq 0$ *for* $i = 1,2,3$



Example:

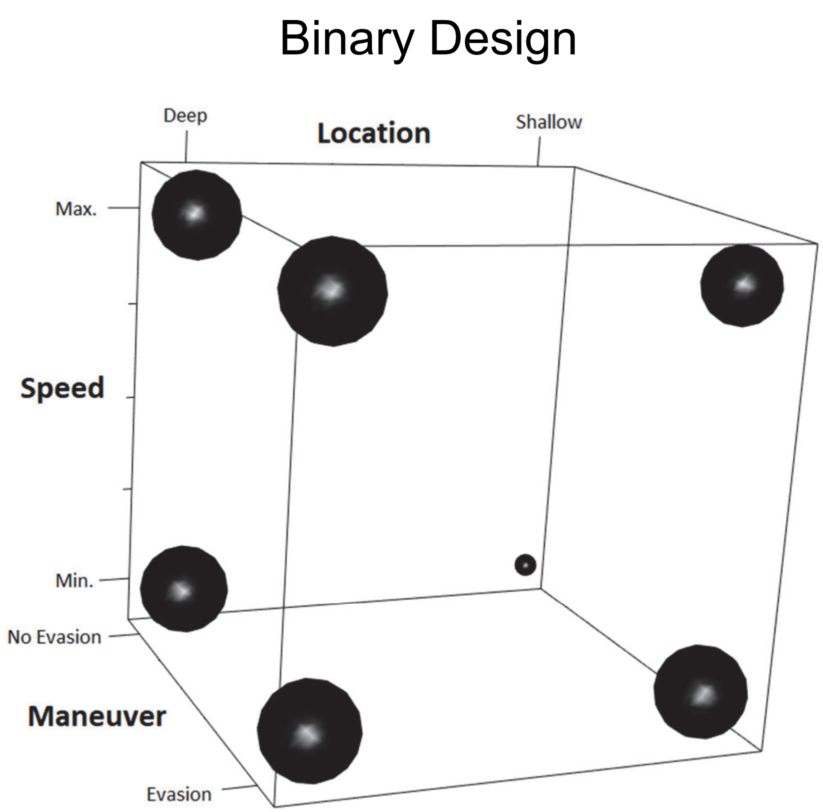
- β_0 corresponds to a baseline hit probability
  - (deep, evasive, minimum speed)
- β_1 corresponds to a change in hit probability for a shallow target boat
- β_2 corresponds to a change in hit probability for a target boat not employing evasive maneuver
- β_3 corresponds to a change in hit probability as the target boat speed increases

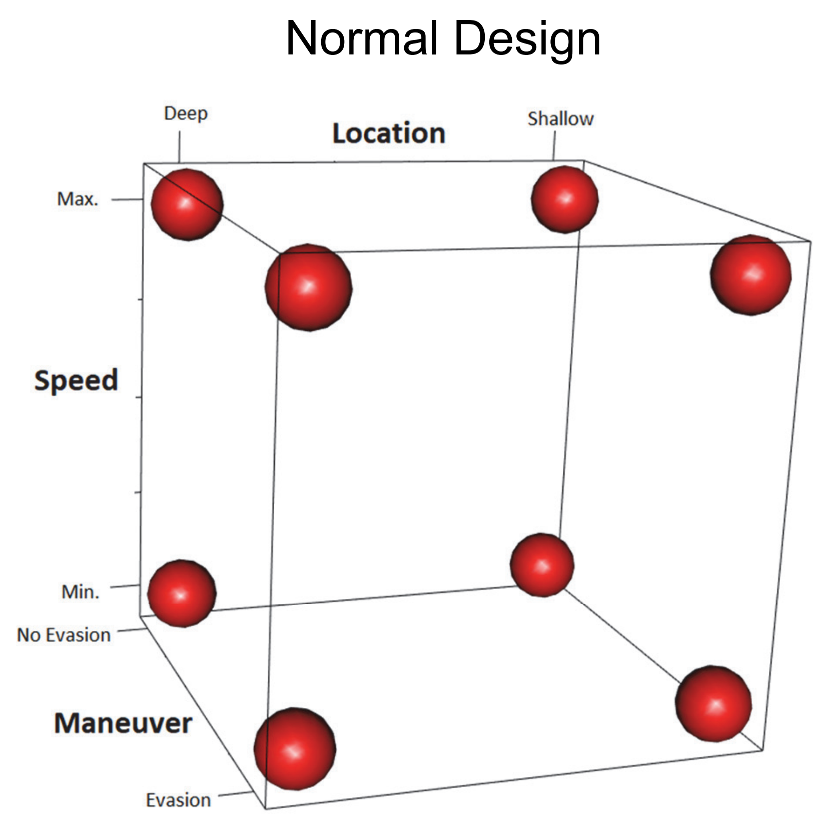
Binary D-optimal design anticipates binary data, whereas the normal D-optimal design does not



	Normal Design	Binary Design
Response Model	$y_i = x_i^T \boldsymbol{\beta} + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$	$y_i \sim Bernoulli(\pi_i)$ where $\pi_i = \frac{\exp(x_i^T \boldsymbol{\beta})}{1+\exp(x_i^T \boldsymbol{\beta})}$
D-Criterion	Maximize the determinant of the information matrix	Maximize the determinant of the information matrix
Information Matrix	$X^T X$	$X^T V_B X$ where V_B depends on $\boldsymbol{\beta}$
Data Collection	Collects binary response variable	Collects binary response variable
Analysis Model	Logistic Regression	Logistic Regression

The binary D-optimal design is unbalanced

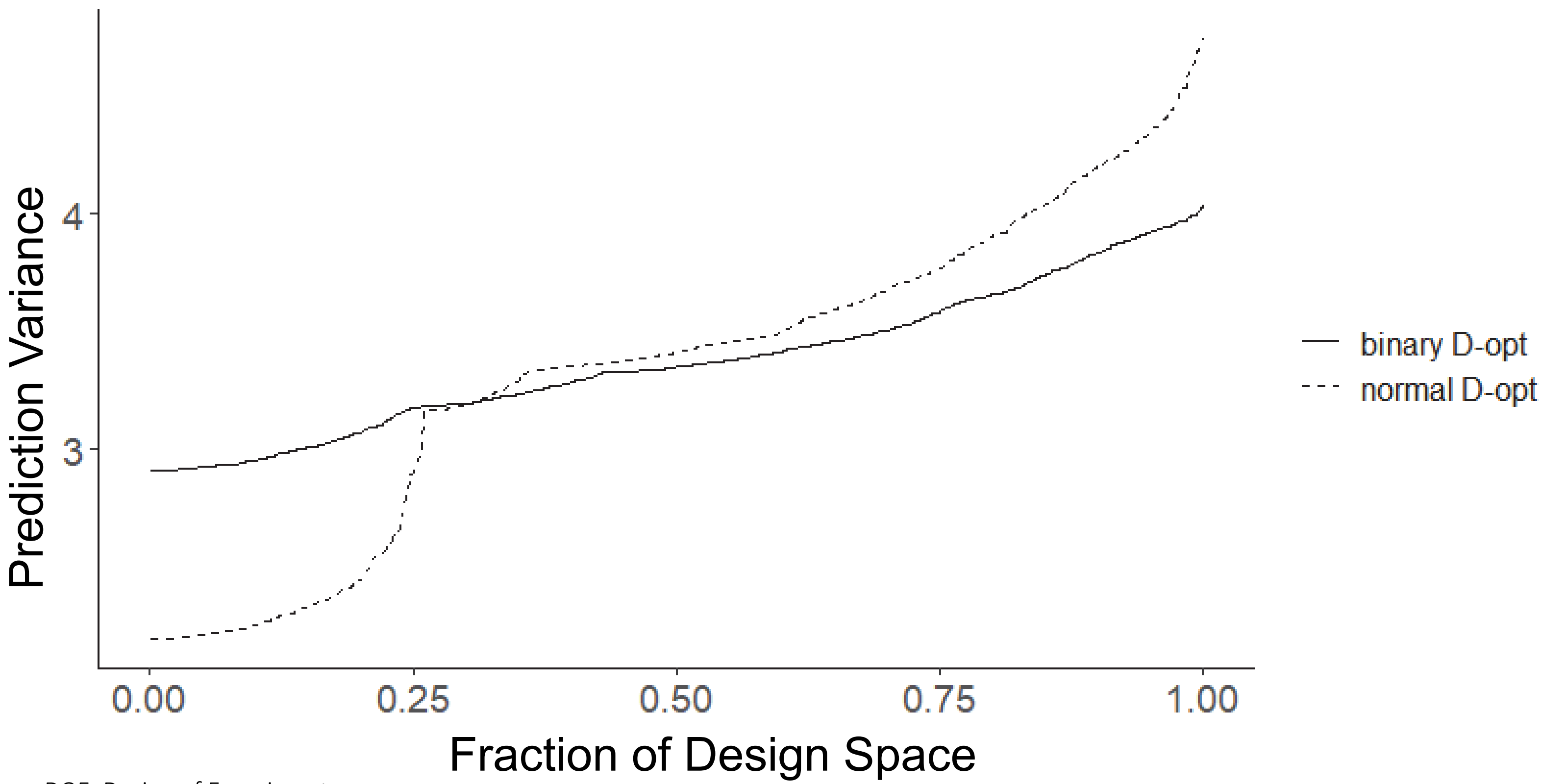




$$\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3) = (-0.5, 1.61, 1.1, 0.205)$$

Standard DOE comparisons favor the binary D-optimal design

- Standard DOE comparisons favor the binary design including:
  - D-efficiency: binary D-optimal is 1.064x more efficient than normal D-optimal
- Fraction of Design Space (FDS) plot



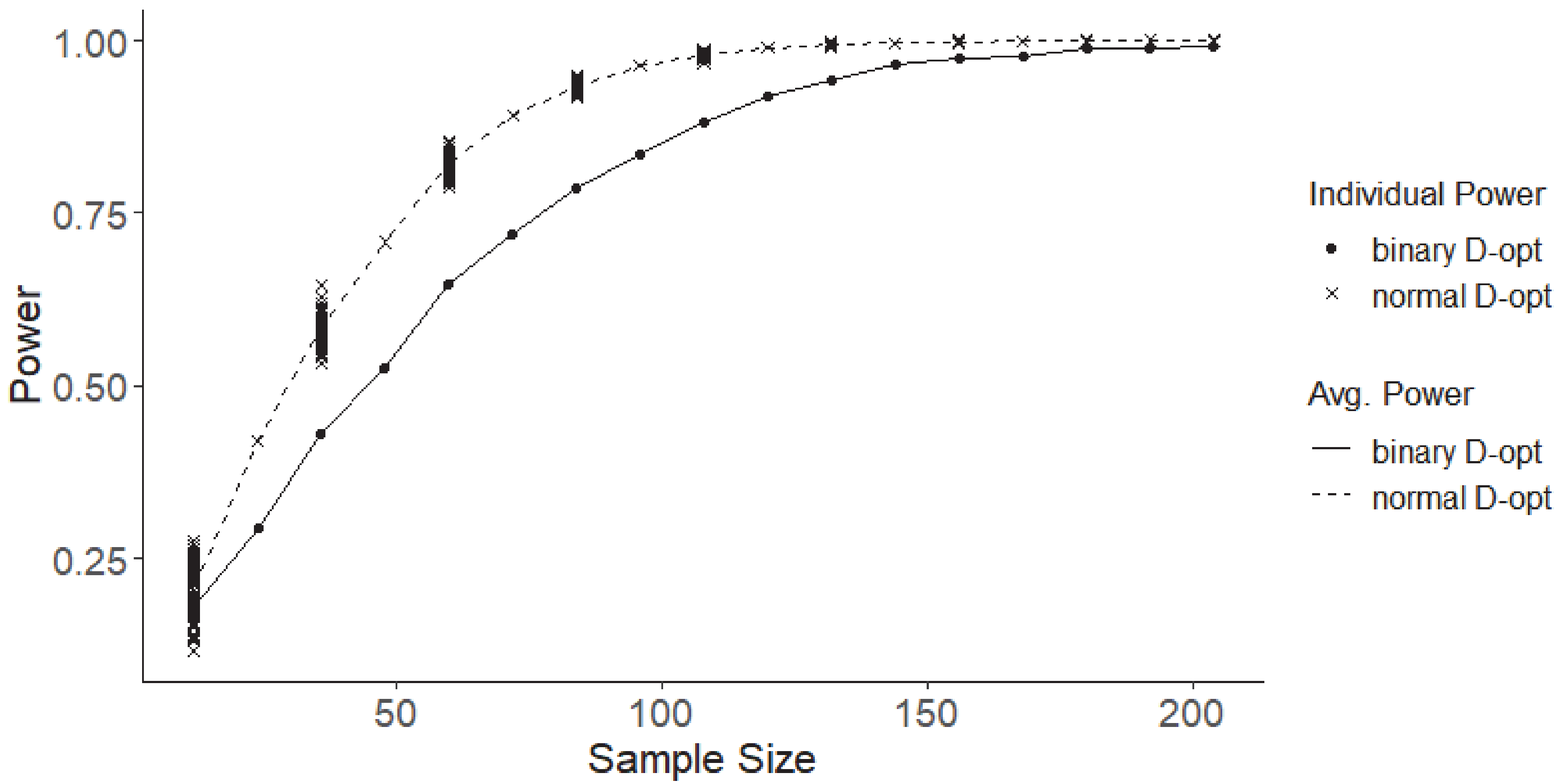
DOE: Design of Experiments

Approximate designs to exact designs

Location	Maneuver	Speed	Normal Design Weight	Binary Design Weight
deep	no evasion	minimum	12.50%	16.42%
deep	no evasion	maximum	12.50%	16.23%
deep	evasion	minimum	12.50%	16.72%
deep	evasion	maximum	12.50%	16.52%
shallow	no evasion	minimum	12.50%	4.40%
shallow	no evasion	maximum	12.50%	0.00%
shallow	evasion	minimum	12.50%	16.12%
shallow	evasion	maximum	12.50%	13.54%

- Approximate designs to exact designs
  - (Pukelsheim, 1993) method
- Statistic used:
  - Likelihood ratio

Binary design underperforms in power analysis



* Using likelihood ratio statistic ($\alpha = 0.05$)

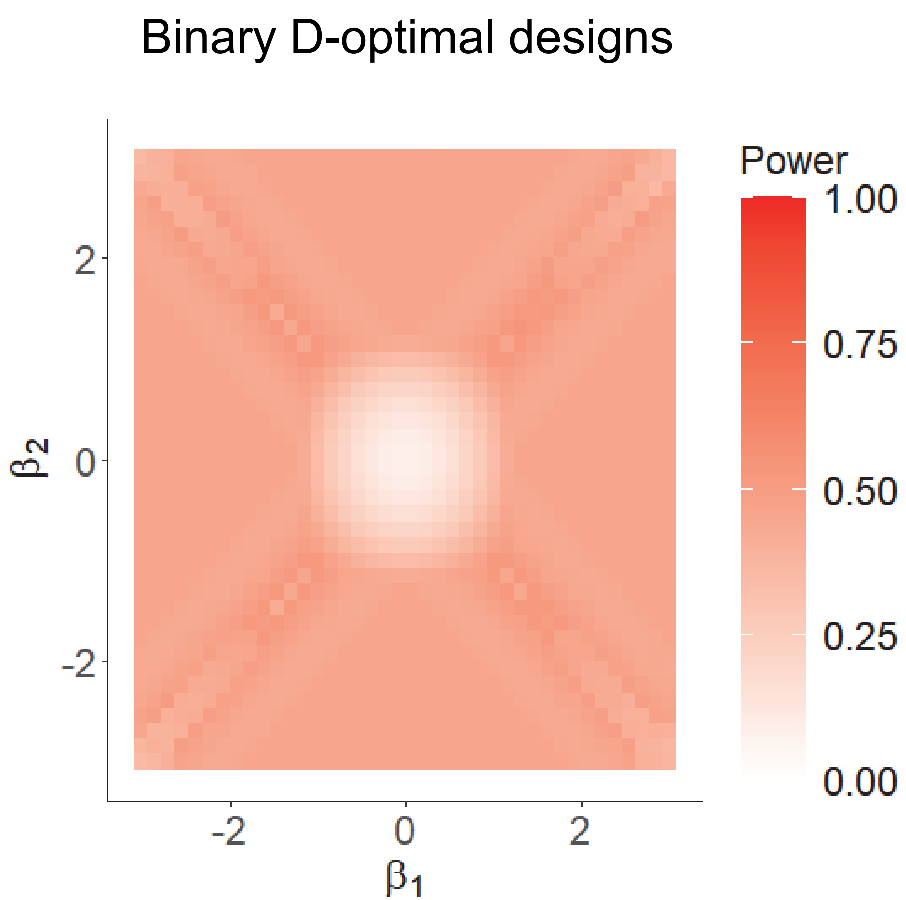
Study power over parameter space

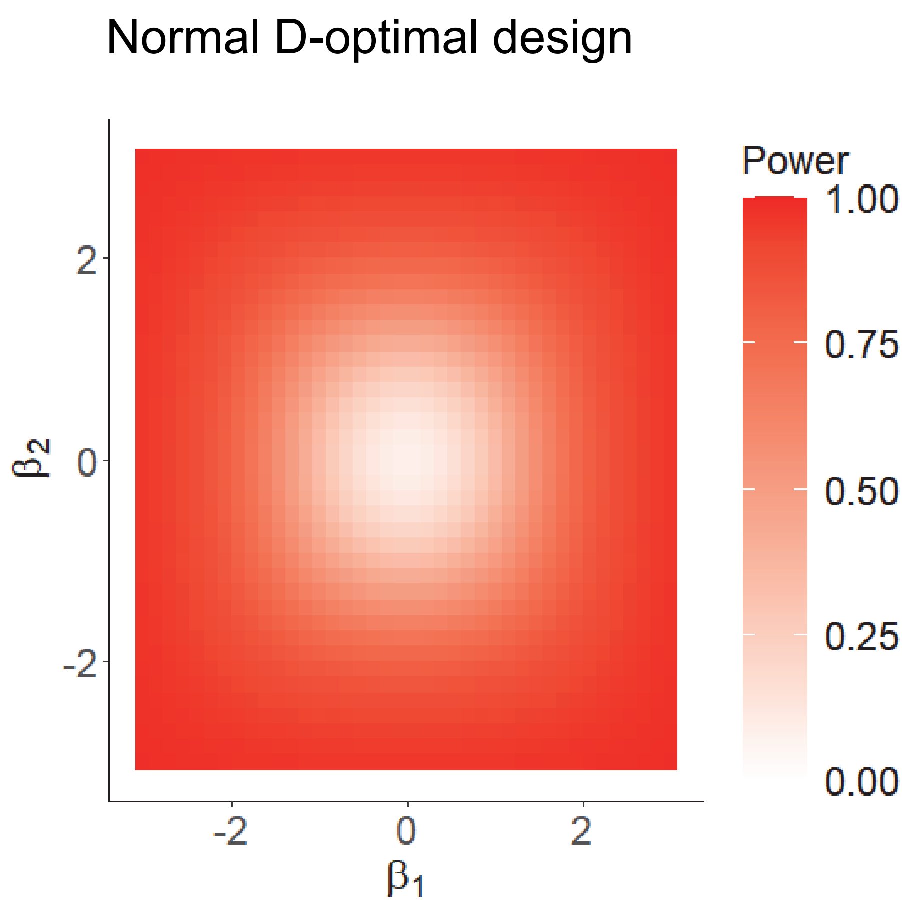
Two continuous factors

- $\eta_i = x_i^T \boldsymbol{\beta}$
- $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)^T$
- $x_i^T = (1, x_1, x_2)$
  - Where x_1 and x_2 are continuous variables in $[-1,1]$

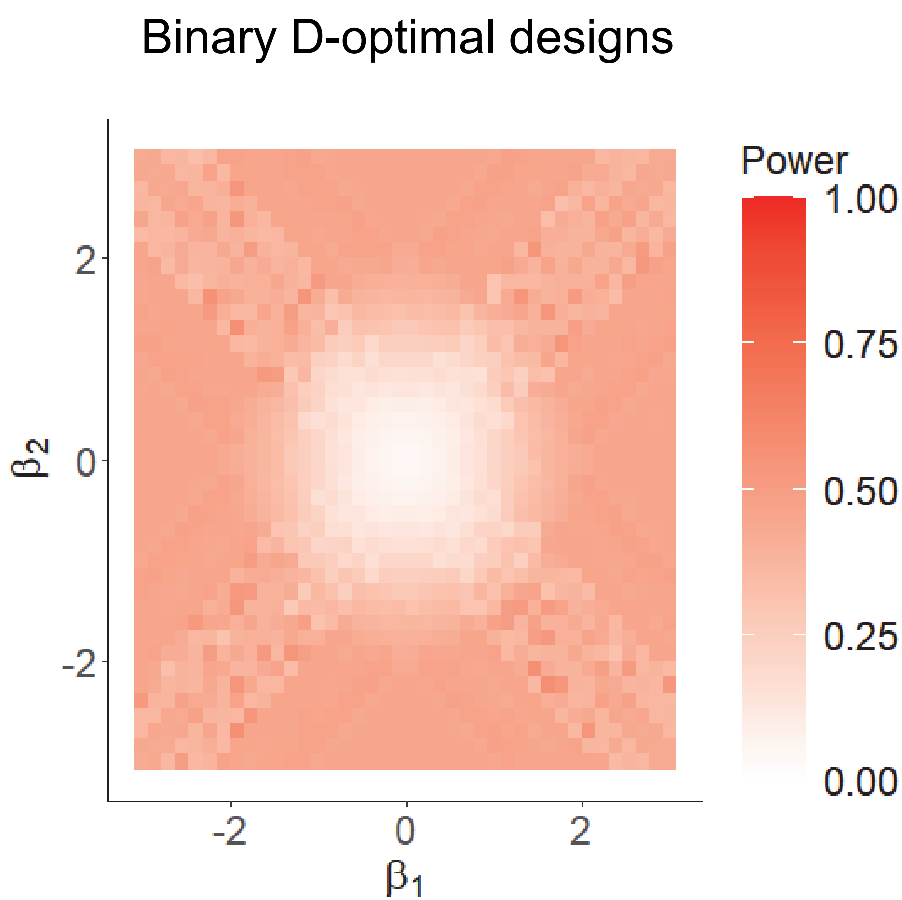
- Fix sample size $n = 12$
- $\boldsymbol{\mathcal{B}} = \{\boldsymbol{\beta}: \beta_0 \in \{0,1,2\}, \beta_1, \beta_2 \in [-3,3]\}$
- For each $\boldsymbol{\beta} \in \boldsymbol{\mathcal{B}}$ over a fine grid,
  - A local binary D-optimal design is found at $\boldsymbol{\beta}$
  - We calculate power for the local binary D-optimal and for the normal design at $\boldsymbol{\beta}$

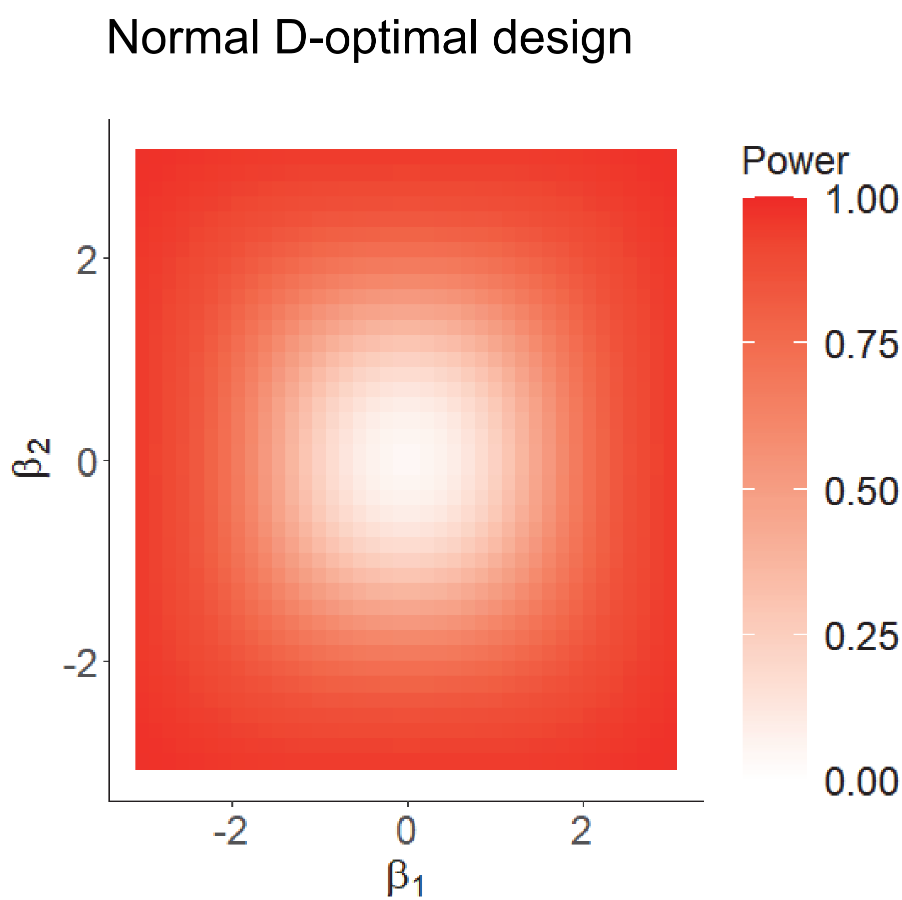
For $\beta_0 = 0$ the normal D-optimal design results in higher power everywhere



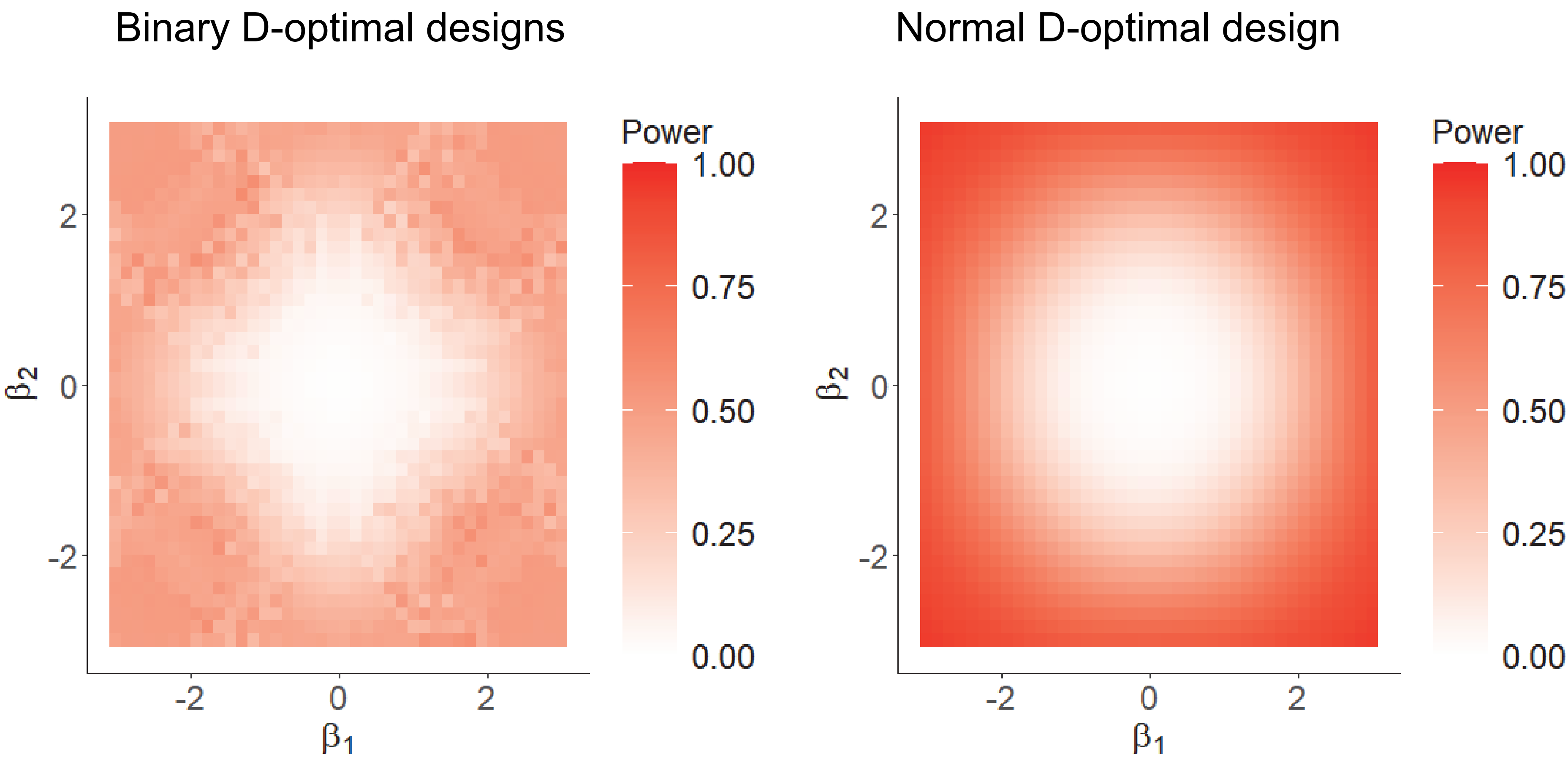


For $\beta_0 = 1$ the normal D-optimal design results in higher power everywhere





For $\beta_0 = 2$ the normal D-optimal design results in higher power nearly everywhere



There is a small, low-power area where local binary D-optimal designs result in slightly higher power than the normal D-optimal design

Where the binary D-optimal design outperforms the normal D-optimal design

Three problems where the binary D-optimal design outperforms the normal D-optimal design:

1. The binary D-optimal design outperforms the normal D-optimal design in low-power regions
2. The binary D-optimal design is constructed using the unknown parameter values
3. As sample size increases, the advantage of the binary D-optimal design dissipates

Fix $\boldsymbol{\beta}$ and increase sample size n

Largest increase for binary D-opt

- $\boldsymbol{\beta}' = (2, -1.31, 1.31)$
- The binary D-optimal design at $\boldsymbol{\beta}'$ results in power ≈ 0.39 at $\boldsymbol{\beta}'$
- The normal D-optimal design results in power ≈ 0.29 at $\boldsymbol{\beta}'$
- Take the approx. designs at $\boldsymbol{\beta}'$ and increase sample size

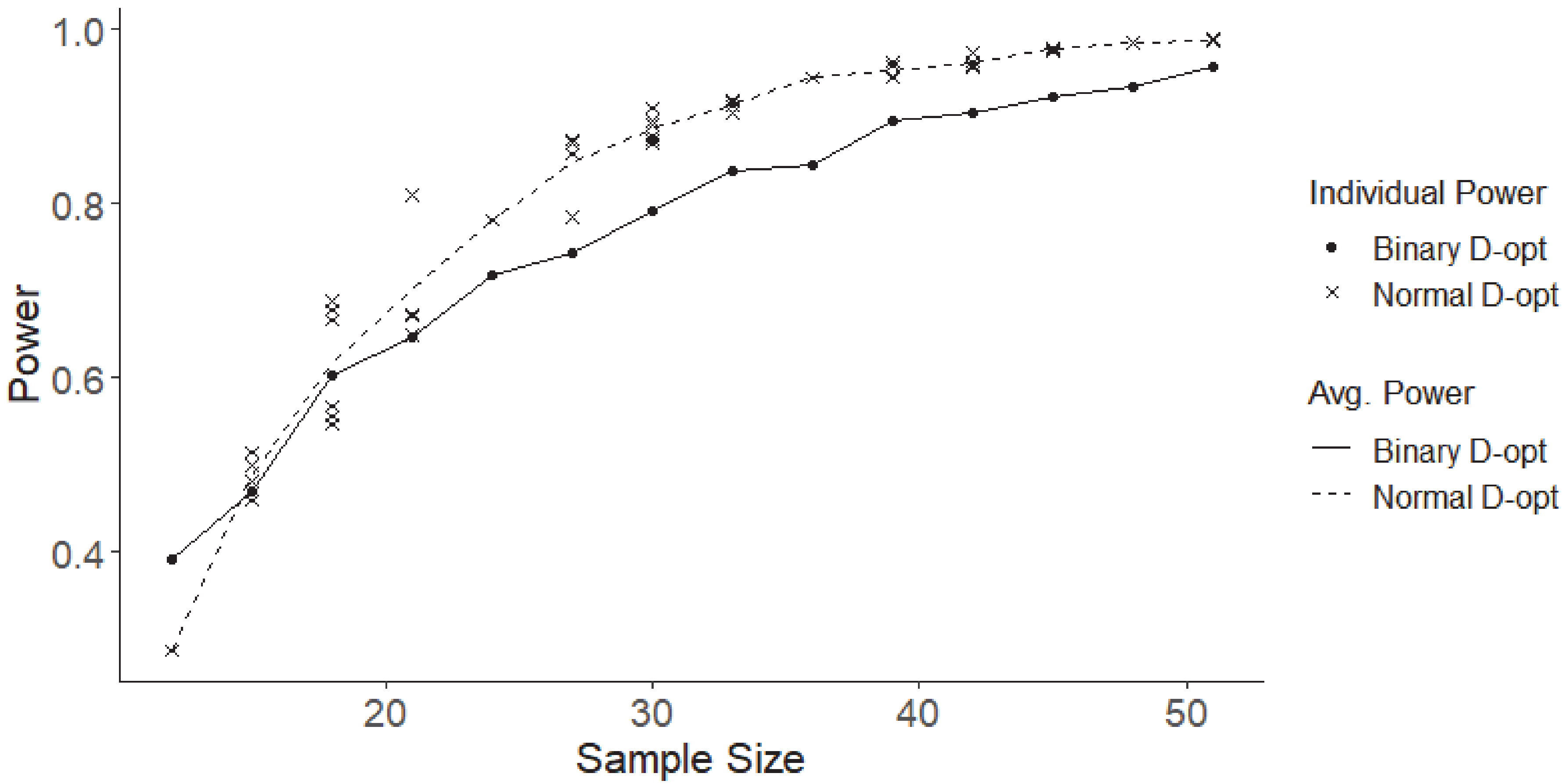
Binary D-optimal design at $\boldsymbol{\beta}'$

Factor 1	Factor 2	Design Weights
1	0.75	1/3
0.64	-1	1/6
1	-0.64	1/6
-0.75	-1	1/3

Normal D-optimal design

Factor 1	Factor 2	Design Weights
1	1	1/4
1	-1	1/4
-1	1	1/4
-1	-1	1/4

The binary D-optimal power advantage disappears with an increasing sample size



Primary Findings and Discussion

- Normal D-optimal designs generally result in higher power than the binary D-optimal designs
- Standard DOE comparisons favor the binary D-optimal design
- Similar findings with interactions and quadratic terms

* Performance is measured by power calculations.

DOE: Design of Experiments

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How Power Analysis was conducted

- Power analysis is conducted via Monte Carlo simulation using the anticipated parameters
- Null Hypothesis: All non-intercept parameters are zero
  - Alternative Hypothesis: At least one non-intercept parameter is non-zero
- Test Statistic is Likelihood Ratio (R package lmtest)
- Parameter estimation is done using the firth correction
  - R package mbest to modify the glm object

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14. ABSTRACT
In many Department of Defense (DoD) Test and Evaluation (T&E) applications, binary response variables are unavoidable. Many have considered D-optimal design of experiments (DOEs) for generalized linear models (GLMs). However, little consideration has been given to assessing how these new designs perform in terms of statistical power for a given hypothesis test. Monte Carlo simulations and exact power calculations suggest that D-optimal designs generally yield higher power than binary D-optimal designs, despite using logistic regression in the analysis after the data have been collected. Results from using statistical power to compare designs contradict traditional DOE comparisons which employ D-efficiency ratios and fractional design space (FDS) plots. Power calculations suggest that practitioners that are primarily interested in the resulting statistical power of a design should use normal D-optimal designs over binary D-optimal designs when logistic regression is to be used in the data analysis after data collection

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