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Tractable Computational Methods for Optimal Control via Fast Dynamic Programming

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14. ABSTRACT
Outcomes delivered by this project include advances in the representation of general solutions of dynamic programming equations and Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs) using idempotent fundamental solution semigroups [P2,8,10–12,22–24,40], new adaptive idempotent methods for solving nonlinear regulator problems [P10, 11, 13], new game representations for
non-quadratic regulator problems [P1, 3, 20, 30, 32, 33, 40], a preliminary theory of stationary control [P4, 8, 15, 21, 23, 27, 31, 36], and a preliminary new approach to curse-of-dimensionality attenuated optimization-based optimal feedback policy synthesis [P9, 17–19]. The project has also supported auxiliary advances in extremum seeking feedback control for dynamical systems evolving on manifolds
[P16], networked control and observer design [P6, 7, 14, 26, 35], quantum control [P28, 37, 38], and stability theory [P5, 25, 29, 34, 39]. These outcomes have been communicated via 40 scholarly publications in international journals and conference proceedings, and 19 invited presentations at national and international meetings. The project contributed direct funding support for the training of two
postdoctoral research fellows in optimal control, via consecutive one year appointments, and partial support for travel associated with participation in conferences and US-based research collaboration (with Professor W.M. McEneaney at UCSD). It facilitated the organization of 4 international minisymposia on the topics of idempotent methods and stationary control [A1, 2, 4, 7], and 3 multi-day
meetings on applied dynamics in Australia [A3, 5, 6].

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Tractable computational methods for optimal control via fast dynamic programming

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Final report

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1 Summary

Feedback control is essential for the robust regulation of dynamical physical and technological processes that underpin power generation and distribution, manufacturing and chemical refining, logistics and transportation, financial markets, etc. The design and synthesis of feedback control systems that achieve robust regulation in an optimal way, as quantified by pre-specified measures of performance, is a very challenging problem. Optimal control theory provides a mathematical framework for feedback control policy design and synthesis, such that the corresponding controlled dynamics are optimal with respect to a user-defined cost or payoff function [R23, 24]. However, these optimal feedback control policies are computationally very expensive to synthesize for practical problems, and often intractably so, due to an attendant curse-of-dimensionality [R9, 10].

This project is concerned with the development of new theory and computational methods for the tractable solution of classes of continuous time optimal control problems [D1–3]. Underlying this development is an investigation of the algebraic properties of the dynamic programming principle (DPP) and associated classes of Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs), yielding various representations and approximations for optimal control problems and their solutions. After three years of funding, outcomes of this project include new theoretical foundations, representations, approximations, computational tools, and applications for optimal control. The research undertaken is documented in 40 scholarly publications, and has been communicated via 19 invited presentations across a range of top international meetings. The project directly supported the training of two postdoctoral researchers, via two one year appointments. It further facilitated the organization of 4 international mini-symposia on the topic of idempotent methods and optimal control, and 3 multi-day meetings on applied dynamics in Australia. A summary of research directions and associated outcomes is provided in Table 1. Further details are provided in Section 2.

Research direction	**Outcomes†**
Idempotent fundamental solution semigroups for optimal control	[P2, 8, 10–12, 22–24, 40], [I1, 2, 6, 11–13, 16, 17], [A2, 7]
Adaptive idempotent methods for non-quadratic regulator problems	[P10, 11, 13], [I1, 2, 4, 11, 17], [A2, 7]
Game representations for non-quadratic regulator problems	[P1, 3, 20, 30, 32, 33, 40], [I2, 10, 15, 17, 18], [A2, 7]
Stationary action and stationary control	[P4, 8, 15, 21–23, 27, 31, 36], [I3, 5, 7, 14, 19] [A1, 4]
Optimization-based optimal feedback synthesis	[P9, 17–19], [I8, 9]
Auxiliary advances	[P5–7, 14, 16, 25, 26, 28, 29, 34, 35, 37–39] [A3, 5, 6]

† See Section 3 for all publications [P*], invited presentations [I*], organized activities [A*], documentation [D*], and references [R*].

Table 1: Summary of research directions and associated outcomes.

2 Description of outcomes

Outcomes delivered by this project include advances in the representation of general solutions of dynamic programming equations and Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs) using idempotent fundamental solution semigroups [P2,8,10–12,22–24,40], new adaptive idempotent methods for solving nonlinear regulator problems [P10, 11, 13], new game representations for non-quadratic regulator problems [P1, 3, 20, 30, 32, 33, 40], a preliminary theory of stationary control [P4, 8, 15, 21, 23, 27, 31, 36], and a preliminary new approach to curse-of-dimensionality attenuated optimization-based optimal feedback policy synthesis [P9, 17–19]. The project has also supported auxiliary advances in extremum seeking feedback control for dynamical systems evolving on manifolds [P16], networked control and observer design [P6, 7, 14, 26, 35], quantum control [P28, 37, 38], and stability theory [P5, 25, 29, 34, 39]. These outcomes have been communicated via 40 scholarly publications in international journals and conference proceedings, and 19 invited presentations at national and international meetings. The project contributed direct funding support for the training of two postdoctoral research fellows in optimal control, via consecutive one year appointments, and partial support for travel associated with participation in conferences and US-based research collaboration (with Professor W.M. McEneaney at UCSD). It facilitated the organization of 4 international mini-symposia on the topics of idempotent methods and stationary control [A1, 2, 4, 7], and 3 multi-day meetings on applied dynamics in Australia [A3, 5, 6].

> *The ongoing research supported by this project received international recognition in 2017 via the awarding of the SIAM SICON Best Paper Prize for the development of an idempotent fundamental solution for the gravitational N-body problem [P40].*

2.1 Background

An optimal control problem is posed with respect to a user-defined real-valued cost function that assigns an order to the set of admissible open-loop controlled behaviours of a dynamical system. The value function associated with an optimal control problem is (usually) defined as the infimum of this cost over the space of admissible open-loop controls (a supremum of a payoff function may alternatively be used). An optimal open-loop control, if it exists, is an element of this admissible control space that renders the cost minimal (or payoff maximal). An optimal feedback control policy is a feedback representation of this optimal open-loop control, i.e. a feedback map from the state dynamics to the control input. The value function is defined on the possibly infinite dimensional state space $\mathcal{X}$ of the dynamics, and is parameterized by the finite or infinite time horizon over which the evolution of the controlled dynamics are of interest. Typical finite and infinite horizon value functions $W_t, W : \mathcal{X} \to \overline{\mathbb{R}}$, $t \in \mathbb{R}_{\geq 0}$, defining optimal control problems with dynamics evolving on $\mathcal{X}$ are defined respectively by

$$W_t(x) \doteq \inf_{u \in \mathcal{U}[0,t]} J_t[\psi](x, u), \qquad W(x) \doteq \lim_{t\to\infty} W_t(x), \tag{1}$$

for all initial states $x \in \mathcal{X}$, provided the limit involved exists. The cost function $J_t[\psi] : \mathcal{X} \times \mathscr{U}[0, t] \to \overline{\mathbb{R}}$ involved is defined on the state and open-loop control spaces $\mathcal{X}$ and $\mathscr{U}[0, t]$, and often takes the form

$$J_t[\psi](x, u) \doteq \int_0^t \ell(x_s) + \tfrac{1}{2} \langle u_s, \mathcal{M}\, u_s \rangle \, ds + \psi(x_t), \tag{2}$$

for all $x \in \mathcal{X}$, $u \in \mathscr{U}[0, t]$, in which $\ell, \psi : \mathcal{X} \to \overline{\mathbb{R}}$ describe running and terminal costs of interest, and $\mathcal{M} \in \mathcal{L}(\mathcal{U})$ is a coercive operator used to weight the control effort expended. The underlying dynamics are often modelled as input affine, using suitable maps $f : \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \mapsto \mathcal{L}(\mathcal{U}; \mathcal{X})$, with the state trajectory $s \mapsto x_s$ satisfying

$$x_s = x + \int_0^s [f(x_r) + g(x_r)\, u_r] \, dr, \tag{3}$$

for all $s \in [0,t]$, in which $u \in \mathscr{U}[0,t]$ is the applied open-loop control. State and open-loop control spaces for a typical finite dimensional problem are $\mathcal{X} \doteq \mathbb{R}^n$ and $\mathscr{U}[0,t] \doteq \mathscr{L}^2([0,t];\mathcal{U})$, $\mathcal{U} \doteq \mathbb{R}^m$.

A solution of the finite horizon optimal control problem of (1) involves the synthesis of either an optimal open-loop control that minimizes $u \mapsto J_t[\psi](x,u)$, given a specific time horizon $t \in \mathbb{R}_{\geq 0}$ and initial state $x \in \mathcal{X}$, or an optimal feedback control policy that evaluates the control $u_s \in \mathcal{U}$ at each time $s \in [0,t]$ as a function of the current state x_s and the time to go $t-s$, i.e. $u_s = K_{t-s}(x_s)$, see (6) below. Similarly, a solution of the infinite horizon optimal control problem in (1) involves the synthesis of a control policy that is independent of the time to go. In either case, a feedback policy is preferred in practice, due to the inherent robustness afforded by the incorporation of feedback.

Dynamic programming is founded on the principle of optimality [R24], and provides a solution path for optimal control problems. It characterizes the temporal evolution of finite horizon value function $t \mapsto W_t$ via a horizon indexed dynamic programming evolution operator, yielding a nonlinear variational equation known as a dynamic programming principle (DPP). Similarly, the infinite horizon value function W is characterized as a fixed point of this horizon indexed dynamic programming evolution operator, for all time horizons. For the finite and infinite horizon optimal control problems of (1), the respective DPPs, and the (common) dynamic programming evolution operator, are given by

$$\begin{gathered} W_{\tau+t}(x) = (\mathcal{S}_\tau\, W_t)(x), \quad W_0(x) = \psi(x), \qquad W(x) = (\mathcal{S}_\tau\, W)(x), \quad W(0) = 0, \\ (\mathcal{S}_\tau\, \psi)(x) \doteq \inf_{u \in \mathcal{U}[0,\tau]} J_\tau[\psi](x,u), \end{gathered} \tag{4}$$

for all $\tau, t \in \mathbb{R}_{\geq 0}$, $x \in \mathcal{X}$. In the limit of a vanishing incremental time horizon $\tau \in \mathbb{R}_{>0}$, the DPPs (4) yield respectively a non-stationary and a stationary Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE), with corresponding boundary data. Under reasonable conditions, the finite and infinite horizon value functions (1) may be characterized as unique viscosity solutions of these HJB PDEs [R10, 11, 15], and these HJB PDEs are given explicitly by

$$0 = \frac{\partial W_t}{\partial t}(x) + H(x, \nabla W_t(x)), \quad W_0(x) = \psi(x), \qquad 0 = H(x, \nabla W(x)), \quad W(0) = 0, \tag{5}$$
$$H(x,p) \doteq \sup_{u \in \mathcal{U}} \{-\ell(x) - \tfrac{1}{2}\langle u,\, \mathcal{M}\, u\rangle - \langle p, f(x) + g(x)\, u\rangle\} = -\ell(x) - \langle p, f(x)\rangle + \tfrac{1}{2}\langle p,\, g(x)\, \mathcal{M}^{-1}\, g(x)'\, p\rangle,$$

for all $t \in \mathbb{R}_{\geq 0}$, $x, p \in \mathcal{X}$. The Hamiltonian $H : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ involved in (5) may be interpreted as the generator of the Lax-Oleinik semigroup of dynamic programming evolution operators $\{\mathcal{S}_\tau\}_{\tau \geq 0}$, defined via the finite horizon DPP (4). The optimal feedback policies involved are characterized via the (possibly generalized) gradient of the respective solutions of these HJB PDEs, i.e.

$$\begin{gathered} K_{t-s}(x) \doteq \mathcal{K}(x, \nabla W_{t-s}(x)), \qquad K(x) \doteq \mathcal{K}(x, \nabla W(x)), \\ \mathcal{K}(x,p) \doteq \arg\max_{u \in \mathcal{U}} \{-\ell(x) - \tfrac{1}{2}\langle u,\, \mathcal{M}\, u\rangle - \langle p, f(x) + g(x)\, u\rangle\} = -\mathcal{M}^{-1}\, g(x)'\, p, \end{gathered} \tag{6}$$

for all $s \in [0,t]$, $x, p \in \mathcal{X}$. The corresponding optimal trajectory is defined as the solution of (3) in the presence of the appropriate finite or infinite horizon optimal feedback policy from (6).

Explicit solutions of the HJB PDEs (5) are exceedingly rare, and numerical methods are typically required to (approximately) evaluate the optimal feedback policies in (6). The computational complexity involved typically scales exponentially with the dimension of the state space $\mathcal{X}$, inflicting a curse-of-dimensionality on the numerical methods involved. This limits their applicability to problems with state space dimension of at most 4 or 5. The curse-of-dimensionality motivates the search for algebraic structure in the DPPs and HJB PDEs at the foundation of optimal control, and the exploitation of this structure in the development of more efficient numerical methods.

2.2 Idempotent fundamental solution semigroups for optimal control

Idempotent fundamental solution semigroups for optimal control exploit the semigroup, semiconvexity, and idempotent linearity properties of the dynamic programming evolution operator in order to provide convolution representations for value function propagation via dynamic programming. For finite or infinite horizon optimal control problems with value functions of the form (1), the relevant idempotent algebra is the min-plus algebra [R4, 10, 12, 13, 16, 19]. This algebra is a commutative semifield over the extended reals $\overline{\mathbb{R}}$, equipped with the addition and multiplication operations $a \oplus b \doteq \min(a,b)$ and $a \otimes b \doteq a + b$, for all $a, b \in \overline{\mathbb{R}}$. As the cost functional $\psi \mapsto J_t[\psi]$ of (2) is min-plus linear, i.e. $J_t[\psi \oplus (c \otimes \phi)](x,u) = J_t[\psi](x,u) \oplus (c \otimes J_t[\phi](x,u))$ for all $c \in \overline{\mathbb{R}}$, $t \in \mathbb{R}_{\geq 0}$, $x \in \mathcal{X}$, $u \in \mathscr{U}[0,t]$, and all terminal costs $\psi, \phi : \mathcal{X} \mapsto \overline{\mathbb{R}}$, an interchange of inf and min implies that the dynamic programming evolution operator $\mathcal{S}_t$ of (4) is likewise min-plus linear. Moreover, the finite horizon value function W_t of (1) has a min-plus convolution representation [P10, 23, 40], defined with respect to a bivariate convolution kernel $G_t : \mathcal{X} \times \mathcal{X} \to \overline{\mathbb{R}}$ and the min-plus delta function $\delta : \mathcal{X} \times \mathcal{X} \to \overline{\mathbb{R}}$ by

$$
\begin{aligned}
W_t(x) = (\mathcal{G}_t^{\oplus}\,\psi)(x) &\doteq \int_{\mathcal{X}}^{\oplus} G_t(x,y) \otimes \psi(y)\, dy \doteq \inf_{y \in \mathcal{X}} \{G_t(x,y) + \psi(y)\}, \\
G_t(x,y) \doteq [\mathcal{S}_t\, \delta(\cdot, y)](x), &\quad \delta(x,y) \doteq \begin{cases} 0, & \|x-y\| = 0, \\ +\infty, & \text{otherwise}, \end{cases}
\end{aligned} \tag{7}
$$

for all $t \in \mathbb{R}_{\geq 0}$, $x, y \in \mathcal{X}$, in which the integration symbol $\int_{\mathcal{X}}^{\oplus}$ denotes the min-plus integral defined via the indicated inf. Dynamic programming and the corresponding DPP (4) imply that $\{\mathcal{G}_\tau^{\oplus}\}_{\tau \geq 0}$ defines a semigroup of min-plus linear min-plus integral operators, referred to as a *min-plus primal space fundamental solution semigroup* [P2, 22–24, 40], [R2, 3, 14]. This semigroup can be used to represent or propagate the value function (1) of a finite horizon optimal control problem to longer time horizons.

An alternative min-plus convolution representation for W_t of (1) follows via a semiconvexity or semiconcavity property of the dynamic programming evolution operator $\mathcal{S}_t$. Spaces of uniformly semiconvex and semiconcave functions on $\mathcal{X}$ are defined with respect to a bi-quadratic basis $\varphi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ by

$$
\begin{aligned}
\mathscr{S}_\varphi^{+} \doteq \left\{ \phi : \mathcal{X} \to \overline{\mathbb{R}} \,\middle|\, \begin{array}{c} x \mapsto \phi(x) + \varphi(x,0) \\ \text{convex, lower closed} \end{array} \right\}, &\quad \mathscr{S}_\varphi^{-} \doteq \left\{ \psi : \mathcal{X} \to \overline{\mathbb{R}} \,\middle|\, -\psi \in \mathscr{S}_\varphi^{+} \right\}, \\
\varphi(x,z) \doteq \tfrac{1}{2} \langle x - z, \mathcal{C}\,(x-z) \rangle, \quad x, z \in \mathcal{X}, &\quad \mathcal{C} \in \mathcal{L}(\mathcal{X}) \text{ fixed a priori.}
\end{aligned} \tag{8}
$$

These spaces are in duality, via either the semiconvex transform $\mathcal{D}_\varphi^{+} : \mathscr{S}_\varphi^{+} \mapsto \mathscr{S}_\varphi^{-}$ or the semiconcave transform $\mathcal{D}_\varphi^{-} : \mathscr{S}_\varphi^{-} \mapsto \mathscr{S}_\varphi^{+}$, see for example [R10], [P10, 24]. These transforms and their inverses satisfy

$$
\begin{aligned}
(\mathcal{D}_\varphi^{+}\,\phi)(z) \doteq -\sup_{x \in \mathcal{X}} \{-\varphi(x,z) - \phi(x)\}, &\qquad ([\mathcal{D}_\varphi^{+}]^{-1}\,\psi)(z) = \sup_{z \in \mathcal{X}} \{-\varphi(x,z) + \psi(z)\}, \\
\mathcal{D}_\varphi^{-}\,\psi \doteq -\mathcal{D}_\varphi^{+}[-\psi] = [\mathcal{D}_\varphi^{+}]^{-1}\,\psi, &\qquad [\mathcal{D}_\varphi^{-}]^{-1}\,\phi = -[\mathcal{D}_\varphi^{+}]^{-1}[-\phi] = \mathcal{D}_\varphi^{+}\,\phi,
\end{aligned} \tag{9}
$$

for all $\phi \in \mathscr{S}_\varphi^{+}$, $\psi \in \mathscr{S}_\varphi^{-}$. It may subsequently be shown, similarly as per [R10], [P10], that

$$
\begin{aligned}
W_t = [\mathcal{D}_\varphi^{-}]^{-1}\, \mathcal{B}_t^{\oplus}\, \mathcal{D}_\varphi^{-}\,\psi, &\quad (\mathcal{B}_t^{\oplus}\, a)(y) \doteq \int_{\mathcal{X}}^{\oplus} B_t(y,z) \otimes a(z)\, dz, \\
B_t(y,z) \doteq (\mathcal{D}_\varphi^{-}\, S_t(\cdot, z))(y), &\quad S_t(y,z) \doteq [\mathcal{S}_t\, \varphi(\cdot, z)](y),
\end{aligned} \tag{10}
$$

for all $t \in \mathbb{R}_{\geq 0}$, $y, z \in \mathcal{X}$. Dynamic programming and the DPP (4) imply that $\{\mathcal{B}_\tau^{\oplus}\}_{\tau \geq 0}$ defines a semigroup of min-plus linear min-plus integral operators, referred to as a *min-plus dual space fundamental solution semigroup* [P10, 40], [R1, 6]. This semigroup can also be used to represent or propagate the finite horizon value function (1) to longer time horizons. Together, (7), (8), (9), (10) yield that

$$\begin{aligned} \mathcal{G}_\tau^\oplus\, \mathcal{G}_t^\oplus &= \mathcal{G}_{\tau+t}, \qquad \mathcal{B}_{\tau+t}^\oplus = \mathcal{B}_\tau^\oplus\, \mathcal{B}_t^\oplus, \\ \mathcal{G}_\tau^\oplus\, \mathcal{G}_t^\oplus\, \psi = \mathcal{G}_{\tau+t}\, \psi &= W_{\tau+t} = [\mathcal{D}_\varphi^-]^{-1}\, \mathcal{B}_{\tau+t}^\oplus\, \mathcal{D}_\varphi^-\, \psi = [\mathcal{D}_\varphi^-]^{-1}\, \mathcal{B}_\tau^\oplus\, \mathcal{B}_t^\oplus\, \mathcal{D}_\varphi^-\, \psi, \end{aligned} \tag{11}$$

for all $\psi \in \mathscr{S}_\varphi^-$ and $\tau \in \mathbb{R}_{\ge 0}$. It may also be shown that $G_t(x,y) = (\mathcal{D}_\varphi^- S_t(x,\cdot))(y)$ for all $x, y \in \mathcal{X}$, analogously to (10), see [2, 10, 24] for an analogous max-plus case.

New idempotent fundamental solution semigroups for a range of optimal control problems have been developed in this project [P2, 8, 10–12, 21–24, 40]. Application of these semigroups facilitate efficient evolution of value functions from their corresponding terminal costs. As indicated in (7), (10), (11), they consist of time horizon indexed min-plus linear min-plus integral operators defined on associated spaces of uniformly semiconcave terminal costs. Each operator acts on its functional argument via an idempotent convolution with a bivariate kernel, over the state space $\mathcal{X}$. Two types of idempotent fundamental solution semigroup have been investigated, corresponding to value function evolution in the primal or dual space as defined by the semiconvex or semiconcave transform, as per (7), (10), (11). Both types of semigroup are equivalent to the Lax-Oleinik semigroup of dynamic programming evolution operators, and their construction exploits idempotent linearity and semiconvexity properties that attend the DPP (4). Computational advantages in their application follow from reduced dimension representation and/or approximation of the attendant bivariate kernels G_t or B_t. These fundamental solution semigroups are the foundation for the development of new approaches to the solution of differential Riccati equations [P24], finite dimensional linear regulator problems with state constraints [P1, 3, 20, 30, 32, 33], nonlinear regulator problems [P10, 11], the gravitational N-body problem [P40], and related two point boundary value problems (TPBVPs) constrained by energy conserving dynamics [P2, 8, 12, 22, 23, 36].

2.3 Adaptive idempotent methods for non-quadratic regulator problems

The spaces $\mathscr{S}_\varphi^+$ and $\mathscr{S}_\varphi^-$ of uniformly semiconvex and semiconcave functions (8) define respectively a max-plus and a min-plus vector space [R10, 12, 13, 16, 19]. Given a countable dense subset $\{z_i\}_{i\in\mathbb{N}}$ of a finite dimensional state space $\mathcal{X}$, the set $\{\psi_i\}_{i\in\mathbb{N}}$, defined via its elements $\psi_i \doteq \varphi(\cdot, z_i)$, $i \in \mathbb{N}$, forms a countable basis for $\mathscr{S}_\varphi^+$ and $\mathscr{S}_\varphi^-$. This countable basis implies the existence of a coordinate representation for functions in these spaces. As the dynamic programming evolution operator $\mathcal{S}_\tau$, $\tau \in \mathbb{R}_{\ge 0}$, is an idempotent linear operator on $\mathscr{S}_\varphi^+$ or $\mathscr{S}_\varphi^-$, it inherits this coordinate representation if it is also endomorphic. This coordinate representation facilitates the development of a coordinate iteration for the value function, via dynamic programming and the dual space fundamental solution, see (10), (11), [R10]. In the case of (1), given some a priori fixed short horizon $\tau \in \mathbb{R}_{>0}$, the coordinate vector e_k associated with the k^{th} such iteration is given by $[e_k]_i \doteq a_k(z_i) \in \overline{\mathbb{R}}$, $a_k \doteq \mathcal{D}_\varphi^-\, W_{k\tau} \in \mathscr{S}_\varphi^+$, $k, i \in \mathbb{N}$. Representation (10) and semigroup (11), along with convergence and other properties of this iteration [R10], [P10, 11], imply that the associated infinite horizon value function W of (1) satisfies

$$\begin{aligned} W(x) = ([\mathcal{D}_\varphi^-]^{-1} a_\infty)(x) = \bigoplus_{j\in\mathbb{N}} \psi_j(x) \otimes [e_\infty]_j, \quad e_\infty \doteq \lim_{k\to\infty} e_k, \\ \text{subject to} \begin{cases} [e_{k+1}]_i = \displaystyle\bigoplus_{j\in\mathbb{N}} B_\tau(z_i, z_j) \otimes [e_k]_j, & k \in \mathbb{Z}_{\ge 0},\ i \in \mathbb{N}, \\ [e_0]_i = (\mathcal{D}_\varphi^-\, \psi)(z_i), & i \in \mathbb{N}, \end{cases} \end{aligned} \tag{12}$$

for all $x \in \mathcal{X}$. Truncation of the basis $\{\psi_i\}_{i\in\mathbb{N}}$ to a subset of finite cardinality $\nu \in \mathbb{N}$ yields a coordinate representation for a corresponding approximation $\widehat{W}$ of the infinite horizon value function W. Interpreting $B_\tau(z_i, z_j)$ as the $(i,j)^{th}$ element of a matrix $B_\tau \in \overline{\mathbb{R}}^{\nu\times\nu}$ yields an idempotent iteration as per [R10], [P10, 11], i.e. $\widehat{e}_{k+1} = B_\tau \otimes \widehat{e}_k$, $k \in \mathbb{Z}_{\geq 0}$, with the approximation $\widehat{W}$ given by

$$\widehat{W}(x) = \bigoplus_{j=1}^{\nu} \psi_j(x) \otimes [\widehat{e}_\infty]_j, \tag{13}$$

for all $x \in \mathcal{X}$. Evaluation of the approximation $\widehat{W}(x)$ for a specific $x \in \mathcal{X}$ corresponds to evaluation of a specific basis function ψ_{j_x} that is *active* at x in (13), shifted by the corresponding coordinate $[\widehat{e}_\infty]_{j_x}$, for the appropriate index $j_x \in \mathbb{N}_{\leq \nu}$, i.e. such that $\psi_{j_x}(x) + [\widehat{e}_\infty]_{j_x} = \bigoplus_{j=1}^{\nu} \psi_j(x) \otimes [\widehat{e}_\infty]_j$. As $\{\psi_j\}_{\mathbb{N}_{\leq\nu}}$ is defined with respect to the bi-quadratic function φ of (8) by $\psi_j \doteq \varphi(\cdot, z_j)$, and φ has state-invariant Hessian $\mathcal{C} \in \mathcal{L}(\mathcal{X})$, the approximation (13) partitions any convex subset $\mathcal{Y} \subset \mathcal{X}$ of the state space into a Voronoi tessellation $\mathscr{T}$ of convex polytopes. For the optimal control problem of (1), for which the min-plus algebra is required, this tessellation is given by

$$\mathscr{T} \doteq \{\overline{\mathcal{Y}}_i\}_{i\leq\nu}, \qquad \mathcal{Y}_i \doteq \left\{ x \in \mathcal{Y} \,\middle|\, \begin{array}{c} \psi_j(x) + [\widehat{e}_\infty]_j > \psi_i(x) + [\widehat{e}_\infty]_i \\ \text{for all } j \in \mathbb{N}_{\leq\nu} \setminus \{i\} \end{array} \right\}. \tag{14}$$

The approximation $\widehat{W}$ of (13) is thus piecewise quadratic, with the pieces involved uniquely determined on the respective polytopes in the tessellation $\mathscr{T}$ of (14), which is itself determined by the idempotent limit $\widehat{e}_\infty \in \overline{\mathbb{R}}^\nu$ of the finite cardinality iteration corresponding to (12). The approximation error associated with $\widehat{W}$, relative to W of (1), may be characterized via a posteriori back-substitution [P13] in the Hamiltonian H of the stationary HJB PDE (5). In particular, given the subset $\mathcal{Z} \doteq \{z_i\}_{i\leq\nu} \subset \mathcal{X}$ of states defining the truncated basis $\{\psi_i\}_{i\leq\nu}$, the worst-case back-substitution error $e(\mathcal{Z})$ on $\mathcal{Y}$ is defined with respect to the back-substitution errors in the constituent polytopes $\mathcal{Y}_i$, $i \in \mathbb{N}_{\leq\nu}$, by

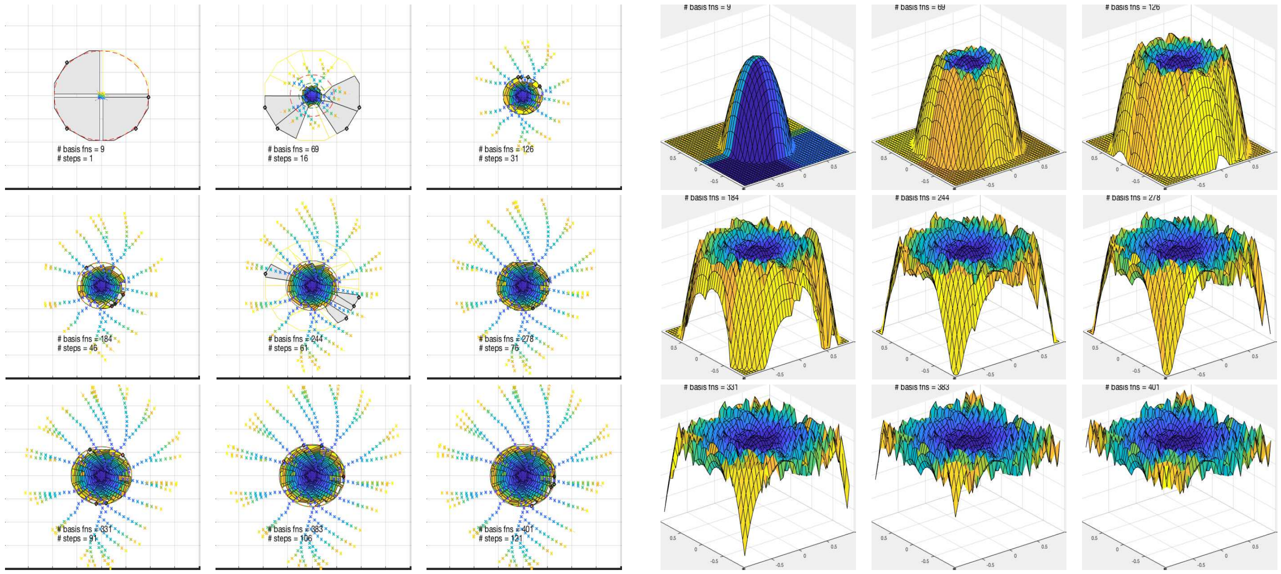
$$e(\mathcal{Z}) \doteq \max_{i\leq\nu} \sup_{x\in\mathcal{Y}_i} |h(x, z_i)|^2, \quad h(x, z_i) \doteq H(x, \nabla\widehat{W}(x)) = H(x, \nabla\psi_i(x)) = H(x, \mathcal{C}\,(x - z_i)), \tag{15}$$

for all $x \in \mathcal{Y}_i$, $i \in \mathbb{N}_{\leq\nu}$. As this back-substitution error is explicitly parameterized by $\mathcal{Z}$, basis adaptation to improve the back-substitution error is possible. The k^{th} iteration of this basis adaptation yields a basis $\mathscr{B}^k$, $k \in \mathbb{N}$, populated with $\nu^k \in \mathbb{N}$ quadratic functions of the form $\psi_i^k \doteq \varphi(\cdot, z_i^k)$, $z_i^k \in \mathcal{Z}^k$, $i \in \mathbb{N}_{\leq\nu^k}$, in which $\nu^k \in \mathbb{N}$ is the corresponding adaptation iteration dependent cardinality. The update step in the adaptation iteration involved is explicitly represented [P10, 11] by

$$\mathscr{B}^{k+1} = (\mathscr{B}^k \setminus \mathscr{B}_-^k) \cup \mathscr{B}_+^k, \tag{16}$$

in which $\mathscr{B}_-^k \subset \mathscr{B}^k$ and $\mathscr{B}_+^k$ denote respectively the subset of existing basis functions that are to be pruned and the set of new basis functions that are to be added. The subset $\mathscr{B}_-^k$ is constructed by collecting those existing basis functions that are unused in the corresponding value function approximation $\widehat{W}^k$ of (13). Other existing basis functions whose corresponding polytopes are of sufficiently small volume may also be included in this prune set. The set $\mathscr{B}_+^k$ of new basis functions is constructed so as to reduce the back-substitution error $e(\mathcal{Z}^k)$ of (15). In particular, new basis functions are added in specific descent directions for $\delta \mapsto e(\mathcal{Z}^k + \delta)$ or, alternatively, in related directions determined by the characteristic flow associated with the Hamiltonian H. Identifying existing basis functions and polytopes that contribute to the worst-case in $e(\mathcal{Z}^k)$ forms part of this construction, see [10, 11]. In this way, iteration (16) simultaneously evolves the basis $\mathscr{B}^k$ and the value function approximation $\widehat{W}^k$, while controlling the growth rate of the basis cardinality ν^k required to achieve a required back-substitution error on $\mathcal{Y}$. The basis evolved in this way is sparse, and the computational effort required to obtain the value function approximation $\widehat{W}^k$ is inherited from [R10], see [P10, 11].

New adaptive idempotent methods have been developed [P10,11] for a general class of finite dimensional infinite horizon nonlinear regulator problems originally considered in [R10], and defined via a value function of similar form to (1). A typical evolution of this basis and the corresponding state dependent back-substitution error is illustrated in Figure 1, see [P10, 11].



(a) Basis and tessellation. (b) A posteriori back-substitution error.

Figure 1: Basis and back-substitution error evolution in an adaptive idempotent method [P10, 11].

2.4 Game representations for non-quadratic regulator problems

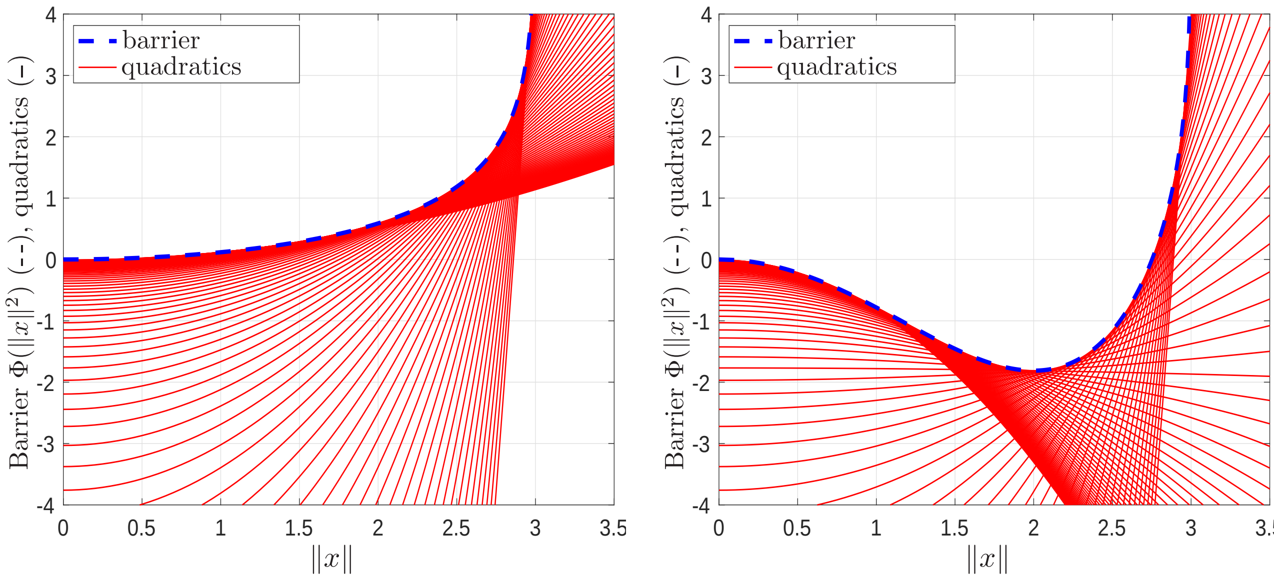
Linear quadratic regulator (LQR) problems are a classical special case of a class of optimal control problems defined via the value functions of (1), for which the dynamics (3) are linear and the cost (2) is quadratic. In that case, these value functions are also quadratic, and the non-stationary and stationary HJB PDEs in (5) reduce to a differential Riccati equation (DRE) and an algebraic Riccati equation (ARE) in the Hessians of these value functions. Numerical methods for the solution of these Riccati equations are well-known, and their application scales tractably with state space dimension.

The introduction of state constraints, non-quadratic cost terms, or nonlinearities in the dynamics underlying an LQR problem fundamentally impacts its solvability. Indeed, the fundamental quadratic structure involved is destroyed, and the solvability of DREs or AREs is no longer relevant or applicable. Instead, the value functions involved are intrinsically non-quadratic, and satisfy only the more general HJB PDEs of (5). However, under certain circumstances, the problematic non-quadratic terms in (2) can be replaced with their equivalent semiconvex relaxations, and the optimal control problem of (1) lifted to an equivalent game problem, see [P1, 3, 20, 30, 32, 33, 40]. This game problem is naturally equipped with quadratic structure by virtue of the semiconvex relaxation involved, allowing its solution to be represented via a family of DREs or AREs that are parameterized by the actions of an introduced maximizing player. This approach is at the foundation of the construction of a new fundamental solution for the gravitational N-body problem in [P40], and the development of a new representation for solutions of finite dimensional state constrained regulator problems [P1, 3, 20, 30, 32, 33].

In the context of finite dimensional state constrained regulator problems, the developments of [P1, 3, 20, 30, 32, 33] focus on a special case where the dynamics (3) are linear, and the trajectory must satisfy state constraints at all times. The state constraints are imposed in the definition (1) of the value function via the introduction of a state dependent extended real-valued barrier term in the running cost function ℓ that contributes to the cost function (2). The barrier function $\Phi : \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ employed is convex and lower closed, thereby guaranteeing the existence of its convex relaxation [R22], and subsequently a semiconvex relaxation of $x \mapsto \Phi(\|x\|^2)$, $x \in \mathcal{X}$, yielding an exact *sup-of-quadratics* representation, see [P3] and (17) below. An approximate sup-of-quadratics representation, corresponding to an approximation $x \mapsto \Phi^M(\|x\|^2)$ to the barrier, indexed by an approximation parameter $M \in \mathbb{R}$, also follows by restricting the family of quadratics used. For a ball state constraint, i.e. requiring $s \mapsto \|x_s\| < b < \infty$ for all $s \in [0, t]$, the respective exact and approximate value functions W_t and W_t^M are as per (1), with the exact and approximate sup-of-quadratic representations contributing via (2) to the respective cost functions by way of the assignments

$$\ell(x) \doteq \tfrac{1}{2}\,\|x\|^2 + \tfrac{1}{2}\,\Phi(\|x\|^2), \qquad \Phi(\|x\|^2) = \sup_{\alpha \geq -\phi(0)} \left\{ a^{-1}(\alpha)\,\|x\|^2 - \alpha \right\} = \begin{cases} \phi(\|x\|^2), & \|x\| < b, \\ +\infty, & \|x\| \geq b, \end{cases}$$
$$\ell^M(x) \doteq \tfrac{1}{2}\,\|x\|^2 + \tfrac{1}{2}\,\Phi^M(\|x\|^2), \quad \Phi^M(\|x\|^2) = \sup_{\alpha \in [-\phi(0), M]} \left\{ a^{-1}(\alpha)\,\|x\|^2 - \alpha \right\}, \quad M \geq -\phi(0), \tag{17}$$

for all $x \in \mathcal{X} \doteq \mathbb{R}^n$. In the sup-of-quadratics representations of (17), $\phi : [0, b) \to \mathbb{R}$ is convex and $a : [-\phi(0), \infty) \to \mathbb{R}$ is its (demonstrably invertible) convex dual, see [P3, Theorem 3.2, Remark 3.3]. Examples of sup-of-quadratics representations obtained for two barrier functions implementing the state constraint ball radius $b \doteq 3$ are illustrated in Figure 2.



(a) Convex barrier $\|x\| \mapsto \Phi(\|x\|^2)$.

(b) Semiconvex barrier $\|x\| \mapsto \Phi(\|x\|^2)$.

Figure 2: Sup-of-quadratics representation for a pair of barrier functions [P3].

Replacing the barrier function Φ with its exact or approximate sup-of-quadratics representation (17) yields respectively a state constrained regulator problem, or a convergent approximation to it. Convergence is demonstrated both in terms of the value functions involved, i.e. $\lim_{M\to\infty} W_t^M = W_t$ pointwise on $\mathcal{X}$, and in terms of the behaviour of the corresponding optimal state trajectories. Using the exact representation guarantees that trajectories are always confined to the closure of the constraint set of interest, and to its interior almost always (a.e. in time). The approximate problem is shown to yield trajectories that converge to this behaviour [P3]. Measurable selection leads to the aforementioned lifting of the exact and approximate regulator problems to a corresponding pair of two player games. For the ball state constraint of (17), the exact game is defined in [P3] via

$$W_t(x) \doteq \inf_{u \in \mathcal{U}[0,t]} J_t[\psi](x, u) = \inf_{u \in \mathcal{U}[0,t]} \sup_{\alpha \in \mathscr{A}[0,t]} \bar{J}_t[\psi](x, u, \alpha) = \sup_{\alpha \in \mathscr{A}[0,t]} \inf_{u \in \mathcal{U}[0,t]} \bar{J}_t[\psi](x, u, \alpha),$$
$$\bar{J}_t[\psi](x, u, \alpha) \doteq \int_0^t \bar{\ell}(x_s, u_s, \alpha_s)\, ds + \psi(x_t), \quad f \text{ linear in (3)}, \tag{18}$$
$$\bar{\ell}(x, \bar{u}, \bar{\alpha}) \doteq \tfrac{1}{2}\,\|x\|^2 + \tfrac{1}{2}\left[a^{-1}(\bar{\alpha})\,\|x\|^2 - \bar{\alpha}\right] + \tfrac{1}{2}\,\|\bar{u}\|^2,$$

for all $t \in \mathbb{R}_{\geq 0}$, $x \in \mathbb{R}^n$, $u \in \mathscr{U}[0, t] \doteq \mathscr{L}^2([0, t]; \mathbb{R}^m)$, $\alpha \in \mathscr{A}[0, t] \doteq \mathscr{L}^2([0, t]; \mathbb{R})$, $\bar{u} \in \mathbb{R}^m$, $\bar{\alpha} \in \mathbb{R}$.

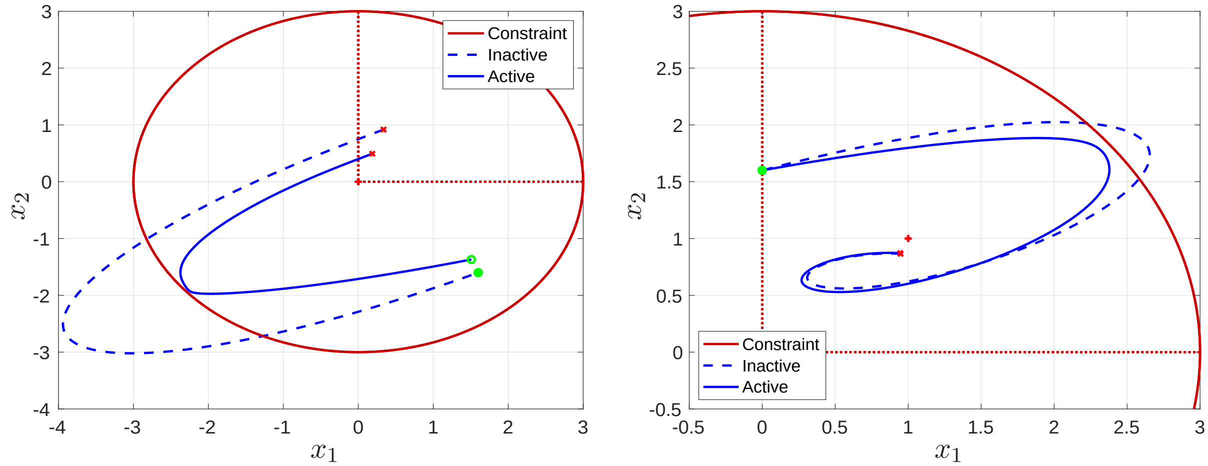


Figure 3: State trajectories, with state constraint inactive and active [P3].

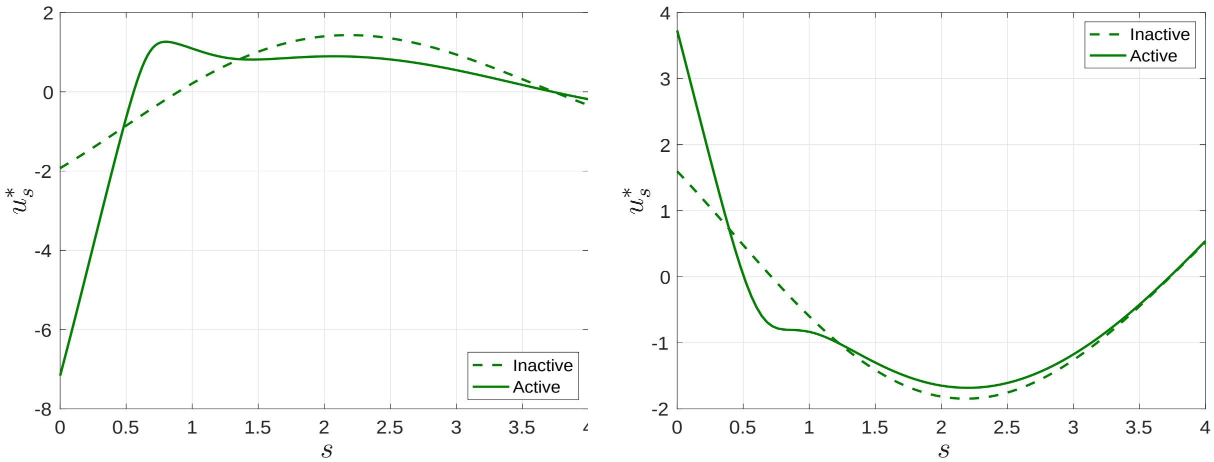


Figure 4: Optimal controls u^* [P3].

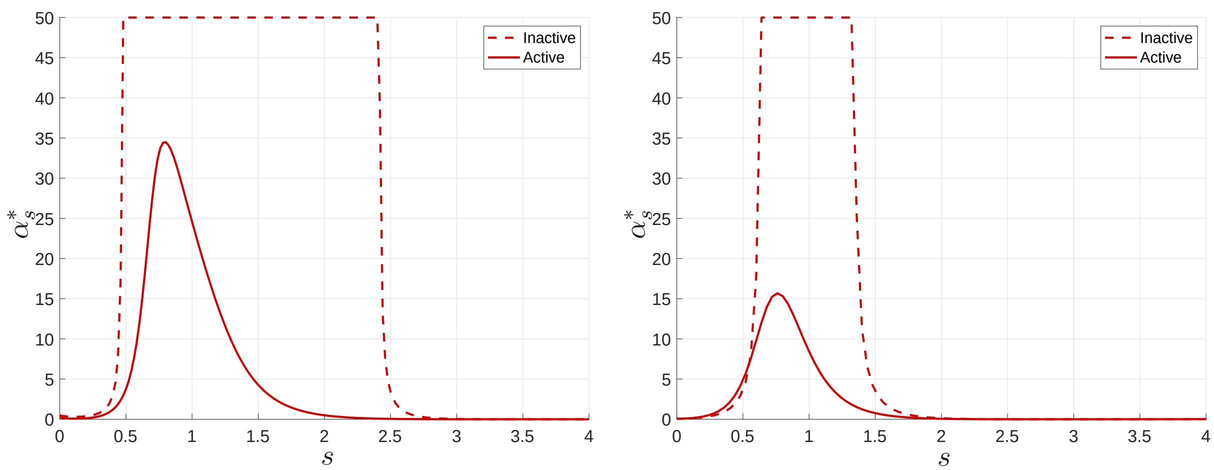


Figure 5: Actions of the maximizing player α^* [P3].

The approximate game value W_t^M is defined analogously, based on the approximate sup-of-quadratics representation in (17). In both games, the minimizing player corresponds to the usual control $s \mapsto u_s$, while the (new) maximizing player is an adversary that negotiates an appropriate quadratic state penalty via its actions $s \mapsto \alpha_s$, given the current state of the trajectory $s \mapsto x_s$ relative to the state constraint. As indicated in (18), the upper value of the game is equivalent to the value (1) of the original regulator problem, in the same quantitative manner regarding the value and the satisfaction of constraints. Moreover, the upper and lower values of the game are equivalent, which is crucial for computational purposes. Consideration of the lower value yields corresponding state feedback characterizations for the optimal policies of both players. These policies are shown to explicitly depend on the state dynamics driven by the optimal control, and solutions of a family of DREs associated with the LQR problems defined by $\inf_{u\in\mathcal{U}[0,t]} \bar{J}_t[\psi](x,u,\alpha)$, for any $\alpha \in \mathscr{A}[0,t]$ and ψ quadratic.

New theory and computational methods have been developed for solving state constrained regulator problems via semiconvex relaxations and lifting to games as outlined via (17), (18), see [P1, 3, 20, 30, 32, 33]. Figures 3, 4, and 5 illustrate optimal trajectories, controls, and actions of the maximizing player corresponding to specific initial and terminal data for a finite horizon state constrained regulator problem. The dashed and solid lines correspond to the constraint being inactive and active, respectively. Actions of the optimal control u^* and maximizing player α^* can be observed to preempt later interactions between the state trajectory and the constraint set boundary, see [P3].

2.5 Stationary action and stationary control

The action principle is a variational principle in modern physics that may be applied to a predefined notion of action to yield the equations of motion of a physical system and its underlying conservation laws. Using a suitable definition of this action, the action principle specializes to Hamilton's action principle [R8, 25], [P2, 23, 40], an important corollary of which states that *any trajectory of an energy conserving system renders the corresponding action functional stationary in the calculus of variation sense*. Hamilton's action principle can be interpreted as providing a characterization of all trajectories of an energy conserving or lossless system, including those constrained by boundary conditions. The action associated with Hamilton's action principle is defined as the integrated Lagrangian over the time horizon $t \in \mathbb{R}_{\geq 0}$ of interest, i.e.

$$\int_0^t T(\dot{x}_s) - V(x_s)\, ds, \qquad T(\dot{x}_s) \doteq \tfrac{1}{2}\, \langle \dot{x}_s, \mathcal{M}\, \dot{x}_s \rangle, \tag{19}$$

in which T and V denote the kinetic and potential energies corresponding to the generalized velocity $\dot{x}_s$ and position x_s, at time $s \in [0,t]$, and $\mathcal{M} \in \mathcal{L}(\mathcal{X})$ is a coercive operator corresponding to the generalized inertia for the energy conserving system of interest. Recalling (2), this form of the action motivates encapsulation of Hamilton's action principle within an optimal control problem of the form (1). To this end, appropriate choices for ℓ and f, g in (2) and (3) are

$$\ell(x) \doteq -V(x), \qquad f(x) \doteq 0,\ \ g(x) \doteq \mathcal{I} \in \mathcal{L}(\mathcal{X}), \tag{20}$$

for all $x \in \mathcal{X} = \mathcal{U}$. Where $\mathcal{X}$ is finite dimensional, given any convex terminal cost ψ, the restriction $u \mapsto J_t[\psi](x,u)$ of the cost (2) typically exhibits a minimum for all sufficiently short strictly positive time horizons $t \in \mathbb{R}_{>0}$, and indeed is often strongly convex [P40]. Consequently, the value function W_t of (1) is well-defined and satisfies the DPP (4) or non-stationary HJB PDE (5), and the optimal control exists and is unique. Moreover, the idempotent primal and dual space fundamental solution semigroups $\{\mathcal{G}_\tau^\oplus\}_{\tau\in[0,t]}$, $\{\mathcal{B}_\tau^\oplus\}_{\tau\in[0,t]}$ of (7), (10), (11) are applicable in its representation and propagation to longer time horizons. Meanwhile, boundary conditions for trajectories of interest can be encapsulated via the initial state $x_0 = x$ and terminal cost ψ, by way of the feedback characterization of the (unique) optimal control provided by (6). For example, with $\psi(x) \doteq \psi_v(x) \doteq \langle x, -\mathcal{M}\, v\rangle$, $x \in \mathcal{X}$, and $v \in \mathcal{X}$

fixed a priori, the optimal trajectory and optimal control corresponding to the generalized position and velocity respectively must satisfy the boundary conditions

$$x_0 = x, \qquad \dot{x}_t = u_t = -\mathcal{M}^{-1}\,\nabla W_0(x_t) = -\mathcal{M}^{-1}\,\nabla\psi_v(x_t) = -\mathcal{M}^{-1}\,(-\mathcal{M}\,v) = v. \tag{21}$$

This formulation of Hamilton's action principle, along with a semiconvex relaxation and lifting to a game problem as per (18), provides the foundation for the seminal development of a min-plus primal space fundamental solution semigroup for the gravitational N-body problem [P40] via (7), (11).

Where $\mathcal{X}$ is infinite dimensional [P2, 22, 23], or where the time horizon $t \in \mathbb{R}_{>0}$ is sufficiently long in the finite dimensional setting [P40], the restriction $u \mapsto J_t[\psi](x, u)$ of (2), (20) is not necessarily convex, and indeed can be concave in particular directions. In this case, stationarity in Hamilton's action principle need not be achieved at a minimum, i.e. via least action, but is instead achieved at a saddle [R8]. Moreover, the existence of such a saddle implies that the optimal control problem of (1), (20) is ill-posed insofar as it does not encapsulate or describe trajectories that render the action (19) stationary in the sense intended by Hamilton's action principle. Instead, in order to capture the desired trajectories, stationarity of the cost (2), (20) must be invoked directly, using an appropriate stationarity operation. Given any continuously Fréchet differentiable map $F : \mathcal{V} \to \mathcal{W}$ between Banach spaces $\mathcal{V}$ and $\mathcal{W}$, the *stat* operation [P2, 22, 23, 27, 31, 36] of interest is defined by

$$\operatorname*{stat}_{v\in\mathcal{V}} F(v) \doteq \left\{ F(\nu) \,\middle|\, \nu \in \operatorname*{arg\,stat}_{v\in\mathcal{V}} F(v) \right\}, \quad \operatorname*{arg\,stat}_{v\in\mathcal{V}} F(v) \doteq \left\{ \nu \in \mathcal{V} \,\middle|\, 0 = \lim_{\eta\to\nu} \frac{\|F(\eta) - F(\nu)\|_{\mathcal{W}}}{\|\eta - \nu\|_{\mathcal{V}}} \right\}. \tag{22}$$

This operation naturally generalizes min and max for continuously Fréchet differentiable functions, as any element of $\arg\min_{v\in\mathcal{V}} F(v)$ or $\arg\max_{v\in\mathcal{V}} F(v)$ must also be an element of $\arg\operatorname{stat}_{v\in\mathcal{V}} F(v)$. In addition, saddle points of F are also included in $\arg\operatorname{stat}_{v\in\mathcal{V}} F(v)$. Under appropriate regularity conditions on the potential V and the terminal payoff ψ, the restriction $u \mapsto J_t[\psi](x, u)$ associated with Hamilton's action principle via (20) is continuously Fréchet differentiable [P31]. Consequently, by choosing $\mathcal{V} \doteq \mathscr{U}[0, t]$, $\mathcal{W} \doteq \mathbb{R}$, and $F \doteq J_t[\psi](x, \cdot) : \mathcal{V} \to \mathcal{W}$ in (22), and imposing additional conditions to ensure that the argstat in (22) is a singleton [P31], the inf appearing in the definition (1) of the encapsulating optimal control problem may be replaced with the corresponding stat operation (22). This yields a finite horizon stationary control problem with value function $\widetilde{W}_t$ defined by

$$\widetilde{W}_t(x) \doteq (\widetilde{\mathcal{S}}_t\,\psi)(x) \doteq \operatorname*{stat}_{u\in\mathscr{U}[0,t]} J_t[\psi](x, u), \tag{23}$$

for all $t \in \mathbb{R}_{\ge 0}$, $x \in \mathcal{X}$, in which $\widetilde{\mathcal{S}}_\tau$ is the corresponding dynamic programming evolution operator. As anticipated by this definition of $\widetilde{\mathcal{S}}_\tau$, dynamic programming extends under some restrictions to cover this new class of stationary control problems, yielding a DPP of analogous form to (4), i.e. $\widetilde{W}_{\tau+t} = \widetilde{\mathcal{S}}_\tau\,\widetilde{W}_t$, for all $\tau, t \in \mathbb{R}_{\ge 0}$, see [P27, Theorem 7]. Similarly, the stationary value function $\widetilde{W}_t$ of (23) is characterized as the solution of a non-stationary HJB PDE analogously to (5), again subject to some restrictions, see [P27, Theorem 8]. Moreover, the semiconvex and semiconcave transforms $\mathcal{D}_\varphi^+$ and $\mathcal{D}_\varphi^-$ of (9) extend and unify into to a corresponding notion of duality, referred to as *static duality*, see [P15]. The associated static dual or transform motivates the definition of a restricted notion of static dual space fundamental solution semigroup, which in principle facilitates propagation of $\widetilde{W}_t$ of (23) to longer time horizons analogously to the idempotent dual space fundamental solution semigroup $\{\mathcal{B}_\tau^\oplus\}_{\tau\ge 0}$ of (10), see for example [P21, Theorem 3.8, (42)].

In characterizing the trajectories of interest via Hamilton's action principle and the value function (23), further insight into the structure of the argstat involved is available if the restriction $u \mapsto J_t[\psi](x, u)$ is convex for all sufficiently short horizons $t \in \mathbb{R}_{>0}$, e.g. where $\mathcal{X}$ is finite dimensional. In that case, any long horizon trajectory can be represented as a temporal concatenation of sufficiently many sufficiently short horizon trajectories, each of which is optimal in an appropriate short horizon optimal control problem (1). Consequently, as the long horizon trajectory cannot be optimal with respect to the long horizon optimal control problem (1), but rather is stationary in the sense of (22), (23), the loss of optimality on the long horizon must be confined to a finite collection of intermediate states that connect the short horizon trajectories. That is, elements of the argstat (22) implicit in (23) can be represented as piecewise optimal short horizon trajectories joined together by intermediate states that are stationary, and not optimal, in an appropriate sense [P23]. This insight is particularly useful in investigating a corresponding infinite dimensional setting for Hamilton's action principle.

In the infinite dimensional setting [R7,17], an illustrative energy conserving system is one which exhibits lossless wave dynamics. A wave equation that exhibits these dynamics is given by [P2,12,22,23],

$$\ddot{x} = -\Lambda\, x, \tag{24}$$

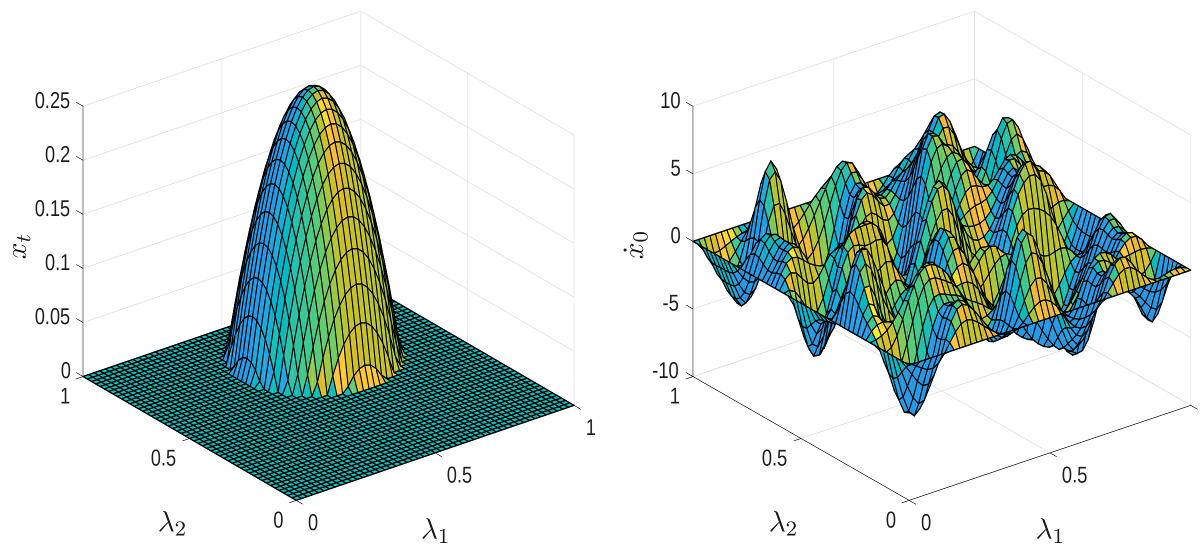
in which Λ is an unbounded, linear, invertible, positive, self-adjoint operator, with domain $\mathcal{H}_2 \doteq \mathrm{dom}\,(\Lambda)$ that is dense in an appropriately defined $\mathscr{L}^2$-space $\mathcal{H}$, and whose inverse $\Lambda^{-1} \in \mathcal{L}(\mathcal{H})$ is compact. Λ has a unique, positive, self-adjoint square-root $\Lambda^{\frac{1}{2}}$, the domain of which is a Hilbert space $\mathcal{H}_1 \doteq \mathrm{dom}\,(\Lambda^{\frac{1}{2}})$ with inner product $\langle x, y\rangle_1 \doteq \langle \Lambda^{\frac{1}{2}}\, x,\, \Lambda^{\frac{1}{2}}\, y\rangle$, $x, y \in \mathcal{H}_1$ derived from $\langle\, ,\rangle$ on $\mathcal{H}$. Motivated by (2), (19), (20), encapsulating optimal and stationary control problems (1), (23) follow via the selection of kinetic and potential energies T and V on $\mathcal{U} \doteq \mathcal{X} \doteq \mathcal{H}_1$, see [P2,12,22], with

$$\ell(x) \doteq -V(x) \doteq -\tfrac{1}{2}\,\|x\|_1^2, \qquad \mathcal{M} \doteq \mathcal{I} \in \mathcal{L}(\mathcal{X}), \qquad T(u) = \tfrac{1}{2}\,\|u\|^2, \tag{25}$$

for all $x, u \in \mathcal{X}$. Unlike the setting where $\mathcal{X}$ is finite dimensional, the restriction $u \mapsto J_t[\psi](x, u)$ obtained is not strongly convex, or even convex, for any sufficiently short time horizon $t \in \mathbb{R}_{>0}$. Indeed, there exist directions in which it is demonstrably concave [P2]. Consequently, the encapsulating optimal control problem of (1), (25) is ill-posed, as per the long horizon finite dimensional case outlined above, insofar as it does not encapsulate or describe trajectories that render the action (19) stationary in the required sense. This necessitates the consideration of a stationary control problem (23).

As indicated in the discussion surrounding (21), boundary conditions can be imposed to restrict attention to trajectories that satisfy TPBVPs, and this extends to the infinite dimensional setting. Moreover, replacing the unbounded operator Λ in (24) with its bounded Yosida approximation [R20] yields a parameterized family of approximate wave equations of the same form. These approximate wave equations are also lossless in an appropriate sense, and the corresponding action can be formalized by replacing the kinetic energy T in (19), (25) with $T^\mu(u) \doteq \frac{1}{2}\,\|u\|^2 + \frac{\mu}{2}\,\|u\|_1^2$, $u \in \mathcal{U} \doteq \mathcal{X} \doteq \mathcal{H}_1$, in which $\mu \in \mathbb{R}_{\geq 0}$ is an approximation parameter. The approximate encapsulating optimal control problem and stationary control problem formulations follow as per (1), (23), with the crucial difference being that the corresponding approximate restriction $u \mapsto J_t^\mu[\psi](x, u)$ of the cost (2) recovers the strong convexity behaviour observed in the finite dimensional setting [P2,12,22,23]. This facilitates the short horizon trajectory concatenation approach outlined in that setting to be applied, yielding trajectories that satisfy TPBVPs constrained by the aforementioned approximate wave equation. The Trotter-Kato theorem [R20] subsequently provides the machinery required to guarantee that these trajectories converge to those of the exact problem in the limit of the approximation parameter tending to zero. The argument involved applies for any boundary conditions, and facilitates the construction of the fundamental solution group generated by Λ. This construction is documented in detail in [P2,23].

Figure 6 illustrates an example in which $-\Lambda$ is the Laplacian on $X \doteq [0,1]^2 \subset \mathbb{R}^2$, with $\mathcal{H} \doteq \mathscr{L}^2(X;\mathbb{R})$, $\mathcal{H}_1 \doteq \mathrm{dom}\,(\Lambda^{\frac{1}{2}})$, $\mathcal{H}_2 \doteq \mathscr{H}_0^2(X;\mathbb{R})$, and $\mathcal{U} \doteq \mathcal{X} \doteq \mathcal{H}_1$. With time horizon $t \doteq \frac{\pi}{3}$, two boundary conditions are specified, i.e. $x_0 \doteq 0 \in \mathcal{X}$, and $\dot{x}_t \in \mathcal{U}$ as illustrated in Figure 6(a), with the remaining initial data $\dot{x}_0$ computed using the above construction [P2] and illustrated in Figure 6(b).



(a) Desired terminal state $\lambda \mapsto x_t(\lambda)$, $\lambda \in X$. (b) Constructed initial velocity $\lambda \mapsto \dot{x}_0(\lambda)$, $\lambda \in X$.

Figure 6: Boundary conditions in a stationary action control problem involving (24), see [P2].

2.6 Optimization-based optimal feedback synthesis

In a standard reparameterization [R15], the finite horizon value function W_t of (1) can be recast as a function of an initial time $s \in [0, t]$ rather than the time horizon $t \in \mathbb{R}_{\geq 0}$, via

$$U_s(x) \doteq W_{t-s}(x) \tag{26}$$

for all $s \in [0, t]$, $x \in \mathcal{X}$, in which the time horizon t is now implicit in U_s. This reparameterized value function is characterized as the solution of a corresponding DPP and as the viscosity solution of a non-stationary HJB PDE, c.f. (4), (5), with the latter given by

$$0 = -\frac{\partial U_s}{\partial s}(x) + H(x, \nabla U_s(x)), \qquad U_t(x) = \psi(x), \tag{27}$$

for all $s \in [0, t]$, $x \in \mathcal{X}$, in which H is the same convex Hamiltonian from (5). The characteristic system [R11, 15, 18, 21, 23] associated with HJB PDE (27) is given by

$$\begin{cases} \dot{x}_r = -\nabla_p H(x_r, p_r), & x_t = y, \\ \dot{p}_r = \nabla_x H(x_r, p_r), & p_t = \nabla \psi(y), \\ \dot{z}_r = -\langle p_r, \nabla_p H(x_r, p_r) \rangle + H(x_r, p_r), & z_t = \psi(y), \end{cases} \tag{28}$$

for all $r \in [0, t]$, $y \in \mathcal{X}$. Using (28), the value function U_s of (26) is equivalently characterized as the unique minimax solution [R18] of HJB PDE (27). In particular, it has the equivalent form

$$\begin{aligned} U_s(x) &= \inf_{y \in \mathcal{Y}_s(x)} J_s^{\sim}[\psi](x, y), \qquad \mathcal{Y}_s(x) \doteq \{ y \in \mathcal{X} \,|\, x_s = x, \; x_t = y \}, \\ J_s^{\sim}[\psi](x, y) &\doteq z_s = \psi(y) + \int_s^t l(x_r) + \tfrac{1}{2} \langle \mathcal{K}(x_r, p_r), \, \mathcal{M}\, \mathcal{K}(x_r, p_r) \rangle \, dr, \quad x_s = x, \; y \in \mathcal{Y}_s(x) \end{aligned} \tag{29}$$

for all $s \in [0, t]$, $x \in \mathcal{X}$, in which $r \mapsto (x_r, p_r, z_r)$ satisfies the characteristic system (28), and the feedback map $\mathcal{K}$ is as per (6). This minimax solution characterization provides an optimization-based representation for the value function U_s of (26), in which the restriction $y \mapsto J_s^{\sim}[\psi](x,y)$ of (29), equivalent to (2), is minimized over all admissible terminal states $y \in \mathcal{Y}_s(x)$ of TPBVP solutions defined with respect to the characteristic system (28). Motivated by (29), see [P17], an approximation $\widehat{U}_s$ for the minimax solution U_s is obtained by reparameterizing the characteristic field of (28) with respect to the initial costate $p_s = p \in \mathcal{X}$, rather than the terminal state $x_t = y \in \mathcal{X}$. In particular,

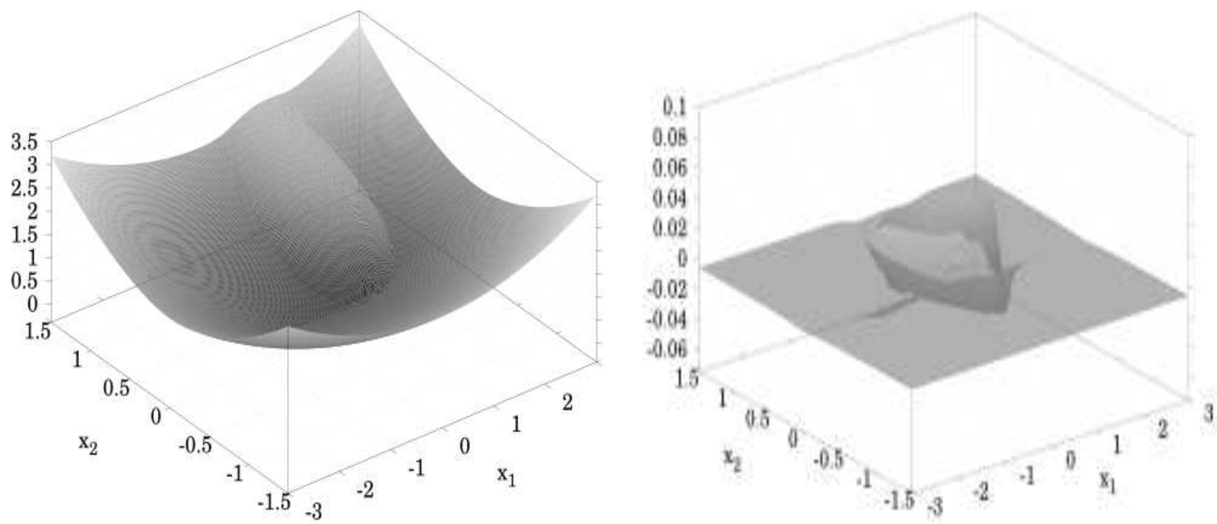
$$\widehat{U}_s(x) \doteq \inf_{p \in \mathcal{X}} \widehat{J}_s[\psi](x,p),$$

$$\widehat{J}_s[\psi](x,p) \doteq \psi(x_t) + \int_s^t l(x_r) + \tfrac{1}{2} \langle \mathcal{K}(x_r,p_r),\, \mathcal{M}\, \mathcal{K}(x_r,p_r) \rangle \, dr, \quad x_s = x,\ p_s = p,$$

for all $s \in [0,t]$, $x, p \in \mathcal{X}$, subject to the dynamics in (28). The corresponding approximation of the value function (1) and associated feedback control policy approximating (6) are given by

$$\widehat{W}_t(x) = \widehat{U}_{t-s}(x), \qquad \widehat{K}_{t-s}(x) \doteq \mathcal{K}\left(x,\, \operatorname*{arg\,min}_{p \in \mathcal{X}} \widehat{J}_s[\psi](x,p)\right), \tag{30}$$

for all $s \in [0,t]$, $x \in \mathcal{X}$. Details concerning the development and application of this approach to standard finite horizon optimal control problems, and in particular its scalability with respect to state space dimension, are documented in [P17]. Figure 7 illustrates the approximate value function $\widehat{W}_t$ of (30) computed for a specific finite horizon optimal control problem with state space dimension $n \doteq 5$, see [P17]. An extension of this approach to the synthesis of control Lyapunov functions via Zubov's method [R5] is explored in [P9, 18, 19].



(a) Value function slice.

(b) A posteriori error slice.

Figure 7: Approximately solving an Eikonal equation with $\mathcal{X} \doteq \mathbb{R}^5$ [P17].

2.7 Auxiliary advances

This project has also supported a range of auxiliary advances, including in extremum seeking feedback control for dynamical systems evolving on manifolds [P16], networked control and observer design, [P6, 7, 14, 26, 35], quantum control [P28, 37, 38], and stability theory [P5, 25, 29, 34, 39]. The details are diverse and are omitted for brevity.

3 References, publications, invited presentations, and organized activities

3.1 References

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3.2 Publications

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[p2] P.M. Dower and W.M. McEneaney. Verifying fundamental solution groups for lossless wave equations via stationary action and optimal control. *Preprint arXiv:1906.03592 (in review)*, 2019.

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3.3 Invited presentations

[I1] P.M. Dower. Sparse approximation methods for optimal control: basis adaptation for max-plus eigenvector methods. In *Monterey Workshop on Computational Control (Monterey CA)*, 2019.

[I2] P.M. Dower. Sparse approximation methods for optimal control. In *AFOSR Dynamics & Control Program Review (Arlington VA, USA)*, 2019.

[I3] P.M. Dower, W.M. McEneaney, and I. Yegorov. Exploiting characteristics in stationary action problems. In *SIAM Conference on Control & Its Applications (Chengdu, China)*, 2019.

[I4] P.M. Dower. Basis adaptation for a max-plus eigenvector method arising in optimal control. In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Hong Kong)*, 2018.

[I5] P.M. Dower. Exploiting characteristics in approximating feedback solutions to optimal control problems. In *AFOSR Dynamics & Control Program Review (Arlington VA, USA)*, 2018.

[I6] P.M. Dower. Fundamental solutions for optimal control. In *Australian Applied Dynamics Workshop (Port Stephens, Australia)*, 2018.

[I7] P.M. Dower and W.M. McEneaney. An action principle for constructing fundamental solution groups for wave equations. In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Hong Kong)*, 2018.

[I8] L. Grüne and P.M. Dower. Hamiltonian based a posteriori error estimation for Hamilton-Jacobi-Bellman equations. In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Hong Kong)*, 2018.

[I9] I. Yegorov, P.M. Dower, and L. Grüne. A characteristics based curse-of-dimensionality-free approach for approximating control Lyapunov functions and feedback stabilization. In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Hong Kong)*, 2018.

[I10] P.M. Dower. Exploiting semiconvex relaxations in optimal control. In *Australian Applied Dynamics Workshop (Geelong, Australia)*, 2017.

[I11] P.M. Dower. Idempotent methods for regulator problems. In *AFOSR Dynamics & Control Program Review (Arlington VA, USA)*, 2017.

[I12] P.M. Dower. Max-plus fundamental solution semigroups for optimal control. In *Workshop on numerical methods for optimal control problems; algorithms, analysis and applications (NUMOC, Rome, Italy)*, 2017.

[I13] P.M. Dower and W.M. McEneaney. Representation of fundamental solution groups for wave equations via stationary action & optimal control. In *IEEE American Control Conference (Seattle WA)*, 2017.

[I14] P.M. Dower, W.M. McEneaney, and R. Zhao. HJBs for two-point boundary value problems constrained by conservative dynamics. In *SIAM Conference on Control & Its Applications (Pittsburgh, PA)*, 2017.

[I15] W.M. McEneaney and P.M. Dower. The principle of least action and fundamental solutions of mass-spring and n-body two-point boundary value problems. In *SIAM Conference on Control & Its Applications (Pittsburgh, PA)*, 2017.

[I16] P.M. Dower. Idempotent methods for worst-case analysis & optimal control. In *AFOSR Dynamics & Control Program Review (Arlington VA, USA)*, 2016.

[I17] P.M. Dower. Semiconvexity and optimal control. In *Australian Applied Dynamics Workshop (Katoomba, Australia)*, 2016.

[I18] P.M. Dower and W.M. McEneaney. A dynamic game approximation for a linear regulator problem with log-barrier state constraint. In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Groningen, The Netherlands)*, 2016.

[I19] W.M. McEneaney and P.M. Dower. The Hamilton-Jacobi equation corresponding to complex-valued stationary action problems. In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Groningen, The Netherlands)*, 2016.

3.4 Organized activities

[A1] P.M. Dower and W.M. McEneaney. Co-organizer, mini-symposium on *Hamilton-Jacobi-Bellman equations in physics and optimal control.* In *SIAM Conference on Control & Its Applications (Chengdu, China)*, 2019.

[A2] P.M. Dower and W.M. McEneaney. Co-organizer, mini-symposium on *Optimal control and Hamilton-Jacobi-Bellman equations.* In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Hong Kong)*, 2018.

[A3] C.M. Kellett, J. Trumpf, M. Cantoni, and P.M. Dower. Organizing committee, *Australian Applied Dynamics Workshop (Newcastle, Australia)*, 2018.

[A4] W.M. McEneaney and P.M. Dower. Co-organizer, mini-symposium on *Stochastic approaches and Hamilton-Jacobi equations*. In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Hong Kong)*, 2018.

[A5] C.M. Kellett, J. Trumpf, M. Cantoni, P.M. Dower, and L. Ntogramatzidis. Organizing committee, *Australian Applied Dynamics Workshop (Geelong, Australia)*, 2017.

[A6] C.M. Kellett, J. Trumpf, M. Cantoni, P.M. Dower, and L. Ntogramatzidis. Organizing committee, *Australian Applied Dynamics Workshop (Katoomba, Australia)*, 2016.

[A7] W.M. McEneaney and P.M. Dower. Co-organizer, mini-symposium on *Max-plus and idempotent techniques in dynamical systems, control & games*. In *International Symposium on Mathematical Theory of Networks and Systems (MTNS, Minneapolis MN)*, 2016.

4 Supported personnel

Vincenzo Basco	Research Fellow, University of Melbourne, Australia	(2019 – 2020)
Ivan Egorov (Yegorov)	Research Fellow, University of Melbourne, Australia	(2017 – 2018)
Peter M. Dower	Associate Professor, University of Melbourne, Australia	(2016 – 2019)

5 Honors and awards

2017 SIAM SICON Best Paper Prize for the seminal development a new min-plus primal space fundamental solution for the gravitational N-body problem, [P40]; see also

`https://sinews.siam.org/Details-Page/prize-spotlight-william-mceneaney-and-peter-dower`.

6 Transitions

Transitions are limited to the scholarly articles, invited presentations, and organised activities listed in Section 3.

7 New discoveries

New discoveries are summarized in Sections 1 and 2.

8 AFRL points of contact

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