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PREDICTING THE TEMPERATURE PROFILE IN A CLOSED VESSEL USING A ONE-DIMENSIONAL ANALYTICAL HEAT CONDUCTION MODEL WITH TIME-VARYING BOUNDARY CONDITIONS

Christopher Houthuysen
Luis Costa
Carlton Adam

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14. ABSTRACT

This report documents the derivation for the temperature profile of an unsteady, one-dimensional, radial-driven heat conduction problem with time-varying boundary conditions for closed vessel applications. The solution was achieved using the shifting function method to develop a closed-form, analytical solution for the aforementioned configuration. An example case is also provided in this report to walkthrough the solution approach following the derivations. The methodology in this report can be leveraged for various closed vessel configurations if the time-varying heat transfer coefficient is fully defined as a function of time.

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Christopher Houthuysen	(973) 724-4551

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INTRODUCTION

In this paper, an analytical expression for the radial temperature profile of a closed vessel is derived assuming unsteady, one-dimensional heat conduction in cylindrical coordinates. The domain is modeled as a hollow cylinder with a time-varying, convective heat transfer boundary condition at the inner radius and a fixed temperature boundary condition at the outer radius. For the time-varying, convective heat transfer boundary, it is assumed that the temperature of the gas remains relatively constant during the deflagration event. Additionally, the choice of a fixed temperature boundary condition is based on the assumption that conductive heat transfer through the vessel walls occurs at a slower rate compared to the deflagration event, thereby maintaining the outer wall at its initial temperature. Furthermore, this paper utilizes the shifting function method demonstrated in reference 1 to develop a closed-form analytical solution for this particular configuration. The closed-form solution obtained is then numerically solved via series expansion as demonstrated later in this report.

DISCUSSION

Governing Equation and Mathematical Representation

Per reference 2, the general equation for radial-driven, unsteady heat conduction in cylindrical coordinates is written in equation 1. Note that it is assumed that the material properties remain uniform and do not vary as a function of position or temperature.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

Where,

T = Temperature (K)
r = Radius (m)
α = Thermal diffusivity (m^2/s)
t = Time (s)

For this derivation, it is assumed that the closed vessel is initially at ambient temperature, and is subjected to unsteady, forced convection at the inner radius ($r = a$) and a fixed temperature at the outer radius ($r = b$). Mathematically, the initial condition is written as equation 2, and the boundary conditions are written as equations 3 and 4. The boundary conditions are also shown pictorially in figure 1.

$$T(r, 0) = T_0(r) = T_\infty \tag{2}$$

Where,

T_∞ = Ambient temperature (K)

$$-k\frac{dT(a,t)}{dr} = h(t)\left(T_g - T(a,t)\right) \tag{3}$$

Where,

k = Thermal conductivity (W/(m-K))
a = Inner radius (m)
$h(t)$ = Time varying, heat transfer coefficient (W/(m^2-K))
T_g = Gas temperature (K)

$$T(b,t) = T_\infty \tag{4}$$

Where,

b = Outer radius (m)

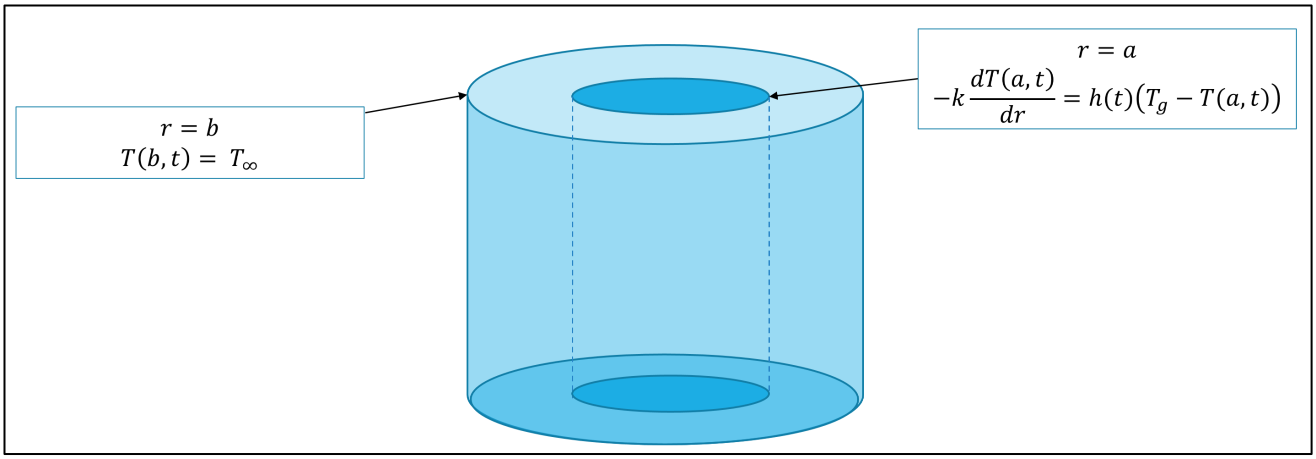

Figure 1
Mathematical representation of closed vessel

To simplify the solution method, the general radius, inner radius, time, temperature, and heat transfer coefficient are nondimensionalized as shown in equations 5 through 9. Note that with the general nondimensionalized radius, the nondimensionalized outer radius is simply 1.

$$R = \frac{r}{b} \tag{5}$$

Where,

R = Nondimensional radius

$$\bar{r} = \frac{a}{b} \tag{6}$$

$$\tau = \frac{\alpha t}{b^2} \tag{7}$$

Where,

τ = Nondimensional time

$$\theta(R,\tau) = \frac{T_g - T(r,t)}{T_{ref}} \tag{8}$$

Where,

θ = Nondimensional temperature profile
T_{ref} = Reference temperature (K)

$$Bi(\tau) = \frac{h(t)b}{k} \tag{9}$$

Where,

$Bi(\tau)$ = Biot function

With these nondimensionalized terms, the governing equation can then be written as equation 10.

$$\frac{\partial^2\theta(R,\tau)}{\partial R^2} + \frac{1}{R}\frac{\partial\theta(R,\tau)}{\partial R} = \frac{\partial\theta(R,\tau)}{\partial\tau} \tag{10}$$

The initial condition can then be written as shown in equation 11. The inner and outer boundary conditions are then written as shown in equations 12 and 13, respectively.

$$\theta_0(R) = \theta(R,0) = \frac{T_g - T_\infty}{T_{ref}} = \Psi \tag{11}$$

Where,

Ψ = Nondimensional initial temperature profile

$$\frac{\partial\theta(\bar{r},\tau)}{\partial R} - Bi(\tau)\theta(\bar{r},\tau) = 0 \tag{12}$$

$$\theta(1,\tau) = \frac{T_g - T_\infty}{T_{ref}} = \Psi \tag{13}$$

To keep the boundary condition of the third kind at the inner surface (i.e., Robin boundary condition), the Biot function is then written as shown in equation 14.

$$Bi(\tau) = \delta + F(\tau) \tag{14}$$

Where,

δ = $Bi(0)$
$F(\tau)$ = $Bi(\tau) - \delta$

Correspondingly, the boundary condition at the inner surface is then rewritten as shown in equation 15.

$$\frac{\partial\theta(\bar{r},\tau)}{\partial R} - \delta\theta(\bar{r},\tau) = F(\tau)\theta(\bar{r},\tau) \tag{15}$$

The Shifting Function Method

To obtain a closed-form solution, the shifting function method is used such that the nondimensional temperature profile, $\theta(R,\tau)$, can be written as shown in equation 16.

$$\theta(R,\tau) = v(R,\tau) + g_1(R)f_1(\tau) + g_2(R)f_2(\tau) \tag{16}$$

Where,

$v(R,\tau)$ = Transformed function
$g_n(R)$ = Shifting functions (to be specified)
$f_1(\tau)$ = Ψ
$f_2(\tau)$ = $F(\tau)\theta(\bar{r},\tau)$

Equation 16 is then substituted into the nondimensional governing equation shown in equation 10 to obtain equation 17.

$$\frac{\partial^2 v(R,\tau)}{\partial R^2} + \frac{\partial^2 g_1(R)}{\partial R^2}\Psi + \frac{\partial^2 g_2(R)}{\partial R^2}F(\tau)\theta(\bar{r},\tau) + \frac{1}{R}\left[\frac{\partial v(\bar{R},\tau)}{\partial R} + \frac{\partial g_1(R)}{\partial R}\Psi + \frac{\partial g_2(R)}{\partial R}F(\tau)\theta(\bar{r},\tau)\right] = \frac{\partial v(\bar{R},\tau)}{\partial \tau} + g_2(R)\frac{\partial(F(\tau)\theta(\bar{r},\tau))}{\partial \tau} \tag{17}$$

With this modification, the initial condition and the inner and outer boundary conditions are once again rewritten as shown in equations 18 through 20, respectively.

$$v(R,0) + g_1(R)\Psi + g_2(R)F(0)\theta(\bar{r},0) = \theta_0(R) \tag{18}$$

$$\frac{\partial v(\bar{r},\tau)}{\partial R} - \delta v(\bar{r},\tau) + \left[\frac{\partial g_1(\bar{r})}{\partial R} - \delta g_1(\bar{r})\right]\Psi + \left[\frac{\partial g_2(\bar{r})}{\partial R} - \delta g_2(\bar{r})\right]F(\tau)\theta(\bar{r},\tau) = F(\tau)\theta(\bar{r},\tau) \tag{19}$$

$$v(1,\tau) + g_1(1)\Psi + g_2(1)F(\tau)\theta(\bar{r},\tau) = \Psi \tag{20}$$

The shifting functions now must be determined such that they make the aforementioned boundary and initial conditions homogeneous. Upon inspection of the constraints, the shifting functions are determined (eq. 21).

$$\left.\begin{aligned} g_1(1) &= 1 \\ g_2(1) &= 0 \\ g_1(\bar{r}) &= 0 \\ g_2(\bar{r}) &= 0 \\ \frac{\partial g_1(\bar{r})}{\partial R} &= 0 \\ \frac{\partial g_2(\bar{r})}{\partial R} &= 1 \end{aligned}\right\} \quad \begin{aligned} g_1(R) &= \left(\frac{\bar{r}-R}{\bar{r}-1}\right)^2 \\ g_2(R) &= \frac{(R-1)(R-\bar{r})}{\bar{r}-1} \end{aligned} \tag{21}$$

Upon substituting the shifting functions into equation 16, it is shown that when $R = \bar{r}$, $\theta(\bar{r},\tau) = v(\bar{r},\tau)$. With this result, the shifting functions are then inserted into equation 17 to obtain equation 22.

$$\frac{\partial^2 v(R,\tau)}{\partial R^2} + \frac{2}{(\bar{r}-1)^2}\Psi + \frac{2}{(\bar{r}-1)}F(\tau)v(\bar{r},\tau) + \frac{1}{R}\left[\frac{\partial v(\bar{R},\tau)}{\partial R} - \frac{2(\bar{r}-R)}{(\bar{r}-1)^2}\Psi + \frac{2R-\bar{r}-1}{\bar{r}-1}F(\tau)v(\bar{r},\tau)\right] = \frac{\partial v(\bar{R},\tau)}{\partial \tau} + \frac{(R-1)(R-\bar{r})}{\bar{r}-1}\frac{\partial(F(\tau)v(\bar{r},\tau))}{\partial \tau} \tag{22}$$

The initial condition is then transformed as shown in equation 23, and the boundary conditions become homogenous as shown in equations 24 and 25. With the boundary conditions now homogeneous, the governing equation now takes the form of the Sturm-Liouville problem where the partial differential equation can be solved via use of separation of variables. Further discussion on Sturm-Liouville theory can be found in advanced mathematics textbooks such as reference 3.

$$v(R,0) = \theta_0(R) - \left(\frac{\bar{r}-R}{\bar{r}-1}\right)^2 \Psi \tag{23}$$

$$\frac{\partial v(\bar{r},\tau)}{\partial R} - \delta v(\bar{r},\tau) = 0 \tag{24}$$

$$v(1,\tau) = 0 \tag{25}$$

Solution to the Spatial Domain

Prior to proceeding, it's worth considering the case where the boundary conditions are constant and homogeneous, and thus the governing differential equation remains of the form provided in equation 10. In this scenario, the governing differential equation is solved by proposing equation 26.

$$\theta(R,\tau) = \varphi_n(R) q_n(\tau) \tag{26}$$

Where,

$\varphi_n(R)$ = Solution to the spatial domain
$q_n(\tau)$ = Solution to the temporal domain

Via substitution of equation 26 into equation 10, equation 27 is obtained.

$$\varphi_n''(R) q_n(\tau) + \frac{1}{R}\varphi_n'(R) q_n(\tau) = \varphi_n(R)\dot{q}_n(\tau) \tag{27}$$

Equation 27 is then divided by $\varphi(R)$ and $q(\tau)$ as shown in equation 28. Since the right hand side of the equation is only a function of space and the left hand side of the equation is only a function of time, both sides of the equation are equal to some constant value, $-\lambda^2$.

$$\frac{1}{\varphi_n(R)}\left(\varphi_n''(R) + \frac{1}{R}\varphi_n'(R)\right) = \frac{\dot{q}_n(\tau)}{q_n(\tau)} = -\lambda^2 \tag{28}$$

With further rearrangement as shown in equation 29, the spatial domain takes the form of the Bessel equation of order 0 (per reference 3) and the temporal domain takes the form of a linear ordinary differential equation.

$$\begin{gathered} \varphi_n''(R) + \frac{1}{R}\varphi_n'(R) + \lambda^2 \varphi_n(R) = 0 \\ \dot{q}_n(\tau) + \lambda^2 q_n(\tau) = 0 \end{gathered} \tag{29}$$

With this aside in mind, the solution to the spatial domain for this application is assumed to take the form of the solution to the Bessel equation of order 0. Thus, the trial function shown in equation 30 is proposed to satisfy the boundary conditions.

$$\varphi_n(R) = c_1 J_0(\lambda_n R) + c_2 Y_0(\lambda_n R) \tag{30}$$

Where,

$c_{1,2}$ = Constants
λ_n = Eigenvalues
J_0 = Bessel function of the first kind of order 0
Y_0 = Bessel function of the seconds kind of order 0

First, the boundary condition at R = 1 is analyzed. By noting that $\varphi_n(1) = 0$, the expression in equation 31 is obtained:

$$c_2 = -c_1 \frac{J_0(\lambda_n)}{Y_0(\lambda_n)} \tag{31}$$

The boundary at the inner radius is then analyzed and is written in the form shown in equation 32:

$$\frac{\partial \varphi_n(\bar{r})}{\partial R} - \delta \varphi_n(\bar{r}) = 0 \tag{32}$$

Via substitution of equations 30 and 31 into equation 32, equation 33 is obtained:

$$\left(-c_1 \lambda_n J_1(\lambda_n \bar{r}) + c_1 \lambda_n \frac{J_0(\lambda_n)}{Y_0(\lambda_n)} Y_1(\lambda_n \bar{r})\right) - \delta \left(c_1 J_0(\lambda_n \bar{r}) - c_1 \frac{J_0(\lambda_n)}{Y_0(\lambda_n)} Y_0(\lambda_n \bar{r})\right) = 0 \tag{33}$$

After algebraic manipulation of equation 33, the transcendental equation as shown in equation 34 is obtained. Note that this equation is used to solve for the eigenvalues pertaining to the spatial domain.

$$\frac{\delta}{\lambda_n} = \frac{J_0(\lambda_n) Y_1(\lambda_n \bar{r}) - J_1(\lambda_n \bar{r}) Y_0(\lambda_n)}{J_0(\lambda_n \bar{r}) Y_0(\lambda_n) - J_0(\lambda_n) Y_0(\lambda_n \bar{r})} \tag{34}$$

The spatial domain now takes the form of the Sturm-Liouville problem, and, thus, the trial functions have the following orthogonality relation as shown in equation 35:

$$\int_{\bar{r}}^{1} R\, \varphi_m(R) \varphi_n(R) dR = \begin{cases} 0, m \neq n \\ N_n, m = n \end{cases} \tag{35}$$

Where,

$$N_n = \int_{\bar{r}}^{1} R\, \varphi_n{}^2(R) dR = \int_{\bar{r}}^{1} R \left(c_1 J_0(\lambda_n R) - c_1 \frac{J_0(\lambda_n)}{Y_0(\lambda_n)} Y_0(\lambda_n R)\right)^2 dR$$

Via integration of N_n over the definite integral, the expression in equation 36 is obtained. Note that equation 36 is left in compact form and must be evaluated by the bounds of the integral.

$$N_n = \left[\left(\frac{c_1{}^2 R^2}{2}\right)\left(J_0{}^2(\lambda_n R) + J_1{}^2(\lambda_n R) - 2\frac{J_0(\lambda_n)}{Y_0(\lambda_n)}[J_0(\lambda_n R) Y_0(\lambda_n R) + J_1(\lambda_n R) Y_1(\lambda_n R)] + \left(\frac{J_0(\lambda_n)}{Y_0(\lambda_n)}\right)^2 [Y_0(\lambda_n R) + Y_1(\lambda_n R)]\right)\right]_{\bar{r}}^{1} \tag{36}$$

Solution to the Temporal Domain

With the current methodology, $v(R,\tau)$ can now be written in terms of an infinite series summation of separable variables pertaining to the temporal and spatial domain as shown in equation 37. Note that this is due to the principle of superposition.

$$v(R,\tau) = \sum_{n=1}^{\infty} \varphi_n(R) q_n(\tau) \tag{37}$$

Equation 37 is then inserted into equation 22 and rearranged as shown in equation 38:

$$\sum_{n=1}^{\infty} \Bigg(\left[\varphi_n''(R) + \frac{1}{R}\varphi_n'(R)\right] q_n(\tau) + \frac{4R-\bar{r}-1}{R(\bar{r}-1)} F(\tau)\varphi_n(\bar{r})q_n(\tau) - \dot{q}_n(\tau)\varphi_n(R) - \frac{(R-1)(R-\bar{r})}{\bar{r}-1}\left(\dot{F}(\tau)\varphi_n(\bar{r})q_n(\tau) + F(\tau)\varphi_n(\bar{r})\dot{q}_n(\tau)\right)\Bigg) = \frac{-4R+2\bar{r}}{R(\bar{r}-1)^2}\Psi \tag{38}$$

The orthogonality principle must now be utilized to obtain the eigenfunction solution. To do this, equation 38 is multiplied by $R\varphi_n(R)$, integrated with respect to R from $\bar{r}$ to 1, and divided by N_n. Prior to proceeding, it is worth noting that $\left[\varphi_n''(R) + \frac{1}{R}\varphi_n'(R)\right]$ is simply equivalent to $-\lambda_n{}^2\varphi_n(R)$. With this approach, equation 38 becomes equation 39:

$$\sum_{n=1}^{\infty}\left(-\lambda_n{}^2 q_n(\tau) + \gamma_n F(\tau)q_n(\tau) - \dot{q}_n(\tau) - \beta_n\left(\dot{F}(\tau)q_n(\tau) + F(\tau)\dot{q}_n(\tau)\right)\right) = \sum_{n=1}^{\infty}\Omega_n \tag{39}$$

Where,

$$\gamma_n = \frac{\varphi_n(\bar{r})}{N_n(\bar{r}-1)}\int_{\bar{r}}^{1}\frac{R\varphi_n(R)(4R-\bar{r}-1)}{R}dR = \frac{\varphi_n(\bar{r})}{(\bar{r}-1)}\left(4\zeta_{J1} - \bar{r}\zeta_{J0} - \zeta_{J0} - \frac{J_0(\lambda_n)}{Y_0(\lambda_n)}(4\zeta_{Y1} - \bar{r}\zeta_{Y0} - \zeta_{Y0})\right)$$

$$\beta_n = \frac{\varphi_n(\bar{r})}{N_n(\bar{r}-1)}\int_{\bar{r}}^{1}R\varphi_n(R)(R-1)(R-\bar{r})dR = \frac{\varphi_n(\bar{r})}{(\bar{r}-1)}\left(\zeta_{J3} - \bar{r}\zeta_{J2} - \zeta_{J2} + \bar{r}\zeta_{J1} - \frac{J_0(\lambda_n)}{Y_0(\lambda_n)}(\zeta_{Y3} - \bar{r}\zeta_{Y2} - \zeta_{Y2} + \bar{r}\zeta_{Y1})\right)$$

$$\Omega_n = \frac{\Psi}{N_n(\bar{r}-1)^2}\int_{\bar{r}}^{1}\frac{R\varphi_n(R)(2\bar{r}-4R)}{R}dR = \frac{\Psi}{(\bar{r}-1)^2}\left(\overline{2r}\zeta_{J0} - 4\zeta_{J1} - \frac{J_0(\lambda_n)}{Y_0(\lambda_n)}(\overline{2r}\zeta_{Y0} - 4\zeta_{Y1})\right)$$

$$\zeta_{Wi} = \frac{1}{N_n}\int_{\bar{r}}^{1}R^i c_1 W_0(\lambda_n R)dR\text{ , } where\ W = J, Y$$

Equation 39 is now rearranged as shown in equation 40. To simplify the nomenclature throughout this section, the summation term is not included; however, it should be noted that all terms still follow the series summation as expressed previously.

$$\dot{q}_n(\tau) + q_n(\tau)\left(\frac{-\lambda_n{}^2 + \gamma_n F(\tau) - \beta_n\dot{F}(\tau)}{-1-\beta_n F(\tau)}\right) = \frac{\Omega_n}{-1-\beta_n F(\tau)} \tag{40}$$

Per equation 40, the solution now takes the form of a first order, linear differential equation that can be solved via methods discussed in ordinary differential equation books such as reference 4. To solve equation 40, an integrating factor is proposed as shown in equation 41:

$$\mu_n = e^{\int\left(\frac{-\lambda_n{}^2+\gamma_n F(\tau)-\beta_n \dot{F}(\tau)}{-1-\beta_n F(\tau)}\right)d\tau} \tag{41}$$

After multiplying both sides of equation 40 by the integrating factor and integrating by τ, the solution to the temporal domain is determined to be as shown in equation 42.

$$q_n(\tau) = \frac{\int\left(\frac{\mu_n \Omega_n}{-1-\beta_n F(\tau)}\right)d\tau}{\mu_n} \tag{42}$$

Solution of the Integrating Factor Assuming a Linear Biot Function

With a general solution obtained for the temporal domain, it is curious to see the closed-form solution for the temporal domain by assuming a given Biot function. As an aside for illustrative purposes of the solution approach, a simple linear Biot function is chosen in the form shown in equation 43:

$$Bi(\tau) = A\tau + B \tag{43}$$

With this form, it is then shown that $F(\tau)$ and $\dot{F}(\tau)$ can be written as shown in equations 44 and 45, respectively.

$$F(\tau) = A\tau \tag{44}$$

$$\dot{F}(\tau) = A \tag{45}$$

Equations 44 and 45 are then plugged into equation 41 to obtain equation 46.

$$\mu_n = e^{\int\left(\frac{-\lambda_n{}^2}{-1-\beta_n A\tau}\right)d\tau} e^{\int\left(\frac{\gamma_n A\tau}{-1-\beta_n A\tau}\right)d\tau} e^{\int\left(\frac{-\beta_n A}{-1-\beta_n A\tau}\right)d\tau} \tag{46}$$

After integrating the terms shown in equation 46, the integrating factor is then determined as shown in equation 47 after some algebraic manipulation. To obtain the closed-form result, equation 47 is inserted into equation 42.

$$\mu_n = e^{\left(\frac{\ln(1+\beta_n A\tau)\left(A\beta_n{}^2+\beta_n\lambda_n{}^2+\gamma_n\right)}{A\beta_n{}^2} - \frac{\gamma_n\tau}{\beta_n}\right)} \tag{47}$$

Solving the Initial Condition

Now that the spatial domain has been solved and a general solution for the temporal domain has been determined, the unknown constants (one per eigenvalue) in the general solution must be determined by solving the initial condition as outlined in reference 5. To do this, equation 23 is utilized and the orthogonality principle is invoked (i.e., multiply both sides by $R\varphi_n(R)$, integrate with respect to R from $\bar{r}$ to 1, and divided by N_n) to obtain equation 48.

$$\sum_{n=1}^{\infty} q_n(0) = \sum_{n=1}^{\infty} \frac{\int_{\bar{r}}^{1} R\varphi_n(R)\left[\theta_0(R) - \left(\frac{\bar{r}-R}{\bar{r}-1}\right)^2 \Psi\right]dR}{N_n} \tag{48}$$

After appropriate substitutions, equation 48 can be expanded into the form shown in equation 49.

$$\sum_{n=1}^{\infty} q_n(0) = \sum_{n=1}^{\infty} \theta_0(R)\left(\zeta_{J1} - \frac{J_0(\lambda_n)}{Y_0(\lambda_n)}\zeta_{Y1}\right) - \frac{\Psi}{(\bar{r}-1)^2}\left[\bar{r}^2\zeta_{J1} - 2\bar{r}\zeta_{J2} + \zeta_{J3} - \frac{J_0(\lambda_n)}{Y_0(\lambda_n)}(\bar{r}^2\zeta_{Y1} - 2\bar{r}\zeta_{Y2} + \zeta_{Y3})\right] \quad (1)$$

The General Solution

Following the methodology outlined herein, the general solution to the temperature profile is obtained via equation 50:

$$T(r,t) = T_g - T_{ref}\left(\sum_{n=1}^{\infty} \varphi_n\left(\frac{r}{b}\right) q_n\left(\frac{\alpha t}{b^2}\right)\right) - T_{ref}\left(\frac{\bar{r}-\frac{r}{b}}{\bar{r}-1}\right)^2 \Psi \quad (50)$$

Example Case

For the configuration discussed throughout this derivation, assume the closed vessel is subjected to the following parameters and has the material properties detailed in table 1.

Table 1
Parameters for example case

Parameter	Value
a	0.003 m
b	0.01 m
k	50 W/m-K
α	1.38274e-05 m^2/s
T_∞	293 K
T_g	2500 K
T_{ref}	$T_g - T_\infty$
$h(t)$	$1{,}382.7t + 10{,}000$ W/(m^2-K)

The parameters shown in table 1 are nondimensionalized and the time varying heat transfer coefficient is converted in terms of the Biot function as shown in equation 51.

$$Bi(\tau) = 2\tau + 2 \quad (51)$$

Following the methodology outlined previously, the eigenvalues for this configuration are computed using the transcendental equation shown in equation 34. For this example, only 13 eigenvalues were used to obtain convergence. The eigenvalues were determined using Newton's method and are plotted in figure 2.

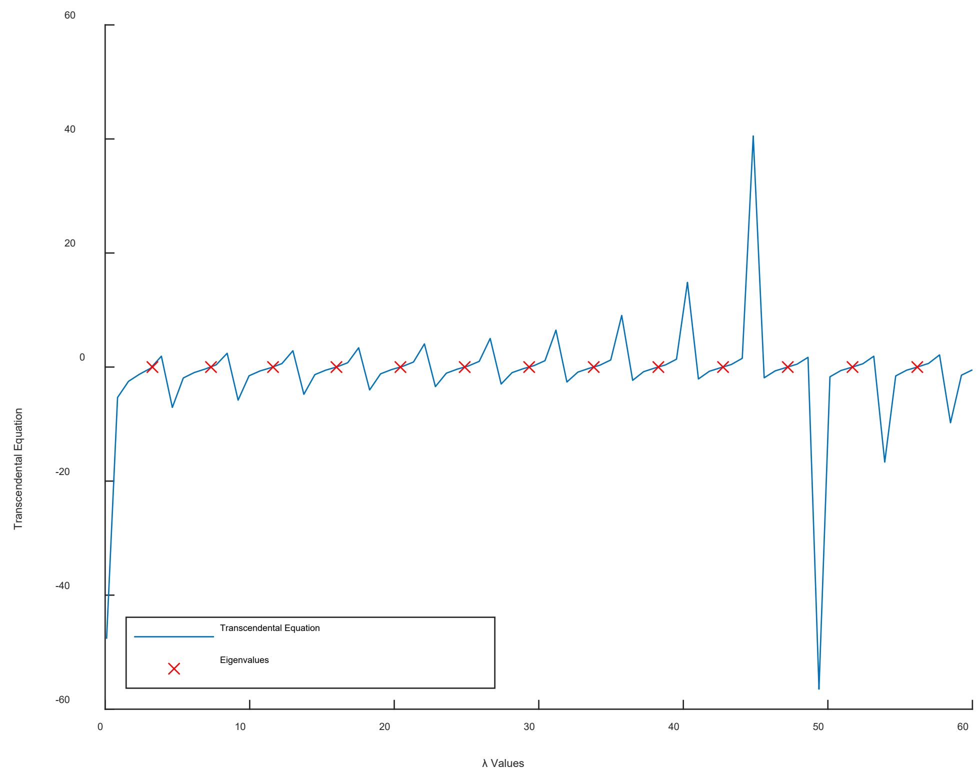

Figure 2
Determining eigenvalues of example case

With the eigenvalues computed, the unknown constants can be computed via solution to the initial condition as shown in equation 49. As stated in the derivation, it is assumed that the entire closed vessel is initially at the ambient temperature. With the initial condition preserved at each radii of interest, the temporal domain can be solved via use of equation 42. The corresponding nondimensionalized temperature profile is then obtained from equation 16 after solving the series summation at each radial position for a given instance in time. The dimensionalized temperature profile is then computed as shown via equation 50.

For this case, the analysis was run for approximately 1.58 sec and the time-varying temperature profiles at discrete radial positions were analyzed. An elongated duration for this example was utilized to analyze how the solution changes radially as a function of time. As shown in figure 3, a steep initial increase in the temperature profile close to the inner radius is achieved and then all interior positions begin to gradually increase with time due to the increased heat flux boundary condition. The results are shown in figure 3.

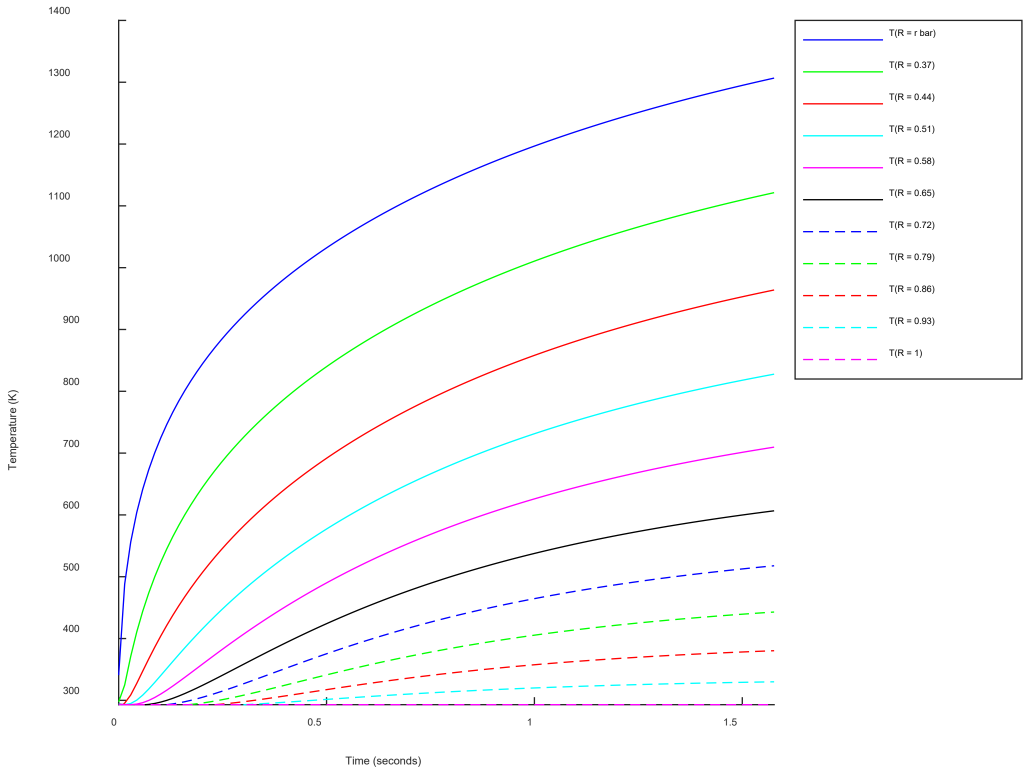

Figure 3
Temperature versus time for example case

It was then curious to see how this case compares with the steady-state solution [assuming a heat transfer coefficient of $10{,}000$ W/(m^2-K)]. As shown in figure 4, the surface temperature is higher due to the gradual increase in the heat transfer coefficient; however, the temperature of the transient solution is lower toward the outer radius of the tube to ensure that the solution remains continuous and differentiable throughout the spatial domain. Note that this is due to the imposed constant temperature boundary condition at the outer radius.

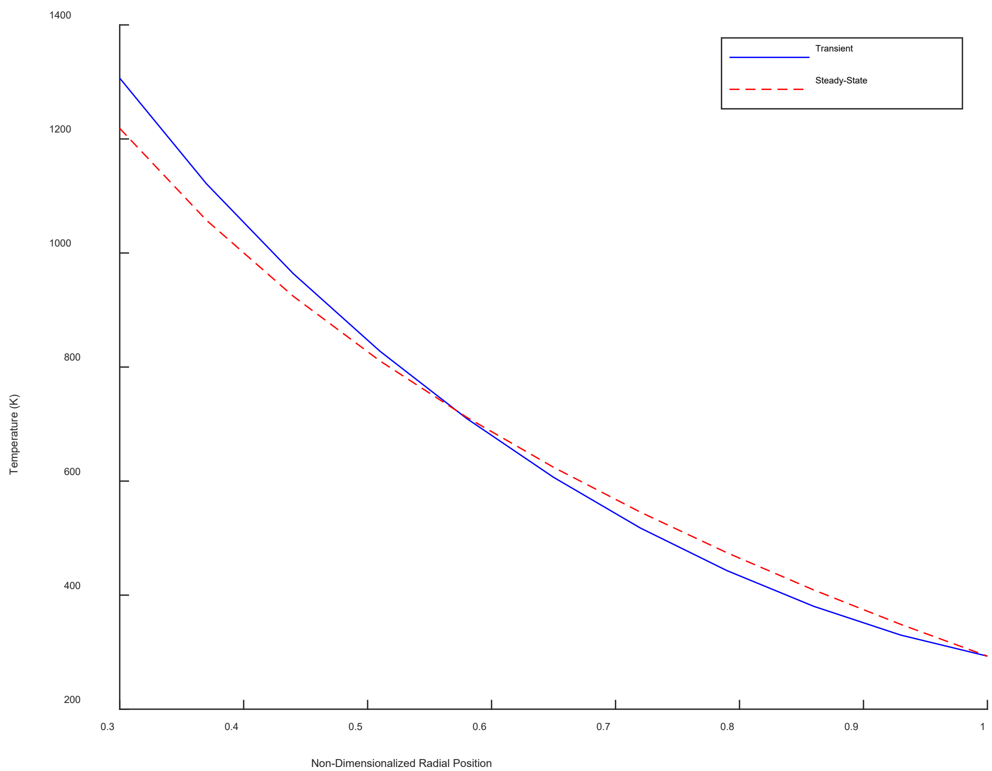

Figure 4
Temperature profile of transient and steady-state solutions for example case

CONCLUSIONS

This report documents the derivation for the radial temperature profile of an unsteady, one-dimensional heat conduction problem with time-varying boundary conditions for closed vessel applications. The solution was achieved using the shifting function method to develop a closed-form analytical solution for the configuration discussed herein. An example case was also provided in this report to walkthrough the solution approach and showcase the results. Furthermore, the methodology outlined in this report can be used to quantitatively determine the temperature profile of closed vessels if the heat transfer coefficient is defined as a function of time.

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Predicting the Temperature Profile in a Gun Barrel Using a One-Dimensional Analytical Heat Conduction Model with Time-Varying Boundary Conditions
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