

ROUTING AND ACTION

MEMORANDUM

ROUTING

TO:(1) Terminal Effects Branch (TE) (Ford, Denise)

Report is available for review

(2) Proposal Files Report No.:

Proposal Number: 71630-TE.12

DESCRIPTION OF MATERIAL

CONTRACT OR GRANT NUMBER: W911NF-18-1-0023

INSTITUTION: University of South Carolina

PRINCIPAL INVESTIGATOR: Addis Kidane

TYPE REPORT: Final Report

DATE RECEIVED: 1/24/23 7:22PM

PERIOD COVERED: 11/29/17 12:00AM through 8/31/22 12:00AM

TITLE: Final Report: Optimal Functionally Graded Cellular Materials (FGCMs) to Mitigate Shock and Impact Loading

ACTION TAKEN BY DIVISION

(x) Report has been reviewed for technical sufficiency and IS [x] IS NOT [] satisfactory.

() Based on my technical review, I have identified no OPSEC or Technology Protection concerns that need to be addressed regarding this report.

(x) Performance of the research effort was accomplished in a satisfactory manner and all other technical requirements have been fulfilled.

(x) Based upon my knowledge of the research project, I agree with the patent information disclosed.

Approved by SSL\DENISE.FORD on 5/23/23 5:45PM

ARO FORM 36-E

REPORT DOCUMENTATION PAGE			Form Approved OMB NO. 0704-0188		
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14. ABSTRACT					
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a. REPORT UU	b. ABSTRACT UU	c. THIS PAGE UU			19b. TELEPHONE NUMBER 803-777-2502

RPPR Final Report

as of 23-May-2023

Agency Code: 21XD

Proposal Number: 71630TE

Agreement Number: W911NF-18-1-0023

INVESTIGATOR(S):

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DUNS Number: 041387846

EIN: 576001153

Report Date: 30-Nov-2022

Date Received: 24-Jan-2023

Final Report for Period Beginning 29-Nov-2017 and Ending 31-Aug-2022

Title: Optimal Functionally Graded Cellular Materials (FGCMs) to Mitigate Shock and Impact Loading

Begin Performance Period: 29-Nov-2017

End Performance Period: 31-Aug-2022

Report Term: 0-Other

Submitted By: Addis Kidane

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Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

STEM Degrees: 1

STEM Participants: 4

Major Goals: Objective

The proposed research will open a new era in understanding cellular materials subjected to high-strain rate loading. It will attempt to answer some fundamental questions such as (a) What are the primary mechanics responsible for failure initiation and propagation in cellular materials subjected to very high-rate loading conditions, (b) What are the important material properties required to develop a physics-based constitutive model that governs the mechanics of cellular material subjected to different loading conditions? In general, the proposed study will focus on understanding multi-scale deformation behavior that leads to the failure of cellular materials under impact and shock loading conditions. Understanding the deformation mechanisms and how they are affected by the increased compressibility expected in cellular materials, the effect of wave speed and strain rate on the elastic and plastic properties of individual struts and global cellular structures of different initial density foams can provide key design concepts in the development of optimal FGMCs that can be used in different applications. Unlike traditional cellular materials, the proposed FGMC material systems will be tailored to absorb more energy when subjected to shock and impact loading while resisting fragmentation. The major goals are divided into three objectives listed below.

Obj.1 Multi-scale failure mechanisms of cellular materials subjected to impact/shock loading.

We propose to use in-situ high-speed multi-scale digital image correlation-based experiments to understand the dynamic deformation mechanisms at different length scales.

Obj.2. Developing a constitutive material model

Obj.3. Design optimal functionally graded cellular materials (FGMCs)

By integrating the multi-scale high-strain rate experiments, proposed material model, and semi-analytical methods, an effort will be made to design optimal FGMCs that can mitigate the impact and shock loading.

Accomplishments: Both experimental, analytical, and numerical studies have been conducted during the project period, and a significant accomplishment has been achieved toward the goals. As discussed below, the accomplishments are listed in three objectives proposed.

Obj. 1 Multi-scale failure mechanisms of cellular materials subjected to impact/shock loading

As proposed, the multiscale failure mechanics was investigated experimentally using a split Hopkinson pressure bar (SHPB), high-speed imaging, and digital image correlation.

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- The dynamic response of nonlinearly graded foam was investigated through the use of SHPB and 2D-DIC.
- Strain rate was held constant, between 600 s⁻¹ - 700 s⁻¹ throughout the investigation so sample geometry can be compared neglecting any strain rate effect.
- Each specimen was impacted from both the low and high-density sides to determine the effect of specimen orientation with respect to the loading direction.

Energy Absorption

- The low, middle, high (LMH) and MHL density arrangements outperformed the others by absorbing on average 7% more energy at 65% global strain.
- Again, the LMH and MHL specimens outperformed the others by absorbing 9% more energy at 50% strain.
- At each yield of the three points, the dynamically loaded specimen absorbed 30%, 43%, and 22% more energy respectively.
- At failure, however, the quasi-static loading on average absorbed 22% more energy.

Numerical

- A 2-D cell-based finite element model has been developed using the Voronoi technique for functionally graded cellular material subjected to dynamic loading.
- The energy absorption performance of graded foam structure with a higher density layer towards the impact side is found to be better than the uniform density foam structure. Alternatively, the foam structure with a lower density layer on the impact end is found to be underperforming.
- The graded foam structure is found to be promising as it has the potential to absorb more energy if tailored in an appropriate configuration.
- For quasi-static loading, the optimization of functionally graded foam has been studied analytically for maximizing energy absorbed per unit volume and energy absorbed per unit mass.
 - o The relative advantage of graded material over uniform-density material has also been evaluated.
- A critical value of relative density, beyond which the functionally graded material is not advantageous, has been determined based on a rigid, perfectly plastic and locking model (R-PP-L).

Obj.2. Developing a constitutive material model

Constituent response:

A three-parameter nonlinear constitutive model capturing the mechanical response of cellular foam materials has been developed. Quasi-static experiments are performed on rigid polyurethane (PUR) foam to obtain the material parameters required for the model. The functional gradation is modeled using power-law variations of relative density. Energy absorbed per unit volume and energy absorbed per unit mass is used as performance metrics. The comparative advantage of the functionally graded material over uniform-density foam material is also studied.

- The optimal density of a uniform-density cellular material is determined based on three objective functions; maximizing energy absorption, minimizing stress, and minimizing weight.
- The constitutive model has been effectively applied to compare the response of functionally graded material with uniform-density material.

Shock propagation:

A theoretical framework for density-graded cellular materials subjected to direct impact loading is developed using shock theory. An explicit and closed-form solution for the stationary impact scenario of density-graded CM is proposed based on the RPPL material model. A rigid, perfectly plastic, and locking (RPPL) material behavior is used for the graded cellular material, with the stationary surface assumed to be rigid. The critical density gradient or the condition for the existence of the double shock mode is obtained as a first step. Afterward, the energy absorption and the stresses at the ends of the cellular material are evaluated for both single and double shock modes. The influence of density gradient on the energy absorption capacity, incident stress, and transmitted stress is studied and finally, the optimal density gradients that provide maximum impact resistance to objects placed at the incident end and transmitted end are determined.

1. Stationary impact scenario

- An explicit and closed-form solution for the stationary impact scenario of density-graded cellular material is developed based on the rigid, perfectly plastic, and locking material model.
- Closed-form expressions are derived for the energy absorption capacity and the incident stress using shock front position as an independent variable.
- The explicit solution captures the dynamic behavior of the material with great accuracy, as shown through the comparison of analytical results with the numerical solutions.
- The analytical solution shows that the density gradation improves the energy absorption capacity, and the enhancement is maximum when the density gradient parameter is maximum.

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- The proposed solution will lead to the faster and more efficient design of density-graded structures for impact mitigation.
- 2. Direct impact scenario
 - Analytical solutions for special cases under direct impact scenarios of density-graded cellular material are obtained based on the rigid, perfectly plastic, and locking material model.
 - The analytical solution shows that the density gradation improves the energy absorption capacity same as that shown for the stationary impact scenario.
 - The peak incident stress and transmitted stress can be significantly reduced by choosing suitable density gradation.
 - Optimal values of density gradient parameters are computed that will maximize the impact resistance for objects placed at the incident end and the transmitter end.

Obj.3. Design optimal functionally graded cellular materials (FGMCs)

As proposed, the optimal density configuration of cellular materials is first explored using the non-linear constitutive model. The optimal density is determined through analytical modeling for the maximum energy absorption performance of the cellular material. The impact response of density-graded 3D Voronoi cellular structures is also studied and compared to its uniform density counterpart. The specimens are fabricated via photopolymer jetting technology, an additive manufacturing technique that enables high accuracy of intricate features and complex shapes that are distinctive of Voronoi cellular structures. The density gradation is achieved by changing the cell size along the impact direction. The foam specimens are impinged by a freely falling rigid mass with the help of a drop tower that allows determining the response at an intermediate strain-rate regime. The performance is analyzed by studying the energy absorption and transmitted force as a function of density gradation. It has been shown that the dynamic responses of the uniform-density and density-graded Voronoi cellular specimens are significantly different. The uniform specimen possesses a flat plateau in its stress-strain curve, whereas the graded specimen has a characteristic increasing slope in the plateau region. These distinctive features result in differing energy absorption capabilities. Although the energy absorption of the uniform specimen is optimized for moderate energy input, its performance deteriorates at low and high energy input levels. On the other hand, the graded specimen outperforms at low and high energy inputs, and its performance does not significantly deteriorate at moderate energy inputs. Thus, density graded Voronoi cellular structures provide all-around energy absorption performance.

Training Opportunities: • One Ph.D. student (Vijendra Gupta) and two M.S students (Abigail Wohlford and Dennis Miller) participated and trained during the project period. Abigail Wohlford participated in a research experience at Army Research Laboratory at Aberdeen Proving Ground, during the project period.

- In addition, one High school student has been trained as part of the project through the AEOP high school apprenticeship program.

Results Dissemination: The find of the project has been disseminated through international journals and conferences. In addition, PI used his website to disseminate the finding. During the project period (2019/202) PI, Addis Kidane has been awarded a sub-Saharan Africa U.S. Fulbright Scholar. He traveled and spent a year in Ethiopia, where he teaches and researched at the Addis Ababa Institute of Technology. Kidane used the opportunity to disseminate his work through seminars and workshops.

Honors and Awards: Fulbright Scholar,

During the project period (2019/202) PI, Addis Kidane has been awarded sub-Saharan Africa U.S. Fulbright Scholar. He traveled and spent a year in Ethiopia, where he teaches and research at Addis Ababa Institute of Technology. Kidane used the opportunity to disseminate his work through seminar and workshop.

PI Kidane has been awarded the 2018 James W. Dally Young Investigator Award, from the society for experimental mechanics.

The 2018 Breakthrough Stars, the Office of the Vice President for Research, The University of South Carolina

Protocol Activity Status:

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- Technology Transfer:** • PI Kidane was invited and gave a presentation on recently developed experimental method at the Los Alamos National Laboratory University collaboration workshop.
- PI, Dr. Kidane has been working with researchers at Army Research Laboratory at Aberdeen Proving Ground.

PARTICIPANTS:

Participant Type: Graduate Student (research assistant)

Participant: Vijendra Gupta

Person Months Worked: 12.00

Funding Support:

Project Contribution:

National Academy Member: N

Participant Type: Graduate Student (research assistant)

Participant: Dennis Miller

Person Months Worked: 12.00

Funding Support:

Project Contribution:

National Academy Member: N

Participant Type: Graduate Student (research assistant)

Participant: Chizoba Onwuka

Person Months Worked: 4.00

Funding Support:

Project Contribution:

National Academy Member: N

Participant Type: Graduate Student (research assistant)

Participant: Abigail Wohlford

Person Months Worked: 12.00

Funding Support:

Project Contribution:

National Academy Member: N

ARTICLES:

RPPR Final Report as of 23-May-2023

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Journal: International Journal of Solids and Structures

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Publication Identifier: 10.1016/j.ijsolstr.2018.02.003

Volume:

Issue:

First Page #: 270

Date Submitted: 8/30/18 12:00AM

Date Published: 5/1/18 4:00AM

Publication Location:

Article Title: Experimental characterization of compaction wave propagation in cellular polymers

Authors: Suraj Ravindran, Behrad Koohbor, Peter Malchow, Addis Kidane

Keywords: Polymer foam Wave propagation, Direct impact, Digital image correlation, Compaction

Abstract: A great deal of literature is available in the study of compaction wave propagation in cellular materials. However, experimental investigations concerning the spatial variation of the deformation features of compaction waves are limited. In this study, the formation and propagation of compaction wave in a low-density polymeric foam under intermediate velocity projectile impact loading are investigated experimentally. The results are discussed in terms of the compaction wave characteristic parameters such as compaction wave velocity, axial strain, particle velocity, etc. In addition, the spatial distribution of inertia stress during compaction wave propagation, and the critical velocity required to form shock are estimated. The stress jump across the compaction wave is also calculated using an inertia stress analysis and compared with the shock theory. It was observed that an elastic precursor propagates, at a velocity of 740 m/s, along the material upon impact, and it decays as it pro

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Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: International Journal of Impact Engineering

Publication Identifier Type: DOI

Publication Identifier: 10.1016/j.ijimpeng.2021.103820

Volume: 150

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Date Submitted: 8/25/21 12:00AM

Date Published:

Publication Location:

Article Title: In Situ Deformation Characterization of Density-Graded Foams in Quasi-Static and Impact Loading Conditions

Authors: Behrad Koohbor, Suraj Ravindran, Addis Kidane

Keywords: Density graded; foam; impact; digital image correlation; inertia

Abstract: Digital image correlation is utilized to characterize deformation and strain fields developed within the layers of density-graded multilayered foam structures subjected to uniaxial quasi-static and dynamic compression. Three-layered graded structures fabricated from rigid polyurethane foams with nominal densities of 160, 240, and 320 kg/m³ are subjected to quasi-static and dynamic loading. The quasi-static measurements show that, irrespective of the loading direction, the densification is initiated in the lowest density layer and propagates into other layers later once the first layer is fully densified. The deformation mechanisms are different in the case of dynamic loading conditions than quasi-static loading. In the case of dynamic loading, the deformation mechanism depends on the sample orientation relative to the direction of the applied load. In cases where the higher density layers are impacted, the propagation of the elastic and compaction waves leads to partial deformation.

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Journal: Theoretical and Applied Mechanics Letters
Publication Identifier Type: DOI **Publication Identifier:** on <https://doi.org/10.1016/j.taml.2021.100288>
Volume: 11 **Issue:** 5 **First Page #:** 100288
Date Submitted: 1/20/23 12:00AM **Date Published:**
Publication Location:
Article Title: Closed-Form Solution for Shock Wave Propagation in Density-Graded Cellular Material Under Impact
Authors: Vijendra Gupta , Addis Kidane, Michael Sutton
Keywords: Functionally graded cellular material, Analytical modeling, Impact response, Closed-form solution, Energy absorption, Density gradient
Abstract: Density-graded cellular materials have tremendous potential in structural applications where impact resistance is required. Cellular materials subjected to high impact loading result in a compaction type deformation, usually modeled using continuum-based shock theory. The resulting governing differential equation of the shock model is nonlinear, and the density gradient further complicates the problem. Earlier studies have employed numerical methods to obtain the solution. In this study, an analytical closed-form solution is proposed to predict the response of density-graded cellular materials subjected to a rigid body impact. Solutions for the velocity of the impinging rigid body mass, energy absorption capacity of the cellular material, and the incident stress are obtained for a single shock propagation. The results obtained are in excellent agreement with the existing numerical solutions found in the literature.
Distribution Statement: 1-Approved for public release; distribution is unlimited.
Acknowledged Federal Support: Y

CONFERENCE PAPERS:

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: 2018 SEM Annual Conference and Exposition on Experimental and Applied Mechanics
Date Received: 18-Sep-2020 **Conference Date:** 06-Jun-2018 **Date Published:** 28-Oct-2018
Conference Location: Greenville SC
Paper Title: Energy Absorption Characteristics of Graded Foams Subjected to High Velocity Loading
Authors: Abigail Wohlford, Suraj Ravindran, Addis Kidane
Acknowledged Federal Support: Y

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: Annual Conference and Exposition on Experimental and Applied Mechanics
Date Received: 18-Sep-2020 **Conference Date:** 03-Jun-2019 **Date Published:**
Conference Location: Reno, Nevada
Paper Title: Numerical and Experimental Investigation of Density Graded Foams Subjected to Dynamic Loading
Authors: Vijendra Gupta#x3b; Dennis Miller#x3b; Addis Kidane
Acknowledged Federal Support: Y

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: Annual Conference and Exposition on Experimental and Applied Mechanics
Date Received: 18-Sep-2020 **Conference Date:** 03-Jun-2019 **Date Published:**
Conference Location: Reno, NV
Paper Title: Dynamic Response of Layered Functionally Graded Polyurethane Foam with Nonlinear Density Variation
Authors: Dennis Miller#x3b; Vijendra Gupta#x3b; Addis Kidane
Acknowledged Federal Support: Y

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Conference Name: Annual Conference and Exposition on Experimental and Applied Mechanics
Date Received: 25-Aug-2021 Conference Date: 11-Jun-2020 Date Published:
Conference Location: Virtual
Paper Title: Optimization for improved energy absorption and effect of density gradation in cellular materials
Authors: Vijendra Gupta, Dennis Miller, Addis Kidane, and Michael Sutton
Acknowledged Federal Support: **Y**

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: Annual Conference and Exposition on Experimental and Applied Mechanics
Date Received: 24-Jan-2023 Conference Date: 13-Jun-2022 Date Published: 24-Aug-2023
Conference Location: Pittsburg
Paper Title: Density-Graded 3D Voronoi Cellular Structures for Improved Impact Performance
Authors: Vijendra Gupta; Addis Kidane, Michael Sutton
Acknowledged Federal Support: **Y**

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: Society for Experimental Mechanics Annual Conference and Exposition
Date Received: 24-Jan-2023 Conference Date: 15-Jul-2021 Date Published:
Conference Location: Virtual
Paper Title: Effect of Impact Velocity on Dynamic Response of Polymeric Foams
Authors: Vijendra Gupta, Chizoba Onwuka, Addis Kidane, and Michael Sutton
Acknowledged Federal Support: **Y**

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: Conference Proceedings of the Society for Experimental Mechanics Series
Date Received: 24-Jan-2023 Conference Date: 13-Jun-2022 Date Published: 02-Jan-2023
Conference Location: Pittsburg
Paper Title: Additive Manufacturing of Graded Voronoi Cellular Structures for Increased Design Flexibility
Authors: Vijendra Gupta, Addis Kidane, and Michael Sutton
Acknowledged Federal Support: **Y**

DISSERTATIONS:

Publication Type: Thesis or Dissertation
Institution: the University of South Carolina
Date Received: 30-Aug-2018 Completion Date: 8/1/18 2:59AM
Title: Effect Of Material Orientation On The Impact Response Of Graded Cellular Materials
Authors: Abigail Wohlford
Acknowledged Federal Support: **Y**

RPPR Final Report
as of 23-May-2023

Partners

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I certify that the information in the report is complete and accurate:

Signature: Addis Kidane

Signature Date: 1/24/23 7:22PM

Project Summary - Grant # W911NF-18-1-0023
(Reporting Period: 11/29/2017 – 08/31/2022)

Effort Title: Optimal Functionally Graded Cellular Materials (FGCMs) to Mitigate Shock and Impact Loading

Addis Kidane
Department of Mechanical Engineering
The University of South Carolina, Columbia SC, 29208

Abstract

The main objective of the proposed work is to develop a well-defined, experimentally-validated material model that can be implemented in the design of optimal functionally graded cellular materials (FGCMs) that can effectively mitigate shock and impact loading. First, different density foams with known cell size and structure will be investigated to improve our understanding of the failure processes that are activated in cellular materials subjected to high strain rate loading. The specimens will be subjected to controlled high strain rate impact and shock loading conditions while using multi-scale imaging and digital image correlation to (a) observe the local deformation processes, (b) quantify the local deformations and strain that are present and (c) quantify the inertia and change in local densities that are experienced by the heterogeneous cellular material system. The observations and measurements will be integrated with an inverse method to quantify the full field stress-strain relation of the material across the specimen section by accounting for the effect of inertia, strain rate, and change in density. The constitutive relation obtained experimentally will be used to develop a material model that governs the response of cellular materials subjected to high strain rate loading as a function of density and loading rate. Finally, using the constitutive relationship and a semi-analytical method, a range of FGMC topologies will be explored to obtain an optimal design for functionality. In this proposal, the PI will focus on the shock and impact mitigation functionally and the optimal design will be selected through energy-absorption and efficiency diagrams. The concepts proposed for this study can be extended to other functionality, including thermal acoustic transport properties. If successful, the work will have a significant contribution to the effort to innovate material and structures that absorb energy, deflect penetration and disperse momentum, processes that are crucial for the mission of the US Army.

Objective

The proposed research will open a new era in understanding cellular materials subjected to high-strain rate loading. It will attempt to answer some fundamental questions such as (a) What are the primary mechanics responsible for failure initiation and propagation in cellular materials subjected to very high rate loading conditions, (b) What are the important material properties required to develop a physics-based constitutive model that governs the mechanics of cellular material subjected to different loading conditions? In general, the proposed study will focus on understanding multi-scale deformation behavior that leads to the failure of cellular materials under impact and shock loading conditions. Understanding the deformation mechanisms and how they are affected by the increased compressibility expected in cellular materials, the effect of wave speed and strain rate on the elastic and plastic properties of individual struts and global cellular structures of different initial density foams can provide key design concepts in the development of optimal FGMCs that can be used in different applications. Unlike traditional cellular materials, the proposed FGMC material systems will be tailored to absorb more energy when subjected to shock and impact loading while resisting fragmentation. The major goals are divided into three objectives listed below.

Obj.1 Multi-scale failure mechanisms of cellular materials subjected to impact/shock loading

Cellular materials are heterogeneous, and understanding their failure mechanism, especially under impact loading, is a very challenging task. The best approach to understanding the failure mechanisms in heterogeneous materials would be to use multi-scale measurements, ranging from individual struts to overall foam deformation. To test the dynamic properties at extremely high strain rate conditions, impacting specimens against a rigid frame and recovering the samples for high magnification postmortem analysis using Secondary Electronic Microcopy (SEM) has been effective. Though the SEM postmortem analysis could be effective in characterizing the failure surface, even in cellular materials, determining failure initiation and damage propagation is impossible. ***We propose to use in-situ high-speed multi-scale digital image correlation-based experiments to understand the dynamic deformation mechanisms at different length scales.*** In the proposed work, the multi-scale nature of deformation under dynamic compression loading will be examined, from individual struts to overall foam deformation. The relation between strain localization in individual struts and deformation bands at the macro-scale as a function of strain rate will be explored. High-speed, in-situ imaging, and digital image correlation (DIC) techniques will be used to explore the onset and propagation of strain localization bands. Finally, the stress strains for a range of initial density cellular materials as a function of strain rates will be measured using both Hopkinson pressure bar experiments and a modified shock tube apparatus with DIC. It is anticipated that this will lead to a clear understanding of the main failure mechanisms in cellular materials as a function of initial density and strain rate. In addition, it will be used in the effort to develop a physics-based material model for cellular materials.

Obj.2. Developing a constitutive material model

Developing a physic-based predictive material model for high strain rate loading conditions is a very challenging task due to the need for obtaining accurate material parameters over a range of strain rates. The challenge is even more significant for cellular materials, as the wave speed is low and the materials are highly compressible. The low wave speed in cellular materials delays the equilibrium time during loading, which makes the deformation nonhomogeneous. In such cases, averaging the stress strain from a single-point measurement would not give the actual material properties. To achieve this objective, the PI will use a state-of-the-art measurement method, 3D digital image correlation, to obtain full-field strain measurements. The measurement will be used with a non-parametric method to obtain full field inertia stress and total stress across the specimen accounting for compressibility. As part of the research objective discussed in section B.1, the PI will identify the important material parameters required for developing the constitutive model. Using his expertise in high-strain experiments and full-field measurements, the PI will (a) extract important material parameters as a function of density and strain rate and (b) develop a physics-based model that can be verified experimentally.

Obj.3. Design optimal functionally graded cellular materials (FGMCs)

By integrating the multi-scale high-strain rate experiments, proposed material model, and semi-analytical methods, an effort will be made to design optimal FGMCs that can mitigate the impact and shock loading. A range of FGCMs will be considered, and their energy absorption characteristics and impact behavior will be determined. The energy absorption performance of each topology FGCM will be characterized through the energy-absorption diagram. The energy absorption -diagram for particular FGCMs can be obtained by plotting the area under the stress-stress curve as a function of density and applied stress. Note that the material model discussed in section B.2 will be used to generate the stress-strain behavior. Finally, the efficiency diagram method will be used to obtain the optimal FGCM design, and selective optimal layered FGCMs will be fabricated using similar polymer foam with different densities. The specimen will be subjected to impact and shock loading, and their energy absorption locus as a function of foam density, strain rate, and applied stress will be developed. The experimental result will be used to validate the model.

Approach

We propose to develop a well-defined, experimentally-validated material model that can be implemented in the design of an optimal FGMC that can effectively mitigate shock and impact loading.

- First, different density closed cell foams with known cell size and structure will be investigated to improve our understanding of the failure processes that are activated in cellular materials subjected to high strain rate loading.
- The specimens will be subjected to control, high strain rate loading conditions while using multi-scale imaging and digital image correlation to (a) observe the local deformation processes, (b) quantify the local deformations and strain that are present, and (c) quantify the inertia and change in local densities that are experienced by the heterogeneous cellular material system.
- The observations and measurements will be integrated with a non-parametric method to quantify the full field stress-strain relation of the material across the specimen section by accounting for the effect of inertia, strain rate, and change in density. The result will be used to develop a material model that governs the response of cellular materials subjected to high strain rate loading as a function of density and loading rate.
- Finally, using the constitutive relationship and a semi-analytical method, a range of FGMC topologies will be explored to obtain an optimal design for functionality.

The density gradation envisioned by the investigator is in only one direction, through the thickness, which is practical for plate-like structures subjected to impact and employed for shock mitigation. The only constraint that is envisioned for the plate-like structure optimization is that the overall thickness will remain the same. The optimization will be based on energy absorbed and strength. The variables are the density of each layer, the thickness of each layer, and the orientation of the layers.

Accomplishments

Both experimental, analytical, and numerical studies have been conducted during the project period, and a significant accomplishment has been achieved toward the goals. As discussed below, the accomplishments are listed in three objectives proposed.

Obj. 1 Multi-scale failure mechanisms of cellular materials subjected to impact/shock loading

As proposed, the multiscale failure mechanics was investigated experimentally using a split Hopkinson pressure bar (SHPB), high-speed imaging, and digital image correlation.

- The dynamic response of nonlinearly graded foam was investigated through the use of SHPB and 2D-DIC.
- A power-law function was used to create specimen gradation, where the continuous function with a power law exponent of 2 and 0.5 was approximated through discrete layers.
- Strain rate was held constant, between 600 s^{-1} - 700 s^{-1} throughout the investigation so sample geometry can be compared neglecting any strain rate effect.
- Each specimen was impacted from both the low and high-density sides to determine the effect of specimen orientation with respect to the loading direction.
- Finding,
 - Concave Power-law Approximation ($m=2$):
 - Produced a global strain valid until $\sim 36\%$ strain, where the fracture occurred at this value
 - The average inertial stress (impacting from the high and low-density sides) was approximately 0.5 MPa ; or $\sim 5\%$ of the global response

- Energy of about 350 MJ/m^3 was absorbed before the fracture occurred at the second yield point.
- Convex Power-law Approximation ($m=0.5$):
 - Produced a global strain valid until $\sim 15\%$ strain, where the fracture occurred at this strain level
 - The average inertial stress (impacting from the high and low-density sides) was approximately 2 MPa; or $\sim 8\%$ of the global response
 - Energy of about 50 MJ/m^3 was absorbed before the fracture at $\sim 15\%$ strain.
- The Response of symmetrically graded foam subjected to intermediate and high dynamic loadings is also investigated experimentally.
- The total energy absorbed and the effect of strain rate are obtained.
- The particle velocity in the material is obtained from the measured displacement fields.
- The elastic wave speed in the material is measured experimentally
- The stress in each layer was obtained and finally, the elastic stress wave transmission through each interface is obtained.
- Experimental Study - Symmetrically graded foams
 - Experimental Information
 - Intermediate dynamic loading on symmetric FGFM to determine the energy absorbing characteristics using a Split Hopkinson Pressure bar, ultra-high-speed cameras, and digital image correlation.
 - High strain rate loading of the same specimens using direct impacts was performed to study the elastic wave propagation, such that impedance analysis of the layer interfaces can be quantified.
 - 6 symmetric specimens were created using 3 different densities, 160, 320, and 480 Kg/m^3 such that only the stacking arrangement is varied
 - Energy Absorption
 - Dynamically loaded at $\sim 800 \text{ 1/s}$, the stress-strain response was achieved such that points of interest and the total energy absorbed can be quantified.
 - Layer yield, densification-to-elastic, and fracture points are used for comparison in conjunction with energy absorption.
 - Dynamic response is compared with the quasi-static response
 - Findings:
 - Quasi-Static
 - For any inversely graded specimen pair, one gradation is superior over its counterpart due to an increase of yield stress for the superior gradation.
 - The low, middle, high (LMH), and MHL density arrangements outperformed the others by absorbing on average 7% more energy at 65% global strain.
 - Dynamically
 - The inversely graded advantage is observed in the dynamic loading.
 - Again, the LMH and MHL specimens outperformed the others by absorbing 9% more energy at 50% strain
 - Q.S vs Dynamic
 - At each yield of the three points, the dynamically loaded specimen absorbed 30%, 43%, and 22% more energy respectively.
 - At failure, however, the Q.S specimens on average absorbed 22% more energy.

Numerical

- A 2-D cell-based finite element model has been developed using the Voronoi technique for functionally graded cellular material subjected to dynamic loading.
- The energy absorption performance of graded foam structure with a higher density layer towards the impact side is found to be better than the uniform density foam structure. Alternatively, the foam structure with a lower density layer on the impact end is found to be underperforming.
- The graded foam structure is found to be promising as it has the potential to absorb more energy if tailored in an appropriate configuration.
- For quasi-static loading, the optimization of functionally graded foam has been studied analytically for maximizing energy absorbed per unit volume and energy absorbed per unit mass.
 - The relative advantage of graded material over uniform-density material has also been evaluated.
- A critical value of relative density, beyond which the functionally graded material is not advantageous, has been determined based on a rigid, perfectly plastic and locking model (R-PP-L).

Obj.2. Developing a constitutive material model

Constituent response

A three-parameter nonlinear constitutive model capturing the mechanical response of cellular foam materials has been developed. Quasi-static experiments are performed on rigid polyurethane (PUR) foam to obtain the material parameters required for the model. The functional gradation is modeled using power-law variations of relative density. Energy absorbed per unit volume and energy absorbed per unit mass is used as performance metrics. The comparative advantage of the functionally graded material over uniform-density foam material is also studied.

- The optimal density of a uniform-density cellular material is determined based on three objective functions; maximizing energy absorption, minimizing stress, and minimizing weight.
- The constitutive model has been effectively applied to compare the response of functionally graded material with uniform-density material.

Shock propagation

A theoretical framework for density-graded cellular materials subjected to direct impact loading is developed using shock theory. An explicit and closed-form solution for the stationary impact scenario of density-graded CM is proposed based on the RPPL material model. A rigid, perfectly plastic, and locking (RPPL) material behavior is used for the graded cellular material, with the stationary surface assumed to be rigid. The critical density gradient or the condition for the existence of the double shock mode is obtained as a first step. Afterward, the energy absorption and the stresses at the ends of the cellular material are evaluated for both single and double shock modes. The influence of density gradient on the energy absorption capacity, incident stress, and transmitted stress is studied and finally, the optimal density gradients that provide maximum impact resistance to objects placed at the incident end and transmitted end are determined.

1. Stationary impact scenario

- An explicit and closed-form solution for the stationary impact scenario of density-graded cellular material is developed based on the rigid, perfectly plastic, and locking material model.
- Closed-form expressions are derived for the energy absorption capacity and the incident stress using shock front position as an independent variable.
- The explicit solution captures the dynamic behavior of the material with great accuracy, as shown through the comparison of analytical results with the numerical solutions.

- The analytical solution shows that the density gradation improves the energy absorption capacity, and the enhancement is maximum when the density gradient parameter is maximum.
- The proposed solution will lead to the faster and more efficient design of density-graded structures for impact mitigation.

2. Direct impact scenario

- Analytical solutions for special cases under direct impact scenarios of density-graded cellular material are obtained based on the rigid, perfectly plastic, and locking material model.
- The analytical solution shows that the density gradation improves the energy absorption capacity same as that shown for the stationary impact scenario.
- The peak incident stress and transmitted stress can be significantly reduced by choosing suitable density gradation.
- Optimal values of density gradient parameters are computed that will maximize the impact resistance for objects placed at the incident end and the transmitter end.

Obj.3. Design optimal functionally graded cellular materials (FGMCs)

As proposed, the optimal density configuration of cellular materials is first explored using the non-linear constitutive model. The optimal density is determined through analytical modeling for the maximum energy absorption performance of the cellular material. The impact response of density-graded 3D Voronoi cellular structures is also studied and compared to its uniform density counterpart. The specimens are fabricated via photopolymer jetting technology, an additive manufacturing technique that enables high accuracy of intricate features and complex shapes that are distinctive of Voronoi cellular structures. The density gradation is achieved by changing the cell size along the impact direction. The foam specimens are impinged by a freely falling rigid mass with the help of a drop tower that allows determining the response at an intermediate strain-rate regime. The performance is analyzed by studying the energy absorption and transmitted force as a function of density gradation. It has been shown that the dynamic responses of the uniform-density and density-graded Voronoi cellular specimens are significantly different. The uniform specimen possesses a flat plateau in its stress-strain curve, whereas the graded specimen has a characteristic increasing slope in the plateau region. These distinctive features result in differing energy absorption capabilities. Although the energy absorption of the uniform specimen is optimized for moderate energy input, its performance deteriorates at low and high energy input levels. On the other hand, the graded specimen outperforms at low and high energy inputs, and its performance does not significantly deteriorate at moderate energy inputs. Thus, density-graded Voronoi cellular structures provide all-around energy absorption performance.

Collaborations and Technology Transfer

- PI Kidane was invited and gave a presentation on the recently developed experimental method at the Los Alamos National Laboratory University collaboration workshop.
- PI, Dr. Kidane has been working with researchers at Army Research Laboratory at Aberdeen Proving Ground.

Resulting Journal Publications during Reporting Period

- Gupta, Vijendra, Addis Kidane, and Michael Sutton. "Additive Manufacturing of Graded Voronoi Cellular Structures for Increased Design Flexibility." *Society for Experimental Mechanics Annual Conference and*

Exposition. Springer, Cham, 2023.

- Gupta, Vijendra, Addis Kidane, and Michael Sutton. "Density-Graded 3D Voronoi Cellular Structures for Improved Impact Performance." *Society for Experimental Mechanics Annual Conference and Exposition*. Springer, Cham, 2023.
- Gupta, Vijendra, et al. "Effect of Impact Velocity on Dynamic Response of Polymeric Foams." *Dynamic Behavior of Materials, Volume 1*. Springer, Cham, 2022. 73-78.
- Gupta, Vijendra, Addis Kidane, and Michael Sutton. "Closed-form solution for shock wave propagation in density-graded cellular material under impact." *Theoretical and Applied Mechanics Letters* 11.5 (2021): 100288.
- Koohbor, Behrad, Suraj Ravindran, and Addis Kidane. "In situ deformation characterization of density-graded foams in quasi-static and impact loading conditions." *International Journal of Impact Engineering* 150 (2021): 103820.
- Gupta, Vijendra, et al. "Optimization for Improved Energy Absorption and the Effect of Density Gradation in Cellular Materials." *Mechanics of Biological Systems and Materials & Micro-and Nanomechanics & Research Applications*. Springer, Cham, 2021. 13-20.
- Miller, Dennis, Vijendra Gupta, and Addis Kidane. "Dynamic response of layered functionally graded polyurethane foam with nonlinear density variation." *Dynamic Behavior of Materials, Volume 1*. Springer, Cham, 2020. 25-30.
- Gupta, Vijendra, Dennis Miller, and Addis Kidane. "Numerical and Experimental Investigation of Density Graded Foams Subjected to Impact Loading." *Dynamic Behavior of Materials, Volume 1*. Springer, Cham, 2020. 31-35.
- Wohlford, Abigail, Suraj Ravindran, and Addis Kidane. "Energy Absorption Characteristics of Graded Foams Subjected to High Velocity Loading." *Dynamic Behavior of Materials, Volume 1*. Springer, Cham, 2019. 229-232.
- Wohlford, Abigail. *Effect of Material Orientation On the Impact Response of Graded Cellular Materials*. Diss. University of South Carolina, 2018. 10842404.

Graduate Students Involved During Reporting Period

- Vijendra Gupta, Ph.D.
- Dennis Miller, M.S.
- Abigail Wohlford M.S
- Chizoba Onwuka, M.S.

Awards, Honors, and Appointments (Insert into “Honors”)

- Fulbright Scholar.
During the project period (2019/202) PI, Addis Kidane has been awarded a sub-Saharan Africa U.S. Fulbright Scholar. He traveled and spent a year in Ethiopia, where he teaches and researched at the Addis Ababa Institute of Technology. Kidane used the opportunity to disseminate his work through seminars and workshops.

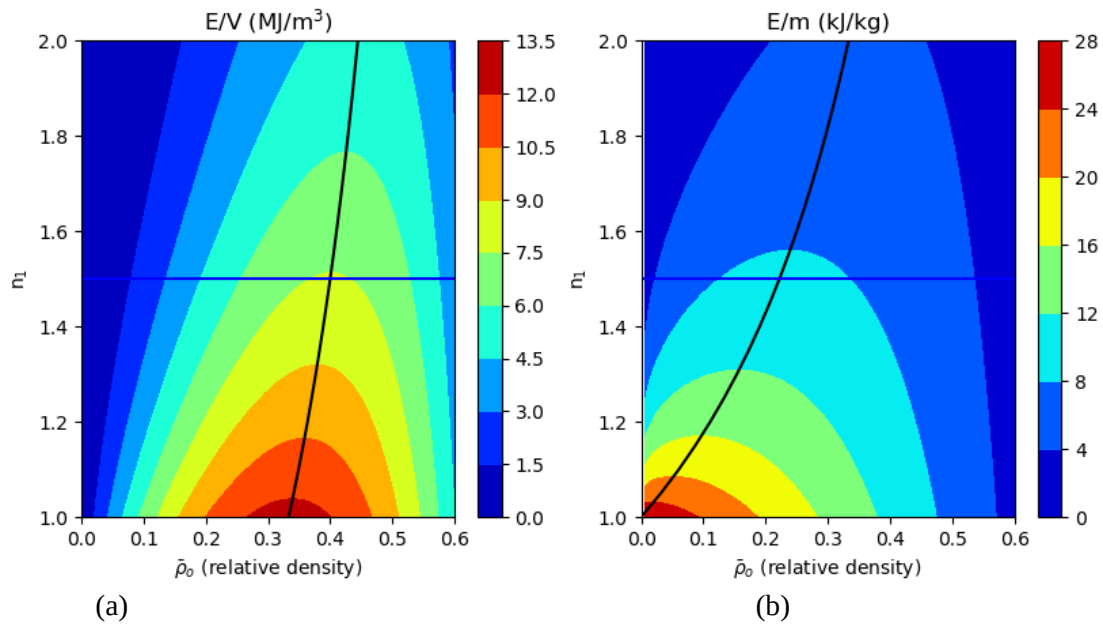


Fig.1. Contour plots of (a) energy absorbed per unit volume and (b) energy absorbed per unit mass as a function of relative density and n_1 (exponent in plateau stress model)

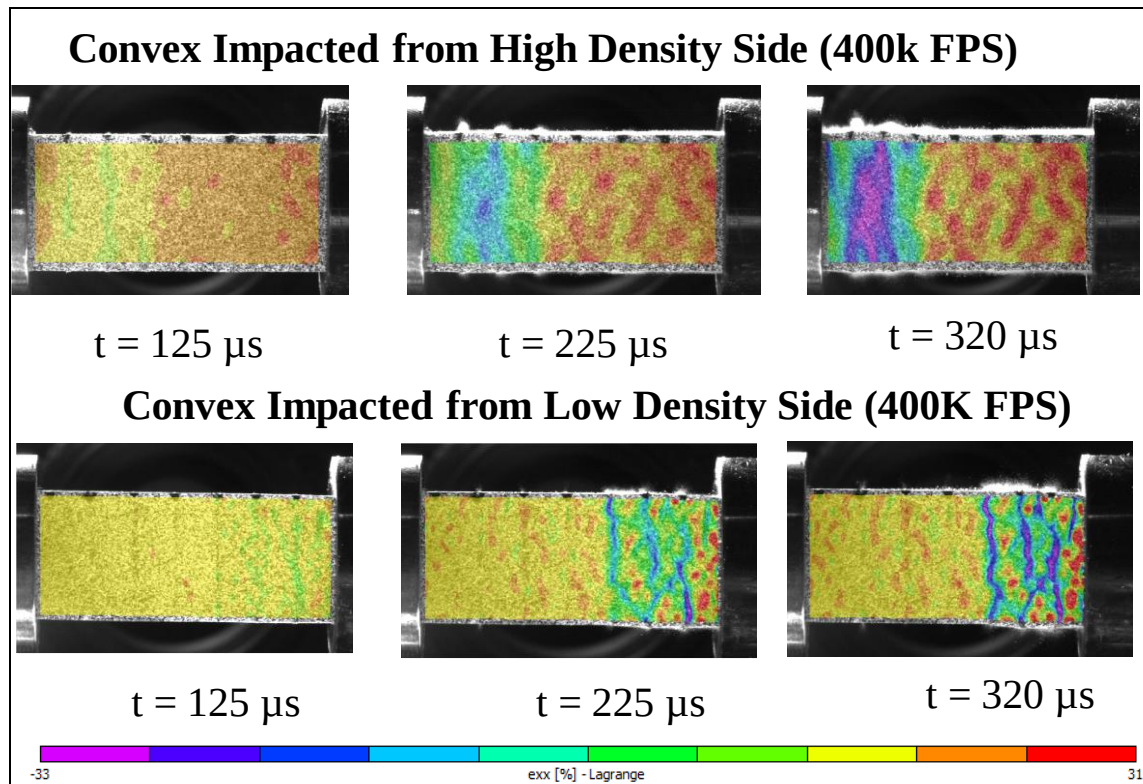


Fig. 2. Strain evolution of FGM foam with convex configuration

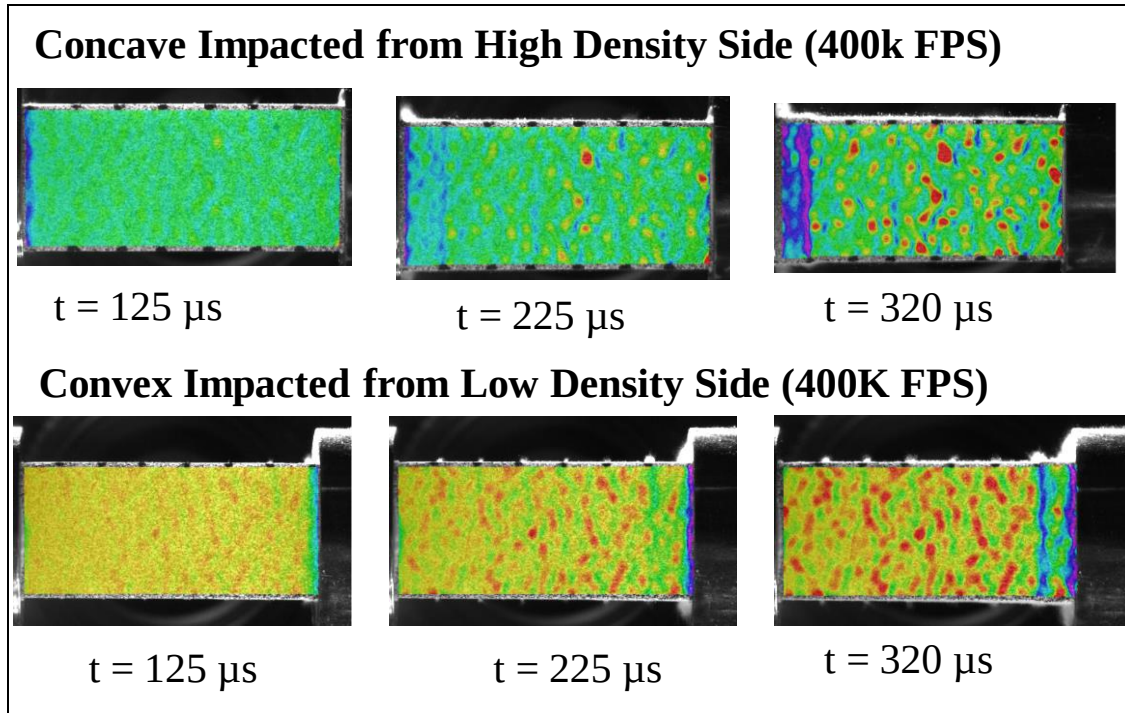


Fig. 3. Strain evolution of FGM foam with concave configuration

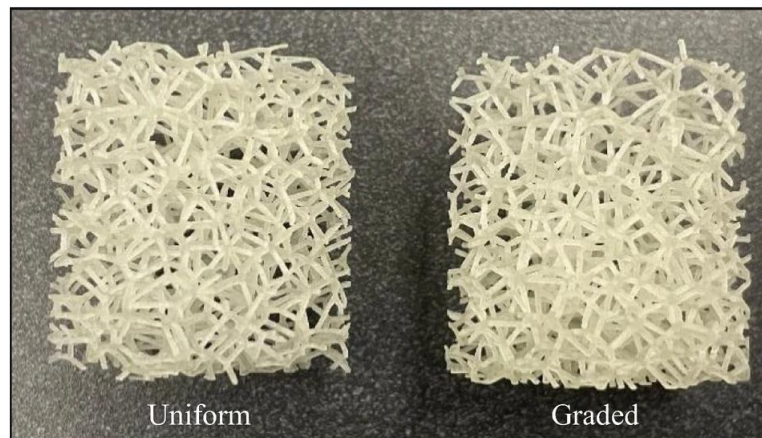


Fig.4 3D Printed uniform and graded density cellular materials



Fig.5. Group pictures at the Society for Experimental Mechanics Southeast Graduate student symposium, in Greenville SC, March 2019,

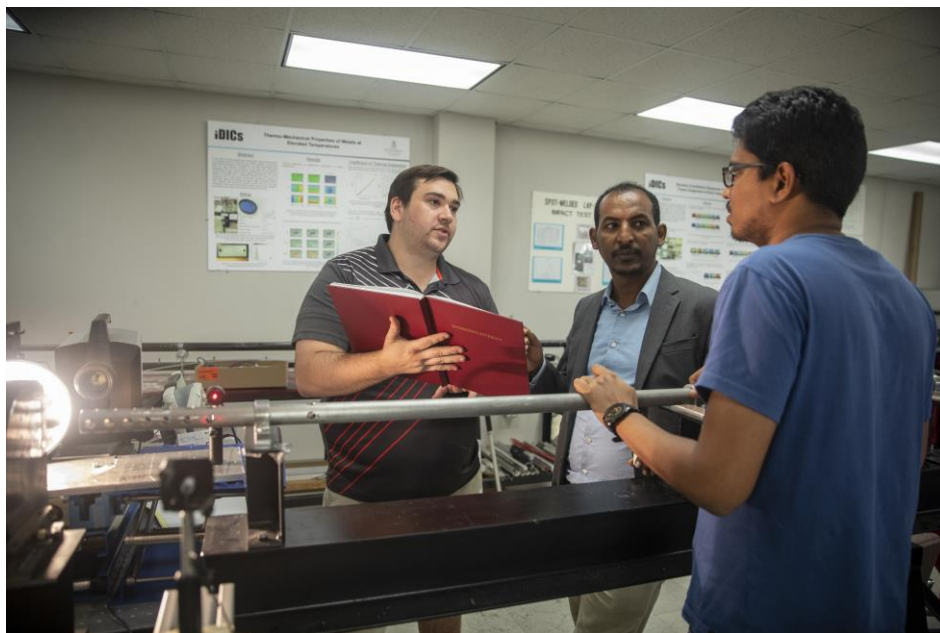


Fig.6. PI and the two graduate students (Dennis Miller and Vijendra Gupta) discussing the experimental result

EFFECT OF MATERIAL ORIENTATION ON THE IMPACT RESPONSE OF GRADED
CELLULAR MATERIALS

by

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ABSTRACT

In this study, the effect of layer stacking on the energy absorption characteristics of density-graded cellular polymers subjected to high velocity impact is investigated experimentally. The focus in the present work is to characterize the constitutive response and deformation mechanisms of these Functionally Graded Foam Materials (FGFM) using full-field measurements. Uniform foam Quasi-static and dynamic testing of uniform foam specimens are completed for each nominal density used in the layered foam samples. The low strain-rate experiments are performed in order to determine the mechanical properties of foam specimens. The high-rate experiments for homogenous foam specimens were completed to further characterize the constitutive response and also to provide baseline data for comparison to the response of FGM specimens. High rate loading experiments are performed using Split Hopkinson Pressure Bar (SHPB) combined with ultra-high-speed imaging to measure in-situ deformations and observe the formation and propagation of elastic and inelastic stress waves during impact. The FGM specimens were fabricated in-house by bonding different bulk density polymeric foam layers in different stacking arrangements. The effect of the orientation of the discrete layers on the dynamic response is quantified using high speed imaging with digital image correlation (DIC). The challenges with dynamic equilibrium due to low mechanical impedance of soft materials are carefully considered. The effects of inertia and material compressibility are included in analysis. The approach uses DIC to gather the full-field data which is used to measure the acceleration and density, later used to estimate the

stress gradients developed in the material. The temporal and spatial distributions of the inertia stress are superimposed with the boundary stress measured from the transmitted signal to satisfy the equation of dynamic stress equilibrium. The local and global responses are both examined in order to assess the overall performance of each gradient sequence. The average stress-strain curves obtained are then used to find the total energy absorbed during loading. Since the desired final goal is to be able to optimize this graded cushioning structure for any specific situation, the “best” arrangement of the FGM system is defined to be the layered system that has the highest energy absorption based on the model being used to characterize response. Recommendations for the extension of this work will be made at the end.

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LIST OF ABBREVIATIONS

ASTM	American Section of the International Association for Testing Materials
DIC.....	Digital Image Correlation
FGFM.....	Functionally Graded Foam Material
FGM.....	Functionally Graded Material
SHPB.....	Split Hopkinson Pressure Bar

CHAPTER 1

INTRODUCTION

1.1 OVERVIEW OF FOAM & FUNCTIONALLY GRADED MATERIALS

Polymeric foams are widely used due to their relatively high specific strength, low density, and superior energy absorption characteristics. They have become increasingly relevant because of their light structural weight, ability to be molded into complex geometries, and their capability to reduce injury or damage caused by an explosion[1]. The ideal energy dissipation of these soft materials is a result of the porous structure of these foams. The deformation mechanisms at the cellular scale are cell wall bending leading up to wall buckling or fracture after yielding; this process is typically how energy is absorbed in such systems[2]. Owing to these desirable characteristics, foam materials are commonly used as cushioning or protective structures in a variety of industries including automotive, aerospace, marine vessels, and consumer product packaging for transport. Since structural foams can be soft and flexible, or relatively stiff, but still deformable, such systems can be especially effective due to the dissipation of energy by permanent deformation and progressive crushing of its microstructure. In addition to typical quasi-static applications, foam systems have great potential in dynamic loading applications including high-velocity impact loading. Though foams have been shown to be good energy absorbers, the ability to predict their dynamic response is complicated by their very high strain-rate dependency, presence of inertial effects, and the change in foam density that occurs during crushing. These effects result in highly

nonlinear mechanical response at a variety of scales in such cellular materials. Here, the continuum scale response is directly linked to the mechanisms of deformation occurring at the level of an individual foam cell, so that the macroscopic mechanical properties of foam are correlated with the base material and the complex microstructure[3].

It is well known that the strength and energy absorption are highly sensitive to the relative density of the polymeric foam. Defining relative density, ρ^*/ρ_s , as the ratio between the density of the cellular material, ρ^* , and the parent solid material from which the cell walls are made, ρ_s , as the relative density increases, the cell walls thicken and the pore space diminishes; this intuitively indicates that the “strength” of the material will increase.

Since the main requirements for an energy-absorbing structure is to dissipate the kinetic energy generated from an impact while ensuring the maximum resisting force is below a certain limit, it is intuitively clear that the cell size and selection of the parent material must be optimized. The foam structure will have a much smaller max force than the parent (matrix) material of equal volume for the same amount of energy dissipated[4]. These porous materials have a low volume fraction of the parent material, which allows for large degrees of plastic crushing to occur at a constant plateau stress until the densification strain is reached[5]. The main difference between foam and a solid dense material is the big volume change; there is less lateral spreading, instead the cells are simply collapsing. The complex cellular microstructure allows for large deformation to occur at a constant nominal stress, so large amounts of energy can be absorbed without generating high stress. Higher density foams show higher strength compared to lower density foams but possess lower densification strain. Thus, with homogeneous foam there

occurs a compromise between energy absorption, strength, and weight with foams of different densities.

In order to possess a high level of all these sought-after characteristics, the concept of graded foam is introduced. Functionally Graded Materials are in the family of advanced engineering composites designed for a specific function or application. The basic idea is to take advantage of the unique properties of the individual components and combine them to tailor a single enhanced structure. FGMs have either a gradual or a stepwise change in material properties along a specific direction. A graded material has better designability and will at the very least have an equal outcome, but will more likely outperform its corresponding uniform material[6]. FGMs originally gained a lot of attention as metal-ceramic composites to have a gradually changing thermal expansion coefficient. FGMs were not invented by engineers though, they appear in nature such as bones or plants, where the cellular structures vary in thickness or size. FGM's are advanced multifunctional materials with spatial gradation in composition, i.e. the mechanical properties.

As noted by Minoo Naebe [7], "The ability to understand and manipulate materials has been fundamental to our technical development over time.". Many researchers have taken an interest in the mechanical response of FGMs concerning the load response to dynamic loading, and the energy absorbing characteristics of cellular graded structures. With functionally graded foams, the desire is to diminish the peak acceleration generated from impact in order to mitigate the stress wave[5]. The stepwise crushing from the lowest density to the highest allows the stress to gradually dampen out.

For the example of a helmet liner, the density gradient could reduce the severity of the injury induced in the person being protected[2].

The capability to optimize a structure by gradually varying the morphology is something most any scientist would be interested in, and offers large possibilities for engineering ingenuity. The prospect of continuously graded materials has been pursued by many to have a material with specific spatial variation of structural properties. The fabrication and manufacturing techniques of continuous functionally graded materials are being developed; there are many challenges still involved in the processes. The utilization of FGMs and their multifunctional properties can reduce stress in the overall structure. The material architecture has great potential to improve structural performance and meet the many demands of advanced materials to solve a wide array of problems. Graded cellular materials are highly developed structures thanks to their evolving solid-volume fractions in the preferred material axis orientation. The continued advancements and research into Functionally graded foam materials will lead to improved light-weight energy absorbing structures for any loading condition.

1.2 OUTLINE AND OBJECTIVE

The structure of this work will comprise of different analyses of homogeneous and heterogeneous cellular materials. The mechanical characterization of uniform and graded foams is presented for various strain rates. The full-field measurements are acquired using Digital Image Correlation. First the constitutive response of the uniform foam is examined for quasi-static and dynamic loading rates. The mechanical properties for each density foam are acquired. Next the linear graded foam structure is compared with the

responses of each uniform foam. Lastly the placement of the different density layers will be rearranged to analyze the effect of the sequence of mechanical properties.

The objective of this present work is to accurately experimentally characterize dynamic response of FGFMs and optimize the gradient for energy absorption. The full-field stress is calculated by superimposing the boundary-measured stress with the calculated inertia stress. The stacking sequence of discretized layers of different bulk densities is varied to determine the ideal gradation for load bearing performance. FGFM samples are constructed and subjected to dynamic loading via SHPB to generate stress in the discretized foam samples. Based on the experimental results the effects of orientation on the response of FGM's are analyzed by observing the stress-strain response. The conclusion of these experiments will serve to improve the understanding of graded energy absorbing structures and optimize protective cushioning structures in their endless array of applications.

CHAPTER 2

LITERATURE REVIEW

Foam materials have attracted a lot of attention for many years especially when energy dissipation and low structural weight are desirable design considerations. In order to use these materials in applications and design, the mechanical response needs to be well defined for any loading condition. The quasi-static response has long well been understood and accepted and used to define these structures. However, it is not a fair assumption to make that the defined quasi-static properties can properly characterize the material in all real-life scenarios. Attributable to the many applications that involve damage protection there is a need for understanding of the response of these low-mechanical-impedance materials under high strain rate loading conditions. But the dynamic mechanical responses of these materials under such conditions are not well understood.

There are many issues that come into play with dynamic testing of polymeric foams. Soft materials possess low stiffness, yield strength, and wave speeds. Low stiffness requires highly sensitive load detection, the low strength causes significant inertia effects, and the low wave speed delays stress equilibrium. Chen[8] addressed the issues with using the Kolsky technique for soft-material characterization and offered remedies to better obtain more accurate results. The use of polymeric bars has been employed to reduce the impedance mismatch thereby improving the accuracy of the transmitted signal [9]. Liu [10] offered a solution using large diameter nylon Hopkinson

bars in conjunction with optical field measurements. Some researchers have explored the transient stress-state to investigate the initial dynamic deformation under impact. A non-parametric method was developed to be able to discount the dynamic equilibrium assumption. The technique [11] deduces the stress-strain relation at any point on the specimen using a force measurement at the specimen boundary and a displacement measurement using Digital Image Correlation. It was found that some parts of the material were strained up to 5% while the distal end wasn't strained at all. Pulse-shaping techniques have been introduced to ensure uniform deformation of low-impedance materials at high strain rates [12]. Hu analyzed the dynamic behavior of hexagonal honeycombs, another species of cellular materials ideal for energy absorption. He found that 90% of the crushing strength is attributed to inertia effects under dynamic crushing conditions. This goes to show why cellular materials possess higher crushing strength under impact loading versus quasi-static conditions [13]. There are many analyses and methods of characterization that have been proposed specifically on energy absorption of soft materials. Avalle [4] examined the energy absorption characteristics of compressive impact loading on polymeric foams using the energy-absorption diagram method.

While the examination of homogeneous foam is nothing new, the investigation of functionally graded foams is still in its early stages. Functionally graded materials first gained a lot of interest as a combination of ceramic and metals, eliminating sharp interfaces that caused severe stress localizations. Virtual FGFM's have been studied by Kiernan[14] using SHPB and the energy dissipation was found to be shaped by the gradient. He found that the propagation of the stress wave was discernibly defined by the variation in density; consequently, the damage inflicted on the object being protected

could be reduced. Finite element simulations showed that FGFMs exhibit superior characteristics over the equivalent uniform foam and that the density range plays a role in the performance of the graded foam[2]. The determination of the ideal density range is dependent on if the impact is high or low energy.

Numerical simulations require the actual physical response to authenticate the material models. Functionally graded syntactic foams were fabricated that were capable of reaching 60-75% compression without significant loss in strength[15]. The compressive strain and strength were found to be dependent on the composition of the material with strong sensitivity to the weakest layer in the structure. The functionally graded syntactic foam was found to have 200-500% increased energy absorption versus the comparable plain foam [16]. Sandwich configurations with middle-high and middle-low density distributions were analyzed with respect to their impact response. They were investigated theoretically and numerically and found to improve the energy absorption [17]. Koohbor [18] employed an analytical approach to study the effect of discrete and continuous density gradation which was validated with experimental data. The optimal gradation is contingent on the amount of deformation that would be exerted on the structure. It is apparent that the gradient effect of these materials has been carefully studied analytically and theoretically but there still lacks experimental work in the literature.

CHAPTER 3

CONSTITUTIVE RESPONSE OF UNIFORM FOAM

3.1 ABSTRACT

Cellular materials are abundantly used in commercial, industry, and military applications therefore there is a necessity for a thorough understanding of their non-linear properties and viscoelastic response. Furthermore, effective testing methods have required innovation for these unique materials. The constitutive behavior of homogeneous polymeric foam is investigated experimentally at various loading rates. Quasi-static and dynamic experiments are performed to observe the materials' response at differing strain rates. MTS and Split Hopkinson Pressure Bar (SHPB) are used along with Digital Image Correlation to perform the experiments. Polyurethane samples of five different densities were examined to observe the effect of the relative density. A non-parametric method based on force measurement at the specimen boundary and the strain field measured from DIC is implemented to identify the stress field.

3.2 INTRODUCTION

Polymeric foams are widely used in energy absorption applications across many industries. Foams make exemplary dampers due to their ability to reduce noise. They are relatively cheap and have great heat insulation along with their many other redeeming properties. Structural foams are commonly used as the core of sandwiches. There are many applications that are leading to the use of these soft materials in high rate and large deformation situations. For that purpose, there has come an increasing amount of study

into the strain rate dependency. Chen [19] modified a SHPB to perform dynamic compressive experiments on polymeric foams and found the dynamic stress levels to be twice as much as the corresponding quasi-static values. Weißenborn adapted an analytical model to predict the strain rate relationship to the stress-strain response for foams with different densities [20]. Validating the results with experimental tests it was found that there is increasing strain rate sensitivity with higher density foams.

The majority of dynamic loading methods are valid only when the stress and strain are homogeneous. But with the low impedance of foam, dynamic equilibrium is difficult to achieve. A new method using the full-field measurements of strain and inertia stress is implemented to account for the delayed stress equilibrium. The inertia stress is superimposed with the measured boundary stress from SHPB using a reconstructed general dynamic stress equilibrium equation. The mechanical properties and constitutive response are acquired for five foams of differing nominal density. The average stress-strain response of each foam will be presented since the materials do not deform uniformly. The stress-strain curve is used to calculate the energy absorbed for each material.

3.3 MATERIAL AND SPECIMEN GEOMETRY

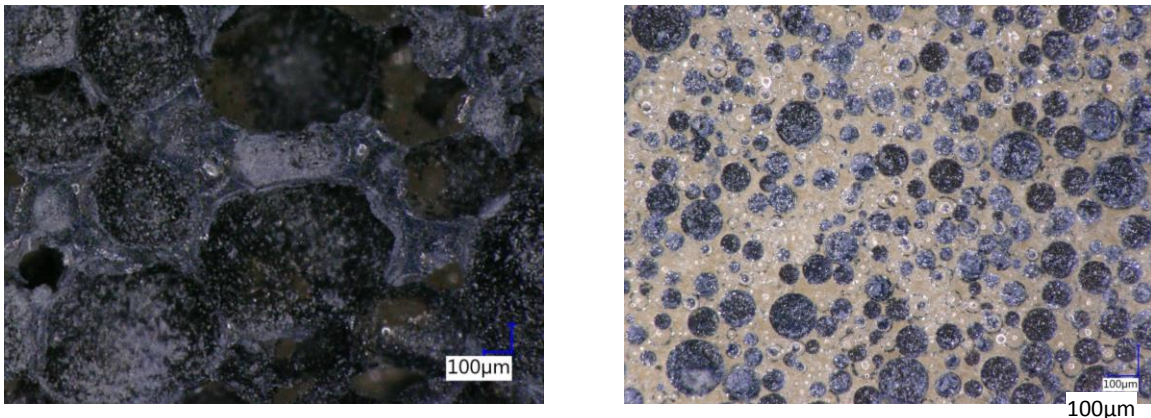


Figure 3.1. Lowest and highest density microstructure

Rigid closed cell foams are used, meaning the gas-filled pores are sealed, or isolated. The foams used for this experiment are considered “soft materials” because using the Kolsky method, a specimen having a strength less than 10 MPa presents issues associated with soft material characterization[8]. Polyurethane foams of 64 kg/m³, 160 kg/m³, 320 kg/m³, 480 kg/m³, and 640 kg/m³ nominal densities were purchased from General Plastics, specifically the FR-7100 Aerospace grade series. So, the matrix material is the same for every sample but the pore size is varying. The difference in the microstructure is shown in Figure 3.1 for the lowest and highest density foams. Rectangular prism specimens of 25.4 x 17 x 17 mm³ for both sets of experiments were fabricated in-house. The samples were extracted from the as-received billets supplied by General Plastics using a band saw, and then final machining was executed using a milling machine to achieve the final desired dimensions. It is important to note that all the samples were cut parallel to rise which has to do with how the gas bubbles rose in the polymeric material during the process of manufacturing. A random high-contrast speckle is applied on the front surface in accordance with the necessary pattern for DIC and shown in Figure 3.2.

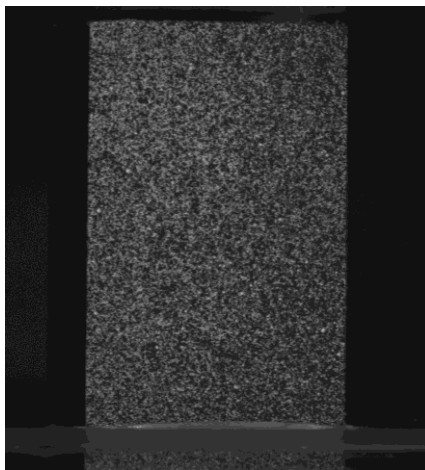


Figure 3.2. Speckle pattern

3.4 EXPERIMENTAL SETUPS

A 5 kN MTS load frame was used for the quasi-static compression experiments. A smaller load cell was needed to obtain more accurate data for the low-density materials but the highest density foam was unable to fully densify using this particular setup. The low rate experiments were conducted under displacement control at an average applied strain rate of 0.001 /s to obtain the loading data. During testing, images were captured at a rate of 1 fps using a 5 MP stereo camera system equipped with 60 mm lenses. The Point-Grey cameras have a full-field resolution of 2448 x 2048 pixel². Stereo DIC was implemented in order to account for lateral deformation of the foam. The cameras were at a distance of about 300mm from the specimen and the angle between the two cameras was 12.3°. The complete experimental setup for the quasi-static experiments is shown in Figure 3.3.

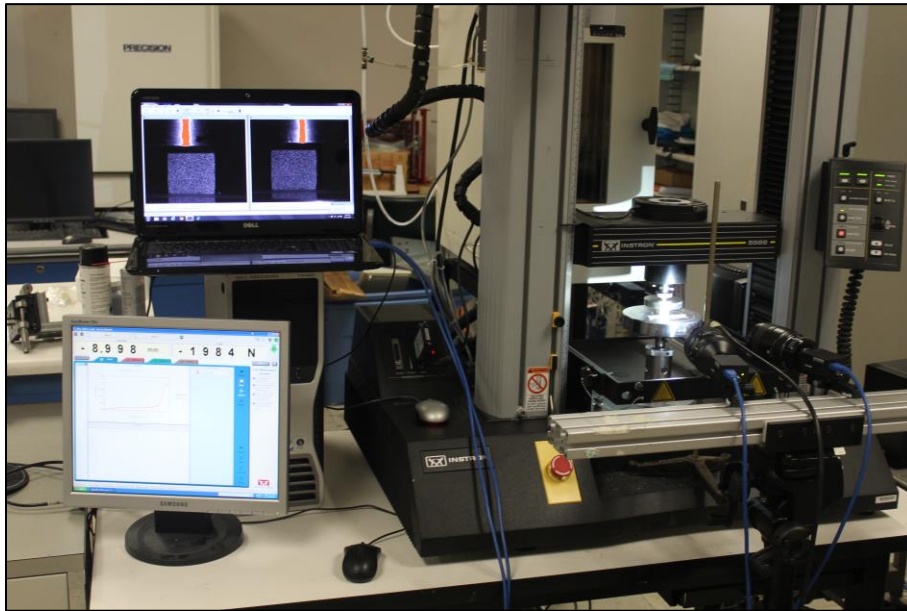


Figure 3.3. Quasi-static experimental setup

The elements of the dynamic testing operation include a dynamic loading device, bar components, data acquisition and recording device, and high-speed imaging system.

The sample was loaded dynamically using a Split Hopkinson Pressure Bar. SHPB is one of the most commonly used experimental setups for strain rates between 10^2 and 10^4 s^{-1} . This dynamic testing method employs stress wave propagation through elastic bars to load a sample and measure the deformation and loading. Standard electrical resistance strain gauges are used to measure the compressive and tensile stress waves propagating through the bars. The schematic drawing and a picture of the experimental setup are shown in Figure 3.4. The SHPB method and its analyses are typically used for testing metals and other high mechanical impedance materials. The uncertainties associated with soft material characterization will be addressed by changing the conventional method. For valid results some modifications had to be made to the standard analyses for testing low mechanical impedance materials, which will be discussed in Section 3.5.2.

Lubrication was used between the sample and bar interfaces to eliminate friction. The low mechanical impedance of the specimen causes the incident bar-specimen interface to act similar to a free surface for the lower density samples. Therefore, most of the incident wave is reflected back and only a small portion of the loading pulse is transmitted through the sample to the transmitter bar. Consequently, the transmitted signal has very small amplitude that can be difficult to differentiate between it and noise. To lessen the impedance mismatch between the specimen and bars, Polycarbonate bars and striker were used amplify the transmitted signal. The use of these viscoelastic bars thereby improves the accuracy of the measurements. The striker bar has a length of 0.45m to achieve large sample deformation. The striker bar is propelled using high pressure helium to generate an elastic stress wave.

To measure the full-field deformation during loading two ultra-high-speed HPV-X2 Shimadzu cameras were used to acquire synchronized images on both sides of the specimen. The cameras were placed at a distance of 0.3 meters from the face of the sample and checked to be planar and parallel to the specimen face. The cameras are capable of acquiring 128 images at the full-field resolution, and for this particular work imaging rate of 200,000 and 625,000 frames per second were utilized, meaning one image was captured every 1.6 and 5 μ s. Since 128 images were taken with both cameras the full-field compressive response can only be captured for 204.8 and 640 μ s which do not capture the entire deformation since the stress wave is about 730 μ s long. The low framerate was used to capture a larger duration of the deformation and the higher framerate was employed to have better temporal resolution, to better capture the displacement during the initial transient time of impact. There is high acceleration during the transient time as the sample goes from being stationary to experiencing high relative motion. This high relative change in particle velocity must be captured for the dynamic evaluation.

The cameras were equipped with a 100-mm macro lens (Tokina) providing an optical resolution of 100 μ m/pixel. A high intensity flash monolight (Photogenic) was used to illuminate the area of interest on the sample after trigger from the strain gages. The entire system was triggered using an oscilloscope. The oscilloscope also collected the wave data from the strain gauges through a signal-conditioning amplifier. The projectile velocity was measured using a laser system which found an average of 16.5 m/s.

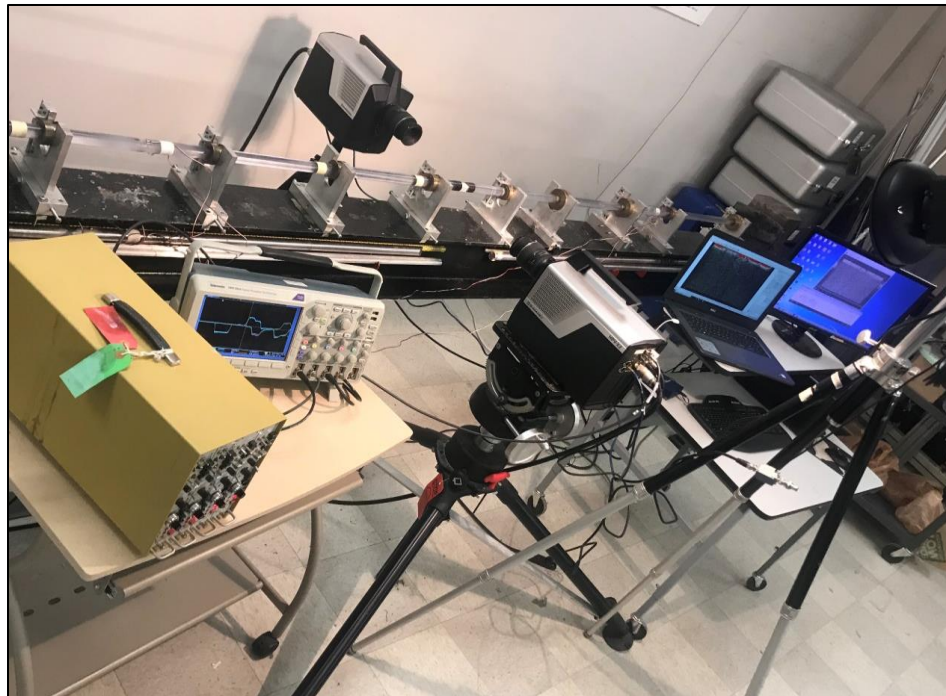
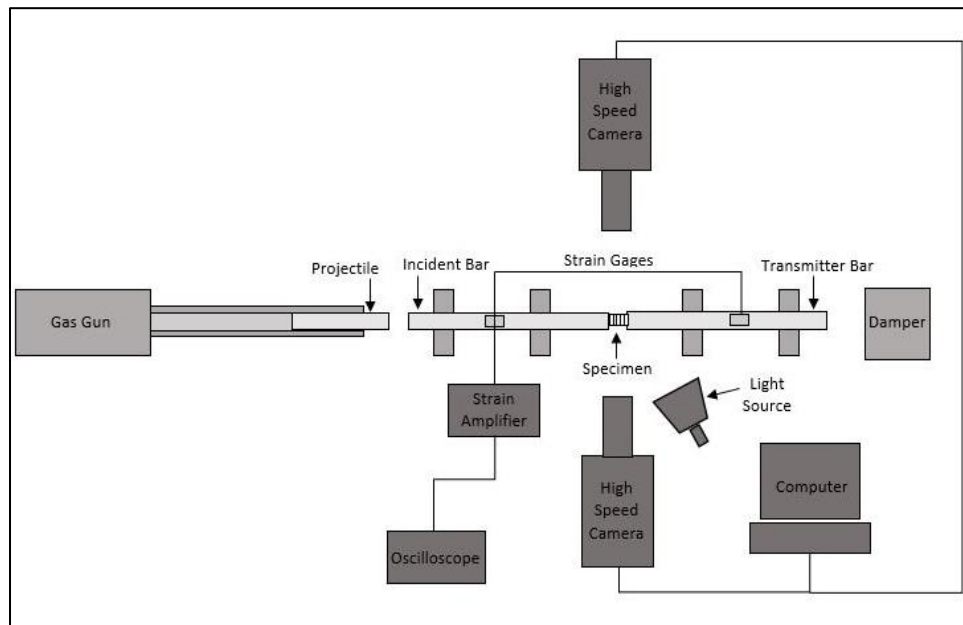


Figure 3.4. Schematic illustration and photograph of dynamic experimental setup

3.5 POST PROCESSING

3.5.1 DIGITAL IMAGE CORRELATION

Digital Image Correlation (DIC) is a non-contact optical technique used to measure the full-field displacement of a surface. A speckle pattern is put on the surface and while being deformed the images taken are used to correlate between the undeformed and deformed state to give the displacement. Then the displacement can be differentiated to find the strain. The images acquired from the quasi-static experiments were processed in the commercial DIC software VIC-3D (Correlated Solutions Inc.). The average speckle size was taken into account to choose the subset and step sizes of 67 and 5. The subset must have sufficient uniqueness and cover at least 6x6 speckles (3 black and 3 white). The center-weighted Gaussian filter was used to derive the strain distribution from the full-field displacement. The correlation criterion was selected to be zero normalized squared difference which is insensitive to the scale of the light intensity, which is necessary since a flash of light is being used. Optimized 8-tap interpolation function was used to convert the discretized point measurements to continuous data.

For the dynamic tests the images were analyzed using VIC-2D (Correlated Solutions Inc.). A subset and step size of 17 and 1 were used for image correlation. The analysis was performed using an incremental algorithm to enable data acquisition for significantly large compressive strains.

3.5.2 FULL FIELD STRESS AND DENSITY CALCULATIONS

The present analysis is based on the assumption of uniaxial deformation. To incorporate the effects of inertia which is known to have a dominating role with soft materials, a new approach is used to define the total axial stress using the full-field

deformation response. Newton's law of motion is manipulated to give the following total stress equation:

$$\sigma_1(x, t) = \frac{F_1(0, t)}{A(0, t)} + \int_{\xi=0}^{\xi=x} \rho(\xi, t) \frac{\partial^2 u_1}{\partial t^2}(\xi, t) d\xi$$

The reconstructed equation shows that the total stress is a location dependent quantity. The first term of the equation is the boundary stress calculated from the transmitted signal. The boundary term is calculated from the transmitted signal using standard one-dimensional wave analysis and time shifted to the bar-specimen interface. The integral term represents the spatial distribution of the inertia stress that is calculated using the displacement data from DIC and material density. The length of the specimen is sliced into 100 segments to acquire the local response. The displacement field is differentiated with respect to space and time to obtain the acceleration field. The acceleration distribution was determined using a finite differences operation and temporally smoothed. The formulation is credible in the event that lateral and shear deformations are negligibly small i.e. the uniaxial deformation assumption is validated.

The foams are highly compressible meaning significant density change is undergone during loading. The density field is calculated as a function of the displacement data obtained from DIC and the initial density, ρ_o . The local density, ρ , at position, x , and time, t , is found to be:

$$\rho(x, t) = J\rho_o$$

Where J is the Jacobian at any point at a given time, which is the transformation from the undeformed to the deformed state. J is the determinant of the deformation gradient, F , which is determined from the DIC measured displacement field, d . More

details on the derivations of these stress and density calculations can be found elsewhere[21].

3.6 RESULTS AND DISCUSSION

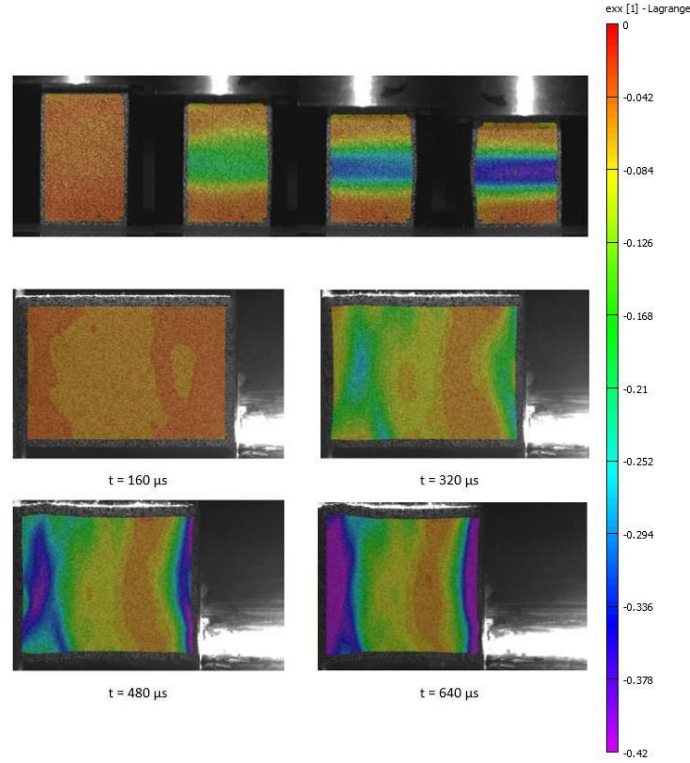


Figure 3.5. Axial strain contour plots of dynamic test of 160 kg/m³

The evolution of the axial strain on the surface of the specimen is shown in Figure 3.5. It is shown that the deformation is mostly symmetric about the centerline of the sample. Though the non-uniform distribution of strain in the axial direction is clearly noted. The quasi-static stress-strain curves for each nominal density is shown in Figure 3.6. The initial slope of the stress-strain curve is the linear elastic regime from which the Elastic Modulus, E , is calculated. This deformation region is controlled by cell wall bending (more specifically, cell face stretching for closed cell foam [22]). The plateau region is associated with the collapsing of the cells. Once all of the cell walls are touching, the cells are completely collapsed and all the void spaces are removed, there is

a rapid increase of stress which is known as the densification stage. This dramatic stiffening of the material happens when the cell wall material itself is being compressed, it acts as a solid.

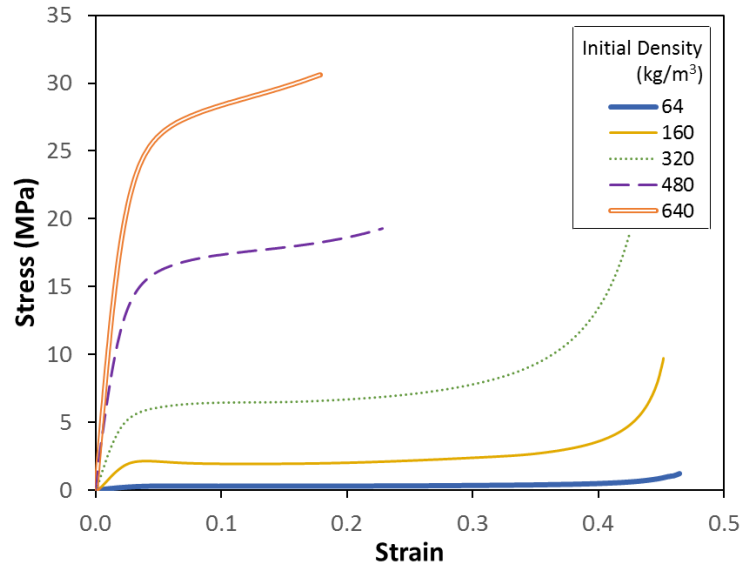


Figure 3.6. Quasi-static compressive stress-strain curve for various densities

It is clear to see that increasing the density of the foam increases the elastic modulus, increases the plateau stress, and reduces the densification strain. Two tests were done for each foam to conclude that the elastic modulus and yield strength values were consistent. The yield point is recognized as the first point on the stress-strain diagram where the strain increases without an increase in strain. It is where the elastic deformation stops and the material begins to deform plastically. The compressive strength is defined to be the stress at the yield point unless it occurs after 10% deformation. If there is no reduction in stress, then the stress at 10% strain is the compressive strength (ASTM standard D1621). The low-density foams exhibit a drop in stress after reaching a peak stress while the higher density foams only show strain hardening. The lowest density

foam has the lowest compressive strength because it has the largest volume of voids, therefore there is a smaller volume of the solid material.

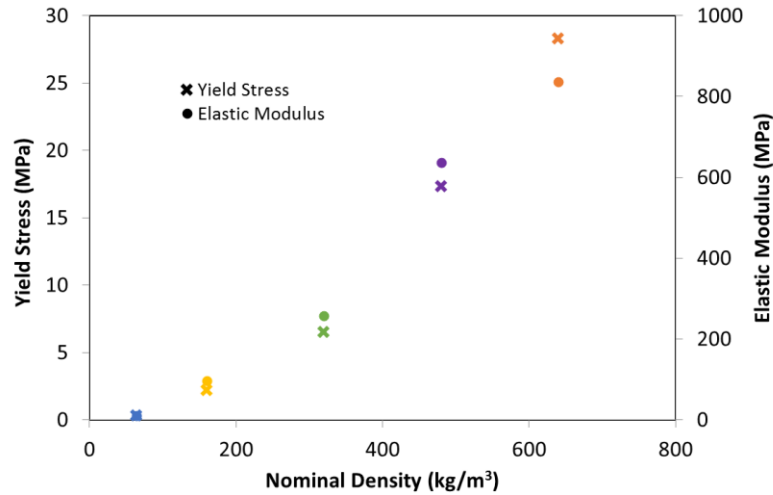


Figure 3.7. Variation of elastic modulus and yield stress as function of density

The distinction between the yield stress and elastic modulus as related to the foam density are depicted in Figure 3.7. A linear relationship is observed for these mechanical properties. It is important to remind the reader, that even though the highest density sees higher yield strength, that does not make it the ideal foam. The end goal is to have the most efficient energy absorbing structure, which means a large area under the stress-strain curve is what is truly desired.

Next the results of the dynamic experiments are investigated. Figure 3.8 shows the incident and transmitted signals from the strain gauges for the lowest and highest density foams. The lowest density has very little transmitted stress, the material is so soft that the interface acts similar to a free surface, almost all of the wave is reflected. Note that the transmitted signal in Figure 3.8.a is magnified x5. Whereas the highest density foam has a very small reflected wave and a very large transmitted wave. Using one wave analysis the boundary stress was calculated from the transmitted signal.

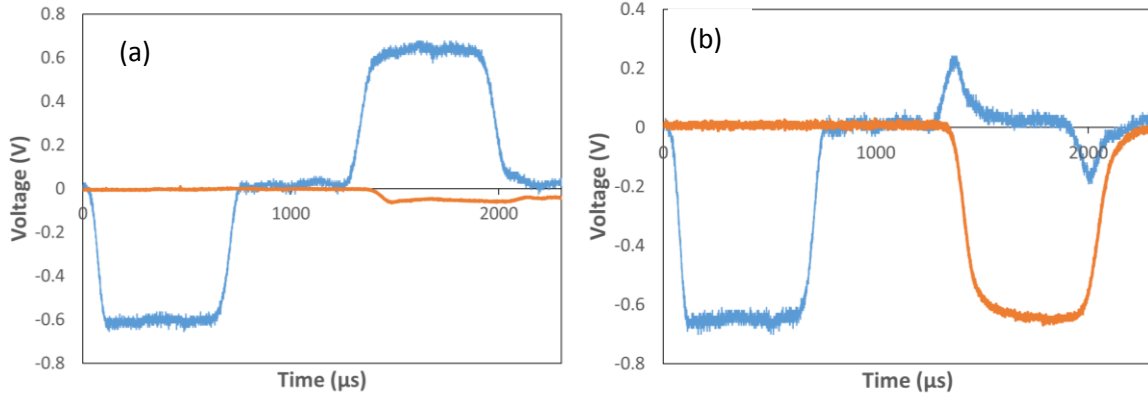


Figure 3.8. Oscilloscope voltage signals of dynamic compression tests for (a) 64 and (b) 640 kg/m³ foam densities

The axial acceleration along the specimen axis was calculated by taking a double derivative of the axial displacement that was recovered from the camera taking images every 1.6 μ s. This high framerate has better accuracy to not filter out any data during the transient loading process. The spatial variation of the axial acceleration for different times is shown in Figure 3.9. The acceleration curves increase everywhere along the length of the specimen up to 30 μ s, and then the acceleration starts to come down. In Figure 3.9.b the acceleration curves are observed to oscillate around the zero axis. This shows that the foam is performing its desired function of damping the peak accelerations.

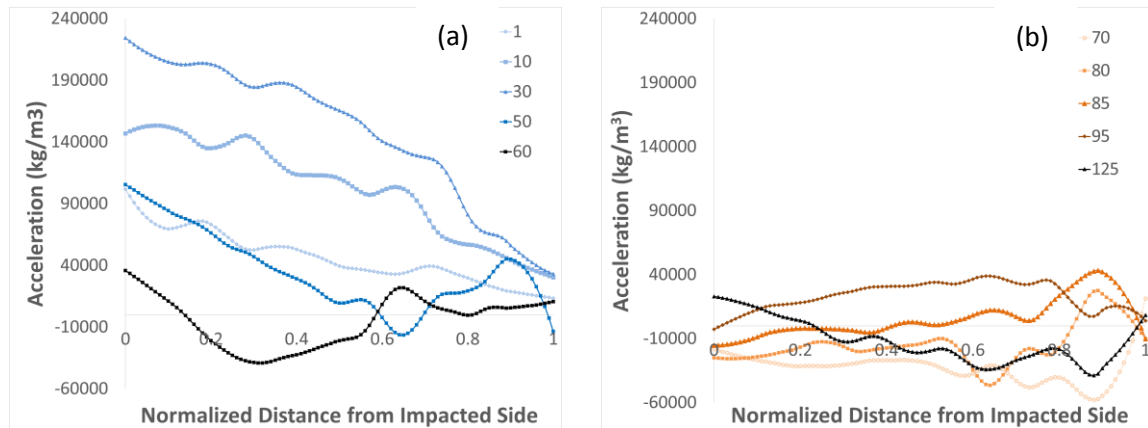


Figure 3.9. Acceleration curves of 160 kg/m³ foam for the (a) first half of the test and (b) the second half

The axial acceleration for all five foams is shown in Figure 3.10. All five of the foams follow a similar trend but the magnitudes slightly vary. Since all of the materials experience the same force for the same time duration, but have differing masses it would be expected that the displacement would vary causing the acceleration to be different. It is shown that the peak acceleration diminishes as time goes on. Those oscillations seen after the peak acceleration are in all probability just noise from the displacement data. When the acceleration becomes negative signifies that the stress wave has reached the end of the length of the specimen and reflects back. It is observed from these acceleration curves that during the transient deformation stage there occurs a rapid increase in particle motion.

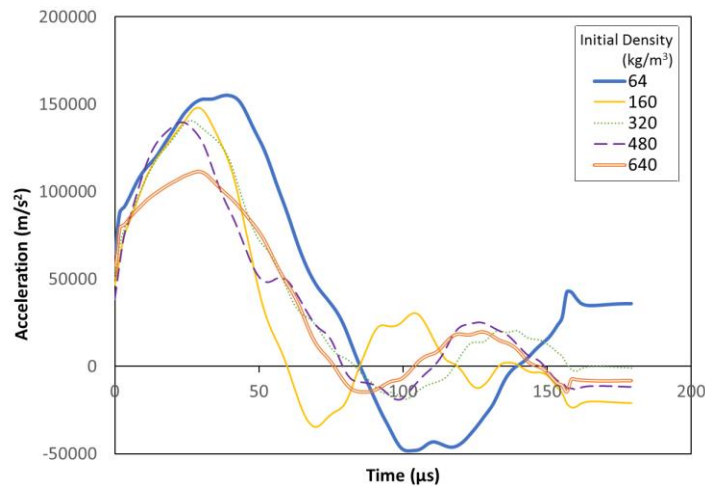


Figure 3.10. Acceleration for all five foams

The spatial distribution of the axial inertial stress is calculated using the previously given equation with the functions of acceleration and local density. The spatial variation of the inertia stress along the length of specimen is depicted in Figure 3.11 for different times. Similar to what was observed with the acceleration curves in Figure 3.9, the maximum peak occurs within the first 30 μs at the impacted end.

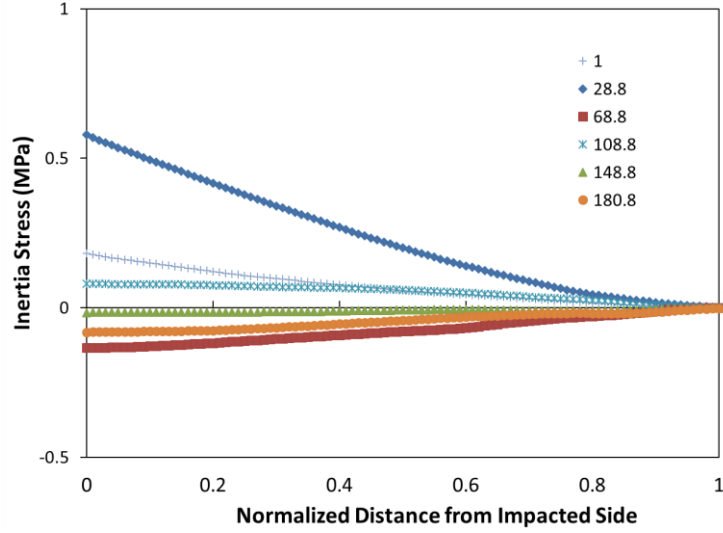


Figure 3.11. Inertia stress for 160 kg/m^3

The positive inertia stress is compressive for this work, which when added to the boundary stress will increase the overall axial stress. The observed inertia stress is clearly significant at the impacted end and cannot be ignored for these foam materials. The inertia stress is added to the boundary stress measured from the transmitter bar using the previously given equation. The added inertia term results in the appearance of a spike in the stress in the initial portion of the stress curve.

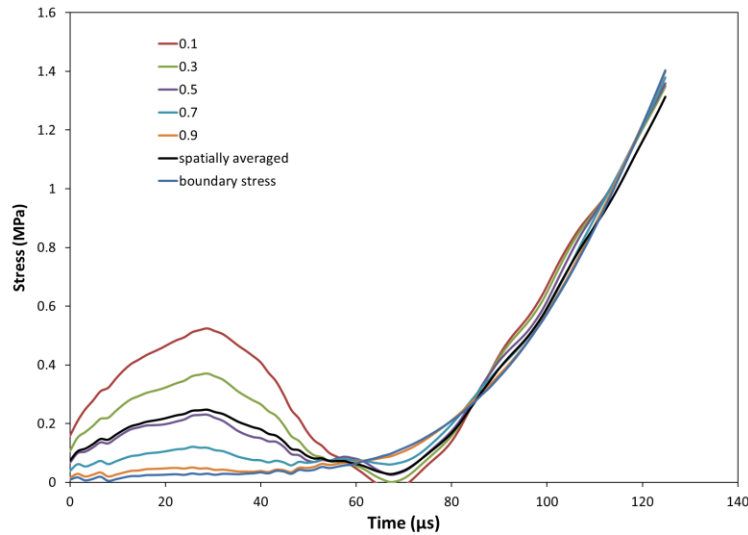


Figure 3.12. Superimposed inertia and boundary stress for each layer for 160 kg/m^3

Figure 3.12 shows the local evolution of the total axial stress at five different points along the specimen length. Substantial variation is observed during the first 80 μs after impact. The spatial variation decreases at longer durations due to the acceleration damping out. Even though the stress and strain vary locally along the specimen length, it is the global stress-strain curve that is needed to determine the specific energy. The local stress curves are spatially averaged and are shown in the graph to give the effective average constitutive response. The total stress-strain curves for all the foams are given in Figure 3.13. The global strain was directly output using DIC. Also, it should be noted that the stress referred to throughout this work is the engineering stress since the true stress cannot be calculated because it assumes that the volume remains constant which is not valid with these compressible materials.

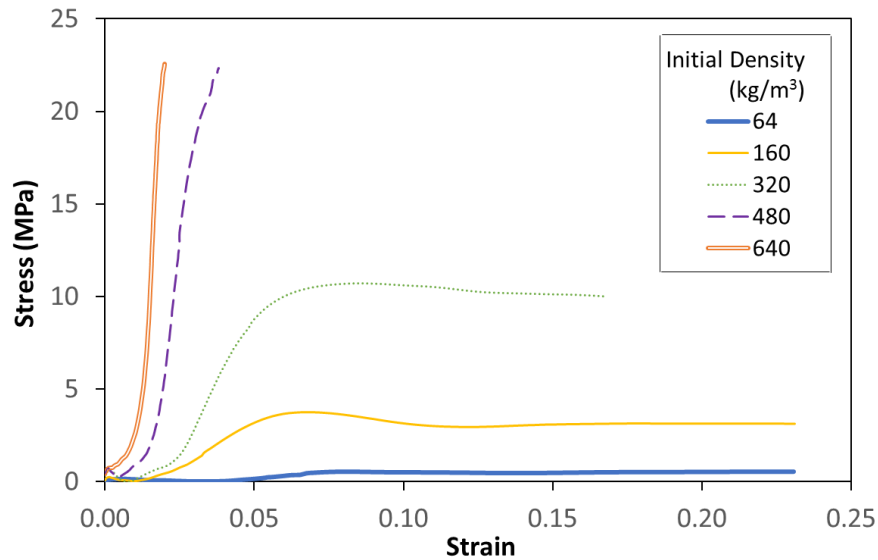


Figure 3.13. Stress-strain curve for dynamic tests

The depicted stress-strain curve is only for the first 640 μs , since that is the duration of the images being captured to find the global strain. The inertia stress intensifies the SHPB measured stress during the early transient time. The plateau stress

plays the biggest role in energy absorption. The highest density reaches the highest maximum stress but densification begins at a very low strain. The lowest density has a very long plateau stress, it experiences very large deformation but at the compromise of a very low maximum force. The strain energy density (the strain energy per unit volume) is calculated by measuring the area under the stress-strain curves. This value is graphed as a function of stress and strain in Figure 3.14.

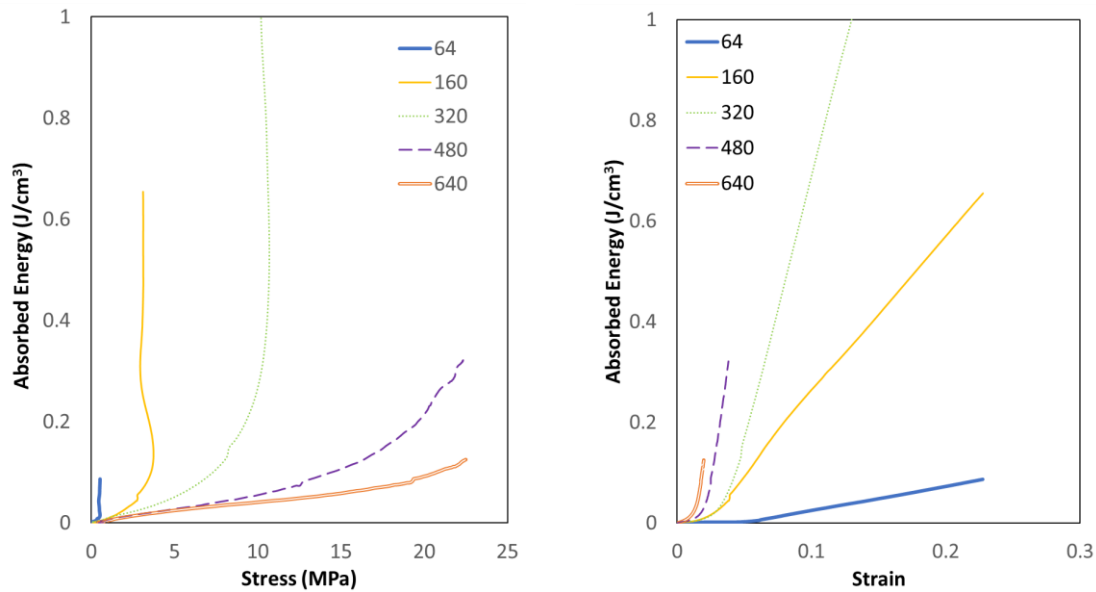


Figure 3.14 Energy absorbed as a function of stress and strain

For a predefined stress design criterion, as the desired stress is increased, the increasing density foam becomes the next ideal foam. For 1 MPa stress, the lowest density foam as the highest amount of energy absorbed, but for a stress of 23.5 MPa, the highest density would be the selected foam. The opposite matter is true for a specific level of strain. For smaller strain, the 640 kg/m³ is “better, but for a large amount of deformation the 64 kg/m³ would be chosen. The increase in plateau stress is shown for the dynamic as compared to the quasi-static tests in Figure 3.15. This apparent strain rate sensitivity is reliant upon the viscoelastic nature of these polymeric foams.

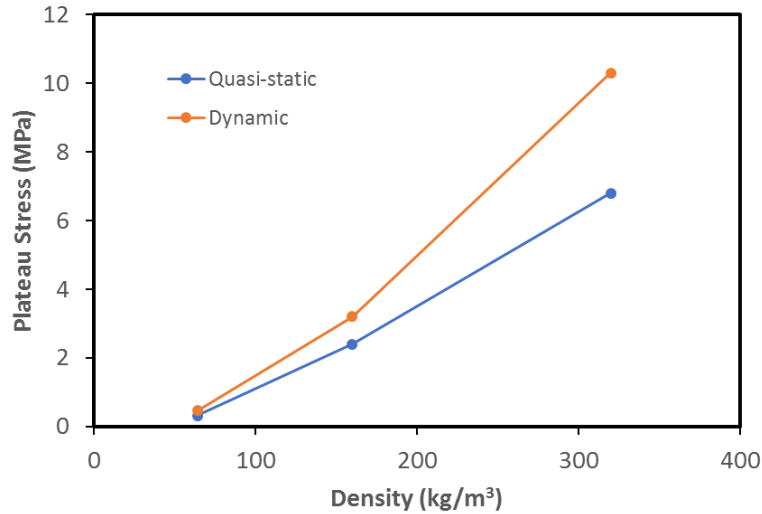


Figure 3.15. Strain rate sensitivity of three lowest density foams

3.7 CONCLUSIONS

The energy absorption of five different density foams is found experimentally for low and high strain rates. A servo-mechanical load frame and Split Hopkinson Pressure bar are used in conjunction with Digital Image Correlation. For accurate results with SHPB the best approach is to have uniform deformation at an equilibrated stress state and constant strain rate. None of these conditions are present for a soft material under dynamic loading. A new method that takes account of the inertia effects and compressibility of the sample is applied in this work. The stress-strain response of each material is calculated to determine the mechanical properties and constitutive response of the homogeneous foams. The stress-strain response clearly has a strong relation to the density of the foam. The elastic modulus and yield strength increase with increasing density, and the densification strain decreases. This stress-strain relationship is also indicative of the correlation between the density and the energy absorption. The material is also shown to have high strain rate sensitivity, the response gets stiffer with increasing strain rate.

CHAPTER 4

DENSITY GRADED FOAM COMPARISON STUDY

4.1 ABSTRACT

The improvement of energy absorption with graded cellular materials is investigated through experimental techniques. Dynamic testing of polymeric foam is performed using Split Hopkinson Pressure Bar (SHPB) and Digital Image Correlation. The validity of dynamic loading conditions depending on homogenous stress distribution is overcome by including the effects of inertia. The dynamic stress equilibrium equation is utilized with a boundary stress measured from the SHPB while the deformation of the entire foam specimen is recorded with high speed photography. The data is mathematically processed in order to draw the characteristic stress-strain curve. Results indicate that the layered structures yield an improved energy absorption compared with monolithic foam under specific design criterion.

4.2 INTRODUCTION

Functionally graded materials are in the family of engineering composites that are designed for a specific function. There has been a considerable amount of numerical work done to characterize the dynamic deformation of FGMs but a lack of experimental study is noticed. This has to do with the many challenges with high strain rate testing of soft materials that make it difficult to achieve an acceptable characterization of this response. So there are already many issues that come into play with dynamic testing of foams, let alone a soft material with density gradation. The non-equilibrated stress is a

result of the axial inertia, which will be taken into account in the analysis of this work. The concept of Functionally Graded Foam Material (FGFM) has been introduced to improve upon certain properties as compared to a homogenous foam. The appeal of FGFM is to enhance the energy absorption characteristics due to the presence of the high-density foam, while maintaining a low structural weight. A graded structure with 5 different densities is examined.

4.3 MATERIAL AND SPECIMEN GEOMETRY

The five distinct layers of different bulk densities form the samples prepared in house, which are made from the uniform foams used in Chapter 3. Each layer of polyurethane foam was machined to $17 \times 17 \times 5 \text{ mm}^3$ and then polished using silicon carbide papers to have a smooth surface, suitable for speckling. The layers were bonded together with a thin layer of highly-flexible polyurethane adhesive. It is vital that the adhesive is flexible to account for small relative lateral deformation of the layers and to minimize shear stress developed within the interface. The final length of the specimens is 25 mm. A schematic of the arrangement of the layers along with a picture are shown in Figure 4.1.

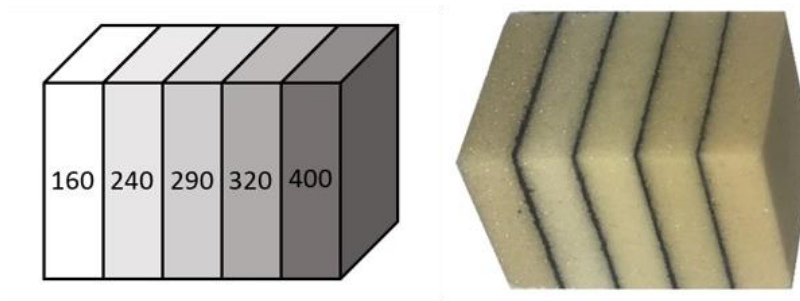


Figure 4.1. Graded foam schematic and photograph

Before speckling, a marker was used to draw a line between the layers to indicate the interfaces. A small piece of tape was used to cover part of the front surface before

speckling to leave a strip unpainted to help distinguish the different layers during deformation. The front surface of the specimens was painted white and then black speckling was applied using conventional spray painting method. The same experimental methods were used for quasi-static and dynamic testing of the five-layered density graded structure.

4.4 RESULTS AND DISCUSSION

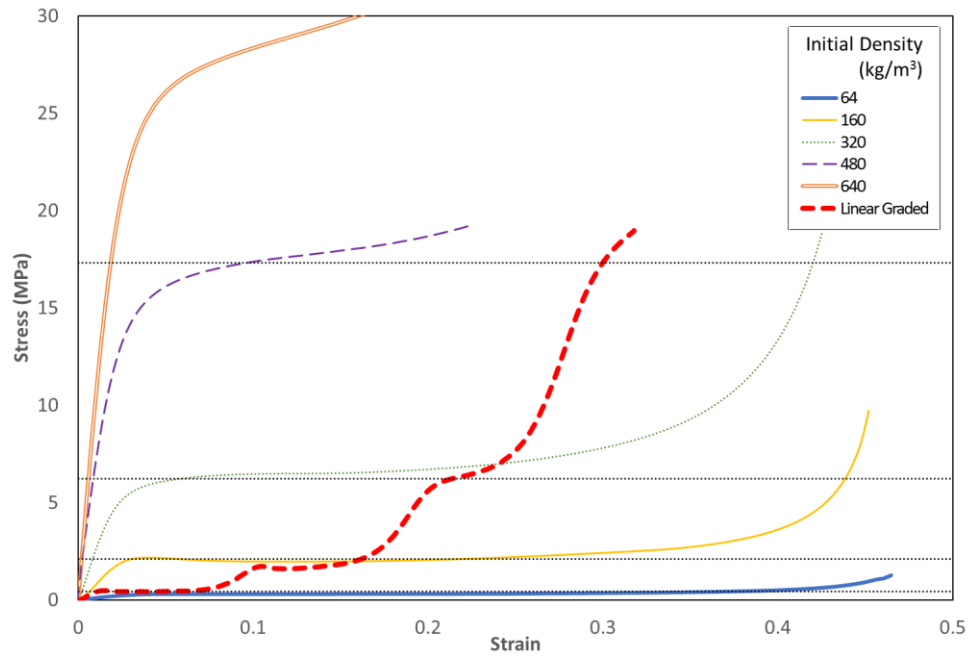


Figure 4.2. Stress-strain curve for quasi-static tests

The quasi-static stress-strain response for the graded structure is shown in relation to the uniform foams in Figure 4.2. Each of the incremental stepwise plateaus correspond to the plateau stress of that corresponding uniform foam. Figure 4.3 investigates the energy absorption compared with the five uniform foams.

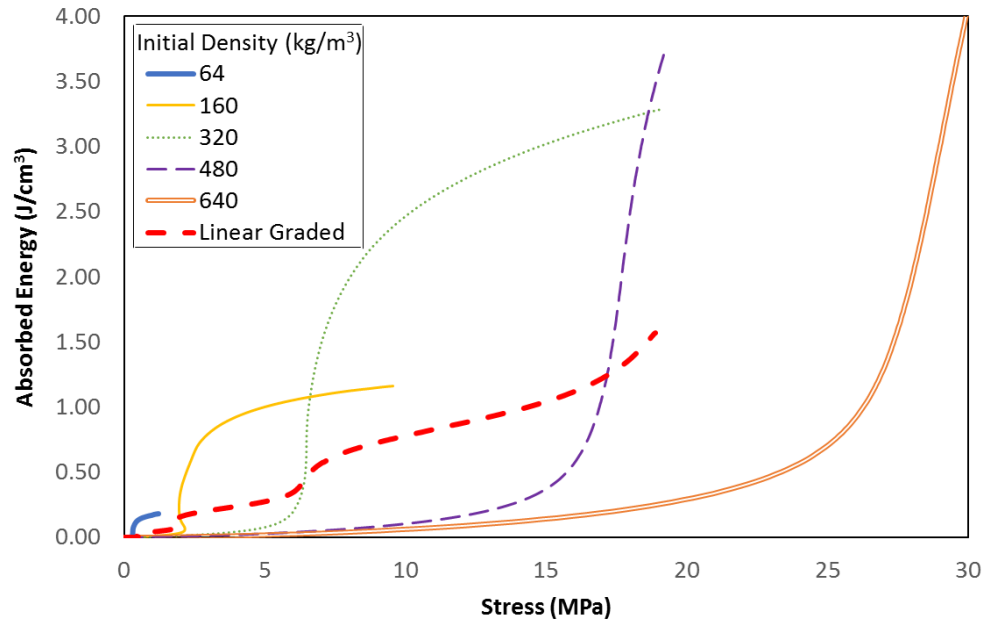


Figure 4.3. Absorbed energy as function of stress for quasi-static experiments

The graded foam shows improved energy absorbance performance compared to each homogenous foam up until the uniform foam curve intersects the graded foam curve and the absorbed energy increases dramatically. This initial improvement could be due to the extended plateaus provided by the low-density layer. It is noted that each time the absorbed energy curve sees a rise it intersects the corresponding uniform foam curve.

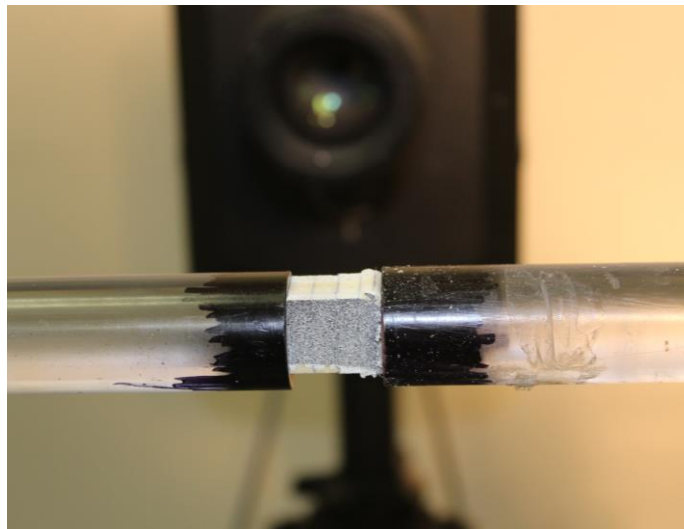


Figure 4.4. Picture of deformed specimen

Next the dynamic response of the graded foam as compared to the uniform foams is presented. A photograph of the specimen after deformation is shown in Figure 4.4 to examine how the layers deformed. It is noted that the entire sample was not completely deformed, only the first three layers completely failed. This was not a characteristic of the structure but rather not a large enough load was applied to crush the highest density layers.

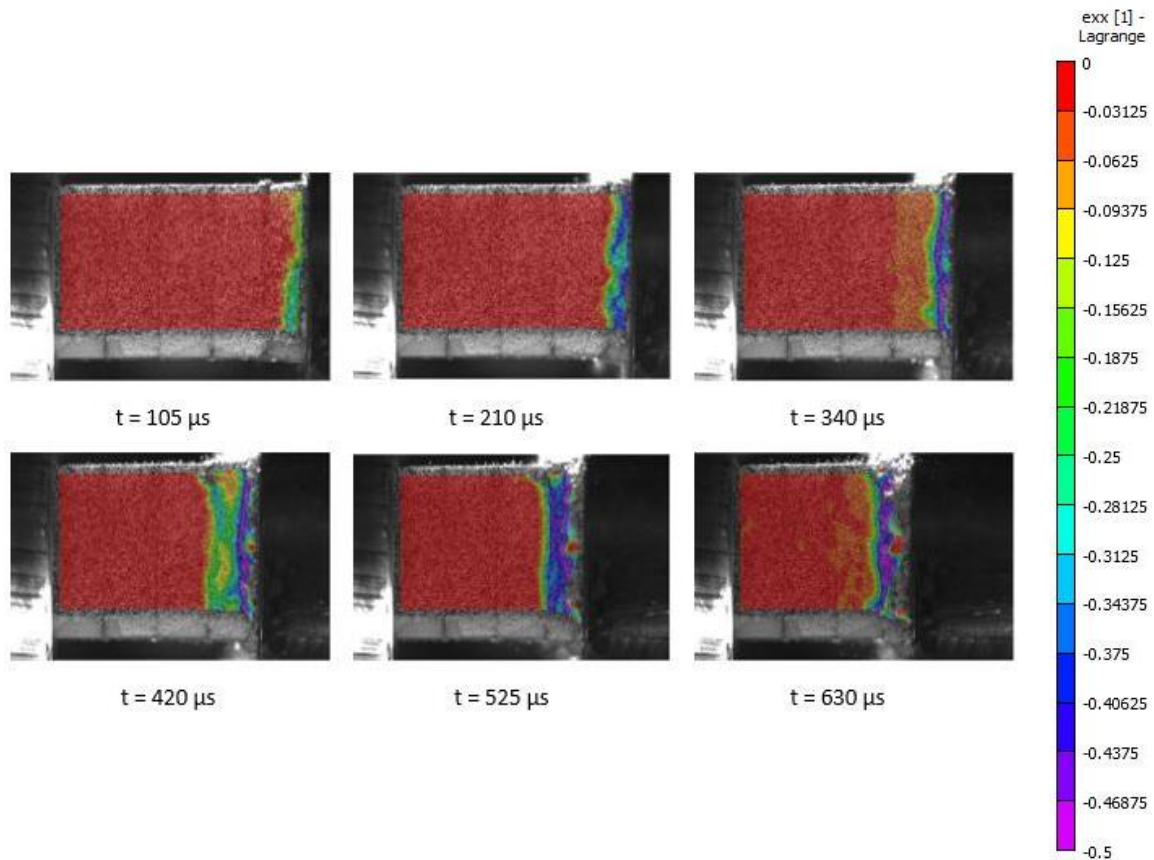


Figure 4.5. Axial strain contour plots

The contour plots show the axial strain distribution in Figure 4.5. The difference between the strains in each layer is easily seen. The lowest density layer completely crushes before the distal end of the specimen has even been exposed to the stress wave. The local strain from each layer is plotted in Figure 4.6. It is evident that the strain is not

homogeneous along the length of the specimen. Each layer is strained in a stepwise order starting with the lowest density. While the first layer is being crushed, the rest of the layers see very little strain, and once it has failed, the next layer begins to strain, a dramatic increase in the strain can be observed. There is very little change in the strain for the two highest density layers.

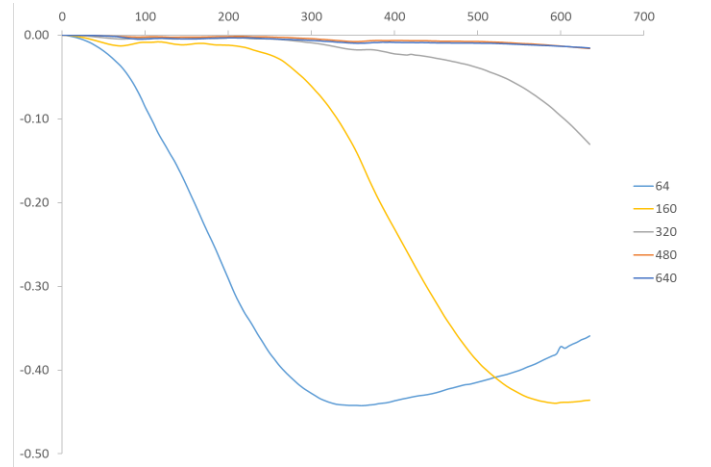


Figure 4.6. Local Strain plot

The stress measured from the transmitter bar using the SHPB technique is displayed in Figure 4.7. The uniform foams from the previous chapter exhibited a near constant plateau stress while the graded foam shows a smoothly rising plateau stress. The stress wave propagating through the layers can be seen as the increasing steps. Also, it is remarked that for the first 80 μ s the stress is near zero because the transmitter bar is not exposed to the stress wave yet, though deformation has already begun to occur at the impacted end. The signal to the strain gauge is delayed due to the super-low impedance of the soft layer.

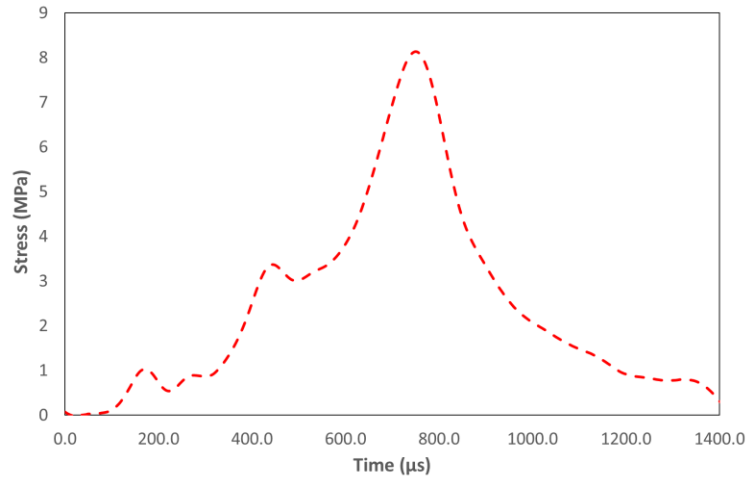


Figure 4.7. Boundary stress measured from strain gauges

The inertia stress is calculated for this transient duration while accounting for the layers of different initial densities. It was added to the boundary stress to give the following total stress-strain curve in Figure 4.8. The graded foam was compared with the two low density foams since those were the two layers impacted and failed. The inertia stress was found to have a slightly higher magnitude as compared to the 64 kg/m³ density foam, which was the first layer impacted for the graded foam.

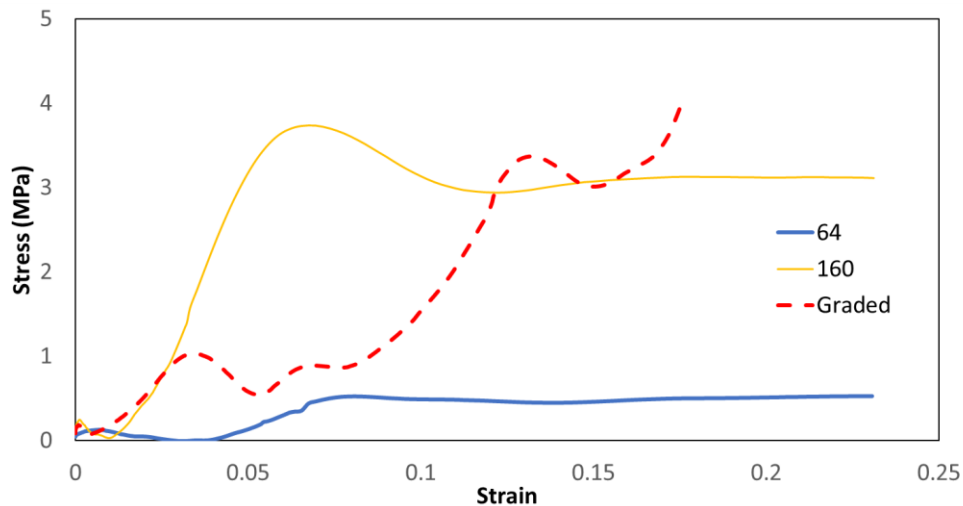


Figure 4.8 Total dynamic stress-strain response compared to uniform

Similar to the quasi-static stress-strain curves, a stepwise increase in the flow stress occurs at certain stress levels. Due to the framerate of the cameras, the entire global strain during deformation could not be captured, so only the lowest density uniform foams are shown in Figure 4.8 for comparison with the graded foam. The peaks in the graded curve seem to correlate with the compressive strength of the uniform foams.

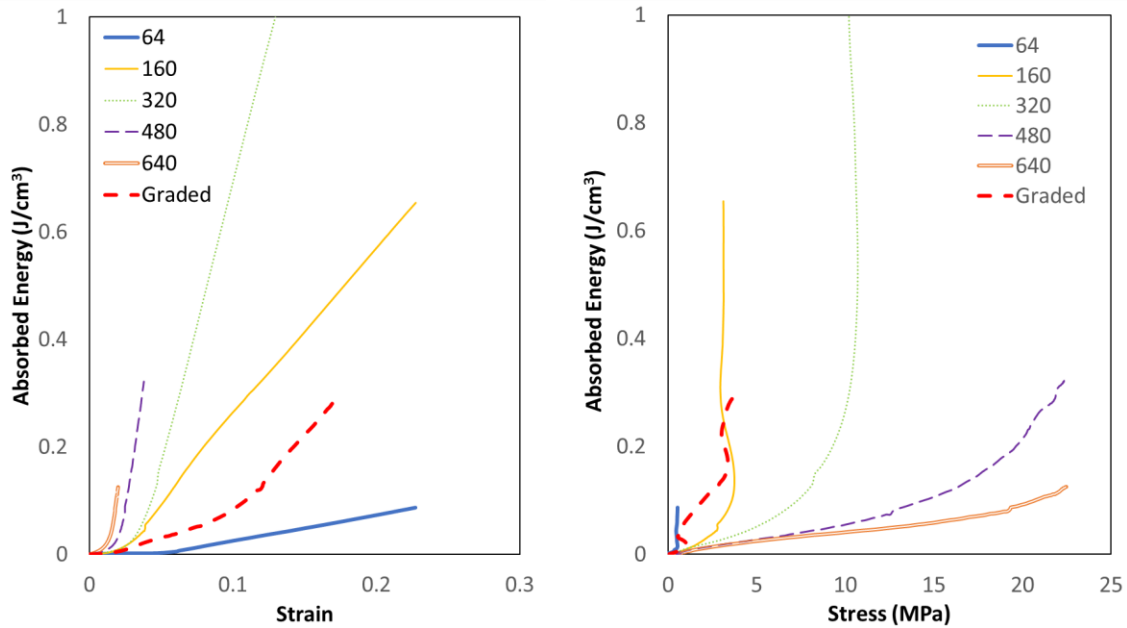


Figure 4.9 Energy absorbed as a function of stress and strain-graded compared to uniform

The absorbed energy in relation to the stress and strain is investigated in Figure 4.9. It is observed that for certain stresses and strains, the graded foam does show improved energy absorption over each homogenous foam. For a very small stress, less than 0.5 MPa, the lowest density shows the highest amount of energy absorbed, but then up to 3 MPa the graded foam exceeds the rest of the foams. For a predefined strain the graded foam appears to only show improvement over the lowest density. Though if the desired amount of deformation is greater than 4% the graded foam would be ideal over

the two highest densities because they see very little strain before the material fully densifies.

4.5 CONCLUSION

The constitutive response of a functionally graded foam material compared to homogeneous cellular materials was studied. Quasi-static and dynamic tests were conducted to investigate the stress-strain response. The temporal evolution of the sequential localized deformation of the material along the axial direction was observed using DIC. Synchronization of the oscilloscope signals with the digital image recordings enables measurement of the total dynamic stress-strain response. This method has made it possible to evaluate the non-equilibrium state of stress for these low impedance materials. The density distribution was seen to have a significant effect on the mechanical response. Depending on the design criterion the graded foam shows improved energy absorption over uniform foams for certain cases. Specifically if the application of the structure requires a small stress, the linear graded foam shows superior energy absorption over the uniform foams.

CHAPTER 5

EFFECTS OF STACKING SEQUENCE ON ENERGY ABSORPTION OF GRADED FOAMS

5.1 ABSTRACT

Different bulk density polymeric foam layers are bonded together in different stacking arrangements and subjected to impact loading. Ultra-high-speed imaging is implemented to measure the deformation and observe the formation and propagation of the stress waves during direct impact. The arranging of the layers' effect on energy absorption is explored. The optimum FGM configuration will depend considerably on the critical design conditions for the specific application.

5.2 INTRODUCTION

Polymeric foams are cellular materials that in many industries such as aerospace, automotive, and military have become of great deal of interest. These cellular structures are ultra-light solids which absorb substantial energy in compression. Their many applications involve absorbing impact and shock mitigation through energy dissipation by progressive local crushing. It is well known that energy absorption is strongly related to the foam density. Functionally Graded Materials (FGM) are advanced engineering materials that enable a material to have the best properties of multiples materials. The significant advantage of functionally graded foam materials is the optimization of strength to weight ratio. It is well established that higher density results in higher

strength. Graded foam has the appeal of higher strength (higher density) combined with lighter weight (lower densities).

The advent of graded foam has mostly been analyzed through simulations and analytical works. The numerical representations of graded foam have for the majority been analyzed through gradient functions. Cui [2] tested the variation of gradation of the foam characteristics with finite element simulations and found improved performance over single-phase foams. Kiernan [5] developed a finite element model of the SHPB to study the wave propagation through FGFM's. The impact response of density graded cellular polymers has been analyzed by observing the propagation of compaction waves using DIC [23]. Zeng [24] studied the influence of the density gradient on the energy absorption capacity of graded polymeric hollow sphere agglomerates. The ideal gradient architecture for optimal design of graded polymeric foams in energy absorbing applications is investigated in the present work. It is shown that the placement of these layers in reference to the side being impacted vs. the side in contact with the object being protected makes a difference in the mechanical response under dynamic loading.

5.3 MATERIALS AND SPECIMEN GEOMETRY

Discretely layered specimens were made with 5 different bulk density foams. The different sequences studied took on either a linear stepwise or sandwich configuration as the following stacking arrangements: 64/160/320/480/640, 640/480/320/160/64, 64/480/640/480/64, and 480/320/64/320/480 kg/m³. The layups were chosen to all have similar weights for comparison. A picture of the two sandwich configurations is shown in Figure 5.1. The linear increasing density was shown in Figure 4.1 and the linear

decreasing gradation is the same sample just flipped in the test frame. The same experimental method for dynamic testing was used as in the previous chapters.



Figure 5.1 Pictures of sandwich orientations

5.4 RESULTS AND DISCUSSION

The contour plots of the axial strain spatial distribution are shown for each orientation at the same global strain in Figure 5.2. It is clearly observed that as would be expected, in every instance, the lowest density foam is the first to fail. After complete densification of the lowest density layer deformation moves on the next highest density layer. The middle-high arrangement had the lowest density layer on both ends and it is worth noting that the impacted end was showing a higher strain value.

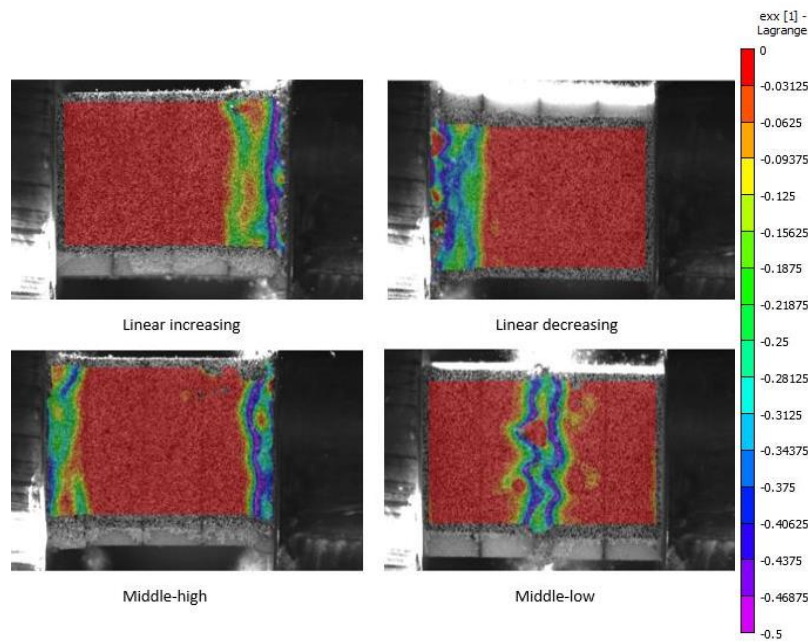


Fig 5.2. Axial strain contour plots

The local strain evolutions obtained from each layer for the two linear gradations are presented in Figure 5.3. The heterogeneity of the strain distribution is clearly observed. There appears to be a small but important time delay for initiation of strain in the lowest density layer when it is farthest from the impacted end. This time delay could improve the failure response of the structure because it could lead to a time delay for damage to initiate. Kiernan [5] observed this same time delay, deducing that a decrease in density (decreasing the yield stress) will allow for greater amount of energy to be absorbed plastically. Also, it is observed in the figure that the second highest density layer is slightly strained before the lowest density layer is fully densified when it is exposed to the stress wave first (decreasing density arrangement). Alternatively, with the linear increasing density, each layer begins to strain once the previous lowest density layer is fully densified.

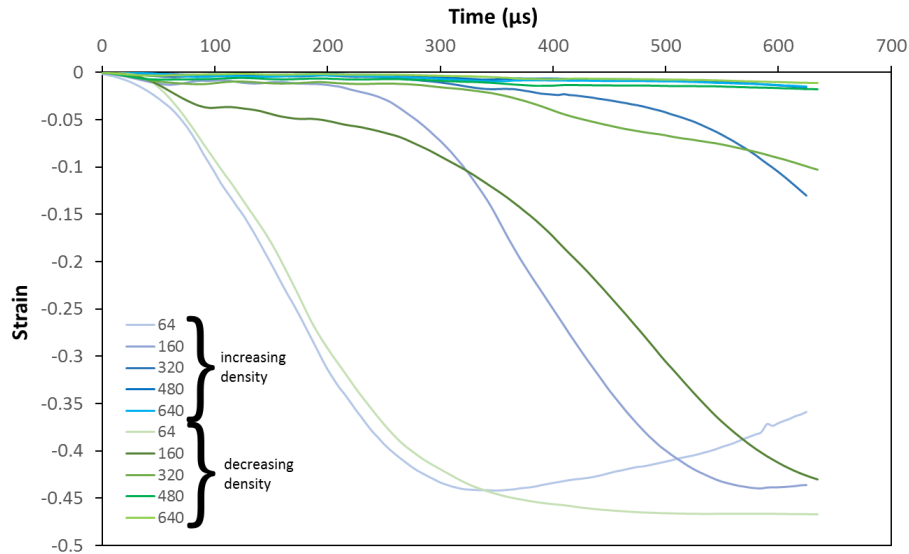


Figure 5.3 Local strain plots for linear gradation

The inertia stress is superimposed with the boundary measured stress to give the total global stress. This stress is plotted with the global axial strain extracted from DIC in

Figure 5.4. Again, it should be noted that the full structure wasn't completely failed and the equilibrium state was not reached. From the figure it can be seen that at the very early stage of the tests there is a large variation of the stress due to the inertia effects. The large plateau of the middle-high sandwich configuration is nearly identical to that of the 64 kg/m³ foam which makes sense because the configuration has two layers of that low-density foam. The initial part of the middle-low curve is very similar as well but at 13% strain there is a dramatic increase in the stress because the 64 kg/m³ layer has fully failed and the next highest density is a much larger step increase of the density as compared to the other orientations. It reaches the smallest total strain because the final strain depends on the crushing level of the next weakest layer. The two sandwich configurations are nearly opposite in their stress-strain response. The middle-high layup sees large strain but small stress, while the middle-low sees small strain and large stress.

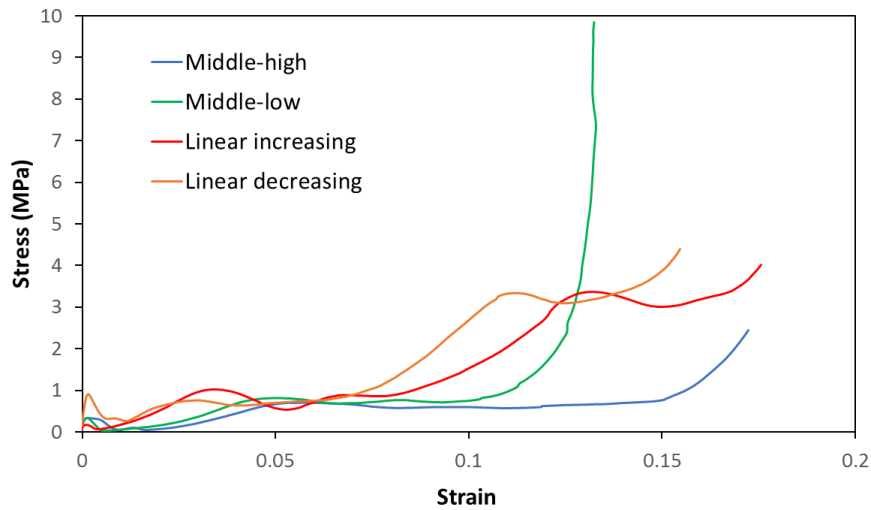


Figure 5.4. Total Stress-Strain

The area under the curve is calculated to find the energy absorbed. Since the total loading and unloading of the specimen is not captured, two different possibilities are offered to compare the four options. If the design criteria are based upon a predefined

stress or strain, the following graphs portray the comparison of energy absorbed for the four different configurations. The absorbed energy as a function of stress is depicted in Figure 5.5 and as a function of strain in Figure 5.6. These plots are used in order to obtain a relative comparison of these stacking arrangements for a fixed time duration.

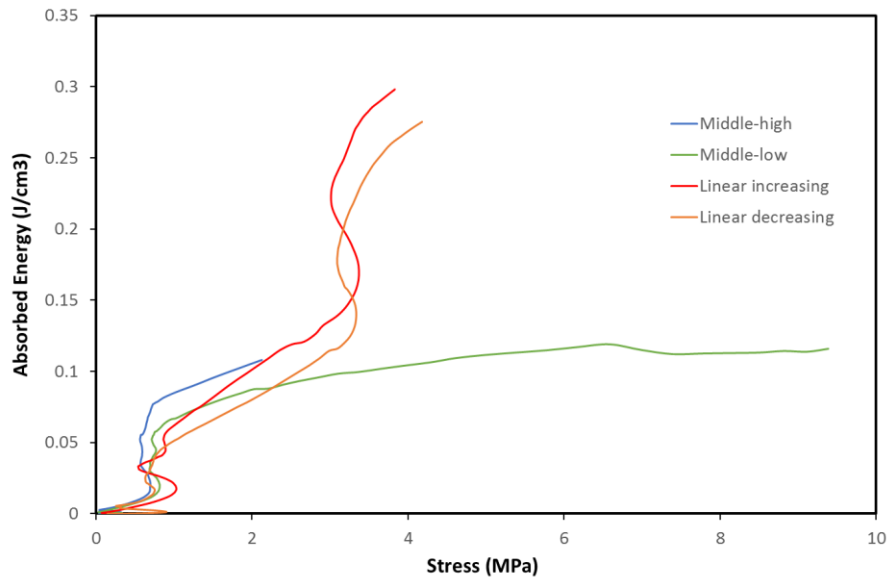


Figure 5.5 Energy absorbed for predefined stress

The middle-high orientation shows higher energy absorption for a small stress, up to about 2 MPa, which can be attributed to the double layer of the lowest density foam. The lowest density layers would support extended plateaus which subsequently promotes higher energy absorption. With an increase in transmitted stress, the middle-low layup sees very little increase in amount of energy absorbed. Overall the two linear gradations see the highest amount of energy absorbed for this duration of impact. If there is a specific amount of deformation that is desired, the following plot shows the comparison of energy absorbed for a defined strain.

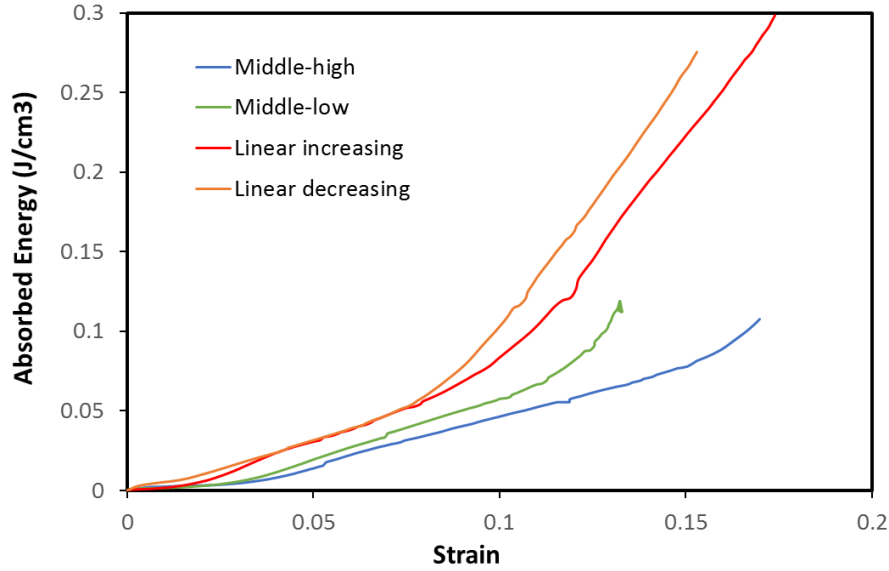


Figure 5.6. Energy absorbed for predefined strain

Obviously the linearly decreasing configuration is the desirable layup for this design criterion up to a strain of 15%. This greater energy absorption could be an effect of the time delay that was observed in Figure 5.2. It is intuitive that the middle-high would see the smallest amount of absorbed energy for a specific strain because the structure sees a very small stress due to the presence of the 64 kg/m^3 layers. Owing to the small stress, even at larger strains, befalls a small specific energy magnitude. If the design criterion would be to attain a certain amount of deformation, for strains less than 17% it is apparent that the sandwich structures would not be the ideal candidates.

5.5 CONCLUSION

The ability to tailor the morphology of these cellular materials to improve energy absorption will increase the safety standards in the many applications that these materials can be used. Ultimately, the final goal is to have a predictive model for graded foam that can be used in design tasks intended for impact loading environments. It is shown that the gradient does in fact play a role in the energy absorption for these heterogeneous

materials. Regardless of the arrangement of the layers, the lowest density layer is always the first to fail. So, one advantage is that by rearrangement of the layers comes the ability to force the initial crushing to occur at a predetermined region. In order to compare four different stacking arrangements, there is a fixed time duration for the resulting data. The strategy to have the most energy absorbed for a small predefined strain would be to have the highest density, the hardest material, at the impacted side and a linearly decreasing gradation to the structure that is being protected. For the case of a stress less than 2 MPa, the middle-high layup would be the most suitable. For these two cases, the desirable trait is to have the softest materials in contact with the material being protected. Intuitively that sounds ideal to have the softest material cushioning to protected object to reduce injury. But taken as a whole, the linear graded arrangement saw the highest magnitude of absorbed energy, regardless of which way it was oriented.

CHAPTER 6

SUMMARY AND RECOMMENDATIONS

6.1. SUMMARY

Functionally graded materials have the advantage of achieving tailored morphologies for desirable structural properties. The goal is to optimize the stacking arrangement of layers of different bulk densities for superior energy absorption. Rigid polyurethane foams were investigated at low and high strain rates. First the uniform foams were individually tested to get each material's constitutive response. High strain rate experiments were conducted on specimens that were fabricated in the lab and deformed using a standard SHPB. The compressive deformation behavior was observed using high-speed cameras and image correlation software. The full-field displacement was implemented to calculate the inertia effects and material compressibility. The inertia stress was shown to be significant during dynamic loading, confirming the idea that the SHPB analysis would be inaccurate since it uses the assumption of zero inertia effects. Also, it was verified that there is substantial spatial variability in stress and strain for these low impedance specimens. The full-field strain maps extracted from DIC were used to evaluate the mechanical response of the graded sample. By rearranging the configuration of the density gradation compared to the loading direction, a variation in the densification and strain progression occurs. These deformation characteristics will be used to tailor the material to a specific load and timeframe.

The relative density is observed to have a distinct influence on the energy absorption. The proposition of graded foams having the advantageous characteristics of different densities was a productive effort. It is found that graded foam materials do have enhanced characteristics over homogenous foams for certain cases. The wanted energy absorption improvement was confirmed for specific loading and deformation conditions using experimental methods.

6.2. RECCOMENDATIONS

It is recommended that this study continue to be explored. The optimization of enhanced energy absorbing foams could serve to improve many engineering applications of cushioning structures. It would also be advisable to use stereovision DIC for the dynamic tests to look into any radial inertia effects. The tests should be extended to include more stacking arrangements and variations of the thickness of each of the layers, as well as all of these orientations should be tested at varying strain rates. It would be interesting to see the effects of strain rate, if at a certain strain rate one configuration becomes better than another. With the advances in fabrication methods, the experiments could eventually include the study of continuous foams.

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Analytical Formulation of Shock Front Characteristics in Density-Graded Cellular Materials for Improved Impact Resistance

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Abstract

Due to their exceptional impact resistance capabilities, density-graded cellular materials have immense potential in applications where crashworthiness requirements are of prime importance. When subjected to impact loading, deformation in density graded cellular materials is characterized by either the single or the double shock mode. Previous studies have utilized numerical techniques to solve the equations governing compaction shock wave propagation. In this study, analytical solutions are formulated for the velocity of the impinging rigid mass, energy absorption capacity, incident stress, and transmitted stress in the single shock mode. The analytical solutions are shown to be in excellent agreement with the numerical solutions. The expression for the critical density gradient parameter that separates single and double shock regimes is obtained. Furthermore, the effect of density gradient on the energy absorption and stresses is studied. The optimal density gradient that maximizes impact protection is determined as a function of impinging rigid mass. It is found that increasing density gradients provide improved impact protection to objects at the incident end. On the other hand, the objects at the transmitted end have maximum protection when the density gradient parameter is equal to its critical value.

Keywords: Functionally graded cellular material, Analytical solution, Direct impact response, Shock model, Energy absorption, Optimal density gradient

1 Highlights

- Analytical solutions for density-graded cellular materials are derived
- Condition for double shock mode in a density-graded cellular material is shown
- Optimal density gradients for improved impact resistance are proposed
- Single shock modes favor superior impact protection

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- The work will promote the design and optimization of density-graded cellular materials

1. Introduction

In recent years, the technological expansion in material design and fabrication has led to the development of increasingly superior and efficient impact-mitigating structures. Cellular materials have dominated this space due to their lightweight and excellent mechanical properties suitable for impact protection [1]. Lately, density-graded cellular materials, where densities vary gradually along the impact direction, have shown improved energy absorption and impact resistance properties compared to uniform-density cellular materials, resulting in increased interest in functionally graded cellular materials for impact-resistant applications [2, 3].

Density-graded cellular material research has made notable progress through a combination of experimental, numerical, and analytical approaches. Most of the experimental investigations have been based on structures with step-wise gradations in density, achieved by combining several uniform density layers due to challenges in the fabrication of continuously graded cellular materials [4–8]. However, recently these manufacturing difficulties have been alleviated with modern manufacturing methods such as additive manufacturing and continuously graded cellular materials can be fabricated [9–11]. On the other hand, cell-based finite element models have been effective in investigating the high-velocity dynamic response of density-graded cellular structures [12–16]. Such models provide insight into the underlying mesoscale mechanisms but at the cost of increased computational time. Other numerical studies have employed continuum-based finite element models with representative continuum-level material properties [17, 18].

Analytical studies on cellular materials usually exploit classical shock theory, where the cellular material is modeled as a continuum, substantially reducing computational time. Shock waves arise in cellular materials when subjected to impact loading due to the progressive compaction of cells. Hence, shock waves in cellular materials are sometimes called compaction waves or, occasionally, densification waves. Although classical shock theory is generally used for modeling shock in dense solids, it has been shown to be effective in predicting the dynamic response of cellular materials. The early work based on classical shock theory presented an analytical solution for the impact response of various wood materials [19]. Afterwards, numerous contributions by many researchers developed the shock model for cellular materials [20–24]. Of late, there has been an effort toward applying the shock theory to explore and develop graded cellular materials. The authors in Ref. [25] employed a plateau stress gradient in their studies of graded cellular rods consisting of hexagonal and circular chains of cells; they identified that two shocks could co-exist in a graded cellular material subjected to impact loading. Later, investigators considered gradients in density for scenarios where a rigid mass strikes a stationary cellular material (referred to as stationary impact scenario) [2, 18]. The

Nomenclature

Roman symbols

${}_2F_1$	hypergeometric function
A	area of cross-section
a, b, d	parameters of the hypergeometric function
c	integration constant
C_1, C_2, C_3, C_4	solid (parent) material constants
E_d	specific energy absorption capacity
k, v, y	variables used for integral evaluation
L, l	total length of cellular material in the undeformed and deformed configurations, respectively
M	mass of rigid body
m	mass of cellular material
n	summation index of series
p	variable used in the integral form of the hypergeometric function
t	time
u	displacement of rigid mass
V	common velocity of the rigid mass and the undeformed region of cellular material
V_0	initial common velocity of the rigid mass and cellular material
V_{d0}	densification velocity of uniform density cellular material
V_d	densification velocity
V_f	common velocity of the rigid mass and forward compacted region of cellular material
V_{sb}	velocity of backward shock front
V_{sf}	velocity of forward shock front
X	undeformed position coordinate along the length of cellular material
X_b, x_b	length of backward compacted material in the undeformed and deformed configurations, respectively
X_f, x_f	length of forward compacted material in the undeformed and deformed configurations, respectively
X_u	length of undeformed compacted material
Y	symbol representing $\bar{V}^2$
z	variable used in the hypergeometric function
IRF_i, IRF_t	impact resistance factor for incident and transmitted end, respectively

Greek symbols

α	symbol representing $\sqrt{4\lambda\bar{M} + (1 + \lambda)^2}$
ϵ_{d0}	densification strain of uniform density cellular material
ϵ_{db}	strain behind backward shock front
ϵ_{df}	strain behind forward shock front
ϵ_d	densification strain
$\hat{\rho}_s$	density of parent (solid) material
λ	density gradient parameter
λ_{cr}	critical value of density gradient parameter
$\lambda_i^{opt}, \lambda_t^{opt}$	optimal density gradient parameters for maximum IRF_i and IRF_t , respectively
ρ	relative density
ρ_0	average relative density
ρ_a	relative density ahead of shock front
ρ_b, ρ_{db}	relative density ahead of and behind backward shock front, respectively
ρ_f, ρ_{df}	relative density ahead of and behind forward shock front, respectively
ρ_i, ρ_t	relative density at the incident and transmitted end, respectively
σ_i, σ_t	stress at the incident and transmitted end, respectively
σ_{p0}	plateau stress corresponding to average relative density
σ_{pb}, σ_{db}	plateau stress ahead of and behind backward shock front, respectively
σ_{pf}, σ_{df}	plateau stress ahead of and behind forward shock front, respectively
σ_{pi}, σ_{pt}	plateau stress of the cellular material at the incident and transmitted end, respectively
σ_p	plateau stress
ξ	any physical quantity describing the state of cellular material

Overbar symbol

$\overline{(\cdot)}$	dimensionless variable for dimensional quantity $(\cdot)$
----------------------	---

Abbreviations

IRF	impact resistance factor
RPPL	rigid, perfectly plastic, and locking

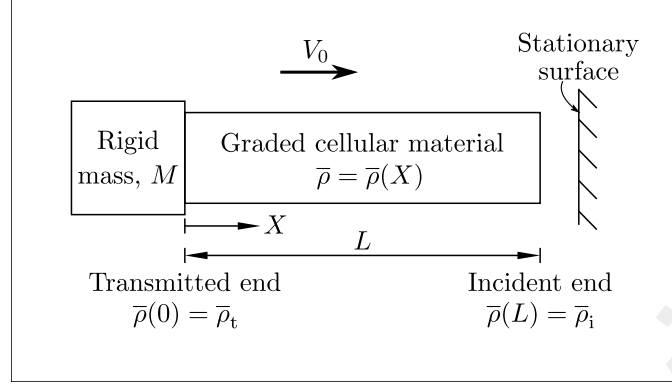


Figure 1: Schematic diagram of the density-graded cellular material backed by rigid mass colliding a stationary surface

collision of a moving cellular material (with or without a rigid backing mass) with a rigid stationary object (referred to as a direct impact scenario and also shown in Figure 1) has also been studied. Integral expressions for the shock front speed and the dynamic crushing stress were obtained for the special cases of nonlinear density profiles with positive density gradients¹ [26]. Additional noticeable theoretical works in the field include, studies on cellular materials with linear density variation [27] and nonlinear density variation [3] as well as the determination of an appropriate density distribution for specific crashworthiness requirements [28, 29]. All the previously mentioned studies of graded cellular material utilized numerical techniques to obtain the solution of the governing differential equations. The analytical solutions clearly divulge the relationship between the influencing parameters apart from being computationally more efficient than their numerical counterparts. Recently, an explicit and closed-form solution for density-graded cellular material with stationary impact scenario was developed [30]. Unfortunately, the existing analytical formulation can not be used for a direct impact scenario of density-graded cellular materials.

In this paper, a theoretical framework for density-graded cellular materials subjected to direct impact loading is developed using shock theory. A rigid, perfectly plastic, and locking (RPPL) material behavior is used for the graded cellular material, with the stationary surface assumed to be rigid. The variation in density along the specimen length is assumed to be linear and both positive and negative density gradients are considered. The critical density gradient or the condition for existence of the double shock mode is obtained as a first step. Afterward, the energy absorption and the stresses at the ends of the cellular material are evaluated for both single and double shock modes. The influence of density gradient on the energy absorption capacity, incident stress, and transmitted stress is studied and finally, the optimal density gradients that provide maximum impact resistance to objects placed at the incident end and transmitted

¹Density gradient is considered positive when density gradually increases from incident end to transmitted end. Conversely, it is negative when density gradually decreases from incident end to transmitted end.

end are determined.

The paper is structured as follows. The theoretical framework is laid out in Section 2. Also, the necessary equations governing the deformation of density-graded cellular materials under impact loading are derived for both single and double shock modes. In Section 3, the analytical solutions are formulated and compared with the numerical solutions. Section 4 that follows later presents the effect of density gradation on impact resistance of cellular structures. The optimal density gradient parameters are also determined in this section. Finally, the concluding remarks are given in Section 5.

2. Governing Equations

2.1. Density gradient parameter

Let ρ_t and ρ_i are the relative densities² at the transmitted and incident ends respectively, density gradient parameter λ is defined as:

$$\lambda = \frac{\rho_t - \rho_i}{2\rho_0} \quad (1)$$

where ρ_0 is the average relative density. The linear variation of relative density ρ along the length is expressed as:

$$\rho = \rho_0 \left[1 + \lambda \left(1 - \frac{2X}{L} \right) \right] \quad (2)$$

where λ is bounded between its extreme values -1 and 1. The uniform density cellular material corresponds to $\lambda = 0$, whereas the density gradient is steepest when $\lambda = \pm 1$. Positive values of λ indicate that density is increasing from the incident end towards the transmitted end. On the other hand, negative values have the opposite density variation. The relative densities at the ends are obtained using Eq. (2) as follows:

$$\rho_t = \rho_0 (1 + \lambda) \quad (3)$$

$$\rho_i = \rho_0 (1 - \lambda) \quad (4)$$

It should be noted that the magnitude of λ cannot be greater than unity since it implies negative density within the cellular material.

²The relative density is the ratio of the density of the cellular material to the density of the parent (solid) material.

2.2. Material model

For RPPL idealization, material behavior is assumed to be rigid up to the plateau stress. Also, the strain can take only two values; it can be either zero or equal to densification strain. Therefore, RPPL material model requires only two material parameters: plateau stress σ_p and densification strain ϵ_d . Both the material parameters vary along the length of cellular material as they are dependent on the relative density. Their variations are modeled using the following equations [1]:

$$\sigma_p = C_1 \rho^{C_2} \quad (5)$$

$$\epsilon_d = C_3 - C_4 \rho \quad (6)$$

where C_1 , C_2 , C_3 , and C_4 are parameters of the parent (solid) material.

2.3. Single shock mode

Consider a graded cellular material backed by a rigid mass M traveling at a velocity V_0 . Its deformation at some time instant t after impact is depicted in Figure 2. The undeformed and deformed configurations are shown in Figure 2a and Figure 2b, respectively. Immediately after the cellular material strikes the stationary surface, a shock front is initiated traveling in the backward direction (from the incident end towards the transmitted end). This shock front is termed a backward shock front. It moves with velocity V_{sb} engulfing the undeformed material and bringing it to rest. The velocity of compacted material behind the shock front is zero. The moving shock front separates undeformed and compacted material. The particle velocity across the shock front is discontinuous, but displacement continuity holds. The lengths of compacted material in the undeformed and the deformed configurations are X_b and x_b , respectively. The displacement of the rigid mass is u and the total length of cellular material is reduced to length l . The length of the remaining undeformed material after impact has occurred is X_u . The variables ahead of and behind the shock front are shown in Figure 2c.

The following equation is obtained by applying the principle of balance of linear momentum to the rigid mass and the undeformed region of cellular material:

$$\sigma_{pb} = - \left(\frac{M}{A} + \hat{\rho}_s \int_0^{L-X_b} \rho(X) dX \right) \frac{dV}{dt} \quad (7)$$

where σ_{pb} is the plateau stress ahead of shock front, A is the cross-sectional area of cellular material, and V is the common velocity of the rigid mass and the undeformed region of cellular material.

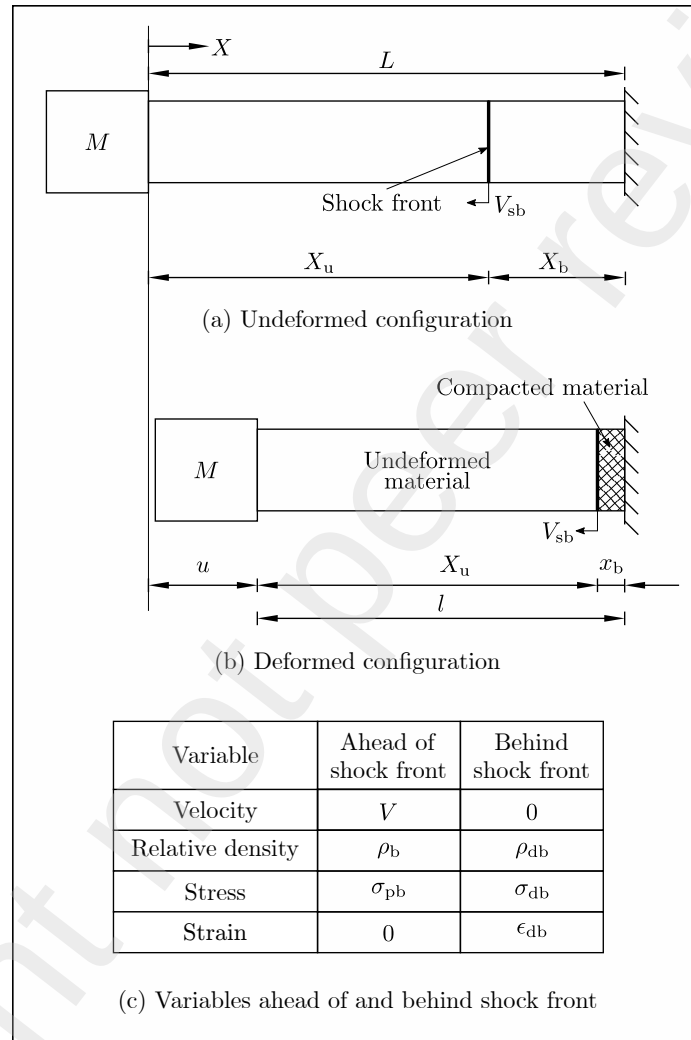


Figure 2: Deformation of graded cellular material with a single shock front (a) undeformed configuration (b) deformed configuration (c) variables ahead of and behind the shock front

Since compressive strain is considered positive by convention, the instantaneous strain ϵ_{db} behind the shock front is given by,

$$\epsilon_{db} = \frac{dX_b - dx_b}{dX_b} \quad (8)$$

where the infinitesimal undeformed length dX_b of the cellular material ahead of shock front is compacted to dx_b behind the shock front. Using the kinematic relationships, $X_b = L - X_u$ and $x_b = L - X_u - u$, ϵ_{db} is written as:

$$\epsilon_{db} = \frac{du}{dX_b} \quad (9)$$

The velocity V is given by the following Eq. (10).

$$V = \frac{du}{dt} \quad (10)$$

Therefore, the following relationship is obtained from Eqs. (9) and (10):

$$dt = \frac{\epsilon_{db}}{V} dX_b \quad (11)$$

Eq. (7) is transformed to the following equation using Eq. (11),

$$\frac{\sigma_{pb}\epsilon_{db}}{V} = - \left(\frac{M}{A} + \hat{\rho}_s \int_0^{L-X_b} \rho(X) dX \right) \frac{dV}{dX_b} \quad (12)$$

Non-dimensionalizing variables using the following relationships,

$$\bar{X}_b = \frac{X_b}{L}; \bar{M} = \frac{M}{m}; \bar{\rho} = \frac{\rho}{\rho_0}; \bar{\sigma}_{pb} = \frac{\sigma_{pb}}{\sigma_{p0}}; \bar{V} = \frac{V}{\sqrt{\sigma_{p0}/(\rho_0 \hat{\rho}_s)}} \quad (13)$$

where m is the mass of the cellular material, σ_{p0} is the plateau stress corresponding to the average relative density ρ_0 , the dimensionless form of Eq. (12) is written:

$$\frac{\bar{\sigma}_{pb}\epsilon_{db}}{\bar{V}} = - \left(\bar{M} + \int_0^{1-\bar{X}_b} \bar{\rho}(\bar{X}) d\bar{X} \right) \frac{d\bar{V}}{d\bar{X}_b} \quad (14)$$

The integral in Eq. (14) is evaluated using Eq. (2). Therefore,

$$\frac{\bar{\sigma}_{pb}\epsilon_{db}}{\bar{V}} = - (\bar{M} + (1 - \bar{X}_b) (1 + \lambda \bar{X}_b)) \frac{d\bar{V}}{d\bar{X}_b} \quad (15)$$

Though Eq. (15) can be solved numerically, the authors developed an analytical solution that is shown in Section 3.

2.4. Condition for double shock mode

Deformation in graded cellular material can also have two shock fronts (double shock mode) depending on the density gradient. If the density gradient parameter λ is below its critical value, λ_{cr} , double shock mode exists. However, if λ is equal to or greater than λ_{cr} , deformation will be characterized by single shock mode. Since shock front formation occurs due to compaction of the cellular material, it is only possible if stress exceeds the plateau stress. Thus, the following inequality must hold for initiation of the second shock front from the transmitted end.

$$\sigma_t > \sigma_{pt} \quad (16)$$

where σ_t and σ_{pt} are the reaction stress and the plateau stress at the transmitted end, respectively. The reaction stress σ_t is determined by applying the principle of balance of linear momentum to the rigid mass as follows:

$$\sigma_t = - \left(\frac{M}{A} \right) \frac{dV}{dt} \quad (17)$$

Using Eq. (7), the time derivative of the velocity in Eq. (17) is eliminated. Therefore,

$$\sigma_t = \frac{M\sigma_{pb}}{M + \hat{\rho}_s A \int_0^{L-X_b} \rho(X) dX} \quad (18)$$

Using Eq. (2), the following equation is obtained.

$$\sigma_t = \frac{M\sigma_{pb}}{M + m(1 - X_b/L)(1 + \lambda X_b/L)} \quad (19)$$

As seen in Eq. (19), the reaction stress σ_t depends on the position of the backward shock front X_b and density gradient parameter λ . A comparison of σ_t and σ_{pt} for representative values of $\lambda \geq \lambda_{cr}$ is shown in Figure 3. As λ is reduced, the gap between σ_t and σ_{pt} diminishes, but σ_t never exceeds σ_{pt} . Therefore, the second shock front at the transmitted end is never formed for $\lambda \geq \lambda_{cr}$, since the inequality in Eq. (16) is not satisfied. However, when $\lambda < \lambda_{cr}$, σ_t exceeds σ_{pt} , and a shock front is formed at the transmitted end. Once the second shock front is formed, σ_t is not given by Eq. (19). Determining the stress requires consideration of double shock mode, which is discussed in subsequent sections. Since $\sigma_t = \sigma_{pt}$, when $\lambda = \lambda_{cr}$, initial conditions (corresponding to $X_b = 0$) can be used in Eq. (19) for determining λ_{cr} . Therefore, from Eqs.

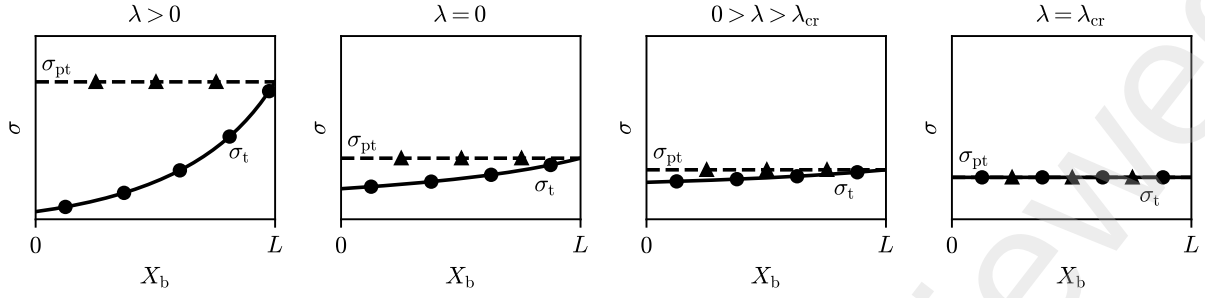


Figure 3: Comparison of reaction stress σ_t and plateau stress σ_{pt} at the transmitted end for $\lambda \geq \lambda_{cr}$

(16) and (19), the following inequality for existence of double shock mode is obtained:

$$\sigma_{pi} > \left(\frac{M+m}{M} \right) \sigma_{pt} \quad (20)$$

where σ_{pi} is the plateau stress of the material at the incident end. Using Eqs. (3), (4) and (5), the condition on λ is obtained.

$$\lambda < - \frac{(\overline{M}+1)^{\frac{1}{c_2}} - \overline{M}^{\frac{1}{c_2}}}{(\overline{M}+1)^{\frac{1}{c_2}} + \overline{M}^{\frac{1}{c_2}}} \quad (21)$$

The right hand side of the inequality in Eq. (21) is nothing but λ_{cr} . Therefore,

$$\lambda_{cr} = - \frac{(\overline{M}+1)^{\frac{1}{c_2}} - \overline{M}^{\frac{1}{c_2}}}{(\overline{M}+1)^{\frac{1}{c_2}} + \overline{M}^{\frac{1}{c_2}}} \quad (22)$$

Figure 4 shows λ_{cr} as a function of $\overline{M}$. For very large values of rigid mass, λ_{cr} tends to zero. This is very similar to the stationary impact scenario where double shock mode exists for all negative values of λ , and single shock mode exists for all non-negative values [2, 18]. On the other hand, in the absence of the rigid mass (i.e. $\overline{M} = 0$), $\lambda_{cr} = -1$, indicating λ must be less than -1 for the existence of double shock mode. However, -1 is the lower bound of λ . Thus, double shock mode is never possible in the absence of the rigid mass, and single shock mode exists for the full range of values of λ . It can also be inferred that single shock mode exists for all non-negative values of λ .

2.5. Double shock mode

The deformation with a double shock mode is shown in Figure 5. In addition to a backward shock front, another shock front is initiated from the transmitted end. This is termed as the forward shock front. Its length is denoted by X_f and x_f in undeformed and deformed configurations, respectively. The shock-jump equations are used to derive the governing differential equations. For forward shock front, the equations

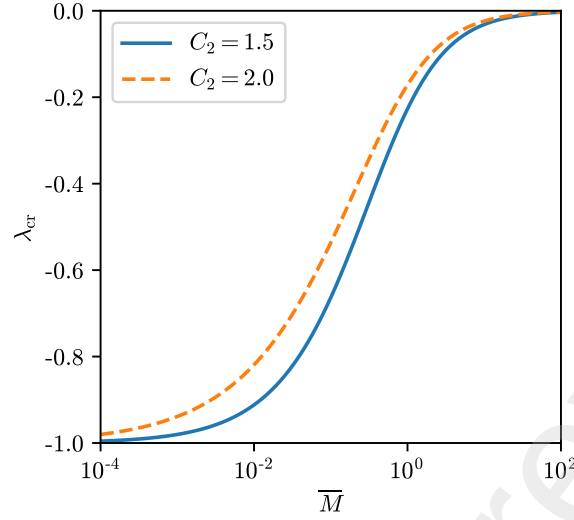


Figure 4: Critical value of density gradient parameter

corresponding to conservation of mass and balance of linear momentum in the Lagrangian form are as follows [31]:

$$-\rho_f V_{sf} \llbracket 1/\rho \rrbracket = \llbracket V \rrbracket \quad (23)$$

$$\rho_f \hat{\rho}_s V_{sf} \llbracket V \rrbracket = \llbracket \sigma \rrbracket \quad (24)$$

where ρ_f is the relative density ahead of forward shock front, and V_{sf} is the Lagrangian shock velocity of forward shock front. The symbol $\llbracket \xi \rrbracket$ indicates a jump in the quantity ξ across the shock front, and it is given by $\llbracket \xi \rrbracket = \xi^+ - \xi^-$ such that ξ^+ and ξ^- are states behind and ahead of shock front, respectively. For consistency with the sign convention for strain during compaction, compressive stress is considered positive in Eq. (24).

The shock velocity V_{sf} is given by the following equation:

$$V_{sf} = \frac{dX_f}{dt} \quad (25)$$

The instantaneous strain ϵ_{df} behind the forward shock front is given by,

$$\epsilon_{df} = \frac{dX_f - dx_f}{dX_f} \quad (26)$$

where dX_f is the undeformed infinitesimal length of cellular material ahead of shock front that is compacted to length dx_f behind the shock front. Using conservation of mass of the infinitesimal length of cellular

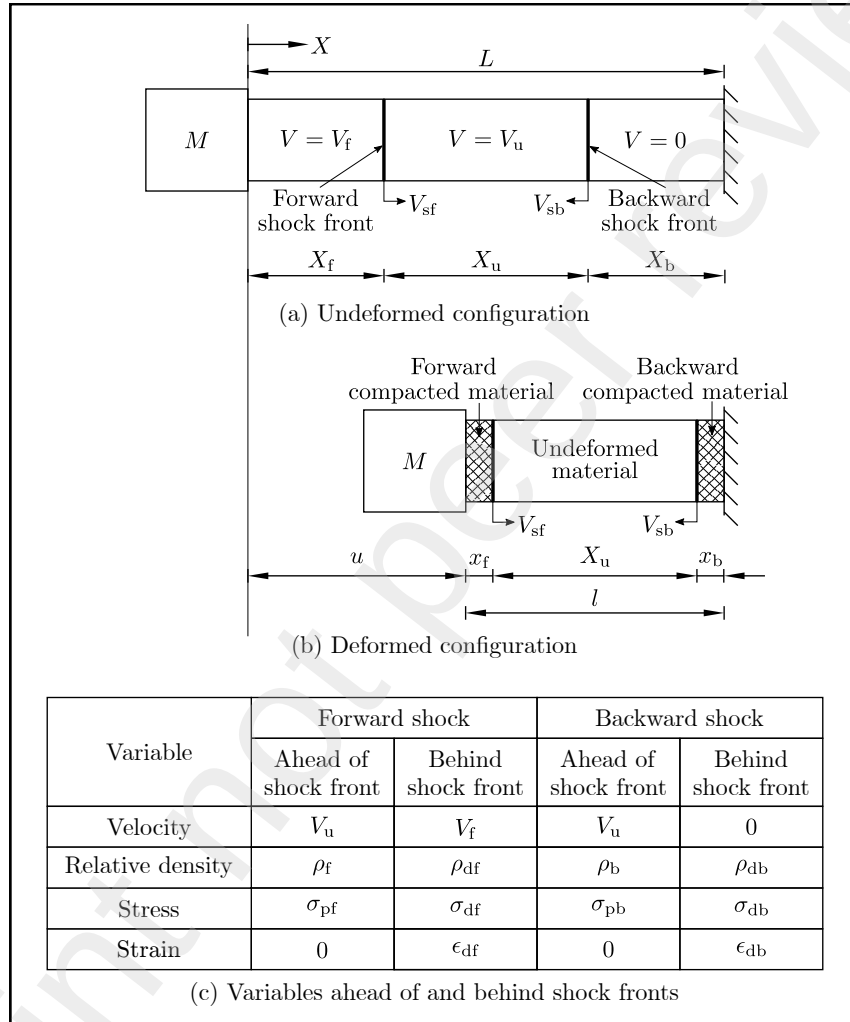


Figure 5: Deformation of graded cellular material with two shock fronts (a) undeformed configuration (b) deformed configuration (c) variables ahead of and behind the shock fronts

material, Eq. (26) is written:

$$\epsilon_{df} = 1 - \frac{\rho_f}{\rho_{df}} \quad (27)$$

Using the material states ahead of and behind forward shock front given in Figure 5, and Eqs. (23), (25) and (27), the following relationship is obtained:

$$dt = \frac{\epsilon_{df}}{V_f - V_u} dX_f \quad (28)$$

where V_f is the common velocity of the rigid mass and forward compacted region, and V_u is the velocity of the undeformed region. From Eqs. (24) (25) and (28), the stress behind a forward shock front σ_{df} is written:

$$\sigma_{df} = \sigma_{pf} + \frac{\rho_f \hat{\rho}_s}{\epsilon_{df}} (V_f - V_u)^2 \quad (29)$$

where σ_{pf} is the plateau stress ahead of forward shock front. Eq. (29) can be generalized to get the stress jump across a shock front as follows.

$$[[\sigma]] = \rho_a \hat{\rho}_s \frac{[[V]]^2}{[[\epsilon]]} \quad (30)$$

where ρ_a represents a generic symbol for the relative density ahead of either forward ($\rho_a = \rho_f$) or backward shock fronts ($\rho_a = \rho_b$).

Subsequently, the following equation is obtained from the principle of balance of linear momentum applied to the rigid mass and forward compacted material.

$$\sigma_{df} = - \left(\frac{M}{A} + \hat{\rho}_s \int_0^{X_f} \rho(X) dX \right) \frac{dV_f}{dt} \quad (31)$$

Combining Eqs. (28), (29) and (31), the following differential equation is obtained:

$$\sigma_{pf} \epsilon_{df} + \rho_f \hat{\rho}_s (V_f - V_u)^2 = - \left(\frac{M}{A} + \hat{\rho}_s \int_0^{X_f} \rho(X) dX \right) (V_f - V_u) \frac{dV_f}{dX_f} \quad (32)$$

Using similar non-dimensionalization as given in Eq. (13), the following equation is obtained:

$$\bar{\sigma}_{pf} \epsilon_{df} + \bar{\rho}_f (\bar{V}_f - \bar{V}_u)^2 = - \left(\bar{M} + \int_0^{\bar{X}_f} \bar{\rho}(\bar{X}) d\bar{X} \right) (\bar{V}_f - \bar{V}_u) \frac{d\bar{V}_f}{d\bar{X}_f} \quad (33)$$

Now, the equations related to backward shock front are derived. For backward shock front, the shock-

179 jump equation corresponding to conservation of mass in the Lagrangian form is given by:

$$-\rho_b V_{sb} \llbracket 1/\rho \rrbracket = \llbracket V \rrbracket \quad (34)$$

180 The shock front velocity of the backward shock front is given by:

$$V_{sb} = -\frac{dX_b}{dt} \quad (35)$$

181 The strain behind the backward shock front can be obtained in a similar manner as that in Eq. (27) for
182 forward shock front.

$$\epsilon_{db} = 1 - \frac{\rho_b}{\rho_{db}} \quad (36)$$

183 Using the material states ahead of and behind backward shock front, and Eqs. (34), (35) and (36), the
184 following relation is obtained:

$$dt = \frac{\epsilon_{db}}{V_u} dX_b \quad (37)$$

185 Principle of balance of linear momentum for the undeformed region gives:

$$\sigma_{pf} - \sigma_{pb} = \left(\hat{\rho}_s \int_{X_f}^{L-X_b} \rho(X) dX \right) \frac{dV_u}{dt} \quad (38)$$

186 Combining Eqs. (37) and (38), and after non-dimensionalization, the following equation is obtained:

$$(\bar{\sigma}_{pf} - \bar{\sigma}_{pb}) \epsilon_{db} = \left(\int_{\bar{X}_f}^{1-\bar{X}_b} \bar{\rho}(\bar{X}) d\bar{X} \right) \bar{V}_u \frac{d\bar{V}_u}{d\bar{X}_b} \quad (39)$$

187 The relationship between $d\bar{X}_f$ and $d\bar{X}_b$ is obtained from Eqs. (28) and (37) and is given by:

$$d\bar{X}_f = \frac{(\bar{V}_f - \bar{V}_u) \epsilon_{db}}{\bar{V}_u \epsilon_{df}} d\bar{X}_b \quad (40)$$

188 Eqs. (33), (39) and (40) form the set of differential equations required for the solution in double shock mode.

189 3. Analytical Solutions

190 The solutions are determined analytically for the single shock mode. The differential equation given by
191 Eq. (15) is solved for special cases depending on the value of λ .

3.1. Uniform density cellular material

For uniform density material ($\lambda = 0$), $\bar{\sigma}_{pb} = 1$ and $\epsilon_{db} = \epsilon_{d0}$, where ϵ_{d0} is the densification strain of the uniform density material. Therefore, Eq. (15) reduces to

$$\frac{\epsilon_{d0}}{\bar{M} + 1 - \bar{X}_b} d\bar{X}_b = -\bar{V} d\bar{V} \quad (41)$$

The solution to differential equation given by Eq. (41) is obtained by integrating and then using initial conditions, $\bar{V} = \bar{V}_0$ and $\bar{X}_b = 0$.

$$\bar{V} = \sqrt{\bar{V}_0^2 + 2\epsilon_{d0} \ln \left(1 - \frac{\bar{X}_b}{\bar{M} + 1} \right)} \text{ if } \lambda = 0 \quad (42)$$

Eq. (42) is the same as the solution originally obtained in [19].

Using Eq. (42), the densification velocity, which is the minimum initial velocity required for complete compaction of the cellular material, can be obtained. It is worthwhile here to note the energy transfer during the compaction process. The initial kinetic energy possessed by the rigid mass and cellular material is dissipated by compaction of the cellular material. An initial velocity less than the densification velocity only partially compacts the cellular material, even though the final velocity reduces to zero. On the contrary, an initial velocity higher than the densification velocity completely compacts the cellular material, but the final velocity is not zero. However, when the initial velocity is equal to the densification velocity, the cellular material is just fully compacted and the final velocity is zero. Therefore, the densification velocity of the uniform density material $\bar{V}_{d0}$ is obtained by using $\bar{V}_0 = \bar{V}_{d0}$ when $\bar{V} = 0$ and $\bar{X}_b = 1$ in Eq. (42). The expression for $\bar{V}_{d0}$ is given by:

$$\bar{V}_{d0} = \sqrt{2\epsilon_{d0} \ln \left(1 + \frac{1}{\bar{M}} \right)} \text{ if } \lambda = 0 \quad (43)$$

3.2. Density-graded cellular material

Now, the solution for the graded cellular material ($\lambda \neq 0$) is determined, which is one of the main objectives of the current work. Since the plateau stress and densification strain have a direct relationship with relative density, expressing the variables in terms of relative density results in simplifying equations. Using Eqs. (2), (5), (6) and (15), the following equation can be obtained:

$$\frac{2\bar{\rho}_b^{C_2} (C_3 - C_4 \rho_b)}{\bar{V}} = - \left(4\lambda \bar{M} + (1 + \lambda)^2 - \bar{\rho}_b^2 \right) \frac{d\bar{V}}{d\bar{\rho}_b} \quad (44)$$

Note that Eq. (44) is not valid for $\lambda = 0$. It is a nonlinear differential equation that can be converted to linear differential equation by using the transformation, $Y = \bar{V}^2$. Therefore, the following form of differential

equation is obtained after rearranging the terms.

$$\frac{dY}{d\bar{\rho}_b} = -\frac{4\bar{\rho}_b^{C_2}(C_3 - C_4\rho_b)}{\alpha^2 - \bar{\rho}_b^2} \quad (45)$$

where $\alpha = \sqrt{4\lambda\bar{M} + (1 + \lambda)^2}$. The solution to this differential equation is obtained by evaluating the integrals in the following equation.

$$Y = \bar{V}^2 = -4 \left(C_3 \int \frac{\bar{\rho}_b^{C_2} d\bar{\rho}_b}{\alpha^2 - \bar{\rho}_b^2} - C_4 \rho_0 \int \frac{\bar{\rho}_b^{C_2+1} d\bar{\rho}_b}{\alpha^2 - \bar{\rho}_b^2} \right) \quad (46)$$

The solutions are obtained for some special cases as follows.

3.2.1. Closed-form solution

The plateau stress of flexible foams are typically modeled with quadratic variation ($C_2 = 2$) with respect to the relative density [32]. In this case, the integrals can be evaluated in closed-form. The exact solution is given as follows:

$$\bar{V} = \sqrt{\bar{V}_0^2 + 4C_3(\bar{\rho}_b - \bar{\rho}_i) - 2\rho_0 C_4(\bar{\rho}_b^2 - \bar{\rho}_i^2) + 2\alpha(C_3 - \alpha\rho_0 C_4) \ln\left(\frac{\alpha - \bar{\rho}_b}{\alpha - \bar{\rho}_i}\right) - 2\alpha(C_3 + \alpha\rho_0 C_4) \ln\left(\frac{\alpha + \bar{\rho}_b}{\alpha + \bar{\rho}_i}\right)} \quad (47)$$

where the constant of integration has been evaluated using the initial conditions, $\bar{V} = \bar{V}_0$ and $\bar{\rho}_b = \bar{\rho}_i$. A direct comparison of the velocity prediction by Eq. (47) with numerical simulation results is shown in Figure 6. The numerical solution is obtained using a forward finite difference method to Eq. (15). Figures 6a and 6c show the dimensionless velocity $\bar{V}$ for $\bar{V}_0 = 1$ and $\bar{V}_0 = 2$, respectively. The corresponding errors in the numerical solution are shown in Figures 6b and 6d. The calculation is repeated for different step increments ΔX_b . The following values are used for the comparison: $\rho_0 = 0.15$, $\bar{M} = 1$, $C_3 = 1$, $C_4 = 1$ and $\lambda = 0.5$. Henceforth, these values are used unless otherwise stated. The numerical solution converges to the closed-form solution as the step increment is reduced. The error in the numerical solution grows larger with the propagation of the shock front as seen in the Figures 6b and 6d. Moreover, it should be noted that in the absence of analytical solution, multiple iterations must be performed to confirm the convergence of the numerical method. For $\bar{V}_0 = 1$, as seen in Figure 6a, even after exhausting the kinetic energy, the shock front does not reach the other (transmitted) end. Only partial compaction of the cellular material indicates that the initial velocity is less than the densification velocity. On the other hand, for $\bar{V}_0 = 2$, the shock front reaches the transmitted end, but some residual kinetic energy is left behind, indicated by the non-zero final velocity in Figure 6c. Residual kinetic energy implies that the initial velocity is greater than the densification

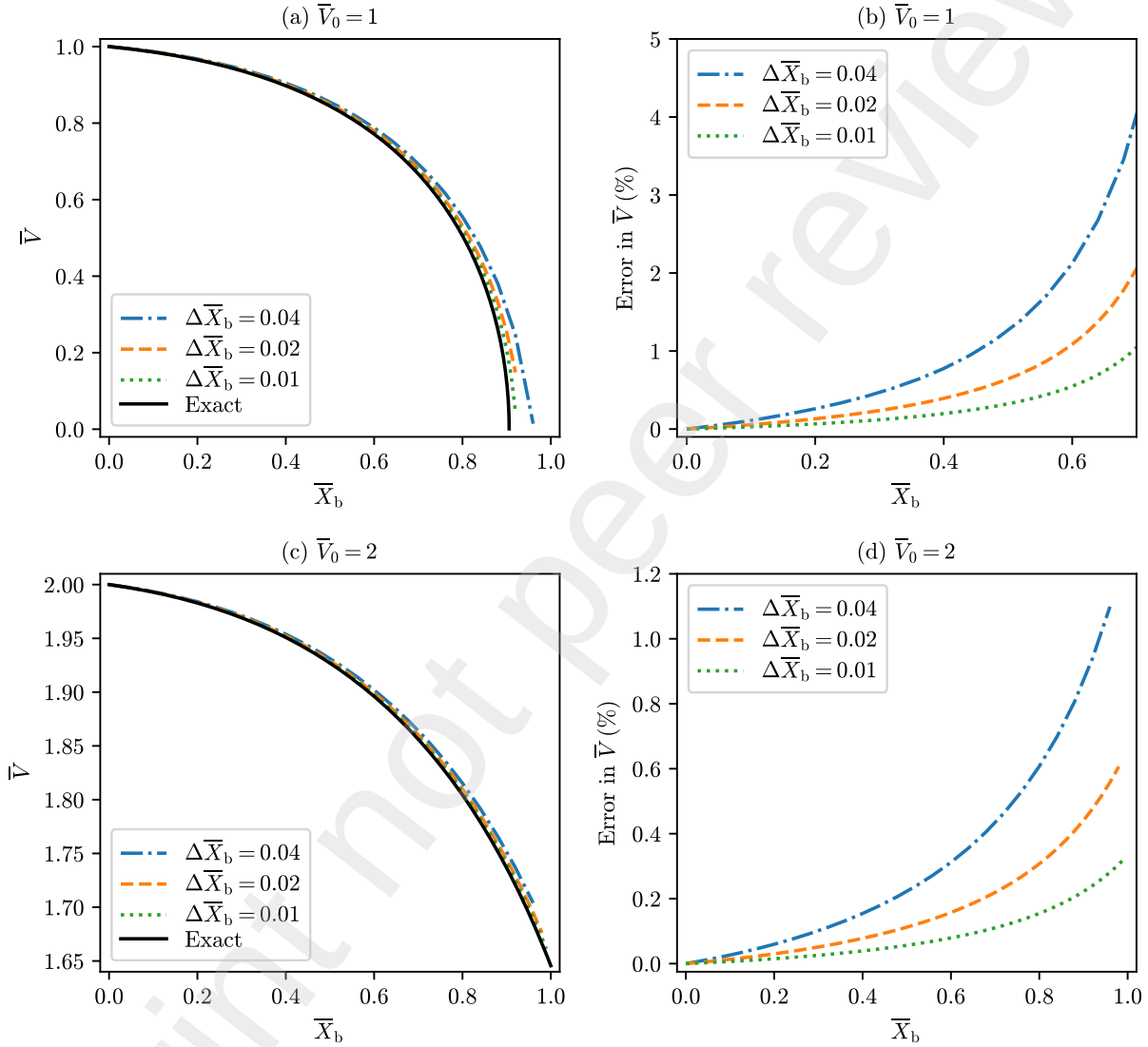


Figure 6: A comparison of rigid mass velocity $\bar{V}$ evaluated using numerical method with that obtained from the exact (closed-form) solution (a) $\bar{V}$ when $\bar{V}_0 = 1$ (b) Error in $\bar{V}$ when $\bar{V}_0 = 1$ (c) $\bar{V}$ when $\bar{V}_0 = 2$ (d) Error in $\bar{V}$ when $\bar{V}_0 = 2$. Here $\bar{\rho}_0 = 0.15$, $\bar{M} = 1$, $C_3 = 1$, $C_4 = 1$ and $\lambda = 0.5$ are used for calculation and the numerical solutions are shown for step increments $\Delta\bar{X}_b = 0.04, 0.02, 0.01$.

238 velocity. The densification velocity ($\bar{V}_d$) can be precisely obtained from Eq. (47) by substituting $\bar{V}_0 = \bar{V}_d$,
 239 $\bar{V} = 0$ and $\bar{\rho}_b = \bar{\rho}_t$ as follows.

$$\bar{V}_d = \sqrt{-4C_3(\bar{\rho}_t - \bar{\rho}_i) + 2\rho_0 C_4(\bar{\rho}_t^2 - \bar{\rho}_i^2) - 2\alpha(C_3 - \alpha\rho_0 C_4) \ln\left(\frac{\alpha - \bar{\rho}_t}{\alpha - \bar{\rho}_i}\right) + 2\alpha(C_3 + \alpha\rho_0 C_4) \ln\left(\frac{\alpha + \bar{\rho}_t}{\alpha + \bar{\rho}_i}\right)} \quad (48)$$

240 3.2.2. Series solution

241 For $\lambda > 0$, irrespective of the value of C_2 , the integrals in Eq. (46) are evaluated in Appendix A in terms
 242 of infinite series. Using the relation given by Eq. (A.9), the solution is obtained as:

$$\bar{V}^2 = -4 \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n}} \left(\frac{C_3 \bar{\rho}_b^{C_2+2n-1}}{C_2+2n-1} - \frac{\rho_0 C_4 \bar{\rho}_b^{C_2+2n}}{C_2+2n} \right) + c \quad (49)$$

where c is the integration constant determined by using the initial conditions. Thus, the final form of the solution is given by,

$$\bar{V} = \sqrt{\bar{V}_0^2 - \sum_{n=1}^{\infty} \frac{4}{(4\lambda\bar{M} + (1+\lambda)^2)^n} \left(C_3 \frac{\bar{\rho}_b^{C_2+2n-1} - \bar{\rho}_i^{C_2+2n-1}}{C_2+2n-1} - \rho_0 C_4 \frac{\bar{\rho}_b^{C_2+2n} - \bar{\rho}_i^{C_2+2n}}{C_2+2n} \right)} \quad \text{if } 0 < \lambda \leq 1 \quad (50)$$

243 This solution is not valid for $\lambda < 0$, since the series solution does not converge (see Appendix A). Although
 244 the solution contains infinite series, it converges with just the first few terms of the series for λ values away
 245 from zero. For example, the velocity is evaluated corresponding to $\lambda = 0.5$ using 1-term, 2-term and 4-term
 246 approximations and plotted in Figure 7a. It is computed for $C_2 = 2$ so that it can be compared to the
 247 available exact solution given by Eq. (47). Figure 7b shows the corresponding errors due to truncation of
 248 series and it is observed that the error for the 4-term solution is less than 1%. Therefore, for $\lambda = 0.5$, a high
 249 degree of accuracy can be achieved with just the first four terms in the infinite series. Similarly, the velocity
 250 is evaluated corresponding to $\lambda = 0.05$ (representative of λ close to zero) using 4-term, 8-term and 16-term
 251 approximations and shown in 7c. It is seen in 7d that the 16-term solution has an error below 1% indicating
 252 the requirement of more terms for values of λ close to zero.

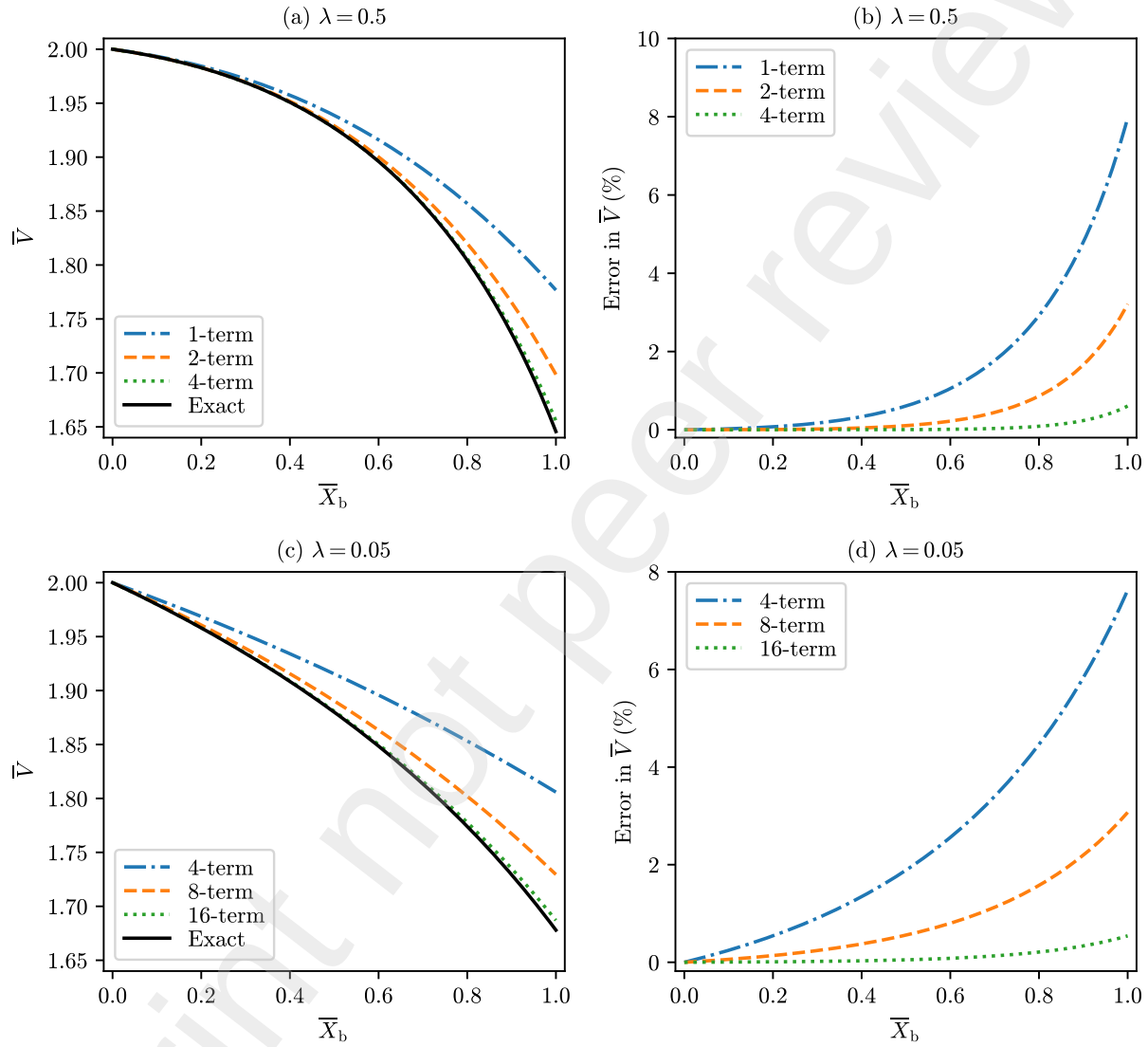


Figure 7: A comparison of rigid mass velocity $\bar{V}$ obtained from series solution with that of the exact solution (a) $\bar{V}$ when $\lambda = 0.5$ (b) Error in $\bar{V}$ when $\lambda = 0.5$ (c) $\bar{V}$ when $\lambda = 0.05$ (d) Error in $\bar{V}$ when $\lambda = 0.05$. Here $\bar{\rho}_0 = 0.15$, $\bar{M} = 1$, $C_2 = 2$, $C_3 = 1$, $C_4 = 1$ and $V_0 = 2$ are used for calculation and the dimensionless velocity $\bar{V}$ is computed by using partial sums of the infinite series solution.

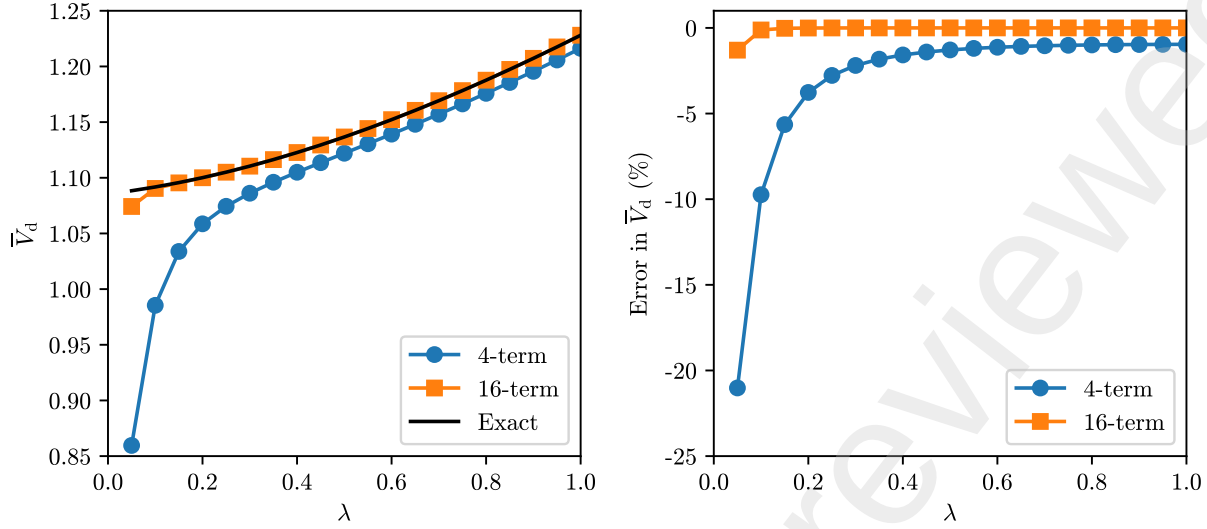


Figure 8: Densification velocity versus density gradient parameter

The densification velocity $\bar{V}_d$ can be obtained from Eq. (50). Using Eqs. (3) and (4), it is given by:

$$\bar{V}_d = \sqrt{\sum_{n=1}^{\infty} \frac{4}{(4\lambda\bar{M} + (1+\lambda)^2)^n} \left(C_3 \frac{(1+\lambda)^{C_2+2n-1} - (1-\lambda)^{C_2+2n-1}}{C_2+2n-1} - \rho_0 C_4 \frac{(1+\lambda)^{C_2+2n} - (1-\lambda)^{C_2+2n}}{C_2+2n} \right)}$$

if $0 < \lambda \leq 1$ (51)

The densification velocity is evaluated using 4-term and 16-term approximations and is shown as a function of λ in Figure 8. The exact solution is also obtained using Eq. (48) and plotted in the figure. The 4-term solution is accurate for values of λ away from the zero. However, for λ close to zero, it results in large errors while the 16-term solution results in significantly better estimations. As seen in Eq. (51), the terms with $(1+\lambda)$ and $(1-\lambda)$ tend to nullify each other when λ approaches zero. Therefore, more higher-order terms are needed to achieve the same level of accuracy.

4. Results and Discussion

The impact resistance performance of the cellular material depends on the energy absorption capacity, incident stress and transmitted stress. All these performance variables are determined for both single and double shock mode using $\bar{\rho}_0 = 0.15$, $C_2 = 2$, $C_3 = 1$ and $C_4 = 1$. As $C_2 = 2$, the exact solution derived in Section 3.2.1 is used for single shock mode. However, for double shock mode, the differential equations given by Eqs. (33), (39) and (40) are solved numerically using a finite difference method. In the numerical calculations, since the finite step increments for the forward shock $\Delta\bar{X}_f$ and the backward shock front $\Delta\bar{X}_b$

are related through the discretized form of Eq. (40), only one of them can be independently specified. An independent finite step increment ($\Delta\bar{X}$) can be taken on one of either forward or backward shock front, and the other should be computed using Eq. (40). Since the backward shock front travels faster than the forward shock front immediately after impact, $\Delta\bar{X}_b$ is greater than the corresponding $\Delta\bar{X}_f$. Therefore, $\Delta\bar{X}_b$ is independently incremented (i.e. $\Delta\bar{X}_b = \Delta\bar{X}$), and $\Delta\bar{X}_f$ is calculated using $\Delta\bar{X}_b$, until $\Delta\bar{X}_b \geq \Delta\bar{X}_f$ holds. Once $\Delta\bar{X}_f$ exceeds $\Delta\bar{X}_b$, $\Delta\bar{X}_f$ is independently incremented (i.e. $\Delta\bar{X}_f = \Delta\bar{X}$), and $\Delta\bar{X}_b$ is calculated using $\Delta\bar{X}_f$. Numerical singularities are avoided by independently incrementing $\Delta\bar{X}_f$ when $\bar{V}_u$ approaches zero (see Eq. (40)). To ensure accurate results, $\Delta\bar{X}$ is taken equal to 0.001 for all step increments.

For double shock mode, the backward shock front can stop propagating when $\bar{V}_u$ reduces to zero [25]. After this point, only the forward shock front exists, and Eq. (33) can still be used to calculate $\bar{V}$ by substituting $\bar{V}_u = 0$. However, Eqs. (39) and (40) are no longer required.

The important variables influencing the impact resistance are discussed as follows.

4.1. Energy absorption capacity

Energy absorption capacity is the energy dissipated by the cellular material when the initial velocity is equal to the densification velocity. It is determined by using the principle of conservation of energy. The total kinetic energy supplied, which is equal to the sum of the kinetic energies of the rigid body and the cellular material before impact, is dissipated by compaction of cellular material. Therefore, the specific energy absorption capacity or energy absorption capacity per unit mass of cellular material is obtained as:

$$\bar{E}_d = \frac{1}{2} (\bar{M} + 1) \bar{V}_d^2 \quad (52)$$

where $\bar{E}_d = E_d \rho_0 \hat{\rho}_s / \sigma_{p0}$ is the dimensionless form of specific energy absorption capacity E_d . For single shock mode, Eqs. (43) and (48) are used to calculate $\bar{V}_d$ in Eq. (52). However, for double shock mode, the computation of $\bar{V}_d$ is intensive. Apart from several iterations on step increment used in numerical solutions, multiple iterations on initial velocity are also needed for reaching closer approximations.

Figure 9 shows $\bar{E}_d$ versus λ , and it is divided into two regions: (i) single shock mode when $\lambda \geq \lambda_{cr}$, (ii) double shock mode when $\lambda < \lambda_{cr}$. The boundary line between these regions is the locus of points corresponding to $\lambda = \lambda_{cr}$, and is shown in the figure with a dashed line. In single shock region, the specific energy absorption capacity increases monotonically with λ and, thus, is maximum when $\lambda = 1$. Therefore, the specific energy absorption capacity of the graded cellular material with $\lambda > 0$ is always greater than that of the uniform density material ($\lambda = 0$). However, it is lower than that of the uniform density material for graded material with λ such that $\lambda_{cr} < \lambda < 0$. In double shock region, for $\lambda < \lambda_{cr}$, it decreases sharply and then increases. The sharp decrease in energy absorption capacity is due to the formation of the second

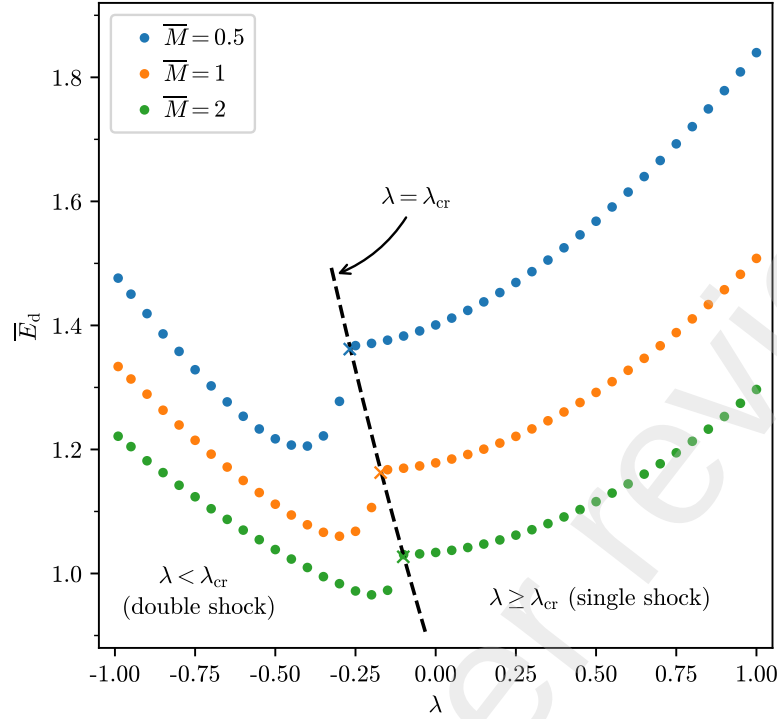


Figure 9: Energy absorption capacity versus density gradient parameter

(forward) shock front, which compacts the cellular material at low velocities. Energy absorption is velocity dependent and less energy is absorbed at low velocities. It should be noted that immediately after impact, the velocity jump, and consequently the shock velocity, is low at the transmitted end since the velocity on both sides of the shock front is the same. In reality, the forward shock front will not be a sharp discontinuity due to a small difference in velocity across the shock. However, in this study, for simplicity, it is treated as a sharp discontinuity. This assumption does not lead to results far from reality and has been used with a great degree of success in previous studies [2, 18]. Further, for steeper density gradients in the double shock region (λ values close to -1), the specific energy absorption capacity improves, as seen in the figure. Even though the compaction due to the second shock front occurs at low velocity, the improvement in energy absorbed is observed because of the compaction of higher density material at higher velocity by the backward shock front. In other words, only a small mass of cellular material is underutilized by compaction at a low velocity near the transmitted end because the density at that end is very low.

4.2. Incident stress

The incident stress is calculated from the stress behind the backward shock front. The backward compacted material is in static equilibrium, since the undeformed material towards the incident end is brought to rest after the shock front passes over it. Therefore, the incident stress σ_i that is the reaction stress at

the incident end is equal to the stress behind the backward shock front σ_{db} . For single shock mode, it is obtained using Eq. (30) as follows:

$$\sigma_i = \sigma_{db} = \sigma_{pb} + \rho_b \hat{\rho}_s \frac{V^2}{\epsilon_{db}} \quad (53)$$

Therefore, in dimensionless form, the incident stress is given by:

$$\bar{\sigma}_i = \bar{\sigma}_{pb} + \frac{\bar{\rho}_b \bar{V}^2}{\epsilon_{db}} \quad (54)$$

For double shock mode, the incident stress is calculated based on the value of V_u . When $V_u > 0$, both shock fronts are propagating, and the incident stress is given by,

$$\bar{\sigma}_i = \bar{\sigma}_{pb} + \frac{\bar{\rho}_b \bar{V}_u^2}{\epsilon_{db}} \quad (55)$$

When $V_u = 0$, the backward shock front does not exist. Therefore, both the undeformed material as well as the backward compacted material are in static equilibrium. Therefore, the incident stress is given by,

$$\bar{\sigma}_i = \bar{\sigma}_{pf} \quad (56)$$

The dimensionless incident stress versus the shock front position for both single shock mode and double shock mode are shown in Figure 10a and Figure 10b, respectively. The plots are shown for the case when the initial velocity is equal to the densification velocity. For single shock mode, the stress profile is continuous for all density gradients. Since the plateau stress and the densification strain are functions of the relative density, the incident stress, according to Eq. (54), depends on the relative density ρ_b and velocity V ahead of the backward shock front. Note that since backward shock front travels backward, its position is given by: $\bar{X} = 1 - \bar{X}_b$. Therefore, the initial stress (at $\bar{X}_b = 0$) corresponds to $\bar{X} = 1$. As shown in Figure 10a, for $\lambda = \lambda_{cr}$ and $\lambda = 0$, the incident stress is initially high and it decreases as the shock front propagates. This trend in incident stress is due to the velocity ahead of the shock front, which is initially high and then gradually reduces because of compaction of the cellular material. This trend is also true for positive values of λ that are very close to zero. For slightly higher positive values of λ , for example, $\lambda = 0.25$, the incident stress remains approximately flat indicating the influence of increasing density gradation. Due to density gradation, the local relative density is low at the incident end and high at the transmitted end. Low density at the incident end results in lower initial incident stress offsetting the gains in stress due to high velocity. Further, for higher positive values of λ , such as $\lambda = 0.5$ and $\lambda = 1.0$, the dominance of density gradation grows. As seen for $\lambda = 0.25, 0.5, 1.0$, the interplay between density gradation and velocity causes the incident

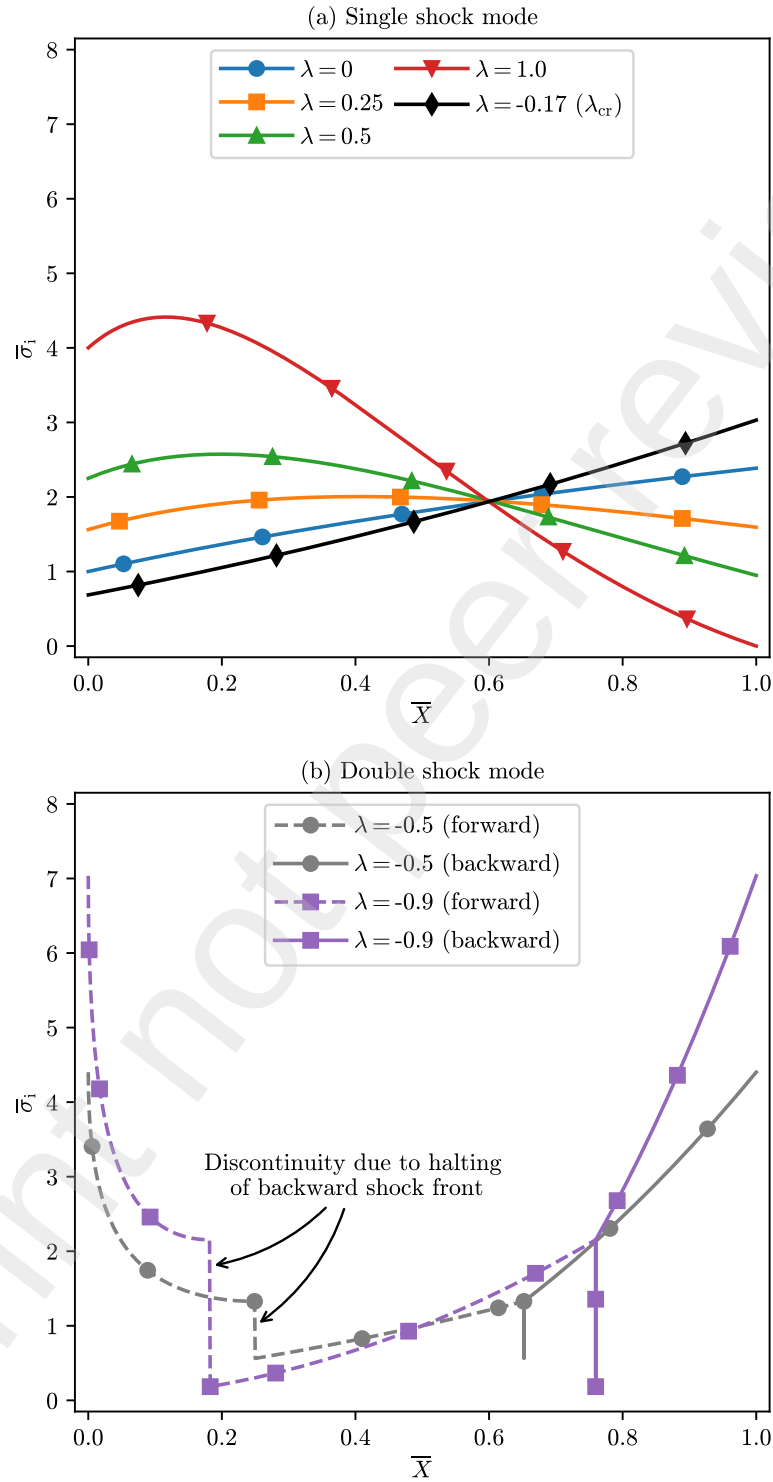


Figure 10: Dimensionless incident stress (a) single shock mode (b) double shock mode

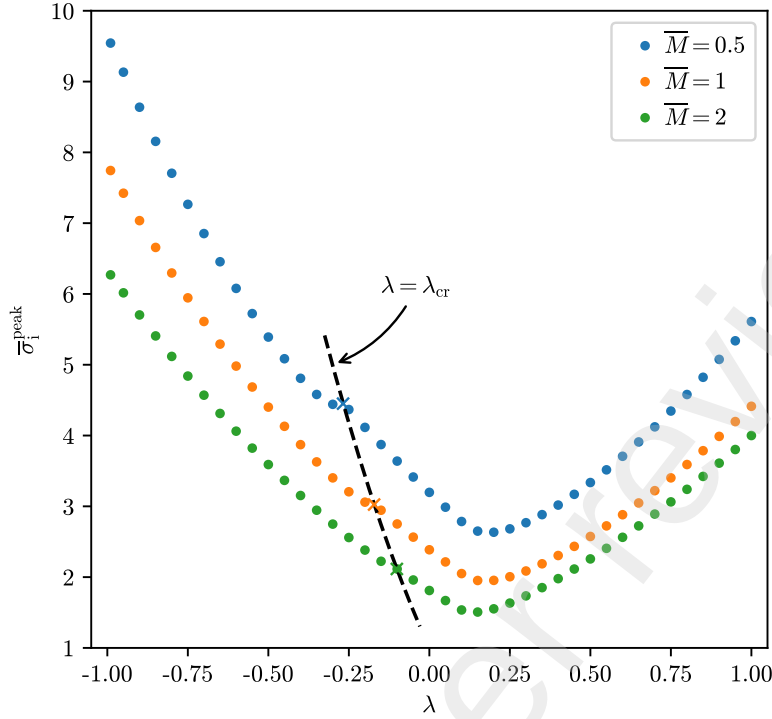


Figure 11: Dimensionless peak incident stress versus density gradient parameter

stress to first increase, reach a peak and then decrease. Both the location and magnitude of peak incident stress vary depending on λ . In certain applications where specific crashworthiness requirements prescribe a particular type of stress profile, a suitable density gradient parameter can be selected based on their response.

Since two shock fronts coexist in double shock mode, the stress profile is much more complicated. The variation in stress is shown as a function of forward and backward shock fronts in Figure 10b. The initial incident stress in this case is given by $\bar{X}_f = 0$ which corresponds to $\bar{X} = 0$, or $\bar{X}_b = 0$ which corresponds to $\bar{X} = 1$. Irrespective of the density gradient parameter, the initial incident stress is also the peak stress. As the shock fronts propagate, the incident stress decreases. A discontinuity occurs when the backward shock front stops propagating, as noted in the figure by a sudden drop in stress. Beyond this point, the incident stress equals the plateau stress of material ahead of the forward shock front. Hence, the stress increases as the forward shock front propagates further because the relative density, and consequently, the plateau stress increase in the forward direction.

The peak incident stress $\bar{\sigma}_i^{\text{peak}}$ is shown as a function of the density gradient parameter in Figure 11 for three different rigid masses ($\bar{M} = 0.5, 1$ and 2). For all the three rigid masses, it is maximum when $\lambda = -1$, and minimum when λ is somewhere between 0 and 0.25 . It can also be seen that the peak incident stress for all the negative values of λ is greater than that of the uniform density material ($\lambda = 0$). On the other

hand, it is lower for smaller positive values of λ but higher for higher positive values of λ .

4.3. Impact resistance factor for incident end

As shown above, the density gradient has a significant effect on the energy absorption capacity as well as the incident stress. Both are important for impact resistance performance. A cellular material that can absorb more energy is favorable. Also, most often there is a limit on the stress that an object can withstand. Therefore, lower peak incident stress, which is the peak stress acting on the object placed at the incident end, is advantageous. Thus, the density gradient that leads to high energy absorption capacity but low peak incident stress is beneficial for impact protection at the incident end. When $\lambda = 1$, although the energy absorption is maximum, the incident stress is at least high, if not a maximum. The incident stress is minimum for small positive values of λ . Note that for quasi-static loading conditions, the density that maximizes the energy absorption efficiency, which is the ratio of the absorbed energy to the stress, is considered optimal [33]. However, for high-speed impact loading conditions, the stresses at the two ends of cellular material are completely different from each other as they are dynamic in nature. Thus, to determine the optimal density gradient parameter for high-speed impact loading conditions, an impact resistance factor (IRF) is defined. For protecting objects placed at the incident end, IRF_i is used, and it is defined as the ratio of the specific energy absorption capacity to the peak incident stress,

$$\text{IRF}_i = \frac{\bar{E}_d}{\sigma_i^{\text{peak}}} \quad (57)$$

The peak incident stress used in the definition corresponds to the case when the initial velocity equals the densification velocity, and it is consistent with the energy absorption capacity since it is also evaluated for the densification velocity.

A higher value of IRF_i indicates superior impact resistance. IRF_i versus λ is shown in Figure 12. It reaches peak value between $\lambda = 0$ and $\lambda = 0.25$ for all the three different $\bar{M}$ values shown. A heavier rigid mass leads to a higher IRF_i . Moreover, in general, graded cellular materials with positive density gradients have higher IRF_i than those with negative density gradients.

4.4. Transmitted stress

For single shock mode, the dimensionless transmitted stress is obtained by using Eq. (19) and is given by,

$$\bar{\sigma}_t = \frac{\bar{M}\bar{\sigma}_{\text{pb}}}{\bar{M} + (1 - \bar{X}_b)(1 + \lambda\bar{X}_b)} \quad (58)$$

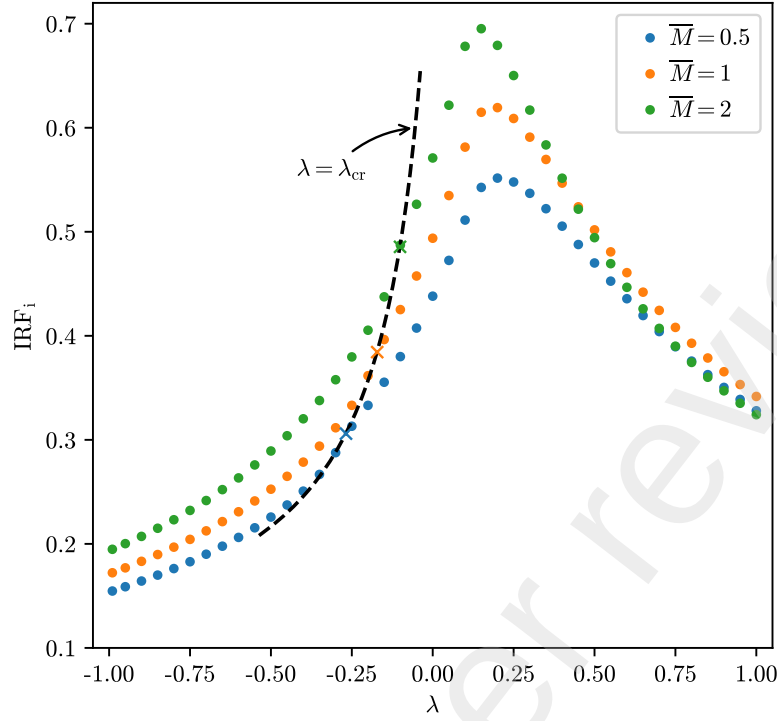


Figure 12: IRF_i versus density gradient parameter

For double shock mode, the transmitted stress is determined by applying the principle of balance of linear momentum to the rigid mass. Thus,

$$\sigma_t = - \left(\frac{M}{A} \right) \frac{dV_f}{dt} \quad (59)$$

The time derivative of V_f in Eq. (59) is eliminated by using Eq. (31). Therefore, from Eq. (29), the following equation can be obtained.

$$\sigma_t = \frac{M}{M + \hat{\rho}_s A \int_0^{X_f} \rho(X) dX} \left(\sigma_{pf} + \frac{\rho_f \hat{\rho}_s}{\epsilon_{df}} (V_f - V_u)^2 \right) \quad (60)$$

Using Eq. (2), the transmitted stress in dimensionless form is given by,

$$\bar{\sigma}_t = \frac{\bar{M}}{\bar{M} + (1 + \lambda) \bar{X}_b - \lambda \bar{X}_b^2} \left(\bar{\sigma}_{pf} + \frac{\bar{\rho}_f}{\epsilon_{df}} (\bar{V}_f - \bar{V}_u)^2 \right) \quad (61)$$

When the backward shock front stops, Eq. (61) can still be used to determine the transmitted stress.

Figure 13 shows the transmitted stress or the reaction stress on the rigid mass. As shown in Figure 13a, for single shock mode, the transmitted stress is independent of the initial velocity, depending only on the

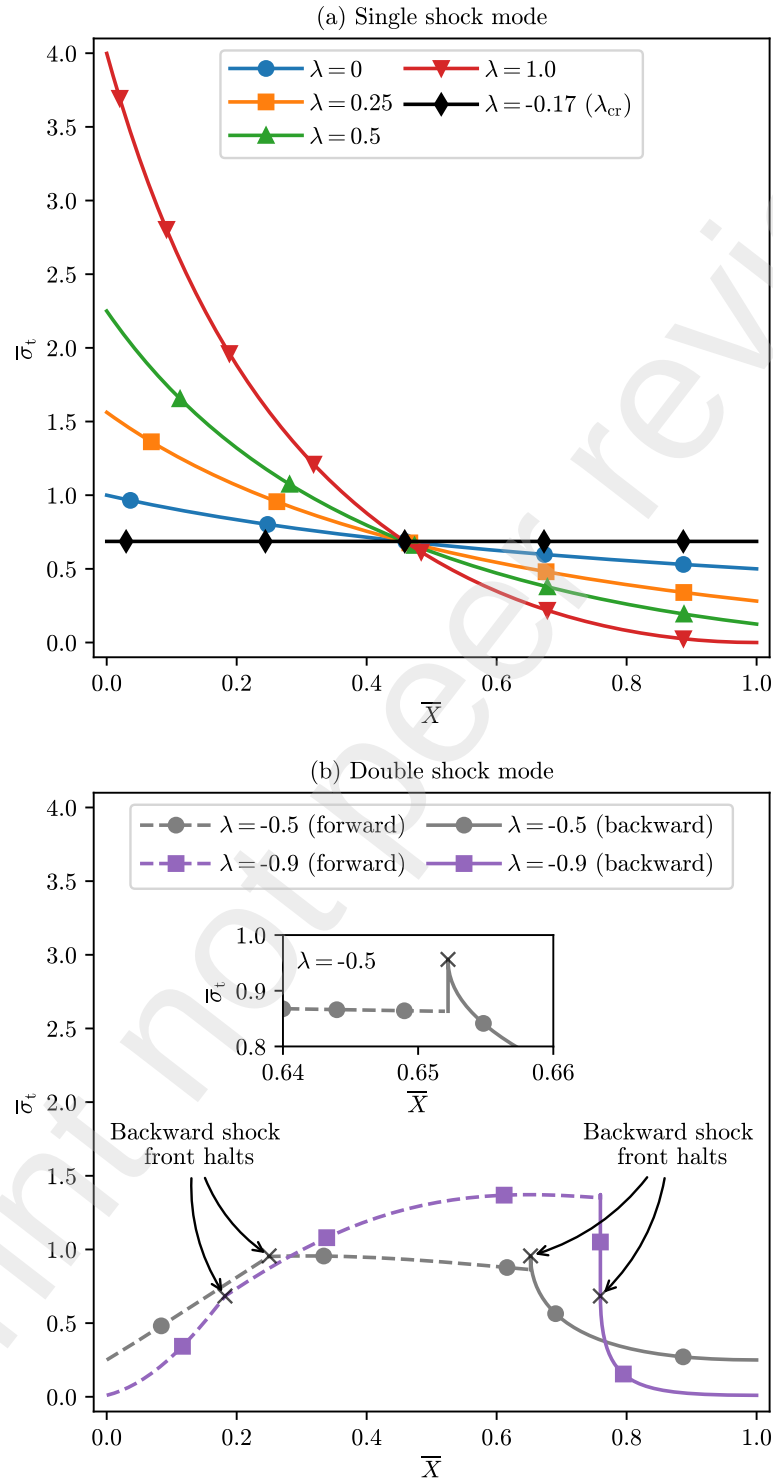


Figure 13: Dimensionless transmitted stress (a) single shock mode (b) double shock mode

rigid mass and the initial density distribution of the cellular material. For most λ values, although the stress is initially low (corresponding to $\bar{X} = 1$), it increases as the shock front propagates. The increase in stress is proportional to the magnitude of the density gradient parameter. Regardless of the density gradient, peak transmitted stress occurs at the very end (corresponding to $\bar{X} = 0$). When $\lambda = \lambda_{cr}$, the stress profile is exactly flat because the reaction stress on the rigid mass is equal to the plateau stress of material at the transmitted end throughout the shock front propagation, as shown in Figure 3. Such a stress profile is desirable in many applications.

As shown in Figure 13b, the transmitted stress is more involved in double shock mode. For both $\lambda = -0.5$ and $\lambda = -0.9$, it is low initially ($\bar{X} = 0$ or $\bar{X} = 1$), and it increases until the point when the backward shock front stops propagating. For $\lambda = -0.5$, it reaches its peak value when backward shock front halts (see magnified inset plot for clarity), but for $\lambda = -0.9$, it continues to increase further. The peak values of transmitted stress $\bar{\sigma}_t^{\text{peak}}$ are shown in Figure 14. For single shock mode, it is observed that peak transmitted stress is independent of the rigid mass as all the three curves corresponding to different $\bar{M}$ coincide with each other in the figure. Rigid mass independence can also be verified by substituting $\bar{X} = 0$ or $\bar{X}_b = 1$ (location for peak transmitted stress for single shock mode) in Eq. (58). It is seen that the peak transmitted stress equals the plateau stress of material at the transmitted end. For double shock mode, lower rigid mass results in lower peak transmitted stress.

4.5. Impact resistance factor for transmitted end

In a similar manner as the incident end, an impact resistance factor for the transmitted end IRF_t is defined as the ratio of the specific energy absorption capacity to the peak transmitted stress,

$$\text{IRF}_t = \frac{\bar{E}_d}{\bar{\sigma}_t^{\text{peak}}} \quad (62)$$

The density gradient that maximizes IRF_t provides maximum impact protection to the object placed at the transmitted end. Figure 15 shows the IRF_t . It is interesting to note that regardless of the rigid mass, IRF_t reaches peak value when $\lambda = \lambda_{cr}$. Moreover, it is higher for lighter rigid mass. Furthermore, in general, negative density gradients have higher IRF_t than the positive density gradients.

4.6. Optimal density gradient

The optimal values of the density gradient parameter λ_i^{opt} and λ_t^{opt} corresponding to maximum IRF_i and IRF_t , respectively are shown in Figure 16. Both λ_i^{opt} and λ_t^{opt} are very close to 0 for very high values of $\bar{M}$. In other words, the uniform density cellular material is optimal for a very heavy rigid mass, irrespective of whether impact protection is required at the incident or transmitted end. However, excluding the narrow

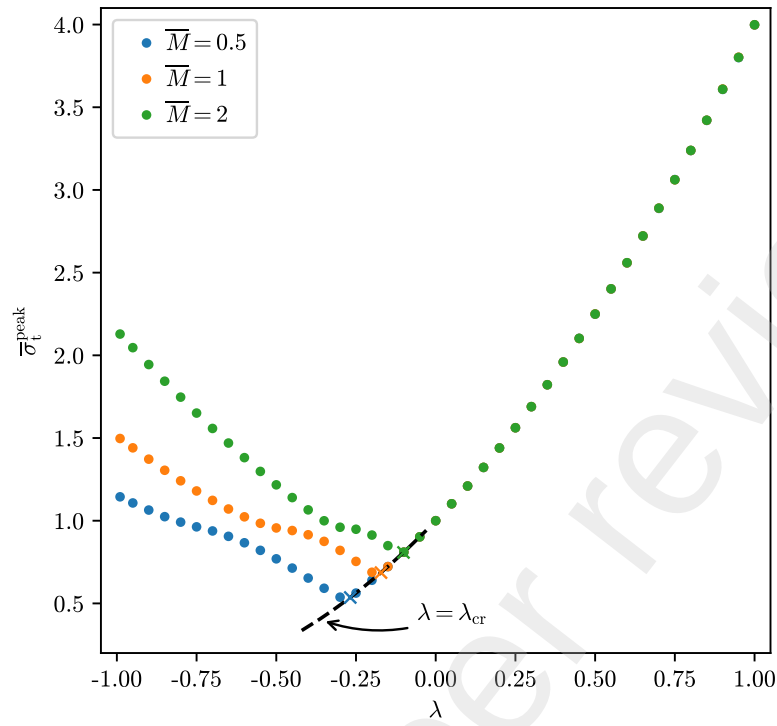


Figure 14: Dimensionless peak transmitted stress versus density gradient parameter

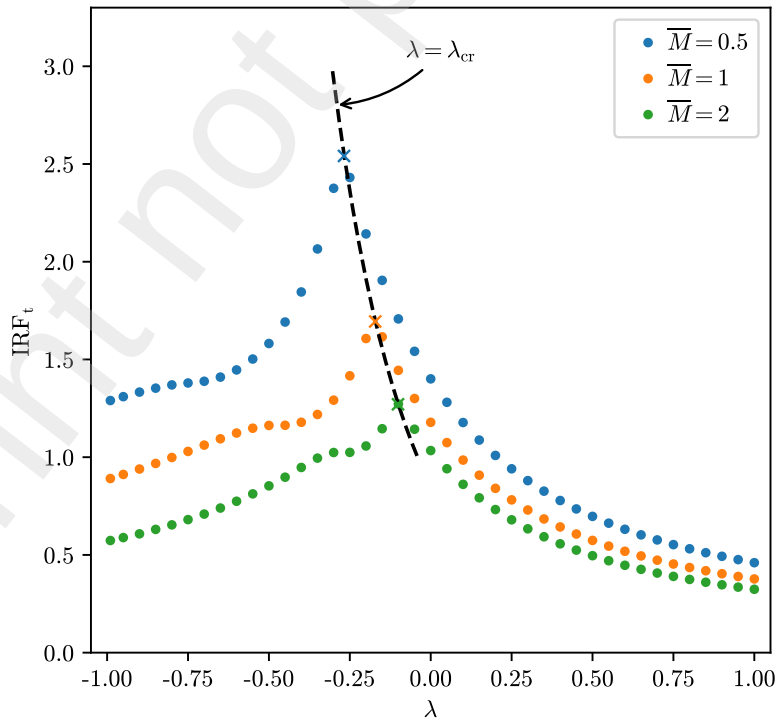


Figure 15: IRF_t versus density gradient parameter

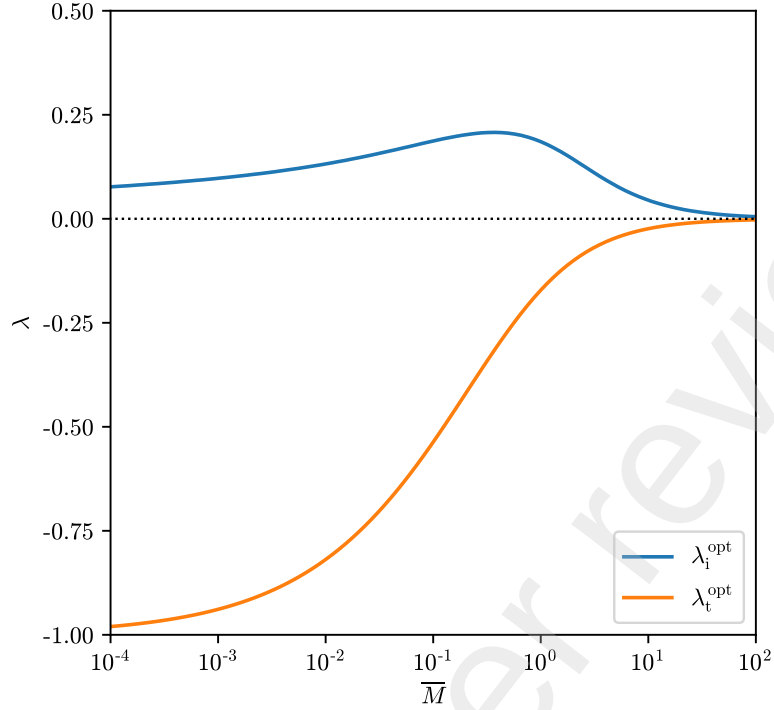


Figure 16: Optimal density gradient parameter

range of heavy masses, both λ_i^{opt} and λ_t^{opt} are non-zero for a broad spectrum of rigid masses. The figure shows the optimal density gradients that should be used for a specific rigid mass; λ_i^{opt} remains between 0 and 0.25, but λ_t^{opt} can have values ranging from -1 to 0. For very small values of rigid mass, λ_t^{opt} is close to -1. Interestingly, all the optimal values of density gradients belong to the single shock regime. The second shock front not only makes the deformation complex but also results in sub-optimal impact resistance performance.

5. Conclusions

The impact response of density-graded cellular material results in either the single or double shock mode depending on whether the density gradient parameter is less or greater than its critical value. The expression for the critical density gradient parameter has been derived, and it is shown that the critical value depends on the mass of the impinged rigid body and the parameter controlling plateau stress variation. The analytical solutions have been formulated for special cases in the single shock mode. A closed-form solution has been derived when the plateau stress has a quadratic variation with relative density. On the other hand, a series solution is proposed for graded cellular materials when the density increases from the incident end towards the transmitted end. The analytical expressions for the instantaneous velocity of rigid mass, densification

velocity, energy absorption capacity, incident stress, and transmitted stress have been derived.

The analytical solutions are shown to be consistent with numerical solutions. The results show that density gradient remarkably governs the impact resistance performance since it influences both the absorbed energy and the peak stress imposed on the object intended to be protected. Non-dimensional metrics IRF_i and IRF_t for the incident and transmitted ends, respectively, are defined to study the impact resistance performance. The metrics results indicate that a graded cellular material should be oriented such that the low-density material is placed towards the incident end for protecting the objects at that end; specifically, smaller density gradients must be chosen for improved impact resistance at incident end. Conversely, in general, graded cellular materials with low-density material towards the transmitted end are prospective candidates for protecting the objects placed at the transmitted end. Precisely, it is advantageous to use a density gradient equal to the critical density gradient parameter when the object at the transmitted end is to be safeguarded. The recommended optimal values of density gradient parameter are determined as a function of impinged rigid mass. All the optimal values lie in the single shock region, indicating that the double shock mode results in not only a complex deformation mode but also inferior impact resistance.

Furthermore, this study should promote the understanding of dynamics of density-graded cellular materials. Particularly, the availability of the analytical solutions derived in this study for single shock mode should lay the foundation for the efficient evaluation of density-graded cellular structures.

CRediT authorship contribution statement

Vijendra Gupta: Conceptualization, Methodology, Analysis, Writing - original draft. Addis Kidane: Supervision, Review, Writing - review & editing. Michael Sutton: Review, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Integral Evaluation Using Hypergeometric Series

The integrals in Eq. (46) are evaluated using hypergeometric series. The hypergeometric function ${}_2F_1$ is defined as [34, 35]:

$${}_2F_1(a, b; d; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(d)_n} \frac{z^n}{n!} \quad (\text{A.1})$$

where $(a)_n \equiv a(a+1)(a+2)(a+3)\dots(a+n-1)$ is the (rising factorial) Pochhammer symbol. Setting $a = 1$ and $d = b + 1$, we get:

$${}_2F_1(1, b; b + 1; z) = \sum_{n=0}^{\infty} \frac{b}{b + n} z^n \quad (\text{A.2})$$

where the relations $(1)_n = n!$ and $(b)_n / (b + 1)_n = b / (b + n)$ are used.

In integral form, the hypergeometric function is given by:

$${}_2F_1(a, b; d; z) = \frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)} \int_0^1 \frac{p^{b-1} (1-p)^{d-b-1}}{(1-zp)^a} dp \quad (\text{A.3})$$

where Γ is the gamma function. Again setting $a = 1$ and $d = b + 1$ we get:

$${}_2F_1(1, b; b + 1; z) = b \int_0^1 \frac{p^{b-1}}{(1-zp)} dp \quad (\text{A.4})$$

where the properties of the gamma function, $\Gamma(1) = 1$ and $\Gamma(b + 1) = b\Gamma(b)$, are used to arrive at Eq. (A.4).

From Eqs. (A.2) and (A.4), the following equation is obtained.

$$\int_0^1 \frac{p^b}{zp - 1} dt = \sum_{n=0}^{\infty} \frac{-z^n}{b + n + 1} \quad (\text{A.5})$$

The integrals in Eq. (46) have the form of the following integral I .

$$I = \int \frac{\bar{\rho}_b^k d\bar{\rho}_b}{\alpha^2 - \bar{\rho}_b^2} \quad (\text{A.6})$$

The integral I is transformed so that Eq. (A.5) can be used. In definite integral form,

$$I = \int_0^{\bar{\rho}_b} \frac{y^k dy}{\alpha^2 - y^2} + c \quad (\text{A.7})$$

where c is the integration constant. Substituting $y^2 = \bar{\rho}_b^2 v$, we get,

$$I = -\frac{\bar{\rho}_b^{k+1}}{2\alpha^2} \int_0^1 \frac{v^{\frac{k-1}{2}} dv}{\frac{\bar{\rho}_b^2}{\alpha^2} v - 1} + c \quad (\text{A.8})$$

Using Eq. (A.5), the integral can be evaluated as:

$$I = \frac{\bar{\rho}_b^{k+1}}{\alpha^2} \sum_{n=0}^{\infty} \frac{\bar{\rho}_b^{2n}}{(k+2n+1)\alpha^{2n}} + c \quad (\text{A.9})$$

After shifting the summation index,

$$I = \int \frac{\bar{\rho}_b^k d\bar{\rho}_b}{\alpha^2 - \bar{\rho}_b^2} = \bar{\rho}_b^{k-1} \sum_{n=1}^{\infty} \frac{\bar{\rho}_b^{2n}}{(k+2n-1)\alpha^{2n}} + c \quad (\text{A.10})$$

The validity of the series expression in Eq. (A.10) is based on the following requirements:

1. The series converges only when $\alpha \geq \bar{\rho}_b$.
2. The condition, $k+2n \neq 1$, must be satisfied.

The deductions for both these requirements are shown as follows.

Since $\alpha = \sqrt{4\lambda\bar{M} + (1+\lambda)^2}$, the first requirement translates to:

$$4\lambda\rho_0^2\bar{M} + \rho_0^2(1+\lambda)^2 \geq \rho_b^2 \quad (\text{A.11})$$

Using Eq. (4), the inequality can be simplified to:

$$4\lambda\rho_0^2\bar{M} + \rho_t^2 \geq \rho_b^2 \quad (\text{A.12})$$

Eq. (A.12) is always satisfied for any $\lambda \geq 0$, since $\rho_t \geq \rho_b$ as well as both $\rho_0 \geq 0$ and $\bar{M} \geq 0$ hold true for any $\lambda \geq 0$. Contrarily, if $\lambda < 0$, the inequality is violated. Thus, Eq. (A.10) converges only for $\lambda \geq 0$. Further, since Eq. (46) is only applicable for density-graded materials, which have $\lambda \neq 0$, the solution given by Eq. (50) obtained using Eq. (A.10) is only valid for $\lambda > 0$. Note that the solution for $\lambda = 0$ is obtained separately in Section 3.1.

The second requirement ($k+2n \neq 1$) is always satisfied. According to the integrals used in Eq. (46), $k = C_2$ or $k = C_2 + 1$. Since $n \geq 1$ (see Eq. (A.10)) and $C_2 > 0$ (see Eq. (5)), $k+2n \neq 1$ is always satisfied.

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Letter

Closed-form solution for shock wave propagation in density-graded cellular material under impact

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ABSTRACT

Density-graded cellular materials have tremendous potential in structural applications where impact resistance is required. Cellular materials subjected to high impact loading result in a compaction type deformation, usually modeled using continuum-based shock theory. The resulting governing differential equation of the shock model is nonlinear, and the density gradient further complicates the problem. Earlier studies have employed numerical methods to obtain the solution. In this study, an analytical closed-form solution is proposed to predict the response of density-graded cellular materials subjected to a rigid body impact. Solutions for the velocity of the impinging rigid body mass, energy absorption capacity of the cellular material, and the incident stress are obtained for a single shock propagation. The results obtained are in excellent agreement with the existing numerical solutions found in the literature. The proposed analytical solution can be potentially used for parametric studies and for effectively designing graded structures to mitigate impact.

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Cellular materials (CMs) have peculiar properties pertinent to impact mitigation [1,2]. They provide exceptional energy absorption (EA) capability and are used as protection material in dynamic impact applications. They are also extremely lightweight as their solid material is distributed sparsely. They are composed of interconnected cells at the mesoscale level. When a CM is subjected to impact loading, a high-velocity is suddenly imposed at the impact end that causes layer-wise collapse of cells [3]. The progressive collapse of cells advances like a propagating shock and the continuum-based classical shock theory is usually used to model the deformation behavior. The authors in Ref. [3] first applied the shock theory to study the uniaxial dynamic crushing of various wood materials. They obtained an analytical solution for the direct impact scenario, where the CM (wood specimen) moving with a rigid backing mass strikes a stationary surface, assuming a rigid, perfectly plastic, locking (RPPL) material response. Later, the analytical solution for a stationary impact scenario, where a rigid mass moving with an initial velocity strikes a stationary CM, was also developed [4–6]. All the aforementioned studies focused on the response of uniform density CM.

Graded CMs with density variation potentially offer further improvement in the EA performance and crashworthiness, and they

have gained attention in recent years [7–14]. Many studies have employed finite element methods to study the response of density-graded CMs subjected to impact loading. Some of the finite element studies have modeled the graded CM as a continuum [7,8], while some other works have made use of cell-based models that concentrate on mesoscale response [15–18]. Although the finite element models seek to capture the material behavior rigorously, they are computationally intensive. There also have been efforts to study the response of the graded CM through experiments, even though manufacturing of continuously graded CMs is challenging. Nevertheless, rapid advancement in manufacturing technology, such as additive manufacturing, has been overcoming this problem [19]. Currently, experimental investigations often approximate the graded structures with multiple uniform density layers of different densities [20–23]. To model continuous density gradation, there have been attempts to take an analytical approach. On this front, initially, a honeycomb structure with a uniform density but a gradient in plateau stress was examined [10]. Although the simplified assumptions in Ref. [10] led to a closed-form solution, the variation in the density was not captured. Thereafter, the density-graded structure was also studied using an analytical approach for the direct impact scenario [24]. Assuming the density of the compacted CM is constant regardless of the initial density, analytical solutions for the shock velocity and the incident stress were obtained for various density profiles in Ref. [24]. There have

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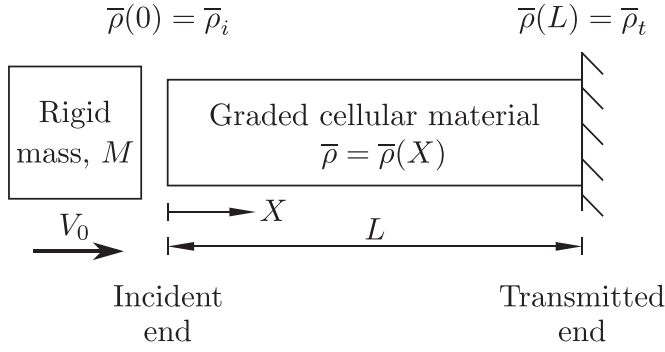


Fig. 1. Schematic diagram of rigid mass striking density-graded CM.

been some works on design strategy to determine the density distribution based on specific crashworthiness requirements [25,26]. Also, several other works [9,11,13] have investigated the stationary impact scenario with a density gradient, but they have used numerical methods for the solution. For better accuracy, numerical methods demand a small step size that results in large number of iterations. A precursory analytical study for density-graded CM under quasi-static loading conditions was performed recently [27]. However, apart from a few attempts discussed above, a general analytical formulation of graded CM under stationary impact scenario does not exist.

In this paper, an explicit and closed-form solution for the stationary impact scenario of density-graded CM is proposed based on the RPPL material model. The solution is facilitated by using the shock front position as an independent variable in the governing shock equations. The effect of density gradation on the total energy absorbed and incident stress is presented. The scope of the study is limited to a linear density variation and a single compaction shock front. A single shock front exists when the density gradient is positive (density increasing in the direction of impact), but two shocks can coexist when the density gradient is negative (density decreasing in the direction of impact) [9,11]. It should also be noted that the solution assumes that the response is governed by the shock deformation mechanism as is the case for impact velocities above a certain critical velocity [4,28] when the inertial effects associated with localized cell collapse dominate. The proposed closed-form expressions provide significant savings in computational time and effort. In general, they present a better description of the physical nature of the problem. Moreover, parametric studies can be performed efficiently with the help of explicit analytical solutions.

The differential equations governing the dynamic behavior are formulated using shock theory by considering CM as a continuum. A density-graded CM of mass \$m\$ and length \$L\$ is shown in Fig. 1. A rigid body of mass \$M\$ strikes the CM at the incident end with an initial velocity \$V_0\$ and the other (transmitted) end of the CM is held stationary. The relative density varies linearly from \$\bar{\rho}_i\$ at the incident end to \$\bar{\rho}_t\$ at the transmitted end. The relative density \$\bar{\rho}\$ is expressed as the ratio of CM density (\$\rho\$) and solid material density (\$\rho_s\$).

A dimensionless density gradient parameter \$\lambda\$ quantifying the gradation is defined as follows:

$$\lambda = \frac{\Delta \bar{\rho}}{2\bar{\rho}_0}, \quad (1)$$

where \$\Delta \bar{\rho} = \bar{\rho}_t - \bar{\rho}_i\$ is the difference between the relative densities at the two ends and \$\bar{\rho}_0 = (\bar{\rho}_i + \bar{\rho}_t)/2\$ is the average relative density. The linear variation in relative density along the length of the CM can be expressed as:

$$\bar{\rho} = \bar{\rho}_0 \left[1 + \lambda \left(\frac{2X}{L} - 1 \right) \right], \quad (2)$$

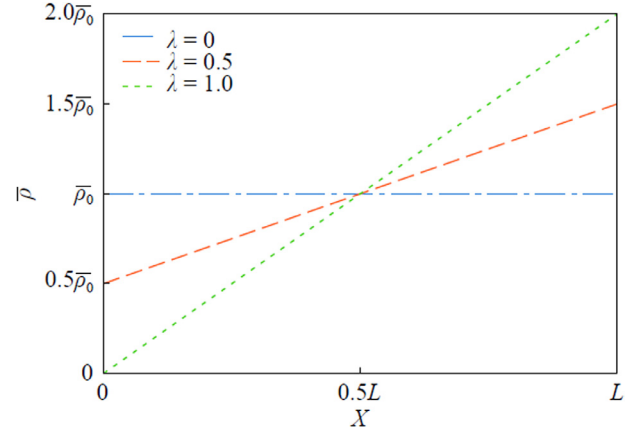


Fig. 2. Relative density variation.

where \$X\$ is the undeformed position coordinate along the length of the CM. The density gradient parameter can have values ranging from -1 to 1, with the zero value corresponding to the uniform density. This study deals with only non-negative values of the density gradient parameter. Negative values of density gradient parameter result in double shock mode, which is out of the scope of the present study. The relative density variation for different density gradient parameters is shown in Fig. 2. Since the relative density cannot take negative values, density gradient parameter can be at most equal to one.

An RPPL material behavior, same as the idealization employed in previous studies [3,5,9,11], is used in the present study. The variation in the plateau stress \$\sigma_p\$ and densification strain \$\epsilon_d\$ (sometimes also called as locking strain) with respect to the relative density \$\bar{\rho}\$ is given by the following equations [1]:

$$\sigma_p = \sigma_s C_1 \bar{\rho}^{C_2}, \quad (3)$$

$$\epsilon_d = C_3 - C_4 \bar{\rho}, \quad (4)$$

where \$\sigma_s\$ is the yield stress of the solid material and \$C_1, C_2, C_3\$ and \$C_4\$ are solid material parameters. Both the plateau stress and densification strain also vary along the length as they are dependent on the relative density. The parent material parameters \$C_1, C_2, C_3\$ and \$C_4\$ are often obtained from either experiments or numerical simulations. In previous studies, the value of \$C_2\$ has been taken to be 1.5 or 2 [1,9,10,29]. The parameter \$C_3\$ is taken equal to \$C_4\$ in some works [9] while both \$C_3\$ and \$C_4\$ are taken specifically equal to unity in some other works [11,13,30]. To maintain generality, the solution in the present study is determined without placing such restrictions on parent material parameters. Specific values are substituted after the solution expression is obtained.

The deformation of the CM after the rigid mass has moved by distance \$u\$ is illustrated in Fig. 3. Both the undeformed and deformed configurations are shown in the figure. At this instant, the shock front has propagated towards the transmitted end by distance \$X_f\$, and its instantaneous velocity is \$V_{sf}\$. The region behind the shock front is compacted to length \$x_f\$, and it moves along with the rigid mass at the instantaneous velocity, \$V_f\$. The material ahead of the shock front of length \$X_u\$ remains undeformed, and hence its velocity is zero. The total length of the CM in deformed configuration is \$l\$. The jump equations across the shock front are derived based on the conservation of mass and linear momentum in the Lagrangian form Ref. [31]. They are given by:

$$-\rho_f V_{sf} [1/\rho] = [V], \quad (5)$$

$$\rho_f V_{sf} [V] = [\sigma], \quad (6)$$

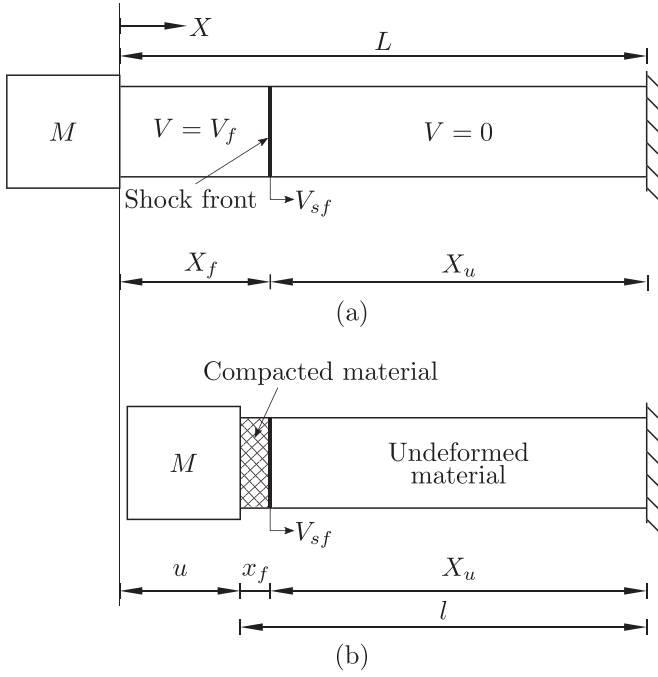


Fig. 3. Deformation in density-graded CM **a** undeformed configuration and **b** deformed configuration.

where ρ_f is the density just ahead of the shock front. The symbol $[\xi] = \xi^+ - \xi^-$ denotes the change in physical quantity ξ across the shock front as it jumps from ξ^- to ξ^+ . The superscripts - and + indicate the quantities just ahead of and behind the shock front, respectively. The compressive stress is taken to be positive in Eq. (6). The states of the material ahead of and behind the shock front are given by Eqs. (7) and (8), respectively.

$$\{\rho^-, \sigma^-, \epsilon^-, V^-\} = \{\rho_f, \sigma_{pf}, 0, 0\}, \quad (7)$$

$$\{\rho^+, \sigma^+, \epsilon^+, V^+\} = \{\rho_{df}, \sigma_{df}, \epsilon_{df}, V_f\}. \quad (8)$$

Eliminating V_{sf} from Eqs. (5) and (6), and using Eqs. (7) and (8) yields,

$$(\sigma_{df} - \sigma_{pf}) \left(1 - \frac{\rho_f}{\rho_{df}}\right) = \rho_f V_f^2. \quad (9)$$

The instantaneous strain ϵ_{df} behind the shock front is given by:

$$\epsilon_{df} = \frac{dX_f - dx_f}{dX_f}, \quad (10)$$

where dX_f is an infinitesimal length of undeformed material compacting to length dx_f after the passage of shock. Note that the compressive strains are considered to be positive. Using conservation of mass, Eq. (10) can be written as:

$$\epsilon_{df} = 1 - \frac{\rho_f}{\rho_{df}}. \quad (11)$$

Combining Eqs. (9) and (11), the stress behind the shock is obtained.

$$\sigma_{df} = \sigma_{pf} + \frac{\rho_f V_f^2}{\epsilon_{df}}. \quad (12)$$

The conservation of linear momentum for the rigid mass and compacted region gives:

$$\sigma_{df} = -\left(\frac{M}{A} + \int_0^{X_f} \rho(X) dX\right) \frac{dV_f}{dt}, \quad (13)$$

where A is the cross-sectional area of CM. Using Eqs. (5) and (11), it can be shown that the shock front velocity is given by:

$$V_{sf} = \frac{dX_f}{dt} = \frac{V_f}{\epsilon_{df}}. \quad (14)$$

The final differential equation was constructed with time as the independent variable in Refs. [9,11] and its solution was obtained numerically. However, in this work, to facilitate the analytical solution process, the differential equation is formulated with shock front position X_f as the independent variable. Therefore, using Eqs. (12)–(14) the following differential equation is derived:

$$\frac{\sigma_{pf} \epsilon_{df}}{V_f} + \rho_f V_f = -\left(\frac{M}{A} + \int_0^{X_f} \rho(X) dX\right) \frac{dV_f}{dX_f}. \quad (15)$$

The following relations are used for non-dimensionalization:

$$\bar{X}_f = \frac{X_f}{L}, \bar{M} = \frac{M}{m}, \bar{\sigma}_{pf} = \frac{\sigma_{pf}}{\sigma_{p0}}, \bar{V}_f = \frac{V_f}{\sqrt{\sigma_{p0}/\rho_0}}, \quad (16)$$

where σ_{p0} is the plateau stress corresponding to the average density ρ_0 , $m = \rho_0 AL$ is the mass of the CM, and $\bar{M}$ is the ratio of rigid mass and CM mass. The following is the dimensionless form of Eq. (15):

$$\frac{\bar{\sigma}_{pf} \epsilon_{df}}{\bar{V}_f} + \frac{\bar{\rho}_f}{\bar{\rho}_0} \bar{V}_f = -\left(\bar{M} + \frac{1}{\bar{\rho}_0} \int_0^{\bar{X}_f} \bar{\rho}(\bar{X}) d\bar{X}\right) \frac{d\bar{V}_f}{d\bar{X}_f}. \quad (17)$$

Using $Y = \bar{V}_f^2$, Eq. (17) can be transformed to the following linear ordinary differential equation:

$$\frac{dY}{d\bar{X}_f} + \frac{2H}{G} Y = \frac{-2F}{G}, \quad (18)$$

where the functions $F(\bar{X}_f)$, $G(\bar{X}_f)$ and $H(\bar{X}_f)$ defined to aid the solution are given as follows.

$$F(\bar{X}_f) = \bar{\sigma}_{pf} \epsilon_{df}, \quad (19)$$

$$G(\bar{X}_f) = \bar{M} + \frac{1}{\bar{\rho}_0} \int_0^{\bar{X}_f} \bar{\rho}(\bar{X}) d\bar{X}, \quad (20)$$

$$H(\bar{X}_f) = \frac{\bar{\rho}_f}{\bar{\rho}_0}. \quad (21)$$

The solution to Eq. (18) is given by:

$$Y(\bar{X}_f) = \bar{V}_f^2 = (-Q + c)/P, \quad (22)$$

where c is a constant to be obtained from the initial conditions and the functions P and Q are given by the following equations:

$$P(\bar{X}_f) = \exp\left(\int \frac{2H}{G} d\bar{X}_f\right), \quad (23)$$

$$Q(\bar{X}_f) = \int \left(\frac{2F}{G} P\right) d\bar{X}_f. \quad (24)$$

The function G given by Eq. (20) can be evaluated by using Eq. (2), and its form in terms of λ is given by:

$$G(\bar{X}_f) = \bar{M} + \bar{X}_f [1 - \lambda(1 - \bar{X}_f)]. \quad (25)$$

Therefore, using Eq. (25), the integral in Eq. (23) can be simplified resulting in:

$$P = \exp\left(2 \int \frac{dG}{G}\right) = G^2. \quad (26)$$

Thus, Eq. (24) also reduces to:

$$Q = 2 \int FGd\bar{X}_f. \quad (27)$$

The integral in Eq. (27) can be evaluated depending on the value of λ . For $\lambda = 0$, the expression for Q takes a simpler form and is expressed as:

$$Q(\bar{X}_f) = \epsilon_{d0}\bar{X}_f(2\bar{M} + \bar{X}_f), \text{ if } \lambda = 0, \quad (28)$$

where ϵ_{d0} is the densification strain corresponding to average density ρ_0 . For $\lambda \neq 0$, using Eqs. (1)–(4), the integral in Eq. (27) can be written in terms of $\bar{\rho}_f$ leading to:

$$Q = \int \frac{C_1\sigma_s\bar{\rho}_f^{C_2}(C_3 - C_4\bar{\rho}_f)}{\sigma_{p0}} \left[\bar{M} + \frac{\bar{\rho}_f^2 - \bar{\rho}_0^2(1 - \lambda)^2}{4\lambda\bar{\rho}_0^2} \right] \frac{d\bar{\rho}_f}{\lambda\bar{\rho}_0}. \quad (29)$$

Further, the integral in Eq. (29) can be evaluated to the following form:

$$Q(\bar{X}_f) = \frac{\bar{\sigma}_{pf}}{4\lambda^2} \left\{ \frac{\bar{\rho}_f}{\bar{\rho}_0} \left(\frac{C_3}{C_2 + 1} - \frac{C_4\bar{\rho}_f}{C_2 + 2} \right) \cdot [4\lambda\bar{M} - (1 - \lambda)^2] + \frac{\bar{\rho}_f^3}{\bar{\rho}_0^3} \left(\frac{C_3}{C_2 + 3} - \frac{C_4\bar{\rho}_f}{C_2 + 4} \right) \right\}, \text{ if } 0 < \lambda \leq 1. \quad (30)$$

It is important to note that Eq. (30) is not defined for $\lambda = 0$ and is only valid when $\lambda \neq 0$.

The constant c is determined from initial conditions. For $\bar{X}_f = 0$, $G = \bar{M}$ and $Y = \bar{V}_0^2$. Thus, c is given by:

$$c = \bar{M}^2\bar{V}_0^2 + Q(0). \quad (31)$$

Therefore, the final solution for V_f is obtained in terms of dimensionless functions $G(\bar{X}_f)$ and $Q(\bar{X}_f)$ as:

$$\bar{V}_f = \frac{1}{G(\bar{X}_f)} \sqrt{\bar{M}^2\bar{V}_0^2 - [Q(\bar{X}_f) - Q(0)]}, \quad (32)$$

where $Q(0)$ is the initial value of function Q , when the shock front is at the incident end. Thus, $Q(0)$ can be obtained by using $\bar{X}_f = 0$ and $\bar{\rho}_f = \bar{\rho}_i$. For $\lambda = 0$, the Eq. (32) after simplification is identical to that shown in the previous work for a uniform density CM [6]. However, Eq. (32) is more general and is applicable to CM with any linear non-negative density gradient including the specific case of uniform density. The function Q is determined by Eq. (28) or Eq. (30) depending on the value of the density gradient parameter.

The numerical solution is obtained with time as an independent variable similar to previous studies [9,11]. Thus, using Eq. (14), the differential equation given by Eq. (17) is transformed to:

$$\frac{d\bar{V}_f}{d\bar{t}} = - \left(\bar{\sigma}_{pf} + \frac{\bar{\rho}_f\bar{V}_f^2}{\bar{\rho}_0\epsilon_{df}} \right) \left[\bar{M} + \frac{1}{\bar{\rho}_0} \int_0^{\bar{X}_f} \bar{\rho}(\bar{X})d\bar{X} \right]^{-1}, \quad (33)$$

where $\bar{t} = (t\sqrt{\sigma_{p0}/\rho_0})/L$. Equations (14) and (33) are used in discretized form to numerically calculate the instantaneous velocity of the rigid mass as a function of shock front location. Forward finite difference method is used to obtain the solution.

The comparison of the velocity of the rigid mass obtained from Eq. (32) with the numerical solutions is shown in Fig. 4. The plots are shown for $\bar{V}_0 = 1$ and $\bar{V}_0 = 2$. Unless otherwise stated, the following representative values are used for the calculations: $\bar{\rho}_0 = 0.15$, $\bar{M} = 1$, $C_2 = 2$, $C_3 = 1$, $C_4 = 1$ and $\lambda = 0.5$. With $\bar{V}_0 = 1$, the input kinetic energy supplied is not sufficient to propagate the shock front to the transmitted end. The rigid mass comes to rest and the CM remains partially compacted (see Fig. 4a). The error in the numerical solution is shown in Fig. 4b. One of the drawbacks of the numerical solution is that the accuracy of the solu-

tion is dependent on the time step $\delta\bar{t}$. Often multiple iterations are required for determining the time step that will yield sufficiently accurate results. It is observed in the figure that the numerical solution slowly converges to the analytical solution as the time step is reduced. The reduction in time step size only increases the computational time. The error due to the finite time step is small in the beginning, but it accumulates with shock front propagation and is maximum in the end. The error is only shown up to $\bar{X}_f = 0.6$ in Fig. 4b because it increases dramatically as the velocity approaches zero. Figure 4c shows the case when $\bar{V}_0 = 2$. In this case, the input energy is more than the minimum energy required for the full compaction of CM. Therefore, the shock front propagates to the other end as seen in the figure, and the velocity of the rigid mass, in the end, remains non-zero even after full compaction of CM. The error in the numerical solution decreases with smaller step sizes (see Fig. 4d) same as the case observed for $\bar{V}_0 = 1$.

There exists an initial velocity of the impinging mass whose associated kinetic energy is just sufficient to compact the CM completely and at the same time, the impinging mass comes to rest. This initial velocity is denoted as densification velocity and is obtained by substituting $\bar{V}_f = 0$ at $\bar{X}_f = 1$ in Eq. (32). The dimensionless densification velocity is given by:

$$\bar{V}_d = \frac{\sqrt{Q(1) - Q(0)}}{\bar{M}}. \quad (34)$$

The input kinetic energy is dissipated by the compaction of CM. The kinetic energy associated with the densification velocity is called the densification energy, and it is equal to the EA capacity of CM according to the principle of conservation of energy. The specific EA capacity is the EA capacity per unit mass of the CM. Thus, the analytical expression for the dimensionless specific EA capacity $\bar{E}_d$ is given by:

$$\bar{E}_d = \frac{1}{2} \bar{M} \bar{V}_d^2 = \frac{Q(1) - Q(0)}{2\bar{M}}. \quad (35)$$

The specific EA capacity is shown in Fig. 5a and the error in the numerical solutions in Fig. 5b. It is clearly seen that the numerical solutions can be misleading if an appropriate time step is not used. A coarser time step ($\delta\bar{t} = 1 \times 10^{-2}$) shows an initial decreasing trend as opposed to the monotonically increasing trend shown by the analytical solution in Fig. 5a. It is also observed that the density gradation improves the EA capacity. The energy absorption occurs through plastic dissipation, and it is greatly enhanced by the shock deformation mechanism when the compaction velocity is high. Note that the compaction velocity is high during the early stage of deformation, and it gradually decreases due to energy absorption. The overall mass of the graded CMs is the same as that of the uniform density CM. However, in graded CMs, the mass is distributed by placing the low-density material towards the incident end and the high-density material towards the transmitted end. The low-density material compacts at high velocity, absorbing more energy due to the shock deformation mechanism. As the CM gets compacted further, the high-density material towards the transmitted end experiences lower velocities resulting in lower energy absorption by the shock deformation mechanism. However, the high density of the material ensures a significant amount of plastic dissipation owing to its high plateau stress. Thus, the variation in density along the length helps in effectively absorbing energy throughout the compaction process. Therefore the overall energy absorption capacity is enhanced in a density-graded CM over the uniform density CM, and the enhancement is maximum for the steepest density gradient.

The incident stress is the stress exerted on the rigid mass by the CM. It is obtained by applying conservation of linear momen-

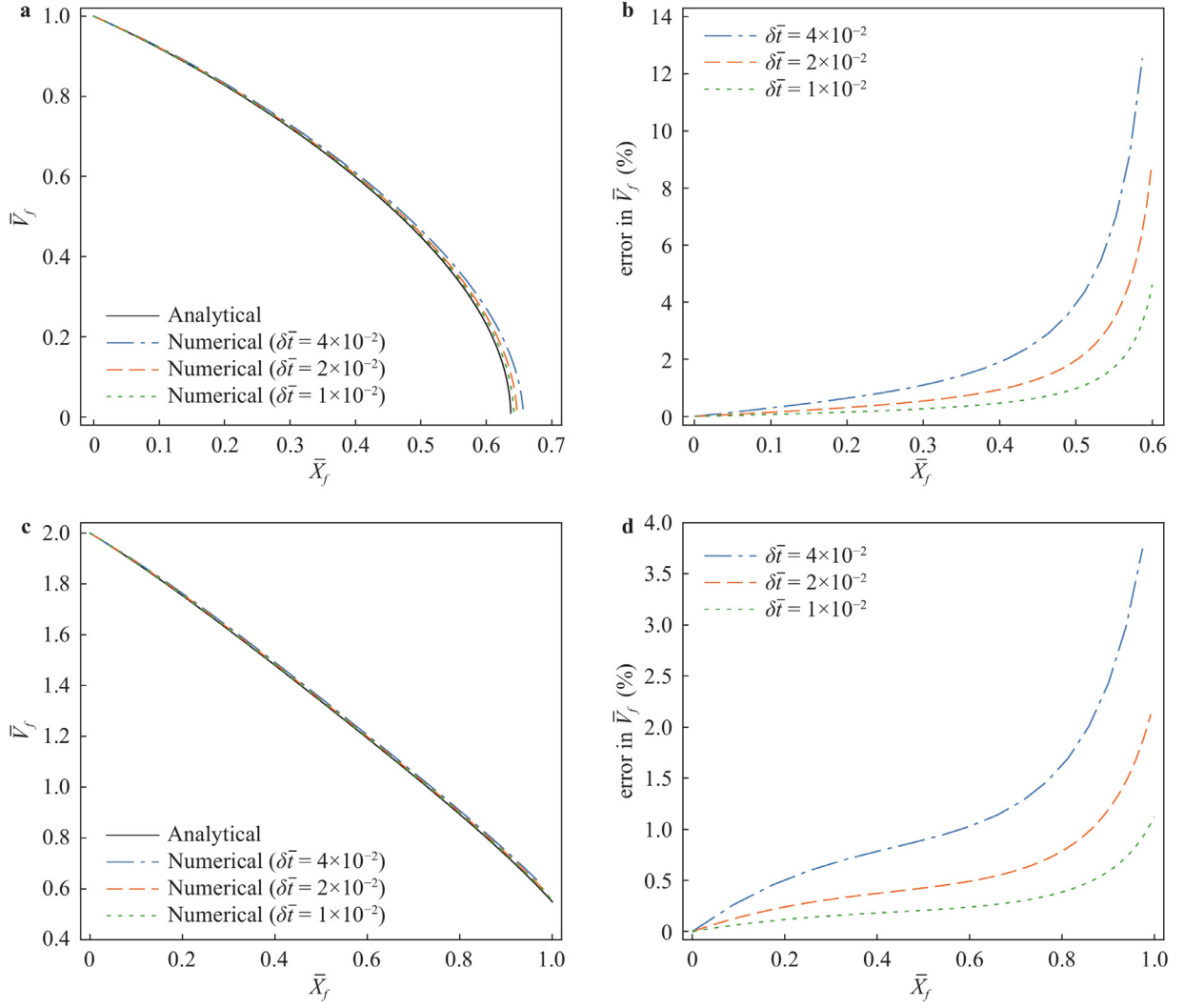


Fig. 4. Instantaneous velocity $\bar{V}_f$ of the rigid mass as a function of $\bar{X}_f$, **a** $\bar{V}_f$ when $\bar{V}_0 = 1$, **b** error in $\bar{V}_f$ when $\bar{V}_0 = 1$, **c** $\bar{V}_f$ when $\bar{V}_0 = 2$, **d** error in $\bar{V}_f$ when $\bar{V}_0 = 2$.

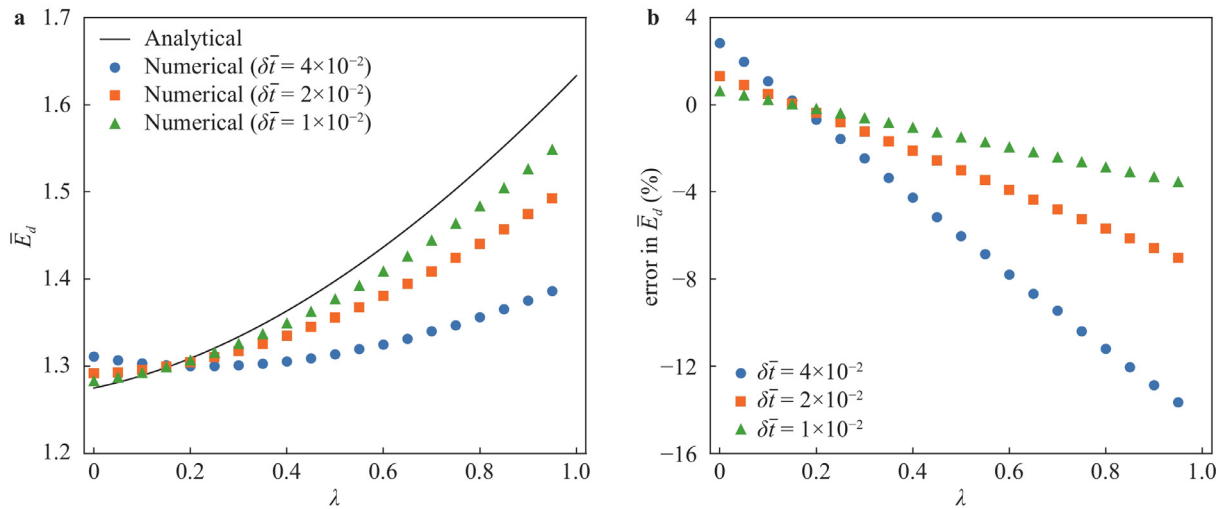


Fig. 5. **a** Specific EA capacity $\bar{E}_d$ as a function of λ ; **b** error in specific EA capacity $\bar{E}_d$.

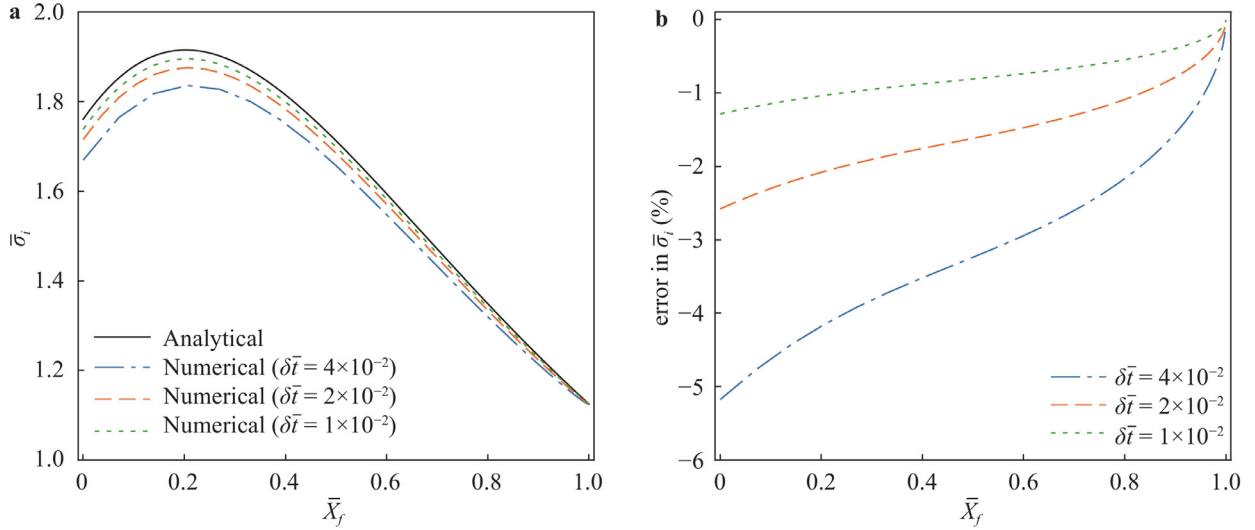


Fig. 6. **a** Dimensionless incident stress $\bar{\sigma}_i$ as a function of $\bar{X}_f$; **b** error in dimensionless incident stress $\bar{\sigma}_i$.

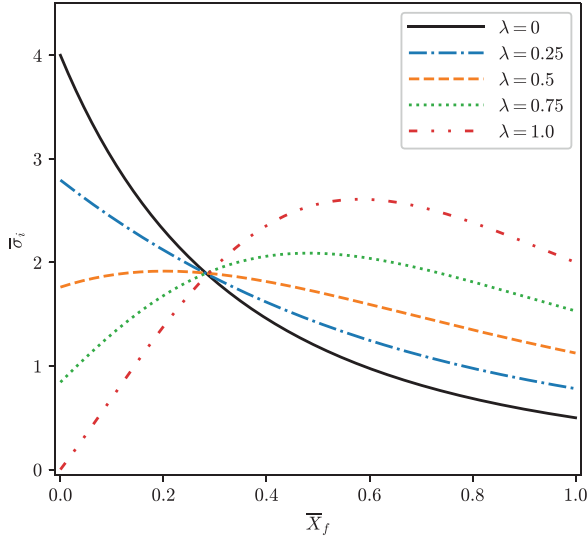


Fig. 7. Dimensionless incident stress $\bar{\sigma}_i$ for different density gradient parameters.

tum to the rigid mass and is expressed as:

$$\sigma_i = -\frac{M}{A} \frac{dV_f}{dt}. \quad (36)$$

Using Eqs. (12), (13), (20) and (36), the dimensionless incident stress is obtained as:

$$\bar{\sigma}_i = \frac{\bar{M}}{G} \left(\bar{\sigma}_{pf} + \frac{\bar{\rho}_f \bar{V}_f^2}{\bar{\rho}_0 \epsilon_{df}} \right). \quad (37)$$

The dimensionless incident stress is shown in Fig. 6a for $\bar{V}_0 = \bar{V}_d$ and $\lambda = 0.5$ along with the error in Fig. 6b. The analytical solution shows a good agreement, as before, with the numerical solution. Figure 7 shows the incident stress obtained analytically for different density gradient parameters. The peak incident stress occurs at $\bar{X}_f = 0$ for $\lambda = 0$ and $\lambda = 0.25$. However, for higher values of λ , it does not occur at $\bar{X}_f = 0$. This is because the density near the incident end is low enough, resulting in smaller plateau stress and

larger densification strain. All these factors (see Eq. (37)) lead to lower incident stress at $\bar{X}_f = 0$ despite compaction at high velocity. In the later stage, as the density of the compacting material increases, the velocity also remains sufficiently high. Therefore, the incident stress increases and reaches a peak value at some location. In the end, since the velocity ($\bar{V}_f$) becomes zero, the incident stress drops to a lower value. Both the location and the magnitude of the peak incident stress are dependent upon the density gradient parameter. This also indicates that the incident stress profile can be designed by choosing a suitable density gradient parameter. This is an extremely useful characteristic that can be exploited when designing impact-resistant structures. Figure 8 shows the dimensionless peak incident stress $\bar{\sigma}_i^{\text{peak}}$ as a function of λ . It is observed that the peak incident stress is minimum around $\lambda = 0.55$. Very high values of λ lead to higher peak incident stress. This is because, for CM with very high λ , the velocity does not drop much after the low-density material near the incident end has compacted. Afterward, the compaction of high-density material at high velocities causes higher peak incident stress.

The theoretical analysis for the stationary graded CM impacted by a rigid mass is presented in this study. RPPL material behavior is assumed, and linear density gradients are considered. An analytical approach that leads to closed-form solutions is demonstrated. Expressions for the velocity of the rigid mass, densification velocity, energy absorption capacity, and incident stress are determined. The proposed solution captures the dynamic behavior of the material with great accuracy. The comparison of analytical results with the numerical solutions validates the accuracy and efficiency of the derived expressions. The analytical solution shows that the density gradation improves the energy absorption capacity, and the enhancement is maximum when the density gradient parameter is maximum. Also, the peak incident stress is observed to be minimum at intermediate values of the density gradient parameter. The effect of each variable on the behavior of CM under impact loading can be determined using the proposed analytical expressions. The solution can also be used to predict the dynamic response of the naturally occurring density-graded CM such as a human skull [32,33]. Furthermore, it could be adapted for modeling a multi-layer structure with discrete values of densities by using a piecewise function for density variation. More importantly, the proposed analytical formulation can be a fast and efficient tool, in general, to design graded CMs for impact mitigation.

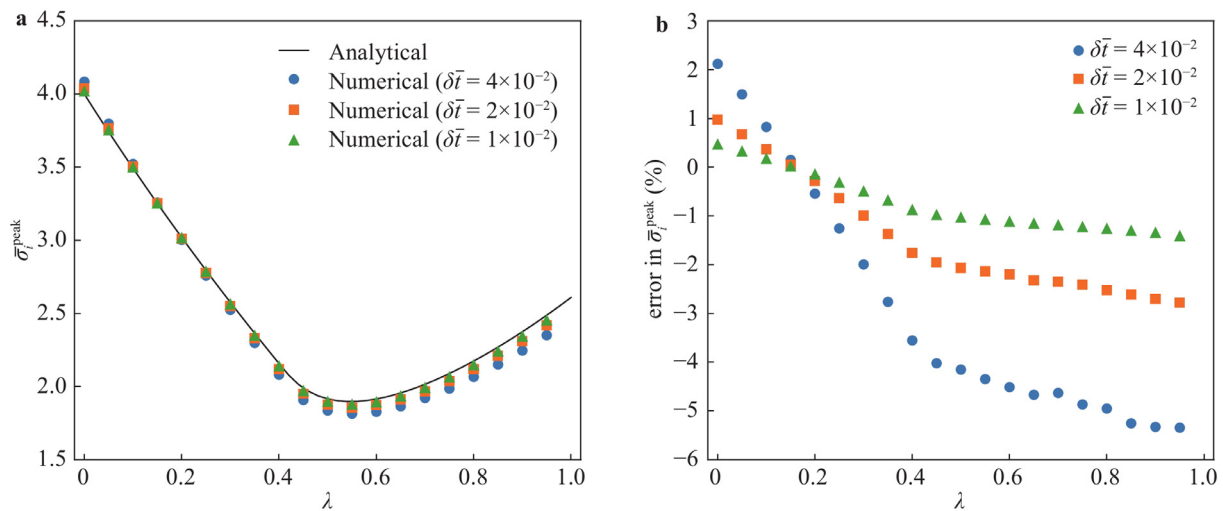


Fig. 8. **a** Dimensionless peak incident stress $\bar{\sigma}_i^{\text{peak}}$ as a function of λ ; **b** error in dimensionless peak incident stress $\bar{\sigma}_i^{\text{peak}}$

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Credit authorship contribution statement

Vijendra Gupta: Conceptualization, Methodology, Formal analysis, Writing –original draft. Addis Kidane: Supervision, Writing –review & editing. Michael Sutton: Supervision, Writing –review & editing.

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