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Tube Shape from Axial Strain Data

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ABSTRACT

This paper shows how the shape of a cylindrical tube can be determined from triplets of axial strain sensors located at numerous axial positions on the outer diameter of the tube.

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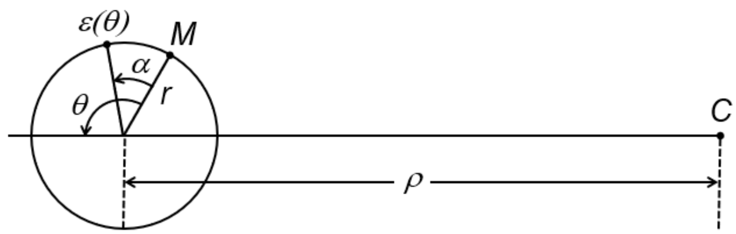
CURVATURE FROM STRAIN DATA

The first challenge is to calculate the local tube curvature from strain data. It will be assumed that at numerous axial positions along the tube there will be a triplet of three fiber optic strain sensors placed at the outer diameter of the tube with a 120° angle between them. P.-L Schaefer et al. derive expressions for the strain for a triplet of finite-width strain sensors [1]. Their results, however, can be applied in the limit for infinitesimally narrow strain sensors, which will be assumed here.

The following sketch shows a cross section of the tube with an axial strain sensor labeled $\varepsilon(\theta)$ located on the outer diameter:



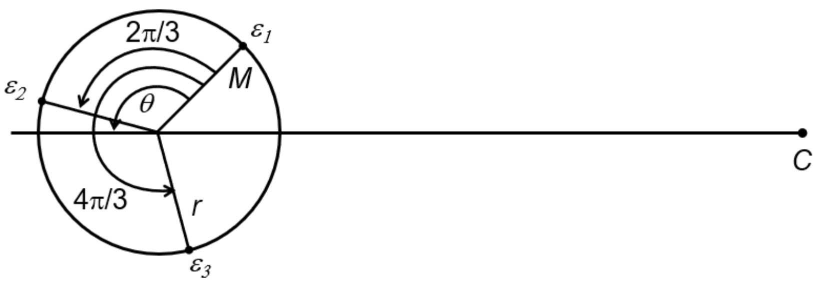
The cross section has radius r and is orthogonal to the plane of bending, with C denoting the center of curvature and ρ the radius of curvature. The axial strain sensor is shown at an angle α to a reference point M in the cross section plane. The plane of bending makes an angle θ with the reference point M.

It can be shown that the axial strain at θ can be written as

$$\varepsilon(\theta) = r\kappa \cos(\theta - \alpha) + \delta$$

Where $\kappa = 1/\rho$ is the curvature and δ is the bias due to deformations other than bending (which we expect to be minimal).

Consider now the case of three axial strain sensors placed at the outer diameter of the tube cross section with a 120° angle between them and oriented so that sensor 1 is at the reference point M as shown here:



Thus $\alpha_1 = 0$, $\alpha_2 = \frac{2\pi}{3}$, and $\alpha_3 = \frac{4\pi}{3}$.

The strains for the three sensors can then be written

$$\varepsilon_1 = r\kappa\cos\theta + \delta$$

$$\varepsilon_2 = r\kappa\cos\left(\theta - \frac{2\pi}{3}\right) + \delta$$

$$\varepsilon_3 = r\kappa\cos\left(\theta - \frac{4\pi}{3}\right) + \delta$$

These three equations can be solved for curvature κ, bending plane angle θ, and bias δ:

$$\kappa = \frac{C}{3r}$$

$$\cos\theta = \frac{2\varepsilon_1 - \varepsilon_2 - \varepsilon_3}{C}$$

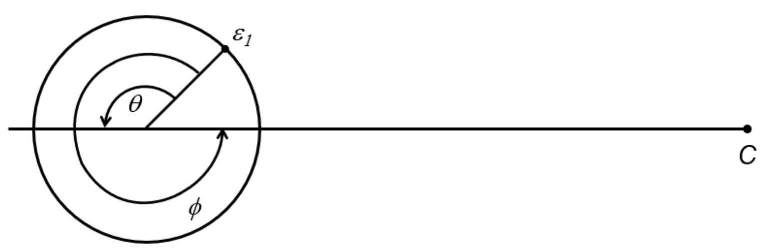
$$\sin\theta = \frac{\sqrt{3}(\varepsilon_2 - \varepsilon_3)}{C}$$

$$\delta = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{C}$$

where

$$C = \sqrt{2[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_1 - \varepsilon_3)^2]}$$

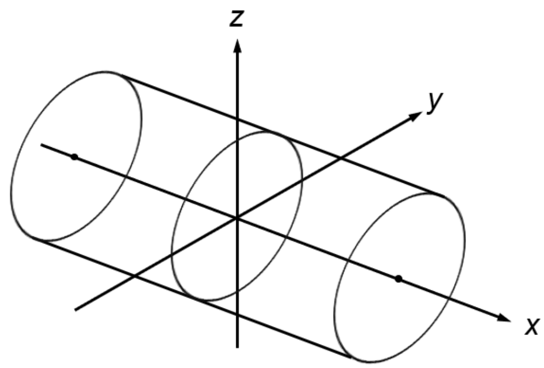
Note that the angle θ above is the angle of the bending plane relative to sensor 1 – but it is the angle of the direction ***away from*** the center of curvature. The sketch below shows ϕ, the angle that the direction of positive curvature makes with sensor 1:



Thus

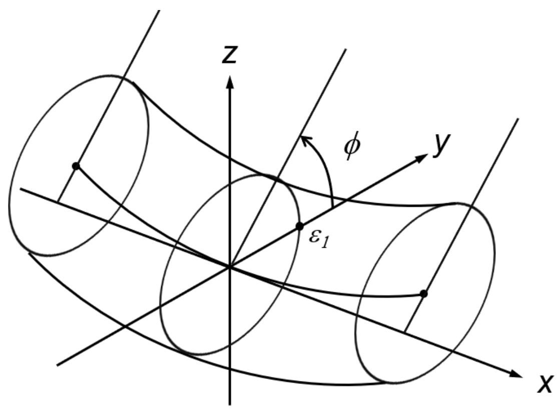
$$\phi = \theta + \pi$$

It will be convenient to consider the tube curvature (and ultimately tube shape) independently in two planes perpendicular to the axis of the tube. The tube axis will be assumed to be x, with the yz-plane perpendicular to the tube axis as shown below:



We will then consider separately the curvature in the xy-plane and the curvature in the xz-plane at the various strain locations along x.

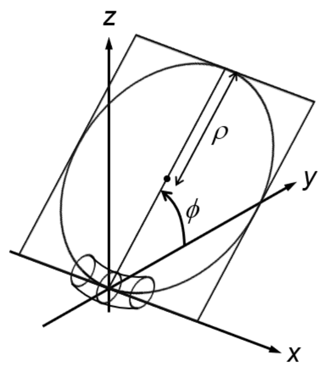
The results above give the curvature κ and the angle ϕ that the center of curvature in the bending plane makes with sensor 1 in the cross section of the tube as shown below. For convenience, we may assume that sensor 1 is on the y-axis.



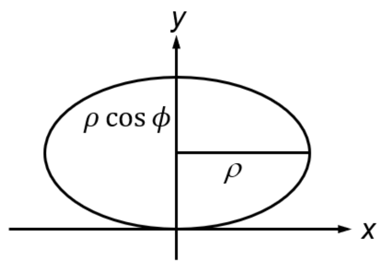
(Although ϕ was shown between π and 2π in the earlier sketch, it can be shown as an angle less than $\pi/2$ without loss of generality.)

It now remains to go from curvature κ in the plane of bending to curvatures κ_y in the xy-plane and κ_z in the xz-plane.

As shown here, the curvature κ is defined by a circle of radius ρ in a plane containing the x-axis (tube axis) but tilted at an angle ϕ to the y-axis:



The projection of the circle in the xy-plane is an ellipse of major axis ρ and minor axis $\rho \cos \phi$ as shown here:



It can be shown that for small values of x, the equation of the ellipse in the vicinity of the tangent point is approximately

$$y = \frac{\cos \phi}{2\rho} x^2$$

The curvature κ_y, equal to the second derivative of y with respect to x, is then

$$\kappa_y = \frac{\cos \phi}{\rho} = \kappa \cos \phi$$

Likewise, the projection of the circle in the xz-plane will be an ellipse of major axis ρ and minor axis $\rho \sin \phi$, and the curvature κ_z can be shown to be

$$\kappa_z = \kappa \sin \phi$$

TUBE SHAPE FROM CURVATURE

The tube shape will be considered independently in the two planes perpendicular to the axis of the tube. The tube axis will be assumed to be x and, in the treatment to follow, the deflection in either of the two transverse directions, y or z, will be v.

This development largely follows that of Glaser, Caccese, and Shahinpoor [2].

The tube will be divided into m segments and $m-1$ nodes between the segments, plus 2 additional nodes at each end of the tube. It is assumed that axial strain sensors at the $m-1$ nodes between elements will provide curvature at each node location.

The transverse displacement of the tube will be approximated as a cubic spline, with each segment represented by a cubic polynomial:

$$v(x) = \sum_{i=1}^{m} \sum_{j=0}^{3} A_{ij} x^j = \sum_{i=1}^{m} (A_{i0} + A_{i1}x + A_{i2}x^2 + A_{i3}x^3)$$

The origin of the x axis can be the left end of the tube for all segments, or each segment can have its own x-axis origin at the left end of the segment. The latter assumption will result in easier conceptualization and a sparser matrix for solution and will be used here. A set of simple axis transformations can provide tube shape relative to a single x-axis origin at the left end of the tube.

The various coefficients A_{ij} can be found by applying the following:

- The continuity equations for $v(x)$, $v'(x)$, and $v''(x)$ at each node between segments,
- The known values of the curvatures $v''(x)$ at each node between segments (where the sensors are located), and
- Four boundary conditions.

For our purposes, two boundary conditions would be an assumed displacement of zero at the left end of the tube and the displacement of the tube at one other point along the tube. The other two boundary conditions would be the zero curvatures assumed at the first and last nodes.

The cubic equation for each individual segment i is

$$v_i(x) = A_{i0} + A_{i1}x + A_{i2}x^2 + A_{i3}x^3$$

The slope is given by

$$v_i'(x) = A_{i1} + 2A_{i2}x + 3A_{i3}x^2$$

The curvature is given by

$$v_i''(x) = 2A_{i2} + 6A_{i3}x$$

Ultimately, the result of applying the continuity equations, curvature data, and boundary conditions will be a matrix equation of the form

$$[S] \cdot \{A\} = \{B\}$$

where

$[S]$ is a system matrix combining the continuity equations, curvature data, and boundary conditions;

$\{A\}$ is the vector containing the coefficients A_{ij}:

$$\{A\} = \{A_{10}, A_{11}, A_{12}, A_{13}, A_{20}, A_{21}, A_{22}, A_{23}, \ldots A_{m0}, A_{m1}, A_{m2}, A_{m3}\}$$

$\{B\}$ is a vector containing (a) zeros for each continuity equation, (b) the measured curvatures at each node between segements, and (c) zeros for each boundary condition.

The detailed forms of $[S]$ and $\{B\}$ will be shown.

CONTINUITY EQUATIONS

To illustrate construction of the $[S]$ matrix and $\{B\}$ vector, start with the continuity equations at the second node, between segments 1 and 2. Let the length of each segment be given by L_i.

Continuity of displacement gives

$$v_1(L_1) = v_2(0)$$

or

$$v_1(L_1) - v_2(0) = 0$$

Using

$$v_1(L_1) = A_{10} + A_{11}L_1 + A_{12}{L_1}^2 + A_{13}{L_1}^3$$

and

$$v_2(0) = A_{20}$$

the continuity equation becomes

$$A_{10} + A_{11}L_1 + A_{12}{L_1}^2 + A_{13}{L_1}^3 - A_{20} = 0$$

Similarly, the continuity of slope at the second node gives

$$v_1'(L_1) - v_2'(0) = 0$$

Using

$$v_1'(L_1) = A_{11} + 2A_{12}L_1 + 3A_{13}{L_1}^2$$

and

$$v_2'(0) = A_{21}$$

the continuity of slope gives

$$A_{11} + 2A_{12}L_1 + 3A_{13}{L_1}^2 - A_{21} = 0$$

Similarly, the continuity of curvature at the second node gives

$$v_1''(L_1) - v_2''(0) = 0$$

Using

$$v_1''(L_1) = 2A_{12} + 6A_{13}L_1$$

and

$$v_2''(0) = 2A_{22}$$

the continuity of curvature gives

$$2A_{12} + 6A_{13}L_1 - 2A_{22} = 0$$

Putting these three relations into matrix form:

$$\begin{bmatrix} 1 & L_1 & {L_1}^2 & {L_1}^3 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2L_1 & 3{L_1}^2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 6L_1 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \\ A_{20} \\ A_{21} \\ A_{22} \\ A_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Likewise, the continuity relations at the third node (between segments 2 and 3) give:

$$\begin{bmatrix} 1 & L_2 & {L_2}^2 & {L_2}^3 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2L_2 & 3{L_2}^2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 6L_2 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} A_{20} \\ A_{21} \\ A_{22} \\ A_{23} \\ A_{30} \\ A_{31} \\ A_{32} \\ A_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In general, then, continuity of displacement, slope, and curvature at each node between segments provides the following matrix equations for $i = 1$ to $m - 1$:

$$\begin{bmatrix} 1 & L_i & {L_i}^2 & {L_i}^3 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2L_i & 3{L_i}^2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 6L_i & 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} A_{i0} \\ A_{i1} \\ A_{i2} \\ A_{i3} \\ A_{i+1,0} \\ A_{i+1,1} \\ A_{i+1,2} \\ A_{i+13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These can be considered as part of the following larger matrix equation:

$$\begin{bmatrix} 1 & L_1 & {L_1}^2 & {L_1}^3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2L_1 & 3{L_1}^2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 6L_1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & L_2 & {L_2}^2 & {L_2}^3 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 2L_2 & 3{L_2}^2 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6L_2 & 0 & 0 & -2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \\ A_{20} \\ A_{21} \\ A_{22} \\ A_{23} \\ A_{30} \\ A_{31} \\ A_{32} \\ A_{33} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

There are three continuity equations at each of the $m - 1$ nodes between segments, leading to a total of $3(m - 1)$ continuity equations.

Thus the larger matrix above would have $3(m - 1)$ rows (corresponding to the number of equations) and $4m$ columns (corresponding to the number of unknown A_{ij} coefficents).

CURVATURE EQUATIONS AT NODES BETWEEN SEGMENTS

At the second node, between the first and second segments, both of the following are true:

$$v_1''(L_1) = 2A_{12} + 6A_{13}L_1 = \kappa_2$$

$$v_2''(0) = 2A_{22} = \kappa_2$$

It is simpler to use the second expression. Then, in matrix form

$$\begin{bmatrix} 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} A_{20} \\ A_{21} \\ A_{22} \\ A_{23} \end{bmatrix} = \kappa_2$$

Similarly, at the third node, between the second and third segments,

$$v_3''(0) = 2A_{32} = \kappa_3$$

In matrix form,

$$\begin{bmatrix} 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} A_{30} \\ A_{31} \\ A_{32} \\ A_{33} \end{bmatrix} = \kappa_3$$

In general, then, the curvature equations at each node between segments provides the following matrix equations for $i = 2$ to m:

$$\begin{bmatrix} 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} A_{i0} \\ A_{i1} \\ A_{i2} \\ A_{i3} \end{bmatrix} = \kappa_i$$

There is one curvature equation at each of the $m - 1$ nodes between segments, leading to a total of $m - 1$ such curvature equations.

DISPLACEMENT BOUNDARY CONDITIONS

The displacement boundary condition at node 1 is that the displacement is 0. Recall that

$$v_1(x) = A_{10} + A_{11}x + A_{12}x^2 + A_{13}x^3$$

Then

$$v_1(0) = A_{10} = 0$$

In matrix form,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = 0$$

The second displacement boundary condition would be the displacement at some point P along the tube. The exact form of the equation would depend in which segment this point lies. For now, we can assume it lies in the first segment. Furthermore, without loss of generality , we can assume the displacement at this point is 0. Then

$$v_1(P) = A_{10} + A_{11}P + A_{12}P^2 + A_{13}P^3 = 0$$

In matrix form,

$$\begin{bmatrix} 1 & P & P^2 & P^3 \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = 0$$

CURVATURE BOUNDARY CONDITIONS

The curvature boundary condition at node 1 is that the curvature is 0. Recall that

$$v_i''(x) = 2A_{i2} + 6A_{i3}x$$

Then

$$v_1''(0) = 2A_{12} = 0$$

In matrix form

$$\begin{bmatrix} 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = 0$$

The curvature boundary condition at the last node, node $m + 1$, would be written in terms of the curvature expression for segment m as follows

$$v_m''(L_m) = 2A_{m2} + 6A_{m3}L_m = 0$$

In matrix form

$$\begin{bmatrix} 0 & 0 & 2 & 6L_m \end{bmatrix} \begin{bmatrix} A_{m0} \\ A_{m1} \\ A_{m2} \\ A_{m3} \end{bmatrix} = 0$$

COMPLETE MATRIX EQUATION

From above:

- Number of continuity equations $= 3(m - 1)$
- Number of curvature equations at nodes between segments $= m - 1$
- Number of boundary conditions $= 4$

Thus the total number of equations

$$= 3(m - 1) + (m - 1) + 4$$

$$= 4m$$

This is equal to the total number of unknown A_{ij} coefficents.

All of the above equations can be combined into a single matrix equation of the following form:

$$\begin{bmatrix} 1 & L_1 & {L_1}^2 & {L_1}^3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 2L_1 & 3{L_1}^2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6L_1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L_2 & {L_2}^2 & {L_2}^3 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2L_2 & 3{L_2}^2 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6L_2 & 0 & 0 & -2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & P & P^2 & P^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2 & 6L_m \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \\ A_{20} \\ A_{21} \\ A_{22} \\ A_{23} \\ A_{30} \\ A_{31} \\ A_{32} \\ A_{33} \\ \cdots \\ A_{m0} \\ A_{m1} \\ A_{m2} \\ A_{m3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ \kappa_2 \\ \kappa_3 \\ \cdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Or

$$[S] \cdot \{A\} = \{B\}$$

The first $3(m-1)$ rows of $[S]$ (the first 6 rows of which are shown) represent the continuity equations. Note that the corresponding elements of $\{B\}$ are all zero.

The next $m-1$ rows of $[S]$ (the first 2 rows of which are shown) represent the curvature equations at nodes between segments. Note that the corresponding elements of $\{B\}$ are the measured curvatures at each of those nodes.

The last four rows represent the two displacement boundary conditions followed by the two curvature boundary conditions. Note that the corresponding elements of $\{B\}$ are all zero.

Since $[S]$ is a square matrix, the above equation can be solved for $\{A\}$ as follows:

$$\{A\} = [S]^{-1} \cdot \{B\}$$

In principle, this can be solved using standard matrix solution algorithms.

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