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Computing the Focusing Contribution to Signal Intensity along a Single HF Ray through an Ionosphere with Fields and Collisions

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14. ABSTRACT
Due to the curved nature of their trajectories, an important part of determining the attenuation or enhancement that high-frequency (HF) radio waves accrue over the course of their trajectories is the focusing factor. A common method to determine the focusing a ray experiences is to model propagating multiple rays around the target ray and finding the average of the final cross-sectional area of this ray tube. Instead, a more reliable and computationally efficient method was proposed in a 1988 Radio Science article. We follow the methodology of that paper and apply it to rays propagating through an ionosphere affected by the Earth's magnetic field and particle collisions. We also expand upon a 1996 Radio Science article which uses the method and describes what equations are necessary for writing it in computer code but does not give the equations themselves. Derivations are included, as well as suggestions to make coding the method easier.

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1. INTRODUCTION

In 1988 and with high-frequency (HF) ray propagation in mind, Dr. L.J. Nickisch published a paper[1] with a new method for calculating focusing in the stationary phase approximation of geometrical optics. Unlike other previous methods involving tracing multiple rays and creating a flux tube, which can be computationally expensive and introduce inaccuracy due to the turbulent nature of the ionosphere, this method only requires tracing one ray, introducing infinitesimal deviations in the ray launch angles. These infinitesimal deviations in launch angles can be traced to their respective landing points by computations along the undeviated ray, using a ray tracing code such as Jones-Stephenson[2] or the more modern MoJo[3] creating a flux tube of sorts, see Figure 1, but one that only requires integrating along one ray. With a traditional flux tube, the deviations in launch angles are not infinitesimal, and for ionospheres with steep gradients, rays with small but finite differences in direction can propagate to widely separated points. Therefore, if one calculates the signal strength as an average over the final cross-sectional area of the tube, it may be a poor approximation to the actual signal strength at the point of interest.

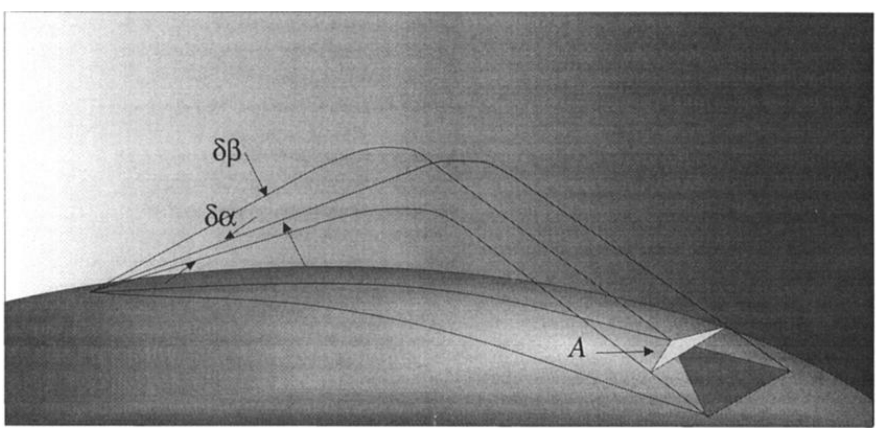

Fig. 1—An example of a flux tube with infinitesimal perturbations, reproduced from [4]. A denotes the area perpendicular to the wave vector at the receiver, $\delta\alpha$ is the azimuthal deviation, and $\delta\beta$ is the deviation in elevation.

The author of [1] demonstrated how the method works in an isotropic, collisionless plasma and gave the expressions for the variables needed for the ray tracing. However, the ionosphere is anisotropic thanks to the presence of the Earth's magnetic field, and there are particle collisions which must be accounted for. The authors of [4] took up the charge to incorporate both of these phenomena into the framework devised by Nickisch. Their effort resulted in a guide on how it might be done, but they did not write out the analogous

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expressions to what is found in [1]. In this paper, we expand upon both works by writing out full expressions for the variables given in the first paper and subsequently writing out the expressions for the partial derivatives presented in the second which are necessary for implementation into a ray tracing code.

2. EXPANDING NICKISCH'S WORK

Both [1] and [2] utilize the framework that Jenifer Haselgrove developed in the 1950's [5] in applying the work of William Rowan Hamilton and what is now known as Hamilton's Principle to geometrical optics. A crucial concept within this framework is the Hamiltonian, $\boldsymbol{H}$, a concise way to represent the total energy of a system in terms of its position and direction of movement. In geometrical optics, the Hamiltonian contains the dispersion relation, which relates the wavelength to the frequency of the wave, allowing one to compute the phase and group velocities of a wave within a medium. In [1], the Hamiltonian is given in (9) for a collisionless, isotropic ionosphere.

Although it is not explicitly spelled out in [1], its (9) is written in Einstein summation notation and can be applied to an isotropic ionosphere with spherical coordinates and be more formally written out as

$$\begin{aligned}\boldsymbol{H}(x^i, p^i) &= \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij} p^i \dot{x}^j - L(x^i, \dot{x}^i) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij} p^i \dot{x}^j - \sum_{i=1}^{3} g_{ii} p^i \dot{x}^i + \frac{\lambda}{2}\left(\xi k^2 - 1 + X\right) \\ &= \frac{\lambda}{2}\left(\xi k^2 - 1 + X\right) \\ &= \frac{\lambda}{2}\left(\xi\left(k_r^2 + k_\theta^2 + k_\phi^2\right) - 1 + X\right),\end{aligned} \tag{1}$$

where L is the Lagrangian, and the dispersion relation appears in the $\xi k^2 - 1 + X$, with $\xi = \frac{c^2}{\omega^2}$. The variable λ is a Lagrange multiplier. In [4], they use the notation $\lambda\xi = ƛ = \frac{c}{\omega}$, which gives λ the value of $\frac{\omega}{c}$, the free space wavenumber. Because one of the purposes of this report is to be an expansion of [1], we retain his notation. Using the Jones-Stephenson notation[2], the Appleton-Hartree dispersion relation to be used with fields and collisions can be written as

$$n^2 = 1 - \frac{2X(1 - iZ - X)}{2(1 - iZ)(1 - iZ - X) - Y^2\sin^2\psi \pm \sqrt{Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1 - iZ - X)^2}}, \tag{2}$$

with

$$X = \frac{\omega_N^2}{\omega^2} \tag{3}$$

(ω_N is the angular plasma frequency),

$$Y = \frac{\omega_H}{\omega} \tag{4}$$

(ω_H is the angular gyro-frequency of the electrons in the ionospheric plasma),

$$Z = \frac{\nu_{eff}}{\omega} \tag{5}$$

(ν_{eff} is the effective collision frequency for the electrons), and ψ is the angle between the wave normal and the earth's magnetic field. These four variables have the dependencies $X(r, \theta, \phi)$, $Y(r, \theta, \phi)$, $Z(r, \theta, \phi)$, and $\psi(r, \theta, \phi, k_r, k_\theta, k_\phi)$. The angle ψ always appears in multiples of $Y \sin\psi$ and $Y \cos\psi$. For this reason, Jones and Stephenson introduced the notation

$$Y_T = Y \sin\psi \tag{6a}$$

and

$$Y_L = Y \cos\psi. \tag{6b}$$

Throughout this paper, we will do the same, although ψ is still present because the derivatives of it are considered instead of the derivatives of Y_T or Y_L. Additionally, as mentioned in [2] and [4], Y_T and Y_L by themselves contain a sign ambiguity. To circumvent that, they will always be seen as multiples of themselves or each other. Furthermore, whenever the exponent of the sine or cosine of ψ is greater than the exponent of its associated Y, that is accounted for by using the substitutions (6) divided by Y. For example, $Y \cos^2\psi = \frac{Y_L^2}{Y}$. Note that Y is the magnitude of the magnetic field vector

$$\boldsymbol{Y} = \frac{e\boldsymbol{B}}{m\omega}, \tag{7}$$

where e and m are the charge and the mass of an electron, respectively, and $\boldsymbol{B}$ is the constant magnetic induction of the earth's field. It should also be noted that technically, only the real parts of the partial derivatives of n^2 are used so that $\boldsymbol{r}$ and $\boldsymbol{k}$ are real. Using the aforementioned notation and substituting (2) into (1) results in the Hamiltonian

$$\boldsymbol{H}\left(x^i, p^i\right) = \frac{\lambda}{2}\left(\xi\left(k_r^2 + k_\theta^2 + k_\phi^2\right) - 1 + \frac{2X(1 - iZ - X)}{2(1 - iZ)(1 - iZ - X) - Y_T^2 \pm \sqrt{Y_T^4 + 4Y_L^2(1 - iZ - X)^2}}\right). \tag{8}$$

Returning to [1], as they are dispersion relation independent, (15) and (16) from the paper, the ray equations in spherical coordinates, still hold:

$$\dot{r} = \frac{\partial \boldsymbol{H}}{\partial k_r}, \tag{9a}$$

$$\dot{\theta} = \frac{1}{r}\frac{\partial \boldsymbol{H}}{\partial k_\theta}, \tag{9b}$$

$$\dot{\phi} = \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{H}}{\partial k_\phi}, \tag{9c}$$

$$\dot{k}_r = -\frac{\partial \boldsymbol{H}}{\partial r} + k_\theta \dot{\theta} + k_\phi \sin\theta \dot{\phi}, \tag{9d}$$

$$\dot{k}_\theta = -\frac{1}{r}\frac{\partial \boldsymbol{H}}{\partial \theta} - \frac{1}{r} k_\theta \dot{r} + k_\phi \cos\theta \dot{\phi}, \tag{9e}$$

and

$$\dot{k}_\phi = -\frac{1}{r\sin\theta}\frac{\partial \boldsymbol{H}}{\partial \phi} - \frac{1}{r} k_\phi \dot{r} - k_\phi \cot\theta \dot{\theta}. \tag{9f}$$

Before writing out the six partial derivatives of $\boldsymbol{H}$ with respect to the independent variables, we introduce a notation shortcut for convenience Let us substitute R for the final term in the denominator of (8), i.e.

$$R\left(r, \theta, \phi, k_r, k_\theta, k_\phi\right) \equiv \pm\sqrt{Y_T^4 + 4Y_L^2 \left(1 - iZ - X\right)^2}. \tag{10}$$

The six first derivatives of R are

$$\begin{aligned} \frac{\partial R}{\partial r} =& \frac{1}{R}\left(2Y_T^3 Y_L \frac{\partial \psi}{\partial r} + 2\frac{Y_T^4}{Y}\frac{\partial Y}{\partial r} + 4Y_L^2\left(1 - iZ - X\right)\left(-i\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\right)\right. \\ &\left. + \left(-4Y_T Y_L \frac{\partial \psi}{\partial r} + 4\frac{Y_L^2}{Y}\frac{\partial Y}{\partial r}\right)\left(1 - iZ - X\right)^2\right), \end{aligned} \tag{11a}$$

$$\begin{aligned} \frac{\partial R}{\partial \theta} =& \frac{1}{R}\left(2Y_T^3 Y_L \frac{\partial \psi}{\partial \theta} + 2\frac{Y_T^4}{Y}\frac{\partial Y}{\partial \theta} + 4Y_L^2\left(1 - iZ - X\right)\left(-i\frac{\partial Z}{\partial \theta} - \frac{\partial X}{\partial \theta}\right)\right. \\ &\left. + \left(-4Y_T Y_L \frac{\partial \psi}{\partial \theta} + 4\frac{Y_L^2}{Y}\frac{\partial Y}{\partial \theta}\right)\left(1 - iZ - X\right)^2\right), \end{aligned} \tag{11b}$$

$$\begin{aligned} \frac{\partial R}{\partial \phi} =& \frac{1}{R}\left(2Y_T^3 Y_L \frac{\partial \psi}{\partial \phi} + 2\frac{Y_T^4}{Y}\frac{\partial Y}{\partial \phi} + 4Y_L^2\left(1 - iZ - X\right)\left(-i\frac{\partial Z}{\partial \phi} - \frac{\partial X}{\partial \phi}\right)\right. \\ &\left. + \left(-4Y_T Y_L \frac{\partial \psi}{\partial \phi} + 4\frac{Y_L^2}{Y}\frac{\partial Y}{\partial \phi}\right)\left(1 - iZ - X\right)^2\right), \end{aligned} \tag{11c}$$

$$\frac{\partial R}{\partial k_r} = \frac{2}{R}\left(Y_T Y_L \frac{\partial \psi}{\partial k_r}\left(Y_T^2 - 2\left(1 - iZ - X\right)^2\right)\right), \tag{11d}$$

$$\frac{\partial R}{\partial k_\theta} = \frac{2}{R}\left(Y_T Y_L \frac{\partial \psi}{\partial k_\theta}\left(Y_T^2 - 2\left(1 - iZ - X\right)^2\right)\right), \tag{11e}$$

and

$$\frac{\partial R}{\partial k_\phi} = \frac{2}{R}\left(Y_T Y_L \frac{\partial \psi}{\partial k_\phi}\left(Y_T^2 - 2\left(1 - iZ - X\right)^2\right)\right). \tag{11f}$$

Thus, (8) can be rewritten as

$$\boldsymbol{H}\left(x^i, p^i\right) = \frac{\lambda}{2}\left(\xi\left(k_r^2 + k_\theta^2 + k_\phi^2\right) - 1 + \frac{2X\left(1 - iZ - X\right)}{2\left(1 - iZ\right)\left(1 - iZ - X\right) - Y_T^2 + R}\right). \tag{12}$$

The denominator in (12) can also be coalesced into one variable to make the forthcoming derivations easier:

$$D\left(r, \theta, \phi, k_r, k_\theta, k_\phi\right) \equiv 2\left(1 - iZ\right)\left(1 - iZ - X\right) - Y_T^2 + R, \tag{13}$$

with the six first derivatives of

$$\frac{\partial D}{\partial r} = 2\left(1 - iZ\right)\left(-i\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\right) - 2i\frac{\partial Z}{\partial r}\left(1 - iZ - X\right) - 2Y_T Y_L \frac{\partial \psi}{\partial r} - 2\frac{Y_T^2}{Y}\frac{\partial Y}{\partial r} + \frac{\partial R}{\partial r}, \tag{14a}$$

$$\frac{\partial D}{\partial \theta} = 2\left(1 - iZ\right)\left(-i\frac{\partial Z}{\partial \theta} - \frac{\partial X}{\partial \theta}\right) - 2i\frac{\partial Z}{\partial \theta}\left(1 - iZ - X\right) - 2Y_T Y_L \frac{\partial \psi}{\partial \theta} - 2\frac{Y_T^2}{Y}\frac{\partial Y}{\partial \theta} + \frac{\partial R}{\partial \theta}, \tag{14b}$$

$$\frac{\partial D}{\partial \phi} = 2\left(1 - iZ\right)\left(-i\frac{\partial Z}{\partial \phi} - \frac{\partial X}{\partial \phi}\right) - 2i\frac{\partial Z}{\partial \phi}\left(1 - iZ - X\right) - 2Y_T Y_L \frac{\partial \psi}{\partial \phi} - 2\frac{Y_T^2}{Y}\frac{\partial Y}{\partial \phi} + \frac{\partial R}{\partial \phi}, \tag{14c}$$

$$\frac{\partial D}{\partial k_r} = -2Y_T Y_L \frac{\partial \psi}{\partial k_r} + \frac{\partial R}{\partial k_r}, \tag{14d}$$

$$\frac{\partial D}{\partial k_\theta} = -2Y_T Y_L \frac{\partial \psi}{\partial k_\theta} + \frac{\partial R}{\partial k_\theta}, \tag{14e}$$

and

$$\frac{\partial D}{\partial k_\phi} = -2Y_T Y_L \frac{\partial \psi}{\partial k_\phi} + \frac{\partial R}{\partial k_\phi}. \tag{14f}$$

With this additional variable, (8) and (12) are simplified as

$$\boldsymbol{H}\left(x^i, p^i\right) = \frac{\lambda}{2}\left(\xi\left(k_r^2 + k_\theta^2 + k_\phi^2\right) - 1 + \frac{2X\left(1 - iZ - X\right)}{D}\right). \tag{15}$$

While it may be tempting to simplify things even further by making a substitution like $F \equiv \frac{2X(1-iZ-X)}{D}$, see Section 3, presently, this would obscure an important pattern seen in (11) and (14) which continues in the first partial derivatives of $\boldsymbol{H}$ itself:

$$\frac{\partial \boldsymbol{H}}{\partial r} = \frac{\lambda}{D^2}\left(D\left(X\left(-i\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\right) + \frac{\partial X}{\partial r}\left(1 - iZ - X\right)\right) - X\left(1 - iZ - X\right)\frac{\partial D}{\partial r}\right), \tag{16a}$$

$$\frac{\partial \boldsymbol{H}}{\partial \theta} = \frac{\lambda}{D^2}\left(D\left(X\left(-i\frac{\partial Z}{\partial \theta} - \frac{\partial X}{\partial \theta}\right) + \frac{\partial X}{\partial \theta}\left(1 - iZ - X\right)\right) - X\left(1 - iZ - X\right)\frac{\partial D}{\partial \theta}\right), \tag{16b}$$

$$\frac{\partial \boldsymbol{H}}{\partial \phi} = \frac{\lambda}{D^2}\left(D\left(X\left(-i\frac{\partial Z}{\partial \phi} - \frac{\partial X}{\partial \phi}\right) + \frac{\partial X}{\partial \phi}\left(1 - iZ - X\right)\right) - X\left(1 - iZ - X\right)\frac{\partial D}{\partial \phi}\right), \tag{16c}$$

$$\frac{\partial \boldsymbol{H}}{\partial k_r} = \lambda \xi k_r - \frac{\lambda}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_r}\right), \tag{16d}$$

$$\frac{\partial \boldsymbol{H}}{\partial k_\theta} = \lambda \xi k_\theta - \frac{\lambda}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\theta}\right), \tag{16e}$$

and

$$\frac{\partial \boldsymbol{H}}{\partial k_\phi} = \lambda \xi k_\phi - \frac{\lambda}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\phi}\right). \tag{16f}$$

Note that in (11), (14), and (16), the first three equations are identical to each other, except that the three independent variables r, θ, and ϕ are switched out. Similarly, the final three equations are identical to one another, except that the three independent variables representing the wave normals, k_r, k_θ, and k_ϕ are switched out. We can use this to our advantage and trim down the number of equations that need to be written out by two-thirds. Borrowing notation from [4], let $\eta = \{r, \theta, \phi\}$. Then, (11) can be written as

$$\begin{aligned}\frac{\partial R}{\partial \eta} =& \frac{1}{R}\left(2Y_T^3 Y_L \frac{\partial \psi}{\partial \eta} + 2\frac{Y_T^4}{Y}\frac{\partial Y}{\partial \eta} + 4Y_L^2\left(1 - iZ - X\right)\left(-i\frac{\partial Z}{\partial \eta} - \frac{\partial X}{\partial \eta}\right)\right. \\ &\left. + \left(-4Y_T Y_L \frac{\partial \psi}{\partial \eta} + 4\frac{Y_L^2}{Y}\frac{\partial Y}{\partial \eta}\right)\left(1 - iZ - X\right)^2\right)\end{aligned} \tag{17a}$$

and

$$\frac{\partial R}{\partial k_\eta} = \frac{2}{R}\left(Y_T Y_L \frac{\partial \psi}{\partial k_\eta}\left(Y_T^2 - 2\left(1 - iZ - X\right)^2\right)\right), \tag{17b}$$

(14) can be rewritten as

$$\frac{\partial D}{\partial \eta} = 2\left(1 - iZ\right)\left(-i\frac{\partial Z}{\partial \eta} - \frac{\partial X}{\partial \eta}\right) - 2i\frac{\partial Z}{\partial \eta}\left(1 - iZ - X\right) - 2Y_T Y_L \frac{\partial \psi}{\partial \eta} - 2\frac{Y_T^2}{Y}\frac{\partial Y}{\partial \eta} + \frac{\partial R}{\partial \eta} \tag{18a}$$

and

$$\frac{\partial D}{\partial k_\eta} = -2Y_T Y_L \frac{\partial \psi}{\partial k_\eta} + \frac{\partial R}{\partial k_\eta}, \tag{18b}$$

and (16) can be rewritten as

$$\begin{aligned}\frac{\partial \boldsymbol{H}}{\partial \eta} =&\frac{\lambda}{D^2}\left(D\left(X\left(-i\frac{\partial Z}{\partial \eta} - \frac{\partial X}{\partial \eta}\right) + \frac{\partial X}{\partial \eta}\left(1 - iZ - X\right)\right) - X\left(1 - iZ - X\right)\frac{\partial D}{\partial \eta}\right)\\ =&\frac{\lambda}{D}\left(X\left(-i\frac{\partial Z}{\partial \eta} - \frac{\partial X}{\partial \eta}\right) + \frac{\partial X}{\partial \eta}\left(1 - iZ - X\right)\right) - \frac{\lambda}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial \eta}\right)\end{aligned} \tag{19a}$$

and

$$\frac{\partial \boldsymbol{H}}{\partial k_\eta} = \lambda \xi k_\eta - \frac{\lambda}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\eta}\right). \tag{19b}$$

Before continuing, we introduce one final notation to make the focusing equations less cumbersome. Varying Hamilton's equations with respect to the six independent variables is given mathematically as (using r as an example)

$$\delta \dot{r} \equiv \frac{\partial \dot{r}}{\partial r}\delta r + \frac{\partial \dot{r}}{\partial \theta}\delta \theta + \frac{\partial \dot{r}}{\partial \phi}\delta \phi + \frac{\partial \dot{r}}{\partial k_r}\delta k_r + \frac{\partial \dot{r}}{\partial k_\theta}\delta k_\theta + \frac{\partial \dot{r}}{\partial k_\phi}\delta k_\phi. \tag{20}$$

Due to the cumbersome nature of what follows, it is prudent to use a similar notation for the dependent variables within the expressions:

$$dX \equiv \frac{\partial X}{\partial r}\delta r + \frac{\partial X}{\partial \theta}\delta \theta + \frac{\partial X}{\partial \phi}\delta \phi, \tag{21a}$$

$$dY \equiv \frac{\partial Y}{\partial r}\delta r + \frac{\partial Y}{\partial \theta}\delta \theta + \frac{\partial Y}{\partial \phi}\delta \phi, \tag{21b}$$

$$dZ \equiv \frac{\partial Z}{\partial r}\delta r + \frac{\partial Z}{\partial \theta}\delta \theta + \frac{\partial Z}{\partial \phi}\delta \phi, \tag{21c}$$

$$d\psi \equiv \frac{\partial \psi}{\partial r}\delta r + \frac{\partial \psi}{\partial \theta}\delta \theta + \frac{\partial \psi}{\partial \phi}\delta \phi + \frac{\partial \psi}{\partial k_r}\delta k_r + \frac{\partial \psi}{\partial k_\theta}\delta k_\theta + \frac{\partial \psi}{\partial k_\phi}\delta k_\phi, \tag{21d}$$

$$dR \equiv \frac{\partial R}{\partial r}\delta r + \frac{\partial R}{\partial \theta}\delta \theta + \frac{\partial R}{\partial \phi}\delta \phi + \frac{\partial R}{\partial k_r}\delta k_r + \frac{\partial R}{\partial k_\theta}\delta k_\theta + \frac{\partial R}{\partial k_\phi}\delta k_\phi, \tag{21e}$$

and

$$dD \equiv \frac{\partial D}{\partial r}\delta r + \frac{\partial D}{\partial \theta}\delta \theta + \frac{\partial D}{\partial \phi}\delta \phi + \frac{\partial D}{\partial k_r}\delta k_r + \frac{\partial D}{\partial k_\theta}\delta k_\theta + \frac{\partial D}{\partial k_\phi}\delta k_\phi. \tag{21f}$$

Using d instead of δ for these is a personal preference so that δ is reserved for the independent variables and their time derivatives and associated perturbations. This notation can also be used for the partial derivatives of the dependent variables. For instance,

$$d\frac{\partial X}{\partial r} \equiv \frac{\partial^2 X}{\partial r^2}\delta r + \frac{\partial^2 X}{\partial \theta \partial r}\delta\theta + \frac{\partial^2 X}{\partial \phi \partial r}\delta\phi. \tag{22}$$

With the preparation out of the way, we are finally able to write out the fields and collision versions of the key (23) and (24) in [1]. Let us look at each of the six variables and their perturbations one at a time.

Combining (9a) and (16a), it is easy to see that

$$\dot{r} = \lambda\xi k_r - \lambda\left(\frac{X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_r}}{D^2}\right). \tag{23}$$

Applying (20)-(21) and factoring out a factor of D, we find that

$$\begin{aligned}\delta\dot{r} =& \lambda\xi\delta k_r - \frac{\lambda}{D^2}\left(X\left(1 - iZ - X\right)d\frac{\partial D}{\partial k_r} + \left(X\left(-idZ - dX\right) + dX\left(1 - iZ - X\right)\right)\frac{\partial D}{\partial k_r}\right)\\ &+ \frac{\lambda}{D^3}\left(2X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_r}dD\right).\end{aligned} \tag{24}$$

Combining (9b) and (16b), we get

$$\dot{\theta} = \frac{\lambda\xi k_\theta}{r} - \frac{\lambda}{r}\left(\frac{X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\theta}}{D^2}\right), \tag{25}$$

which means

$$\begin{aligned}\delta\dot{\theta} =& \frac{\lambda\xi}{r}\delta k_\theta - \frac{\lambda\xi k_\theta}{r^2}\delta r + \frac{\lambda}{r^2 D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\theta}\right)\delta r\\ &- \frac{\lambda}{rD^2}\left(X\left(1 - iZ - X\right)d\frac{\partial D}{\partial k_\theta} + \left(X\left(-idZ - dX\right) + dX\left(1 - iZ - X\right)\right)\frac{\partial D}{\partial k_\theta}\right)\\ &+ \frac{\lambda}{rD^3}\left(2X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\theta}dD\right).\end{aligned} \tag{26}$$

Next, combining (9c) and (16c),

$$\dot{\phi} = \frac{\lambda\xi k_\phi}{r\sin\theta} - \frac{\lambda}{r\sin\theta}\left(\frac{X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\phi}}{D^2}\right), \tag{27}$$

and thus,

$$
\begin{aligned}
\delta\dot{\phi} =& \frac{\lambda\xi}{r\sin\theta}\delta k_\phi - \lambda\xi k_\phi\left(\frac{r\cos\theta\delta\theta + \sin\theta\delta r}{r^2\sin^2\theta}\right) \\
&+ \frac{\lambda}{D^2}\left(\frac{r\cos\theta\delta\theta + \sin\theta\delta r}{r^2\sin^2\theta}\right)\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\phi}\right) \\
&- \frac{\lambda}{rD^2\sin\theta}\left(X\left(1 - iZ - X\right)d\frac{\partial D}{\partial k_\phi} + \left(X\left(-idZ - dX\right) + dX\left(1 - iZ - X\right)\right)\frac{\partial D}{\partial k_\phi}\right) \\
&+ \frac{\lambda}{rD^3\sin\theta}\left(2X\left(1 - iZ - X\right)\frac{\partial D}{\partial k_\phi}dD\right).
\end{aligned}
\tag{28}
$$

Next, we move onto the three vector components of the wave number. Using (9b), (9c), (9d), and (16a),

$$
\begin{aligned}
\dot{k}_r =& -\frac{\lambda}{D^2}\left(D\left(X\left(-i\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\right) + \frac{\partial X}{\partial r}\left(1 - iZ - X\right)\right) - X\left(1 - iZ - X\right)\frac{\partial D}{\partial r}\right) \\
&+ \frac{\lambda\xi}{r}\left(k_\theta^2 + k_\phi^2\right) - \frac{\lambda}{rD^2}\left(X\left(1 - iZ - X\right)\left(k_\theta\frac{\partial D}{\partial k_\theta} + k_\phi\frac{\partial D}{\partial k_\phi}\right)\right) \\
=& -\frac{\lambda}{D}\left(X\left(-i\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\right) + \frac{\partial X}{\partial r}\left(1 - iZ - X\right)\right) + \frac{\lambda}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial r}\right) \\
&+ \frac{\lambda\xi}{r}\left(k_\theta^2 + k_\phi^2\right) - \frac{\lambda}{rD^2}\left(X\left(1 - iZ - X\right)\left(k_\theta\frac{\partial D}{\partial k_\theta} + k_\phi\frac{\partial D}{\partial k_\phi}\right)\right).
\end{aligned}
\tag{29}
$$

To make it as digestible as possible, we write $\delta \dot{k}_r$ in terms corresponding to different orders of $\frac{1}{r}$ and $\frac{1}{D}$:

$$
\begin{aligned}
\delta \dot{k}_r =& -\frac{\lambda}{D}\left[X\left(-id\frac{\partial Z}{\partial r} - d\frac{\partial X}{\partial r}\right) + dX\left(-i\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\right) + \frac{\partial X}{\partial r}\left(-idZ - dX\right)\right. \\
& \left. + d\frac{\partial X}{\partial r}\left(1 - iZ - X\right)\right] \\
& - \frac{\lambda}{D^2}\left[dD\left(X\left(-i\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\right) + \frac{\partial X}{\partial r}\left(1 - iZ - X\right)\right) - X\left(1 - iZ - X\right)d\frac{\partial D}{\partial r}\right. \\
& + \left(X\left(-idZ - dX\right) + dX\left(1 - iZ - X\right)\right)\frac{\partial D}{\partial r} \\
& \left. - 2dD\left(X\left(-i\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\right) + \frac{\partial X}{\partial r}\left(1 - iZ - X\right)\right)\right] \\
& - \frac{2\lambda}{D^3}X\left(1 - iZ - X\right)\frac{\partial D}{\partial r}dD + \frac{2\lambda\xi}{r}\left(k_\theta \delta k_\theta + k_\phi \delta k_\phi\right) - \frac{\lambda\xi}{r^2}\left(k_\theta^2 + k_\phi^2\right)\delta r \\
& - \frac{\lambda}{rD^2}\left[X\left(1 - iZ - X\right)\left(k_\theta d\frac{\partial D}{\partial k_\theta} + \frac{\partial D}{\partial k_\theta}\delta k_\theta + k_\phi d\frac{\partial D}{\partial k_\phi} + \frac{\partial D}{\partial k_\phi}\delta k_\phi\right)\right. \\
& \left. + \left(X\left(-idZ - dX\right) + dX\left(1 - iZ - X\right)\right)\left(k_\theta\frac{\partial D}{\partial k_\theta} + k_\phi\frac{\partial D}{\partial k_\phi}\right)\right] \\
& + \frac{\lambda}{r^2D^2}X\left(1 - iZ - X\right)\left(k_\theta\frac{\partial D}{\partial k_\theta} + k_\phi\frac{\partial D}{\partial k_\phi}\right)\delta r \\
& + \frac{2\lambda}{rD^3}X\left(1 - iZ - X\right)\left(k_\theta\frac{\partial D}{\partial k_\theta} + k_\phi\frac{\partial D}{\partial k_\phi}\right)dD.
\end{aligned} \tag{30}
$$

For $\dot{k}_\theta$, the relevant equations are (9a), (9c), (9e), and (16b). From these, we find that

$$
\begin{aligned}
\dot{k}_\theta =& -\frac{\lambda}{rD^2}\left(D\left(X\left(-i\frac{\partial Z}{\partial \theta} - \frac{\partial X}{\partial \theta}\right) + \frac{\partial X}{\partial \theta}\left(1 - iZ - X\right)\right) - X\left(1 - iZ - X\right)\frac{\partial D}{\partial \theta}\right) \\
& + \frac{\lambda\xi}{r}\left(-k_r k_\theta + k_\phi^2 \cot\theta\right) + \frac{\lambda}{rD^2}\left(X\left(1 - iZ - X\right)\left(k_\theta\frac{\partial D}{\partial k_r} - k_\phi \cot\theta\frac{\partial D}{\partial k_\phi}\right)\right) \\
=& -\frac{\lambda}{rD}\left(X\left(-i\frac{\partial Z}{\partial \theta} - \frac{\partial X}{\partial \theta}\right) + \frac{\partial X}{\partial \theta}\left(1 - iZ - X\right)\right) + \frac{\lambda}{rD^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial \theta}\right) \\
& + \frac{\lambda\xi}{r}\left(-k_r k_\theta + k_\phi^2 \cot\theta\right) + \frac{\lambda}{rD^2}\left(X\left(1 - iZ - X\right)\left(k_\theta\frac{\partial D}{\partial k_r} - k_\phi \cot\theta\frac{\partial D}{\partial k_\phi}\right)\right).
\end{aligned} \tag{31}
$$

Arranging it in terms of different orders of $\frac{1}{r}$ and $\frac{1}{D}$,

$$
\begin{aligned}
\delta\dot{k}_\theta = &-\frac{\lambda}{rD}\left[X\left(-id\frac{\partial Z}{\partial\theta}-d\frac{\partial X}{\partial\theta}\right)+dX\left(-i\frac{\partial Z}{\partial\theta}-\frac{\partial X}{\partial\theta}\right)+\frac{\partial X}{\partial\theta}\left(-idZ-dX\right)\right.\\
&\left.+d\frac{\partial X}{\partial\theta}\left(1-iZ-X\right)\right]\\
&-\frac{\lambda}{rD^2}\left[dD\left(X\left(-i\frac{\partial Z}{\partial\theta}-\frac{\partial X}{\partial\theta}\right)+\frac{\partial X}{\partial\theta}\left(1-iZ-X\right)\right)-X\left(1-iZ-X\right)d\frac{\partial D}{\partial\theta}\right.\\
&-\left(X\left(-idZ-dX\right)+dX\left(1-iZ-X\right)\right)\frac{\partial D}{\partial\theta}\\
&\left.-2dD\left(X\left(-i\frac{\partial Z}{\partial\theta}-\frac{\partial X}{\partial\theta}\right)+\frac{\partial X}{\partial\theta}\left(1-iZ-X\right)\right)\right]\\
&-\frac{2\lambda}{rD^3}dD\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\theta}\right)\\
&+\lambda\xi\left(\frac{-k_r\delta k_\theta-k_\theta\delta k_r+2k_\phi\cot\theta\delta k_\phi}{r}-\frac{k_\phi^2}{r\sin^2\theta}\delta\theta+\frac{k_rk_\theta-k_\phi^2\cot\theta}{r^2}\delta r\right)\\
&+\frac{\lambda}{r^2D}\left(X\left(-i\frac{\partial Z}{\partial\theta}-\frac{\partial X}{\partial\theta}\right)+\frac{\partial X}{\partial\theta}\left(1-iZ-X\right)\right)\delta r\\
&-\frac{\lambda}{r^2D^2}\left(X\left(1-iZ-X\right)\left(\frac{\partial D}{\partial\theta}+k_\theta\frac{\partial D}{\partial k_r}-k_\phi\cot\theta\frac{\partial D}{\partial k_\phi}\right)\right)\delta r\\
&+\frac{\lambda}{rD^2}\left[X\left(1-iZ-X\right)\left(k_\theta d\frac{\partial D}{\partial k_r}+\frac{\partial D}{\partial k_r}\delta k_\theta-\left(k_\phi d\frac{\partial D}{\partial k_\phi}+\frac{\partial D}{\partial k_\phi}\delta k_\phi\right)\cot\theta\right)\right.\\
&\left.+\left(X\left(-idZ-dX\right)+dX\left(1-iZ-X\right)\right)\left(k_\theta\frac{\partial D}{\partial k_r}-k_\phi\cot\theta\frac{\partial D}{\partial k_\phi}\right)\right]\\
&-\frac{\lambda}{rD^3}\left(X\left(1-iZ-X\right)\left(k_\theta\frac{\partial D}{\partial k_r}-k_\phi\cot\theta\frac{\partial D}{\partial k_\phi}\right)dD\right)\\
&+\frac{\lambda k_\phi}{rD^2\sin^2\theta}X\left(1-iZ-X\right)\frac{\partial D}{\partial k_\phi}\delta\theta.
\end{aligned}
\tag{32}
$$

The two terms in (32) that are $O\left(\frac{1}{rD^2}\right)$ are kept separated so that the entire expression is configured similarly to the way the expression in (30) is arranged. Finally, we arrive at $\dot{k}_\phi$. Using (9a), (9b), (9f), and (16c),

$$
\begin{aligned}
\dot{k}_\phi = &-\frac{\lambda}{rD^2\sin\theta}\left(D\left(X\left(-i\frac{\partial Z}{\partial\phi}-\frac{\partial X}{\partial\phi}\right)+\frac{\partial X}{\partial\phi}\left(1-iZ-X\right)\right)-X\left(1-iZ-X\right)\frac{\partial D}{\partial\phi}\right)\\
&-\frac{\lambda\xi}{r}\left(k_rk_\phi+k_\theta k_\phi\cot\theta\right)+\frac{\lambda}{rD^2}\left(X\left(1-iZ-X\right)\left(k_\phi\frac{\partial D}{\partial k_r}+k_\phi\cot\theta\frac{\partial D}{\partial k_\theta}\right)\right)\\
= &-\frac{\lambda}{rD\sin\theta}\left(X\left(-i\frac{\partial Z}{\partial\phi}-\frac{\partial X}{\partial\phi}\right)+\frac{\partial X}{\partial\phi}\left(1-iZ-X\right)\right)+\frac{\lambda}{rD^2\sin\theta}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\phi}\right)\\
&-\frac{\lambda\xi}{r}\left(k_rk_\phi+k_\theta k_\phi\cot\theta\right)+\frac{\lambda}{rD^2}\left(X\left(1-iZ-X\right)\left(k_\phi\frac{\partial D}{\partial k_r}+k_\phi\cot\theta\frac{\partial D}{\partial k_\theta}\right)\right).
\end{aligned}
\tag{33}
$$

Once again arranging things mostly in terms of different orders of $\frac{1}{r}$ and $\frac{1}{D}$,

$$
\begin{aligned}
\delta\dot{k}_\phi = &-\frac{\lambda}{rD\sin\theta}\left[X\left(-id\frac{\partial Z}{\partial\phi}-d\frac{\partial X}{\partial\phi}\right)+dX\left(-i\frac{\partial Z}{\partial\phi}-\frac{\partial X}{\partial\phi}\right)+\frac{\partial X}{\partial\phi}\left(-idZ-dX\right)\right.\\
&\left.+d\frac{\partial X}{\partial\phi}\left(1-iZ-X\right)\right]\\
&-\frac{\lambda}{rD^2\sin\theta}\left[dD\left(X\left(-i\frac{\partial Z}{\partial\phi}-\frac{\partial X}{\partial\phi}\right)+\frac{\partial X}{\partial\phi}\left(1-iZ-X\right)\right)-X\left(1-iZ-X\right)d\frac{\partial D}{\partial\phi}\right.\\
&\left.-\left(X\left(-idZ-dX\right)+dX\left(1-iZ-X\right)\right)\frac{\partial D}{\partial\phi}\right]\\
&+\lambda\left(\frac{r\cos\theta\delta\theta+\sin\theta\delta r}{r^2D\sin^2\theta}\right)\left(X\left(-i\frac{\partial Z}{\partial\phi}-\frac{\partial X}{\partial\phi}\right)+\frac{\partial X}{\partial\phi}\left(1-iZ-X\right)\right)\\
&-\lambda\left(\frac{r\cos\theta\delta\theta+\sin\theta\delta r}{r^2D^2\sin^2\theta}\right)\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\phi}\right)\\
&+\frac{2\lambda dD}{rD^2\sin\theta}\left(X\left(-i\frac{\partial Z}{\partial\phi}-\frac{\partial X}{\partial\phi}\right)+\frac{\partial X}{\partial\phi}\left(1-iZ-X\right)\right)\\
&-\frac{2\lambda dD}{rD^3\sin\theta}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\phi}\right)\\
&+\lambda\xi\left(\frac{-k_r\delta k_\phi-k_\phi\delta k_r-\left(k_\theta\delta k_\phi+k_\phi\delta k_\theta\right)\cot\theta}{r}+\frac{k_\theta k_\phi}{r\sin^2\theta}\delta\theta+\frac{k_rk_\phi+k_\theta k_\phi\cot\theta}{r^2}\delta r\right)\\
&+\frac{\lambda}{rD^2}\left[X\left(1-iZ-X\right)\left(k_\phi d\frac{\partial D}{\partial k_r}+\frac{\partial D}{\partial k_r}\delta k_\phi+\left(k_\phi d\frac{\partial D}{\partial k_\theta}+\frac{\partial D}{\partial k_\theta}\delta k_\phi\right)\cot\theta\right)\right.\\
&\left.+\left(X\left(-idZ-dX\right)+dX\left(1-iZ-X\right)\right)\left(k_\phi\frac{\partial D}{\partial k_r}+k_\phi\cot\theta\frac{\partial D}{\partial k_\theta}\right)\right]\\
&-\frac{\lambda}{rD^3}\left(X\left(1-iZ-X\right)\right)\left(k_\phi\frac{\partial D}{\partial k_r}+k_\phi\cot\theta\frac{\partial D}{\partial k_\theta}\right)dD\\
&-\frac{\lambda}{r^2D^2}\left(X\left(1-iZ-X\right)\right)\left(k_\phi\frac{\partial D}{\partial k_r}+k_\phi\cot\theta\frac{\partial D}{\partial k_\theta}\right)\delta r\\
&-\frac{\lambda}{rD^2\sin^2\theta}\left(X\left(1-iZ-X\right)\right)k_\phi\frac{\partial D}{\partial k_\theta}\delta\theta.
\end{aligned}
\tag{34}
$$

Without fields and collisions, $Z = 0$ and D simplifies to $2(1-X)$. Equations (24), (26), (28), (30), (32), and (34) become Nickisch's (23-24) with these substitutions.

3. A FURTHER SIMPLIFICATION

Recall the discussion on Page 6 about simplifying (15) using

$$F\left(r,\theta,\phi,k_r,k_\theta,k_\phi\right) \equiv \frac{2X\left(1-iZ-X\right)}{D} \tag{35}$$

to get

$$\boldsymbol{H}\left(x^i,p^i\right) = \frac{\lambda}{2}\left(\xi\left(k_r^2+k_\theta^2+k_\phi^2\right)-1+F\right). \tag{36}$$

For $\eta = \{r, \theta, \phi\}$,

$$\frac{\partial F}{\partial \eta} = \frac{2}{D}\left(X\left(-i\frac{\partial Z}{\partial \eta} - \frac{\partial X}{\partial \eta}\right) + \frac{\partial X}{\partial \eta}(1 - iZ - X)\right) - \frac{2}{D^2}\left(X(1 - iZ - X)\frac{\partial D}{\partial \eta}\right), \tag{37a}$$

and

$$\frac{\partial F}{\partial k_\eta} = -\frac{2}{D^2}\left(X(1 - iZ - X)\frac{\partial D}{\partial k_\eta}\right). \tag{37b}$$

Also, (19) becomes

$$\frac{\partial \boldsymbol{H}}{\partial \eta} = \frac{\lambda}{2}\frac{\partial F}{\partial \eta} \tag{38a}$$

and

$$\frac{\partial \boldsymbol{H}}{\partial k_\eta} = \lambda \xi k_\eta + \frac{\lambda}{2}\frac{\partial F}{\partial k_\eta}. \tag{38b}$$

With these derivatives, it is possible to rewrite (23)-(34) in a much simpler fashion. Doing so may be optimal for programming, and with that in mind, we do so here. As with the other variables,

$$dF \equiv \frac{\partial F}{\partial r}\delta r + \frac{\partial F}{\partial \theta}\delta\theta + \frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial k_r}\delta k_r + \frac{\partial F}{\partial k_\theta}\delta k_\theta + \frac{\partial F}{\partial k_\phi}\delta k_\phi. \tag{39}$$

Therefore,

$$\dot{r} = \lambda\xi k_r + \frac{\lambda}{2}\frac{\partial F}{\partial k_r}, \tag{40}$$

$$\delta\dot{r} = \lambda\xi\delta k_r + \frac{\lambda}{2} d\frac{\partial F}{\partial k_r}, \tag{41}$$

$$\dot{\theta} = \frac{\lambda\xi k_\theta}{r} + \frac{\lambda}{2r}\frac{\partial F}{\partial k_\theta}, \tag{42}$$

$$\delta\dot{\theta} = \frac{\lambda\xi}{r}\delta k_\theta - \frac{\lambda\xi k_\theta}{r^2}\delta r - \frac{\lambda}{2r^2}\frac{\partial F}{\partial k_\theta}\delta r + \frac{\lambda}{2r} d\frac{\partial F}{\partial k_\theta}, \tag{43}$$

$$\dot{\phi} = \frac{\lambda\xi k_\phi}{r\sin\theta} + \frac{\lambda}{2r\sin\theta}\frac{\partial F}{\partial k_\phi}, \tag{44}$$

$$\delta\dot{\phi} = \frac{\lambda\xi}{r\sin\theta}\delta k_\phi - \lambda\xi k_\phi\left(\frac{r\cos\theta\delta\theta + \sin\theta\delta r}{r^2\sin^2\theta}\right) - \frac{\lambda}{2}\left(\frac{r\cos\theta\delta\theta + \sin\theta\delta r}{r^2\sin^2\theta}\right)\frac{\partial F}{\partial k_\phi} + \frac{\lambda}{2r\sin\theta} d\frac{\partial F}{\partial k_\phi}, \tag{45}$$

$$\dot{k}_r = -\frac{\lambda}{2}\frac{\partial F}{\partial r} + \frac{\lambda\xi}{r}\left(k_\theta^2 + k_\phi^2\right) + \frac{\lambda}{2r}\left(k_\theta\frac{\partial F}{\partial k_\theta} + k_\phi\frac{\partial F}{\partial k_\phi}\right), \tag{46}$$

$$\begin{aligned}\delta\dot{k}_r =& -\frac{\lambda}{2}d\frac{\partial F}{\partial r} + \frac{2\lambda\xi}{r}\left(k_\theta\delta k_\theta + k_\phi\delta k_\phi\right) - \frac{\lambda\xi}{r^2}\left(k_\theta^2 + k_\phi^2\right)\delta r\\ &+ \frac{\lambda}{2r}\left(k_\theta d\frac{\partial F}{\partial k_\theta} + \frac{\partial F}{\partial k_\theta}\delta k_\theta + k_\phi d\frac{\partial F}{dk_\phi} + \frac{\partial F}{\partial k_\phi}\delta k_\phi\right) - \frac{\lambda}{2r^2}\left(k_\theta\frac{\partial F}{\partial k_\theta} + k_\phi\frac{\partial F}{\partial k_\phi}\right)\delta r,\end{aligned} \tag{47}$$

$$\dot{k}_\theta = -\frac{\lambda}{2r}\frac{\partial F}{\partial\theta} - \frac{\lambda\xi}{r}k_r k_\theta - \frac{\lambda}{2r}k_\theta\frac{\partial F}{\partial k_r} + \frac{\lambda\xi}{r}k_\phi^2\cot\theta + \frac{\lambda}{2r}k_\phi\cot\theta\frac{\partial F}{\partial k_\phi}, \tag{48}$$

$$\begin{aligned}\delta\dot{k}_\theta =& \frac{\lambda\xi}{r}\left(-\left(k_r\delta k_\theta + k_\theta\delta k_r\right) + 2k_\phi\cot\theta\delta k_\phi\right) - \frac{\lambda\xi}{r\sin^2\theta}k_\phi^2\delta\theta\\ &+ \frac{\lambda}{2r}\left(-d\frac{\partial F}{\partial\theta} - \frac{\partial F}{\partial k_r}\delta k_\theta - k_\theta d\frac{\partial F}{\partial k_r} + k_\phi\cot\theta\frac{\partial F}{\partial k_\phi} + \cot\theta\frac{\partial F}{\partial k_\phi}\delta k_\phi\right) - \frac{\lambda}{2r\sin^2\theta}k_\phi\frac{\partial F}{\partial k_\phi}\delta\theta\\ &+ \frac{\lambda\xi}{r^2}\left(k_r k_\theta - k_\phi^2\cot\theta\right)\delta r + \frac{\lambda}{2r^2}\left(\frac{\partial F}{\partial\theta} + k_\theta\frac{\partial F}{\partial k_r} - k_\phi\cot\theta\frac{\partial F}{\partial k_\phi}\right)\delta r,\end{aligned} \tag{49}$$

$$\dot{k}_\phi = -\frac{\lambda}{2r\sin\theta}\frac{\partial F}{\partial\phi} - \frac{\lambda\xi}{r}k_r k_\phi - \frac{\lambda}{2r}k_\phi\frac{\partial F}{\partial k_r} - \frac{\lambda\xi}{r}k_\theta k_\phi\cot\theta - \frac{\lambda}{2r}k_\phi\cot\theta\frac{\partial F}{\partial k_\theta}, \tag{50}$$

and

$$\begin{aligned}\delta\dot{k}_\phi =& -\frac{\lambda\xi}{r}\left(k_r\delta k_\phi + k_\phi\delta k_r + \left(k_\theta\delta k_\phi + k_\phi\delta k_\theta\right)\cot\theta\right) + \frac{\lambda\xi}{r\sin^2}k_\theta k_\phi\delta\theta\\ &+ \frac{\lambda}{2r}\left(-k_\phi d\frac{\partial F}{\partial k_r} - \frac{\partial F}{\partial k_r}\delta k_\phi + \cot\theta\frac{\partial F}{\partial k_\theta}\delta k_\phi\right) - \frac{\lambda}{2r\sin\theta}d\frac{\partial F}{\partial\phi}\\ &+ \frac{\lambda}{2r\sin^2\theta}\left(\cos\theta\frac{\partial F}{\partial\phi}\delta\theta - k_\phi\frac{\partial F}{\partial k_\theta}\delta\theta\right) + \frac{\lambda\xi}{r^2}\left(k_r k_\phi\delta r + k_\theta k_\phi\cot\theta\delta r\right)\\ &+ \frac{\lambda}{2r^2}\left(k_\phi\frac{\partial F}{\partial k_r} - k_\phi\cot\theta\frac{\partial F}{\partial k_\theta}\right)\delta r + \frac{\lambda}{2r^2\sin\theta}\frac{\partial F}{\partial\phi}\delta r.\end{aligned} \tag{51}$$

The Västberg and Lundborg paper make a further generalization to the perturbations in their (24-29), which we reproduce here for convenience. Using our notation and $n^2 = 1 - F$:

$$\delta\dot{r} = \lambda\xi\delta k_r - \frac{1}{2}d\frac{\partial n^2}{\partial k_r}, \tag{52}$$

$$\delta\dot{\theta} = \frac{1}{r}\left(\lambda\xi\delta k_\theta - \frac{1}{2}d\frac{\partial n^2}{\partial k_\theta} - \dot{\theta}\delta r\right), \tag{53}$$

$$\delta\dot{\phi} = \frac{1}{r\sin\theta}\left(\lambda\xi\delta k_\phi - \frac{1}{2}d\frac{\partial n^2}{\partial k_\phi}\right) - \frac{1}{r}\dot{\phi}\delta r - \cot\theta\dot{\phi}\delta\theta, \tag{54}$$

$$\delta\dot{k}_r = \frac{1}{2}d\frac{\partial n^2}{\partial r} + k_\phi \cos\theta\dot{\phi}\delta\theta + \dot{\theta}\delta k_\theta + \sin\theta\dot{\phi}\delta k_\phi + k_\theta\delta\dot{\theta} + k_\phi \sin\theta\delta\dot{\phi}, \tag{55}$$

$$\delta\dot{k}_\theta = \frac{1}{2r}d\frac{\partial n^2}{\partial\theta} - \frac{1}{r^2}\left(\frac{1}{2}\frac{\partial n^2}{\partial\theta} - k_\theta\dot{r}\right)\delta r - k_\phi \sin\theta\dot{\phi}\delta\theta - \frac{1}{r}\dot{r}\delta k_\theta + \cos\theta\dot{\phi}\delta k_\phi - \frac{k_\theta}{r}\delta\dot{r} + k_\phi\cos\theta\delta\dot{\phi}, \tag{56}$$

and

$$\begin{aligned}\delta\dot{k}_\phi =& \frac{1}{2r\sin\theta}\left(d\frac{\partial n^2}{\partial\phi} - \frac{\partial n^2}{\partial\phi}\left(\frac{1}{r}\delta r + \cot\theta\delta\theta\right)\right) + \frac{k_\phi}{r^2}\dot{r}\delta r + \frac{k_\phi}{\sin^2\theta}\dot{\theta}\delta\theta - \left(\frac{1}{r}\dot{r} + \cot\theta\dot{\theta}\right)\delta k_\phi \\ &- \frac{k_\phi}{r}\delta\dot{r} - k_\phi\cot\theta\delta\dot{\theta}.\end{aligned} \tag{57}$$

Conversely, by substituting $n^2 = 1 - F$, the derivatives in the following section can all easily be written in terms of F. We leave it to the reader to use whichever set of equations is the most convenient to his or her needs.

While there may be some use in seeing the equations written out as in Section 2 since it gives a good comparison to the simpler equations in [1], the authors of [4] had a good reason to write the six perturbation derivatives seen in (52)-(57) using only the generalized square of the refraction index, n^2, and not expanding things further. Although it may not be immediately clear at first glance, even the simplest of these equations, (52) contains six *second* derivatives which must be computed thanks to the d operator defined in (21). In Section 2, the six first derivatives of D are given in terms of the six first derivatives of R, which are in turn given in terms of the six first derivatives of X, Y, Z, and ψ. Adding F to the mix makes things look nicer but just adds another layer of derivatives to deal with. For each of the six first derivatives, there are six second derivatives that must be found. Many of those are redundant thanks to the symmetry of second partial derivatives, but it still leaves an intimidating amount to write into any code. In order to make it more approachable, Västberg and Lundborg took a different approach, which is the focus of the next section.

4. EXPANDING THE WORK OF VÄSTBERG AND LUNDBORG

In a 1996 paper[4], Anders Västberg and Bengt Lundborg pick up from where Nickisch left off and did something similar to what is given in Section 2 (although not quite the same), and the results can be seen in Section 3. As stated above, (52)-(57) each contain six second derivatives which must be computed. Doing so in layers as seen in 2 is unfeasible, so instead they used the Chain Rule to write out exactly what the derivatives would need to be in terms of X, Y, Z, and ψ. Eliminating redundancies, there are twenty-one different second derivatives which must be computed, but they can be expressed in just three formulas. Letting η and ζ be any

r, θ, or ϕ, we reproduce their (32) here for the six purely spatial second derivatives:

$$\begin{aligned}
\frac{\partial^2 n^2}{\partial\eta\partial\zeta} =& \frac{\partial X}{\partial\eta}\frac{\partial X}{\partial\zeta}\frac{\partial^2 n^2}{\partial X^2} + \left(\frac{\partial X}{\partial\eta}\frac{\partial Y}{\partial\zeta} + \frac{\partial Y}{\partial\eta}\frac{\partial X}{\partial\zeta}\right)\frac{\partial^2 n^2}{\partial X\partial Y} + \left(\frac{\partial X}{\partial\eta}\frac{\partial Z}{\partial\zeta} + \frac{\partial Z}{\partial\eta}\frac{\partial X}{\partial\zeta}\right)\frac{\partial^2 n^2}{\partial X\partial Z} \\
&+ \left(\frac{\partial X}{\partial\eta}\frac{\partial\psi}{\partial\zeta} + \frac{\partial\psi}{\partial\eta}\frac{\partial X}{\partial\zeta}\right)\frac{\partial^2 n^2}{\partial X\partial\psi} + \frac{\partial Y}{\partial\eta}\frac{\partial Y}{\partial\zeta}\frac{\partial^2 n^2}{\partial Y^2} + \left(\frac{\partial Y}{\partial\eta}\frac{\partial Z}{\partial\zeta} + \frac{\partial Z}{\partial\eta}\frac{\partial Y}{\partial\zeta}\right)\frac{\partial^2 n^2}{\partial Y\partial Z} \\
&+ \left(\frac{\partial Y}{\partial\eta}\frac{\partial\psi}{\partial\zeta} + \frac{\partial\psi}{\partial\eta}\frac{\partial Y}{\partial\zeta}\right)\frac{\partial^2 n^2}{\partial Y\partial\psi} + \frac{\partial Z}{\partial\eta}\frac{\partial Z}{\partial\zeta}\frac{\partial^2 n^2}{\partial Z^2} + \left(\frac{\partial Z}{\partial\eta}\frac{\partial\psi}{\partial\zeta} + \frac{\partial\psi}{\partial\eta}\frac{\partial Z}{\partial\zeta}\right)\frac{\partial^2 n^2}{\partial Z\partial\psi} \\
&+ \frac{\partial\psi}{\partial\eta}\frac{\partial\psi}{\partial\zeta}\frac{\partial^2 n^2}{\partial\psi^2} + \frac{\partial^2 X}{\partial\eta\partial\zeta}\frac{\partial n^2}{\partial X} + \frac{\partial^2 Y}{\partial\eta\partial\zeta}\frac{\partial n^2}{\partial Y} + \frac{\partial^2 Z}{\partial\eta\partial\zeta}\frac{\partial n^2}{\partial Z} + \frac{\partial^2 \psi}{\partial\eta\partial\zeta}\frac{\partial n^2}{\partial\psi}.
\end{aligned} \tag{58}$$

Västberg and Lundborg then break down the second derivatives with respect to ψ to ensure the Y_T and Y_L are properly incorporated, following what Jones and Stephenson did with the first derivatives in their code. However, although the second derivatives are broken down, they are not written out. We will do so here, as well as the other second derivatives that appear in (58). As before, we write the expressions out in terms of different orders of $\frac{1}{D}$ or $\frac{1}{R}$. For full transparency, derivations that require additional steps to arrive at the final expressions are given in Appendix A.

We will first tackle the non-ψ derivatives as the ones involving ψ are given special consideration in the paper's (33)-(39) whereas these are not. Let us start with the first derivatives. For X,

$$\frac{\partial n^2}{\partial X} = -\frac{2}{D}(1 - iZ - 2X) + \frac{2}{D^2}\left(X(1 - iZ - X)\frac{\partial D}{\partial X}\right), \tag{59a}$$

where

$$\frac{\partial D}{\partial X} = -2(1 - iZ) + \frac{\partial R}{\partial X}, \tag{59b}$$

and

$$\frac{\partial R}{\partial X} = \pm\frac{-4Y_L^2(1 - iZ - X)}{\sqrt{Y_T^4 + 4Y_L^2(1 - iZ - X)^2}} = -\frac{4}{R}\left(Y_L^2(1 - iZ - X)\right). \tag{59c}$$

For Y,

$$\frac{\partial n^2}{\partial Y} = \frac{2}{D^2}\left(X(1 - iZ - X)\frac{\partial D}{\partial Y}\right), \tag{60a}$$

where

$$\frac{\partial D}{\partial Y} = -2Y\sin^2\psi + \frac{\partial R}{\partial Y} = \frac{-2Y_T^2}{Y} + \frac{\partial R}{\partial Y}, \tag{60b}$$

and

$$\frac{\partial R}{\partial Y} = \frac{Y_T^4 + R^2}{YR}. \tag{60c}$$

For Z,

$$\frac{\partial n^2}{\partial Z} = \frac{2}{D} iX + \frac{2}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial Z}\right), \tag{61a}$$

where

$$\frac{\partial D}{\partial Z} = -2i\left(2 - 2iZ - X\right) + \frac{\partial R}{\partial Z}, \tag{61b}$$

and

$$\frac{\partial R}{\partial Z} = \pm\frac{-4iY_L^2\left(1 - iZ - X\right)}{\sqrt{Y_T^4 + 4Y_L^2\left(1 - iZ - X\right)^2}} = \frac{-4iY_L^2\left(1 - iZ - X\right)}{R}. \tag{61c}$$

For the second order derivatives, let us first start with $\frac{\partial^2 n^2}{\partial X^2}$.

$$\begin{aligned}\frac{\partial^2 n^2}{\partial X^2} =& \frac{4}{D} + \frac{4}{D^2}\left(1 - iZ - 2X\right)\frac{\partial D}{\partial X} + \frac{2}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial^2 D}{\partial X^2}\right) \\ &- \frac{4}{D^3}\left(X\left(1 - iZ - X\right)\left(\frac{\partial D}{\partial X}\right)^2\right).\end{aligned} \tag{62a}$$

where

$$\frac{\partial^2 D}{\partial X^2} = \frac{\partial^2 R}{\partial X^2}, \tag{62b}$$

and

$$\frac{\partial^2 R}{\partial X^2} = \frac{4Y_T^4 Y_L^2}{R^3}. \tag{62c}$$

Because $\frac{\partial^2 n^2}{\partial X \partial Y} = \frac{\partial^2 n^2}{\partial Y \partial X}$, (and similarly for other variables) we only need to find one of these derivatives, so let us use (60) to write out $\frac{\partial^2 n^2}{\partial X \partial Y}$:

$$\begin{aligned}\frac{\partial^2 n^2}{\partial X \partial Y} =& \frac{2}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial^2 D}{\partial X \partial Y} + \left(1 - iZ - 2X\right)\frac{\partial D}{\partial Y}\right) \\ &- \frac{4}{D^3}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial X}\frac{\partial D}{\partial Y}\right),\end{aligned} \tag{63a}$$

where

$$\frac{\partial^2 D}{\partial X \partial Y} = \frac{\partial^2 R}{\partial X \partial Y}, \tag{63b}$$

and

$$\frac{\partial^2 R}{\partial X \partial Y} = \frac{16Y_L^4\left(1 - iZ - X\right)^3}{YR^3}. \tag{63c}$$

Next we come to $\frac{\partial^2 n^2}{\partial X \partial Z}$. Using (61),

$$\begin{aligned}\frac{\partial^2 n^2}{\partial X \partial Z} =& \frac{2i}{D} + \frac{2}{D^2}\left(-iX\frac{\partial D}{\partial X} + X\left(1 - iZ - X\right)\frac{\partial^2 D}{\partial X \partial Z} + \left(1 - iZ - 2X\right)\frac{\partial D}{\partial Z}\right) \\ &- \frac{4}{D^3}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial Z}\frac{\partial D}{\partial X}\right),\end{aligned} \tag{64a}$$

where

$$\frac{\partial^2 D}{\partial X \partial Z} = 2iX + \frac{\partial^2 R}{\partial X \partial Z}, \tag{64b}$$

and

$$\frac{\partial^2 R}{\partial X \partial Z} = \frac{4iY_T^4 Y_L^2}{R^3}. \tag{64c}$$

For $\frac{\partial^2 n^2}{\partial Y^2}$, we use algebraic manipulation to ensure that ψ does not appear, much like what was done in the derivation seen in (60):

$$\frac{\partial^2 n^2}{\partial Y^2} = \frac{2}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial^2 D}{\partial Y^2}\right) - \frac{4}{D^3}\left(X\left(1 - iZ - X\right)\left(\frac{\partial D}{\partial Y}\right)^2\right), \tag{65a}$$

where

$$\frac{\partial^2 D}{\partial Y^2} = -2\sin^2\psi + \frac{\partial^2 R}{\partial Y^2} = \frac{-2Y_T^2}{Y^2} + \frac{\partial^2 R}{\partial Y^2}, \tag{65b}$$

and

$$\frac{\partial^2 R}{\partial Y^2} = \frac{2Y_T^8 + 12Y_T^4 Y_L^2\left(1 - iZ - X\right)^2}{Y^2 R^3}. \tag{65c}$$

Similarly, the algebraic manipulation can be used again for $\frac{\partial^2 n^2}{\partial Y \partial Z}$ to collect the appearances of ψ in Y_T and Y_L:

$$\frac{\partial^2 n^2}{\partial Y \partial Z} = \frac{2}{D^2}\left(-iX\frac{\partial D}{\partial Y} + X\left(1 - iZ - X\right)\frac{\partial^2 D}{\partial Y \partial Z}\right) - \frac{4}{D^3}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial Z}\frac{\partial D}{\partial Y}\right), \tag{66a}$$

where

$$\frac{\partial^2 D}{\partial Y \partial Z} = \frac{\partial^2 R}{\partial Y \partial Z}, \tag{66b}$$

and

$$\frac{\partial^2 R}{\partial Y \partial Z} = \frac{-16iY_L^4\left(1 - iZ - X\right)^3}{Y R^3}. \tag{66c}$$

Finally, we get to $\frac{\partial^2 n^2}{\partial Z^2}$:

$$\frac{\partial^2 n^2}{\partial Z^2} = \frac{2}{D^2}\left(-2iX\frac{\partial D}{\partial Z} + X\left(1 - iZ - X\right)\frac{\partial^2 D}{\partial Z^2}\right) - \frac{4}{D^3}\left(X\left(1 - iZ - X\right)\left(\frac{\partial D}{\partial Z}\right)^2\right). \tag{67a}$$

where

$$\frac{\partial^2 D}{\partial Z^2} = -4 + \frac{\partial^2 R}{\partial Z^2}, \tag{67b}$$

and

$$\frac{\partial^2 R}{\partial Z^2} = \frac{4Y_T^4 Y_L^2}{R^3}. \tag{67c}$$

With these second derivatives all written out, what is left from (58) are the second derivatives involving ψ. Västberg and Lundborg spell out how to derive them in their (33)-(39) but do not actually do the derivations. We do that here. Much like was done with the derivatives with respect to Y, we preform some algebraic manipulation on the derivatives with respect to ψ to be able to write out the expressions without having to use ψ itself. For the first derivative,

$$\frac{\partial n^2}{\partial \psi} = \frac{2}{D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial \psi}\right), \tag{68a}$$

where

$$\frac{\partial D}{\partial \psi} = -Y^2 \sin\psi\cos\psi + \frac{\partial R}{\partial \psi} = -Y_T Y_L + \frac{\partial R}{\partial \psi}, \tag{68b}$$

and

$$\frac{\partial R}{\partial \psi} = \frac{2Y_T^3 Y_L - 4Y_T Y_L\left(1 - iZ - X\right)^2}{R}. \tag{68c}$$

One more restriction needs to be noted. In (58), derivatives of ψ itself appear. As noted in (51)-(53) in [2], these derivatives are computed based on known quantities only as a multiple of $Y_T Y_L$, and we present them that way in this report. See Appendix B for more details.

We now go through Equations (33)-(39) in Västberg and Lundborg's paper and write out the second derivatives found in them. In these equations, we follow the same convention established above in (17), (18), and (19): η can be used to represent r, θ, or ϕ, and ζ can also be r, θ, or ϕ when needed. Using this notational shortcut allows us to not have to write out multiple expressions that are identical except for a variable swap. First, their (33) states

$$\frac{\partial \psi}{\partial \eta}\frac{\partial^2 n^2}{\partial X \partial \psi} = \left(Y_T Y_L \frac{\partial \psi}{\partial \eta}\right)\cdot\frac{\partial}{\partial X}\left(\frac{1}{Y_T Y_L}\frac{\partial n^2}{\partial \psi}\right).$$

The second derivative here comes from the second term:

$$\frac{\partial}{\partial X}\left(\frac{1}{Y_TY_L}\frac{\partial n^2}{\partial\psi}\right) = \frac{2}{Y_TY_LD^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial X\partial\psi}+\left(1-iZ-2X\right)\frac{\partial D}{\partial\psi}\right) - \frac{4}{Y_TY_LD^3}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial X}\right), \tag{69a}$$

where

$$\frac{\partial^2 D}{\partial X\partial\psi} = \frac{\partial^2 R}{\partial X\partial\psi}, \tag{69b}$$

and

$$\frac{\partial^2 R}{\partial X\partial\psi} = 8Y_TY_L\left(1-iZ-X\right)\left(\frac{1}{R}+\frac{1}{R^3}\left(Y_L^2\left(Y_T^2-2\left(1-iZ-X\right)^2\right)\right)\right). \tag{69c}$$

Thus,

$$\frac{\partial\psi}{\partial\eta}\frac{\partial^2 n^2}{\partial X\partial\psi} = Y_TY_L\frac{\partial\psi}{\partial\eta}\left(\frac{2}{Y_TY_LD^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial X\partial\psi}+\left(1-iZ-2X\right)\frac{\partial D}{\partial\psi}\right) - \frac{4}{Y_TY_LD^3}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial X}\right)\right). \tag{69d}$$

Next, their (34) states

$$\frac{\partial\psi}{\partial\eta}\frac{\partial^2 n^2}{\partial Y\partial\psi} = \left(Y_TY_L\frac{\partial\psi}{\partial\eta}\right)\frac{1}{Y_TY_L}\frac{\partial}{\partial Y}\left(Y_TY_L\left(\frac{1}{Y_TY_L}\frac{\partial n^2}{\partial\psi}\right)\right).$$

We can see from (68) that the Y_TY_L cancels out inside the outer derivative, (It seems it was only written this way due to (50) in [2])

$$\frac{\partial}{\partial Y}\left(Y_TY_L\left(\frac{1}{Y_TY_L}\frac{\partial n^2}{\partial\psi}\right)\right) = \frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial Y\partial\psi}\right) - \frac{4}{D^3}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial Y}\right), \tag{70a}$$

where

$$\frac{\partial^2 D}{\partial Y\partial\psi} = -2Y\sin\psi\cos\psi + \frac{\partial^2 R}{\partial Y\partial\psi} = -\frac{2Y_TY_L}{Y} + \frac{\partial^2 R}{\partial Y\partial\psi}, \tag{70b}$$

and

$$\frac{\partial^2 R}{\partial Y\partial\psi} = \frac{4Y_TY_L\left(Y_T^6+8Y_T^2Y_L^2\left(1-iZ-X\right)^2-4Y_L^2\left(1-iZ-X\right)^4\right)}{YR^3}. \tag{70c}$$

Thus,

$$\frac{\partial\psi}{\partial\eta}\frac{\partial^2 n^2}{\partial Y\partial\psi} = \left(Y_TY_L\frac{\partial\psi}{\partial\eta}\right)\frac{1}{Y_TY_L}\left(\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial Y\partial\psi}\right) - \frac{4}{D^3}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial Y}\right)\right). \tag{70d}$$

Next comes their (35):

$$\frac{\partial\psi}{\partial\eta}\frac{\partial^2 n^2}{\partial Z\partial\psi} = \left(Y_T Y_L \frac{\partial\psi}{\partial\eta}\right)\cdot\frac{\partial}{\partial Z}\left(\frac{1}{Y_T Y_L}\frac{\partial n^2}{\partial\psi}\right).$$

In a similar manner to (69), we focus on the second term:

$$\begin{aligned}\frac{\partial}{\partial Z}\left(\frac{1}{Y_T Y_L}\frac{\partial n^2}{\partial\psi}\right) =& \frac{2}{Y_T Y_L D^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial Z\partial\psi} - iX\frac{\partial D}{\partial\psi}\right)\\ &- \frac{4}{Y_T Y_L D^3}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial Z}\right),\end{aligned} \tag{71a}$$

where

$$\frac{\partial^2 D}{\partial Z\partial\psi} = \frac{\partial^2 R}{\partial Z\partial\psi}, \tag{71b}$$

and

$$\frac{\partial^2 R}{\partial Z\partial\psi} = 8iY_T Y_L\left(1-iZ-X\right)\left(\frac{1}{R} + \frac{1}{R^3}\left(Y_L^2\left(Y_T^2 - 2\left(1-iZ-X\right)^2\right)\right)\right). \tag{71c}$$

Thus,

$$\begin{aligned}\frac{\partial\psi}{\partial\eta}\frac{\partial^2 n^2}{\partial Z\partial\psi} =& \left(Y_T Y_L\frac{\partial\psi}{\partial\eta}\right)\frac{1}{Y_T Y_L}\left(\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial Z\partial\psi} - iX\frac{\partial D}{\partial\psi}\right)\right.\\ &\left.-\frac{4}{D^3}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial Z}\right)\right).\end{aligned} \tag{71d}$$

Their (36) deals with the second derivative with respect to ψ:

$$\frac{\partial\psi}{\partial\eta}\frac{\partial\psi}{\partial\zeta}\frac{\partial^2 n^2}{\partial\psi^2} = \left(Y_T Y_L\frac{\partial\psi}{\partial\eta}\right)\left(Y_T Y_L\frac{\partial\psi}{\partial\zeta}\right)\left(\frac{1}{Y_T^2 Y_L^2}\frac{\partial^2 n^2}{\partial\psi^2}\right).$$

Writing out the second derivative,

$$\frac{\partial^2 n^2}{\partial\psi^2} = \frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial\psi^2}\right) - \frac{4}{D^3}\left(X\left(1-iZ-X\right)\left(\frac{\partial^2 D}{\partial\psi^2}\right)\right), \tag{72a}$$

where

$$\frac{\partial^2 D}{\partial\psi^2} = -Y^2\left(-\sin^2\psi + \cos^2\psi\right) + \frac{\partial^2 R}{\partial\psi^2} = Y_T^2 - Y_L^2 + \frac{\partial^2 R}{\partial\psi^2}, \tag{72b}$$

and

$$\frac{\partial^2 R}{\partial\psi^2} = \frac{-2Y_T^8 + 2Y_T^6 Y_L^2 + 4\left(Y_T^4 Y_L^2 + 6Y_T^2 Y_L^4 + Y_T^6\right)\left(1-iZ-X\right)^2 - 16Y_L^4\left(1-iZ-X\right)^4}{R^3}. \tag{72c}$$

Thus,

$$\frac{\partial\psi}{\partial\eta}\frac{\partial\psi}{\partial\zeta}\frac{\partial^2 n^2}{\partial\psi^2} = \left(Y_T Y_L \frac{\partial\psi}{\partial\eta}\right)\left(Y_T Y_L \frac{\partial\psi}{\partial\zeta}\right)\left(\frac{2}{Y_T^2 Y_L^2 D^2}\left(X\left(1 - iZ - X\right)\frac{\partial^2 D}{\partial\psi^2}\right)\right.$$
$$\left. - \frac{4}{Y_T^2 Y_L^2 D^3}\left(X\left(1 - iZ - X\right)\left(\frac{\partial^2 D}{\partial\psi^2}\right)\right)\right). \tag{72d}$$

In their (37), the second derivative is $\frac{\partial^2\psi}{\partial\eta\partial\zeta}$:

$$\frac{\partial^2\psi}{\partial\eta\partial\zeta}\frac{\partial n^2}{\partial\psi} = \left(Y_T Y_L \frac{\partial^2\psi}{\partial\eta\partial\zeta}\right)\left(\frac{1}{Y_T Y_L}\frac{\partial n^2}{\partial\psi}\right).$$

In order to write that second derivative in terms of known quantities, we must introduce the vector $\boldsymbol{V}$ from [2] as a vector in the direction of the wave normal with a magnitude of $\Re\left(n^2\right)$. Given this notation,

$$Y_T Y_L \frac{\partial^2\psi}{\partial\eta\partial\zeta} = \frac{Y_L^2}{Y}\frac{\partial^2 Y}{\partial\eta\partial\zeta} - \left(\boldsymbol{V}\cdot\frac{\partial^2\boldsymbol{Y}}{\partial\eta\partial\zeta}\right)\left(\frac{Y_L}{V}\right) - \frac{Y_T Y_L \frac{\partial\psi}{\partial\eta} Y_T Y_L \frac{\partial\psi}{\partial\zeta}}{Y_T^2} - \frac{Y_T Y_L}{Y}\frac{\partial\psi}{\partial\zeta}\frac{\partial Y}{\partial\eta} - \frac{Y_T Y_L}{Y}\frac{\partial\psi}{\partial\eta}\frac{\partial Y}{\partial\zeta}, \tag{73}$$

where

$$\boldsymbol{V}\cdot\frac{\partial^2\boldsymbol{Y}}{\partial\eta\partial\zeta} \equiv V_r\frac{\partial^2 Y_r}{\partial\eta\partial\zeta} + V_\theta\frac{\partial^2 Y_\theta}{\partial\eta\partial\zeta} + V_\phi\frac{\partial^2 Y_\phi}{\partial\eta\partial\zeta}. \tag{74}$$

Therefore,

$$\frac{\partial^2\psi}{\partial\eta\partial\zeta}\frac{\partial n^2}{\partial\psi} = \left(\frac{Y_L^2}{Y}\frac{\partial^2 Y}{\partial\eta\partial\zeta} - \left(\boldsymbol{V}\cdot\frac{\partial^2\boldsymbol{Y}}{\partial\eta\partial\zeta}\right)\left(\frac{Y_L}{V}\right) - \frac{Y_T Y_L \frac{\partial\psi}{\partial\eta} Y_T Y_L \frac{\partial\psi}{\partial\zeta}}{Y_T^2}\right.$$
$$\left. - \frac{Y_T Y_L}{Y}\frac{\partial\psi}{\partial\zeta}\frac{\partial Y}{\partial\eta} - \frac{Y_T Y_L}{Y}\frac{\partial\psi}{\partial\eta}\frac{\partial Y}{\partial\zeta}\right)\left(\frac{2}{Y_T Y_L D^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial\psi}\right)\right). \tag{75}$$

The derivation of (73) is given in Appendix B.

While the above covers only six different second derivatives of n^2, the other fifteen can be written as two considerably shorter equations. Nine of them come from their (38), in which there are two (technically six) second derivatives that we must write out:

$$\frac{\partial^2 n^2}{\partial\eta\partial k_\zeta} = \left(Y_T Y_L \frac{\partial\psi}{\partial k_\zeta}\right)\frac{\partial}{\partial\eta}\left(\frac{1}{Y_T Y_L}\frac{\partial n^2}{\partial\psi}\right) + \left(\frac{1}{Y_T Y_L}\frac{\partial n^2}{\partial\psi}\right)\frac{\partial}{\partial\eta}\left(Y_T Y_L \frac{\partial\psi}{\partial k_\zeta}\right).$$

For $\eta = \{r, \theta, \phi\}$,

$$\begin{aligned}\frac{\partial}{\partial\eta}\left(\frac{1}{Y_T Y_L}\frac{\partial n^2}{\partial\psi}\right) =& \frac{1}{Y_T Y_L}\left[\frac{2}{D^2}\left(X(1-iZ-X)\frac{\partial^2 D}{\partial\eta\partial\psi} + \left(X\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right)\right.\right.\right.\\ &\left.\left.\left. + \frac{\partial X}{\partial\eta}(1-iZ-X)\right)\frac{\partial D}{\partial\psi}\right) - \frac{4}{D^3}\left(X(1-iZ-X)\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial\eta}\right)\right]\\ &+ \frac{2}{Y_L^2 D^2}\left(X(1-iZ-X)\frac{\partial D}{\partial\psi}\frac{\partial\psi}{\partial\eta}\right) - \frac{2}{Y_T^2 D^2}\left(X(1-iZ-X)\frac{\partial D}{\partial\psi}\frac{\partial\psi}{\partial\eta}\right)\\ &- \frac{4}{YY_T Y_L D^2}\left(X(1-iZ-X)\frac{\partial D}{\partial\psi}\frac{\partial Y}{\partial\eta}\right),\end{aligned} \tag{76a}$$

where

$$\frac{\partial D}{\partial\eta} = 2(1-iZ)\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right) - 2iZ(1-iZ-X) - 2Y_T Y_L\frac{\partial\psi}{\partial\eta} - \frac{2Y_T^2}{Y}\frac{\partial Y}{\partial\eta} + \frac{\partial R}{\partial\eta}, \tag{76b}$$

$$\begin{aligned}\frac{\partial R}{\partial\eta} =& \left[2YY_T^3 Y_L\frac{\partial\psi}{\partial\eta} + 2Y_T^4\frac{\partial Y}{\partial\eta} + 4YY_L^2(1-iZ-X)\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right)\right.\\ &\left. + \left(-4YY_T Y_L\frac{\partial\psi}{\partial\eta} + 4Y_L^2\frac{\partial Y}{\partial\eta}\right)(1-iZ-X)^2\right]/(YR),\end{aligned} \tag{76c}$$

$$\frac{\partial^2 D}{\partial\eta\partial\psi} = \left(Y_T^2 - Y_L^2\right)\frac{\partial\psi}{\partial\eta} - \frac{2Y_T Y_L}{Y}\frac{\partial Y}{\partial\eta} + \frac{\partial^2 R}{\partial\eta\partial\psi}, \tag{76d}$$

and

$$\begin{aligned}\frac{\partial^2 R}{\partial\eta\partial\psi} =& \frac{2}{YR}\left(\left(-YY_T^4 + 3YY_T^2 Y_L^2 + 2YY_T^2(1-iZ-X)^2 - 2YY_L^2(1-iZ-X)^2\right)\frac{\partial\psi}{\partial\eta}\right.\\ &\left. + \left(4Y_T^3 Y_L - 4Y_T Y_L(1-iZ-X)^2\right)\frac{\partial Y}{\partial\eta} - 4YY_T Y_L(1-iZ-X)\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right)\right)\\ &- \frac{4}{YR^3}Y_T Y_L\left(Y_T^2 - 2(1-iZ-X)^2\right)\left(\left(YY_T^3 Y_L - 2Y_T Y_L\right)\frac{\partial\psi}{\partial\eta}\right.\\ &\left. + \left(Y_T^4 + 2Y_L^2(1-iZ-X)^2\right)\frac{\partial Y}{\partial\eta} + 2YY_L^2(1-iZ-X)\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right)\right)\end{aligned} \tag{76e}$$

Also, for $\zeta = \{r, \theta, \phi\}$,

$$\frac{\partial}{\partial\eta}\left(Y_T Y_L\frac{\partial\psi}{\partial k_\zeta}\right) = Y_T Y_L\frac{\partial^2\psi}{\partial\eta\partial k_\zeta} + \left(-\left(Y_T^2 - Y_L^2\right)\frac{\partial\psi}{\partial\eta} + \frac{2Y_T Y_L}{Y}\frac{\partial Y}{\partial\eta}\right)\frac{\partial\psi}{\partial k_\zeta}. \tag{76f}$$

Thus,

$$
\begin{aligned}
\frac{\partial^2 n^2}{\partial\eta\partial k_\zeta} =& \frac{\partial\psi}{\partial k_\zeta}\left[\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial\eta\partial\psi}+\left(X\left(-i\frac{\partial Z}{\partial\eta}-\frac{\partial X}{\partial\eta}\right)\right.\right.\right. \\
&\left.\left.+\frac{\partial X}{\partial\eta}\left(1-iZ-X\right)\right)\frac{\partial D}{\partial\psi}\right)-\frac{4}{D^3}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial\eta}\right) \\
&\left.+\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\right)\left(\frac{Y_T}{Y_L}\frac{\partial\psi}{\partial\eta}-\frac{Y_L}{Y_T}\frac{\partial\psi}{\partial\eta}-\frac{2}{Y}\frac{\partial Y}{\partial\eta}\right)\right] \\
&+\frac{2}{Y_TY_LD^2}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial\psi}\right)\left[Y_TY_L\frac{\partial^2\psi}{\partial\eta\partial k_\zeta}\right. \\
&\left.+\left(-\left(Y_T^2-Y_L^2\right)\frac{\partial\psi}{\partial\eta}+\frac{2Y_TY_L}{Y}\frac{\partial Y}{\partial\eta}\right)\frac{\partial\psi}{\partial k_\zeta}\right].
\end{aligned}
\tag{76g}
$$

In order not to make (76g) longer than it is, we did not expand the $Y_TY_L\frac{\partial^2\psi}{\partial\eta\partial k_\zeta}$ that comes from (76f) within it, but, via the derivation in Appendix B, the term can be written as

$$
Y_TY_L\frac{\partial^2\psi}{\partial\eta\partial k_\zeta}\frac{Y_L^2}{V^2}\frac{V_\zeta}{Y}\frac{c}{\omega}\frac{\partial Y}{\partial\eta}-\frac{c}{\omega}\frac{\partial Y_\zeta}{\partial\eta}\frac{Y_L}{V}-\frac{Y_TY_L\frac{\partial\psi}{\partial\eta}Y_TY_L\frac{\partial\psi}{\partial k_\zeta}}{Y_T^2}-\frac{1}{Y}Y_TY_L\frac{\partial\psi}{\partial k_\zeta}\frac{\partial Y}{\partial\eta}-\frac{c}{\omega}\frac{V_\zeta}{V^2}Y_TY_L\frac{\partial\psi}{\partial\eta}. \tag{77}
$$

Additionally,

$$
Y_TY_L\frac{\partial\psi}{\partial k_\zeta}=\frac{Y_L^2}{V^2}\frac{c}{\omega}V_\zeta-\frac{c}{\omega}Y_\zeta\frac{Y_L}{V}. \tag{78}
$$

Finally, we get to the six remaining second derivatives of n^2 in their (39):

$$
\frac{\partial^2 n^2}{\partial k_\eta\partial k_\zeta}=\left(Y_TY_L\frac{\partial\psi}{\partial k_\zeta}\right)\frac{\partial}{\partial k_\eta}\left(\frac{1}{Y_TY_L}\frac{\partial n^2}{\partial\psi}\right)+\left(\frac{1}{Y_TY_L}\frac{\partial n^2}{\partial\psi}\right)\frac{\partial}{\partial k_\eta}\left(Y_TY_L\frac{\partial\psi}{\partial k_\zeta}\right).
$$

Expanding the second part of the first term, for $\eta=\{r,\theta,\phi\}$,

$$
\frac{\partial}{\partial k_\eta}\left(\frac{1}{Y_TY_L}\frac{\partial n^2}{\partial\psi}\right)=\frac{2}{Y_TY_L}\left(X\left(1-iZ-X\right)\left(\frac{1}{D^2}\left(\frac{\partial^2 D}{\partial k_\eta\partial\psi}+\frac{Y_T^2-Y_L^2}{Y_TY_L}\frac{\partial D}{\partial\psi}\frac{\partial\psi}{\partial k_\eta}\right)-\frac{2}{D^3}\frac{\partial D}{\partial\psi}\frac{\partial D}{\partial k_\eta}\right)\right), \tag{79a}
$$

where

$$
\frac{\partial D}{\partial k_\eta}=-2Y^2\sin\psi\cos\psi\frac{\partial\psi}{\partial k_\eta}+\frac{\partial R}{\partial k_\eta}=-2Y_TY_L\frac{\partial\psi}{\partial k_\eta}+\frac{\partial R}{\partial k_\eta}, \tag{79b}
$$

$$
\frac{\partial R}{\partial k_\eta}=\frac{2Y_T^3Y_L\frac{\partial\psi}{\partial k_\eta}-4Y_TY_L\frac{\partial\psi}{\partial k_\eta}\left(1-iZ-X\right)^2}{R}, \tag{79c}
$$

$$\frac{\partial^2 D}{\partial k_\eta \partial \psi} = -Y^2\left(-\sin^2\psi + \cos^2\psi\right)\frac{\partial \psi}{\partial k_\eta} + \frac{\partial^2 R}{\partial k_\eta \partial \psi} = \left(Y_T^2 - Y_L^2\right)\frac{\partial \psi}{\partial k_\eta} + \frac{\partial^2 R}{\partial k_\eta \partial \psi}, \tag{79d}$$

and

$$\begin{aligned}\frac{\partial^2 R}{\partial k_\eta \partial \psi} =& \frac{2}{R}\left(\left(-Y_T^4 + 3Y_T^2Y_L^2\right)\frac{\partial \psi}{\partial k_\eta} + 2\left(Y_T^2 - Y_L^2\right)\frac{\partial \psi}{\partial k_\eta}\left(1 - iZ - X\right)^2\right) \\ &- \frac{4}{R^3}\left(Y_T^3Y_L - 2Y_TY_L\left(1 - iZ - X\right)^2\right)\left(Y_T^3Y_L\frac{\partial \psi}{\partial k_\eta} - 2Y_TY_L\frac{\partial \psi}{\partial k_\eta}\left(1 - iZ - X\right)^2\right).\end{aligned} \tag{79e}$$

Also, for $\zeta = \{r, \theta, \phi\}$,

$$\frac{\partial}{\partial k_\eta}\left(Y_TY_L\frac{\partial \psi}{\partial k_\zeta}\right) = Y_TY_L\frac{\partial^2 \psi}{\partial k_\eta \partial k_\zeta} - \left(Y_T^2 - Y_L^2\right)\frac{\partial \psi}{\partial k_\eta}\frac{\partial \psi}{\partial k_\zeta}. \tag{79f}$$

Thus,

$$\begin{aligned}\frac{\partial^2 n^2}{\partial k_\eta \partial k_\zeta} =& 2\left(X\left(1 - iZ - X\right)\left(\frac{1}{D^2}\left(\frac{\partial^2 D}{\partial k_\eta \partial \psi} + \frac{Y_T^2 - Y_L^2}{Y_TY_L}\frac{\partial D}{\partial \psi}\frac{\partial \psi}{\partial k_\eta}\right) - \frac{2}{D^3}\frac{\partial D}{\partial \psi}\frac{\partial D}{\partial k_\eta}\right)\right)\frac{\partial \psi}{\partial k_\zeta} \\ &+ \frac{2}{Y_TY_LD^2}\left(X\left(1 - iZ - X\right)\frac{\partial D}{\partial \psi}\right)\left(Y_TY_L\frac{\partial^2 \psi}{\partial k_\eta \partial k_\zeta} - \left(Y_T^2 - Y_L^2\right)\frac{\partial \psi}{\partial k_\eta}\frac{\partial \psi}{\partial k_\zeta}\right).\end{aligned} \tag{79g}$$

Again, there is a second derivative of ψ that appears in (79f) and (79g). From Appendix B, we find that

$$Y_TY_L\frac{\partial^2 \psi}{\partial k_\eta \partial k_\zeta} = -\frac{c^2}{\omega^2}V_\zeta V_\eta\frac{Y_L^2}{V^4} - \frac{Y_TY_L\frac{\partial \psi}{\partial k_\eta}Y_TY_L\frac{\partial \psi}{\partial k_\zeta}}{Y_T^2} - \frac{c}{\omega}\frac{V_\eta}{V^2}Y_TY_L\frac{\partial \psi}{\partial k_\zeta} - \frac{c}{\omega}\frac{V_\zeta}{V^2}Y_TY_L\frac{\partial \psi}{\partial k_\eta}. \tag{80}$$

We have now written out all the necessary components of Västberg and Lundborg's (32)-(39) in order to compute the second order derivatives of n^2 found within our (52)-(57).

5. CONCLUSION

In this paper, we have built upon the work done by L.J. Nickisch in [1] by applying his methodology to an anisotropic medium with particle collisions. Accounting for more of the phenomena present in the ionosphere allows for more realistic predictions of how the strength of a propagated signal changes due to focusing effects than what was presented in the isotropic, collisionless ionosphere example given in the aforementioned paper. When applying this approach, it is important to be consistent. If a ray is traced through an anisotropic ionosphere with collisions, then the in-line focusing approach must be used with the equations given here and not those in [1].

We have also written out expressions for the derivatives within Section 2.2 of [4]. The Västberg and Lundborg paper is a follow-up on [1] much like the current paper. Although the authors of that paper do give expressions for $\delta\dot{r}$, $\delta\dot{\theta}$, $\delta\dot{\phi}$, $\delta\dot{k}_r$, $\delta\dot{k}_\theta$, and $\delta\dot{k}_\phi$, they do so in the most generalized terms. We have written those expressions out much as Nickisch did, albeit applied to rays in an ionosphere with fields and collisions. The many second derivatives needed to compute all of the expressions are given in [4], and we have written out the expressions for them, which that paper failed to do.

Taken together, the derivations performed for this paper fill a gap in the literature and present a fuller picture of the mathematics needed to include the necessary physical phenomena of the Earth's magnetic field and particle collisions within the framework of the algorithm devised by L.J. Nickisch for focusing in the stationary phase approximation. His method is computationally more efficient than integrating over a ray tube, as well as more accurate, as outlying rays on the "tube" may skew the computed attenuation. The equations presented here allow for further applications of the method.

There are a number of follow-up investigations possible for the equations and derivations presented in this report. Once implemented into a raytracing code such as MoJo, it will be possible to run test cases to see how much of a difference in signal intensity there is between using the method and only considering $\frac{1}{r^2}$ divergence (where r is the slant range). A comparison between this method and a traditional flux tube approach is also warranted, both in computational efficiency and accuracy. Including fields and collisions introduces many more derivatives that need to be computed than the simpler case that Nickisch investigated, and even if the method may be more accurate than a flux tube, it is unclear if there is still a noticeable improvement in computational efficiency, as well. A third comparison study can be done comparing the results of using the full physics like in this paper to using the simpler version of the equations found in [1]. Not accounting for fields and collisions in the focusing while accounting for them in the ray tracing will produce incorrect results, but it would be interesting to quantify the differences. Finally, it may be possible to validate the method presented here experimentally, and that is a current subject of research.

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Appendix A

SELECTED DERIVATIONS FROM SECTION 4

In this appendix, we expand derivations missing from the main text that we believe will save the reader time if attempting to reproduce the results.

Derivation of (60c):

$$\begin{aligned}\frac{\partial R}{\partial Y} &= \pm\frac{2Y^3\sin^4\psi + 4Y\cos^2\psi\,(1-iZ-X)^2}{\sqrt{Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2}}\\ &= \pm\frac{2Y_T^4 + 4Y_L^2\,(1-iZ-X)^2}{Y\sqrt{Y_T^4+4Y_L^2\,(1-iZ-X)^2}}\\ &= \frac{2Y_T^4 + 4Y_L^2\,(1-iZ-X)^2}{YR}\\ &= \frac{Y_T^4+R^2}{YR}.\end{aligned} \tag{A1}$$

Derivation of (62a):

$$\begin{aligned}\frac{\partial^2 n^2}{\partial X^2} =& -\frac{2}{D}(-2) + \frac{2}{D^2}(1-iZ-2X)\frac{\partial D}{\partial X} + \frac{2}{D^2}\left(X(1-iZ-X)\frac{\partial^2 D}{\partial X^2}\right.\\ &\left.+(-X+1-iZ-X)\frac{\partial D}{\partial X}\right) - \frac{4}{D^3}\left(X(1-iZ-X)\left(\frac{\partial D}{\partial X}\right)^2\right)\\ =&\frac{4}{D} + \frac{4}{D^2}(1-iZ-2X)\frac{\partial D}{\partial X} + \frac{2}{D^2}\left(X(1-iZ-X)\frac{\partial^2 D}{\partial X^2}\right)\\ &-\frac{4}{D^3}\left(X(1-iZ-X)\left(\frac{\partial D}{\partial X}\right)^2\right).\end{aligned} \tag{A2}$$

Derivation of (62c):

$$\begin{aligned}\frac{\partial^2 R}{\partial X^2} &= \pm\frac{4Y_L^2\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right) - 16Y_L^4(1-iZ-X)^2}{\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &= \pm\frac{4Y_T^4Y_L^2}{\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &=\frac{4Y_T^4Y_L^2}{R^3}.\end{aligned} \tag{A3}$$

Derivation of (63c):

$$\begin{aligned}\frac{\partial^2 R}{\partial X \partial Y} &= \pm \frac{1}{Y\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right)^{\frac{3}{2}}}\left(-8Y_L^2\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right)(1-iZ-X)\right.\\ &\quad \left.+4Y_L^2\left(2Y_T^4+4Y_L^2(1-iZ-X)^2\right)(1-iZ-X)\right)\\ &= \pm \frac{16Y_L^4(1-iZ-X)^3}{Y\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &= \frac{16Y_L^4(1-iZ-X)^3}{YR^3}.\end{aligned} \tag{A4}$$

Derivation of (64a):

$$\begin{aligned}\frac{\partial^2 n^2}{\partial X \partial Z} &= \frac{2}{D^2}\left(iD+iX\frac{\partial D}{\partial X}+X(1-iZ-X)\frac{\partial^2 D}{\partial X \partial Z}+(1-iZ-2X)\frac{\partial D}{\partial Z}\right)\\ &\quad -\frac{4}{D^3}\left(iDX+X(iZ-X)\frac{\partial D}{\partial Z}\right)\frac{\partial D}{\partial X}\\ &= \frac{2i}{D}+\frac{2}{D^2}\left(-iX\frac{\partial D}{\partial X}+X(1-iZ-X)\frac{\partial^2 D}{\partial X \partial Z}+(1-iZ-2X)\frac{\partial D}{\partial Z}\right)\\ &\quad -\frac{4}{D^3}\left(X(1-iZ-X)\frac{\partial D}{\partial Z}\frac{\partial D}{\partial X}\right),\end{aligned} \tag{A5}$$

Derivation of (64c):

$$\begin{aligned}\frac{\partial^2 R}{\partial X \partial Z} &= \pm \frac{4iY_L^2\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right)-16iY_L^4(1-iZ-X)^2}{\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &= \pm \frac{4iY_T^4Y_L^2}{\left(Y_T^4+4Y_L^2(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &= \frac{4iY_T^4Y_L^2}{R^3}.\end{aligned} \tag{A6}$$

Derivation of (65c):

$$\begin{aligned}\frac{\partial^2 R}{\partial Y^2} &= \pm\Big[\left(Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2\right)\left(6Y^2\sin^4\psi + 4\cos^2\psi\,(1-iZ-X)^2\right)\\ &\quad -\left(2Y^3\sin^4\psi + 4Y\cos^2\psi\,(1-iZ-X)^2\right)\left(2Y^3\sin^4\psi + 4Y\cos^2\psi\,(1-iZ-X)^2\right)\Big]\\ &\quad /\left(Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2\right)^{\frac{3}{2}}\\ &= \pm\,\frac{2Y^6\sin^8\psi + 12Y^4\sin^4\psi\cos^2\psi\,(1-iZ-X)^2}{\left(Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &= \pm\,\frac{2Y_T^8 + 12Y_T^4Y_L^2\,(1-iZ-X)^2}{Y^2\left(Y_T^4 + 4Y_L^2\,(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &=\frac{2Y_T^8 + 12Y_T^4Y_L^2\,(1-iZ-X)^2}{Y^2R^3}.\end{aligned} \tag{A7}$$

Derivation of (66a):

$$\begin{aligned}\frac{\partial^2 n^2}{\partial Y\partial Z} =&\frac{2}{D^2}\left(iX\frac{\partial D}{\partial Y} + X\,(1-iZ-X)\,\frac{\partial^2 D}{\partial Y\partial Z}\right) - \frac{4}{D^2}iX\frac{\partial D}{\partial Y}\\ &- \frac{4}{D^3}\left(X\,(1-iZ-X)\,\frac{\partial D}{\partial Z}\frac{\partial D}{\partial Y}\right)\\ =&\frac{2}{D^2}\left(-iX\frac{\partial D}{\partial Y} + X\,(1-iZ-X)\,\frac{\partial^2 D}{\partial Y\partial Z}\right) - \frac{4}{D^3}\left(X\,(1-iZ-X)\,\frac{\partial D}{\partial Z}\frac{\partial D}{\partial Y}\right),\end{aligned} \tag{A8}$$

Derivation of (66c):

$$\begin{aligned}\frac{\partial^2 R}{\partial Y\partial Z} &= \pm\Big[\left(Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2\right)\left(-8iY\cos^2\psi\,(1-iZ-X)\right)\\ &\quad +4iY^2\cos^2\psi\,(1-iZ-X)\left(2Y^3\sin^4\psi + 4Y\cos^2\psi\,(1-iZ-X)^2\right)\Big]\\ &\quad /\left(Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2\right)^{\frac{3}{2}}\\ &= \pm\,\frac{-16iY^3\cos^4\psi\,(1-iZ-X)^3}{\left(Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &= \pm\,\frac{-16iY_L^4\,(1-iZ-X)^3}{Y\left(Y_T^4 + 4Y_L^2\,(1-iZ-X)^2\right)^{\frac{3}{2}}}\\ &=\frac{-16iY_L^4\,(1-iZ-X)^3}{YR^3}.\end{aligned} \tag{A9}$$

Derivation of (67a):

$$\begin{aligned}\frac{\partial^2 n^2}{\partial Z^2} =&\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial Z^2}\right)-\frac{4}{D^2}iX\frac{\partial D}{\partial Z}-\frac{4}{D^3}\left(X\left(1-iZ-X\right)\left(\frac{\partial D}{\partial Z}\right)^2\right)\\ =&\frac{2}{D^2}\left(-2iX\frac{\partial D}{\partial Z}+X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial Z^2}\right)-\frac{4}{D^3}\left(X\left(1-iZ-X\right)\left(\frac{\partial D}{\partial Z}\right)^2\right).\end{aligned} \tag{A10}$$

Derivation of (67c):

$$\begin{aligned}\frac{\partial^2 R}{\partial Z^2} &= \pm\,\frac{4Y_L^2\left(Y_T^4+4Y_L^2\left(1-iZ-X\right)^2\right)-16Y_L^4\left(1-iZ-X\right)^2}{\left(Y_T^4+4Y_L^2\left(1-iZ-X\right)^2\right)^{\frac{3}{2}}}\\ &= \pm\,\frac{4Y_T^4Y_L^2}{\left(Y_T^4+4Y_L^2\left(1-iZ-X\right)^2\right)^{\frac{3}{2}}}\\ &=\frac{4Y_T^4Y_L^2}{R^3}.\end{aligned} \tag{A11}$$

Derivation of (68c):

$$\begin{aligned}\frac{\partial R}{\partial \psi} &= \pm\,\frac{4Y^4\sin^3\psi\cos\psi-8Y^2\sin\psi\cos\psi\left(1-iZ-X\right)^2}{2\sqrt{Y^4\sin^4\psi+4Y^2\cos^2\psi\left(1-iZ-X\right)^2}}\\ &= \pm\,\frac{2Y_T^3Y_L-4Y_TY_L\left(1-iZ-X\right)^2}{\sqrt{Y_T^4+4Y_L^2\left(1-iZ-X\right)^2}}\\ &=\frac{2Y_T^3Y_L-4Y_TY_L\left(1-iZ-X\right)^2}{R}.\end{aligned} \tag{A12}$$

Derivation of (69c):

$$\begin{aligned}\frac{\partial^2 R}{\partial X\partial \psi} =& \pm\,\frac{1}{\left(Y_T^4+4Y_L^2\left(1-iZ-X\right)^2\right)^{\frac{3}{2}}}\left[8Y_TY_L\left(1-iZ-X\right)\left(Y_T^4+4Y_L^2\left(1-iZ-X\right)^2\right)\right.\\ &\left.+4Y_L^2\left(1-iZ-X\right)\left(2Y_T^3Y_L-4Y_TY_L\left(1-iZ-X\right)^2\right)\right]\\ =& \pm\,\frac{8Y_TY_L\left(1-iZ-X\right)\left(Y_T^4+Y_T^2Y_L^2+2Y_L^2\left(1-iZ-X\right)^2\right)}{\left(Y_T^4+4Y_L^2\left(1-iZ-X\right)^2\right)^{\frac{3}{2}}}\\ =&\frac{8Y_TY_L\left(1-iZ-X\right)\left(R^2+Y_L^2\left(Y_T^2-2\left(1-iZ-X\right)^2\right)\right)}{R^3}\\ =&8Y_TY_L\left(1-iZ-X\right)\left(\frac{1}{R}+\frac{1}{R^3}\left(Y_L^2\left(Y_T^2-2\left(1-iZ-X\right)^2\right)\right)\right).\end{aligned} \tag{A13}$$

Derivation of (70c):

$$
\begin{aligned}
\frac{\partial^2 R}{\partial Y \partial \psi} =& \pm \Big[\Big(8Y^3 \sin^3\psi \cos\psi - 8Y \sin\psi \cos\psi \,(1-iZ-X)^2\Big) \\
&\cdot \Big(Y^4 \sin^4\psi + 4Y^2 \cos^2\psi \,(1-iZ-X)^2\Big) \\
&- \Big(2Y^4 \sin^3\psi \cos\psi - 4Y^2 \sin\psi \cos\psi \,(1-iZ-X)^2\Big) \\
&\cdot \Big(2Y^3 \sin^4\psi + 4Y \cos^2\psi \,(1-iZ-X)^2\Big)\Big] \\
&/\Big(Y^4 \sin^4\psi + 4Y^2 \cos^2\psi \,(1-iZ-X)^2\Big)^{\frac{3}{2}} \\
=& \pm \Big[\Big(8Y_T^3 Y_L - 8Y_T Y_L \,(1-iZ-X)^2\Big)\Big(Y_T^4 + 4Y_L^2 \,(1-iZ-X)^2\Big) \\
&- \Big(2Y_T^3 Y_L - 4Y_T Y_L \,(1-iZ-X)^2\Big)\Big(2Y_T^4 + 4Y_L^2 \,(1-iZ-X)^2\Big)\Big] \\
&/\Big(Y\Big(Y_T^4 + Y_L^2 \,(1-iZ-X)^2\Big)^{\frac{3}{2}}\Big) \\
=& \pm \frac{4Y_T^7 Y_L + 24Y_T^3 Y_L^3 \,(1-iZ-X)^2 - 16Y_T Y_L^3 \,(1-iZ-X)^4}{Y\Big(Y_T^4 + Y_L^2 \,(1-iZ-X)^2\Big)^{\frac{3}{2}}} \\
=& \frac{4Y_T^7 Y_L + 24Y_T^3 Y_L^3 \,(1-iZ-X)^2 - 16Y_T Y_L^3 \,(1-iZ-X)^4}{YR^3} \\
=& \frac{4Y_T Y_L\Big(Y_T^6 + 8Y_T^2 Y_L^2 \,(1-iZ-X)^2 - 4Y_L^2 \,(1-iZ-X)^4\Big)}{YR^3}.
\end{aligned}
\tag{A14}
$$

Derivation of (71c):

$$
\begin{aligned}
\frac{\partial^2 R}{\partial Z \partial \psi} =& \pm \frac{1}{\Big(Y_T^4 + 4Y_L^2 \,(1-iZ-X)^2\Big)^{\frac{3}{2}}} \Big[8iY_T Y_L \,(1-iZ-X)\Big(Y_T^4 + 4Y_L^2 \,(1-iZ-X)^2\Big) \\
&+4iY_L^2 \,(1-iZ-X)\Big(2Y_T^3 Y_L - 4Y_T Y_L \,(1-iZ-X)^2\Big)\Big] \\
=& \pm \frac{8iY_T Y_L \,(1-iZ-X)\Big(Y_T^4 + Y_T^2 Y_L^2 + 2Y_L^2 \,(1-iZ-X)^2\Big)}{\Big(Y_T^4 + 4Y_L^2 \,(1-iZ-X)^2\Big)^{\frac{3}{2}}} \\
=& \frac{8iY_T Y_L \,(1-iZ-X)\Big(R^2 + Y_L^2\Big(Y_T^2 - 2\,(1-iZ-X)^2\Big)\Big)}{R^3} \\
=& 8iY_T Y_L \,(1-iZ-X)\left(\frac{1}{R} + \frac{1}{R^3}\Big(Y_L^2\Big(Y_T^2 - 2\,(1-iZ-X)^2\Big)\Big)\right).
\end{aligned}
\tag{A15}
$$

Derivation of (72c):

$$\begin{aligned}
\frac{\partial^2 R}{\partial\psi^2} &= \Big[\Big(2Y^4\sin^2\psi\left(-\sin^2\psi+3\cos^2\psi\right)-4Y^2\left(-\sin^2\psi+\cos^2\psi\right)(1-iZ-X)^2\Big)\\
&\quad\cdot\left(Y^4\sin^4\psi+4Y^2\cos^2\psi\,(1-iZ-X)^2\right)\\
&\quad-\left(2Y^4\sin^3\psi\cos\psi-4Y^2\sin\psi\cos\psi\,(1-iZ-X)^2\right)^2\Big]\\
&\quad/\left(Y^4\sin^4\psi+4Y^2\cos^2\psi\,(1-iZ-X)^2\right)^{\frac{3}{2}}\\
&=\pm\Big[2Y^8\sin^6\psi\left(-\sin^2\psi+\cos^2\psi\right)\\
&\quad+4Y^6\sin^2\psi\left(\sin^2\psi\cos^2\psi+6\cos^4\psi+\sin^4\psi\right)(1-iZ-X)^2\\
&\quad-16Y^4\cos^4\psi\,(1-iZ-X)^4\Big]/\left(Y^4\sin^4\psi+4Y^2\cos^2\psi\,(1-iZ-X)^2\right)^{\frac{3}{2}}\\
&=\pm\frac{-2Y_T^8+2Y_T^6Y_L^2+4\left(Y_T^4Y_L^2+6Y_T^2Y_L^4+Y_T^6\right)(1-iZ-X)^2-16Y_L^4\,(1-iZ-X)^4}{\left(Y_T^4+4Y_L^2\,(1-iZ-X)^2\right)^{\frac{3}{2}}}\\
&=\frac{-2Y_T^8+2Y_T^6Y_L^2+4\left(Y_T^4Y_L^2+6Y_T^2Y_L^4+Y_T^6\right)(1-iZ-X)^2-16Y_L^4\,(1-iZ-X)^4}{R^3}.
\end{aligned} \tag{A16}$$

Derivation of (76a):

$$\begin{aligned}
\frac{\partial}{\partial\eta}\left(\frac{1}{Y_TY_L}\frac{\partial n^2}{\partial\psi}\right) &= \frac{1}{Y_TY_L}\frac{\partial^2 n^2}{\partial\eta\partial\psi}+\frac{\partial}{\partial\eta}\left(\frac{1}{Y_TY_L}\right)\frac{\partial n^2}{\partial\psi}\\
&=\frac{1}{Y_TY_L}\frac{\partial^2 n^2}{\partial\eta\partial\psi}\\
&\quad+\left(\frac{-\left(-Y^2\sin^2\psi\frac{\partial\psi}{\partial\eta}+\left(Y^2\cos^2\psi\frac{\partial\psi}{\partial\eta}+2Y\sin\psi\frac{\partial Y}{\partial\eta}\right)\cos\psi\right)}{Y^4\sin^2\psi\cos^2\psi}\right)\frac{\partial n^2}{\partial\psi}\\
&=\frac{1}{Y_TY_L}\frac{\partial^2 n^2}{\partial\eta\partial\psi}+\left(\frac{1}{Y_L^2}\frac{\partial\psi}{\partial\eta}-\frac{1}{Y_T^2}\frac{\partial\psi}{\partial\eta}-\frac{2}{YY_TY_L}\frac{\partial Y}{\partial\eta}\right)\frac{\partial n^2}{\partial\psi}\\
&=\frac{1}{Y_TY_L}\Bigg[\frac{2}{D^2}\Bigg(X(1-iZ-X)\frac{\partial^2 D}{\partial\eta\partial\psi}+\Bigg(X\left(-i\frac{\partial Z}{\partial\eta}-\frac{\partial X}{\partial\eta}\right)\\
&\quad+\frac{\partial X}{\partial\eta}(1-iZ-X)\Bigg)\frac{\partial D}{\partial\psi}\Bigg)-\frac{4}{D^3}\left(X(1-iZ-X)\left(\frac{\partial D}{\partial\psi}\right)^2\right)\Bigg]\\
&\quad+\frac{2}{Y_L^2D^2}\left(X(1-iZ-X)\frac{\partial D}{\partial\psi}\frac{\partial\psi}{\partial\eta}\right)\\
&\quad-\frac{2}{Y_T^2D^2}\left(X(1-iZ-X)\frac{\partial D}{\partial\psi}\frac{\partial\psi}{\partial\eta}\right)\\
&\quad-\frac{4}{YY_TY_LD^2}\left(X(1-iZ-X)\frac{\partial D}{\partial\psi}\frac{\partial Y}{\partial\eta}\right),
\end{aligned} \tag{A17}$$

Derivation of (76b):

$$\begin{aligned}\frac{\partial D}{\partial \eta} =&2\left(1-iZ\right)\left(-i\frac{\partial Z}{\partial \eta}-\frac{\partial X}{\partial \eta}\right)-2iZ\left(1-iZ-X\right)-2Y^2\sin\psi\cos\psi\frac{\partial \psi}{\partial \eta}-2Y\frac{\partial Y}{\partial \eta}\sin^2\psi+\frac{\partial R}{\partial \eta}\\ =&2\left(1-iZ\right)\left(-i\frac{\partial Z}{\partial \eta}-\frac{\partial X}{\partial \eta}\right)-2iZ\left(1-iZ-X\right)-2Y_TY_L\frac{\partial \psi}{\partial \eta}-\frac{2Y_T^2}{Y}\frac{\partial Y}{\partial \eta}+\frac{\partial R}{\partial \eta},\end{aligned} \tag{A18}$$

Derivation of (76c):

$$\begin{aligned}\frac{\partial R}{\partial \eta} =& \pm\frac{1}{\sqrt{Y^4\sin^4\psi+4Y^2\cos^2\psi\left(1-iZ-X\right)^2}}\left[2Y^4\sin^3\psi\cos\psi\frac{\partial \psi}{\partial \eta}+2Y^3\frac{\partial Y}{\partial \eta}\sin^4\psi\right.\\ &+4Y^2\cos^2\psi\left(1-iZ-X\right)\left(-i\frac{\partial Z}{\partial \eta}-\frac{\partial X}{\partial \eta}\right)\\ &\left.+\left(-4Y^2\sin\psi\cos\psi\frac{\partial \psi}{\partial \eta}+4Y\frac{\partial Y}{\partial \eta}\cos^2\psi\right)\left(1-iZ-X\right)^2\right]\\ =& \pm\left[2YY_T^3Y_L\frac{\partial \psi}{\partial \eta}+2Y_T^4\frac{\partial Y}{\partial \eta}+4YY_L^2\left(1-iZ-X\right)\left(-i\frac{\partial Z}{\partial \eta}-\frac{\partial X}{\partial \eta}\right)\right.\\ &\left.+\left(-4YY_TY_L\frac{\partial \psi}{\partial \eta}+4Y_L^2\frac{\partial Y}{\partial \eta}\right)\left(1-iZ-X\right)^2\right]\Big/\left(Y\sqrt{Y_T^4+4Y_L^2\left(1-iZ-X\right)^2}\right)\\ =&\left[2YY_T^3Y_L\frac{\partial \psi}{\partial \eta}+2Y_T^4\frac{\partial Y}{\partial \eta}+4YY_L^2\left(1-iZ-X\right)\left(-i\frac{\partial Z}{\partial \eta}-\frac{\partial X}{\partial \eta}\right)\right.\\ &\left.+\left(-4YY_TY_L\frac{\partial \psi}{\partial \eta}+4Y_L^2\frac{\partial Y}{\partial \eta}\right)\left(1-iZ-X\right)^2\right]\Big/\left(YR\right),\end{aligned} \tag{A19}$$

Derivation of (76d):

$$\begin{aligned}\frac{\partial^2 D}{\partial \eta\partial \psi} =& -Y^2\left(-\sin^2\psi+\cos^2\psi\right)\frac{\partial \psi}{\partial \eta}-2Y\frac{\partial Y}{\partial \eta}\sin\psi\cos\psi+\frac{\partial^2 R}{\partial \eta\partial \psi}\\ =&\left(Y_T^2-Y_L^2\right)\frac{\partial \psi}{\partial \eta}-\frac{2Y_TY_L}{Y}\frac{\partial Y}{\partial \eta}+\frac{\partial^2 R}{\partial \eta\partial \psi},\end{aligned} \tag{A20}$$

Derivation of (76e):

$$
\begin{aligned}
\frac{\partial^2 R}{\partial\eta\partial\psi} =& \frac{\partial}{\partial\eta}\left(\frac{2Y^4\sin^3\psi\cos\psi - 4Y^2\sin\psi\cos\psi\,(1-iZ-X)^2}{R}\right)\\
=&\frac{1}{R}\left[2Y^4\left(-\sin^4\psi\frac{\partial\psi}{\partial\eta} + 3\sin^2\psi\cos^2\psi\frac{\partial\psi}{\partial\eta}\right) + 8Y^3\frac{\partial Y}{\partial\eta}\sin^3\psi\cos\psi\right.\\
&-\left(4Y^2\sin\psi\left(2\cos\psi\,(1-iZ-X)\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right) - \sin\psi\frac{\partial\psi}{\partial\eta}\,(1-iZ-X)^2\right)\right.\\
&\left.\left.+\left(4Y^2\cos\psi\frac{\partial\psi}{\partial\eta} + 8Y\frac{\partial Y}{\partial\eta}\sin\psi\right)\cos\psi\,(1-iZ-X)^2\right)\right]\\
&-\frac{1}{R^2}\left(2Y^4\sin^3\psi\cos\psi - 4Y^2\sin\psi\cos\psi\,(1-iZ-X)^2\right)\frac{\partial R}{\partial\eta}\\
=&\frac{1}{R}\left(-2Y_T^4\frac{\partial\psi}{\partial\eta} + 6Y_T^2Y_L^2\frac{\partial\psi}{\partial\eta} + 8Y_T^3\frac{\partial Y}{\partial\eta}\cos\psi - 8Y_TY_L\,(1-iZ-X)\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right)\right.\\
&\left.+4Y_T^2\,(1-iZ-X)^2\frac{\partial\psi}{\partial\eta} - 4Y_L^2\,(1-iZ-X)^2\frac{\partial\psi}{\partial\eta} - 8Y_T\frac{\partial Y}{\partial\eta}\,(1-iZ-X)^2\cos\psi\right)\\
&-\frac{1}{R^2}\left(2Y_T^3Y_L - 4Y_TY_L\,(1-iZ-X)^2\right)\frac{\partial R}{\partial\eta}\\
=&\frac{2}{YR}\left(\left(-YY_T^4 + 3YY_T^2Y_L^2 + 2YY_T^2\,(1-iZ-X)^2 - 2YY_L^2\,(1-iZ-X)^2\right)\frac{\partial\psi}{\partial\eta}\right.\\
&\left.+\left(4Y_T^3Y_L - 4Y_TY_L\,(1-iZ-X)^2\right)\frac{\partial Y}{\partial\eta} - 4YY_TY_L\,(1-iZ-X)\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right)\right)\\
&-\frac{4}{YR^3}Y_TY_L\left(Y_T^2 - 2\,(1-iZ-X)^2\right)\left(\left(YY_T^3Y_L - 2Y_TY_L\right)\frac{\partial\psi}{\partial\eta}\right.\\
&\left.+\left(Y_T^4 + 2Y_L^2\,(1-iZ-X)^2\right)\frac{\partial Y}{\partial\eta} + 2YY_L^2\,(1-iZ-X)\left(-i\frac{\partial Z}{\partial\eta} - \frac{\partial X}{\partial\eta}\right)\right).
\end{aligned}
\tag{A21}
$$

Derivation of (79a):

$$
\begin{aligned}
\frac{\partial}{\partial k_\eta}\left(\frac{1}{Y_T Y_L}\frac{\partial n^2}{\partial \psi}\right) =& \frac{1}{Y_T Y_L}\frac{\partial^2 n^2}{\partial k_\eta \partial \psi} + \frac{\partial}{\partial k_\eta}\left(\frac{1}{Y_T Y_L}\right)\frac{\partial n^2}{\partial \psi} \\
=& \frac{1}{Y_T Y_L}\left(\frac{\partial}{\partial k_\eta}\left(\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial \psi}\right)\right)\right) \\
&+ \frac{\partial}{\partial k_\eta}\left(\frac{1}{Y_T Y_L}\right)\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial \psi}\right) \\
=& \frac{1}{Y_T Y_L}\left(\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial^2 D}{\partial k_\eta \partial \psi}\right)\right. \\
&\left. - \frac{4}{D^3}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial \psi}\frac{\partial D}{\partial k_\eta}\right)\right) \\
&+ \frac{Y_T^2 - Y_L^2}{Y_T^2 Y_L^2}\frac{\partial \psi}{\partial k_\eta}\frac{2}{D^2}\left(X\left(1-iZ-X\right)\frac{\partial D}{\partial \psi}\right) \\
=& \frac{2}{Y_T Y_L}\left(X\left(1-iZ-X\right)\left(\frac{1}{D^2}\left(\frac{\partial^2 D}{\partial k_\eta \partial \psi} + \frac{Y_T^2 - Y_L^2}{Y_T Y_L}\frac{\partial D}{\partial \psi}\frac{\partial \psi}{\partial k_\eta}\right)\right.\right. \\
&\left.\left. - \frac{2}{D^3}\frac{\partial D}{\partial \psi}\frac{\partial D}{\partial k_\eta}\right)\right),
\end{aligned}
\tag{A22}
$$

Derivation of (79c):

$$
\begin{aligned}
\frac{\partial R}{\partial k_\eta} &= \pm\frac{2Y^4 \sin^3 \cos\psi \frac{\partial \psi}{\partial k_\eta} - 4Y^2 \sin\psi \cos\psi \frac{\partial \psi}{\partial k_\eta}\left(1-iZ-X\right)^2}{\sqrt{Y^4 \sin^4\psi + 4Y^2 \cos^2\psi \left(1-iZ-X\right)^2}} \\
&= \pm\frac{2Y_T^3 Y_L \frac{\partial \psi}{\partial k_\eta} - 4Y_T Y_L \frac{\partial \psi}{\partial k_\eta}\left(1-iZ-X\right)^2}{\sqrt{Y_T^4 + Y_L^2\left(1-iZ-X\right)^2}} \\
&= \frac{2Y_T^3 Y_L \frac{\partial \psi}{\partial k_\eta} - 4Y_T Y_L \frac{\partial \psi}{\partial k_\eta}\left(1-iZ-X\right)^2}{R},
\end{aligned}
\tag{A23}
$$

Derivation of (79e):

$$
\begin{aligned}
\frac{\partial^2 R}{\partial k_\eta \partial\psi} = \pm & \Bigg[\Bigg(\left(-2Y^4\sin^4\psi + 6Y^4\sin^2\psi\cos^2\psi\right)\frac{\partial\psi}{\partial k_\eta} \\
& -4Y^2\left(-\sin^2\psi+\cos^2\psi\right)\frac{\partial\psi}{\partial k_\eta}(1-iZ-X)^2\Bigg) \\
& \cdot\left(Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2\right) \\
& -\left(2Y^4\sin^3\psi\cos\psi - 4Y^2\sin\psi\cos\psi\,(1-iZ-X)^2\right)\Bigg(2Y^4\sin^3\psi\cos\psi\frac{\partial\psi}{\partial k_\eta} \\
& -4Y^2\sin\psi\cos\psi\frac{\partial\psi}{\partial k_\eta}(1-iZ-X)^2\Bigg)\Bigg] \\
& /\left(Y^4\sin^4\psi + 4Y^2\cos^2\psi\,(1-iZ-X)^2\right)^{\frac{3}{2}} \\
= \pm & \Bigg[\Bigg(\left(-2Y_T^4 + 6Y_T^2Y_L^2\right)\frac{\partial\psi}{\partial k_\eta} \\
& +4\left(Y_T^2 - Y_L^2\right)\frac{\partial\psi}{\partial k_\eta}(1-iZ-X)^2\Bigg)\left(Y_T^4 + 4Y_L^2(1-iZ-X)^2\right) \\
& -\left(2Y_T^3Y_L - 4Y_TY_L(1-iZ-X)^2\right)\left(2Y_T^3Y_L\frac{\partial\psi}{\partial k_\eta} - 4Y_TY_L\frac{\partial\psi}{\partial k_\eta}(1-iZ-X)^2\right)\Bigg] \\
& /\left(Y_T^4 + 4Y_L^2(1-iZ-X)^2\right)^{\frac{3}{2}} \\
= & \Bigg[\Bigg(\left(-2Y_T^4 + 6Y_T^2Y_L^2\right)\frac{\partial\psi}{\partial k_\eta} + 4\left(Y_T^2 - Y_L^2\right)\frac{\partial\psi}{\partial k_\eta}(1-iZ-X)^2\Bigg)R^2 \\
& -\left(2Y_T^3Y_L - 4Y_TY_L(1-iZ-X)^2\right) \\
& \cdot\left(2Y_T^3Y_L\frac{\partial\psi}{\partial k_\eta} - 4Y_TY_L\frac{\partial\psi}{\partial k_\eta}(1-iZ-X)^2\right)\Bigg]/R^3 \\
= & \frac{2}{R}\left(\left(-Y_T^4 + 3Y_T^2Y_L^2\right)\frac{\partial\psi}{\partial k_\eta} + 2\left(Y_T^2 - Y_L^2\right)\frac{\partial\psi}{\partial k_\eta}(1-iZ-X)^2\right) \\
& -\frac{4}{R^3}\left(Y_T^3Y_L - 2Y_TY_L(1-iZ-X)^2\right)\left(Y_T^3Y_L\frac{\partial\psi}{\partial k_\eta} - 2Y_TY_L\frac{\partial\psi}{\partial k_\eta}(1-iZ-X)^2\right).
\end{aligned}
\tag{A24}
$$

Appendix B

HOW TO DEAL WITH DERIVATIVES OF ψ

B.1 Västberg and Lundborg's (37)

In Västberg and Lundborg's (33)-(39), they frequently use the terms $\left(Y_T Y_L \frac{\partial\psi}{\partial\eta}\right)$ and $\left(Y_T Y_L \frac{\partial\psi}{\partial\zeta}\right)$ where $\eta = \{r, \theta, \phi\}$ and $\zeta = \{r, \theta, \phi\}$. This is because these can be determined from known quantities thanks to (51)-(53) in the Jones-Stephenson paper. For instance,

$$Y_T Y_L \frac{\partial\psi}{\partial\eta} = \frac{Y_L^2}{Y}\frac{\partial Y}{\partial\eta} - \left(\boldsymbol{V}\cdot\frac{\partial\boldsymbol{Y}}{\partial\eta}\right)\left(\frac{Y_L}{V}\right), \tag{B1}$$

where $\boldsymbol{V}$ is a vector in the wave normal direction with magnitude $\Re\left(n^2\right)$, and

$$\boldsymbol{V}\frac{\partial\boldsymbol{Y}}{\partial\eta} = V_r\frac{\partial Y_r}{\partial\eta} + V_\theta\frac{\partial Y_\theta}{\partial\eta} + V_\phi\frac{\partial Y_\phi}{\partial\eta}. \tag{B2}$$

Note that $V_\eta = \frac{c}{\omega}k_\eta$ for all η. However, there is a second order derivative term in their (37) that must be addressed: $\left(Y_T Y_L \frac{\partial^2\psi}{\partial\eta\partial\zeta}\right)$. It can also be written with known quantities in a similar way to the first derivatives albeit with some added terms. The key to its derivation is how (B1) is derived, which we reveal here and then adapt the method to the second derivatives. Because ψ is the angle between the vectors $\boldsymbol{V}$ and $\boldsymbol{Y}$, by definition of the dot product,

$$\boldsymbol{V}\cdot\boldsymbol{Y} = VY\cos\psi. \tag{B3}$$

Taking the derivative of this with respect to η,

$$\frac{\partial}{\partial\eta}\left(\boldsymbol{V}\cdot\boldsymbol{Y}\right) = \frac{\partial}{\partial\eta}\left(VY\cos\psi\right) = -VY\sin\psi\frac{\partial\psi}{\partial\eta} + \left(V\frac{\partial Y}{\partial\eta} + \frac{\partial V}{\partial\eta}Y\right)\cos\psi. \tag{B4}$$

Because the components of $\boldsymbol{V}$ are just a multiple of the components of $\boldsymbol{k}$, they are constant with respect to position, and thus the magnitude of $\boldsymbol{V}$ is also constant with respect to position, meaning $\frac{\partial V}{\partial\eta} = 0$. So,

$$\frac{\partial}{\partial\eta}\left(\boldsymbol{V}\cdot\boldsymbol{Y}\right) = -VY\sin\psi\frac{\partial\psi}{\partial\eta} + V\frac{\partial Y}{\partial\eta}\cos\psi. \tag{B5}$$

But, by the Product Rule,

$$\frac{\partial}{\partial\eta}\left(\boldsymbol{V}\cdot\boldsymbol{Y}\right) = \boldsymbol{V}\cdot\frac{\partial\boldsymbol{Y}}{\partial\eta} + \frac{\partial\boldsymbol{V}}{\partial\eta}\cdot\boldsymbol{Y}. \tag{B6}$$

$\frac{\partial \boldsymbol{V}}{\partial \eta} = 0$, and combining (B5) and (B6), we get

$$-VY \sin\psi \frac{\partial \psi}{\partial \eta} + V \frac{\partial Y}{\partial \eta} \cos\psi = \boldsymbol{V} \cdot \frac{\partial \boldsymbol{Y}}{\partial \eta}, \tag{B7a}$$

and thus

$$VY \sin\psi \frac{\partial \psi}{\partial \eta} = V \frac{\partial Y}{\partial \eta} \cos\psi - \left(\boldsymbol{V} \cdot \frac{\partial \boldsymbol{Y}}{\partial \eta} \right). \tag{B7b}$$

Multiplying (B7b) by $\frac{Y_L}{V}$ gives us

$$Y_T Y_L \frac{\partial \psi}{\partial \eta} = Y_L \frac{\partial Y}{\partial \eta} \cos\psi - \left(\boldsymbol{V} \cdot \frac{\partial \boldsymbol{Y}}{\partial \eta} \right) \left(\frac{Y_L}{V} \right). \tag{B8}$$

To arrive at (B1), multiply the first term on the right hand side of (B8) by $\frac{Y}{Y}$. The $\frac{Y_L}{V}$ factor is a known quantity from (42) in [2]:

$$\frac{Y_L}{V} = \frac{\boldsymbol{V} \cdot \boldsymbol{Y}}{V^2}. \tag{B9}$$

The procedure to determine $\left(Y_T Y_L \frac{\partial^2 \psi}{\partial \eta \partial \zeta} \right)$ is identical except this time we take the second derivative of $\boldsymbol{V} \cdot \boldsymbol{Y}$ two different ways. Using the cosine and eliminating terms known to be zero,

$$\frac{\partial^2}{\partial \eta \partial \zeta} (\boldsymbol{V} \cdot \boldsymbol{Y}) = -VY \left(\sin\psi \frac{\partial^2 \psi}{\partial \eta \partial \zeta} + \cos\psi \frac{\partial \psi}{\partial \eta} \frac{\partial \psi}{\partial \zeta} \right) - V \sin\psi \frac{\partial Y}{\partial \eta} \frac{\partial \psi}{\partial \zeta} - V \sin\psi \frac{\partial Y}{\partial \zeta} \frac{\partial \psi}{\partial \eta} + V \cos\psi \frac{\partial^2 Y}{\partial \eta \partial \zeta}, \tag{B10}$$

and using the Product Rule and eliminating terms known to be zero,

$$\frac{\partial^2}{\partial \eta \partial \zeta} (\boldsymbol{V} \cdot \boldsymbol{Y}) = \boldsymbol{V} \cdot \frac{\partial^2 \boldsymbol{Y}}{\partial \eta \partial \zeta}. \tag{B11}$$

Combining (B10) and (B11),

$$VY \sin\psi \frac{\partial^2 \psi}{\partial \eta \partial \zeta} = V \cos\psi \frac{\partial^2 Y}{\partial \eta \partial \zeta} - \left(\boldsymbol{V} \cdot \frac{\partial^2 \boldsymbol{Y}}{\partial \eta \partial \zeta} \right) - VY \cos\psi \frac{\partial \psi}{\partial \eta} \frac{\partial \psi}{\partial \zeta} - V \sin\psi \frac{\partial \psi}{\partial \zeta} \frac{\partial Y}{\partial \eta} - V \sin\psi \frac{\partial \psi}{\partial \eta} \frac{\partial Y}{\partial \zeta}. \tag{B12}$$

Multiplying both sides by $\frac{Y_L}{V}$ and multiplying terms by $\frac{Y}{Y}$ when appropriate give

$$Y_T Y_L \frac{\partial^2 \psi}{\partial \eta \partial \zeta} = \frac{Y_L^2}{Y} \frac{\partial^2 Y}{\partial \eta \partial \zeta} - \left(\boldsymbol{V} \cdot \frac{\partial^2 \boldsymbol{Y}}{\partial \eta \partial \zeta} \right) - Y_L^2 \frac{\partial \psi}{\partial \eta} \frac{\partial \psi}{\partial \zeta} - \frac{Y_T Y_L}{Y} \frac{\partial \psi}{\partial \zeta} \frac{\partial Y}{\partial \eta} - \frac{Y_T Y_L}{Y} \frac{\partial \psi}{\partial \eta} \frac{\partial Y}{\partial \zeta}. \tag{B13}$$

Finally, in order to get it into a form that can be used, the third term on the right hand side must be multiplied by $\frac{Y_T^2}{Y_T^2}$, resulting in

$$Y_T Y_L \frac{\partial^2 \psi}{\partial \eta \partial \zeta} = \frac{Y_L^2}{Y} \frac{\partial^2 Y}{\partial \eta \partial \zeta} - \left(\boldsymbol{V} \cdot \frac{\partial^2 \boldsymbol{Y}}{\partial \eta \partial \zeta} \right) \left(\frac{Y_L}{V} \right) - \frac{Y_T Y_L \frac{\partial \psi}{\partial \eta} Y_T Y_L \frac{\partial \psi}{\partial \zeta}}{Y_T^2} - \frac{Y_T Y_L}{Y} \frac{\partial \psi}{\partial \zeta} \frac{\partial Y}{\partial \eta} - \frac{Y_T Y_L}{Y} \frac{\partial \psi}{\partial \eta} \frac{\partial Y}{\partial \zeta}. \tag{B14}$$

B.2 Derivatives of ψ with respect to k_ζ

While the Jones-Stephenson paper does not write out expansions for $\left(Y_T Y_L \frac{\partial \psi}{\partial k_\zeta}\right)$, you will find this term in Västberg and Lundborg's (38) and (39). Similarly to the way the above equations were derived, this term can be determined by writing out $\frac{\partial}{\partial k_\zeta}(\boldsymbol{V}\cdot\boldsymbol{Y})$ two different ways. First,

$$\frac{\partial}{\partial k_\zeta}(\boldsymbol{V}\cdot\boldsymbol{Y}) = \frac{\partial}{\partial k_\zeta}(VY\cos\psi) = -VY\sin\psi\frac{\partial\psi}{\partial k_\zeta} + \left(V\frac{\partial Y}{\partial k_\zeta} + \frac{\partial V}{\partial k_\zeta}Y\right)\cos\psi. \tag{B15}$$

The earth's magnetic field is only dependent on position and not the movement of any ray, so $\frac{\partial Y}{\partial k_\zeta} = 0$ for all ζ. Thus,

$$\frac{\partial}{\partial k_\zeta}(\boldsymbol{V}\cdot\boldsymbol{Y}) = -VY\sin\psi\frac{\partial\psi}{\partial k_\zeta} + \frac{\partial V}{\partial k_\zeta}Y\cos\psi. \tag{B16}$$

From the Product Rule, we find

$$\begin{aligned}\frac{\partial}{\partial k_\zeta}(\boldsymbol{V}\cdot\boldsymbol{Y}) =& \boldsymbol{V}\cdot\frac{\partial \boldsymbol{Y}}{\partial k_\zeta} + \frac{\partial \boldsymbol{V}}{\partial k_\zeta}\cdot\boldsymbol{Y} \\ =& \frac{\partial \boldsymbol{V}}{\partial k_\zeta}\cdot\boldsymbol{Y}.\end{aligned} \tag{B17}$$

Equating (B16) with (B17) and multiplying by $\frac{Y_L}{V}$, one gets

$$Y_T Y_L\frac{\partial\psi}{\partial k_\zeta} = \frac{Y_L^2}{V}\frac{\partial V}{\partial k_\zeta} - \left(\frac{\partial \boldsymbol{V}}{\partial k_\zeta}\cdot\boldsymbol{Y}\right)\left(\frac{Y_L}{V}\right). \tag{B18}$$

B.3 Other derivatives of ψ needed for Västeberg and Lundborg's (38) and (39)

Although they do not appear explicitly in the paper's (38) and (39), there are other second derivatives of ψ that are present when writing out all of the terms, as seen in (76f) and (79f). These terms can be derived in a similar way as the previous derivatives. For $Y_T Y_L \frac{\partial^2 \psi}{\partial \eta \partial k_\zeta}$,

$$\begin{aligned}
\frac{\partial^2}{\partial \eta \partial k_\zeta} (\boldsymbol{V} \cdot \boldsymbol{Y}) =& \frac{\partial^2}{\partial \eta \partial k_\zeta} (VY \cos \psi) \\
=& \frac{\partial}{\partial \eta} \left(-VY \sin \psi \frac{\partial \psi}{\partial k_\zeta} + \left(V \frac{\partial Y}{\partial k_\zeta} + \frac{\partial V}{\partial k_\zeta} Y \right) \cos \psi \right) \\
=& \frac{\partial}{\partial \eta} \left(-VY \sin \psi \frac{\partial \psi}{\partial k_\zeta} + \frac{\partial V}{\partial k_\zeta} Y \cos \psi \right) \\
=& -VY \left(\sin \psi \frac{\partial^2 \psi}{\partial \eta \partial k_\zeta} + \cos \psi \frac{\partial \psi}{\partial \eta} \frac{\partial \psi}{\partial k_\zeta} \right) + \left(-V \frac{\partial Y}{\partial \eta} - \frac{\partial V}{\partial \eta} Y \right) \sin \psi \frac{\partial \psi}{\partial k_\zeta} \\
& - \frac{\partial V}{\partial k_\zeta} Y \sin \psi \frac{\partial \psi}{\partial \eta} + \left(\frac{\partial V}{\partial k_\zeta} \frac{\partial Y}{\partial \eta} + \frac{\partial^2 V}{\partial \eta \partial k_\zeta} Y \right) \cos \psi \\
=& -VY \left(\sin \psi \frac{\partial^2 \psi}{\partial \eta \partial k_\zeta} + \cos \psi \frac{\partial \psi}{\partial \eta} \frac{\partial \psi}{\partial k_\zeta} \right) - V \frac{\partial Y}{\partial \eta} \sin \psi \frac{\partial \psi}{\partial k_\zeta} - \frac{\partial V}{\partial k_\zeta} Y \sin \psi \frac{\partial \psi}{\partial \eta} \\
& + \frac{\partial V}{\partial k_\zeta} \frac{\partial Y}{\partial \eta} \cos \psi,
\end{aligned} \tag{B19}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial \eta \partial k_\zeta} (\boldsymbol{V} \cdot \boldsymbol{Y}) =& \frac{\partial}{\partial \eta} \left(\boldsymbol{V} \cdot \frac{\partial \boldsymbol{Y}}{\partial k_\zeta} + \frac{\partial \boldsymbol{V}}{\partial k_\zeta} \cdot \boldsymbol{Y} \right) \\
=& \frac{\partial}{\partial \eta} \left(\frac{\partial \boldsymbol{V}}{\partial k_\zeta} \cdot \boldsymbol{Y} \right) \\
=& \frac{\partial \boldsymbol{V}}{\partial k_\zeta} \cdot \frac{\partial \boldsymbol{Y}}{\partial \eta} + \frac{\partial^2 \boldsymbol{V}}{\partial \eta \partial k_\zeta} \cdot \boldsymbol{Y} \\
=& \frac{\partial \boldsymbol{V}}{\partial k_\zeta} \frac{\partial \boldsymbol{Y}}{\partial \eta}.
\end{aligned} \tag{B20}$$

Combining (B19) and (B20), rearranging the terms, multiplying both sides by $\frac{Y_L}{V}$, and ensuring that $\frac{\partial \psi}{\partial \eta}$ and $\frac{\partial \psi}{\partial k_\zeta}$ are always multiplied by $Y_T Y_L$, one arrives at

$$\begin{aligned}
Y_T Y_L \frac{\partial^2 \psi}{\partial \eta \partial k_\zeta} =& \frac{Y_L^2}{VY} \frac{\partial V}{\partial k_\zeta} \frac{\partial Y}{\partial \eta} - \left(\frac{\partial \boldsymbol{V}}{\partial k_\zeta} \cdot \frac{\partial \boldsymbol{Y}}{\partial \eta} \right) \left(\frac{Y_L}{V} \right) - \frac{Y_T Y_L \frac{\partial \psi}{\partial \eta} Y_T Y_L \frac{\partial \psi}{\partial k_\zeta}}{Y_T^2} - \frac{Y_T Y_L}{V} \frac{\partial \psi}{\partial \eta} \frac{\partial V}{\partial k_\zeta} \\
& - \frac{Y_T Y_L}{Y} \frac{\partial \psi}{\partial k_\zeta} \frac{\partial Y}{\partial \eta}.
\end{aligned} \tag{B21}$$

For $Y_T Y_L \frac{\partial^2 \psi}{\partial k_\eta \partial k_\zeta}$,

$$
\begin{aligned}
\frac{\partial^2}{\partial k_\eta \partial k_\zeta}(\boldsymbol{V}\cdot\boldsymbol{Y}) =& \frac{\partial^2}{\partial k_\eta \partial k_\zeta}(VY\cos\psi) \\
=& \frac{\partial}{\partial k_\eta}\left(-VY\sin\psi\frac{\partial\psi}{\partial k_\zeta} + \left(V\frac{\partial Y}{\partial k_\zeta} + \frac{\partial V}{\partial k_\zeta}Y\right)\cos\psi\right) \\
=& \frac{\partial}{\partial k_\eta}\left(-VY\sin\psi\frac{\partial\psi}{\partial k_\zeta} + \frac{\partial V}{\partial k_\zeta}Y\cos\psi\right) \\
=& -VY\left(\sin\psi\frac{\partial^2\psi}{\partial k_\eta \partial k_\zeta} + \cos\psi\frac{\partial\psi}{\partial k_\eta}\frac{\partial\psi}{\partial k_\zeta}\right) + \left(-V\frac{\partial Y}{\partial k_\eta} + \frac{\partial V}{\partial k_\eta}Y\right)\sin\psi\frac{\partial\psi}{\partial k_\zeta} \\
&- \frac{\partial V}{\partial k_\zeta}Y\sin\psi\frac{\partial\psi}{\partial k_\eta} + \left(\frac{\partial V}{\partial k_\zeta}\frac{\partial Y}{\partial k_\eta} + \frac{\partial^2 V}{\partial k_\eta \partial k_\zeta}Y\right)\cos\psi \\
=& -VY\left(\sin\psi\frac{\partial^2\psi}{\partial k_\eta \partial k_\zeta} + \cos\psi\frac{\partial\psi}{\partial k_\eta}\frac{\partial\psi}{\partial k_\zeta}\right) - \frac{\partial V}{\partial k_\eta}Y\sin\psi\frac{\partial\psi}{\partial k_\zeta} - \frac{\partial V}{\partial k_\zeta}Y\sin\psi\frac{\partial\psi}{\partial k_\eta} \\
&+ \frac{\partial^2 V}{\partial k_\eta \partial k_\zeta}Y\cos\psi,
\end{aligned}
\tag{B22}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial k_\eta \partial k_\zeta}(\boldsymbol{V}\cdot\boldsymbol{Y}) =& \frac{\partial}{\partial k_\eta}\left(bmV\cdot\frac{\partial\boldsymbol{Y}}{\partial k_\zeta} + \frac{\partial\boldsymbol{V}}{\partial k_\zeta}\cdot\boldsymbol{Y}\right) \\
=& \frac{\partial}{\partial k_\eta}\left(\frac{\partial\boldsymbol{V}}{\partial k_\zeta}\cdot\boldsymbol{Y}\right) \\
=& \frac{\partial\boldsymbol{V}}{\partial k_\zeta}\cdot\frac{\partial\boldsymbol{Y}}{\partial k_\eta} + \frac{\partial^2\boldsymbol{V}}{\partial k_\eta \partial k_\zeta}\cdot\boldsymbol{Y} \\
=& \frac{\partial^2\boldsymbol{V}}{\partial k_\eta \partial k_\zeta}\cdot\boldsymbol{Y}.
\end{aligned}
\tag{B23}
$$

Combining (B22) and (B23), rearranging the terms, multiplying both sides by $\frac{Y_L}{V}$, and ensuring that $\frac{\partial\psi}{\partial k_\eta}$ and $\frac{\partial\psi}{\partial k_\zeta}$ are always multiplied by $Y_T Y_L$, one arrives at

$$
\begin{aligned}
Y_T Y_L\frac{\partial^2\psi}{\partial k_\eta \partial k_\zeta} =& \frac{Y_L^2}{V}\frac{\partial^2 V}{\partial k_\eta \partial k_\zeta} - \left(\frac{\partial^2\boldsymbol{V}}{\partial k_\eta \partial k_\zeta}\cdot\boldsymbol{Y}\right)\left(\frac{Y_L}{V}\right) - \frac{Y_T Y_L\frac{\partial\psi}{\partial k_\eta}Y_T Y_L\frac{\partial\psi}{\partial k_\zeta}}{Y_T^2} - \frac{Y_T Y_L}{V}\frac{\partial\psi}{\partial k_\zeta}\frac{\partial V}{\partial k_\eta} \\
&- \frac{Y_T Y_L}{V}\frac{\partial\psi}{\partial k_\eta}\frac{\partial V}{\partial k_\zeta}.
\end{aligned}
\tag{B24}
$$

B.4 Derivatives of V

While the derivatives of the components of $\boldsymbol{Y}$ and the scalar Y can be computed with the magnetic field model being used, derivatives of the components of V are generally not computed. However, they are needed for the derivations found in B.2 and B.3. On page 93 of [2], we find that $V_\eta = \frac{c}{\omega} k_\eta$, as seen above in the text under (B2). So,

$$\frac{\partial V_\eta}{\partial k_\zeta} = \begin{cases} \frac{c}{\omega} & \text{if } \eta = \zeta \\ 0 & \text{if } \eta \neq \zeta. \end{cases} \tag{B25}$$

Also, it is easy to see that $\frac{\partial V_\eta}{\partial \zeta} = 0$ for all η and ζ, and all second derivatives of V_η are equal to zero, as well. So, the dot products with derivatives of $\boldsymbol{V}$ are easy to compute since, for example,

$$\frac{\partial^2 \boldsymbol{V}}{\partial k_\eta \partial k_\zeta} \cdot \boldsymbol{Y} = \frac{\partial^2 V_r}{\partial k_\eta \partial k_\zeta} Y_r + \frac{\partial^2 V_\theta}{\partial k_\eta \partial k_\zeta} Y_\theta + \frac{\partial^2 V_\phi}{\partial k_\eta \partial k_\zeta} Y_\phi = 0. \tag{B26}$$

But what about the derivatives of V? We don't know V, but we do know $V^2 = V_r^2 + V_\theta^2 + V_\phi^2$. Let's let $\eta = r$ as an example.

$$\frac{\partial}{\partial k_r} V^2 = 2V \frac{\partial V}{\partial k_r} \rightarrow \frac{\partial V}{\partial k_r} = \frac{1}{2V} \frac{\partial}{\partial k_r} V^2. \tag{B27}$$

But,

$$\frac{\partial}{\partial k_r} V^2 = \frac{\partial}{\partial k_r} \left(V_r^2 + V_\theta^2 + V_\phi^2 \right) = \frac{\partial}{\partial k_r} V_r^2 = 2V_r \frac{\partial V_r}{\partial k_r} = 2V_r \frac{c}{\omega}. \tag{B28}$$

Therefore,

$$\frac{\partial V}{\partial k_r} = \frac{c}{\omega} \frac{V_r}{V}. \tag{B29}$$

So,

$$\frac{\partial V}{\partial k_\zeta} = \frac{c}{\omega} \frac{V_\zeta}{V} \tag{B30}$$

for $\zeta = \{r, \theta, \phi\}$, and

$$\frac{\partial^2 V}{\partial k_\eta \partial k_\zeta} = -\frac{c^2}{\omega^2} \frac{V_\zeta V_\eta}{V^3}. \tag{B31}$$

These have the unknown quantities V and V^3 in them, but this problem goes away when they are put in their context within the equations in the previous two subsections. Doing so, we get:

$$Y_T Y_L \frac{\partial \psi}{\partial k_\zeta} = \frac{Y_L^2}{V^2} \frac{c}{\omega} V_\zeta - \frac{c}{\omega} Y_\zeta \frac{Y_L}{V}, \tag{B32}$$

$$Y_T Y_L \frac{\partial^2 \psi}{\partial \eta \partial k_\zeta} = \frac{Y_L^2}{V^2} \frac{V_\zeta}{Y} \frac{c}{\omega} \frac{\partial Y}{\partial \eta} - \frac{c}{\omega} \frac{\partial Y_\zeta}{\partial \eta} \frac{Y_L}{V} - \frac{Y_T Y_L \frac{\partial \psi}{\partial \eta} Y_T Y_L \frac{\partial \psi}{\partial k_\zeta}}{Y_T^2} - \frac{1}{Y} Y_T Y_L \frac{\partial \psi}{\partial k_\zeta} \frac{\partial Y}{\partial \eta} - \frac{c}{\omega} \frac{V_\zeta}{V^2} Y_T Y_L \frac{\partial \psi}{\partial \eta}, \tag{B33}$$

and

$$Y_T Y_L \frac{\partial^2 \psi}{\partial k_\eta \partial k_\zeta} = -\frac{c^2}{\omega^2} V_\zeta V_\eta \frac{Y_L^2}{V^4} - \frac{Y_T Y_L \frac{\partial \psi}{\partial k_\eta} Y_T Y_L \frac{\partial \psi}{\partial k_\zeta}}{Y_T^2} - \frac{c}{\omega} \frac{V_\eta}{V^2} Y_T Y_L \frac{\partial \psi}{\partial k_\zeta} - \frac{c}{\omega} \frac{V_\zeta}{V^2} Y_T Y_L \frac{\partial \psi}{\partial k_\eta}. \tag{B34}$$