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SATELLITE LAUNCHING TRAJECTORY CALCULATIONS

GENERAL DESCRIPTION OF NAREC PROGRAMS FOR THE TWO-DIMENSIONAL CASE

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ABSTRACT

This report describes in a fairly detailed manner the common characteristics of programs used to calculate satellite launching vehicle trajectories on the Laboratory's electronic digital computer (NAREC). Equations of motion and auxiliary formulas are given along with a description of program organization. The formulations are two-dimensional ones and are largely concerned with powered flight for the first two stages. The report is intended to be a source of specific information for the problem originators and others who use these programs. It should also be useful to anyone engaged in the design of similar programs.

PROBLEM STATUS

This is an interim report; work is continuing.

AUTHORIZATION

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INTRODUCTION

This report describes in a fairly detailed manner common characteristics of programs used to calculate satellite launching vehicle trajectories on the Laboratory's electronic digital computer (NAREC). Equations of motion and auxiliary formulas are given along with definitions and units of the physical quantities involved. A brief description of the structure and logical organization of the programs is also given. An attempt will be made as discussion proceeds to indicate some of the ways in which these programs can be easily modified or extended. The report is intended to be a source of specific information for the problem originators and others who use the programs. It should also be useful to anyone interested in the design of similar programs.

The formulations given are not necessarily those which are advantageous for computational work, but are theoretically equivalent to them. The forms given are the result of factoring for convenient presentation. Scaling considerations necessitated by the fact that NAREC is a fixed point machine have also been omitted.

EQUATIONS OF MOTION

The equations given below are solved numerically to obtain vehicle trajectories for the time period from launching of the vehicle through burnout of its second-stage engine. The launching vehicle is treated as a point mass subject to the forces of thrust, lift, drag, and gravitation. The earth is represented as spherical and nonrotating with mass concentrated at its center. A two-dimensional, rectangular coordinate system is used with origin considered to be at the center of the earth and y-axis passing through the vehicle launching point (Fig. 1).

The equations of motion are as follows:

$$\ddot{y} = -\frac{Gy}{r^3} + \frac{g_R}{w - \dot{w}t}\left[F \sin \bar{\theta} + \frac{1}{2} \rho v^2 S \left(- C_D \sin \theta + C_L \alpha \cos \theta\right)\right]$$

$$\ddot{x} = -\frac{Gx}{r^3} + \frac{g_R}{w - \dot{w}t}\left[F \cos \bar{\theta} + \frac{1}{2} \rho v^2 S \left(- C_D \cos \theta - C_L \alpha \sin \theta\right)\right]$$

where

$$F = F_R + A_e \left(p_R - p\right)$$

$$\rho = 0.0839239\ (p/T)$$

$$\theta = \tan^{-1}(\dot{y}/\dot{x})$$

$$\alpha = \bar{\theta} - \theta$$

$$G = g_R R^2$$

$$v^2 = \dot{y}^2 + \dot{x}^2$$

$$r^2 = y^2 + x^2$$

$$\dot{w} = \frac{F_R}{I_R}.$$

Also needed are

$$h = r - R$$

$$M = \frac{v}{a}.$$

Two quantities which are often of interest but not required in the solution of the above equations are

$$\dot{\theta} = \frac{1}{v^2}(\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x})$$

$$s = R \tan^{-1}(x/y).$$

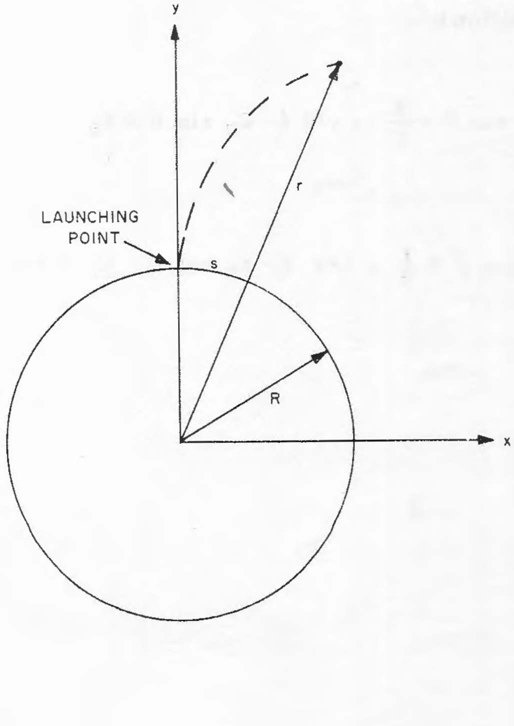

Fig. 1 - Coordinate system

Definitions of the above quantities follow, with physical units given in parentheses. The orientation of vector quantities is shown in Fig. 2.

t (seconds) - time. For solution of the above equations t equals zero at the launching of the vehicle, at the beginning of the delay period between first-stage burnout and second-stage ignition, at second-stage ignition, and at the beginning of the coasting period after second-stage burnout.

w (pounds) - gross weight of the vehicle when t equals zero as defined above.

$\dot{w}$ (pounds per second) - rate of weight loss of the vehicle.

r (feet) - distance from center of the earth.

R (feet) - a special value of r, namely the value taken for the radius of the earth. When used as a subscript it denotes the sea-level value of the given quantity.

h (feet) - altitude.

s (feet) - corresponds to a great circle distance on the earth from the launching point to the vehicle subpoint.

v (feet per second) - speed.

θ (degrees or radians) - angle between the tangent to the trajectory and the x-axis.

$\bar{\theta}$ (degrees or radians) - angle between the thrust vector and the x-axis. No formula was given for this since it may be specified in several ways depending on the purpose of the calculations. In one program $\bar{\theta}$ is taken as equal to θ; such trajectories are known as gravity turn, or zero lift, trajectories. In another case $\dot{\bar{\theta}}$ is given as a step function of time and $\bar{\theta}$ calculated from this. Other possibilities can be easily accommodated and $\bar{\theta}$ may be specified differently in the two stages.

α (degrees) - angle of attack.

g (feet per second^2) - acceleration due to gravity.

F (pounds) - thrust.

I (seconds) - specific impulse.

A_e (inches^2) - nozzle exit area.

S (feet^2) - vehicle frontal area.

M (dimensionless) - Mach number.

C_D (dimensionless) - drag coefficient specified as a function of Mach number only. This is presented to the computer as a table of functional values in which four-point, Lagrangian polynominal interpolation is performed to obtain desired values. It can be changed rather easily. Less than thirty points are required to satisfactorily represent the usual aerodynamical graph of this function.

C_L (degrees^{-1}) - coefficient of lift specified as a function of Mach number only. This is also presented to the computer as a table of function values easily changed. Only straight lines are fitted between the tabulated points, however.

a (feet per second) - speed of sound in air. This is presented to the computer as a tabular function of altitude with straight lines fitted between tabulated points.

T (degrees Rankine) - temperature of the air. This is also a tabular function of altitude handled in the same manner as the speed of sound.

p (pounds per inch^2) - atmospheric pressure as a function of altitude. This is another tabular function. Four-point, Lagrangian polynomial interpolation is used to obtain desired values.

ρ (pound - seconds^2 per foot^4, or equivalently, slugs per foot^3) - atmospheric density as a function of altitude. This function is calculated from p and T as shown above.

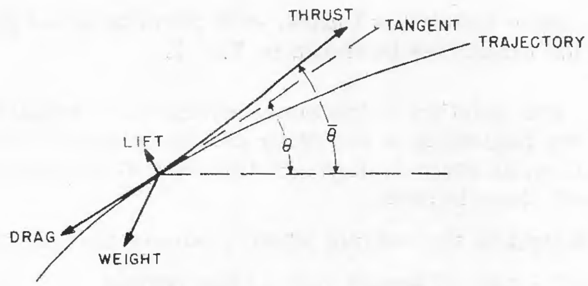

Fig. 2 - Orientation of vector quantities
(not drawn to scale)

In general, any function which is presented to the computer in the form of a table can be fairly easily changed.

NUMERICAL SOLUTION OF EQUATIONS OF MOTION

In this section successive values of t will be designated t_i, t_{i+1}, t_{i+2}, etc. Corresponding functional values will be designated x_i, x_{i+1}, ...; y_i, y_{i+1}, ...; $\dot{x}_i$, $\dot{x}_{i+1}$, ...; etc. For convenience, define k as $t_{i+1} - t_i$. The equations of motion are a particular case of a general class of equations to which the method of solution applies. The general form is

$$\ddot{y} = Y(t, y, x, \dot{y}, \dot{x})$$

$$\ddot{x} = X(t, y, x, \dot{y}, \dot{x}).$$

Given the values y_i, x_i, $\dot{y}_i$, and $\dot{x}_i$ at t_i, the procedure yields corresponding information at t_{i+1}. The process is repeated, advancing all subscripts by unity, for as many intervals as desired.

To advance the solution from t_i to t_{i+1} certain particular values of the second derivatives are required. They are obtained by evaluating the equations of motion three times using the arguments shown in Table 1.

TABLE 1
Arguments for Second Derivatives

Y_1, X_1	Y_2, X_2	Y_3, X_3
t_i	$t_i+\frac{2}{3}k$	$t_i+\frac{2}{3}k$
y_i	$y_i+\frac{2}{3}k(\dot{y}_i+\frac{1}{3}kY_1)$	$y_i+\frac{2}{3}k(\dot{y}_i+\frac{1}{3}kY_1)$
x_i	$x_i+\frac{2}{3}k(\dot{x}_i+\frac{1}{3}kX_1)$	$x_i+\frac{2}{3}k(\dot{x}_i+\frac{1}{3}kX_1)$
$\dot{y}_i$	$\dot{y}_i+\frac{2}{3}kY_1$	$\dot{y}_i+\frac{2}{3}kY_2$
$\dot{x}_i$	$\dot{x}_i+\frac{2}{3}kX_1$	$\dot{x}_i+\frac{2}{3}kX_2$

Simple formulas yield the required information at t_{i+1}:

$$y_{i+1} = y_i + k\left[\dot{y}_i + \frac{k}{8}\,(2Y_1 + Y_2 + Y_3)\right]$$

$$x_{i+1} = x_i + k\left[\dot{x}_i + \frac{k}{8}\,(2X_1 + X_2 + X_3)\right]$$

$$\dot{y}_{i+1} = \dot{y}_i + \frac{k}{8}\left[2\,Y_1 + 3\,(Y_2 + Y_3)\right]$$

$$\dot{x}_{i+1} = \dot{x}_i + \frac{k}{8}\left[2\,X_1 + 3\,(X_2 + X_3)\right].$$

This will be recognized as a process of the Runge-Kutta type.* The error is $O(k^4)$.

The trajectory which is calculated from the equations of motion may represent as many as five separate solutions to the given equations. Trajectories are divided into five sections as follows:

1. Short period of vertical flight following the launching of the vehicle. During this period $\bar{\theta}$ and θ are held equal to 90 degrees. The duration of vertical flight is normally from ten to twenty seconds.

2. From end of vertical flight to burnout of the first stage. During this period $\bar{\theta}$ follows the law specified for the first stage. If a gravity turn trajectory is specified, it is necessary to impose a starting value other than 90 degrees for $\bar{\theta}$ and θ at the end of the vertical flight. This might be interpreted as instantaneously tilting the vehicle a small amount from vertical.

3. Delay period between first-stage burnout and ignition of the second-stage engine. Initial conditions for this section are the same as conditions at first-stage burnout. The first-stage engine, tanks, etc., are assumed dropped at burnout, and the vehicle frontal area is reduced. Thrust is held zero, and no other mass change is allowed.

4. Second-stage ignition to dropping of nose cone. Initial conditions for this section are the same as conditions at end of the third section. Second-stage vehicle and engine parameters are supplied to the equations of motion. A different law may be followed by $\bar{\theta}$ than was used during first stage.

5. From nose-cone drop to second-stage burnout. At the start of this section the vehicle weight is abruptly decreased by the weight of the cone. In all other respects the solution in this section is a continuation of the previous one.

By use of certain items specified on the data tape some of the sections of the trajectory may be omitted if desired. The second stage may be omitted altogether; hence, the programs may be used directly for a single-stage vehicle. It is possible to neglect the effects of dropping the nose cone without using the subterfuge of dropping a cone whose weight is zero. The elapsed time between first-stage burnout and second-stage ingnition may be taken as zero.

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*E. J. Nyström, "Über die numerische Integration von Differentialgleichungen," Acta Societatis Scientiarum Fennicae 50, No. 13, 1925

For the duration of any one of the five trajectory sections the size of the time interval, k , is held constant. The one exception to this is that if the length of time of any section is not an exact multiple of k, the size of the final step in the section is adjusted. The size of k in each section is specified on the data tape corresponding to a given vehicle under study.

PRINTING

Provision must be made as part of the computer programs to print out information corresponding to calculated points on the trajectories. Obvious things which might be printed are t_i , y_i , x_i, $\dot{y}_i$, and $\dot{x}_i$. Thrust and drag as functions of time or position, or both, may also be of interest. The printing sections in present programs have been organized so that they may be fairly easily modified so long as the printed material consists of information that is accumulated during the solution process, or is readily obtained therefrom. The same printing scheme is used from launching through second-stage burnout, that is, through all five sections described above.

The frequency with which printing will take place may be regulated for each section in terms of the interval size, k , used. This is done by specifying on the data tapes that printing will take place at every interval, every two intervals, etc. Provision is made, however, to print at the end of every trajectory section regardless of the printing frequency. Hence, a minimum amount of printing can be obtained by specifying that printing take place after a sufficiently large number of intervals.

APOGEE OF VEHICLE ASSUMING THIRD STAGE IS NOT FIRED

The trajectory that is obtained between burnout of the second stage and ignition of the third stage will be referred to as "the coasting flight." No numerical solution of the equations of motion is obtained for this phase of the launching vehicle trajectory, nor is a solution obtained for the third-stage flight. Present programs have been constructed with a view to facilitating their extension to include solution for the coasting and third-stage phases.

The coasting flight of the remainder of the vehicle after second-stage burnout is not completely ignored in present programs, however. Formulas are used to calculate the apogee of this flight (farthest distance from center of the earth), total velocity at apogee, coasting time from second-stage burnout to apogee, and ground range to apogee (great circle distance from launching point to apogee subpoint).

For calculations extending through second-stage burnout a nonrotating earth was assumed. It is necessary to take into account, however, the fact that a velocity in the x direction was imparted to the vehicle at launching, and that its effects continue throughout the powered flight. A fairly good, partial, correction can be made to the second-stage burnout conditions of position and velocity as follows:

$$ {}^{c}x_0 = {}^{2}x_b + \left({}^{1}t_b + {}^{d}t_f + {}^{2}t_b\right) v_{te} \sin L $$

$$ {}^{c}\dot{x}_0 = {}^{2}\dot{x}_b + v_{te} \sin L. $$

Quantities involved are defined, and units given, below:

${}^{c}x_0$ (feet) - x at zero time of the coasting period.

${}^{c}\dot{x}_0$ (feet per second) - $\dot{x}$ at zero time of the coasting period.

$^{2}x_{b}$ (feet) - x at burnout of second stage.

$^{2}\dot{x}_{b}$ (feet per second) - $\dot{x}$ at burnout of second stage.

$^{1}t_{b}$ (seconds) - first-stage burning time.

$^{d}t_{f}$ (seconds) - delay time between first-stage burnout and second-stage ignition.

$^{2}t_{b}$ (seconds) - second-stage burning time.

v_{te} (feet per second) - velocity of a point on the earth's surface in a due easterly direction at launching latitude.

L (degrees or radians) - firing angle measured from North in the plane tangent to the earth's surface at the launching point.

No correction is applied in the y direction. Hence

$$^{c}y_{0} = {}^{2}y_{b}$$

$$^{c}\dot{y}_{0} = {}^{2}\dot{y}_{b}.$$

The coasting time to apogee, the apogee, the total velocity at apogee, and the range to apogee can be calculated from the conditions given at the beginning of coasting flight. It is assumed in the following formulas for these quantities that positions and velocities required are those at $^{c}t_{0}$, except when subscript "a" indicates that the quantity is taken corresponding to the time at which apogee occurs in the coasting flight. The formulas* are

$$^{c}t_{a} = d\left(\frac{d}{G}\right)^{1/2} (\pi - U + e \sin U)$$

$$^{c}r_{a} = d\,(1 + e)$$

$$^{c}v_{a} = \frac{1}{^{c}r_{a}}\; r^{2}\,\dot{\phi}$$

$$^{c}s_{a} = R\,(\eta + \phi) - \left(^{1}t_{b} + {}^{d}t_{f} + {}^{2}t_{b} + {}^{c}t_{a}\right) v_{te} \sin L$$

where

$$U = 2 \tan^{-1}\left(\frac{1 - e}{1 - \cos\eta} \times \frac{1 + \cos\eta}{1 + e}\right)^{1/2}, \quad 0 \leqq U \leqq \pi$$

$$\cos\eta = \frac{1}{e}\left[1 - \frac{r}{G}\,(r\,\dot{\phi})^{2}\right]$$

$$e^{2} = 1 - \frac{1}{Gd}\,(r\dot{\phi})^{2}\, r^{2}$$

$$d = \frac{G}{\frac{2G}{r} - v^{2}}$$

$$r\dot{\phi} = \frac{1}{r}\,(\dot{x}\,y - \dot{y}\,x)$$

$$\phi = \tan^{-1}(x/y).$$

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*J. M. J. Kooy and J. W. H. Uytenbogaart, "Ballistics of the Future," New York:McGraw-Hill, 1946

VERTICAL TOLERANCE

If it is assumed that the altitude and speed of the launching vehicle are more than sufficient to place the satellite in an elliptic orbit which does not pass closer to the center of the earth than a specified amount, then the launching of the satellite need not take place in the direction of the local horizontal at the projection point. Vertical tolerance is defined as the maximum tolerable angular deviation from local horizontal.

Present programs do not include third-stage calculations. A reasonable calculation of vertical tolerance is made, however, by assuming that the distance from the center of the earth at burnout of the third stage is equal to the apogee of the coasting flight following second-stage burnout. In addition, the velocity contribution that is expected from the third stage is simply added to the velocity at apogee of the coasting flight.

Vertical tolerance σ may be obtained from

$$\sin^2 \sigma = 1 - \frac{1 - e_1^2}{1 - (\delta^2 + 2\delta)^2}$$

where

$$e_1 = 1 + \frac{P}{{}^3r_b} (\delta^2 + 2\delta - 1)$$

$$\delta = {}^3v_b \left(\frac{{}^3r_b}{G} \right)^{1/2} - 1$$

${}^3r_b$ (feet) - distance from center of the earth at third-stage burnout.

${}^3v_b$ (feet per second) - speed of launching vehicle at third-stage burnout.

P (feet) - minimum tolerable distance of any point on the satellite orbit from the center of the earth.

GENERAL OUTLINE OF PROGRAMS

Coding for the programs has been broken up into blocks, chiefly according to function. Each block has been assigned space in the storage system of NAREC in an amount felt to be adequate in the light of foreseeable possibilities. The blocks are largely self-contained with the necessary interconnections well defined. Within the space limitations assigned it is a fairly simple matter to change the contents of a block completely with a minimum of disturbance to coding in other blocks. This approach has proved itself in practice; a variety of special computing requirements have been met without once having to go through extensive code checking procedures.

The organization of a typical program may be more easily understood by use of a simplified flow diagram (Fig. 3). A typical program contains considerably more functional blocks than are depicted. The diagram, however, will help to emphasize important non-computational aspects which it is the purpose of this section to describe. Briefly, it represents the following situation: The Director initiates the calculation of trajectories and may systematically vary parameters until it ascertains that trajectories with desired characteristics are being obtained. The Trajectory Monitor "supervises" the calculation of one trajectory, setting up each section in turn for Runge-Kutta solution and Printing through the Integrating and Printing Monitor. Acceleration components are evaluated on the initiative of RK which supplies the necessary arguments.

The equations of motion are evaluated in A. Values of t, x, y, $\dot{x}$, $\dot{y}$, and vehicle characteristics must be supplied to it by other portions of the program. Curves for the

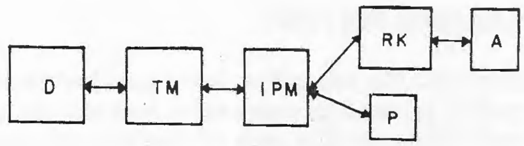

Fig. 3 - Simplified flow diagram

atmospheric model for C_D and for C_L ordinarily are changed infrequently and have therefore been made part of block A. Control of the computer passes to A from RK and back again.

Block RK contains the Runge-Kutta solution procedure for one time interval. If it is given t_i, x_i, y_i, $\dot{x}_i$, $\dot{y}_i$, $\ddot{x}_i$, and $\ddot{y}_i$, and if A has available to it other needed information then RK will construct required arguments for the equations of motion, control their evaluations, and finally yield t_{i+1}, x_{i+1}, y_{i+1}, $\dot{x}_{i+1}$, $\dot{y}_{i+1}$, $\ddot{x}_{i+1}$, and $\ddot{y}_{i+1}$. Control of the computer is originally passed to RK from IPM; it returns to IPM after solution for one interval has been completed.

Printing of trajectory information is accomplished in P. Information pertaining to apogee of the coasting flight is not handled here. The action of P can be suppressed by block D. Control is passed between IPM and P.

Block IPM handles the entire numerical solution for one trajectory section. It assumes that the initial conditions and integration interval size for the section are available to RK and that proper vehicle data are available to A. Block IPM requires, in addition, the time at which the given section ends and the frequency with which printing is desired. After solution for one time interval has been completed, control of the computer is returned from RK to IPM, where it is determined whether this is the last full-size interval and whether printing is desired for the interval just completed. If solution for the last full-size interval has been obtained, the size of the interval is modified to take care of the fractional portion, and control is given to RK which completes the solution for the section and returns control. Control goes to P at this point and then back to IPM. During solution for the rest of the section, control is sent to P only as often as indicated on the data tape for the given section. For the purposes of block IPM it is assumed that any given section will end at a time not a multiple of the interval size; this may mean that k is practically zero for the final step. Control originally goes to IPM from TM and returns to TM when a trajectory section is completed.

The TM block "supervises" calculation of a single trajectory from launching of the vehicle through second-stage burnout. It then computes the quantities related to apogee of the coasting flight and the vertical tolerance. These are printed independently of block P. It is in block TM that proper initial conditions, vehicle parameters, $\overline{\theta}$ programs, etc., are supplied for the computation of each trajectory section in turn. After a section is set up, control passes from TM to IPM where it remains until the section is completed. Control is then returned to TM. Information put in with the data is examined here to determine if certain sections are to be omitted. Control originally passes to TM from block D and returns when a trajectory has been completed.

Block D treats trajectories as functional units. It has its simplest form when only one trajectory is calculated for a given set of data. It can be used, however, to vary a parameter (or parameters) systematically until a trajectory with specified characteristics is obtained. This block can suppress the printing in block P until it is established that the desired trajectory will be obtained. The small amount that is printed in TM for each trajectory serves to monitor the workings of the program in such cases. An example of this is the case where the tilt angle, imposed in the gravity turn trajectories after the initial vertical flight, is varied until a desired apogee in the coasting flight is obtained.

AUTOMATIC STORAGE LOADING SYSTEM

Insertion of information into the computer from punched paper tapes is checked as follows: Each block of coding is read in separately and stored in its assigned space; the number of items in each block and the sum of those items are compared with known figures. This practically guarantees that computations cannot be started if tapes or input system are defective. Furthermore, the checking system is completely automatic, with each program constructed having its own checking system. The computer operators are required only to place tapes on the reader in required order. Automatic features of the checking system also give some protection against improper tape combinations.

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