



AFRL-AFOSR-VA-TR-2024-0146

**Statistics of Optical and Quasi-Optical Line-of-Sight Propagation through the
Stably Stratified, Optically Clear Atmosphere**

**Muschinski, Andreas
NORTHWEST RESEARCH ASSOCIATES INC.
4118 148 AVE NE
REDMOND, WA, 98052
US**

**02/07/2024
Final Technical Report**

DISTRIBUTION A: Distribution approved for public release.

Air Force Research Laboratory
Air Force Office of Scientific Research
Arlington, Virginia 22203
Air Force Materiel Command

REPORT DOCUMENTATION PAGE

PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ORGANIZATION.

1. REPORT DATE 20240207		2. REPORT TYPE Final		3. DATES COVERED	
				START DATE 20180801	END DATE 20230731
4. TITLE AND SUBTITLE Statistics of Optical and Quasi-Optical Line-of-Sight Propagation through the Stably Stratified, Optically Clear Atmosphere					
5a. CONTRACT NUMBER FA9550-18-C-0011		5b. GRANT NUMBER		5c. PROGRAM ELEMENT NUMBER 61102F	
5d. PROJECT NUMBER		5e. TASK NUMBER		5f. WORK UNIT NUMBER	
6. AUTHOR(S) Andreas Muschinski					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NORTHWEST RESEARCH ASSOCIATES INC. 4118 148 AVE NE REDMOND, WA 98052 US					8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Air Force Office of Scientific Research 875 N. Randolph St. Room 3112 Arlington, VA 22203				10. SPONSOR/MONITOR'S ACRONYM(S) AFRL/AFOSR RTB1	11. SPONSOR/MONITOR'S REPORT NUMBER(S) AFRL-AFOSR-VA-TR-2024-0146
12. DISTRIBUTION/AVAILABILITY STATEMENT A Distribution Unlimited: PB Public Release					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT The purpose of this project was to develop, refine, and validate physics-based models of the spatiotemporal statistics of clear-air refractive-index fluctuations and to apply them to canonical, line-of-sight propagation scenarios and imaging-system performance metrics. In our approach, we combined theoretical analysis, field data analysis (both in-situ and remote), and computer simulations (both fluid mechanics and electromagnetics). Our activities focused on the following research topics: non-Kolmogorov turbulence and inner-scale effects; anisotropy and outer-scale effects; non-Gaussianity of turbulent refractive-index and phase increments; and the relative merits of the geometrical-optics method, the Rytov method, and the phase-screen method for modeling optical propagation through the turbulent atmosphere. Our main findings can be summarized as follows: 1) At small scales, the Obukhov-Corrsin similarity theory is robust and reliable, which questions the usefulness and need of "non-Kolmogorov turbulence" models, at least for turbulence near the ground; 2) the Hill-Frehlich model predicts inner-scale effects reliably; 3) the probability density of turbulent refractive-index and phase increments tend to be Laplacian and not Gaussian; 4) in the strong-fluctuation regime, the geometrical-optics method (ray-tracing) is more reliable than the Rytov method.					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:				17. LIMITATION OF ABSTRACT	
a. REPORT U	b. ABSTRACT U	c. THIS PAGE U	UU		18. NUMBER OF PAGES 122
19a. NAME OF RESPONSIBLE PERSON ARJE NACHMAN					19b. PHONE NUMBER (Include area code) 426-8427

Standard Form 298 (Rev.5/2020)
Prescribed by ANSI Std. Z39.18

Award Number: FA9550-18-C-0011

Report Type: Final

Reporting Period: August 1, 2018 through July 31, 2023

Distribution Statement: Distribution A – Approved for Public Release

Program Officer Name: Dr. Arje Nachman

Principal Investigator Name: Dr. Andreas Muschinski

Project Title: Statistics of optical and quasi-optical line-of-sight propagation through the stably stratified, optically clear atmosphere

Abstract: The purpose of this project was to develop, refine, and validate physics-based models of the spatiotemporal statistics of clear-air refractive-index fluctuations and to apply them to canonical, line-of-sight propagation scenarios and imaging-system performance metrics. In our approach, we combined theoretical analysis, field data analysis (both in-situ and remote), and computer simulations (both fluid mechanics and electromagnetics). Our activities focused on the following research topics: non-Kolmogorov turbulence and inner-scale effects; anisotropy and outer-scale effects; non-Gaussianity of turbulent refractive-index and phase increments; and the relative merits of the geometrical-optics method, the Rytov method, and the phase-screen method for modeling optical propagation through the turbulent atmosphere. Our main findings can be summarized as follows: 1) At small scales, the Obukhov-Corrsin similarity theory is robust and reliable, which questions the usefulness and need of "non-Kolmogorov turbulence" models, at least for turbulence near the ground; 2) the Hill-Frehlich model predicts inner-scale effects reliably; 3) the probability density of turbulent refractive-index and phase increments tend to be Laplacian and not Gaussian; 4) in the strong-fluctuation regime, the geometrical-optics method (ray-tracing) is more reliable than the Rytov method.

Section 1

Participants

Andreas Muschinski, Senior Research Scientist
10 months
Principal Investigator

Other Partners or Collaborators

Eric Wagner, NWRA, Senior Support Engineer
Design and field testing of fine-wire sensors

Charles Martin, NWRA, Senior Research Engineer
Design and integration of data acquisition systems

Carter Mak, NWRA, undergraduate engineering student
Field experiment and data analysis support

Nathan Kuczun, NWRA, undergraduate engineering student
Field experiment and data analysis support

Ryan Schatzle, NWRA, undergraduate physics student
Field experiment and data analysis support

Maxwell Anderson, NWRA, undergraduate engineering student
Field experiment and data analysis support

Marcus Schaller, NWRA, undergraduate physics student
Field experiment and data analysis support

David Voelz, NMSU, Professor of Electrical Engineering
Collaborator on computational electromagnetics

Erandi Wijerathna, NMSU, postdoctoral researcher
Collaborator on computational electromagnetics

Andrea Baca, NMSU, graduate student
Field experiment support

Robert Hull, NMSU, Technical staff
Field experiment support

Hanyu Zhan, NMSU, postdoctoral researcher
Collaborator on computational electromagnetics

Gregory Rieker, CU, Associate Professor of Mechanical Engineering
Collaborator on field experiment

Emily Caldwell, CU, graduate engineering student
Collaborator on field experiment

Laura Sinclair, NIST, staff scientist
Collaborator on field experiment

William Swann, NIST, staff scientist
Collaborator on field experiment

Stephen M. de Bruyn Kops, UMass, professor of mechanical engineering
Collaborator on computational fluid mechanics

Journal Articles

Emily D. Caldwell, William C. Swann, Jennifer L. Ellis, Martha I. Bodine, Carter Mak, Nathan Kuczun, Nathan R. Newbury, Laura C. Sinclair, Andreas Muschinski, and Gregory B. Rieker. "Optical timing jitter due to atmospheric turbulence: comparison of frequency comb measurements to predictions from micrometeorological sensors." *Optics Express* 28, no. 18 (2020): 26661-26675.

<https://doi.org/10.1364/OE.400434>

Andreas Muschinski. "Phase-factor spectra of turbulent phase screens." *JOSA A* 38, no. 9 (2021): 1339-1348. <https://doi.org/10.1364/JOSAA.429928>

Andreas Muschinski and Eric L. Wagner. "First field measurements of optical turbulence near the ground with a newly developed fine-wire thermometer," *Waves in Random and Complex Media*, 33, 1234-1249, 2023. <https://doi.org/10.1080/17455030.2022.2076957>

David G. Voelz, Erandi Wijerathna, Andreas Muschinski, and Xifeng Xiao. "Computer simulations of optical turbulence in the weak- and strong-scattering regime: angle-of-arrival fluctuations obtained from ray optics and wave optics," *Opt. Eng.* 57, 104102, 2018.

<https://doi.org/10.1117/1.OE.57.10.104102>

Erandi Wijerathna, Hanyu Zhan, David Voelz, and Andreas Muschinski. "Low-wavenumber compensation with Zernike tilt for non-Kolmogorov turbulence phase screens," *Appl. Opt.*, 62, 1253-1262, 2023.

<https://doi.org/10.1364/AO.475825>

Conference Papers and Presentations

Andreas Muschinski. "Non-classical models of optical turbulence." Invited oral Presentation. In *Unconventional Imaging and Adaptive Optics 2022*, vol. 12239, p. 1223902. SPIE, 2022.

<https://doi.org/10.1117/12.2633193>

Andreas Muschinski and Stephen M. de Bruyn Kops. "Phase Screens of Optical Turbulence Generated by Means of Direct Numerical Simulation of First Principles of Fluid Mechanics." In *Propagation Through and Characterization of Atmospheric and Oceanic Phenomena*, pp. PTu5F-4. Optica Publishing Group, 2018. <https://doi.org/10.1364/PCAOP.2018.PTu5F.4>

Andreas Muschinski, D. G. Voelz, A. Baca, R. Hull, 2023: Scintillation observed in an 84-km propagation experiment across White Sands Missile Range. In *Propagation Through and Characterization of Atmospheric and Oceanic Phenomena* (Boston, MA, 14–17 August, 2023), extended abstract, pp. PTh2F-4. <https://doi.org/10.1364/PCAOP.2023.PTh2F.4>

Section 2

Award number: FA9550-18-C-0011

Report Type: Final progress report

Performance Period: August 1, 2018 through July 31, 2023

Sponsor: AFOSR Physics of Sensing program (PO Dr. Arje Nachman)

Project Title: Statistics of optical and quasi-optical line-of-sight propagation through the stably stratified, optically clear atmosphere

Authors: Andreas Muschinski

Awardee Organization: NorthWest Research Associates (NWRA), Boulder, CO

Date: January 24, 2024

Abstract:

The purpose of this project was to develop, refine, and validate physics-based models of the spatiotemporal statistics of clear-air refractive-index fluctuations and to apply them to canonical, line-of-sight propagation scenarios and imaging-system performance metrics. In our approach, we combined theoretical analysis, field data analysis (both in-situ and remote), and computer simulations (both fluid mechanics and electromagnetics). Our activities focused on the following research topics: non-Kolmogorov turbulence and inner-scale effects; anisotropy and outer-scale effects; non-Gaussianity of turbulent refractive-index and phase increments; and the relative merits of the geometrical-optics method, the Rytov method, and the phase-screen method for modeling optical propagation through the turbulent atmosphere. Our main findings can be summarized as follows: 1) At small scales, the Obukhov-Corrsin similarity theory is robust and reliable, which questions the usefulness and need of “non-Kolmogorov turbulence” models, at least for turbulence near the ground; 2) the Hill-Frehlich model predicts inner-scale effects reliably; 3) the probability density of turbulent refractive-index and phase increments tend to be Laplacian and not Gaussian; 4) in the strong-fluctuation regime, the geometrical-optics method (ray-tracing) is more reliable than the Rytov method.

1 Accomplishments

It is well known that electromagnetic propagation through the optically clear atmosphere is affected by random fluctuations of the air’s refractive index. These effects are important for a wide range of science and engineering disciplines, including atmospheric remote sensing, optical free-space communication, terrestrial and extraterrestrial imaging, adaptive optics, and directed-energy technology. The progress of the physical understanding of these effects during the recent decades has been documented in numerous monographs, collections, and review articles (e.g., Tatarskii, 1961; Strohbehn, 1968; Tatarskii, 1971; Uscinski, 1977; Ishimaru, 1978; Strohbehn, 1978; Roddier, 1981; Uscinski, 1986; Andreas, 1990; Beland, 1993; Roggemann and Welsh, 1996; Kopeika, 1998; Wheelon, 2001, 2003; Andrews and Phillips, 2005; Korotkova, 2014; Goodman, 2015, 2020; Korotkova, 2022).

The purpose of this project was to develop, refine, and validate physics-based models of the spatiotemporal statistics of clear-air refractive-index fluctuations and to apply them to canonical, line-of-sight propagation scenarios and imaging-system performance metrics. We conducted our research by means of a combination of theoretical, computational, and observational methods, and we published our findings in five journal papers (Voelz et al., 2018; Caldwell et al., 2020; Muschinski, 2021; Muschinski and Wagner, 2023; Wijerathna et al., 2023) and three conference papers (Muschinski and de Bruyn Kops, 2018; Muschinski, 2022; Muschinski et al., 2023). In the following, we summarize our results with respect to three main topics: non-Kolmogorov turbulence and

inner-scale effects; anisotropy and outer-scale effects; and electromagnetic propagation through the turbulent atmosphere.

1.1 Non-Kolmogorov turbulence and inner-scale effects

The classical Obukhov-Corrsin similarity theory (Obukhov, 1949; Corrsin, 1951) predicts that the 3D spectrum of the temperature fluctuations in fully developed, locally homogeneous and isotropic turbulence has the form

$$\Phi_T(\boldsymbol{\kappa}) = \beta_3 \chi \varepsilon^{-1/3} \kappa^{-11/3} h(\kappa \ell_0), \quad (1)$$

where β_3 is the dimensionless, universal Obukhov-Corrsin constant for the 3D spectrum, χ is the dissipation rate of the temperature variance, ε is the dissipation rate of the turbulent kinetic energy per unit mass, ℓ_0 is the inner scale (Hill and Clifford, 1978; Muschinski, 2015; Muschinski and Wagner, 2023), and the wavenumber κ is the magnitude of the 3D wave vector, $\boldsymbol{\kappa}$. Tatarskii (1961) restated (1) as

$$\Phi_T(\boldsymbol{\kappa}) = \frac{\Gamma(8/3) \sin(\pi/3)}{4\pi^2} C_T^2 \kappa^{-11/3} h(\kappa \ell_0), \quad (2)$$

where C_T^2 is the temperature structure parameter.

The form (2), as opposed to (1), has two advantages: first, the dimensionless coefficient

$$\Gamma(8/3) \sin(\pi/3) / 4\pi^2 \approx 0.033 \quad (3)$$

is a mathematical constant, in contrast to β_3 which is a physical coefficient that can be determined only empirically; second, C_T^2 can be measured much more easily than χ and ε .

For many applications, it can be safely assumed that the refractive-index fluctuation is directly proportional to the temperature fluctuation (e.g., Muschinski and Wagner, 2023, p. 1242), which implies that the 3D refractive-index spectrum $\Phi_n(\boldsymbol{\kappa})$ has the same form as the 3D temperature spectrum $\Phi_T(\boldsymbol{\kappa})$:

$$\Phi_n(\boldsymbol{\kappa}) = \frac{\Gamma(8/3) \sin(\pi/3)}{4\pi^2} C_n^2 \kappa^{-11/3} h(\kappa \ell_0), \quad (4)$$

where C_n^2 is the refractive-index structure parameter. The “tail function” $h(\kappa \ell_0)$, however, is the same in (1), (2), and (4). The Obukhov-Corrsin spectrum given in (1), (2), or (4) is, somewhat misleadingly, often referred to as the “Kolmogorov spectrum”.

Specifics of the functional form of $h(\kappa \ell_0)$ were not predicted by Obukhov or Corrsin, except that $h(\kappa \eta) \rightarrow 1$ for $\kappa \eta \ll 1$ and $h(\kappa \eta) \rightarrow 0$ for $\kappa \eta \rightarrow \infty$. Frehlich (1992) found an additional constraint for $h(\kappa \ell_0)$, namely,

$$\int_0^\infty h(y) y^{1/3} dy = \frac{27\Gamma(1/3)}{10} = 7.233. \quad (5)$$

Muschinski (2015) showed that most of the $h(\kappa \ell_0)$ published in the literature since the early 1990s violate Frehlich’s equation. To the best of our knowledge, the only post-1990 $h(\kappa \ell_0)$ model that does not violate (5) is the Hill-Frehlich model (Hill, 1978; Frehlich, 1992); see Muschinski (2015) and Muschinski and de Bruyn Kops (2015) for details.

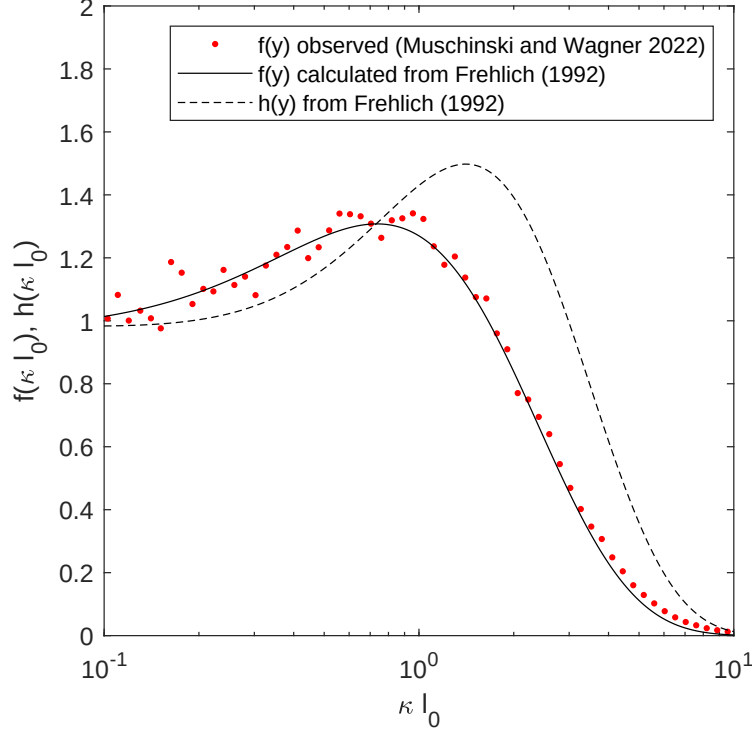


Figure 1: *Model and field measurements of the “Hill bump” (Muschinski, 2022). Dashed line: 3D tail function $h(\kappa l_0)$ according to Frehlich (1992, Eqs. 23 and 24 on pp. 1496f.); solid line: 1D tail function $f(\kappa l_0)$ calculated from Frehlich’s $h(\kappa l_0)$; red dots: 1D tail function $f(\kappa l_0)$ retrieved from fine-wire temperature measurements described in Muschinski and Wagner (2023).*

During the last two or three decades, a large body of literature on “non-Kolmogorov turbulence” has emerged, referring to models of the scalar turbulence spectrum that deviate from the Obukhov-Corrsin spectrum, or “Kolmogorov spectrum”, given in (1), (2), or (4). A widely used spectral model for non-Kolmogorov turbulence is the one suggested by Toselli et al. (2008), where the power-law exponent is allowed to deviate from $-11/3$, the value predicted by the Obukhov-Corrsin similarity theory.

However, our previous direct numerical simulations of turbulence in stratified shear flow (Muschinski and de Bruyn Kops, 2015) as well as our in-situ measurements by means of ultrasonic anemometer-thermometers (Muschinski, 2017; Muschinski and Wagner, 2023) and fine-wire thermometers (Muschinski and Wagner, 2023) indicate that the classical $\kappa^{-11/3}$ power law is valid. This finding, which is consistent with the comment by Tatarskii and Zavorotny (1993) from thirty years ago, questions the usefulness or need of “non-Kolmogorov” turbulence models, at least for fully developed turbulence.

Recently (Muschinski, 2022), we determined $h(\kappa l_0)$ based on our fine-wire temperature measurements and found very good agreement with the Hill-Frehlich model (Hill, 1978; Frehlich, 1992); see Fig. 1. This is an additional piece of empirical evidence that corroborates the classical Obukhov-Corrsin theory.

While we have not found empirical evidence for the existence of non-Kolmogorov turbulence, that is, for the absence of an inertial-convective range characterized by a $\kappa^{-11/3}$ power law at small scales, it would be premature to conclude that non-Kolmogorov turbulence does not exist. We still

expect non-Kolmogorov turbulence to occur far above the surface layer under conditions of stable stratification and very low wind shear. It appears that this issue can be conclusively resolved only on the basis of high-resolution in-situ measurements.

1.2 Anisotropy and outer-scale effects

The Obukhov-Corrsin similarity theory predicts the existence of a universal refractive-index spectrum for wavenumbers large compared to a certain wavenumber κ_0 which is determined by a certain outer scale length $L_0 \sim 1/\kappa_0$. For κ comparable to and smaller than κ_0 , the 3D spectrum $\Phi_n(\kappa)$ deviates from the Obukhov-Corrsin spectrum, usually such that the actual $\Phi_n(\kappa)$ is smaller than the spectral density predicted by the Obukhov-Corrsin spectrum. Therefore, we can write

$$\Phi_n(\kappa) = f(\kappa L_0) \times \frac{\Gamma(8/3) \sin(\pi/3)}{4\pi^2} C_n^2 \kappa^{-11/3} h(\kappa \ell_0), \quad (6)$$

where $f(\kappa L_0)$ is a dimensionless function of the dimensionless wave vector κL_0 , such that $f(\kappa L_0) \rightarrow 1$ for $\kappa L_0 \gg 1$ and $f(\kappa L_0) \rightarrow 0$ for $\kappa L_0 \ll 1$. Qualitatively, this asymptotic behavior corresponds to the empirical fact that 1D temperature spectra observed with ultrasonic thermometers follow an $f^{-5/3}$ power law in the inertial-convective subrange but decrease with frequency f less rapidly than $f^{-5/3}$ at lower frequencies; see, e.g., Muschinski (2017) and Muschinski and Wagner (2023, p. 1246, figure 10). Also in our collocated field measurements of timing-jitter fluctuations observed with a frequency-comb laser system and refractive-index fluctuations observed with ultrasonic anemometer-thermometers and a microbarometer, we see a low-frequency “roll-off” both in the retrieved timing-jitter spectra and refractive-index spectra (Caldwell et al., 2020, pp. 26,669ff.).

The Obukhov-Corrsin theory *per se* predicts neither the magnitude of L_0 nor the turbulence statistics at scales larger than L_0 . Tatarskii (1961) constructed a defining equation for L_0 on the basis of the heuristic argument that L_0 is the scale at which the deterministic vertical potential-temperature (θ) increment, and the turbulent potential-temperature increment are of the same magnitude:

$$L_0 = \left(\frac{C_\theta^2}{[\partial\langle\theta\rangle/\partial z]^2} \right)^{3/4} \quad (7)$$

(Tatarskii, 1961, p. 51, eq. 5.29”).

Several other models for the outer scale have been suggested, such as the Ozmidov length, the Thorpe length, and the length scale associated with the wavenumber κ_0 in the von-Kármán spectrum,

$$\Phi_n(\kappa) = 0.033 C_n^2 (\kappa^2 + \kappa_0^2)^{-11/6} \quad (8)$$

(Andrews and Phillips, 2005, p. 68); for details and further discussion see, e.g., Muschinski et al. (2004), Almalkie and de Bruyn Kops (2012), de Bruyn Kops (2015), and references therein.

There is a defining equation for each of these outer-scale models, and the physics behind them continues to be controversial (e.g., Basu and Holtslag, 2022). The lack of a consensus about an unambiguous and universal defining equation for L_0 is exemplified on pp. 68 and 69 in the standard text by Andrews and Phillips (2005), where three different values for $\kappa_0 L_0$ are offered: $\kappa_0 L_0 = 2\pi$, $\kappa_0 L_0 = 4\pi$, and $\kappa_0 L_0 = 1$.

Moreover, it is sometimes overlooked that the outer scale marks the length scale beyond which atmospheric turbulence becomes anisotropic. In the stably stratified atmosphere, turbulent “eddies”

larger than the outer scale tend to be pancake-shaped, such that the 3D refractive-index spectral density $\Phi_n(\kappa_x, \kappa_y, \kappa_z)$ decreases less rapidly along the vertical (κ_z) direction than along the horizontal directions and is no longer a function only of $\kappa = \sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}$. Obviously, anisotropic turbulence cannot be modeled by means of the traditional von-Kármán spectrum given in Eq. (8), which is per definition isotropic.

It has been known since the 1970s that anisotropic turbulence in the free atmosphere is the cause of the strong aspect sensitivity of the VHF clear-air radar reflectivity at near-zenith directions (Gage and Green, 1978; Röttger and Liu, 1978; Gage, 1990; Muschinski, 1996, 2004). A clear-air radar detects backscatter from clear-air turbulence at radial scales with a wavelength $\lambda/2$, the “Bragg wavelength” (where λ is the radar wavelength), which in the case of a 50-MHz VHF radar amounts to 3 m. UHF clear-air radars with Bragg wavelengths of less than 1 m, however, usually detect no such anisotropy.

In recent years, the optical-propagation community has become more interested in anisotropic optical turbulence (e.g., Gudimetla et al., 2012; Toselli, 2014; Gudimetla et al., 2014; Muschinski, 2015; Bos et al., 2015; Beason et al., 2018).

1.3 Electromagnetic propagation through the turbulent atmosphere

The propagation of electromagnetic waves through the optically clear atmosphere is affected mainly through refraction and diffraction. These effects are complicated further if the wavelength dependence of the refractive index (dispersion) is non-negligible. Random fluctuations of the clear-air refractive index are caused by turbulence, gravity waves, and other multi-scale phenomena (Kelvin-Helmholtz instability, Rayleigh-Benard convection, etc.), and the development and validation of statistical models of the associated velocity and scalar (temperature, humidity, density) fluctuations continues to be a key goal in fluid mechanics, boundary-layer meteorology, micrometeorology, aeronomy, and related fields.

Under a set of assumptions that are often fulfilled (Tatarskii, 1961), for a given realization of a fluctuating 3D refractive-index field and a given electromagnetic source field (e.g., a plane wave, a spherical wave, or a beam wave), the resulting 3D electromagnetic phasor field can be solved by means of solving the Helmholtz equation. Standard methods for solving the Helmholtz equation are the geometrical-optics method (Sommerfeld and Runge, 1911; Tatarskii, 1961), the Rytov method (Tatarskii, 1961), and the “phase-screen method” (Goodman, 1968; Schmidt, 2010; Voelz, 2011; Goodman, 2015, 2017, 2020) which relies on the Fourier-Fresnel diffraction integral. It has long been known that the geometrical-optics method (GOM) does not account for diffraction effects and that the Rytov method (RM) does not properly account for strong-fluctuation effects. However, the phase-screen method (PSM), particularly the split-step PSM, does not suffer from either of these problems.

Motivated by an earlier theoretical study (Cheon and Muschinski, 2007) on the relative merits of the GOM and the RM to predict the variance of the aperture-averaged angle-of-arrival of plane and spherical waves propagating through classical turbulence in the weak-fluctuation regime, we (Voelz et al., 2018) used a split-step PSM algorithm to quantitatively study the limitations of the GOM and the RM not only for the weak-fluctuation regime but also for the strong-fluctuation regime. Our computer simulations confirmed the earlier analytical results (Cheon and Muschinski, 2007) for the weak-fluctuation regime; for the strong-fluctuation regime, however, we found that the GOM performs better than the RM.

A standard assumption that is generally used in PSM computer simulations is the assumption that turbulent phase screens are normally (Gaussian) distributed. However, there is both computational (Muschinski and de Bruyn Kops, 2018) and observational evidence (for details see the Annual Status Report #2, September 2020) that turbulent refractive-index and phase increments are often Laplacian-distributed and not Gaussian-distributed. In a theoretical study (Muschinski, 2021), we analyzed effects of the probability density of turbulent phase increments (Gaussian vs. Laplacian) on the second-order statistics of the phase factor (that is, the phasor in the case of negligible amplitude fluctuations) depends not only on the phase spectrum but also on the probability density of the phase increments. At length scales (transverse to the propagation direction) small compared to the Fried length, however, the effect of the non-Gaussianity of phase increments is negligible.

One systemic problem of computational wave optics is associated with the difficulty to capture phase fluctuations with beam-transverse length scales comparable to and larger than the beam-transverse model domain size. This problem is caused by the “built-in” periodicity of 2D fields represented by 2D Fourier series. The problem can be mitigated in various ways, which include increasing the model domain (which is computationally expensive), adding randomly generated subharmonics, or expanding the optical field in terms of Zernike polynomials. We (Wijerathna et al., 2023) investigated analytically and computationally a hybrid method that computationally generates phase screens by combining FFT-generated phase screens with low-wavenumber “Zernike tilts”. We showed that this hybrid method is computationally efficient and can be applied to any statistical model of statistically homogeneous and isotropic refractive-index turbulence.

Bibliography

- Almalkie, S. and S. M. de Bruyn Kops, 2012: Kinetic energy dynamics in forced, homogeneous, and axisymmetric stably stratified turbulence. *J. Turbulence*, **13**, 1–29.
- Andreas, E. L., (Ed.) , 1990: *Turbulence in a refractive medium*, SPIE Milestone Series, Vol. MS 25. SPIE Optical Engineering Press, Bellingham, Washington, USA, 693 pp.
- Andrews, L. C. and R. L. Phillips, 2005: *Laser beam propagation through random media*. 2d ed., SPIE Press, Bellingham, Washington, 783 pp.
- Basu, S. and A. A. M. Holtslag, 2022: Revisiting and revising Tatarskii’s formulation for the temperature structure parameter (C_T^2) in atmospheric flow. *Env. Fluid Mech.*, **22**, 1107–1119.
- Beason, M., et al., 2018: Near ground measure and theoretical model of plane wave covariance of intensity in anisotropic turbulence. *Opt. Lett.*, **43**, 2607–2610.
- Beland, R. R., 1993: Propagation through atmospheric optical turbulence. *The infrared and electro-optical systems handbook, Vol. 2 (Atmospheric propagation of radiation)*, F. G. Smith, Ed., SPIE Optical Engineering Press, Bellingham, Washington, USA, 157–232.
- Bos, J. P., M. C. Roggemann, and V. S. R. Gudimetla, 2015: Anisotropic non-Kolmogorov turbulence phase screens with variable orientation. *Appl. Opt.*, **54**, 2039–2045.
- Caldwell, E. D., et al., 2020: Optical timing jitter due to atmospheric turbulence: comparison of frequency comb measurements to predictions from micrometeorological sensors. *Opt. Exp.*, **28**, 26,661–26,675.
- Cheon, Y. and A. Muschinski, 2007: Closed-form approximations for the angle-of-arrival variance

- of plane and spherical waves propagating through homogeneous and isotropic turbulence. *J. Opt. Soc. Am. A*, **24**, 415–422.
- Corrsin, S., 1951: On the spectrum of isotropic temperature fluctuations in an isotropic turbulence. *J. Appl. Phys.*, **22**, 469–473.
- de Bruyn Kops, S. M., 2015: Classical turbulence scaling and intermittency in strongly stratified Boussinesq turbulence. *J. Fluid Mech.*, **775**, 436–463.
- Frehlich, R. G., 1992: Laser scintillation measurements of the temperature spectrum in the atmospheric surface layer. *J. Atmos. Sci.*, **49**, 1494–1509.
- Gage, K. S., 1990: Radar observations of the free atmosphere: structure and dynamics. *Radar in meteorology*, D. Atlas, Ed., Amer. Meteor. Soc., Boston, 534–565.
- Gage, K. S. and J. L. Green, 1978: Evidence for specular reflection from monostatic VHF radar observations of the stratosphere. *Radio Sci.*, **13**, 991–1001.
- Goodman, J. W., 1968: *Introduction to Fourier optics*. 1st ed., McGraw-Hill, Boston, Massachusetts, 287 pp.
- Goodman, J. W., 2015: *Statistical optics*. 2d ed., John Wiley & Sons, Hoboken, New Jersey, 516 pp.
- Goodman, J. W., 2017: *Introduction to Fourier optics*. 4th ed., W. H. Freeman and Company, New York, New York, 546 pp.
- Goodman, J. W., 2020: *Speckle phenomena in optics*. 2d ed., SPIE Press, Bellingham, Washington, United States, 452 pp.
- Gudimetla, V. S., R. B. Holmes, and J. F. Riker, 2012: Analytical expressions for the log-amplitude correlation function for plane wave propagation in anisotropic non-Kolmogorov refractive turbulence. *J. Opt. Soc. Am. A*, **29**, 2622–2627.
- Gudimetla, V. S., R. B. Holmes, and J. F. Riker, 2014: Analytical expressions for the log-amplitude correlation function for spherical wave propagation through anisotropic non-Kolmogorov atmosphere. *J. Opt. Soc. Am. A*, **31**, 148–154.
- Hill, R. J., 1978: Models of the scalar spectrum for turbulent advection. *J. Fluid Mech.*, **88**, 541–562.
- Hill, R. J. and S. F. Clifford, 1978: Modified spectrum of atmospheric temperature fluctuations and its application to optical propagation. *J. Opt. Soc. Am.*, **68**, 892–899.
- Ishimaru, A., 1978: *Wave propagation and scattering in random media, Vol. 2, Scattering in random media*. Academic Press, Boston, Massachusetts, 251–572 pp.
- Kopeika, N. S., 1998: *A system engineering approach to imaging*. SPIE, Bellingham, Washington, 653 pp.
- Korotkova, O., 2014: *Random light beams: theory and applications*. CRC Press, Boca Raton, 361 pp.
- Korotkova, O., 2022: *Theoretical statistical optics*. World Scientific Publishing, Hackensack, New Jersey, 319 pp.

- Muschinski, A., 1996: Possible effect of Kelvin-Helmholtz instability on VHF radar observations of the mean vertical wind. *J. Appl. Meteor.*, **35**, 2210–2217.
- Muschinski, A., 2004: Local and global sampling of clear-air Doppler radar signals. *Radio Sci.*, **39**, RS1008.
- Muschinski, A., 2015: Temperature variance dissipation equation and its relevance for optical turbulence modeling. *J. Opt. Soc. Am. A*, **32**, 2195–2200.
- Muschinski, A., 2017: Non-Kolmogorov turbulence. *Propagation Through and Characterization of Atmospheric and Oceanic Phenomena*, Optical Society of America, PW2D–1.
- Muschinski, A., 2021: Phase-factor spectra of turbulent phase screens. *J. Opt. Soc. Am. A*, **38**, 1339–1348.
- Muschinski, A., 2022: Non-classical models of optical turbulence. *Unconventional Imaging and Adaptive Optics 2022*, Bellingham, WA, SPIE, Proceedings of SPIE, Vol. 12239, 1223902.
- Muschinski, A. and S. M. de Bruyn Kops, 2015: Investigation of Hill’s optical turbulence model by means of direct numerical simulation. *J. Opt. Soc. Am. A*, **32**, 2423–2430.
- Muschinski, A. and S. M. de Bruyn Kops, 2018: Phase screens of optical turbulence generated by means of direct numerical simulation of first principles of fluid mechanics. *Propagation Through and Characterization of Atmospheric and Oceanic Phenomena*, Optical Society of America, PTu5F–4.
- Muschinski, A., R. G. Frehlich, and B. B. Balsley, 2004: Small-scale and large-scale intermittency in the nocturnal boundary layer and the residual layer. *J. Fluid Mech.*, **515**, 319–351.
- Muschinski, A., D. G. Voelz, A. Baca, and R. Hull, 2023: Scintillation observed in an 84-km propagation experiment across White Sands Missile Range. *Propagation Through and Characterization of Atmospheric and Oceanic Phenomena*, Optical Society of America, 2 pp., submitted.
- Muschinski, A. and E. L. Wagner, 2023: First field measurements of optical turbulence near the ground with a newly developed fine-wire thermometer. *Waves Rand. Comp. Med.*, **33**, 1234–1249.
- Obukhov, A. M., 1949: The structure of the temperature field in a turbulent flow. *Izv. Akad Nauk SSSR, Ser. Geogr. i Geofiz.*, **13**, 58–69.
- Roddier, F., 1981: The effects of atmospheric turbulence in optical astronomy. *Prog. Opt.*, **19**, 281–376.
- Roggemann, M. C. and B. M. Welsh, 1996: *Imaging through turbulence*. CRC Press, Boca Raton, Florida, 320 pp.
- Röttger, J. and C. H. Liu, 1978: Partial reflection and scattering of VHF radar signals from the clear atmosphere. *Geophys. Res. Lett.*, **5**, 357–360.
- Schmidt, J. D., 2010: *Numerical simulation of optical wave propagation*. SPIE Press, Bellingham, Washington, 196 pp.
- Sommerfeld, A. and J. Runge, 1911: Anwendung der Vektorrechnung auf die Grundlagen der geometrischen Optik. *Ann. Phys.*, **340**, 277–298.
- Strohbehn, J. W., 1968: Line-of-sight wave propagation through the turbulent atmosphere. *Proc. IEEE*, **56**, 1301–1318.

- Strohbehn, J. W., (Ed.) , 1978: *Laser beam propagation in the atmosphere*, Topics in Applied Physics, Vol. 25. Springer, Berlin, 325 pp.
- Tatarskii, V. I., 1961: *Wave propagation in a turbulent medium*. McGraw-Hill, New York, 285 pp.
- Tatarskii, V. I., 1971: *The effects of the turbulent atmosphere on wave propagation*. Israel Program for Scientific Translation, Jerusalem, Israel, 472 pp.
- Tatarskii, V. I. and V. U. Zavorotny, 1993: Atmospheric turbulence and the resolution limits of large ground-based telescopes: comment. *J. Opt. Soc. Am. A*, **10**, 2410–2414.
- Toselli, I., 2014: Introducing the concept of anisotropy at different scales for modeling optical turbulence. *J. Opt. Soc. Am. A*, **31**, 1868–1875.
- Toselli, I., L. C. Andrews, R. L. Phillips, and V. Ferrero, 2008: Free-space optical system performance for laser beam propagation through non-Kolmogorov turbulence. *Opt. Eng.*, **47**, 026 003.
- Uscinski, B. J., 1977: *The elements of wave propagation in random media*. McGraw-Hill, New York, 153 pp.
- Uscinski, B. U., (Ed.) , 1986: *Wave propagation and scattering*. Clarendon Press, Oxford, United Kingdom, 381 pp.
- Voelz, D. G., 2011: *Computational Fourier optics*. SPIE Press, Bellingham, Washington, 232 pp.
- Voelz, D. G., E. Wijerathna, A. Muschinski, and X. Xiao, 2018: Computer simulations of optical turbulence in the weak- and strong-scattering regime: angle-of-arrival fluctuations obtained from ray optics and wave optics. *Opt. Eng.*, **57**, 104 102.
- Wheelon, A. D., 2001: *Electromagnetic scintillation — I. Geometrical optics*. Cambridge University Press, Cambridge, United Kingdom, 455 pp.
- Wheelon, A. D., 2003: *Electromagnetic scintillation — II. Weak scattering*. Cambridge University Press, Cambridge, United Kingdom, 440 pp.
- Wijerathna, E., H. Zhan, D. Voelz, and A. Muschinski, 2023: Low-wavenumber compensation with Zernike tilt for non-Kolmogorov turbulence phase screens. *Appl. Opt.*, **62**, 1253–1262.

2 Impacts

2.1 Impact on the principal disciplines of the project

We have investigated various characteristics of wave propagation through the turbulent atmosphere by means of observations (using optical sensors and in-situ sensors such as ultrasonic anemometer-thermometers, fine-wire thermometers, and microbarometers), theoretical analysis, and computer simulations.

Our in-situ measurements show no compelling evidence of the existence of “non-Kolmogorov turbulence” in the atmospheric surface layer. With “non-Kolmogorov turbulence,” we denote turbulent temperature fluctuations characterized by an inertial-convective range 3D temperature spectrum that follows a power law that differs from the classical $\kappa^{-11/3}$ power law predicted by the Obukhov-Corrsin theory (Obukhov, 1949; Corrsin, 1951; Tatarskii, 1961). Even in the stably stratified atmospheric surface layer close to the ground, both ultrasonic anemometers and fine-wire thermometers reveal robust, fully developed “Kolmogorov turbulence”.

While our in-situ observations do not support the popular belief that atmospheric turbulence is often “non-Kolmogorov,” they do provide incontrovertible evidence that turbulent temperature increments are strongly non-Gaussian, typically two-sided exponential. To what extent the central limit theorem “converts” the non-Gaussian temperature increments to Gaussian phase increments (a standard assumption in the physics of optical turbulence) depends on the number of statistically independent phase screens along the propagation path. This number is generally large for propagation over long horizontal paths but it may be very small (on the order of 1) for vertical or slanted propagation paths if the turbulence is concentrated within a single turbulent layer.

We introduced several novel developments related to computational wave optics simulations that should be of considerable utility for other researchers investigating atmospheric turbulence and free space optical systems. Along with other advancements, we incorporated rays within the wave simulation to investigate the applicability of geometrical-optics method, and we validated a simple and effective method for consistently representing low-wavenumber components of turbulent phase screens.

2.2 Impact on other disciplines

The physics of optical turbulence plays an important role for various science and engineering disciplines in the civilian and military arenas, such as optical remote sensing, free-space optical communication, optical imaging of terrestrial and extraterrestrial objects and adaptive optics, as well as target-acquisition, beam-control and directed-energy technologies. The concepts, theoretical models, model assumptions, and the observational and computational tools and techniques that were applied and investigated in this project are of critical importance for the design and operation of a wide variety of systems used by researchers and engineers working in these disciplines.

2.3 Impact on the development of human resources

While this project did not provide financial support for students, it provided intellectual support for various graduate and undergraduate students, some of which have found employment in aerospace engineering and related fields.

During the lifetime of this project, four undergraduate students earned their Bachelor degrees at the University of Colorado Boulder: Ryan Schaetzle in physics, Carter Mak and Nathan Kuczun

in aerospace engineering, and Maxwell Anderson in mechanical engineering. Now, Ryan Schaetzle works in quality control at Ball Aerospace, Boulder, CO; Carter Mak develops software for satellite simulations at the NASA Jet Propulsion Laboratory, Pasadena, CA; Nathan Kuczun is a systems engineer at Freefly Systems, Woodinville, WA; and Maxwell Anderson, who in the meantime has earned his Master's degree from CU's Mechanical Engineering Dept., has recently joined the Deep Space Gateway team at Lockheed Martin, Littleton, CO.

Erandi Wijerathna received her PhD in Electrical Engineering at NMSU, Las Cruces, NM partway through this project. She continued supporting the final year of the project as a postdoctoral scholar and is now a postdoctoral researcher for a project sponsored by the Intelligence Advanced Research Projects Activity (IARPA) involving imaging through turbulence. Currently, Dr. Wijerathna is a postdoctoral researcher at the University of California, Riverside, CA. Andrea Baca entered the Master's program in Electrical Engineering at NMSU.

2.4 Impact on teaching and educational experiences

While Dr. Muschinski did not teach university courses during the lifetime of this project, his many discussions and informal collaboration with NMSU's Dr. Voelz did have impacts on teaching.

Specific concepts and tools developed in this project were introduced into several graduate classes at NMSU. These classes included "Fourier Methods in Electro-Optics" where some of the wave optics simulations developments were presented, "Statistical Optics" where new understanding related to turbulence theory was discussed, and "Optical System Design" where, for example, the optical design for the light sources used in the long-range optical propagation experiment at White Sands Missile Range was assigned as a final project.

Weekly meetings of the optics research group led by Dr. Voelz at NMSU allowed students working on other activities to learn about this project. For example, the project provided an opportunity for undergraduate students in the group to observe the work and assist on occasion. For example, several students learned how to configure the light sources and operate camera systems for the long-range optical propagation experiment. The undergraduate students were also helpful in running some of the simulation codes and organizing data.

2.5 Impact on physical, institutional, and information resources that form infrastructure

While this project provided financial support mainly for Dr. Muschinski's analytical work, there was significant synergy with two other of Dr. Muschinski's AFOSR-sponsored research projects which focused more on instrument development, field experiments, and computer simulations.

During the lifetime of this project, the NWRA team integrated and refined optical and in-situ sensors and sensing techniques. A unique fine-wire turbulence measurement system was developed in-house *ab initio*. All these sensors are portable and will be available for future field experiments, even in remote locations "off the grid".

Computer simulation codes and computational tools developed during the lifetime of this project are applicable in many other problems involving laser propagation and imaging through turbulence. The introduction of the new long-range optical propagation test area across White Sands Missile Range near NMSU provides a unique capability for optical propagation and atmospheric testing in the future.

3 Attachments

The following five journal papers (Voelz et al., 2018; Caldwell et al., 2020; Muschinski, 2021; Muschinski and Wagner, 2023; Wijerathna et al., 2023) and three conference papers (Muschinski and de Bruyn Kops, 2018; Muschinski, 2022; Muschinski et al., 2023), all published between 2018 and 2023, are attached at the end of this report.

Computer simulations of optical turbulence in the weak- and strong-scattering regime: angle-of-arrival fluctuations obtained from ray optics and wave optics

David Voelz,^{a,*} Erandi Wijerathna,^a Andreas Muschinski,^{b,c} and Xifeng Xiao^a

^aNew Mexico State University, Klipsch School of Electrical and Computer Engineering, Las Cruces, New Mexico, United States

^bNorthWest Research Associates, Boulder, Colorado, United States

^cUniversity of Colorado Boulder, Aerospace Engineering Department, Boulder, Colorado, United States

Abstract. It is known that certain geometrical-optics predictions often agree well with optical turbulence field observations even though theoretical constraints for ignoring diffraction may be violated. Geometrical optics assumptions can simplify analyses, and ray optics can significantly reduce simulation computation time. Here, an investigation into angle-of-arrival fluctuations is presented involving wave optics and geometrical (ray) optics computer simulations of a plane wave of visible light propagating through a turbulent refractive-index field. The simulation and Rytov-based theory results for the variances of aperture-filtered angle-of-arrival fluctuations generally agree well for weak scattering (Rytov variance, $\sigma_R^2 \lesssim 0.2$), but for increasing Rytov variance, the simulation results demonstrate a positive slope that can be significantly shallower than that predicted by the theory. For weak-to-moderate scattering regimes ($\sigma_R^2 \lesssim 2.67$), a comparison of the ray and wave results show they match for aperture diameters greater than about two Fresnel lengths. This result is consistent with a previous theoretical analysis by Cheon and Muschinski. For the strongest scattering case studied ($\sigma_R^2 = 26.7$), the wave and ray simulations match for aperture diameters greater than about 10 Fresnel lengths. For smaller apertures, we attribute the disparity between the wave and ray simulation results to a Fresnel filtering effect. © The Authors. Published by SPIE under a Creative Commons Attribution 3.0 Unported License. Distribution or reproduction of this work in whole or in part requires full attribution of the original publication, including its DOI. [DOI: 10.1117/1.OE.57.10.104102]

Keywords: atmospheric turbulence; Rytov variance; angle of arrival; wave optics; split-step simulation; ray optics.

Paper 180577 received Apr. 23, 2018; accepted for publication Aug. 29, 2018; published online Oct. 5, 2018.

1 Introduction

Understanding the physics of optical wave front distortions caused by turbulence in the atmospheric refractive-index field is essential for the design and operation of optical systems used in free-space optical communication, surveillance, navigation, remote sensing, astronomy, and directed-energy technologies.^{1–6} If particle scattering, absorption, coupling, and depolarization effects are negligible then there are two phenomena that determine the optical field: refraction and diffraction. The combined effects of refraction and diffraction lead to phase-gradient fluctuations (angle-of-arrival fluctuations) and intensity fluctuations (scintillation), both of which limit the performance of a variety of optical systems.

The analysis of propagation scenarios through both deterministic and stochastic refractive-index fields may be substantially simplified if diffraction effects can be neglected, and considerable theoretical effort has been devoted to establishing criteria for the negligibility of diffraction effects.^{1–3,7,8} These theoretical constraints, however, tend to be excessively conservative, as pointed out by Strohbehn 50 years ago.⁹ In other words, certain geometrical-optics predictions often agree well with field observations, but it remains unclear, from the theoretician's perspective, why that is so.

In addition to simplifying theoretical analyses, geometrical optics and ray tracing can also substantially reduce computation time for numerical simulations of propagation

through turbulence. The utility of geometrical and ray optics is exemplified in a variety of publications from the last several decades. For example, Churnside and Lataitis¹⁰ presented a simple geometrical optics equation for the variance of beam displacements caused by propagation through weak turbulence, and geometrical optics simulation models have been developed by Yuksel et al.¹¹ to investigate effects of aperture averaging on intensity fluctuations. More recently, Lachinova et al.¹² have compared the performance of a brightness function-based numerical simulation method with a wave optics-based method for incoherent imaging through turbulence. The brightness function at the receiving pupil is estimated using ray-like trajectories that are perturbed by the turbulent refractive index gradients. These authors found that the brightness approach can reduce computation times by several orders of magnitude. Other recent examples include ray-tracing methods applied for simulating laser beam-steering effects in turbulent media,¹³ the derivation of turbulence strength profiles from radiometer temperature measurements with a geometrical ray-tracing analysis,¹⁴ and the application of ray tracing through phase screens for the simulation of images from optical survey telescopes.¹⁵

Since the 1980s, powerful phase-screen techniques have been developed^{16–19} that allow optical propagation through atmospheric turbulence to be accurately simulated well into the “strong-scattering” regime, that is, for scenarios where the negligibility of diffraction effects and the “weak-scattering” assumption become invalid, or at least questionable. Such simulations can be used to test and refine the

*Address all correspondence to: David Voelz, E-mail: davvoelz@nmsu.edu

traditional, often excessively conservative, criteria for the negligibility of diffraction effects.

Here, we present and discuss computer simulations of plane waves of visible light ($\lambda = 500$ nm) propagating through fully turbulent, homogeneous, and isotropic refractive-index fields. We study the optical angle-of-arrival (AOA) fluctuations by means of both a wave-optics propagation simulation and a ray-tracing simulation through a turbulent path that is characterized by a sequence of phase screens with Kolmogorov statistics. The path length assumed is $L = 2$ km and the refractive-index structure parameter C_n^2 ranges from $10^{-16} \text{ m}^{-2/3}$ to $10^{-13} \text{ m}^{-2/3}$, which corresponds to Rytov variances σ_R^2 that span from 0.0267 (weak scattering) to 26.7 (strong scattering), where $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$ and $k = 2\pi/\lambda$ is the optical wave number. For the two simulation methods, we examine the variance and histograms of the aperture-filtered (aperture-averaged) AOA fluctuations for aperture diameters ranging from 0.6 to 13 Fresnel lengths. We compare the simulation results with theoretical predictions²⁰ based on the Rytov theory, which accounts for both refraction and diffraction effects but does not capture strong-scattering effects. Our specific interests are: (1) to better understand under what conditions the Rytov theory for AOA fluctuations becomes invalid, (2) to reveal the behavior of the AOA fluctuations in strong scattering conditions, and

(3) to investigate the consequences of ignoring diffraction and the applicability of ray tracing for modeling AOA fluctuations.

2 Approach

A plane wave is propagated through a set of split-step phase screens that model atmospheric turbulence [Fig. 1(a)] and similarly, a bundle of parallel rays is traced through the same screens [Fig. 1(b)]. The wave optics simulation provides a coherent wave result that accounts for refraction, diffraction, and interference. On the other hand, the ray-tracing simulations provide the trajectories of a large number of individual rays, where the phases and amplitudes are left undefined.

The simulation parameters of interest are defined in Table 1 and the values used for the simulations are listed. The turbulence along the path is assumed to be statistically homogeneous and isotropic. The path is divided into equal length segments and a phase screen is positioned at the center of each segment. The screens are created with the Fourier filtering method and have Kolmogorov statistics. To model an infinite outer scale, a random tilt component is added to each screen that compensates for turbulent wave-front tilts with length scales greater than the grid width (see Ref. 18 and references therein). Zero inner scale is assumed for

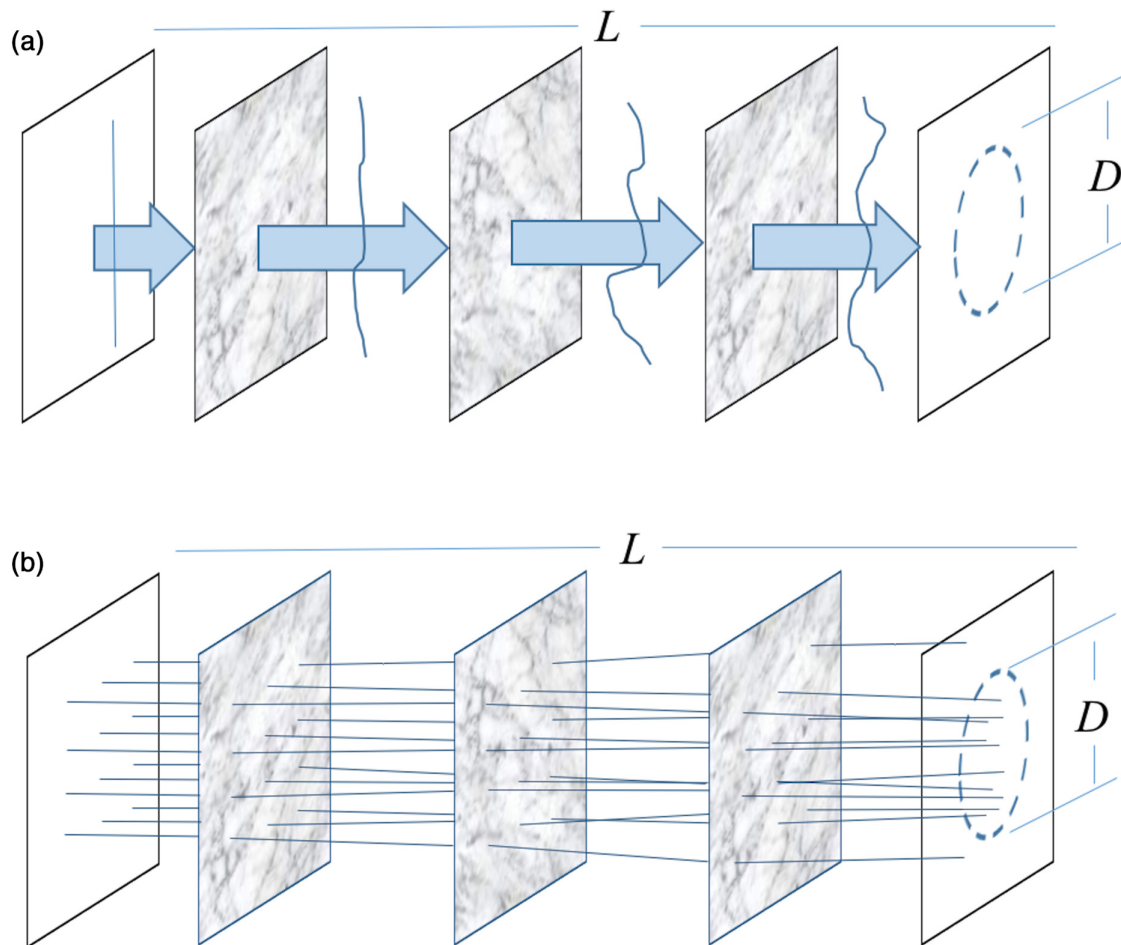


Fig. 1 Illustration of the split-step phase screens that model turbulence in (a) wave and (b) ray simulations. L is the propagation path length and D is the diameter of the aperture that is applied in the observation plane for the aperture-averaged AOA.

Table 1 Simulation parameter values.

Parameter	Notation	Value
Grid physical size	$S \times S$	1 m $\times$ 1 m
Grid samples		1000 $\times$ 1000
Grid spacing	Δx	1 mm
Wavelength	λ	0.5 μ m
Path length	L	2 km
Fresnel length	$\sqrt{\lambda L}$	3.2 cm
Aperture diameter	D	0.02 to 0.4 m
Turbulence spectrum		Kolmogorov
Structure parameter	C_n^2	10^{-16} to 10^{-13} m $^{-2/3}$
Rytov variance	σ_R^2	0.0267 to 26.7
Number of phase screens	N	6 (for 10^{-16} m $^{-2/3}$); 6 (for 10^{-15} m $^{-2/3}$); 10 (for 10^{-14} m $^{-2/3}$); 22 (for 10^{-13} m $^{-2/3}$)

the turbulence spectrum; however, the period of the highest spatial frequency of a screen is twice the grid spacing, so the modeled inner scale is no smaller than 2 mm.

The number of turbulence screens N required along the path is an integer and should be greater than $(10\sigma_R^2)^{6/11}$, which ensures each screen corresponds to weak scattering, and therefore phase-only screens are adequate.¹⁹ The minimum value of N for the strongest scattering case is 22 screens. Fewer screens can be used for the weaker scattering cases, which reduces computational time. However, we implemented more than the required minimum number of screens for the weak scattering cases to provide a better representation of the continuous volume turbulence that is assumed along the path.

The grid parameters are chosen to provide a grid spacing of $\Delta x = \lambda L/S$, where S is the grid side length. This sampling choice has been shown to make artifacts that arise from the discrete Fourier transform implementation of the Fresnel diffraction integral to be sufficiently small to be negligible.²¹ The resulting grid spacing Δx of 1 mm allows about $3\times$ sampling of the minimum Fried parameter (r_0) value encountered in our study. The propagation length L , wavelength λ , and four choices of turbulence structure parameter C_n^2 (10^{-16} , 10^{-15} , 10^{-14} , and 10^{-13} m $^{-2/3}$) correspond to Rytov variances σ_R^2 of 0.0267, 0.267, 2.67, and 26.7, which range in scattering strength from weak ($\sigma_R^2 \ll 1$) to strong ($\sigma_R^2 > 1$). Corresponding Fried diameter [$r_0 = (0.423 k^2 C_n^2 L)^{-3/5}$] values for the four cases are calculated to be 21.3, 5.3, 1.33, and 0.33 cm.

Ultimately, our interest is to examine the variance and histogram of the fluctuations of the aperture-averaged AOA. This requires running the simulation many times for independent turbulence realizations, obtaining the aperture-averaged AOA for each realization, and then computing the variance.

2.1 Wave Optics Simulation

The initial plane wave at the entrance plane, $z = 0$, is a unit-amplitude, zero-phase field that is propagated in steps between the entrance plane and the observation plane using a discretized form of the Fresnel diffraction integral.²¹ At the i 'th intermediate plane, the phase screen ψ_i is applied in the usual way as $u'_i = u_i \exp(j\psi_i)$, where u_i is the incident field and u'_i is the field that exits the plane. The discrete Fourier transform-based simulation produces a periodicity of the electric field in the x - y plane across the grid boundaries. In addition, the compensating random tilt component in the turbulence phase screens creates some "ringing" artifacts in the receiving plane near the grid boundaries. To minimize unwanted periodicity and edge effects, the aperture is only applied to the center 0.4×0.4 fraction of the receiving plane grid.

One approach to find the aperture-averaged AOA for a given turbulence realization is to numerically compute the field phase gradients at the observation plane grid points and average the results within the aperture region. However, direct calculation of the phase gradients at the grid points is problematic for strong scattering because of branch points (implying discontinuous phase changes from 2π to 0, or 0 to 2π) that appear in the wave front and correspond to theoretically infinite phase gradients.

For the results presented here, we use an approach where the wave front within the aperture for a given turbulence realization is numerically focused with a focus transmittance function.²¹ The x - and y -direction centroid positions of the intensity spot at the focal plane are found and the aperture-averaged AOA is the angle defined by the ratio of the centroid displacement (from the optical axis) and the focal length. This AOA is computed for many independent turbulence realizations and the variance is calculated. This approach has the advantage that it is representative of a practical AOA measurement that can be made with a lens and camera system.²² It also effectively weighs the angle contributions by the spatial field amplitude in the pupil, where, for example, the amplitude is vanishing near a branch point so the sometimes extremely large gradients associated with these features are not significant for the AOA variance calculation.

2.2 Ray Tracing Simulation

For the ray simulation, as shown in Fig. 1(b), the source is comprised of a set of rays where each ray is positioned at a grid point corresponding to the wave simulation and is initially directed parallel to the optical axis. For a given turbulence realization, the rays are traced from one plane to the next where at the i 'th plane, the phase screen changes the x -direction ray angle as

$$\theta'_i = \theta_i + \frac{1}{k} \frac{d\psi_i}{dx}, \quad (1)$$

where θ_i and θ'_i are the incoming and outgoing ray angles and $\frac{d\psi_i}{dx}$ is the phase-screen gradient in the x -direction. Ray angle changes in the y -direction are computed similarly using the y -direction gradients. A three-point calculation is used to compute the screen gradients, and the gradients are interpolated at the position where the ray strikes the plane. The x - and y -direction angles for the rays that eventually fall

within the aperture region at the observation plane are averaged to obtain the aperture-averaged AOA. Our approach for tracing rays through the split-step phase screens is similar to the method presented by Lachinova et al.,¹² however, we do not compute the brightness function that is part of their incoherent image simulation.

3 Theory

The variance of the aperture-filtered AOA fluctuations for a circular aperture and for a plane wave propagated through homogeneous and locally isotropic inertial subrange turbulence is given by Cheon and Muschinski²⁰ as

$$\langle \overline{\theta^2} \rangle_P = \gamma_P(q) C_n^2 L D^{-\frac{1}{3}} = 0.81 \gamma_P(q) \sigma_R^2 k^{-\frac{7}{6}} L^{-\frac{5}{6}} D^{-\frac{1}{3}}, \quad (2)$$

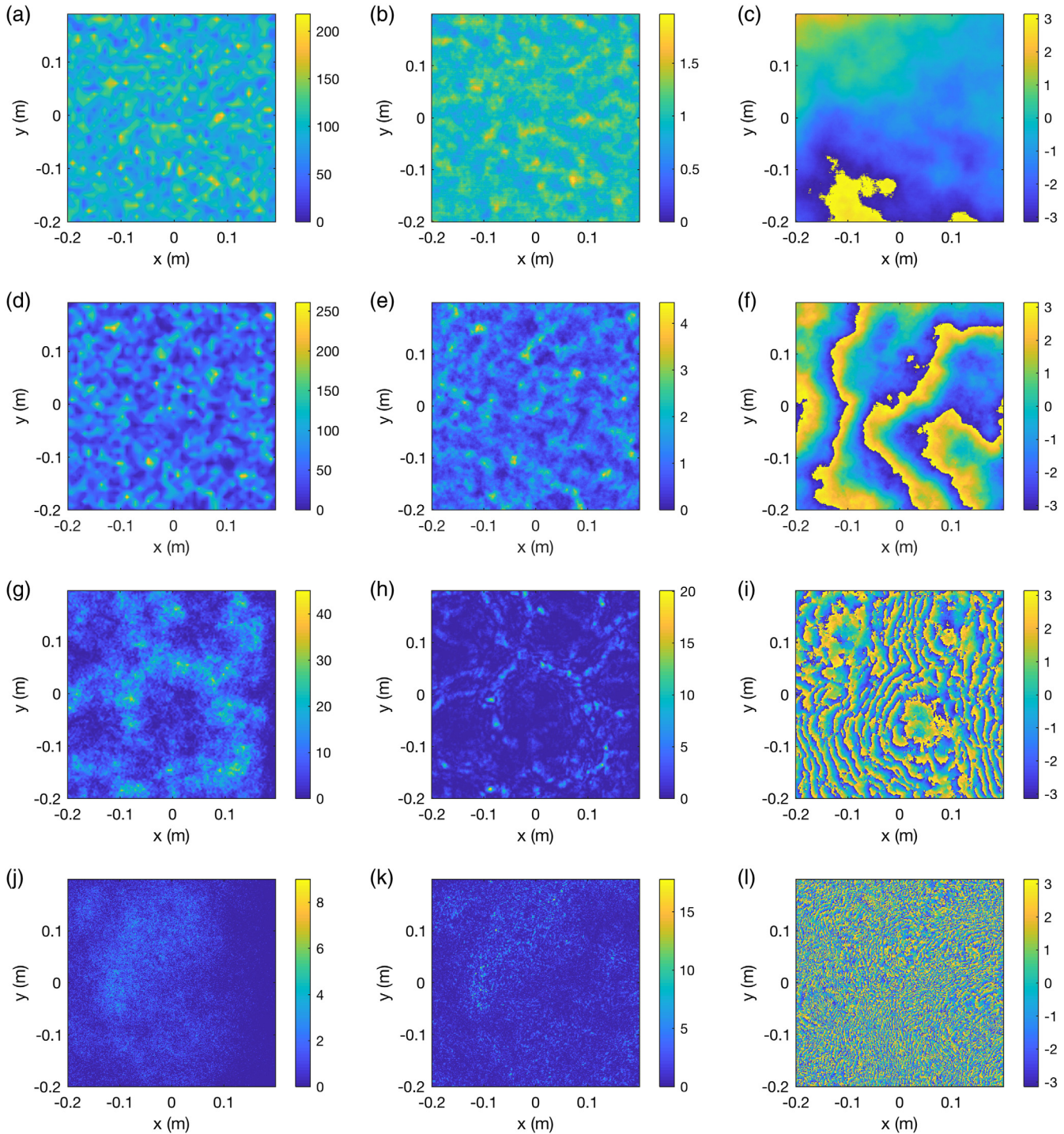


Fig. 2 Observation plane examples. Left-to-right in each row: ray-density, wave-intensity, and wave-phase cross-sections. Turbulent scattering strength increasing from top-to-bottom along columns: (a–c) $\sigma_R^2 = 0.0267$, $C_n^2 = 10^{-16} \text{ m}^{-2/3}$, $r_0 = 21.3 \text{ cm}$; (d–f) $\sigma_R^2 = 0.267$, $C_n^2 = 10^{-15} \text{ m}^{-2/3}$, $r_0 = 5.3 \text{ cm}$; (g–i) $\sigma_R^2 = 2.67$, $C_n^2 = 10^{-14} \text{ m}^{-2/3}$, $r_0 = 1.33 \text{ cm}$; (j–l) $\sigma_R^2 = 26.7$, $C_n^2 = 10^{-13} \text{ m}^{-2/3}$, $r_0 = 0.33 \text{ cm}$. Ray density binning: (a) 10×10 bins, (d) 8×8 bins, (g) 3×3 bins, and (j) 1×1 bins.

where

$$\gamma_P(q) = \frac{\sqrt{3}}{16} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{8}{3}\right) \left(\frac{\beta}{2}\right)^{-\frac{1}{3}} \left\{ 1 + \frac{6}{5} \left(\frac{\pi}{2}\right)^{\frac{1}{6}} \beta^{\frac{1}{3}} q^{\frac{1}{3}} \right. \\ \left. \times \left[1 + \frac{\pi^2}{4} \beta^4 q^4 \right]^{\frac{5}{12}} \sin \left[\frac{5}{6} \arctan \left(\frac{2}{\pi \beta^2 q^2} \right) \right] \right\}, \quad (3)$$

where D is the aperture diameter, $q = D/\sqrt{\lambda L}$ is the aperture diameter normalized by the Fresnel length, $\beta = 0.5216$ is a constant, and Γ is the gamma function. Equations (2) and (3) were developed assuming the Rytov approximation for the propagating field. This result is applicable for all values of q and assumes infinite outer scale and zero inner scale.

4 Results

4.1 Observation Plane Illustration

Before presenting the AOA results, we first examine the character of the rays and waves arriving at the observation plane for a single turbulence realization. Figure 2 shows single realization examples within the central $0.4 \text{ m} \times 0.4 \text{ m}$ area for the scattering scenarios corresponding to $\sigma_R^2 = 0.0267, 0.267, 2.67$, and 26.7 . Shown for comparison are: (1) ray-density fields obtained from the ray simulation and (2) intensity and phase fields obtained from the wave optics simulation. The ray density fields were obtained by counting the rays within areas defined by the binning values. The binning is relative to the grid spacing Δx , for example, 10×10 binning indicates an area of $10 \text{ mm} \times 10 \text{ mm}$. The bin size was selected to emphasize the most apparent characteristic feature sizes within the frame. A contour plot algorithm was also applied to the ray density plot to reduce the blockiness of the display.

Common spatial features are apparent in the intensity and ray-density fields as shown in Fig. 2. The corresponding features are related to the effects of refraction as ray tracing does not account for the interference and diffractive effects provided in the wave simulation. The phase cross-sections, displayed modulo- 2π , illustrate the increasing wave front complexity as a function of the Rytov variance. As noted previously, after the ray and wave fields are generated at the observation plane as shown in Fig. 2, a circular aperture of diameter D is imposed and the methods described in Sec. 2 are applied to find the aperture-averaged AOA field for the particular turbulence realization.

4.2 Variances of Aperture-Filtered Angle-of-Arrival Fluctuations

The variances of aperture-averaged AOA fluctuations are computed from many independent turbulence realizations. Figure 3 shows results for the theoretical [Eq. (2)] and simulated aperture-averaged AOA variances as a function of the normalized aperture diameter q for the four values of C_n^2 listed in Table 1. Each data point was estimated from 1000 turbulence realizations. The general behavior is that the AOA variance decreases as a function of increasing aperture diameter. Both simulation results generally agree well with the AOA theory²⁰ for weak scattering (e.g., $\sigma_R^2 \lesssim 0.2$) but decrease significantly below the theory curve as the Rytov variance increases. The results also show the agreement between theory and simulation is better in stronger

scattering for larger apertures (e.g., $\sigma_R^2 = 2.67$ and $q \gtrsim 10$). The over-prediction of the AOA variances by the Rytov theory is likely a result of the well-known failure of the Rytov approximation in the strong-moderate- to strong-scattering regime. We note that the Rytov theory is also known to over-predict intensity scintillation as a function of the Rytov variance in the moderate- and strong-scattering regimes.

In general, the ray-based and wave-based simulation variances compare favorably in Fig. 3. The largest discrepancy is a 15% higher variance for the rays in strong scattering with a small aperture ($\sigma_R^2 = 26.7$ and $q = 0.63$). For the weak-to-moderate scattering cases [Figs. 3(a)–3(c)], the ray and wave results match well for $q \gtrsim 2$. That is, diffraction effects are negligible in the weak- and moderate-scattering regime if the aperture diameter is larger than about twice the Fresnel length. This result is consistent with the theoretical analysis by Cheon and Muschinski²⁰ who found that for weak scattering, the AOA variance predicted by geometrical optics deviates from that predicted by the Rytov theory by $<1\%$ if $q > 1.65$. For smaller q , the wave simulations predict a lower AOA variance than the ray simulations. For the strongest scattering case [Fig. 3(d)], the wave and ray variance values converge for $q \gtrsim 10$.

Figure 4 shows an alternative presentation of the results where aperture-averaged AOA variances for the small aperture ($q = 0.63$; $D = 0.02 \text{ m}$) and the large aperture ($q = 12.6$; $D = 0.4 \text{ m}$) are plotted as a function of the Rytov variance, σ_R^2 . The Rytov theory curves demonstrate the linear dependence described by Eq. (2). For the small aperture [Fig. 4(a)], the simulations show a positive slope as a function of scattering strength that is significantly shallower than the theory line. For large aperture diameters, the Rytov-based theory is expected to collapse to the geometrical optics solution.²⁰ This behavior is suggested in the large aperture case [Fig. 4(b)], where the simulation results follow the theory line much more closely.

Figure 4 also shows a small difference between the ray-based and wave-based results for the small aperture ($q = 0.63$), but essentially no difference for the large aperture ($q = 12.6$). As noted previously, geometrical optics is known to be questionable for $q < 1$, which is consistent with the fact that our wave-optics simulations produce smaller AOA variances than the AOA variances produced by our ray-tracing simulations. We attribute this to “Fresnel filtering.” The Fresnel-filter kernel in the wave-number representation of the AOA variance for plane waves predicted by the Rytov theory has the form $K_F(x) = 1 + (2\pi/x^2) \sin(x^2/2\pi)$,²⁰ where $x = \kappa(\lambda L)^{1/2}$ is the wave number normalized by the Fresnel wave number $1/(\lambda L)^{1/2}$. The case $x \ll 1$ (Fresnel length is small compared with κ^{-1} , the wave-front tilt length scale under consideration) is the geometrical-optics limit where the Fresnel length has no effect on the AOA variance. For x comparable to and >1 , the Fresnel length (and therefore diffraction) does play a role: $K_F(x)$ becomes smaller than $K_F(x)$ in the geometrical-optics limit. Therefore, diffraction causes the AOA variance predicted by wave optics to be smaller than the AOA variance predicted by geometrical optics.

That is, a finite Fresnel length acts as a characteristic filter scale similar to the filtering effects due to a finite inner turbulence length scale or a finite aperture diameter. Essentially, diffraction causes wave-front tilt components with length scales comparable to and smaller than the Fresnel length

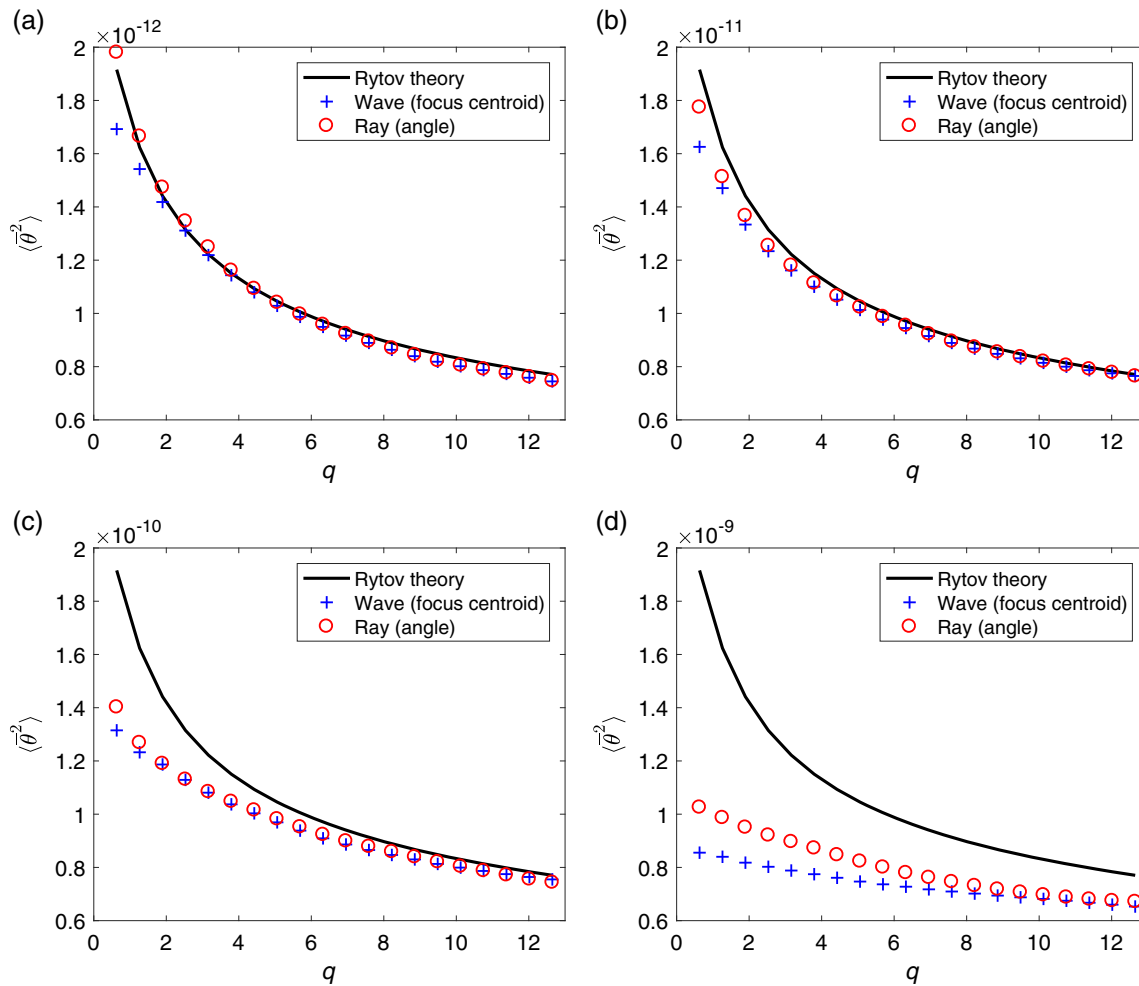


Fig. 3 Aperture-averaged AOA simulation results and Rytov-based theory as a function of normalized aperture size q : (a) $\sigma_R^2 = 0.0267$, $C_n^2 = 10^{-16} \text{ m}^{-2/3}$, (b) $\sigma_R^2 = 0.267$, $C_n^2 = 10^{-15} \text{ m}^{-2/3}$, (c) $\sigma_R^2 = 2.67$, $C_n^2 = 10^{-14} \text{ m}^{-2/3}$, and (d) $\sigma_R^2 = 26.7$, $C_n^2 = 10^{-13} \text{ m}^{-2/3}$.

scales to be smoothed out. The Fresnel-filtering effect is significant when the aperture size and the inner scale of turbulence are both comparable to or smaller than the Fresnel length, and it is negligible otherwise. The Fresnel filtering

is accounted for in the wave-optics simulations but not in the ray-tracing simulations.

To examine the AOA distribution results in detail, we generated histogram estimates of the probability density

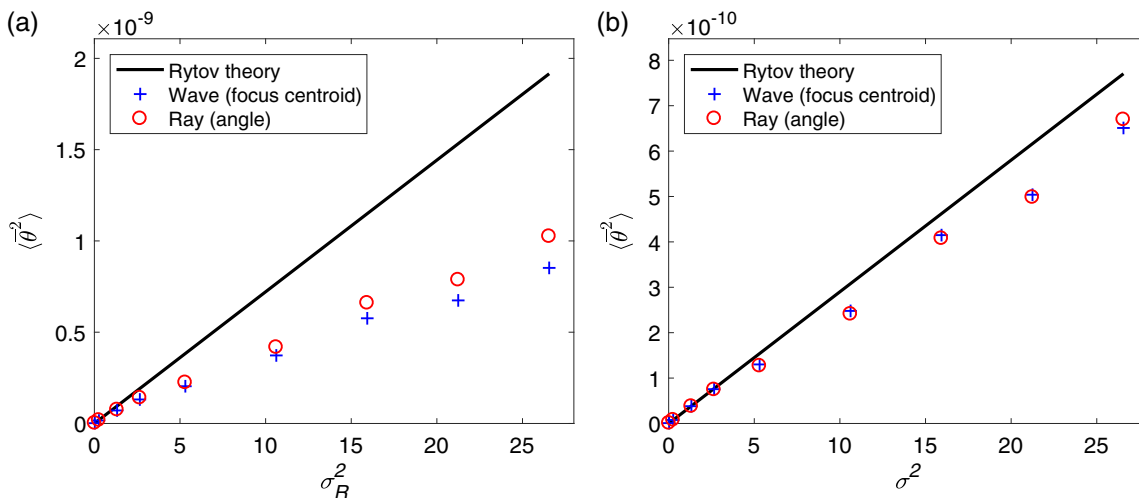


Fig. 4 Aperture-averaged AOA simulation results and Rytov-based theory as a function of σ_R^2 for: (a) $q = 0.63$; $D = 0.02 \text{ m}$ and (b) $q = 12.6$; $D = 0.4 \text{ m}$.

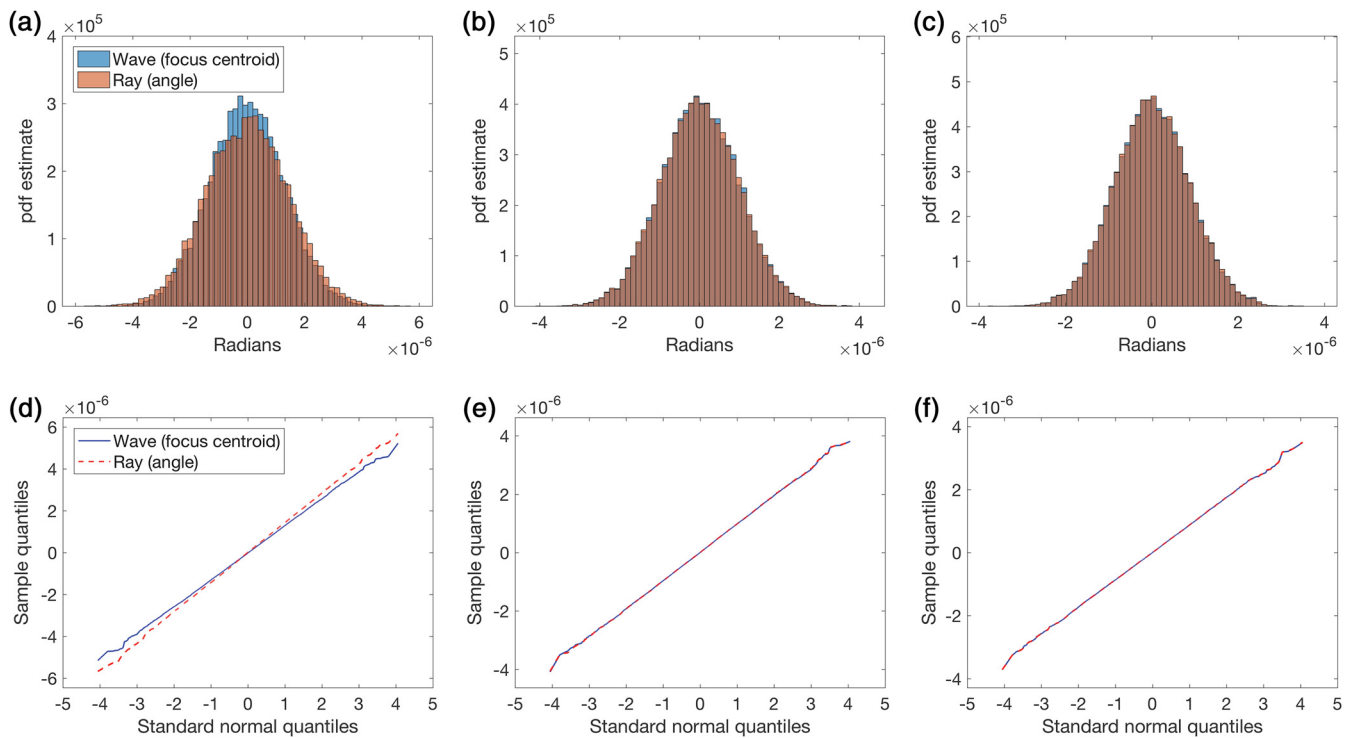


Fig. 5 Statistics of aperture-filtered AOA fluctuations for weak scattering, $\sigma_R^2 = 0.0267$. Left-to-right in each row: $q = 0.63$ ($D = 0.02$ m), $q = 6.3$ ($D = 0.2$ m), and $q = 12.6$ ($D = 0.4$ m). (a–c) pdf estimates and (d–f) corresponding q–q plots.

functions (pdfs). Figures 5(a)–5(c) show the histogram plots for the weakest scattering case and for three aperture diameters. To reduce noise in the histogram, we used 10,000 realizations for these results. We also generated corresponding

quantile–quantile (q–q) plots²³ shown in Figs. 5(d)–5(f). The q–q plots compare the simulation sample distribution to a standard normal (Gaussian) distribution. In all cases, the ray and wave samples create straight lines that indicate

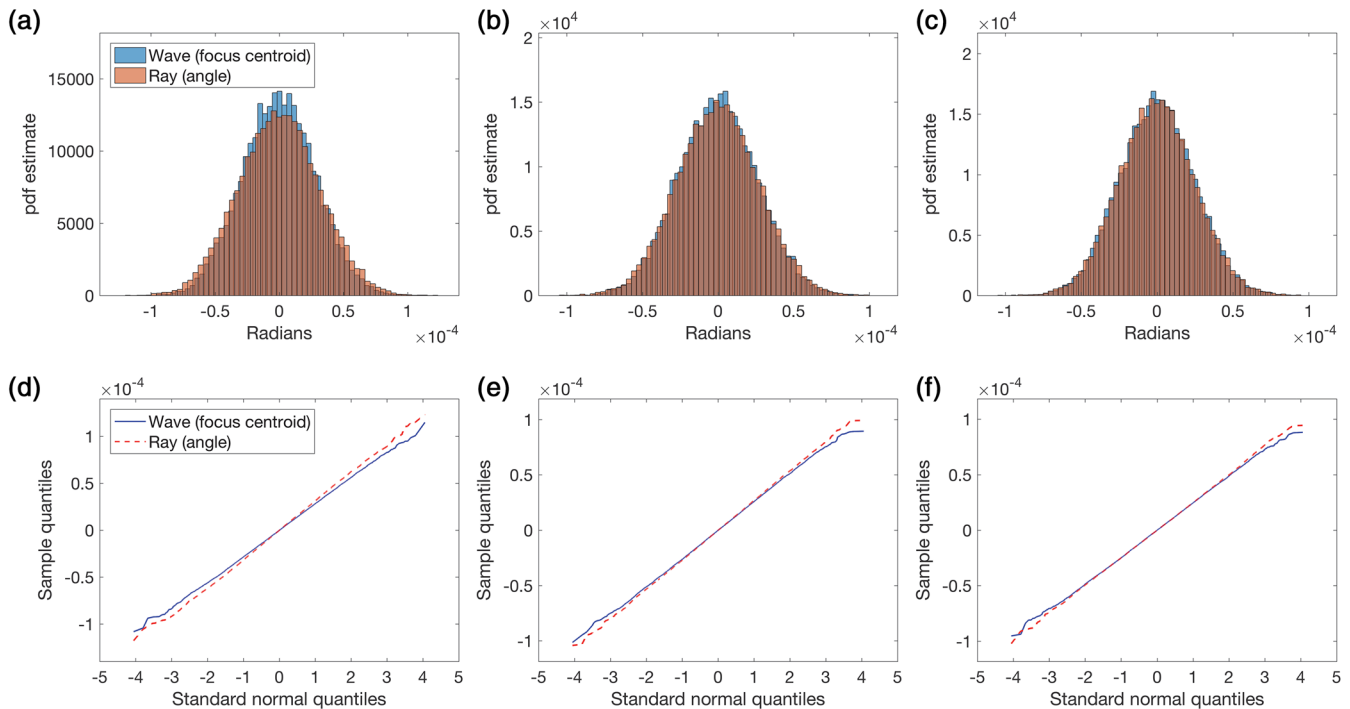


Fig. 6 Statistics of aperture-filtered AOA fluctuations for strong scattering, $\sigma_R^2 = 26.7$. Left-to-right in each row: $q = 0.63$ ($D = 0.02$ m), $q = 6.3$ ($D = 0.2$ m), and $q = 12.6$ ($D = 0.4$ m). (a–c) pdf estimates and (d–f) corresponding q–q plots.

Gaussian distributions. The two different slopes in the q - q curves in Fig. 5(d) correspond to a difference in the AOA variance for the wave and ray results, which was discussed previously for small q . Histogram and q - q plots for strong scattering are shown in Fig. 6. These distributions are also Gaussian and q - q plot slope differences in Figs. 6(d)–6(f) are consistent with the variance results in Fig. 3(d). We expect a Gaussian distribution for the aperture-averaged AOA if the propagation through turbulence and aperture averaging represents a large sum of independent tilt components such that the central limit theorem applies.

5 Conclusions

The simulations illustrate that after propagation through homogeneous turbulence, wave and ray results share obvious intensity/ray-density and AOA characteristics related to the effects of refraction. We find that both the wave and ray simulation results generally predict variances of the aperture-averaged AOA fluctuations that agree well with the Rytov theory for weak scattering (e.g., $\sigma_R^2 \lesssim 0.2$), but as scattering strength (i.e., the Rytov variance) increases, the simulation results demonstrate a positive slope that can be significantly shallower than that predicted by the weak-scattering (Rytov) theory. This over-prediction by the Rytov theory is likely a result of the failure of the Rytov approximation in the strong-scattering regime.

Some insights related to the negligibility of diffraction effects are gained from a comparison of the wave and ray simulation results. For weak-to-moderate scattering regimes ($\sigma_R^2 \lesssim 2.67$), the ray and wave results match well for apertures where $q \gtrsim 2$. This result is consistent with the theoretical analysis by Cheon and Muschinski²⁰ who found that for weak scattering, the AOA variance predicted by geometrical optics deviates from that predicted by the Rytov theory by $<1\%$ if $q > 1.65$. For the strongest scattering case studied here ($\sigma_R^2 = 26.7$), the wave and ray variance values agree well for $q \gtrsim 10$. For smaller apertures, we find the wave simulation variances are smaller than those produced by the ray simulations, and we attribute this to a “Fresnel filtering” effect associated with diffraction. Both the wave and ray AOA distributions are found to be Gaussian.

Acknowledgments

Earlier results of this work appeared in the conference proceedings paper: D. Voelz, E. Wijerathna, X. Xiao, A. Muschinski, “Angle-of-arrival variance of waves and rays in strong atmospheric scattering: split-step simulation results,” Proc. SPIE 10410, Unconventional and Indirect Imaging, Image Reconstruction, and Wavefront Sensing 2017, 104100Y (6 September 2017). This work was funded by Air Force Office of Science Research (AFOSR) (FA9550-17-C-0021 and FA9550-18-C-0011), Air Force Office of Science Research (AFOSR) Multidisciplinary University Research Initiatives (MURI) Program (FA9550-12-1-0449 and FA9550-18-1-0009), National Science Foundation (NSF) Physical and Dynamics Meteorology Program (AGS-1547476), and U.S. Department of Defense (DOD) High Performance Computing Modernization Program (Frontier Project FP-CFD-FY14-007).

References

1. V. I. Tatarskii, *Wave Propagation in a Turbulent Medium*, pp. 285, McGraw-Hill, New York (1961).
2. V. I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation*, pp. 472, Israel Program for Scientific Translation, Jerusalem (1971).
3. A. D. Wheelon, *Electromagnetic Scintillation I—Geometric Optics*, pp. 455, Cambridge University Press, Cambridge, United Kingdom (2001).
4. A. D. Wheelon, *Electromagnetic Scintillation II—Weak Scattering*, pp. 440, Cambridge University Press, Cambridge, United Kingdom (2003).
5. L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media*, 2nd ed., pp. 783, SPIE Press, Bellingham, Washington (2005).
6. O. Korotkova, *Random Light Beams: Theory and Applications*, pp. 361, CRC Press, Boca Raton, Florida (2014).
7. A. Sommerfeld and I. Runge, “Anwendung der Vektorrechnung auf die Grundlagen der geometrischen Optik,” *Ann. Phys.* **340**, 277–298 (1911).
8. D. L. Fried, “Optical resolution through a randomly inhomogeneous medium for very long and very short exposures,” *J. Opt. Soc. Am.* **56**, 1372–1379 (1966).
9. J. W. Strohbehn, “Line-of-sight wave propagation through the turbulent atmosphere,” *Proc. IEEE* **56**, 1301–1318 (1968).
10. J. H. Churnside and R. J. Lataitis, “Wander of an optical beam in the turbulent atmosphere,” *Appl. Opt.* **29**, 926–930 (1990).
11. H. Yuksel, W. Atia, and C. C. Davis, “A geometrical optics approach for modeling atmospheric turbulence,” *Proc. SPIE* **5891**, 589109 (2005).
12. S. L. Lachinova et al., “Comparative analysis of numerical simulation techniques for incoherent imaging of extended objects through atmospheric turbulence,” *Opt. Eng.* **56**, 071509 (2017).
13. Y. Wang and W. D. Kulatilaka, “Optical ray tracing method for simulating beam-steering effects during laser diagnostics in turbulent media,” *Appl. Opt.* **56**, E106–E115 (2017).
14. B. E. Vynhale, “Path profiles of C_n^2 derived from radiometer temperature measurements and geometrical ray tracing,” *Proc. SPIE* **10096**, 100961G (2017).
15. J. R. Peterson et al., “Simulation of astronomical images from optical survey telescopes using a comprehensive photon Monte Carlo approach,” *Astrophys. J. Suppl. Ser.* **218**, 14 (2015).
16. D. L. Knepp, “Multiple phase-screen calculation of the temporal behavior of stochastic waves,” *Proc. IEEE* **71**, 722–737 (1983).
17. J. M. Martin and S. M. Flatté, “Intensity images and statistics from numerical simulation of wave propagation in 3-D random media,” *Appl. Opt.* **27**, 2111–2126 (1988).
18. J. D. Schmidt, *Numerical Simulation of Optical Wave Propagation*, pp. 149–182, SPIE Press, Bellingham, Washington (2010).
19. X. Xiao and D. Voelz, “On-axis probability density function and fade behavior of partially coherent beams propagating through turbulence,” *Appl. Opt.* **48**, 167–175 (2009).
20. Y. Cheon and A. Muschinski, “Closed-form approximations for the angle-of-arrival variance of plane and spherical waves propagating through homogeneous and isotropic turbulence,” *J. Opt. Soc. Am. A* **24**, 415–422 (2007).
21. D. G. Voelz, *Computational Fourier Optics*, pp. 63–96, SPIE Press, Bellingham, Washington (2011).
22. Y. Cheon et al., “Angle-of-arrival anemometry by means of a large-aperture Schmidt–Cassegrain telescope equipped with a CCD camera,” *J. Opt. Soc. Am. A* **24**, 3478–3492 (2007).
23. W. L. Martinez and A. R. Martinez, *Computational Statistics Handbook with MATLAB*, 3rd ed., pp. 124–132, CRC Press, Boca Raton, Florida (2016).

David Voelz is a professor in the Klipsch School of Electrical and Computer Engineering at New Mexico State University. His research interests include spectral and polarization sensing, laser beam propagation through atmospheric turbulence, laser communications, imaging theory, and astronomical instrumentation development. He earned his PhD EE degree from the University of Illinois in 1987. He is a fellow of SPIE.

Erandi Wijerathna is a PhD student in the Klipsch School of Electrical and Computer Engineering at New Mexico State University. Her research interests include laser beam propagation through atmospheric turbulence, imaging theory, and polarimetric lidar. She earned her BS (honors) in physics from the University of Colombo, Sri Lanka, in 2011. She received her MS degree in physics and in electrical engineering from New Mexico State University in 2016.

Andreas Muschinski (MS in physics, TU Braunschweig, Germany, 1990; PhD in meteorology, U Hannover, Germany, 1992; habilitation

in meteorology, U Hannover, Germany, 1998) is an atmospheric physicist. He was a CIRES research scientist at the University of Colorado at Boulder, Colorado (1998–2004) and a professor of electrical engineering at UMass Amherst (2004–2011). Since 2011, he has been a senior research scientist at NorthWest Research Associates, Boulder, Colorado. During the last 28 years, he has conducted research on atmospheric turbulence and atmospheric wave propagation.

Xifeng Xiao received her BS and MS degrees in physics from Xiamen University Fujian, China, in 1998 and 2001, respectively. She earned her MS and PhD degrees in electrical engineering from New Mexico State University in 2004 and 2008, respectively. Currently, she is a software engineer at Ball Aerospace, Albuquerque, NM. Her research interests include simulation and modeling of free-space laser communication, liquid-crystal polarization, and demonstration and implementation of AOTF technologies.

Phase Screens of Optical Turbulence Generated by Means of Direct Numerical Simulation of First Principles of Fluid Mechanics

Andreas Muschinski¹ and Stephen M. de Bruyn Kops²

¹NorthWest Research Associates, Boulder, CO, 3380 Mitchell Lane, USA
andreas@nwra.com

²Dept. of Mechanical and Industrial Engineering, University of Massachusetts Amherst, 160 Governors Drive, Amherst, MA 01003-2210, USA
debk@umass.edu

Abstract: Split-step Fourier-Fresnel algorithms based on phase screens are powerful tools to computationally simulate optical propagation through atmospheric and oceanic turbulence. Usually, phase screens are generated by means of idealized turbulence models. Here we present and discuss phase screens resulting from direct numerical simulation of the Navier-Stokes equations and the scalar transport equation. © 2018 The Author(s)

OCIS codes: 010.0010, 010.7060, 0104030.

1. Introduction

The standard method to computationally simulate optical propagation through turbulent refractive-index fields in the atmosphere or the ocean is the split-step Fourier-Fresnel technique [1, 2]. This technique solves the Helmholtz equation by means of sequences of “phase screens” placed along the propagation path. In contrast to the geometrical-optics method, which ignores diffraction effects, the Fourier-Fresnel technique captures both refraction and diffraction effects; and in contrast to the Rytov approximation, which assumes weak scattering, the multi phase-screen technique is applicable also far into the strong-scattering regime.

Usually, phase screens are generated by means of 2D, Gaussian-distributed, uncorrelated fields, which are then filtered in 2D wave-number space by means of a model of the 3D wave-number spectrum of scalar turbulence, such as the Obukhov-Corrsin model [3, 4] or variances thereof, as discussed in [5]. In addition to the choice of the spectral model, this method of generating phase screens relies on two assumptions: (1) the phase perturbations are Gaussian-distributed; (2) the phase perturbations are statistically homogeneous in planes perpendicular to the propagation direction. Both assumptions are questionable for layered and intermittent turbulence in atmospheric and oceanic flows.

Here, we present and discuss phase screens simulated by means of direct numerical simulation of the first principles of fluid mechanics, that is, the Navier-Stokes equations and the scalar transport equation.

2. Split-step Fourier-Fresnel propagation and phase screens

Let the z -axis coincide with the propagation direction, and let us refer to the x - y planes at z_1 and at z_2 (where $z_2 > z_1$) as the entrance and exit planes, respectively. The complex optical fields in the entrance and exit planes are $E_{\text{in}}(x, y) \equiv E(x, y, z_1)$ and $E_{\text{out}}(x, y) \equiv E(x, y, z_2)$, respectively. Based on a set of assumptions that are not very restrictive, $E_{\text{out}}(x, y)$ can be computed from $E_{\text{in}}(x, y)$ in two steps: first, propagation from z_1 to z_2 through the vacuum; second, the application of a phase change (which in general varies with x and y) in order to account for spatial refractive-index variations of the medium between the entrance and exit planes. The first step, which captures diffraction effects and is often modeled based on the Fresnel approximation, can be written as a 2D convolution in the x - y domain,

$$E_1(x, y) = \iint E_{\text{in}}(x, y) K(x, y) dx dy, \quad (1)$$

where $K(x, y)$ is the propagation kernel [6, p. 73, eq. 4-23]. The second step, which accounts for refractive effects, is a simple multiplication in the x - y domain:

$$E_2(x, y) = E_1(x, y) \exp[\phi(x, y)], \quad (2)$$

where the 2D phase field $\phi(x, y)$ is referred to as the “phase screen”. Therefore, the input-output relationship for the split-step Fourier-Fresnel propagator is

$$E_{\text{out}}(x, y) = \exp[\phi(x, y)] \iint E_{\text{in}}(x, y) K(x, y) dx dy. \quad (3)$$

Details of how to computationally implement split-step Fourier-Fresnel propagation algorithms can be found in [1] and [2].

The phase screen is the 1D integral, along the z -direction, of the 3D refractive-index fluctuation field, $n_1(x, y, z)$, between the entrance and exit planes:

$$\phi(x, y) = k \int_{z_1}^{z_2} n_1(x, y, z) dz, \quad (4)$$

where $k = 2\pi/\lambda$ is the optical wavelength. Tatarskii [7, p. 101, eq. 6.34] showed that, for homogeneous turbulence, the 2D phase-screen spectrum $F_\phi(\kappa_x, \kappa_y)$ can be written in terms of the 3D refractive-index spectrum $\Phi_n(\kappa_x, \kappa_y, \kappa_z)$ taken at $\kappa_z = 0$:

$$F_\phi(\kappa_x, \kappa_y) = 2\pi k^2 L \Phi_n(\kappa_x, \kappa_y, 0), \quad (5)$$

where $L = z_2 - z_1$ is the distance between the entrance and exit planes.

3. Phase screens from direct numerical simulation of the first principles of fluid mechanics

In this presentation, we compute and analyze phase screens from turbulent refractive-index fields obtained by numerical simulation (DNS) of first principles of fluid mechanics, specifically the Navier-Stokes equations for an incompressible fluid and the scalar transport equation; for details, see [8] and [9]. We present and discuss probability densities, 2D spectra, and structure functions of these DNS-generated phase screens and compare the results with theoretical models. We analyze phase screens obtained from DNS-generated, turbulent refractive-index fields for Prandtl numbers 0.7 (air) and 7 (water). Moreover, we consider both isotropic and stably stratified turbulence.

4. Acknowledgements

A. M.’s contributions to this study were partially supported by the National Science Foundation (NSF) Physical and Dynamic Meteorology Program (AGS-1547476); the Air Force Office of Scientific Research (AFOSR) Multidisciplinary University Research Initiative (MURI) “Integrated Measurement and Modeling Characterization of Stratospheric Turbulence” (FA9550-18-1-0009); and the AFOSR Remote Sensing and Imaging Physics Program (FA9550-17-C-0021). S. M. de B. K.’s and A. M.’s contributions were partially supported by the U.S. Department of Defense (DOD) High Performance Computing Modernization Program (HPCMP) Frontier Program (FP-CFD-FY14-007).

References

1. J. M. Martin and S. M. Flatté, “Intensity images and statistics from numerical simulation of wave propagation in 3-D random media,” *Appl. Opt.* **27**, 2111–2126 (1988).
2. J. D. Schmidt, *Numerical simulation of optical wave propagation* (SPIE Press, Bellingham, Washington, 2010).
3. A. M. Obukhov, “The structure of the temperature field in a turbulent flow,” *Izv. Akad Nauk SSSR, Ser. Geogr. i Geofiz.* **13**, 58–69 (1949).
4. S. Corrsin, “On the spectrum of isotropic temperature fluctuations in an isotropic turbulence,” *J. Appl. Phys.* **22**, 469–473 (1951).
5. A. Muschinski, “Temperature variance dissipation equation and its relevance for optical turbulence modeling,” *J. Opt. Soc. Am. A* **32**, 2195–2200 (2015).
6. J. W. Goodman, *Introduction to Fourier optics* (McGraw-Hill, Boston, Massachusetts, 1996), 3rd ed.
7. V. I. Tatarskii, *Wave propagation in a turbulent medium* (McGraw-Hill, New York, 1961).
8. S. M. de Bruyn Kops, “Classical turbulence scaling and intermittency in strongly stratified Boussinesq turbulence,” *J. Fluid Mech.* **775**, 436–463 (2015).
9. A. Muschinski and S. M. de Bruyn Kops, “Investigation of Hill’s optical turbulence model by means of direct numerical simulation,” *J. Opt. Soc. Am. A* **32**, 2423–2430 (2015).

Phase Screens of Optical Turbulence Generated by Means of Direct Numerical Simulation of First Principles of Fluid Mechanics

¹Andreas Muschinski and ²Stephen de Bruyn Kops

¹NorthWest Research Associates (NWRA), Boulder, CO
Aerospace Dept., University of Colorado Boulder, Boulder, CO

²Mechanical and Industrial Eng. Dept., Univ. of Massachusetts, Amherst, MA

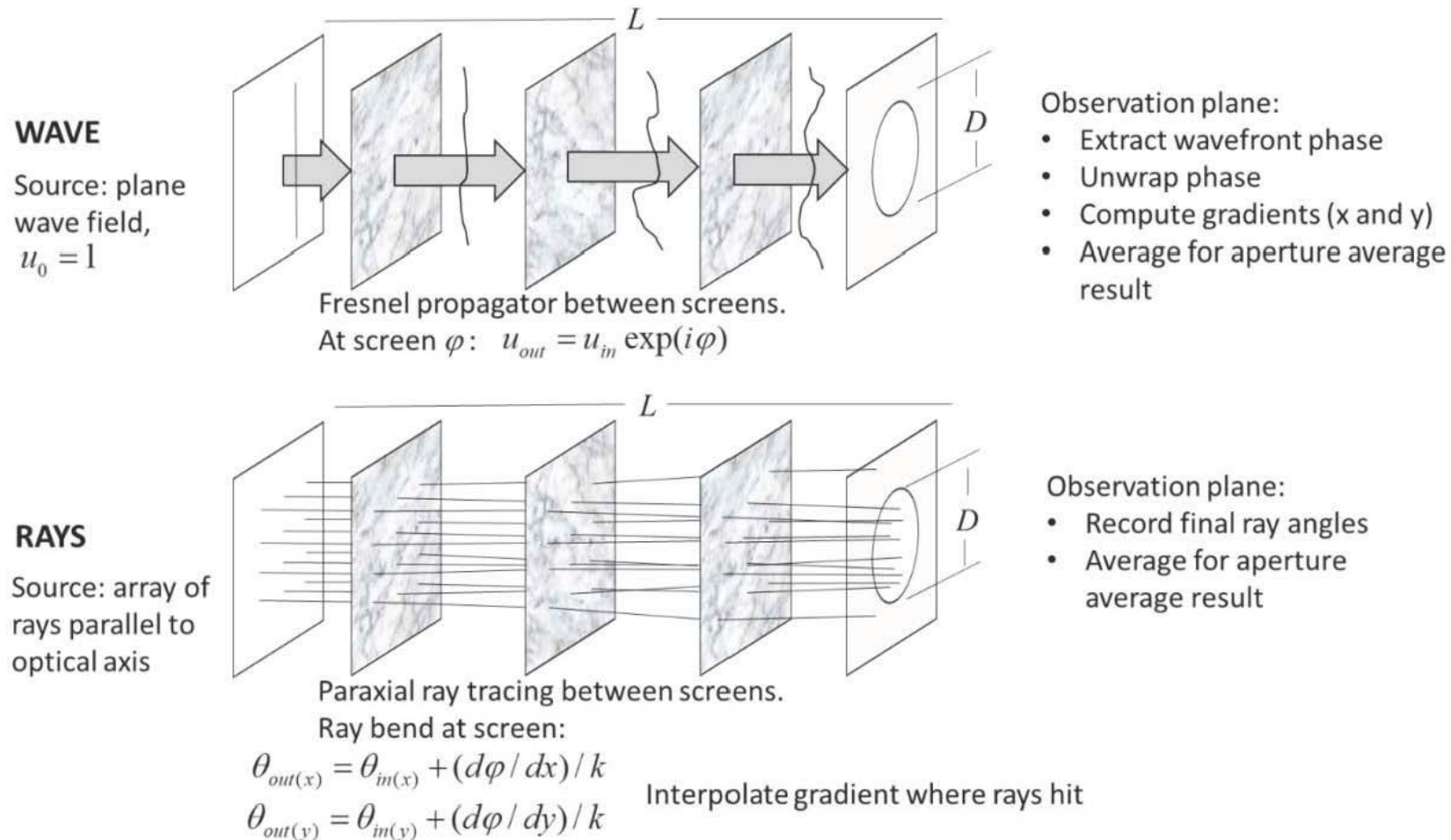
**OSA Imaging and Applied Optics Congress
Wyndham Orlando International Drive
Orlando, Florida
June 25-28, 2018**

Overview

- Eikonal fluctuations and phase screens
- Direct numerical simulation of turbulent phase screens
- Wave-number spectra of DNS-generated phase screens
- Histograms of spatial increments of DNS-generated phase screens
- Summary and conclusions

Eikonal fluctuations and phase screens

Split-step Fourier-Fresnel propagator



From: Voelz, Wijerathni, Muschinski and Xiao, 2017: Angle-of-arrival variance of waves and rays in strong atmospheric scattering: split-step simulation results. *SPIE*, **10410**, pp. 104100Y-1ff.

Eikonal fluctuations and phase screens

Eikonal fluctuations

The eikonal is the optical path length, here for propagation in the y -direction:

$$\psi(x, z, L) = \int_0^L n(x, y, z) dy. \quad (1)$$

(Note: In vacuo, $\psi(x, z, L) = L$.)

Eikonal fluctuations:

$$\psi_1(x, z, L) = \int_0^L n(x, y, z) dy - \left\langle \int_0^L n(x, y, z) dy \right\rangle. \quad (2)$$

Therefore,

$$\psi_1(x, z, L) = \int_0^L n_1(x, y, z) dy, \quad (3)$$

where

$$n_1(x, z, y) = n(x, y, z) - \langle n(x, y, z) \rangle. \quad (4)$$

Phase screen

The phase fluctuations are the eikonal fluctuations multiplied by the optical wave number k_0 . Therefore, the turbulent “phase screen” is

$$S_1(x, z, L) = k_0 \int_0^L n_1(x, y, z) dy. \quad (5)$$

Note that, per this definition, the ensemble average of the phase screen is zero:

$$\langle S_1(x, z, L) \rangle = 0. \quad (6)$$

Tatarskii (1961) showed that the 3D spectrum of the refractive-index fluctuations in fully developed, scalar turbulence (Obukhov 1949, Corrsin 1951) is

$$\Phi_n(k) = 0.033 C_n^2 k^{-11/3} \quad (7)$$

and that the corresponding 2D phase-screen spectrum for fully developed, scalar turbulence is

$$F_S(k_\perp) = 2\pi k_0^2 L \Phi_n(k_\perp), \quad (8)$$

where $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ and $k_\perp = \sqrt{k_x^2 + k_y^2}$.

Direct numerical simulation of turbulent phase screens

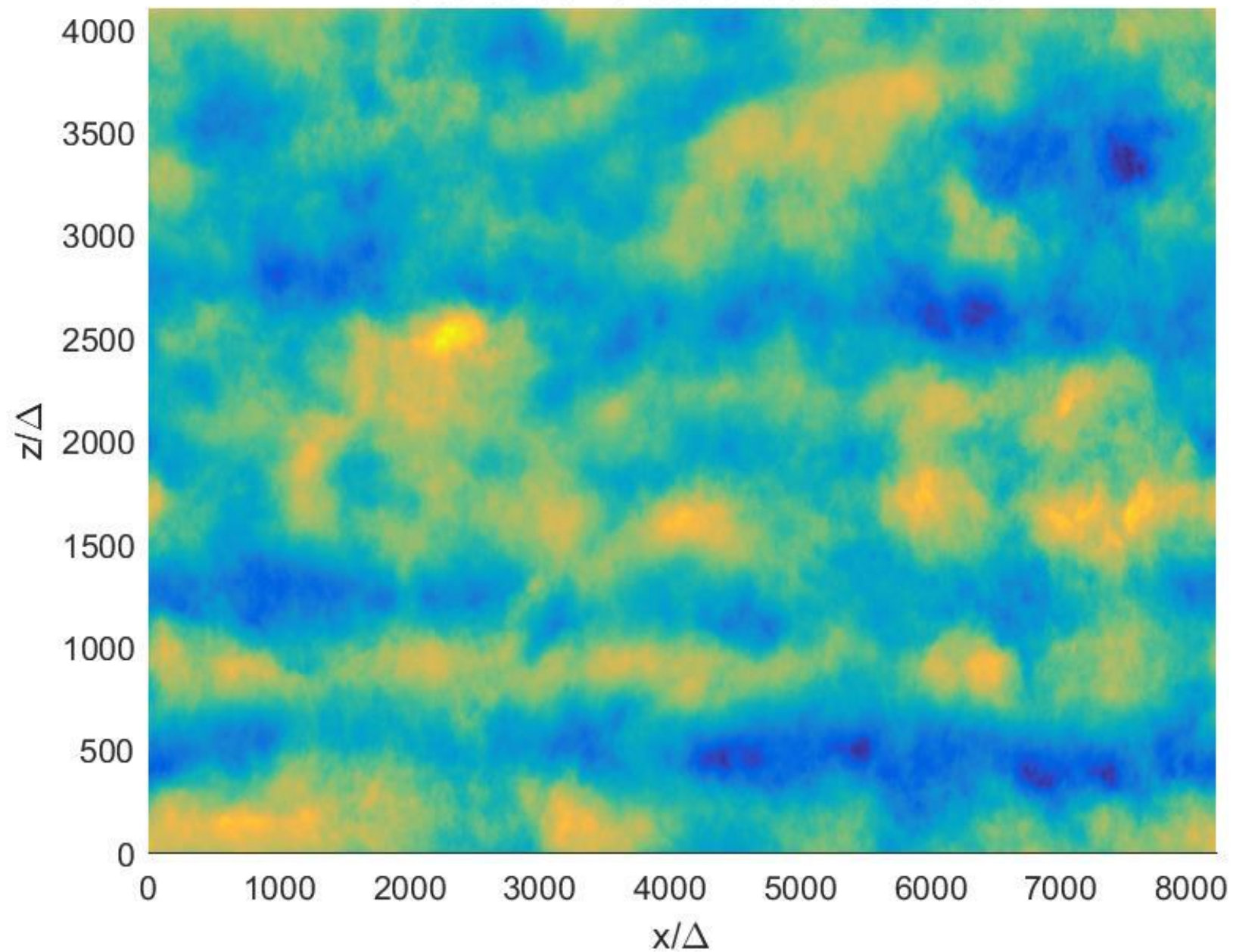
Simulations

We have simulated temporally evolving fields of velocity fluctuations and scalar fluctuations in isotropic and stratified turbulence for Prandtl numbers resembling air ($Pr = 0.7$) and water ($Pr = 7.0$).

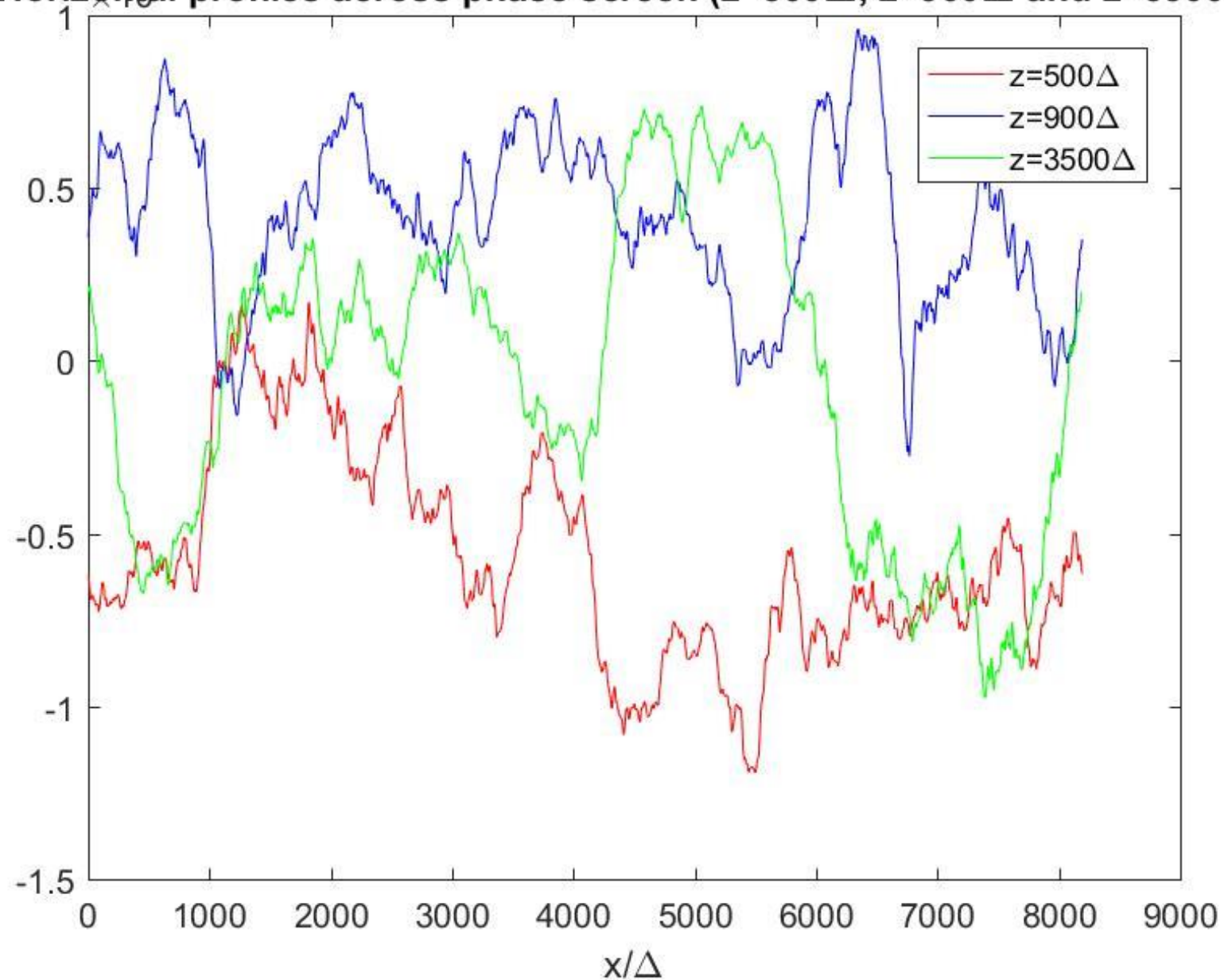
Here, we consider a moderately stable case:

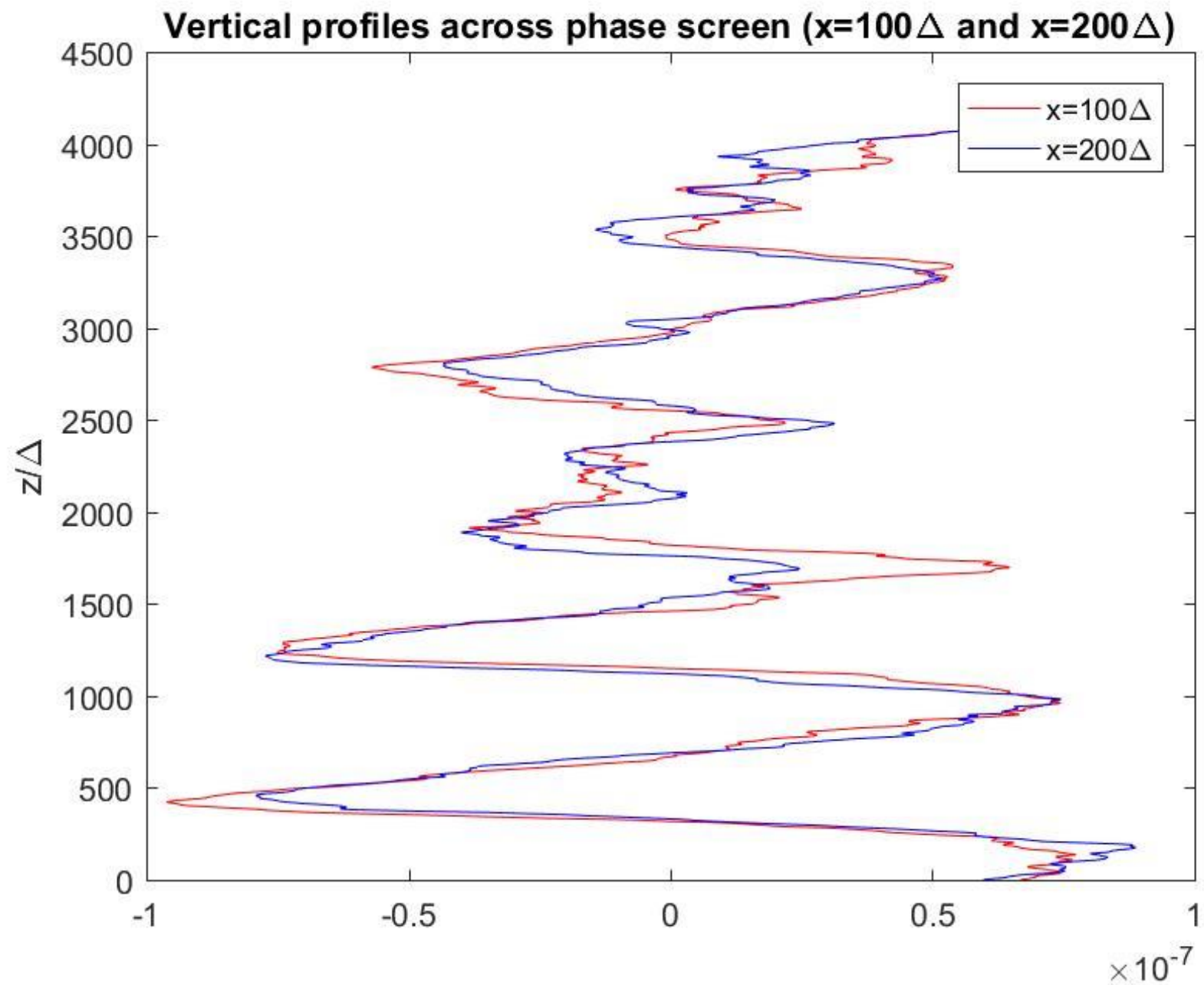
- Number of grid points: $8192 \times 8192 \times 4096$ (274 billion grid points)
- Buoyancy Reynolds number: 200
- Prandtl number: 1.0
- Kolmogorov length: $\eta = 1.5 \Delta$.

Phase screen of stratified turbulence

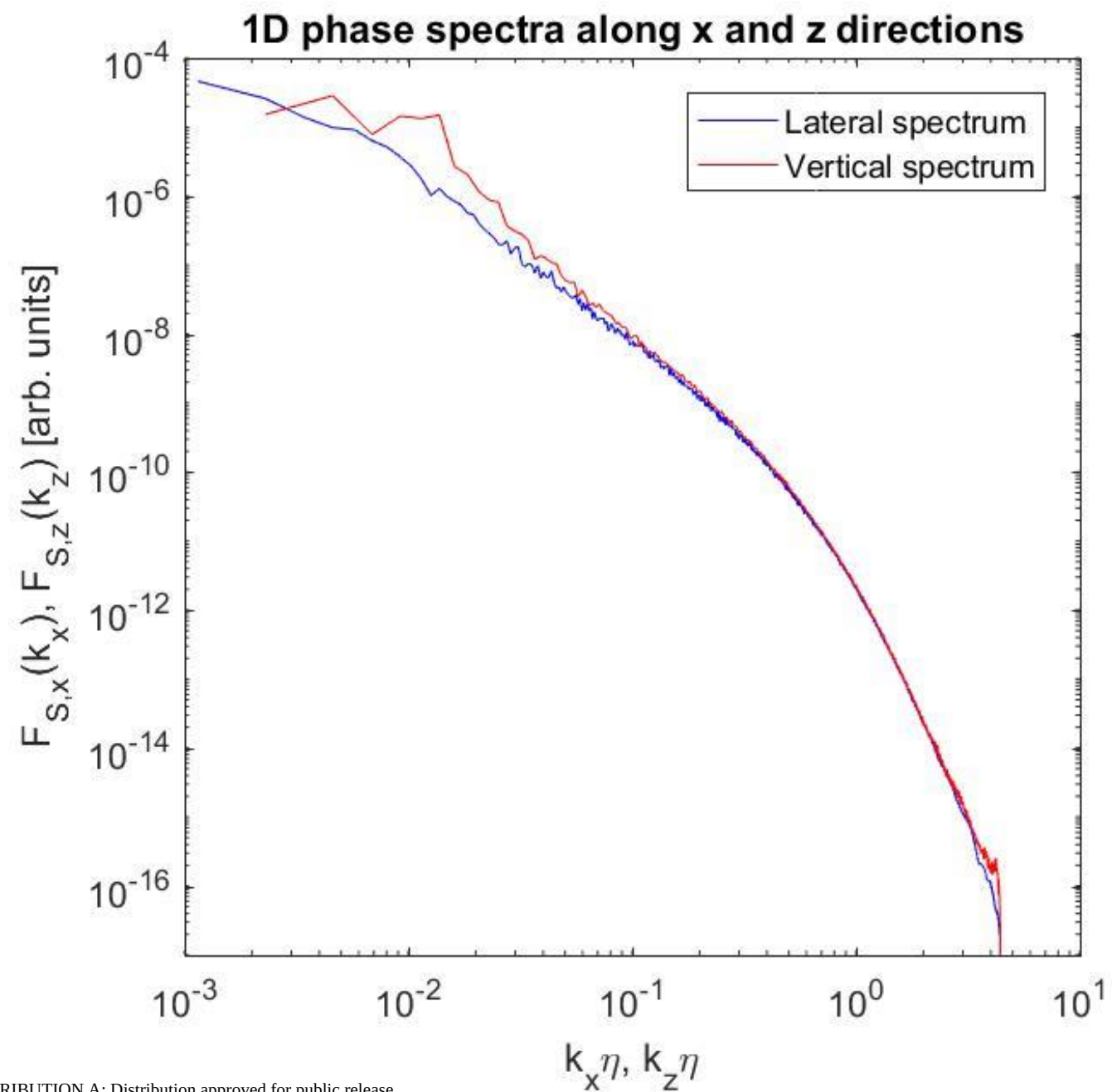


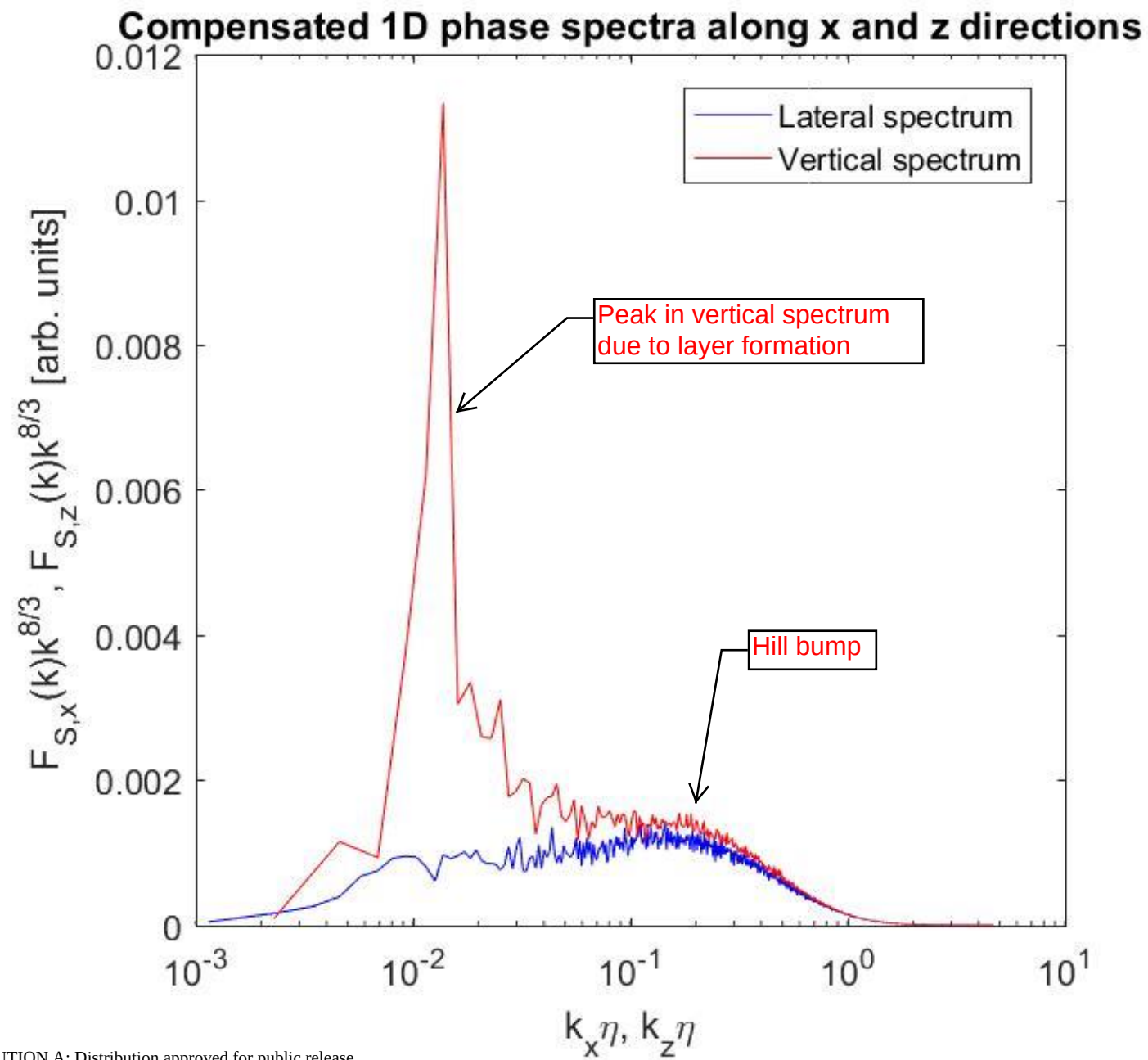
Horizontal profiles across phase screen ($z=500\Delta$, $z=900\Delta$ and $z=3500\Delta$)



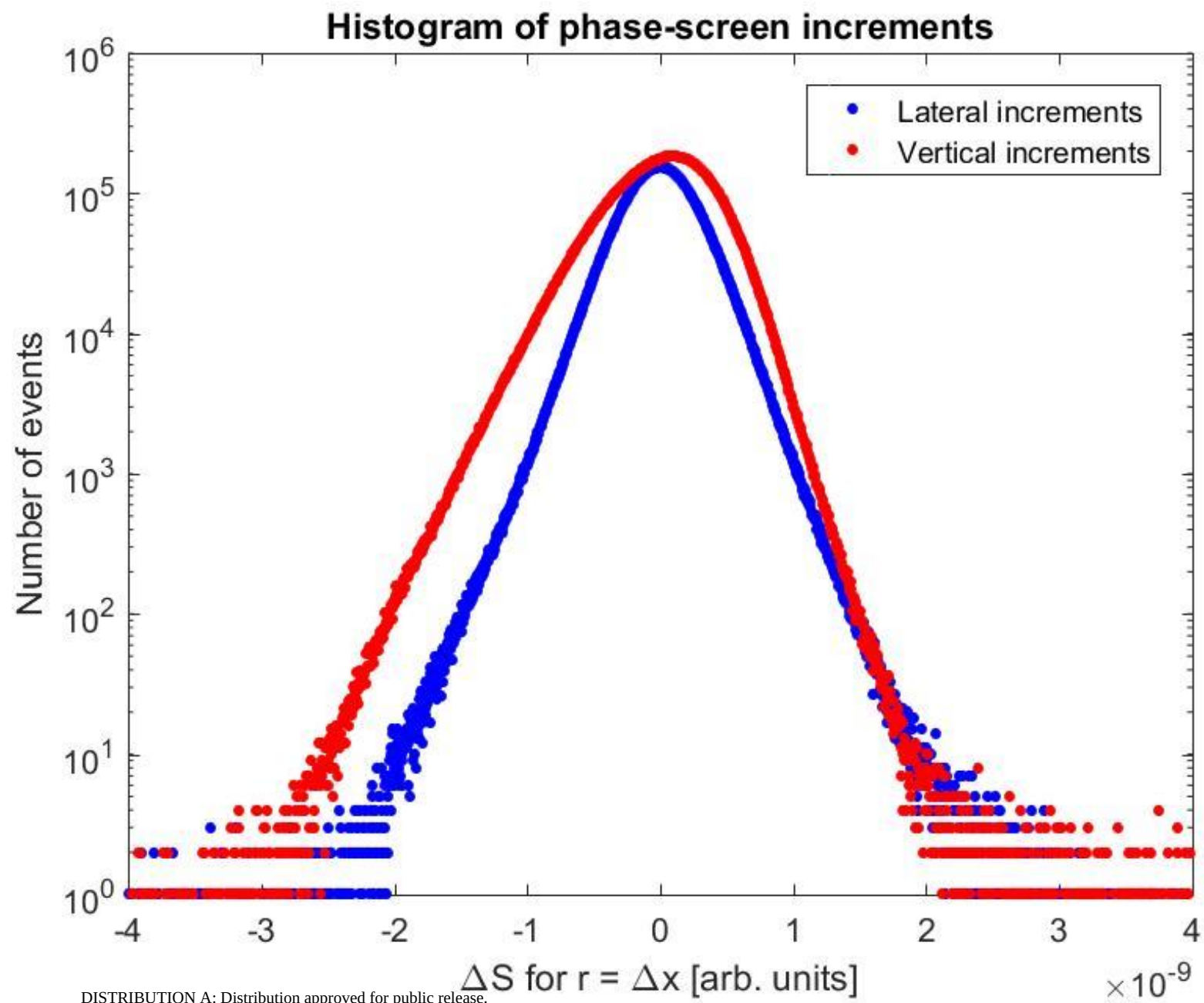


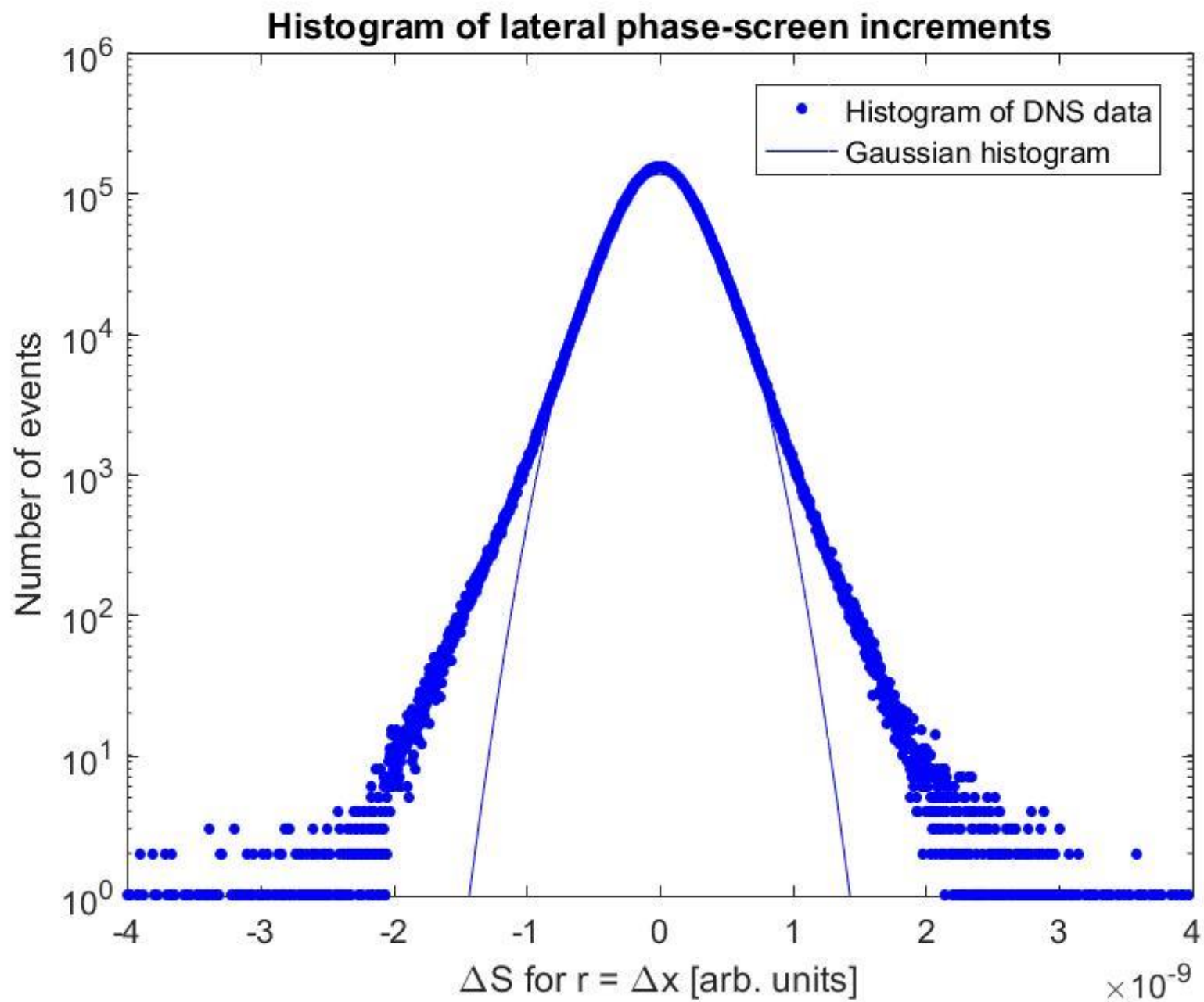
Wave-number spectra of DNS-generated phase screens

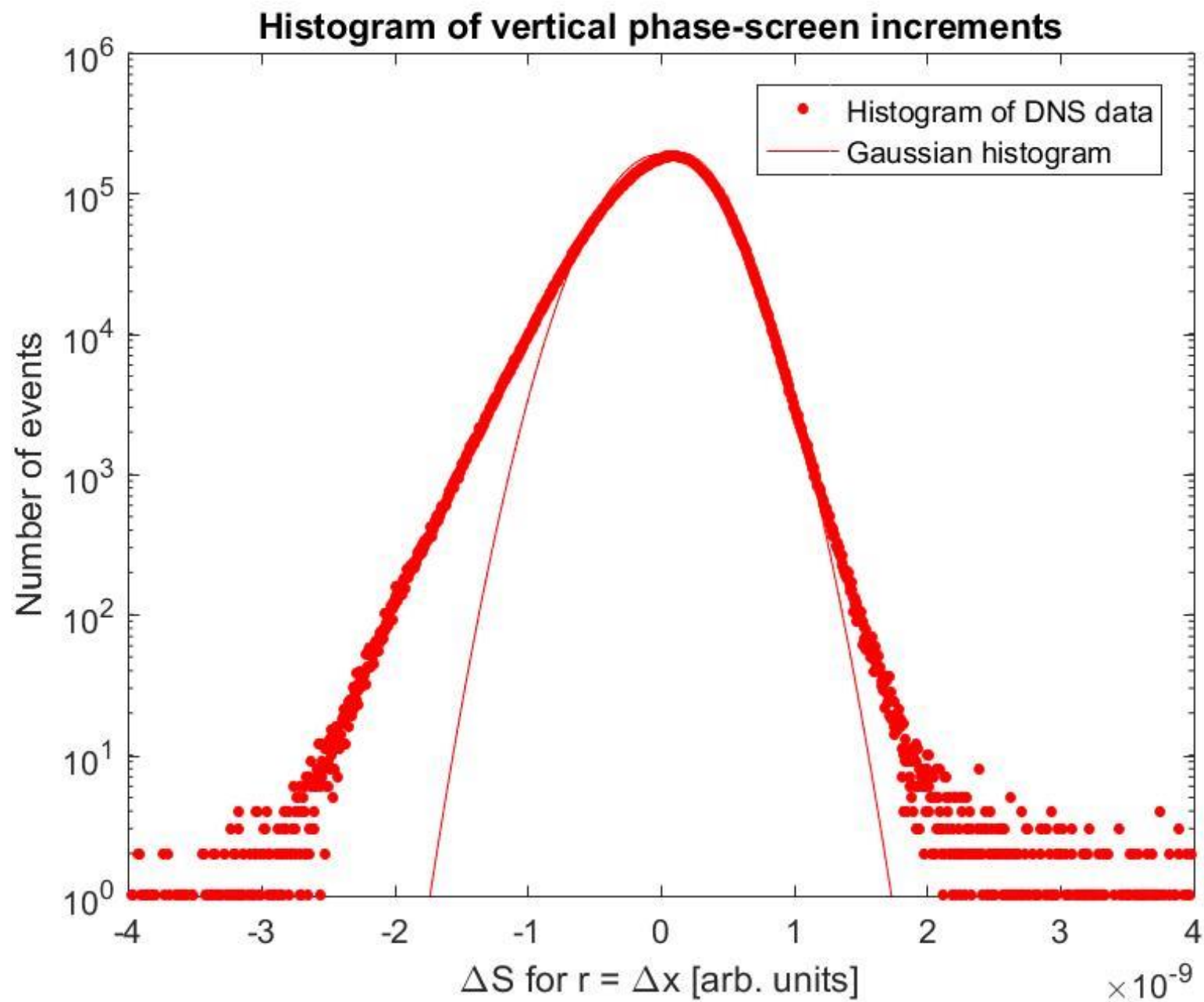




Histograms of spatial increments of DNS-generated phase screens







Summary and conclusions

- DNS can be used to generate turbulent phase screens from first principles of fluid mechanics
- DNS-generated phase screens can be used to develop realistic models of phase screens for stratified turbulence
- Probability densities of spatial increments of DNS-generated phase screens are non-Gaussian
- The non-Gaussianity of DNS-generated phase-screen increments questions the validity of the traditional approach of generating phase screens, which assumes Gaussianity.



Optical timing jitter due to atmospheric turbulence: comparison of frequency comb measurements to predictions from micrometeorological sensors

EMILY D. CALDWELL,^{1,2,*} WILLIAM C. SWANN,² JENNIFER L. ELLIS,² MARTHA I. BODINE,² CARTER MAK,^{1,3} NATHAN KUCZUN,^{1,3} NATHAN R. NEWBURY,² LAURA C. SINCLAIR,² ANDREAS MUSCHINSKI,^{1,3} AND GREGORY B. RIEKER¹

¹University of Colorado, 1111 Engineering Drive, Boulder, CO 80305, USA

²National Institute of Standards and Technology, 325 Broadway, Boulder, CO 80309, USA

³NorthWest Research Associates, 3380 Mitchell Lane, Boulder, Colorado 80301, USA

*emily.hannah@colorado.edu

Abstract: During propagation through atmospheric turbulence, variations in the refractive index of air cause fluctuations in the time-of-flight of laser light. These timing jitter fluctuations are a major noise source for precision laser ranging, optical time transfer, and long-baseline interferometry. While there exist models that estimate the turbulence-induced timing jitter power spectra using parameters obtainable from conventional micrometeorological instruments, a direct and independent comparison of these models to measured timing jitter data has not been done. Here we perform this comparison, measuring turbulence-induced optical pulse timing jitter over a horizontal, near-ground path using frequency comb lasers while independently characterizing the turbulence along the path using a suite of micrometeorological sensors. We compare the power spectra of measured optical pulse timing jitter to predictions based on the measured micrometeorological data and standard turbulence theory. To further quantitatively compare the frequency comb data to the micrometeorological measurements, we extract and compare the refractive index structure parameter, C_n^2 , from both systems and find agreement to within a factor of 5 for wind speed >1 m/s, and further improvement is possible as wind speed increases. These results validate the use of conventional micrometeorological instruments in predicting optical timing jitter statistics over co-located laser beam paths.

© 2020 Optical Society of America under the terms of the [OSA Open Access Publishing Agreement](#)

1. Introduction

Optical turbulence in the atmosphere arises due to the turbulent nature of the air which results in eddies of variable refractive index [1]. As light propagates through the atmosphere, optical turbulence modulates its intensity and phase. Over the past 60 years, the theoretical and experimental attention of the optical turbulence community has focused on understanding intensity fluctuations, often referred to as scintillation, due to their particularly deleterious effects on optical imaging and free-space optical communication [2,3]. In efforts to improve astronomical observation, the field of adaptive optics additionally grew our understanding of atmospheric phase fluctuations in the higher order aberrative modes (tilt, defocus, astigmatism, etc.) [4–6]. In part because it is not important for adaptive optics, the lowest order phase fluctuation, called the piston mode, has experienced comparatively less experimental attention.

The piston mode aberration manifests as phase noise or timing jitter, and is an important noise source in laser ranging [7–9], long-baseline and stellar interferometry [10,11], and free-space optical time transfer [12–18]. While theoretical descriptions of atmospheric phase noise have

existed since the publication of Tatarskii's work in 1961 [19], experimental characterizations of phase fluctuations over wide length scales have proven to be challenging.

Early measurements of turbulent fluctuations of optical phase in the 1970s investigated the differential phase power spectrum and phase structure function by interfering two points from the same continuous-wave laser beam [20–23]. To directly measure phase fluctuations rather than differential phase, the traditional approach has been to use continuous wave interferometry [8,10,24–26]. While these interferometric experiments are highly sensitive, the fringes must be continuously visible to track phase changes over time. Loss of beam power due to scintillation, beam steering, or beam obstruction could therefore cause phase slips, reducing the duration of phase-continuous measurements.

The invention of the optical frequency comb has enabled hours-long measurements that do not suffer from 2π phase ambiguities after full signal loss [27]. Using frequency combs and linear optical sampling to measure the timing jitter of pulses, researchers have measured turbulence-induced phase noise across a 2 km outdoor slant-path, finding order-of-magnitude agreement to theoretical models [13]. However, the phase power spectra showed reduced power laws in the inertial subrange and contained no evidence of the turbulence outer scale. These measurements were conducted over complex terrain including multiple buildings and a steeply sloped hill. This topography coupled with the structures could very well have induced non-homogenous turbulence along the beam path. Furthermore, no micrometeorological sensors were deployed along the propagation path, so the atmospheric parameters in the models could not be independently determined. Using mode-locked lasers and balanced optical cross correlation, other researchers have measured timing jitter due to indoor atmospheric turbulence and found reasonable agreement to the Kolmogorov power law between 100 mHz and 10 Hz [28].

To build on the previous work described above, we measured optical timing jitter due to atmospheric turbulence across a uniform outdoor path while independently characterizing the turbulent conditions using co-located micrometeorological sensors in the form of ultrasonic anemometer-thermometers (hereafter referred to as “sonics”) and a quartz-crystal barometer. We use the data from the micrometeorological sensors to empirically determine the free parameters in three theoretical models for the timing jitter power spectral density and compare these models to the optical timing spectra, finding the best agreement to the Greenwood-Tarazano model [1,29]. We also quantitatively compare the refractive index structure function, C_n^2 , derived from each the optical timing and micrometeorological data, finding agreement at moderate to high wind speeds (>1 m/s). These results show that micrometeorological data can help estimate timing jitter across nearby optical paths with the caveat that lower wind speeds will cause systematic under-prediction of the timing jitter power spectrum at all frequencies.

The structure of this paper is as follows. In Sec. 2 we present the established theory of turbulence-induced optical timing jitter. Next in Sec. 3, we present the experimental setup and in Sec. 4 we present the results from a weeklong measurement campaign. In Sec. 5 we compare these results to theory and present a direct comparison of C_n^2 extracted from the frequency comb and micrometeorological sensor measurements. Finally, in Sec. 6 we conclude with a summary of our findings and consider their implications.

2. Theory

In this section we outline several models for atmospheric timing jitter following the work of Tatarskii [30]. These models will be used with C_n^2 and wind speed input from the micrometeorological instruments for a direct comparison with the timing jitter measured with the frequency combs. In Sec. 2.1 we begin with a discussion of the 3D spatial spectrum of optical turbulence. Then, in Sec. 2.2 we outline the derivation used to calculate the path-integrated temporal turbulence spectrum from the spatial spectrum. Finally, in Sec. 2.3, we present the

models discussed in Sec. 2.1 reformulated as functions of time, not space, along an integrated path. Ultimately, it will be these temporal spectra that we compare directly to the data.

2.1. Turbulence spectra

The refractive index of air is a function of the air temperature, pressure, and moisture content [31]. Turbulent atmospheric mixing creates eddies of varying refractive indices. For isotropic turbulence the statistics of these eddies are described by the spatial power spectrum of the refractive index, $\Phi_n(\kappa)$, where the independent variable κ is the magnitude of the three-dimensional wavenumber $\kappa = (\kappa_1^2 + \kappa_2^2 + \kappa_3^2)^{1/2}$ [rad/m]. The simplest model of this spectrum is the so-called Kolmogorov model given by:

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3}, \quad 1/L_0 < \kappa < 1/l_0, \quad (1)$$

where the bounds are set by the outer and inner scale lengths, L_0 and l_0 respectively. Physically these lengths can be thought of as the largest and smallest turbulent eddy size where energy is injected and dissipated in a turbulent system [1].

To loosen the length scale restrictions on κ , several alternatives to Eq. (1) have been developed that account for the effects of the outer scale with filter functions. One such spectrum is the von Kármán spectrum [32]:

$$\Phi_n(\kappa) = 0.033 C_n^2 [\kappa^2 + (1/L_0)^2]^{-11/6}, \quad 0 < \kappa < 1/l_0. \quad (2)$$

This spectrum rolls off from the $-11/3$ power law to a constant magnitude at frequencies below $\kappa < 1/L_0$. Note we have omitted the exponential term in Eq. (2) that accounts for the inner scale effects to simplify this analysis. The frequency comb system noise floor obscures the turbulence spectrum at the higher Fourier frequencies making measurements of the inner scale inaccessible.

While Eq. (2) presents a convenient analytical form, experimental work over several decades has shown this spectrum often underestimates the energy at low wavenumbers [7,11,29]. This prompted Greenwood *et al.* [29] to propose a new, empirically-derived spectrum based on microtemperature measurements near the ground, called the Greenwood-Tarazano spectrum:

$$\Phi_n(\kappa) = 0.033 C_n^2 [\kappa^2 + (\kappa/L_0)]^{-11/6}, \quad 0 < \kappa < 1/l_0, \quad (3)$$

The Greenwood-Tarazano model exhibits a more gradual rolloff below $\kappa = 1/L_0$ than the von Kármán spectrum. Physically this indicates that length scales larger than the nominal outer scale still influence the timing jitter [1].

It should be noted that there are different conventions in the literature regarding the exact forms of Eqs. (2) and (3), with some sources [1,22,31] using $2\pi/L_0$ in place of $1/L_0$. Here we have chosen the latter convention used in Refs. [13,29,32] to maintain consistency with the Greenwood-Tarazano spectrum originally proposed in [29].

2.2. Conversion from spatial spectra to path integrated temporal spectra

Whereas Eqs. (1), (2), and (3) define the spatial statistics of volumetric turbulence, the timing jitter data generated by our frequency comb system is in time series form and measures timing fluctuations accumulated as the light propagates. We therefore need a temporal power spectrum that accounts for integrated fluctuations through three-dimensional turbulence for a direct comparison.

Tatarskii derives the temporal spectrum of phase fluctuations accumulated during propagation [30, Ch. 4, Sec. 52]. Specifically, Eq. (52.33) from Ref. [30] gives the temporal spectrum as the

integral of the 2D spectrum:

$$S_\phi(f) = 8\pi/V_\perp \int_0^\infty F_\phi(\kappa_\perp, L, k) d\kappa_1, \quad (4)$$

where F_ϕ is the 2D spectrum which is related to the 3D spectrum by a Fourier transform in one dimension [30, pg. 30], $\kappa_\perp = (\kappa_1^2 + \kappa_2^2)^{1/2}$ is the 2D wavenumber perpendicular to the beam path, L is the length of the propagation path, k is the optical wavenumber, and $V_\perp$ is the magnitude of the wind vector perpendicular to the beam path. F_ϕ is given in Ch. 3, Sec. 46, Eq. (37a) in Ref. [30] as

$$F_\phi(\kappa_\perp) = \pi k^2 L g(\kappa_\perp) \Phi_n(\kappa_\perp), \quad (5)$$

where $g(\kappa_\perp, L)$ is a filter function to account for diffraction written as

$$g(\kappa_\perp) = 1 + \frac{k}{\kappa_\perp^2 L} \sin\left(\frac{\kappa_\perp^2 L}{k}\right). \quad (6)$$

At the length scales accessible to our measurements, $(\kappa_\perp^2 L)/k \ll 1$, so we can approximate $g(\kappa_\perp, L) \approx 2$. It is also worth noting that replacing the scalar 3D wavenumber, κ , with the 2D wavenumber, $\kappa_\perp$, in Φ_n , as was done for Eq. (5), is only valid if we assume locally isotropic turbulence.

To convert to a temporal spectrum, we must also apply Taylor's Frozen Turbulence Hypothesis, which posits that turbulent eddies remain unchanged within the time it takes for wind to move them laterally across the beam path [33]. This allows us to equate a single spatial frequency orthogonal to the beam path, to a temporal frequency according to $\kappa_2 = 2\pi f/V_\perp$ where f is a temporal frequency in Hz. Using this assumption as well as the locally isotropic assumption described above we can substitute Eq. (5) into Eq. (4) to get the one-sided spectrum:

$$S_\phi(f) = \frac{16\pi^2 k^2 L}{V_\perp} \int_0^\infty \Phi\left(\sqrt{\kappa_1^2 + (2\pi f/V_\perp)^2}\right) d\kappa_1. \quad (7)$$

2.3. Phase and timing jitter power spectral density

Having shown the steps for switching between spatial and temporal spectra, we now present temporal phase and timing jitter models associated with the Kolmogorov, von Kármán and Greenwood-Tarazano spectra. Substituting Eq. (1) for Φ_n in Eq. (7) we get the Kolmogorov model for the phase power spectral density, equivalent to Ch. 4, Sec. 52, Eq. (35) in [30]:

$$S_\phi(f) = 0.0326 k^2 C_n^2 L f^{-8/3} V_\perp^{5/3}, \quad (8)$$

Similarly, the von Kármán spectrum, Eq. (2), gives

$$S_\phi(f) = 4.384 \frac{k^2 C_n^2 L}{V_\perp} \left(\frac{4\pi^2 f^2}{V_\perp^2} + \frac{1}{L_0^2} \right)^{-4/3}, \quad (9)$$

and the Greenwood-Tarazano spectrum, Eq. (3), gives

$$S_\phi(f) = 5.211 \frac{k^2 C_n^2 L}{V_\perp} \int_0^\infty \left[\frac{4\pi^2 f^2}{V_\perp^2} + \kappa_1^2 + L_0^{-1} \left(\frac{4\pi^2 f^2}{V_\perp^2} + \kappa_1^2 \right)^{1/2} \right]^{-11/6} d\kappa_1. \quad (10)$$

To directly compare to our data, we must convert Eqs. (8), (9), and (10) from phase spectra to timing jitter spectra using a dimensional term:

$$S_{jitter} = \left(\frac{n_g \lambda_0}{2\pi c} \right)^2 S_\phi(f), \quad (11)$$

where n_g is the group refractive index approximated to 1 hereafter, $\lambda_0 = 2\pi/k$ is the central wavelength, and c is the speed of light.

Finally, to account for correlated turbulence along our folded (out-and-back) experimental path (discussed in Sec. 3), we multiply the temporal spectrum by a factor of 4. The factor of 4 comes from treating the timing jitter accumulated over the path to and from the retroreflector as two separate random variables, X and Y , and the total measured timing jitter for the round trip as the sum of the two, $Z = X + Y$. Since $\text{var}(Z) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$ and the turbulence is perfectly correlated over the timescales of the pulse propagation, $\text{cov}(X, Y) = \text{var}(X)$ and $\text{var}(Z) = 4\text{var}(X)$. Variance is the integral of the power spectrum so we can multiply the one-way temporal spectrum by 4 to get the two-way temporal spectrum. For the Kolmogorov spectrum this gives the final form of S_{jitter} :

$$S_{\text{jitter}}(f) = (4)(0.0326)c^{-2}L C_n^2 f^{-8/3} V_{\perp}^{5/3}. \quad (12)$$

Equation (12) is shown in Fig. 1 alongside the equivalent S_{jitter} forms of the von Kármán and Greenwood-Tarazano models. Hereafter we refer to these S_{jitter} models as the Kolmogorov, von Kármán, and Greenwood-Tarazano models respectively.

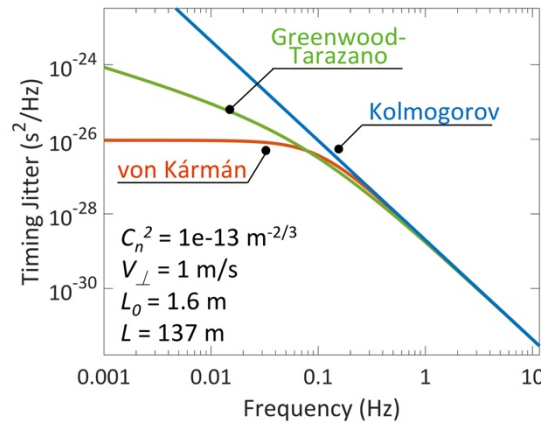


Fig. 1. Comparison of timing jitter power spectral density models, S_{jitter} . Notice the low-frequency variation between S_{jitter} calculated using the Kolmogorov (blue), Greenwood-Tarazano (green) and von Kármán (red) spectra.

3. Experimental setup

The experiment was set up in a field at the Department of Commerce Table Mountain Test Facility 10 miles north of Boulder, Colorado, shown in Fig. 2(a). The field was nearly flat and covered with a mixture of soil, stone, and grass interspersed with low shrubs. The mobile comb laboratory, pictured in Fig. 2(b), housed most of the equipment including the combs and their control electronics. Light launched from an off-axis parabolic mirror just outside the mobile comb lab was directed to a retroreflector 137 m away. To characterize the turbulence with conventional micrometeorological instruments, we set up a meteorological tower approximately 40 meters from the mobile laboratory, about 2 m from the beam path. While the trailer hosting the laboratory surely disrupted the inherent turbulence conditions of the otherwise minimally featured field, the need to minimize and stabilize the optical fiber connecting the combs and free-space launching optics limited the separation between them.

3.1. Frequency comb laser measurements

To measure the atmospheric timing jitter, we used two self-referenced optical frequency comb lasers (“combs”), phase-locked to a field-deployable cavity stabilized laser. The combs used here

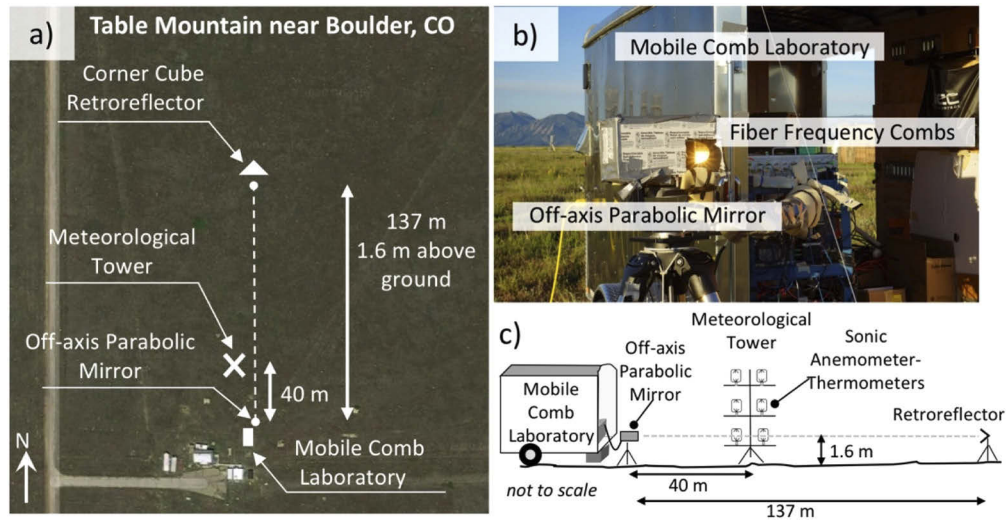


Fig. 2. Experimental setup. a) Aerial diagram of the experiment at Table Mountain showing the orientation of the laser path to the mobile comb laboratory, meteorological tower, and surrounding buildings. b) Photograph showing the mobile comb laboratory housing the fiber frequency combs and the relative location of the off-axis parabolic mirror collimator. c) Side-view diagram of the experiment showing the elements in a).

follow Ref. [34], with pulse repetition rates of approximately 200 MHz. One laser is designated as the signal comb and the other as the local oscillator or “LO” comb. We split the light from the signal comb into two branches, both ultimately bound for heterodyne detection with the LO comb. We sent half the light over a shorted fiber branch and the other half over the atmospheric link that consisted of an off-axis parabolic mirror collimator to launch and receive the light and a retroreflector. We band-pass filtered the light from the signal comb to 14nm centered at 1560nm for both branches. A schematic of this experimental setup is shown in Fig. 3.

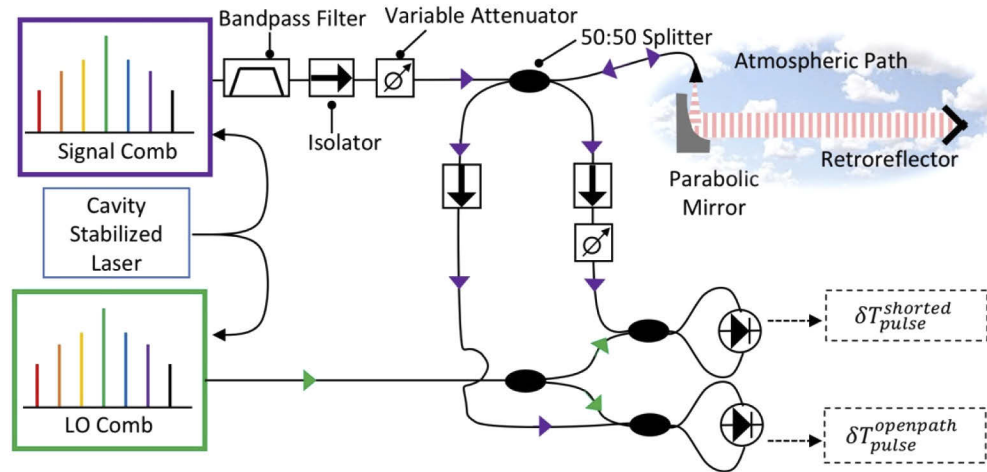


Fig. 3. Schematic of experimental setup to measure atmospheric timing jitter $\delta T_{atmosphere}$. Both the signal and local oscillator frequency combs are phase-locked to the cavity-stabilized laser. Colored arrows indicate the direction light over single-mode PM fiber (black lines). The free-space portion of the setup is shown in striped red and white.

The pulses of the signal comb were detected using linear optical sampling [35,36]. By detuning the repetition rate local oscillator comb ($f_{r,LO}$) to be ~ 2 kHz higher than the repetition rate of the signal comb ($f_{r,S}$), the mixed light from the two combs formed interferogram signals repeating every $1/(f_{r,LO} - f_{r,S}) = 1/(\Delta f_r)$ seconds as the pulses from the two combs stepped through one another. The generated interferogram signals can be interpreted as down-sampled versions of the signal comb pulses. The points comprising the interferograms are measured by balanced detection and the interferogram arrival time (nominally given at the occurrence of the interferogram's peak amplitude) is extracted as described in Ref. [37]. Linear optical sampling allows us to map timing jitter in these interferogram arrival times, δt_{IGM} , to the timing jitter in the arrival of time of the signal pulses, δT_{pulse} , (with respect to the LO comb pulses) according to [13]

$$\delta T_{pulse} = (\Delta f_r / f_{r,S}) \delta t_{IGM}. \quad (13)$$

We simultaneously measure the timing jitter over a fiber-shortened path, denoted $\delta T_{pulse}^{shorted}$, and over atmospheric open path, denoted $\delta T_{pulse}^{openpath}$. Their difference is insensitive to timing jitter from the combs themselves and from most path length variations within the detection system, and yields the atmospheric timing jitter as

$$\delta T_{atmosphere} = \delta T_{pulse}^{openpath} - \delta T_{pulse}^{shorted}. \quad (14)$$

From Fig. 3, it is clear the fiber shortened branch is only partially common to the atmospheric-bound branch, so we measured the entire-system noise floor both before and after the measurement campaign by placing the retroreflector < 1 m from the parabolic mirror and enclosing the air between the two to minimize turbulence. Figure 4 compares the timing jitter measured over the full 280-m optical path to this system noise floor. There is an overall system drift on the order of 0.5 picoseconds over 2.5 hours that is likely due to a combination of the tripods settling and temperature variations seen by the optical fiber despite efforts at insulation. Nevertheless, the noise floor PSD is more than a decade below the typical atmospheric piston noise for Fourier frequencies from 10 Hz to below 10 mHz.

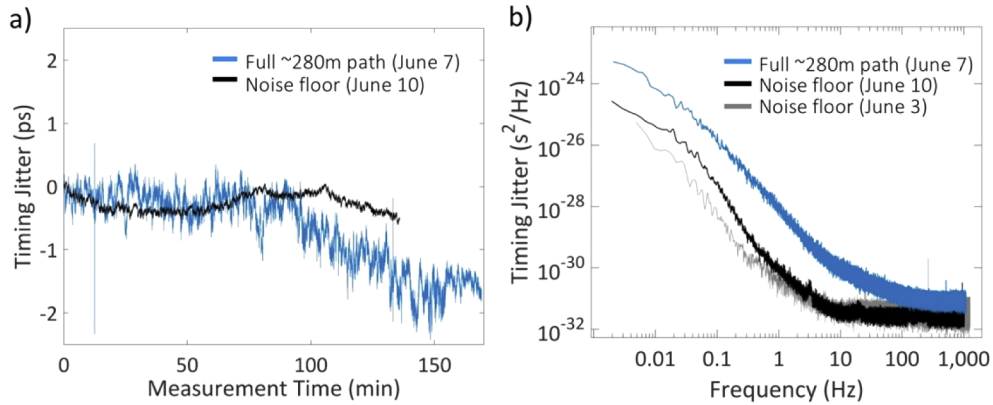


Fig. 4. a) Time series of system noise floor timing jitter (black) shown with atmospheric timing jitter (blue) for scale. b) Corresponding power spectral densities for the data in a) of system noise floor measurements (grey and black) and a typical atmospheric measurement (blue).

We should also mention that a cycle slip in the phase lock of either comb would result in 5 fs jumps in the timing data. While we see no evidence of cycle slips in our data, we cannot verify their absence during moments of signal loss due to scintillation. During these times we rely on the low probability of cycle slips in our system, which has demonstrated continuous slip-free operation over several weeks [27,38].

3.2. Micrometeorological measurements

Four ultrasonic anemometers-thermometers (“sonics”) and one quartz-crystal barometer were deployed approximately 40 m north of the frequency-comb system and 2 m west of the propagation path. The sonics were mounted on a 6 m tall portable meteorological tower. The sonic used to measure the wind vector and determine C_n^2 for this work had the center of its measurement volume located at $z_1 = 1.59$ m, approximately the height of the laser beam path. The setup of the sonics was similar to earlier optical-propagation field studies [39–41]. The barometer was connected to a Quad Disk [42], a device that mitigates wind noise in the measurements of the static pressure.

The sonics provided time series of air temperature and wind velocity at a rate of 32 Hz. According to the manufacturer, the wind measurements have a precision of 1 cm/s and an absolute accuracy of 5 cm/s. The precision of the temperature measurements is 0.01 K, and the absolute accuracy is better than 2 K.

In this paper we consider three wind velocity components: $V_{\parallel}$ the component parallel to the laser beam path, $V_{\perp,H}$, the horizontal component perpendicular to the laser beam path, and $V_{\perp,V}$, the vertical component which is always perpendicular to the laser beam path. The magnitude of the perpendicular wind vector, $V_{\perp}$ introduced in Sec. 2.2, can be written using these components according to $V_{\perp} = (V_{\perp,H}^2 + V_{\perp,V}^2)^{1/2}$.

The barometer was set up to sample at a sampling rate of 10 Hz. According to the manufacturer, its absolute accuracy is 10 Pa (0.01% of the atmospheric pressure). In a study on atmospheric infrasound [43], the precision of the 10 Hz samples were verified to about 1 μ bar (0.1 Pa).

We used the micrometeorological measurements to retrieve the refractive index structure parameter, C_n^2 , through the relationship

$$C_n^2 = \frac{\langle [n(t + \tau) - n(t)]^2 \rangle}{(U\tau)^{2/3}}, \quad (15)$$

where

$$n(t) = 1 + a \frac{\langle P(t) \rangle}{T(t)} \quad (16)$$

is the refractive index of air, P is the static air pressure in units of Pa, T is absolute temperature in K, $U = (\langle V_{\perp,H}^2 \rangle + \langle V_{\parallel}^2 \rangle)^{1/2}$ is the time-averaged magnitude of the mean horizontal wind vector, τ is a time lag, $a = 7.8 \times 10^{-7}$ K/Pa is a constant [44, p. 10], and the angle brackets represent the time average over 1 min. We chose τ such that for each 1 min interval, the spatial separation (by invoking Taylor’s frozen-turbulence hypothesis) was $r = U\tau = 1$ m. This value of r was chosen to be small compared to $2z_1 = 3.18$ m, the diameter of the largest isotropic turbulent eddies centered at z_1 [45], and large compared to the size of the sonic’s sampling volume of about 10 cm. For $r = 1$ m, a typical wind speed of $U = 1$ m/s corresponds to a time lag of $\tau = 1$ s.

4. Results

During the weeklong measurement campaign, we gathered around 4.5 hours of frequency comb timing jitter data, with the longest continuous period being 169 minutes. The optical timing jitter data was taken between 8 am and 12:30 pm MDT on three consecutive days as thunderstorms prevented afternoon measurements. The air temperature, averaged over 20 min, ranged from 8 °C before sunrise on June 1 to 27 °C in the afternoon of June 7. Winds were light and variable, with typical wind speeds ranging between 0 and 2 m/s as shown in Fig. 5.

Figure 6 shows the time series of the turbulence induced timing jitter, $\delta T_{\text{atmosphere}}$. In the longest measurement, taken on June 7th, we detect ~ 2 ps of variation in the time of flight over almost three hours with higher frequency structure in the signal as expected from turbulence.

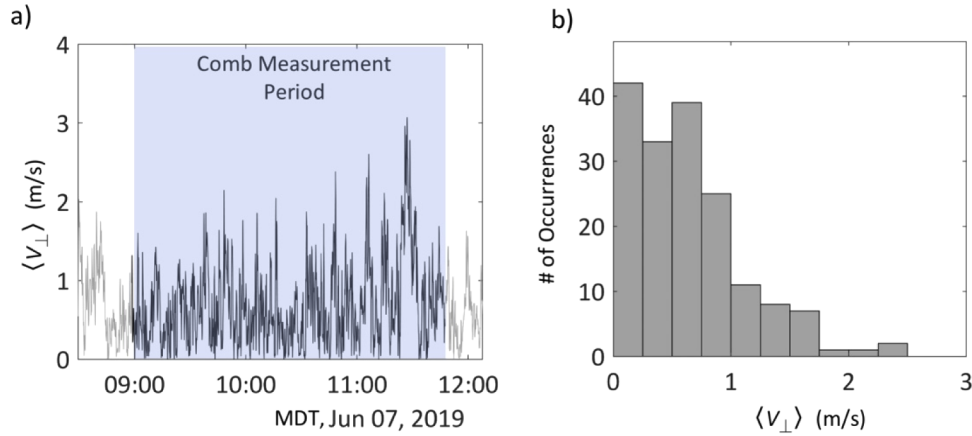


Fig. 5. Wind speeds during the comb measurement period on June 7. a) Time series of 1 minute averages of the absolute value of the perpendicular wind speed vector, $\langle V_{\perp} \rangle$. (b) Histogram of $\langle V_{\perp} \rangle$ for the time series shown in a).

The received optical power varied from below 2 nW (our detection threshold) up to ~ 60 μ W because of scintillation.

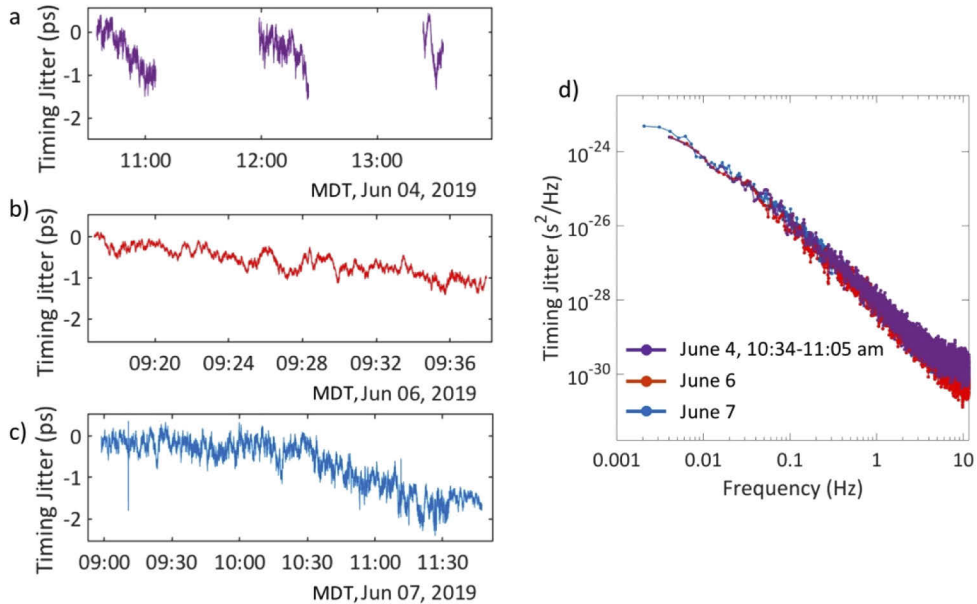


Fig. 6. Time series of $\delta T_{atmosphere}$ measured over the 280 m path. The starting value of each phase-continuous measurement is set to 0 ps. a) Measurements on June 4th 2019. b) Measurements on June 6th 2019. c) Measurements on June 7th 2019.

4.1. Characteristics of timing jitter power spectral density

To compare our results to the theory described in Sec. 2, we estimate power spectral densities (PSDs) from the $\delta T_{atmosphere}$ time series using Welch's method of averaged, modified periodograms

[46] with a Hamming window. A representative timing jitter PSD estimated from Fig. 6(c) is shown in Fig. 7.

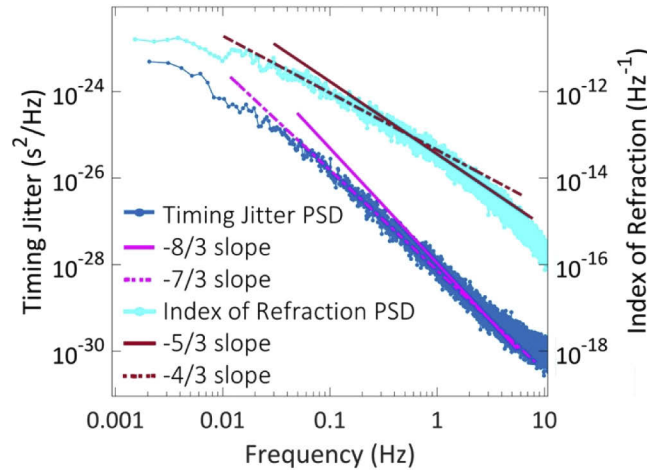


Fig. 7. Power spectral densities of the timing jitter, taken with the frequency comb system, (dark blue) and index of refraction variation, derived from the sonic anemometer-thermometer and barometer data (teal). The time series used to create the spectra were taken concurrently. Also shown are $-8/3$ (solid pink) and $-5/3$ (solid red) power laws predicted by theory in Ref. [30] for the timing jitter and index respectively. Since we see an apparent reduction in the power-law of the timing jitter data we show the $-7/3$ power law (dashed pink) on the timing jitter data along with the similarly reduced $-4/3$ power law (dashed red) on the index data. The curves are vertically offset for clarity and power law lines are a visual aid, not lines of best fit.

For frequencies, f , below 5 Hz, the PSD approximately follows the $-8/3$ power law predicted by theory before rolling off at ~ 0.1 Hz. However, careful examination of the slope in the $0.1 \text{ Hz} < f < 5 \text{ Hz}$ region shows an apparent reduction in the power-law exponent. For the spectrum shown in Fig. 7, the line of best fit is $\propto f^{-6.7/3}$. This is consistent with Ref. [13] and the review of similarly reduced slopes therein. The potential causes of this reduced slope are numerous. Reference [13] points to a breakdown of Taylor's frozen-turbulence hypothesis which might be possible for our measurements due to the low perpendicular wind speeds. For example, during the June 7th run we measured the standard deviation of $V_{\perp}$ divided by its time average to be 1.09, which is much greater than the value 0.33 empirically set as the upper limit for the validity of Taylor's Hypothesis by Ref. [47].

We investigate the power law behavior of the index of refraction PSD where $n(t)$ was calculated from Eq. (16), to see if it shares the same reduced slope as the frequency comb measurements. The same scaling laws that predicts a $-8/3$ power-law for the pulse timing would predict a $-5/3$ power-law for the index of refraction PSD in the inertial subrange [30]. We can see from Fig. 7 that indeed both PSDs show a reduction in their apparent power law at lower frequencies. The index spectrum follows a $-5/3$ power law until ~ 0.5 Hz, where it rolls off to almost $-4/3$. Note that the finite length of the sonics' ultrasonic paths leads to a spatial low-pass filtering and therefore to a frequency spectrum steeper than $-5/3$ at greater than 3 Hz. As described above, the timing jitter data sees a reduced slope over most of its range.

From Fig. 4(a) we know the system noise floor begins to dominate our measurement above 100 Hz. When measuring over the full ~ 280 m path, there is additional noise in the 10 Hz to 100 Hz region that is correlated with times when the return laser beam signal is temporarily lost. The timing jitter measurement is robust against temporary loss of signal, but the loss of signal still

introduces gaps in the timing jitter data ranging from $\sim 500 \mu\text{s}$ to $\sim 3 \text{ s}$ in length. To generate our power spectral densities, we need continuous time series and these gaps must be removed by either concatenation or interpolation of the data. Here, we interpolate and simulations show that this leads to an artificial increase in noise at frequencies above 10 Hz. We also believe some coupling between the angle-of-arrival and piston modes may be an additional source of noise above 5 Hz.

5. Discussion

As described in Sec. 2, there are multiple spectral models of timing jitter with varying low-frequency behavior. In Sec. 5.1 we compare these models to our data, and in Sec. 5.2 we establish a quantitative comparison between the frequency comb and micrometeorological measurements through a comparison of C_n^2 extracted from each system.

5.1. Comparison to existing spectral models

We are interested in the agreement of the measured timing jitter spectra to theoretical curves generated with empirical parameters from the sonics. For the wind parameter we used the average perpendicular wind speed during the measurement period, $\langle V_\perp \rangle$. For the C_n^2 parameter we used a time-averaged C_n^2 value extracted from the micrometeorological data using the direct structure-function approach with a separation of 1 m. The only other free parameter in the models is L_0 which we have set to the beam height above ground (1.6 m) [4,21,22,26].

We first compare to the Kolmogorov model. As shown in Fig. 8(a), this model predicts the timing jitter PSD to within a factor of 10 at 10 Hz dropping to a factor of $\sim 2\times$ between 0.1 Hz and 1 Hz. In Sec. 2 we discussed two theoretical models that account for outer scale effects: the Greenwood-Tarazano model and the von Kármán model. Looking at the low-frequency ($f < 0.1 \text{ Hz}$) region our data in Fig. 8(a), we see the gradual rolloff of the Greenwood-Tarazano better fits the data than the more dramatic von Kármán model. Figure 8(b) shows the effect of varying the outer scale value between 1 m and 10 m for the Greenwood-Tarazano model. These results indicate that use of the von Kármán model could lead to underestimation of timing jitter and timing variance over a turbulent path as described theoretically in Refs. [15–17].

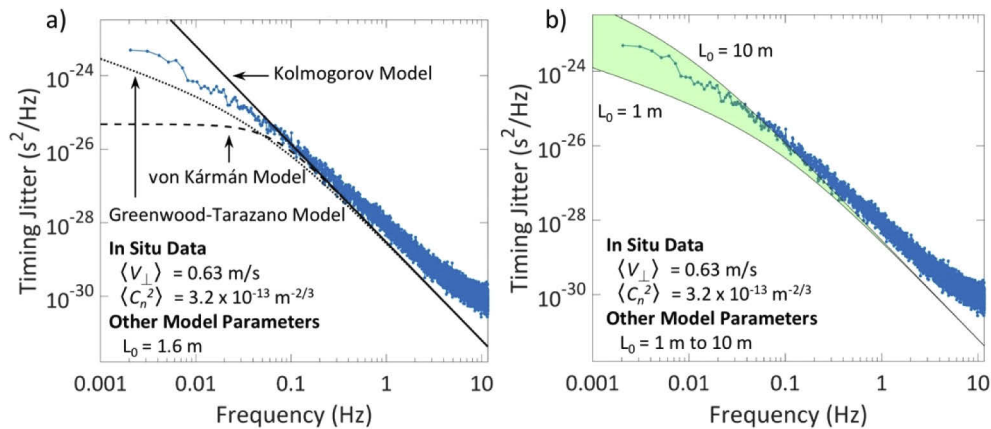


Fig. 8. a) Power spectral density of timing jitter from June 7th with Kolmogorov, von Kármán and Greenwood-Tarazano models using $V_\perp$ and C_n^2 from micrometeorological sensors and L_0 chosen to be height above ground, 1.6 m. b) Same power spectral density showing only the Greenwood-Tarazano model for L_0 ranging from 1 m to 10 m.

5.2. C_n^2 comparison

While we showed qualitative agreement between the models and optical timing jitter data in Sec. 5.1, to quantitatively compare the frequency comb and micrometeorological sensor turbulence measurements over time, it is simplest to consider a single parameter. For this comparison, we extract C_n^2 from both datasets in one-minute time intervals.

To retrieve C_n^2 from the optical timing jitter data, we generate PSDs for each minute of data. We then average the PSD over logarithmically spaced frequency bins. A typical 1-minute PSD (blue line) and corresponding log-spaced average points (black dots) are shown in Fig. 9(a). The PSD is fit to the Kolmogorov model over the frequency range $0.19 \text{ Hz} < f < 5 \text{ Hz}$ with the wind speed fixed from the sonic data and only C_n^2 varied. We chose to fit to the Kolmogorov model since the three models described in Sec. 2 converge for frequencies higher than 0.2 Hz and the Kolmogorov model has the simplest form. The method used to gather C_n^2 from the sonic data is described in Sec. 3.2.

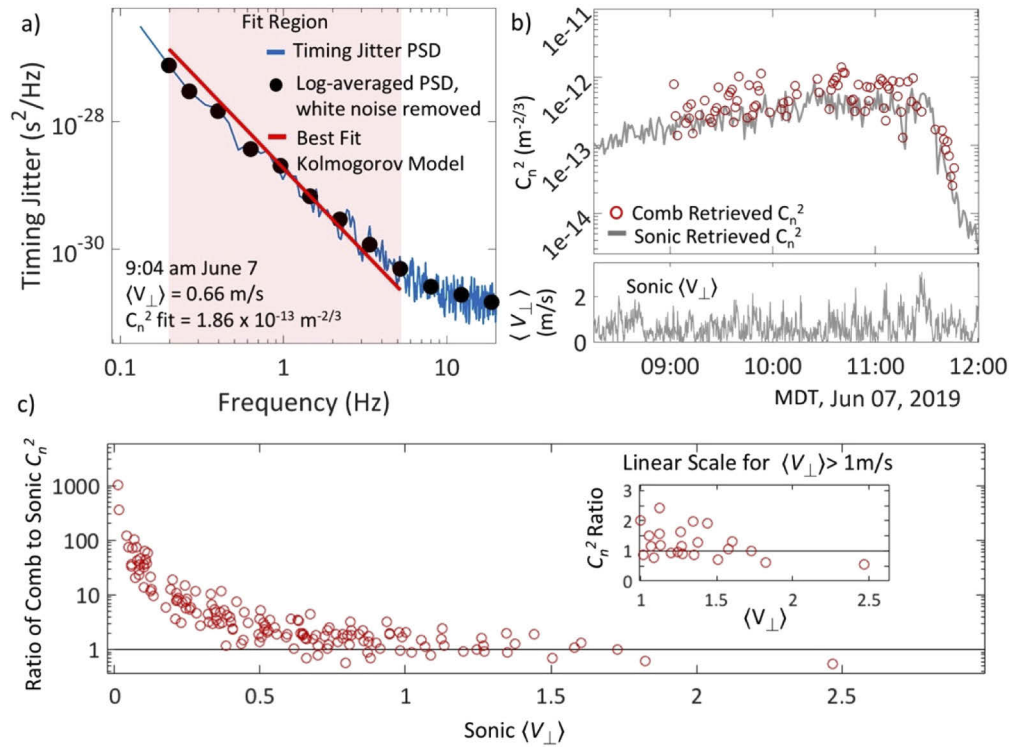


Fig. 9. a) Representative timing jitter power spectral density used to retrieve C_n^2 . The PSD (blue line) is generated from 1-minute long time series. To evenly weight the fitting in logarithmic frequency space we log-average the PSD (black dots). We then use these points in fitting the Kolmogorov model (red line) between $0.19 \text{ Hz} < f < 5 \text{ Hz}$. b) Time series of C_n^2 retrieved from the sonics (gray line) and comb timing jitter data (red circles). Only C_n^2 points for minutes with $\langle V_{\perp} \rangle > 0.5 \text{ m/s}$ are shown for the comb-retrieved C_n^2 . Below the C_n^2 time series is a time series of the 1-minute averaged $\langle V_{\perp} \rangle$ from the sonics. c) Ratio of the comb-retrieved C_n^2 to sonic-retrieved C_n^2 as a function of $\langle V_{\perp} \rangle$.

A time series of the retrieved C_n^2 from both the sonics (grey line) and frequency comb data (red circles) for June 7th is shown in Fig. 9(b). The comb-retrieved C_n^2 values track the slow trends in C_n^2 such as the gradual rise from 9:00am to 11:00am and the rapid decline from 11:00am to 11:45am. It is also clear that the comb-retrieved C_n^2 is systematically larger than the

sonic-retrieved C_n^2 . Plotting the ratio of comb-retrieved C_n^2 to sonic-retrieved C_n^2 as a function of $\langle V_\perp \rangle$ helps illuminate this discrepancy (Fig. 9(c)).

The ratio of comb-retrieved C_n^2 to sonic-retrieved C_n^2 is asymptotically high at $\langle V_\perp \rangle = 0$ m/s, settling into agreement within a factor of ~ 5 for $\langle V_\perp \rangle > 1$ m/s, and with some indication that agreement may continue to improve with higher wind speed. This points to a breakdown of the model's ability to predict timing jitter at low wind speeds (< 1 m/s). It should be noted that the direct structure function approach to calculate C_n^2 also relies on a wind speed term, U . However, looking at the one minute averages during our measurement period on June 7th, U was greater than $\langle V_\perp \rangle$ 98% of the time, with U averaging 1.4 m/s versus 0.7 m/s for $\langle V_\perp \rangle$ over that period. This systematic difference between U and $\langle V_\perp \rangle$ is due to the contribution of $V_\parallel$ to U . The higher values of U would cause the zero-wind-speed asymptotic behavior less frequently in the direct structure function approach than in the optical timing jitter retrieval. There are also interesting questions about the nature of atmospheric turbulence at zero wind speed, for example the breakdown of Taylor's hypothesis that underpins these analyses, but such investigations are beyond the scope of this work.

6. Conclusion

In this paper we presented results of pulse timing jitter over a uniform outdoor path due to atmospheric piston noise. Adjacent to the path we installed a meteorological tower with ultrasonic anemometers and a barometer to simultaneously characterize the turbulence. Consistent with previous measurements using a similar frequency-comb laser system [13], we saw a reduced power law at frequencies corresponding to the inertial range. Unlike these previous measurements, however, we see evidence of an outer scale, and in comparing our results to existing theory, we found the Greenwood-Tarazano model best describes the low frequency behavior of the timing jitter power spectral density. In contrast, we found the von Kármán model underestimates the low-frequency power spectrum. Finally, we compared the refractive index structure parameter, C_n^2 , retrieved from the optical pulse timing noise as measured by the frequency comb system to C_n^2 derived from the micrometeorological instruments and found agreement between the two, which improves with increasing wind speed. These results confirm that conventional micrometeorological sensors can accurately estimate timing jitter over nearby paths for wind speeds > 1 m/s.

Funding

National Science Foundation Graduate Research Fellowship Program (DGE 1650115); National Science Foundation (AGS-1547476); Air Force Office of Scientific Research (FA9550-12-C-007, FA9550-18-1-0009, FA9550-18-C-0011).

Acknowledgments

We thank Jacob Friedlein and Franklyn Quinlan for useful discussions.

Disclosures

The authors declare no conflicts of interest.

References

1. A. D. Wheelon, *Electromagnetic Scintillation: Volume 1*, Geometrical Optics (Cambridge University, 2001).
2. J. W. Strohbehn, "Line-of-sight wave propagation through the turbulent atmosphere," *Proc. IEEE* **56**(8), 1301–1318 (1968).
3. M. A. Khalighi and M. Uysal, "Survey on Free Space Optical Communication: A Communication Theory Perspective," *IEEE Communications Surveys Tutorials* (2014).
4. V. P. Lukin, *Atmospheric Adaptive Optics* (SPIE Press, 1995).

5. J. W. Hardy, *Adaptive Optics for Astronomical Telescopes* (Oxford University, 1998).
6. J. M. Beckers, "Adaptive Optics for Astronomy: Principles, Performance, and Applications," *Annu. Rev. Astron. Astrophys.* **31**(1), 13–62 (1993).
7. C. S. Gardner, "Effects of random path fluctuations on the accuracy of laser ranging systems," *Appl. Opt.* **15**(10), 2539–2545 (1976).
8. K. D. Ridley, "Measurements of laser phase fluctuations induced by atmospheric turbulence over 2 km and 175 km distances," *Appl. Opt.* **50**(26), 5085 (2011).
9. L. Kral, "Optical signal path delay fluctuations caused by atmospheric turbulence," *Opt. Letters* **30**, (n.d.).
10. M. Bester, W. C. Danchi, C. G. Degiacomi, L. J. Greenhill, and C. H. Townes, "Atmospheric fluctuations - Empirical structure functions and projected performance of future instruments," *Astrophys. J.* **392**, 357 (1992).
11. D. F. Buscher, J. T. Armstrong, C. A. Hummel, A. Quirrenbach, D. Mozurkewich, K. J. Johnston, C. S. Denison, M. M. Colavita, and M. Shao, "Interferometric seeing measurements on Mt. Wilson: power spectra and outer scales," *Appl. Opt.* **34**(6), 1081–1096 (1995).
12. K. Djerroud, O. Acef, A. Clairon, P. Lemonde, C. N. Man, E. Samain, and P. Wolf, "Coherent optical link through the turbulent atmosphere," *Opt. Lett.* **35**(9), 1479 (2010).
13. L. C. Sinclair, F. R. Giorgetta, W. C. Swann, E. Baumann, I. Coddington, and N. R. Newbury, "Optical phase noise from atmospheric fluctuations and its impact on optical time-frequency transfer," *Phys. Rev. A* **89**(2), 023805 (2014).
14. L. C. Sinclair, H. Bergeron, W. C. Swann, E. Baumann, J.-D. Deschênes, and N. R. Newbury, "Comparing Optical Oscillators across the Air to Milliradians in Phase and 10^{17} in Frequency," *Phys. Rev. Lett.* **120**(5), 050801 (2018).
15. A. Belmonte, M. T. Taylor, L. Hollberg, and J. M. Kahn, "Effect of atmospheric anisoplanatism on earth-to-satellite time transfer over laser communication links," *Opt. Express* **25**(14), 15676–15686 (2017).
16. W. C. Swann, M. I. Bodine, I. Khader, J.-D. Deschênes, E. Baumann, L. C. Sinclair, and N. R. Newbury, "Measurement of the impact of turbulence anisoplanatism on precision free-space optical time transfer," *Phys. Rev. A* **99**(2), 023855 (2019).
17. M. T. Taylor, A. Belmonte, L. Hollberg, and J. M. Kahn, "Effect of atmospheric turbulence on timing instability for partially reciprocal two-way optical time transfer links," *Phys. Rev. A* **101**(3), 033843 (2020).
18. H. Bergeron, L. C. Sinclair, W. C. Swann, I. Khader, K. C. Cossel, M. Cermak, J.-D. Deschênes, and N. R. Newbury, "Femtosecond time synchronization of optical clocks off of a flying quadcopter," *Nat. Commun.* **10**(1), 1819 (2019).
19. V. I. Tatarskii, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, 1961).
20. M. Bertolotti, M. Carnevale, L. Muzii, and D. Sette, "Interferometric Study of Phase Fluctuations of a Laser Beam Through the Turbulent Atmosphere," *Appl. Opt.* **7**(11), 2246 (1968).
21. S. F. Clifford, G. M. B. Bouricius, G. R. Ochs, and M. H. Ackley, "Phase Variations in Atmospheric Optical Propagation," *J. Opt. Soc. Am.* **61**(10), 1279 (1971).
22. V. P. Lukin and V. V. Pokasov, "Optical wave phase fluctuations," *Appl. Opt.* **20**(1), 121–135 (1981).
23. H. Matsumoto and K. Tsukahara, "Effects of the atmospheric phase fluctuation on long-distance measurement," *Appl. Opt.* **23**(19), 3388–3394 (1984).
24. G. M. B. Bouricius and S. F. Clifford, "Experimental Study of Atmospherically Induced Phase Fluctuations in an Optical Signal," *J. Opt. Soc. Am.* **60**(11), 1484 (1970).
25. G. N. Gibson, J. Heyman, J. Lugten, W. Fitelson, and C. H. Townes, "Optical path length fluctuations in the atmosphere," *Appl. Opt.* **23**(23), 4383–4389 (1984).
26. G. K. Born, R. Bogenberger, K. D. Erben, F. Frank, F. Mohr, and G. Sepp, "Phase-front distortion of laser radiation in a turbulent atmosphere," *Appl. Opt.* **14**(12), 2857–2863 (1975).
27. D. Herman, S. Droste, E. Baumann, J. Roslund, D. Churin, A. Cingoz, J.-D. Deschênes, I. H. Khader, W. C. Swann, C. Nelson, N. R. Newbury, and I. Coddington, "Femtosecond Timekeeping: Slip-Free Clockwork for Optical Timescales," *Phys. Rev. Appl.* **9**(4), 044002 (2018).
28. J. Kang, J. Shin, C. Kim, K. Jung, S. Park, and J. Kim, "Few-femtosecond-resolution characterization and suppression of excess timing jitter and drift in indoor atmospheric frequency comb transfer," *Opt. Express* **22**(21), 26023–26031 (2014).
29. D. P. Greenwood and D. O. Tarazano, "Proposed form for the atmospheric turbulence spatial spectrum at large scales," *J. Opt. Soc. Am. A* **25**(6), 1349 (2008).
30. V. I. Tatarskii, "The effects of the turbulent atmosphere on wave propagation," Jerusalem: Israel Program for Scientific Translations, 1971 (1971).
31. L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media*, 2nd ed. (SPIE, 2005).
32. A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic Press, 1978).
33. G. I. Taylor, "The spectrum of turbulence," *Proc. R. Soc. A* **164**(919), 476–490 (1938).
34. L. C. Sinclair, I. Coddington, W. C. Swann, G. B. Rieker, A. Hati, K. Iwakuni, and N. R. Newbury, "Operation of an optically coherent frequency comb outside the metrology lab," *Opt. Express* **22**(6), 6996–7006 (2014).
35. C. Dorner, D. C. Kilper, H. R. Stuart, G. Raybon, and M. G. Raymer, "Linear optical sampling," *IEEE Photonics Technol. Lett.* **15**(12), 1746–1748 (2003).
36. I. Coddington, W. C. Swann, and N. R. Newbury, "Coherent linear optical sampling at 15 bits of resolution," *Opt. Lett.* **34**(14), 2153 (2009).

37. L. C. Sinclair, H. Bergeron, W. C. Swann, I. Khader, K. C. Cossel, M. Cermak, N. R. Newbury, and J.-D. Deschênes, "Femtosecond optical two-way time-frequency transfer in the presence of motion," *Phys. Rev. A* **99**(2), 023844 (2019).
38. L. C. Sinclair, J.-D. Deschênes, L. Sonderhouse, W. C. Swann, I. H. Khader, E. Baumann, N. R. Newbury, and I. Coddington, "Invited Article: A compact optically coherent fiber frequency comb," *Rev. Sci. Instrum.* **86**(8), 081301 (2015).
39. Y. Cheon, V. Hohreiter, M. Behn, and A. Muschinski, "Angle-of-arrival anemometry by means of a large-aperture Schmidt-Cassegrain telescope equipped with a CCD camera," *J. Opt. Soc. Am. A* **24**(11), 3478–3492 (2007).
40. S. Tichkule and A. Muschinski, "Optical anemometry based on the temporal cross-correlation of angle-of-arrival fluctuations obtained from spatially separated light sources," *Appl. Opt.* **51**(21), 5272–5282 (2012).
41. S. Tichkule and A. Muschinski, "Effects of wind-driven telescope vibrations on measurements of turbulent angle-of-arrival fluctuations," *Appl. Opt.* **53**(21), 4651–4660 (2014).
42. R. T. Nishiyama and A. J. Bedard, "A 'Quad Disc' static pressure probe for measurement in adverse atmospheres: With a comparative review of static pressure probe designs," *Rev. Sci. Instrum.* **62**(9), 2193–2204 (1991).
43. G. K. Subramanian and A. Muschinski, "First Observations of Microbaroms with Single Absolute Barometers," *J. Atmos. Ocean Tech.* **28**(7), 933–943 (2011).
44. S. F. Clifford, "The classical theory of wave propagation in a turbulent medium," in *Laser Beam Propagation in the Atmosphere*, J. W. Strohbehn, ed. (Springer, Berlin Heidelberg, 1978), pp. 9–43.
45. A. Muschinski and R. Roth, "A local interpretation of Heisenberg's transfer theory," *Contr. Atmos. Phys.* **66**, 335–346 (1993).
46. P. Welch, "The use of fast Fourier transform for the estimation of power spectra: A method based on time averaging over short, modified periodograms," *IEEE Trans. Audio Electroacoust.* **15**(2), 70–73 (1967).
47. H. A. Panofsky, H. E. Cramer, and V. R. K. Rao, "The relation between Eulerian time and space spectra," *Q. J. R. Meteorol. Soc.* **84**(361), 270–273 (1958).

Phase-factor spectra of turbulent phase screens

ANDREAS MUSCHINSKI^{1,2,*}

¹NorthWest Research Associates, 3380 Mitchell Lane, Boulder, Colorado 80301, USA

²Ann and H. J. Smead Aerospace Engineering Sciences, University of Colorado Boulder, 3775 Discovery Drive, Boulder, Colorado 80303, USA

*Corresponding author: andreas@nwra.com

Received 29 April 2021; revised 14 July 2021; accepted 21 July 2021; posted 23 July 2021 (Doc. ID 429928); published 20 August 2021

The optical phase ϕ is a key quantity in the physics of light propagating through a turbulent medium. In certain respects, however, the statistics of the phase factor, $\psi = \exp(i\phi)$, are more relevant than the statistics of the phase itself. Here, we present a theoretical analysis of the 2D phase-factor spectrum $F_\psi(\kappa)$ of a random phase screen. We apply the theory to four types of phase screens, each characterized by a power-law phase structure function, $D_\phi(r) = (r/r_c)^\gamma$ (where r_c is the phase coherence length defined by $D_\phi(r_c) = 1 \text{ rad}^2$), and a probability density function $p_\alpha(\alpha)$ of the phase increments for a given spatial lag. We analyze phase screens with turbulent ($\gamma = 5/3$) and quadratic ($\gamma = 2$) phase structure functions and with normally distributed (i.e., Gaussian) versus Laplacian phase increments. We find that there is a pronounced bump in each of the four phase-factor spectra $F_\psi(\kappa)$. The precise location and shape of the bump are different for the four phase-screen types, but in each case it occurs at $\kappa \sim 1/r_c$. The bump is unrelated to the well-known “Hill bump” and is not caused by diffraction effects. It is solely a characteristic of the refractive-index statistics represented by the respective phase screen. We show that the second-order ψ statistics (covariance function, structure function, and spectrum) characterize a random phase screen more completely than the second-order ϕ counterparts. © 2021 Optical Society of America under the terms of the OSA Open Access Publishing Agreement

<https://doi.org/10.1364/JOSAA.429928>

1. INTRODUCTION

The modern physics of optical wave propagation through turbulent media, pioneered by Tatarskii and coworkers in the 1950s and 1960s [1,2], relies on Maxwell’s electromagnetic field theory, the Obukhov–Corrsin theory [3,4] of fully developed scalar turbulence, and the mathematics of stochastic processes and random fields [5]. The starting point of Tatarskii’s propagation theory is the scalar Helmholtz equation. By means of the geometrical-optics method and Rytov’s “method of smooth perturbations,” Tatarskii derived analytical and empirically testable predictions for a wide range of optical observables, such as the spectra and variances of irradiance fluctuations and angle-of-arrival fluctuations. It was clear that the geometrical-optics method cannot account for diffraction effects, and, initially, it was hoped that the Rytov method described the interplay between refraction and diffraction accurately for most cases of practical relevance. After the discovery of the “saturation of scintillation” [6,7], however, the majority of researchers in the field have become aware that the Rytov method is reliable only if the irradiance fluctuations are small compared to the mean irradiance. This regime is usually referred to as the “weak-scattering regime” or “weak-fluctuation regime.” The theoretical and observational progress in the field since the 1970s has been documented in various standard texts and collections (e.g., [8–18]).

An alternative approach to Tatarskii’s methods is to represent turbulent layers as “phase screens” and to solve the problem of optical propagation through random phase screens by means of the Huygens–Fresnel principle. This method was pioneered by Booker, Ratcliffe, and other radio scientists in the late 1940s and 1950s [19,20]. In the 1960s, Goodman [21] showed that the Huygens–Fresnel principle can be approximated as a 2D input–output relationship for an optical wave passing through a linear, shift-invariant optical system, such that the optical output field is the 2D spatial convolution of the optical input field and the optical system’s 2D spatial impulse response function. Goodman’s optical systems approach is the 2D equivalent to the standard theory of linear, time-invariant systems, or “LTI systems” (e.g., [22,23]). Goodman applied this concept first to deterministic optical systems [21,24] and later to random optical systems, such as a slab of turbulent air through which light propagates [25,26]. In 1985, Booker, Ferguson, and Vats [27] analyzed, for the weak-scattering regime, the scintillation index σ_I^2 of a plane wave propagating through classical, fully developed turbulence. They showed that the values of σ_I^2 predicted by the Huygens–Fresnel principle and by the Rytov theory are equal to each other if the phase screen representing the distributed 3D turbulence is placed half-way between the source (input) plane and the observation (output) plane.

In recent decades, the system approach pioneered by Goodman has been widely used in imaging physics [28,29] and computational optics [30–32].

All these methods to analyze and predict optical wave propagation through turbulent media have in common is that it is usually not the phase ϕ that matters but the *phase factor* ψ , where ψ is the complex exponential of the phase, $\psi = \exp(i\phi)$. Nevertheless, most theoretical and observational research has been done on the basis of phase statistics rather than *phase-factor* statistics.

The purpose of this paper is to present and discuss phase-factor statistics for 2D random phase screens. The three main statistics are the phase-factor covariance function, $B_\psi(\mathbf{r})$, the phase-factor structure function, $D_\psi(\mathbf{r})$, and, most importantly, the 2D phase-factor spectrum, $F_\psi(\boldsymbol{\kappa})$. We will show that, in contrast to the phase counterparts $B_\phi(\mathbf{r})$, $D_\phi(\mathbf{r})$, and $F_\phi(\boldsymbol{\kappa})$, the three phase-factor statistics $B_\psi(\mathbf{r})$, $D_\psi(\mathbf{r})$, and $F_\psi(\boldsymbol{\kappa})$ are “well-behaved” at all spatial lags $\mathbf{r}$ and all wave vectors $\boldsymbol{\kappa}$.

The paper is organized as follows. Section 2 develops the three main statistics of the phase factor of a random phase screen: the covariance function, the structure function, and the spatial spectrum. In Section 3, we analyze four specific phase screens, each of which is characterized by a power-law phase structure function and by a probability density function of the phase increments. We consider phase structure functions with power-law exponents $\gamma = 5/3$ or $\gamma = 2$ and phase increments with either Gaussian or Laplacian probability densities. The results are discussed in Section 4. Summary and conclusions are given in Section 5.

2. GENERAL THEORY

A. Applying the Huygens–Fresnel Principle to a Random Phase Screen

Goodman ([21], p. 60) applied the Huygens–Fresnel principle to optical propagation through a phase screen and showed that the scalar, complex, optical field $U(\mathbf{x})$ in a certain observation plane (the “output field”) can be approximated (assuming the Fresnel approximation is valid) as the 2D convolution of an input field $U_0(\mathbf{x})$ and a 2D spatial impulse response function $h(\mathbf{x})$:

$$U(\mathbf{x}) = \iint U_0(\mathbf{x}') h(\mathbf{x} - \mathbf{x}') d\mathbf{x}', \quad (1)$$

where $h(\mathbf{x})$ characterizes the optical system behind the phase screen, such as a slab of turbulent or non-turbulent air or a system consisting of lenses, mirrors, and apertures.

An important special case is an initially unperturbed plane wave propagating through a thin turbulent layer and thereby undergoing phase distortions $\phi(\mathbf{x})$, such that the input field (the optical field immediately after propagating through the turbulent layer) is

$$U_0(\mathbf{x}) = A\psi(\mathbf{x}). \quad (2)$$

Here, A is the complex optical field of the unperturbed wave, and

$$\psi(\mathbf{x}) = \exp[i\phi(\mathbf{x})] \quad (3)$$

is the phase factor, where $\phi(\mathbf{x})$ is the random phase screen representing the phase distortions due to the turbulent layer. Therefore, $U(\mathbf{x})$ is the convolution of the phase factor and of the impulse response function:

$$U(\mathbf{x}) = A \iint \psi(\mathbf{x}') h(\mathbf{x} - \mathbf{x}') d\mathbf{x}'. \quad (4)$$

If $\psi(\mathbf{x})$ is a zero-mean random field, then $U(\mathbf{x})$ is also a zero-mean random field.

If $U(\mathbf{x})$ and $\psi(\mathbf{x})$ are statistically homogeneous (that is, if all statistics of U and of ψ are independent of $\mathbf{x}$), then we can write $U(\mathbf{x})$ and $\psi(\mathbf{x})$ as stochastic Fourier–Stieltjes integrals,

$$U(\mathbf{x}) = \iint \exp(i\boldsymbol{\kappa} \cdot \mathbf{x}) dZ_U(\boldsymbol{\kappa}) \quad (5)$$

and

$$\psi(\mathbf{x}) = \iint \exp(i\boldsymbol{\kappa} \cdot \mathbf{x}) dZ_\psi(\boldsymbol{\kappa}), \quad (6)$$

where the stochastic Fourier amplitudes $dZ_U(\boldsymbol{\kappa})$ and $dZ_\psi(\boldsymbol{\kappa})$ are each statistically orthogonal, such that ([33], p. 19, Eq. 11.53)

$$\langle dZ_U(\boldsymbol{\kappa}) dZ_U(\boldsymbol{\kappa}') \rangle = F_U(\boldsymbol{\kappa}) \delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}') d\boldsymbol{\kappa} d\boldsymbol{\kappa}' \quad (7)$$

and

$$\langle dZ_\psi^*(\boldsymbol{\kappa}) dZ_\psi(\boldsymbol{\kappa}') \rangle = F_\psi(\boldsymbol{\kappa}) \delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}') d\boldsymbol{\kappa} d\boldsymbol{\kappa}', \quad (8)$$

where $F_U(\boldsymbol{\kappa})$ and $F_\psi(\boldsymbol{\kappa})$ are the 2D wavenumber spectra of U and ψ , respectively.

It can be easily shown that

$$dZ_U(\boldsymbol{\kappa}) = AH(\boldsymbol{\kappa}) dZ_\psi(\boldsymbol{\kappa}), \quad (9)$$

where $H(\boldsymbol{\kappa})$ is the 2D transfer function, that is, the 2D Fourier transform of the 2D impulse response function $h(\mathbf{x})$:

$$H(\boldsymbol{\kappa}) = \iint \exp(-i\boldsymbol{\kappa} \cdot \mathbf{x}) h(\mathbf{x}) d\mathbf{x}. \quad (10)$$

Therefore, the 2D power spectrum of U is

$$F_U(\boldsymbol{\kappa}) = |A|^2 F_\psi(\boldsymbol{\kappa}) |H(\boldsymbol{\kappa})|^2. \quad (11)$$

We see that for a given optical system, which is characterized by $H(\boldsymbol{\kappa})$, it is the *phase-factor* spectrum $F_\psi(\boldsymbol{\kappa})$ and not the phase spectrum $F_\phi(\boldsymbol{\kappa})$ that determines the spectral characteristics of U in the general case. In the following, we express $F_\psi(\boldsymbol{\kappa})$ in terms of the phase structure function $D_\phi(\mathbf{r})$ and of the probability density function $p_\alpha(\alpha)$ of the spatial phase increments α .

Before we do so, however, we briefly consider the simplest case of an optical system, namely, a slab of non-turbulent air. Let z be the distance between the input plane (which we assume to coincide with the phase-screen plane) and the observation (output) plane. Then, according to the Fresnel approximation,

$$h(\mathbf{x}) = \frac{\exp(ikz)}{i\lambda z} \exp\left(i \frac{k}{2z} |\mathbf{x}|^2\right) \quad (12)$$

([21], p. 59, Eq. 4-7), where $k = 2\pi/\lambda$ is the optical wavenumber (and λ is the wavelength), the Fourier transform of $h(\mathbf{x})$ is

$$H(\kappa) = \exp(ikz) \exp\left(-i \frac{\lambda z}{4\pi} |\kappa|^2\right) \quad (13)$$

([21], p. 60, Eq. 4-11). This means that, for propagation through the slab of non-turbulent air with thickness z , we have

$$|H(\kappa)|^2 = 1. \quad (14)$$

This somewhat counter-intuitive result leads to

$$F_U(\kappa) = |A|^2 F_\psi(\kappa), \quad (15)$$

which means that Fresnel diffraction along the path of length z in non-turbulent air has no effect on the 2D spectrum $F_U(\kappa)$ of the complex optical field, as has been pointed out by Roddier in his 1981 review paper ([34], p. 294). In other words, $F_U(\kappa)$ is solely determined and equal to (apart from the constant factor $|A|^2$) the phase-factor spectrum $F_\psi(\kappa)$, regardless of the length of the propagation path after passing through the phase screen. Still, $F_\psi(\kappa)$ and $F_U(\kappa)$ depend on the wavelength λ , but λ affects only the phase screen itself, not the propagation through the non-turbulent “slab” between the phase-screen plane and the observation plane at the distance z .

B. Phase-Factor Covariance Function

Consider the covariance function of $\psi(\mathbf{x})$:

$$B_\psi(\mathbf{x}, \mathbf{x}') = \langle \psi^*(\mathbf{x}) \psi(\mathbf{x}') \rangle. \quad (16)$$

If $\psi(\mathbf{x})$ is statistically homogeneous, then $B_\psi(\mathbf{x}, \mathbf{x}')$ depends only on the spatial lag vector

$$\mathbf{r} = \mathbf{x}' - \mathbf{x}, \quad (17)$$

such that

$$B_\psi(\mathbf{r}) = \langle \exp[i\alpha(\mathbf{r})] \rangle, \quad (18)$$

where

$$\alpha(\mathbf{x}, \mathbf{r}) = \phi(\mathbf{x} + \mathbf{r}) - \phi(\mathbf{x}) \quad (19)$$

is the random, zero-mean, spatial phase increment, which is a random function of both $\mathbf{x}$ and $\mathbf{r}$. Throughout this paper, we assume that the phase screen is statistically homogeneous, such that the phase structure function

$$D_\phi(\mathbf{r}) = \langle [\phi(\mathbf{x} + \mathbf{r}) - \phi(\mathbf{x})]^2 \rangle \quad (20)$$

is independent of $\mathbf{x}$. Obviously, the (second-order) phase structure function defined in (20) is the variance of the phase increments, such that

$$D_\phi(\mathbf{r}) = \langle [\alpha(\mathbf{x}, \mathbf{r})]^2 \rangle. \quad (21)$$

Now, let $p_\alpha(\alpha)$ be the probability density of $\alpha(\mathbf{r})$ for a given spatial lag $\mathbf{r}$. Then,

$$B_\psi(\mathbf{r}) = \int_{-\infty}^{\infty} \exp(i\alpha) p_\alpha(\alpha) d\alpha. \quad (22)$$

We see that $B_\psi(\mathbf{r})$ is affected by higher-order statistical moments of the phase increments α , while $D_\phi(\mathbf{r})$ is a purely second-order statistic (the variance) of α . This has profound

implications for the usefulness of $D_\phi(\mathbf{r})$ for the analysis and prediction of higher-order statistics of the complex optical field U , such as the scintillation index or other statistics of irradiance fluctuations.

C. Phase Screens with Gaussian versus Laplacian Phase Increments

It is a standard assumption (e.g., [1,34]) that the turbulent phase increment for a fixed $\mathbf{r}$ is a zero-mean Gaussian random variable, such that

$$p_\alpha(\alpha) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\alpha^2}{2\sigma^2}\right) \quad (23)$$

(e.g., [35], p. 100, Eq. 4-38), where

$$\sigma^2 = D_\phi(\mathbf{r}). \quad (24)$$

After inserting (23) into (22), using Euler's formula, $\exp(i\alpha) = \cos(\alpha) + i \sin(\alpha)$, recognizing that $p_\alpha(\alpha)$ is even, and using the identity

$$\int_0^\infty \exp(-\beta x^2) \cos(bx) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{b^2}{4\beta}\right), \quad (25)$$

where $\text{Re}\beta > 0$ ([36], p. 480, #3.896-4), we obtain the well-known [e.g., ([34], p. 293, Eq. 3.7)] relationship

$$B_\psi(\mathbf{r}) = \exp\left[-\frac{1}{2} D_\phi(\mathbf{r})\right]. \quad (26)$$

While the assumption of Gaussian phase increments is widely accepted, its physical justification relies on the central limit theorem, which is not universally applicable. Therefore, the possibility of phase screens with non-Gaussian phase increments should be taken into account; see, e.g., [37]. Here, we consider, as a counter-example, the two-sided exponential, or Laplacian, probability density function:

$$p_\alpha(\alpha) = \frac{1}{\sqrt{2}\sigma} \exp\left(-\frac{\sqrt{2}}{\sigma} |\alpha|\right). \quad (27)$$

Inserting (27) into (22), using Euler's formula, recognizing that $p_\alpha(\alpha)$ is even, and using the identity

$$\int_0^\infty \exp(-px) \cos(qx + \lambda) dx = \frac{1}{p^2 + q^2} (p \cos \lambda - q \sin \lambda), \quad (28)$$

where $p > 0$ ([36], p. 477, #3.893-2) gives

$$B_\psi(\mathbf{r}) = \frac{1}{1 + \frac{1}{2} D_\phi(\mathbf{r})}. \quad (29)$$

That is, $B_\psi(\mathbf{r})$ for Laplacian increments is different from $B_\psi(\mathbf{r})$ for Gaussian increments, even if the phase structure functions are the same.

D. Phase-Factor Structure Function

The second-order structure function of the phase factor is defined as

$$D_\psi(\mathbf{r}) = \langle |\psi(\mathbf{x} + \mathbf{r}) - \psi(\mathbf{x})|^2 \rangle. \quad (30)$$

Expanding the magnitude squared in (30) gives

$$D_\psi(\mathbf{r}) = \langle [\psi(\mathbf{x} + \mathbf{r}) - \psi(\mathbf{x})]^* [\psi(\mathbf{x} + \mathbf{r}) - \psi(\mathbf{x})] \rangle. \quad (31)$$

Using Euler's formula, $\psi = \exp(i\phi) = \cos \phi + i \sin \phi$, yields, after elementary manipulations,

$$D_\psi(\mathbf{r}) = 2 - 2\langle \cos[\phi(\mathbf{x} + \mathbf{r}) - \phi(\mathbf{x})] \rangle. \quad (32)$$

Now, using the Taylor series of the cosine,

$$\cos \alpha = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \alpha^{2n}, \quad (33)$$

we obtain

$$D_\psi(\mathbf{r}) = 2 - 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \langle [\phi(\mathbf{x} + \mathbf{r}) - \phi(\mathbf{x})]^{2n} \rangle. \quad (34)$$

This gives

$$D_\psi(\mathbf{r}) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(2n)!} D_\phi^{(2n)}(\mathbf{r}), \quad (35)$$

where

$$D_\phi^{(m)}(\mathbf{r}) = \langle [\phi(\mathbf{x} + \mathbf{r}) - \phi(\mathbf{x})]^m \rangle \quad (36)$$

is the m th-order phase structure function.

If the separation $|\mathbf{r}|$ is sufficiently small, the first term ($n = 1$) in (35) is the leading term, which gives

$$D_\psi(\mathbf{r}) \approx D_\phi(\mathbf{r}). \quad (37)$$

That is, for small $|\mathbf{r}|$, the second-order phase-factor structure function is approximately equal to the second-order phase structure function. This is to be expected because, for sufficiently small $|\mathbf{r}|$, the phase increments are small compared to 1, such that $\psi = \exp i\phi \approx 1 + i\phi$, and we obtain $D_\psi(\mathbf{r}) \approx D_\phi(\mathbf{r})$ directly. In general, however, all even moments of the phase increments, that is, all even-order phase structure functions $D_\phi^{(2n)}(\mathbf{r})$, contribute to the second-order phase-factor structure function $D_\psi(\mathbf{r})$.

An important, very general relationship between structure function and covariance function of a random function is

$$D_\psi(\mathbf{r}) = 2[B_\psi(\mathbf{0}) - B_\psi(\mathbf{r})] \quad (38)$$

(e.g., [1], p. 10, Eq. 1-14).

Equation (38) leads to specific expressions for $D_\psi(\mathbf{r})$ in terms of $D_\phi(\mathbf{r})$, depending on the respective probability density function of the phase increments. For Gaussian phase increments, inserting (26) into (38) gives

$$D_\psi(\mathbf{r}) = 2 - 2 \exp \left[-\frac{1}{2} D_\phi(\mathbf{r}) \right]. \quad (39)$$

For Laplacian phase increments, inserting (29) into (38) gives

$$D_\psi(\mathbf{r}) = 2 - \frac{2}{1 + \frac{1}{2} D_\phi(\mathbf{r})}. \quad (40)$$

For both Gaussian and Laplacian phase increments, we obtain $D_\psi(\mathbf{r}) \approx D_\phi(\mathbf{r})$ for very small $|\mathbf{r}|$ and $D_\psi(\mathbf{r}) \approx 2$ for very large $|\mathbf{r}|$. For intermediate $|\mathbf{r}|$, however, $D_\psi(\mathbf{r})$ depends on the probability density of the phase increments.

E. Phase-Factor Spectrum

According to the Wiener-Khintchine theorem, the 2D phase-factor spectrum $F_\psi(\boldsymbol{\kappa})$ is the Fourier transform of $(2\pi)^{-2} B_\psi(\mathbf{r})$:

$$F_\psi(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^2} \iint \exp(-i\boldsymbol{\kappa} \cdot \mathbf{r}) B_\psi(\mathbf{r}) d\mathbf{r}. \quad (41)$$

If the phase screen is statistically isotropic, $D_\phi(\mathbf{r})$ and $B_\psi(\mathbf{r})$ are functions only of $r = |\mathbf{r}|$. Then, changing to polar coordinates,

$$\boldsymbol{\kappa} \cdot \mathbf{r} = \kappa r \cos \theta, \quad d\mathbf{r} = r dr d\theta, \quad (42)$$

gives

$$F_\psi(\kappa) = \frac{1}{(2\pi)^2} \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \exp(-i\kappa r \cos \theta) B_\psi(r) r dr d\theta. \quad (43)$$

Here, $\kappa = |\boldsymbol{\kappa}|$ is the wavenumber (the magnitude of the 2D wave vector). Integrating over θ by means of the identity

$$\int_{\theta=0}^{2\pi} \exp(-i\kappa r \cos \theta) d\theta = 2\pi J_0(\kappa r), \quad (44)$$

which is a special case of a more general identity given by Watson ([38], p. 20, Eq. 5), gives

$$F_\psi(\kappa) = \frac{1}{2\pi} \int_{r=0}^{\infty} J_0(\kappa r) B_\psi(r) r dr. \quad (45)$$

Now, we know that the variance of ψ is

$$\langle |\psi|^2 \rangle = 1. \quad (46)$$

On the other hand, according to Parseval's theorem, $\langle |\psi|^2 \rangle$ is equal to the integral of $F_\psi(\kappa_x, \kappa_y)$ over the entire κ_x - κ_y plane. Therefore,

$$\iint F_\psi(\boldsymbol{\kappa}) d\boldsymbol{\kappa} = 1. \quad (47)$$

In the case of isotropy in the x - y space, changing to polar coordinates, $d\boldsymbol{\kappa} = \kappa d\kappa d\theta = 2\pi \kappa d\kappa$, gives

$$\int_0^{\infty} F_\psi(\kappa) \kappa d\kappa = \frac{1}{2\pi}. \quad (48)$$

F. Dimensionless Phase-Factor Spectrum

It is useful to present $F_\psi(\kappa)$ in a dimensionless form. First, we normalize the spatial lag r with the phase coherence length r_c , which we define by the relationship

$$D_\phi(r_c) = 1. \quad (49)$$

That is, we define r_c as the spatial lag for which the standard deviation of the phase increments is equal to 1 rad. Equation (49) defines r_c unambiguously if $D_\phi(r_c)$ increases strictly with

r , which is the case, for example, for power-law functions with exponents γ larger than zero.

Introducing the dimensionless spatial lag

$$x = \frac{r}{r_c} \quad (50)$$

and substituting x for r in (45) gives

$$F_\psi(K) = \frac{r_c^2}{2\pi} f_\psi(K), \quad (51)$$

where

$$f_\psi(K) = \int_0^\infty J_0(Kx) B_\psi(x) x dx \quad (52)$$

is the dimensionless phase-factor spectrum, and where we have introduced the dimensionless wavenumber

$$K = \kappa r_c. \quad (53)$$

Substituting $x' = Kx$ and renaming the new integration variable x' back to x leads to an alternative equation for $f_\psi(K)$:

$$f_\psi(K) = \frac{1}{K^2} \int_0^\infty J_0(x) B_\psi\left(\frac{x}{K}\right) x dx. \quad (54)$$

Later in the paper, we will use (54) for the numerical evaluation of $f_\psi(K)$.

3. ANALYSIS OF FOUR KINDS OF RANDOM PHASE SCREENS

We have shown that, in general, the phase-factor spectrum $F_\psi(\kappa)$ of a random phase screen depends on both $D_\phi(\mathbf{r})$ and $p_\alpha(\alpha)$. In the following, we will analyze four specific random phase screens, each of which is characterized by a power-law phase structure function (with exponents $\gamma = 5/3$ or $\gamma = 2$) and a phase-increment distribution (either Gaussian or Laplacian).

A. Power-Law Phase Structure Functions

A power-law phase structure function has the form

$$D_\phi(r) = \left(\frac{r}{r_c}\right)^\gamma, \quad (55)$$

where the exponent γ is a dimensionless constant. Note that (55) is consistent with (49), the defining equation for r_c for any strictly increasing $D_\phi(r)$.

Turbulent phase screens are characterized by $\gamma = 5/3$, as shown by Tatarskii [1] on the grounds of the Obukhov–Corrsin theory [3,4] of fully developed, scalar turbulence. For mathematical expediency, turbulent phase structure functions are sometimes (e.g., [34]) approximated by quadratic phase structure functions ($\gamma = 2$). It has to be kept in mind, however, that a quadratic $D_\phi(r)$ represents plane, randomly tilted wavefronts, while wavefronts characterizing optical turbulence are not only randomly tilted but also randomly curved. Therefore, certain aspects of optical turbulence cannot be captured by means of quadratic phase structure functions; see Charnotskii's ([39], pp. 714ff.) critical discussion.

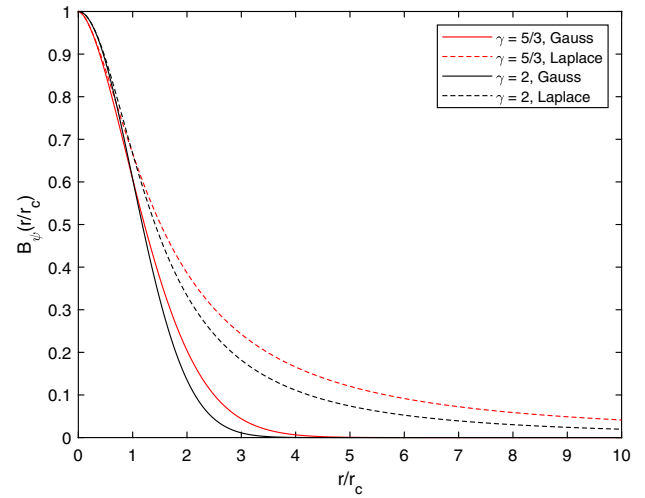


Fig. 1. Covariance function $B_\psi(r/r_c)$ of the phase factor $\psi = \exp(i\phi)$. Red: turbulent phase structure function, $D_\phi(r/r_c) = (r/r_c)^{5/3}$. Black: quadratic phase structure function, $D_\phi(r/r_c) = (r/r_c)^2$. Solid lines: Gaussian phase increments. Dashed lines: Laplacian phase increments.

B. Phase-Factor Covariance Functions

Figure 1 shows the covariance functions $B_\psi(r/r_c)$ of the phase factor ψ for four different phase-screen models: phase screens with an $r^{5/3}$ and r^2 power-law phase structure function, each with either Gaussian- or Laplacian-distributed phase increments. The explicit functional forms of these four $B_\psi(r/r_c)$ models are

$$B_\psi(r/r_c) = \exp\left[-\frac{1}{2}\left(\frac{r}{r_c}\right)^\gamma\right] \quad (56)$$

for Gaussian phase increments, and

$$B_\psi(r/r_c) = \frac{1}{1 + \frac{1}{2}\left(\frac{r}{r_c}\right)^\gamma} \quad (57)$$

for Laplacian phase increments, where $\gamma = 5/3$ for turbulent phase screens, and $\gamma = 2$ for phase screens with quadratic phase structure functions.

All four $B_\psi(r/r_c)$ have in common that $B_\psi(r/r_c) \rightarrow 1$ for $r/r_c \rightarrow 0$ and $B_\psi(r/r_c) \rightarrow 0$ for $r/r_c \rightarrow \infty$. The two $B_\psi(r/r_c)$ for Gaussian phase increments drop more rapidly to zero than the two $B_\psi(r/r_c)$ for Laplacian phase increments.

The nature of the decrease of $B_\psi(r/r_c)$ with increasing r/r_c can be more clearly seen in Fig. 2, where the four $B_\psi(r/r_c)$ are shown in double-logarithmic scaling. The two $B_\psi(r/r_c)$ with Laplacian increments show power-law asymptotes with exponents $-5/3$ and -2 , respectively, consistent with (57). The two $B_\psi(r/r_c)$ with Gaussian increments, however, decrease exponentially, consistent with (56). That is, whether $B_\psi(r/r_c)$ decreases exponentially or follows a power law for $r \gg r_c$ is determined by $p_\alpha(\alpha)$, not by the value of γ .

C. Phase-Factor Structure Functions

Figure 3 shows the phase-factor structure functions $D_\psi(r/r_c)$ for the four phase-screen models. As expected,

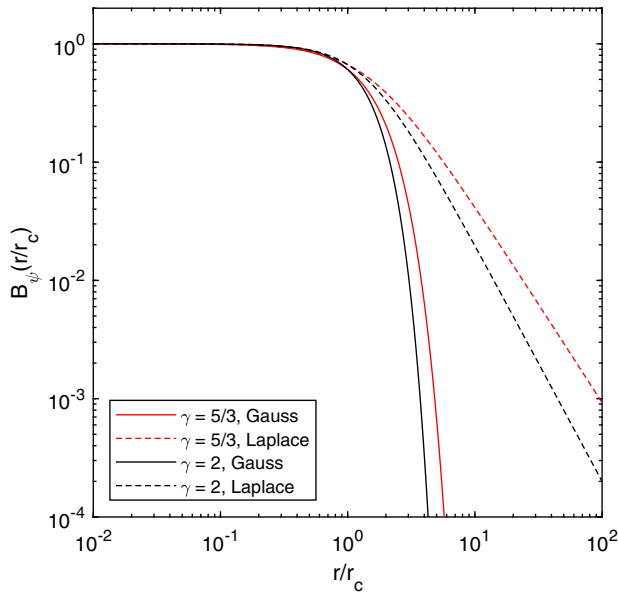


Fig. 2. Same as Fig. 1 but as a log-log plot, emphasizing the power-law behavior of $B_\psi(r/r_c)$ for phase screens with Laplacian increments.

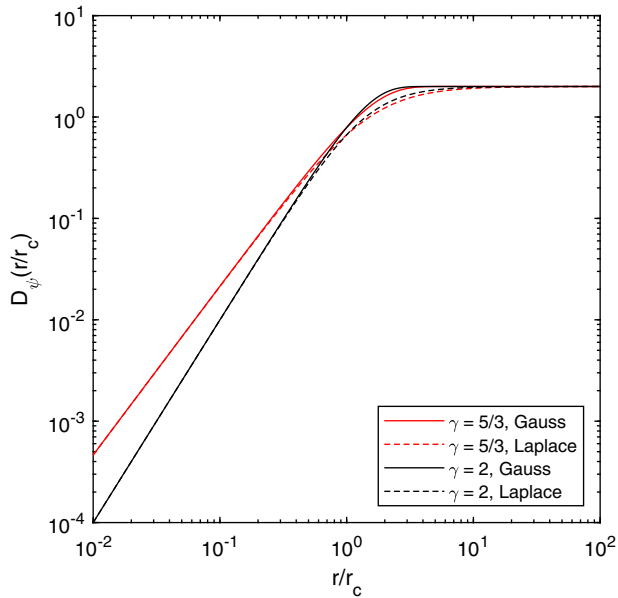


Fig. 3. Phase-factor structure functions $D_\psi(r/r_c)$ for the four types of random phase screens. Line styles and line colors are the same as in Figs. 1 and 2.

$D_\psi(r/r_c) \approx D_\phi(r/r_c)$ for $r/r_c \ll 1$. That is, for small r , $D_\psi(r/r_c)$ follows the same r^γ power law, regardless of whether the phase increments are Gaussian or Laplacian. In all four cases, $D_\psi(r/r_c) \rightarrow 2$ for $r/r_c \rightarrow \infty$, in contrast to $D_\phi(r/r_c)$, which increases without bound for $r/r_c \rightarrow \infty$.

In the intermediate range, where r is comparable to r_c , the four $D_\psi(r/r_c)$ differ significantly from each other.

D. Phase-Factor Spectra

In this section, we evaluate numerically the dimensionless phase-factor spectrum $f_\psi(K)$ introduced in Section 2.F for

the four phase-screen categories that we have specified, and we compare the four spectra with the high-wavenumber asymptote obtained from Tatarskii's classical theory [1].

Inserting (56) and (57) into (54) gives, respectively,

$$f_\psi(K) = \frac{1}{K^2} \int_0^\infty J_0(x) \exp\left[-\frac{1}{2}\left(\frac{x}{K}\right)^\gamma\right] x dx \quad (58)$$

for phase screens with Gaussian increments, and

$$f_\psi(K) = \frac{1}{K^2} \int_0^\infty J_0(x) \left[1 + \frac{1}{2}\left(\frac{x}{K}\right)^\gamma\right]^{-1} x dx \quad (59)$$

for phase screens with Laplacian increments. As before, γ is the exponent of the underlying power-law phase structure function of the form given in (55).

We have numerically evaluated $f_\psi(K)$ given in (58) and (59) for $\gamma = 5/3$ and for $\gamma = 2$ by means of the composite Simpson rule ([40], p. 96, Eq. 3.6.4), and the results are shown in Fig. 4.

Additionally, we show in Fig. 4 (blue line) the high-wavenumber $K^{-11/3}$ asymptote that we have retrieved independently from Tatarskii's classical theory [1] as follows. Starting with our definition of the dimensionless phase spectrum,

$$f_\phi(K) = \frac{2\pi}{r_c^2} F_\phi(\kappa), \quad (60)$$

we use Tatarskii's classical models for the 2D phase spectrum ([1], p. 101, Eq. 6.34; p. 25, Eq. 1.58),

$$F_\phi(\kappa) = 2\pi k^2 L \times 0.033 C_n^2 \kappa^{-11/3}, \quad (61)$$

and, for the phase structure function ([1], p. 155, Eq. 7.101),

$$D_\phi(r) = 2.91 k^2 L C_n^2 r^{5/3}, \quad (62)$$

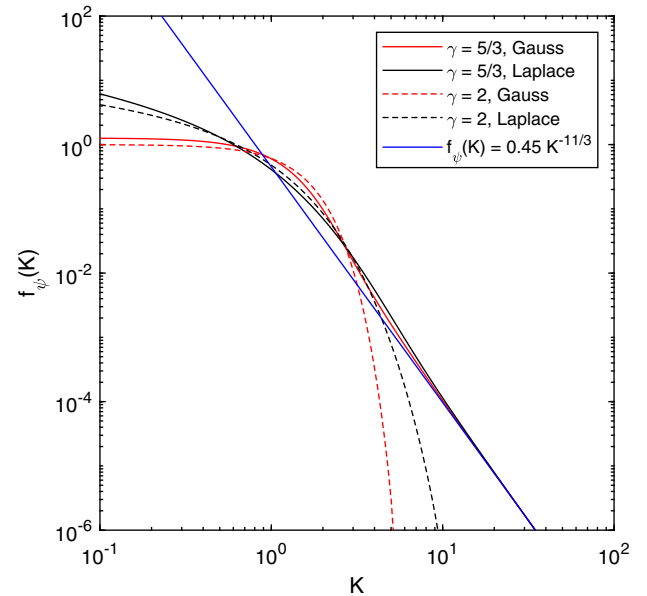


Fig. 4. Dimensionless phase-factor spectra $f_\psi(K)$ for the four types of random phase screens. Here, $K = \kappa r_c$ is a dimensionless wavenumber. Line styles and line colors are the same as in Figs. 1–3. The fifth graph (blue line) is the $K \gg 1$ asymptote of $f_\psi(K)$ that we retrieved from Tatarskii's classical theory [1]; see Section 3.D for details.

where $k = 2\pi/\lambda$ is the optical wavenumber, L is the thickness of the layer assumed to be filled with fully developed, homogeneous and isotropic turbulence, and C_n^2 is the refractive-index structure parameter. We equate (49) and (62) and obtain an expression for the phase coherence length,

$$r_c = (2.91 k^2 L C_n^2)^{-5/3}, \quad (63)$$

and (60) gives

$$f_\phi(K) = 0.45 K^{-11/3}. \quad (64)$$

That is, the parameter $k^2 L C_n^2$ cancels out when we make the transition from $F_\phi(\kappa)$ to its dimensionless counterpart, $f_\phi(K)$, and γ is the only parameter that determines $f_\phi(K)$.

In the discussion that follows, we distinguish between three wavenumber regimes: a low-wavenumber regime, where $K \ll 1$, a high-wavenumber regime, where $K \gg 1$, and a transition regime, where $K \sim 1$.

1. Low-Wavenumber Regime

At low wavenumbers, $K \ll 1$, the spectra for Gaussian phase increments are flat, while the behavior of the spectra for Laplacian phase increments is less clear.

Now, we evaluate $\lim_{K \rightarrow 0} f_\psi(K)$ analytically for Gaussian phase increments. Because $B_\psi(x)$ decreases exponentially for $x > 1$ and drops below 10^{-4} for $x > 6$ (see Fig. 2) and because $|J_0(Kx)| \leq 1$ for all x , we know that only x values small compared to 6 can significantly contribute to the integral in (52). This means that we can safely approximate $J_0(Kx) = 1$ if K is small compared to 6. Then, (52) gives

$$\lim_{K \rightarrow 0} f_\psi(K) = \int_0^\infty B_\psi(x) x dx. \quad (65)$$

Note that this reasoning is valid only for the case of Gaussian phase increments. For Laplacian increments, $B_\psi(x) \approx 2x^{-\gamma}$ for $x \gg 1$, such that $B_\psi(x)x \approx 2x^{1-\gamma}$ for $x \gg 1$, and the integral in (65), which relies on the approximation $J_0(Kx) \approx 1$, diverges unless $\gamma > 2$. That is, the approximation of (65) is not applicable for Laplacian phase screens with $\gamma = 5/3$ or $\gamma = 2$.

For Gaussian phase increments, inserting (56) into (65) gives

$$\lim_{K \rightarrow 0} f_\psi(K) = \int_0^\infty \exp\left(-\frac{x^\gamma}{2}\right) x dx. \quad (66)$$

This integral can be solved by means of the identity

$$\int_0^\infty x^{v-1} \exp(-\mu x^p) dx = \frac{1}{|p|} v^{-v/p} \Gamma\left(\frac{v}{p}\right), \quad (67)$$

where $\text{Re}\mu > 0$ and $\text{Re}v > 0$ ([36], p. 342, #3.478-1). In our case, $v = 2$, $\mu = 1/2$, and $p = \gamma$, which gives

$$\lim_{K \rightarrow 0} f_\psi(K) = \frac{1}{\gamma} 4^{1/\gamma} \Gamma\left(\frac{2}{\gamma}\right), \quad (68)$$

such that $\lim_{K \rightarrow 0} f_\psi(K) = 1$ for $\gamma = 2$ and $\lim_{K \rightarrow 0} f_\psi(K) = (3/5)2^{6/5} \Gamma(6/5) = 1.2656$ for $\gamma = 5/3$. These values are consistent with Fig. 4, substantiating the integrity of our numerical results.

2. High-Wavenumber Regime

Figure 4 shows clearly that for $K \gg 1$ our numerical $f_\psi(K)$ results for $\gamma = 5/3$ converge to the $K^{-11/3}$ power-law phase spectrum retrieved from Tatarskii's classical theory and given in (64), regardless of whether the phase increments are Gaussian or Laplacian distributed. This is consistent with the expectation that the functional form of $p_\alpha(\alpha)$ is irrelevant for $f_\psi(K)$ at high wavenumbers and that $f_\psi(K) \approx f_\phi(K)$ for $K \gg 1$.

For phase screens with quadratic phase structure functions ($\gamma = 2$), the phase-factor spectra drop exponentially at a steeper rate for Gaussian than for Laplacian phase increments.

3. Transition Regime

A remarkable feature of the four $f_\psi(K)$ graphs shown in Fig. 4 is that all of them exceed the Tatarskii asymptote in the transition regime where $K \sim 1$. In order to examine these spectral “bumps” more closely, in Fig. 5, we show the compensated dimensionless phase-factor spectra,

$$g_\psi(K) = f_\psi(K) K^{11/3}, \quad (69)$$

for turbulent phase screens ($\gamma = 5/3$) with Gaussian and Laplacian phase increments. (We do not show the compensated spectra for the Laplacian phase screens because they do not show $K^{-11/3}$ power-law “tails”.) The constant Tatarskii asymptote, $g_\psi(K) = 0.45$, is shown as well.

The bumps in $f_\psi(K)$ shown in Fig. 4 stand out as pronounced maxima of $g_\psi(K)$ in Fig. 5. For Gaussian phase increments, $g_\psi(K)$ reaches a maximum value of 1.26 at $K = 1.8$, and, for Laplacian phase increments, the $g_\psi(K)$ maximum occurs at $K = 2.8$, and its value is 0.98. That is, the Gaussian $f_\psi(K)$ and the Laplacian $f_\psi(K)$ exceed the spectral densities predicted by the Tatarskii asymptote $0.45 K^{-11/3}$ by factors up to 2.8 and 2.2, respectively.

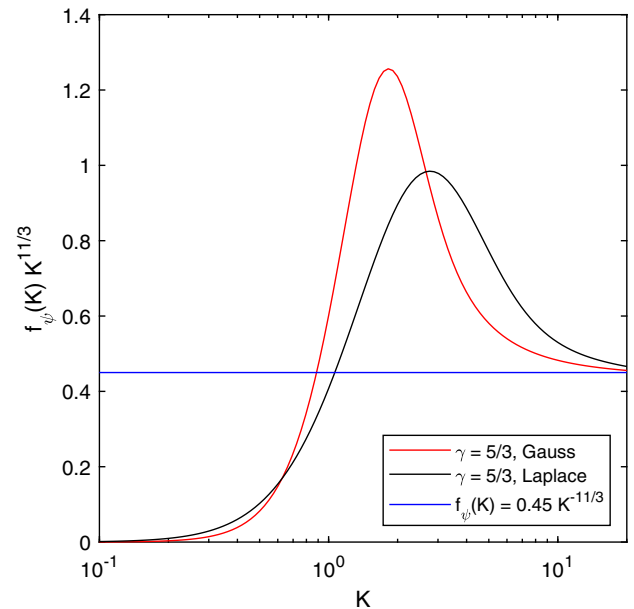


Fig. 5. Same as Fig. 4, except that compensated spectra $f_\psi(K) K^{11/3}$ are shown and that the spectra for quadratic phase structure functions ($\gamma = 2$) are omitted.

4. DISCUSSION

A. Phase-Factor Statistics versus Phase Statistics

Before we discuss the phase-factor spectrum $F_\psi(\kappa)$ in more detail, a few general remarks on phase-factor statistics versus phase statistics are in order.

The analytical treatment of optical turbulence is complicated by the fact that a power-law phase structure function $D_\phi(r)$ increases without bound for an increasing spatial lag r . Therefore, the phase covariance function $B_\phi(r)$ does not exist in this case, and the Wiener–Khinchine relationship, which states that the phase spectrum $F_\phi(\kappa)$ is the Fourier transform of $B_\phi(r)$, is not directly applicable. While it is possible to calculate $F_\phi(\kappa)$ from the structure function, it remains a problem that $F_\phi(\kappa) \rightarrow \infty$ for $\kappa \rightarrow 0$.

In contrast to the phase statistics $D_\phi(r)$, $B_\phi(r)$, and $F_\phi(\kappa)$, the *phase-factor* counterparts $D_\psi(r)$, $B_\psi(r)$, and $F_\psi(\kappa)$ are all “well-behaved” for all values of r and κ , respectively. In particular, $D_\psi(r)$ and $B_\psi(r)$ are finite for $r = 0$ and $r \rightarrow \infty$, and $F_\psi(\kappa)$ is finite for $\kappa = 0$ and $\kappa \rightarrow \infty$.

B. Bump in the Phase-Factor Spectrum

Perhaps the most important result of our analysis is that there is a pronounced bump in the phase-factor spectrum at wavenumbers comparable to the phase-coherence wavenumber

$$\kappa_c = \frac{1}{r_c}. \quad (70)$$

It is to be expected, and we have confirmed numerically, that $F_\psi(\kappa)$ for turbulent phase screens ($\gamma = 5/3$) varies like $\kappa^{-11/3}$ for $\kappa \gg \kappa_c$ and approaches a constant spectral density for $\kappa \ll \kappa_c$, regardless of whether the phase increments are Gaussian or Laplacian. This means that the compensated dimensionless spectrum, $g_\psi(K) = f_\psi(K)K^{11/3}$, increases like $K^{11/3}$ for $K \ll 1$ and approaches a constant value for $K \gg 1$. Our numerical results show, for both Gaussian and Laplacian phase increments, that $\lim_{K \rightarrow \infty} g_\psi(K) = 0.45$, the universal constant that we retrieved analytically from Tatarskii’s classical predictions.

It is somewhat surprising, however, that $g_\psi(K)$ has a pronounced maximum in the transition region, where K is of order 1 (and κ is of order $1/r_c$).

It is not immediately clear what causes this bump in $f_\psi(K)$ and the maximum in $g_\psi(K)$. The bump is certainly not related to the well-known “Hill bump” in the compensated refractive-index spectrum $\Phi_n(\kappa)\kappa^{11/3}$ [41,42] because throughout this paper we have assumed that the phase structure function $D_\phi(r)$ follows a power law, which implies that $\Phi_n(\kappa)$ also follows a power law.

Diffraction effects can also be excluded as a possible cause of the bump in $F_\psi(\kappa)$, simply because all the phase-screen statistics, including $F_\psi(\kappa)$, were derived from relationships in the geometrical-optics limit, i.e., $r \gg \sqrt{L\lambda}$, where L is the thickness of the “slab” of turbulence that the phase screen represents. Moreover, a phase screen is *a priori* solely determined by path-length fluctuations associated with refractive-index fluctuations. In other words, diffraction effects play no role in the $f_\psi(K)$ bumps (Fig. 4), the $g_\psi(K)$ maxima (Fig. 5), or in

any of the other phase-screen statistics that we have discussed in this paper.

C. Phase Coherence Length and Fried Length

We have shown that the phase coherence length for turbulent ($\gamma = 5/3$) phase screens defined in (55) is of the form

$$r_c = (2.91k^2LC_n^2)^{-5/3} \quad (71)$$

if we apply (55) to Tatarskii’s phase structure function model, (62). Our r_c is physically equivalent to the Fried length r_0 , except that Fried ([43], p. 1430, Eq. 6.3) defined his r_0 as

$$D_\phi(r) = 6.88 \left(\frac{r}{r_0} \right)^{5/3}, \quad (72)$$

where, in Fried’s words ([43], p. 1430), “the apparently arbitrary constant 6.88 was chosen on the basis of the analysis of the performance of an optical heterodyne detection system.” In (55), we used the coefficient 1 instead of Fried’s coefficient 6.88. By equating Fried’s $6.88(r/r_0)^{5/3}$ with our $(r/r_c)^{5/3}$, we find that Fried’s length r_0 is $6.88^{3/5} = 3.18$ times larger than our r_c . In a different paper ([44], p. 1376f.), Fried writes “The significance of the factor 6.88, which is more precisely given by $2\{(24/5)\Gamma(6/5)\}^{5/6}$, is contained in the fact that it makes the knee of curve A ... occur at $D = r_0$.” Here, “curve A” is a graph of a normalized telescope resolution as function of the normalized aperture diameter D/r_0 .

It is important to note, as we have emphasized in the previous subsection, that r_c and r_0 are pure geometrical-optics characteristics, which are unaffected by diffraction effects. This is contrary to Fried’s remark ([44], p. 1376, after Eq. 5.1): “The theory, it should be noted, is based on the Rytov approximation to solve the wave equation ...,” which could be misread as suggesting that diffraction effects do play a role in any of the relationships that lead to the equations for r_c or r_0 as stated above.

D. Gaussian versus Laplacian Phase Screens

It is well-established that spatial increments of turbulent temperature fluctuations have “exponential tails” (e.g., [45]) consistent with the Laplace distribution. However, the central limit theorem is usually invoked (e.g., [34]) to argue that, in spite of the Laplacian temperature increments, the phase increments tend to be normally (Gaussian) distributed. This argument relies on the assumption that a large number of statistically independent (or quasi-independent) refractive-index fluctuations of comparable magnitude contribute to the phase fluctuations, and therefore also to the spatial phase increments. This assumption is questionable if the “slab” of turbulent air represented by the phase screen has a thickness that is comparable to, or smaller than, the outer scale of turbulence.

In this paper, we present no empirical evidence that real-world phase screens are non-Gaussian. Rather, we have used the Laplacian probability density as a counter-example and have evaluated the resulting phase-screen statistics for Gaussian versus Laplacian phase screens. We have shown that only in the high-wavenumber regime ($\kappa \gg r_c^{-1}$) are the phase-factor

statistics (covariances, structure functions, and spectra) the same for Gaussian versus Laplacian phase increments.

5. SUMMARY AND CONCLUSIONS

We have analyzed the covariance function $B_\psi(r)$, the structure function $D_\psi(r)$, and the 2D spectrum $F_\psi(\kappa)$ of the phase factor $\psi = \exp(i\phi)$, where ϕ is the phase for four different types of random, 2D phase screens $\phi(\mathbf{x})$. Each of these phase screens is characterized by a power-law phase structure function (with exponents $\gamma = 5/3$ or $\gamma = 2$) and a model for the probability density function of the spatial phase increments (either Gaussian or Laplacian). Our main results are as follows:

1. The phase-factor statistics $B_\psi(r)$, $D_\psi(r)$, and $F_\psi(\kappa)$ are all “well-behaved.” In particular, in contrast to their phase counterparts $B_\phi(r)$, $D_\phi(r)$, and $F_\phi(\kappa)$, they show no singularities at $r = 0$, $r \rightarrow \infty$, $\kappa = 0$, and $\kappa \rightarrow \infty$ if the underlying phase structure function $D_\phi(r)$ follows the power laws $r^{5/3}$ or r^2 .
2. We have defined a phase coherence length, r_c , by requiring $D_\phi(r_c) = 1 \text{ rad}^2$. For $\gamma = 5/3$, our r_c is physically equivalent to Fried’s coherence length r_0 , except for the numerical factor $r_0/r_c = 6.88^{3/5} = 3.18$. We show that both r_c and r_0 are pure phase-screen statistics based on optical path-length (“eikonal”) fluctuations. Both r_c and r_0 are unaffected by any assumptions or approximations regarding the propagation after passing through the phase screen.
3. We have formulated the second-order phase-factor structure function $D_\psi(r)$ as an infinite sum of second- and higher-order (of even order) phase structure functions, $D_\phi^{(2n)}(r)$. For $r \ll r_c$, we find $D_\psi(r) \approx D_\phi(r)$. For $r \gg r_c$, however, $D_\psi(r)$ saturates to the universal value of 2, while $D_\phi(r)$ increases without bound.
4. For turbulent ($\gamma = 5/3$) phase screens, at high wavenumbers ($\kappa \gg r_c^{-1}$), the dimensionless phase-factor spectra $f_\psi(K)$ (where $K = \kappa r_c$ is a dimensionless wavenumber) for Gaussian and Laplacian phase increments converge to the same power law, $f_\psi(K) = 0.45 K^{-11/3}$. The only parameter that affects this result is $\gamma = 5/3$.
5. Independently, we have retrieved the $K \gg 1$ asymptote of $f_\psi(K)$ from Tatarskii’s [1] relationships $F_\psi(\kappa) = 2\pi k^2 L \times 0.033 C_n^2 \kappa^{-11/3}$ and $D_\phi(r) = 2.91 k^2 L C_n^2 r^{5/3}$, and we find the same power law and the same coefficient, $f_\psi(K) = 0.45 K^{-11/3}$.
6. For phase screens with quadratic ($\gamma = 2$) phase structure functions, $f_\psi(K)$ drops exponentially for $K \gg 1$, more rapidly for Gaussian than for Laplacian phase increments.
7. At low wavenumbers ($\kappa \ll r_c^{-1}$, $K \ll 1$), all four ψ spectra become flat, which is consistent with the fact that ψ becomes uncorrelated for $r \gg r_c$.
8. At intermediate wavenumbers ($\kappa \sim r_c^{-1}$, $K \sim 1$), each of the four $f_\psi(K)$ shows a “bump,” in the case of $\gamma = 5/3$ exceeding the high-wavenumber asymptote, $f_\psi(K) = 0.45 K^{-11/3}$, by a factor of up to 2.8 for Gaussian phase increments and a factor of up to 2.2 for Laplacian phase increments.

As we have seen, the second-order phase-factor (ψ) statistics (covariance function, structure function, and 2D spectrum) exhibit a rich variety of phase-screen characteristics that are not as directly, or not at all, captured by means of second-order phase (ϕ) statistics characterizing the same phase screen. There are two obvious advantages of characterizing phase screens by means of ψ statistics rather than ϕ statistics. First, the (second-order) ψ statistics account for the probability density $p_\alpha(\alpha)$ of the phase increments α , while the (second-order) ϕ statistics are invariant with respect to the functional form of $p_\alpha(\alpha)$; second, the second-order ψ statistics account explicitly for effects resulting from a finite phase coherence length r_c , while the second-order ϕ statistics are simply the second-order path-length (eikonal) statistics re-scaled with the constant factor k^2 .

Funding. Air Force Office of Scientific Research (FA9550-17-C-0021, FA9550-18-1-0009, FA9550-18-C-0011); National Science Foundation (AGS-1547476).

Acknowledgment. The author thanks the two anonymous reviewers for their constructive comments and David Voelz for fruitful discussions.

Disclosures. The author declares that there are no conflicts of interest related to this paper.

Data Availability. No data were generated or analyzed in the presented research.

REFERENCES

1. V. I. Tatarskii, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, 1961).
2. V. I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation* (Israel Program for Scientific Translation, 1971).
3. A. M. Obukhov, “The structure of the temperature field in a turbulent flow,” *Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz.* **13**, 58–69 (1949).
4. S. Corrsin, “On the spectrum of isotropic temperature fluctuations in an isotropic turbulence,” *J. Appl. Phys.* **22**, 469–473 (1951).
5. A. M. Yaglom, *An Introduction to the Theory of Stationary Random Functions* (Prentice-Hall, 1962).
6. M. E. Gracheva and A. S. Gurvich, “Strong fluctuations in the intensity of light propagated through the atmosphere close to the earth,” *Sov. Radiophys.* **8**, 511–515 (1965).
7. M. E. Gracheva, “Investigation of the statistical properties of strong fluctuations in the intensity of light propagated through the atmosphere near the earth,” *Izv. VUZ Radiofiz.* **10**, 775–788 (1967).
8. B. J. Uscinski, *The Elements of Wave Propagation in Random Media* (McGraw-Hill, 1977).
9. A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, 1978), Vol. **2**.
10. J. W. Strohbehn, ed., *Laser Beam Propagation in the Atmosphere*, Topics in Applied Physics (Springer, 1978), Vol. **25**.
11. B. U. Uscinski, ed., *Wave Propagation and Scattering* (Clarendon, 1986).
12. E. L. Andreas, ed., *Turbulence in a Refractive Medium*, SPIE Milestone Series (SPIE Optical Engineering, 1990), Vol. **MS 25**.
13. V. I. Tatarskii, A. Ishimaru, and V. U. Zavarotny, eds., *Wave Propagation in Random Media (Scintillation)* (SPIE Optical Engineering, 1993).
14. A. D. Wheelon, *Electromagnetic Scintillation — I. Geometrical Optics* (Cambridge University, 2001).
15. A. D. Wheelon, *Electromagnetic Scintillation — II. Weak Scattering* (Cambridge University, 2003).
16. S. M. Rytov, Y. A. Kravtsov, and V. I. Tatarskii, *Principles of Statistical Radio Physics—4. Wave Propagation through Random Media* (Springer, 1989).
17. L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media*, 2nd ed. (SPIE, 2005).

18. O. Korotkova, *Random Light Beams: Theory and Applications* (CRC Press, 2014).
19. H. G. Booker, J. A. Ratcliffe, and D. H. Shinn, "Diffraction from an irregular screen with applications to ionospheric problems," *Philos. Trans. R. Soc. London* **242**, 579–607 (1950).
20. J. A. Ratcliffe, "Some aspects of diffraction theory and their application to the atmosphere," *Rep. Prog. Phys.* **19**, 188–267 (1956).
21. J. W. Goodman, *Introduction to Fourier Optics*, 1st ed. (McGraw-Hill, 1968).
22. B. P. Lathi, *Linear Systems and Signals*, 2nd ed. (Oxford University, 2005).
23. M. J. Roberts, *Fundamentals of Signals and Systems* (McGraw-Hill, 2008).
24. J. W. Goodman, *Introduction to Fourier Optics*, 4th ed. (W. H. Freeman and Company, 2017).
25. J. W. Goodman, *Statistical Optics*, 2nd ed. (Wiley, 2015).
26. J. W. Goodman, *Speckle Phenomena in Optics*, 2nd ed. (SPIE, 2020).
27. H. G. Booker, J. A. Ferguson, and H. O. Vats, "Comparison between the extended-medium and the phase-screen scintillation theories," *J. Atmos. Terr. Phys.* **47**, 381–399 (1985).
28. M. C. Roggemann and B. M. Welsh, *Imaging through Turbulence* (CRC Press, 1996).
29. N. S. Kopeika, *A System Engineering Approach to Imaging* (SPIE, 1998).
30. J. M. Martin and S. M. Flatté, "Intensity images and statistics from numerical simulation of wave propagation in 3-D random media," *Appl. Opt.* **27**, 2111–2126 (1988).
31. J. D. Schmidt, *Numerical Simulation of Optical Wave Propagation* (SPIE, 2010).
32. D. G. Voelz, *Computational Fourier Optics* (SPIE, 2011).
33. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (The MIT, 1975), Vol. 2.
34. F. Roddier, "The effects of atmospheric turbulence in optical astronomy," *Prog. Opt.* **19**, 281–376 (1981).
35. A. Papoulis, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, 1965).
36. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed. (Academic, 1980).
37. M. Charnotskii, "Optical phase under deep turbulence conditions," *J. Opt. Soc. Am. A* **31**, 1766–1772 (2014).
38. G. N. Watson, *Treatise on the Theory of Bessel Functions* (Cambridge University, 1944).
39. M. Charnotskii, "Common omissions and misconceptions of wave propagation in turbulence: discussion," *J. Opt. Soc. Am. A* **29**, 711–721 (2012).
40. F. B. Hildebrand, *Introduction to Numerical Analysis* (Dover, 1987).
41. R. J. Hill, "Models of the scalar spectrum for turbulent advection," *J. Fluid Mech.* **88**, 541–562 (1978).
42. A. Muschinski and S. M. de Bruyn Kops, "Investigation of Hill's optical turbulence model by means of direct numerical simulation," *J. Opt. Soc. Am. A* **32**, 2423–2430 (2015).
43. D. L. Fried, "Statistics of a geometric representation of wavefront distortion," *J. Opt. Soc. Am.* **55**, 1427–1435 (1965).
44. D. L. Fried, "Optical resolution through a randomly inhomogeneous medium for very long and very short exposures," *J. Opt. Soc. Am.* **56**, 1372–1379 (1966).
45. S. T. Thoroddsen and C. W. Van Atta, "Exponential tails and skewness of density-gradient probability density functions in stably stratified turbulence," *J. Fluid Mech.* **244**, 547–566 (1992).

Non-Classical Models of Optical Turbulence

Andreas Muschinski

NorthWest Research Associates, Inc., Boulder, CO

SPIE Optics + Photonics 2022

San Diego, CA

21-25 August, 2022

Modeling/simulating propagation through the turbulent atmosphere:

A two-step process

1. Model the **spatial statistics of the turbulent refractive-index field**
2. Solve the **Helmholtz equation** by means of the geometrical-optics method, the Rytov method, or the Huygens-Fresnel principle applied to turbulent phase screens

Classical models of optical turbulence (Tatarskii 1961, 1971):

1. 3D Corrsin-Obukhov spectrum (“Kolmogorov spectrum”): $\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3} h(\kappa l_0)$
2. 3D Corrsin-Obukhov spectrum with Gaussian “tail function” $h(\kappa l_0)$
3. 2D Phase spectrum for thin turbulent phase screens: $F_\phi(\kappa_x, \kappa_y) = 2\pi k^2 L \Phi_n(\kappa_x, \kappa_y, 0)$
4. Phase increments are Gaussian-distributed

Collocated sonic and fine-wire temperature measurements

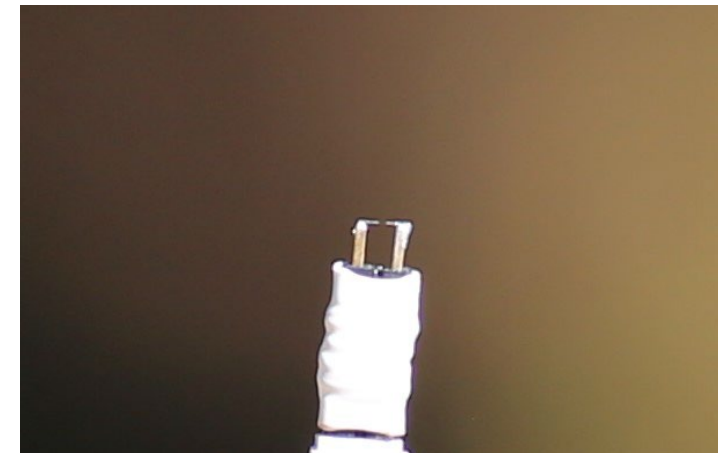


Ultrasonic anemometer-thermometer (“sonic”):

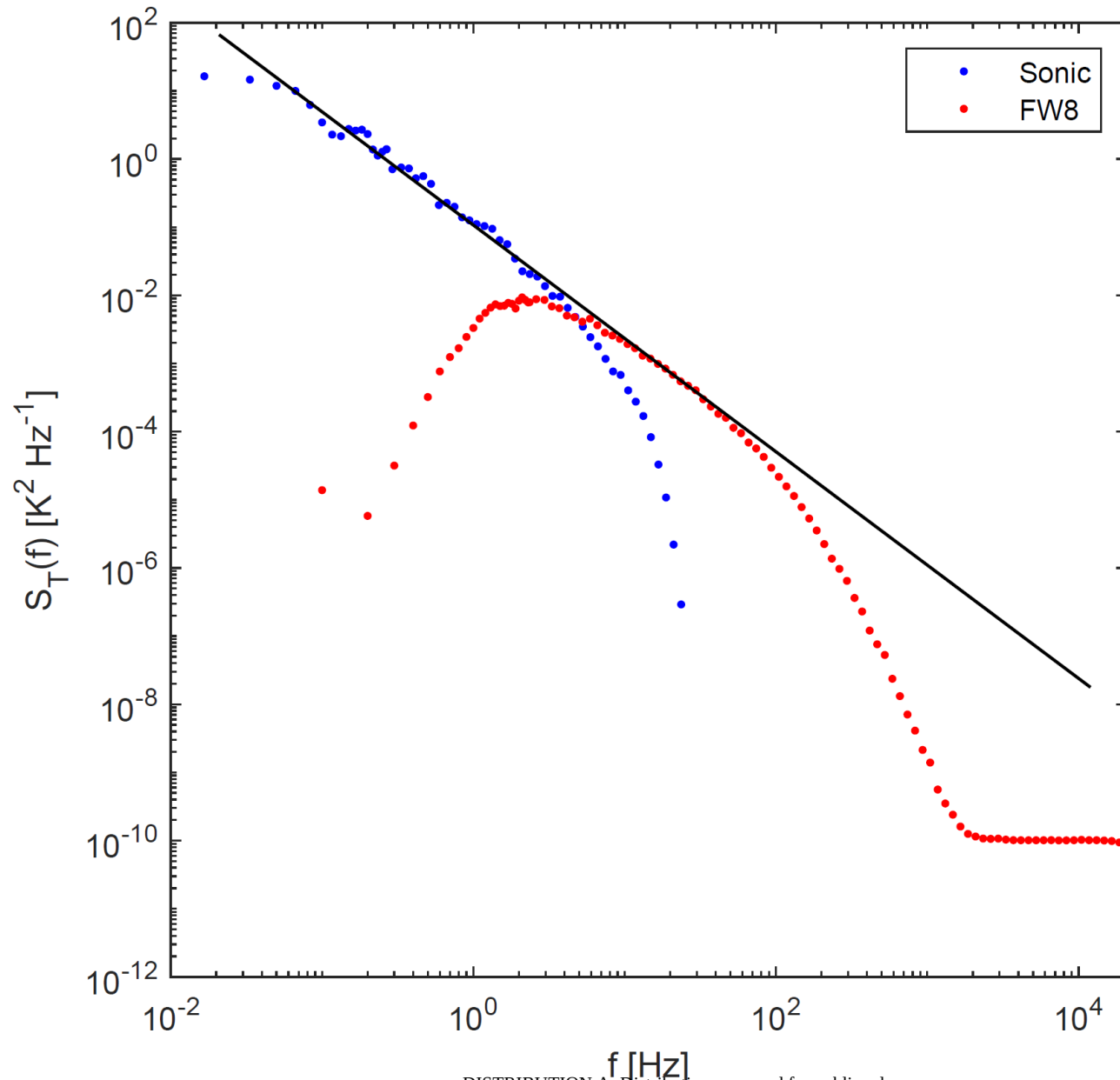
Sampling volume size: 15 cm
Sampling frequency: 32 Hz
Temperature precision: 10 mK
Wind velocity precision: 1 cm/s

Fine-wire thermometers:

Designed and manufactured in-house
by Eric Wagner, NWRA
Active wire length: 0.7 mm
Wire diameter: 0.64 μm
Frequency response: > 10 kHz
Sampling frequency: 44.1 kHz
Precision: 1.5 mK for 20 kHz bandwidth



Collocated sonic and fine-wire temperature measurements



DISTRIBUTION A: Distribution approved for public release.

Black: Theoretical $-5/3$ power-law spectrum

The classical $-5/3$ power law for the temperature spectrum in the inertial-convective subrange was predicted by Obukhov (1949) and Corrsin (1951).

Colored dots: Measurements

Collocated sonic and fine-wire measurements were collected on 5 October, 2021 in the Rocky Mountain foothills near Boulder, CO. Measurement episode: **10 minutes, starting 11:36 MDT**

Blue: Sonic spectrum (estimated from **18,000 samples**)

$-5/3$ spectrum from 0.1 Hz through 3 Hz

$f < 0.1$ Hz: flattening due to finite outer scale

$f > 3$ Hz: roll-off due to spatial filtering (large observation volume)

Red: Fine-wire spectrum (estimated from **26 Mio. samples**)

$-5/3$ spectrum from 20 Hz through 50 Hz:

$f < 20$ Hz: roll-off because of built-in highpass filter

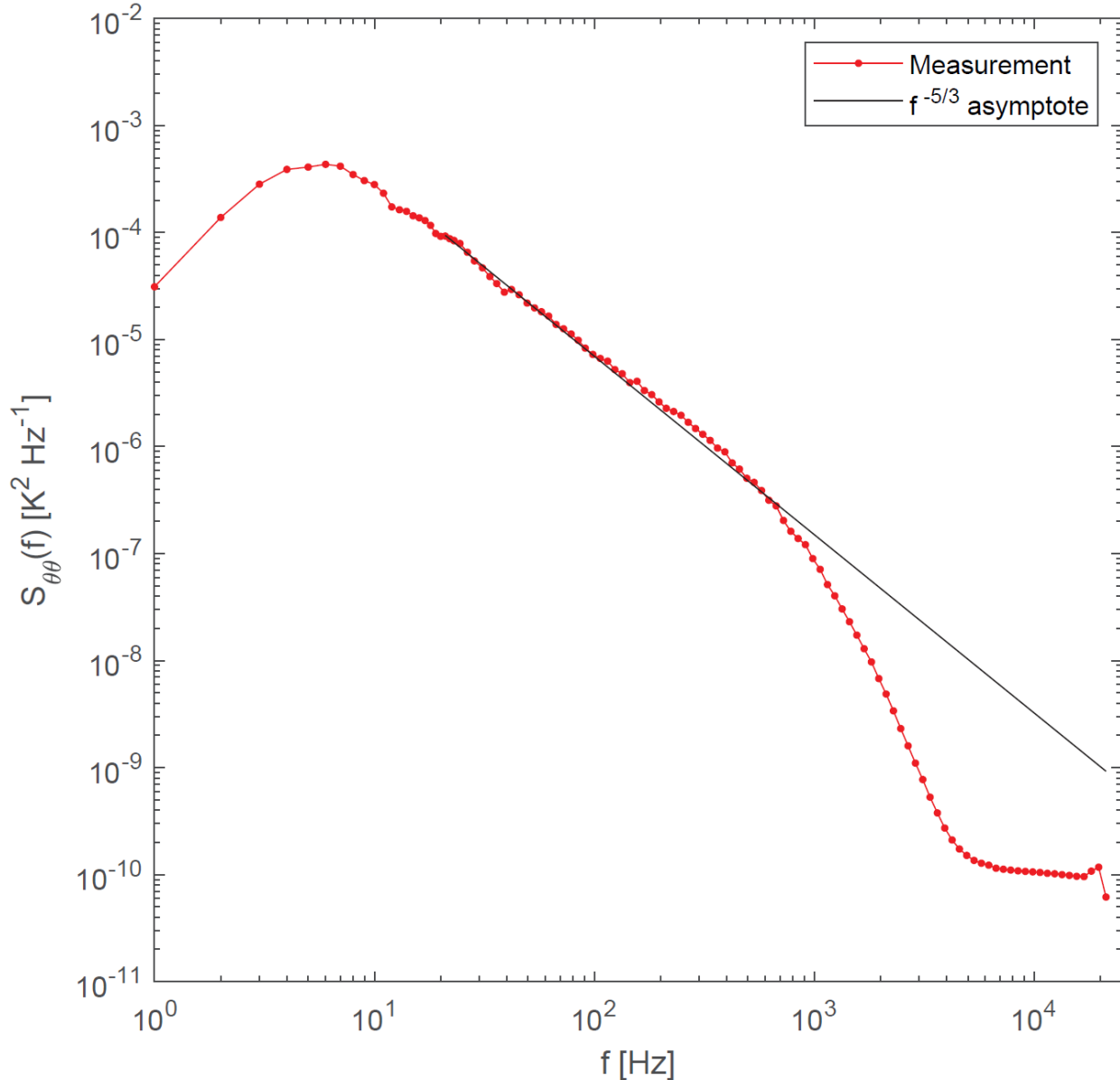
$f > 50$ Hz: transition from inertial-convective subrange to viscous-diffusive subrange

$f > 2$ kHz: uncorrelated noise (rms value 1.5 mK)

The combined sonic/fine-wire spectrum spans 11 decades in spectral density!

**Locally homogeneous and isotropic turbulence:
“non-Kolmogorov” power laws, inner scale, and Hill bump**

Fine-wire temperature measurements out of a car at 40 mph



DISTRIBUTION A: Distribution approved for public release.

Black: Theoretical -5/3 power-law spectrum

The classical -5/3 power law for the temperature spectrum in the inertial-convective subrange was predicted by Obukhov (1949) and Corrsin (1951).

Red: Measured spectrum, estimated from 26 Mio. samples

- The fine-wire measurements were collected on 15 Sep 2020 out of a car traveling at 40 mph near Boulder, CO.
- Measurement episode: 3 min 12 s, starting 22:38 MDT
- -5/3 spectrum (black line) from 10 Hz through 100 Hz:
- $f < 10$ Hz: roll-off because of built-in highpass filter
- $10 \text{ Hz} < f < 100 \text{ Hz}$: inertial-convective range
- $100 < f < 600 \text{ Hz}$: Hill bump (viscous-convective range)
- $f > 600 \text{ Hz}$: transition to the viscous-diffusive subrange
- $f > 4 \text{ kHz}$: uncorrelated noise (rms value 1.5 mK)

From: Muschinski, A., and E. L. Wagner, 2022: First field measurements of optical turbulence near the ground with a newly developed fine-wire thermometer. *Wav. Rand. Comp. Med.*, **32**, in press.

Modeling the tail function of the universal scalar spectrum

Universal scalar spectrum (Obukhov 1949, Corrsin 1951, Tatarskii 1961):

$$\Phi_\theta(\kappa) = 0.033 C_\theta^2 \kappa^{-11/3} h(\kappa \ell_0), \quad (24)$$

where the inner scale ℓ_0 is defined as the r value at which the inertial-convective asymptote and the viscous-diffusive asymptote of the structure function $D_\theta(r)$ intersect. Analysis (Hill and Clifford 1978) gives

$$\ell_0 = \left[\frac{27}{5} \Gamma\left(\frac{1}{3}\right) \beta \text{Pr} \right]^{3/4} \eta = 7.4 \eta, \quad (25)$$

where

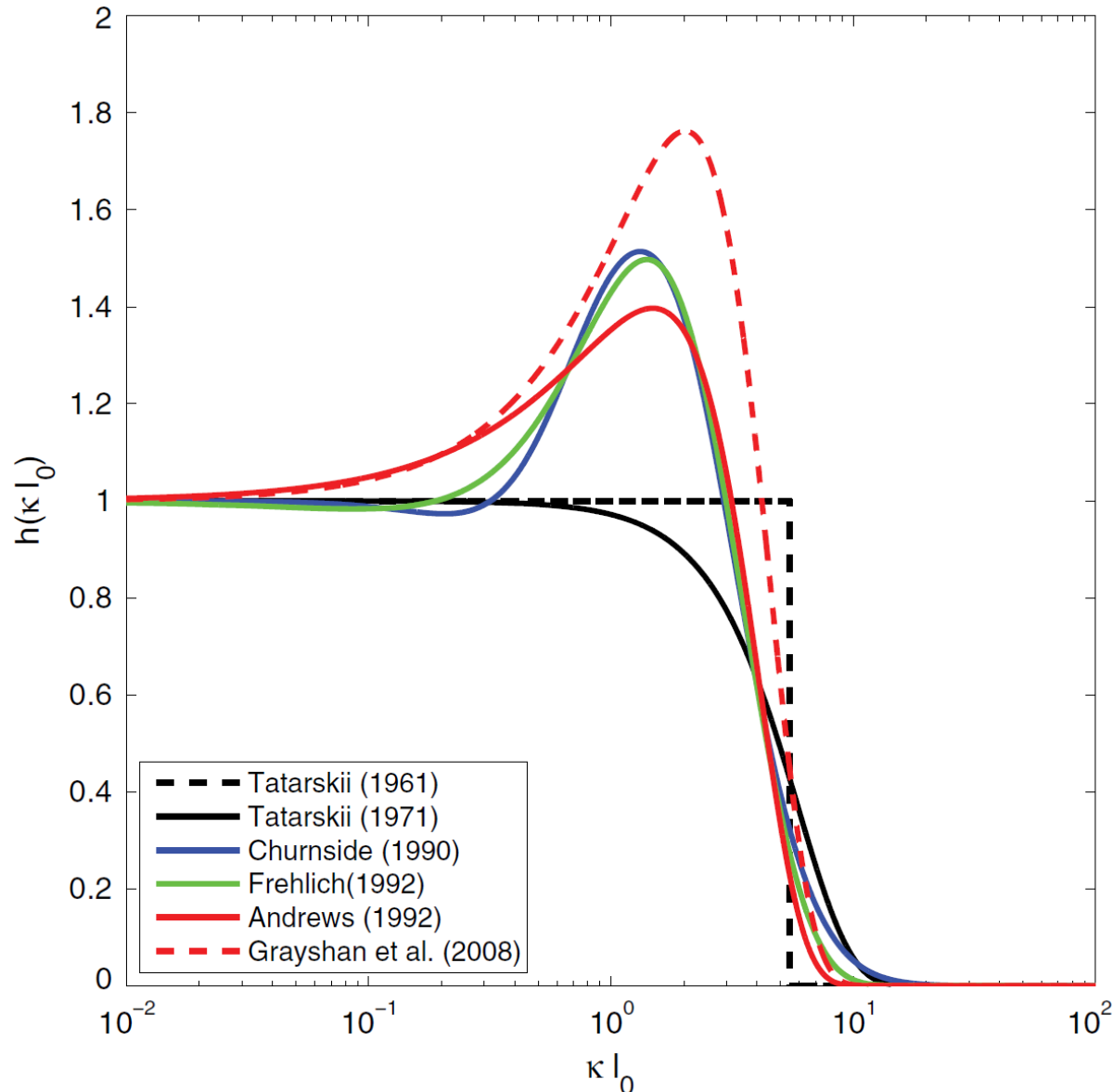
$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} \quad (26)$$

is the Kolmogorov length.

From the scalar transport equation, a “scalar dissipation constraint” for $h(y)$ follows (Frehlich 1992, Muschinski 2015):

$$\int_0^\infty h(y) y^{1/3} dy = \frac{27 \Gamma(1/3)}{10} = 7.233. \quad (27)$$

Six models of the tail function $h(\kappa l_0)$



The two classical models:

- 1) Tatarskii's (1961) cut-off model
 - 2) Tatarskii's (1971) Gaussian model
- The Tatarskii models fulfill the dissipation constraint
 - The Tatarskii models do not account for the "Hill bump"

Four non-classical models:

- All four models account for the "Hill bump"
- All these models, except Frehlich's (1992) model, violate the dissipation constraint

From: Muschinski, A., 2015: Temperature variance dissipation equation and its relevance for optical turbulence modeling. *J. Opt. Soc. Am. A*, **32**, 2195-2200.

3D versus 1D tail function

3D spectrum and tail function:

$$\Phi_{\theta}(\kappa) = 0.033 C_{\theta}^2 \kappa^{-11/3} h(\kappa \ell_0). \quad (28)$$

1D spectrum and tail function:

$$F_{\theta}(\kappa_1) = 0.249 C_{\theta}^2 \kappa^{-5/3} f(\kappa_1 \ell_0). \quad (29)$$

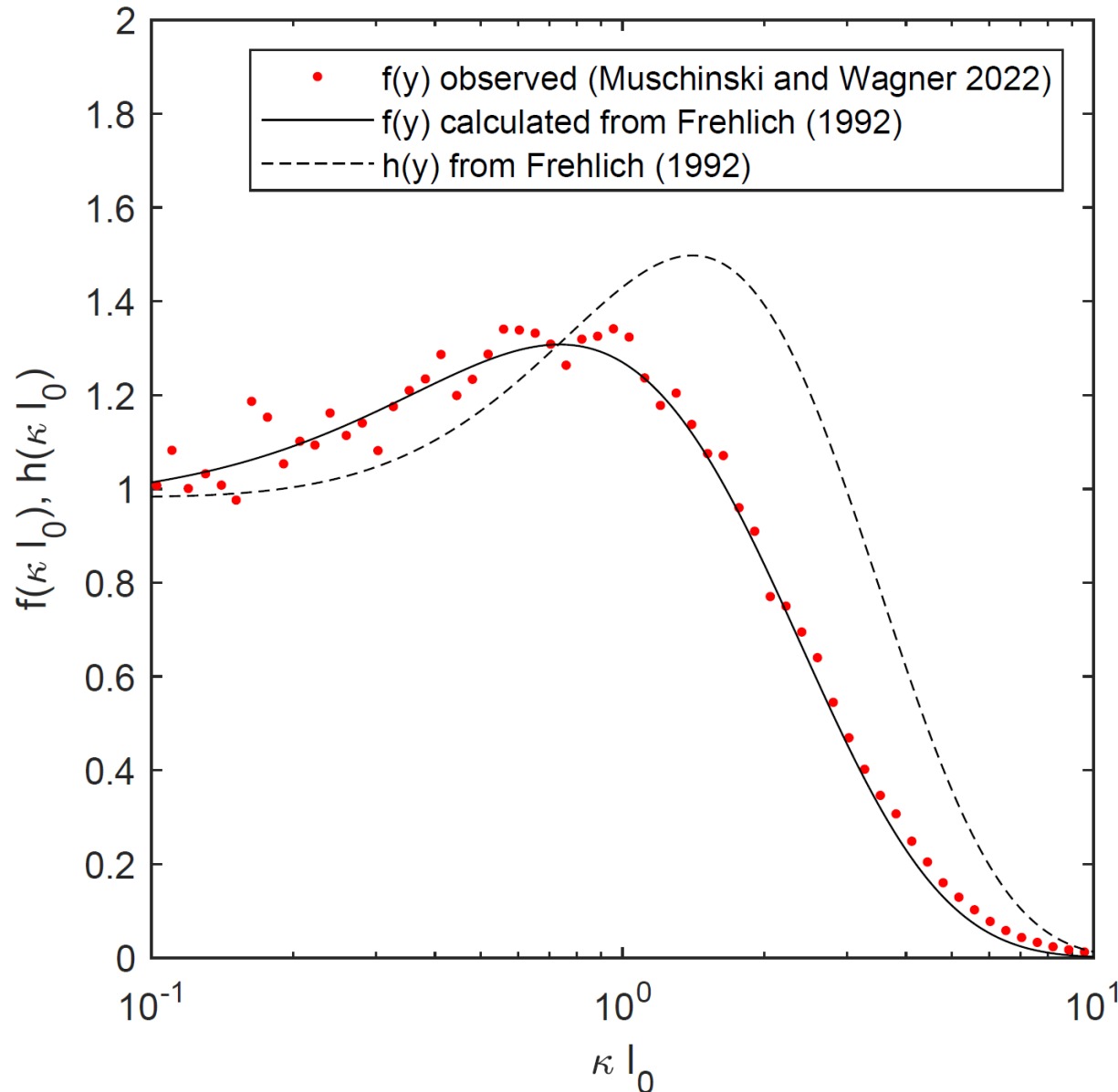
Relationship between 3D spectrum and (one-sided) 1D spectrum:

$$\Phi_{\theta}(\kappa) = -\frac{1}{4\pi\kappa} \left. \frac{dF(\kappa_1)}{d\kappa_1} \right|_{\kappa_1=\kappa}. \quad (30)$$

Basic calculus gives $f(y)$ in terms of $h(y)$:

$$f(y) = \frac{5}{3} y^{5/3} \int_{y'=y}^{\infty} y'^{-8/3} h(y') dy'. \quad (31)$$

Modeled and observed tail functions



Observed 1D tail function $f(y)$:

Retrieved from fine-wire thermometer measurements (Muschinski and Wagner, 2022)

Modeled 3D tail function $h(y)$:

From Frehlich (1992), accounting for Hill bump

Modeled 1D tail function $f(y)$:

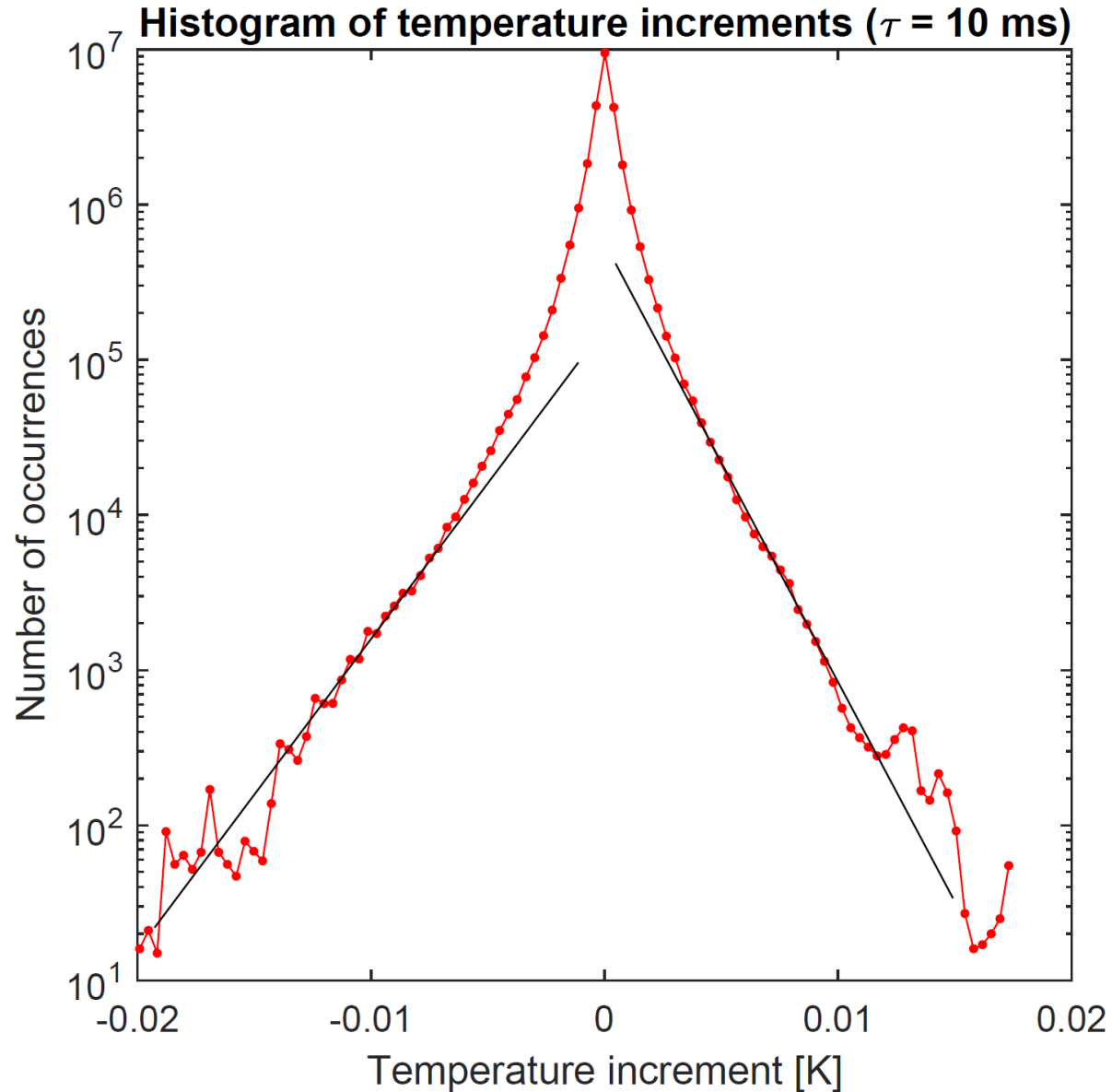
Calculated from Frehlich's (1992) closed-form model of $h(y)$.

$h(y)$ model from: Frehlich, R., 1992: Laser scintillation measurements of the temperature spectrum in the atmospheric surface layer. *J. Atmos Sci.*, **49**, 1494-1509.

Observations from: Muschinski, A., and E. L. Wagner, 2022: First field measurements of optical turbulence near the ground with a newly developed fine-wire thermometer. *Waves in Random and Complex Media*, **32**, in press.

Non-Gaussian temperature and phase increments

Fine-wire temperature measurements out of a car at 40 mph



DISTRIBUTION A: Distribution approved for public release.

Histogram of temporal temperature increments

Measured with a single fine-wire thermometer

Sampling frequency: 44.1 kHz

Temporal lag: 10 ms (441 samples), $d = 20$ cm

Total number of samples: 26 Mio. (10 minutes)

The histogram is “pointy” near the origin
and is **two-sided exponential (“Laplacian”)**
in the flanks – not Gaussian!

Phase screens from first-principle fluid dynamics

Direct numerical simulation of turbulent temperature field and wind vector field in 3D space

First principles: Navier-Stokes equation and scalar transport equation

Flow type: Life cycle of a Kelvin-Helmholtz instability in a stably stratified atmosphere

Model domain size: $N_x \times N_y \times N_z = 3240 \times 9720 \times 3240$

That is, **102 billion grid points in 3D space** for each timestep (tens of thousands of timesteps)

(Data created and provided by David Fritts and coworkers, GATS, Boulder, CO, as part of an AFOSR-sponsored MURI)

Phase screens from first-principle fluid dynamics

Turbulent phase screen:

$$\phi(x, y) = k \int_0^L n_1(x, y, z) dz, \quad (1)$$

where

$$n_1(x, y, z) = n(x, y, z) - \langle n(x, y, z) \rangle \quad (2)$$

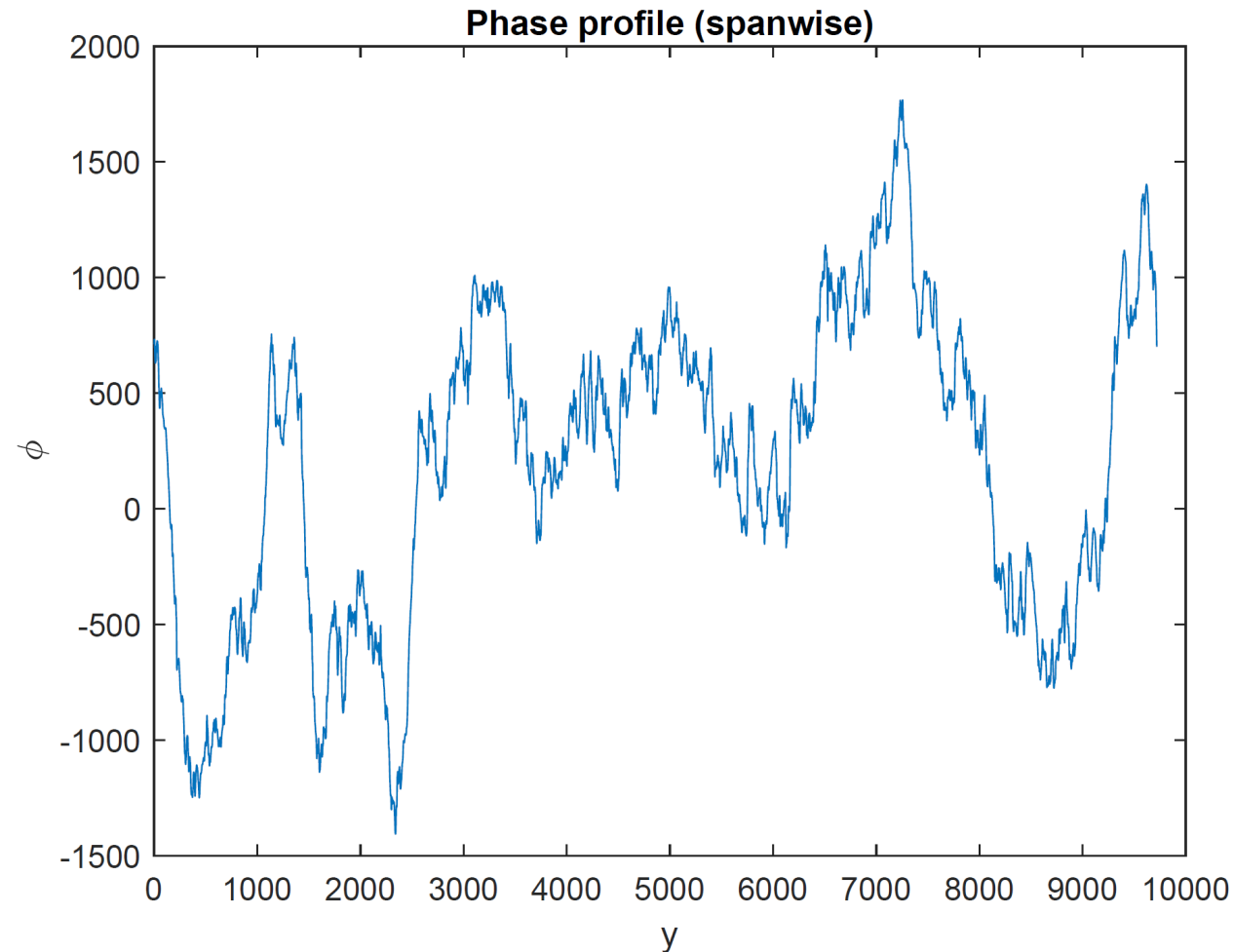
are the refractive-index fluctuations, L is the thickness of the “slab” of turbulent air represented by the phase screen, and k is the optical wavenumber.

If the turbulence is homogeneous (not necessarily isotropic), the 2D spectrum of the phase screen is (Tatarskii, 1961)

$$F_\phi(\kappa_x, \kappa_y) = 2\pi k^2 L \Phi_n(\kappa_x, \kappa_y, 0), \quad (3)$$

where $\Phi_n(\kappa_x, \kappa_y, \kappa_z)$ is the 3D (in the general case anisotropic) refractive-index spectrum.

Phase screens from first-principle fluid dynamics



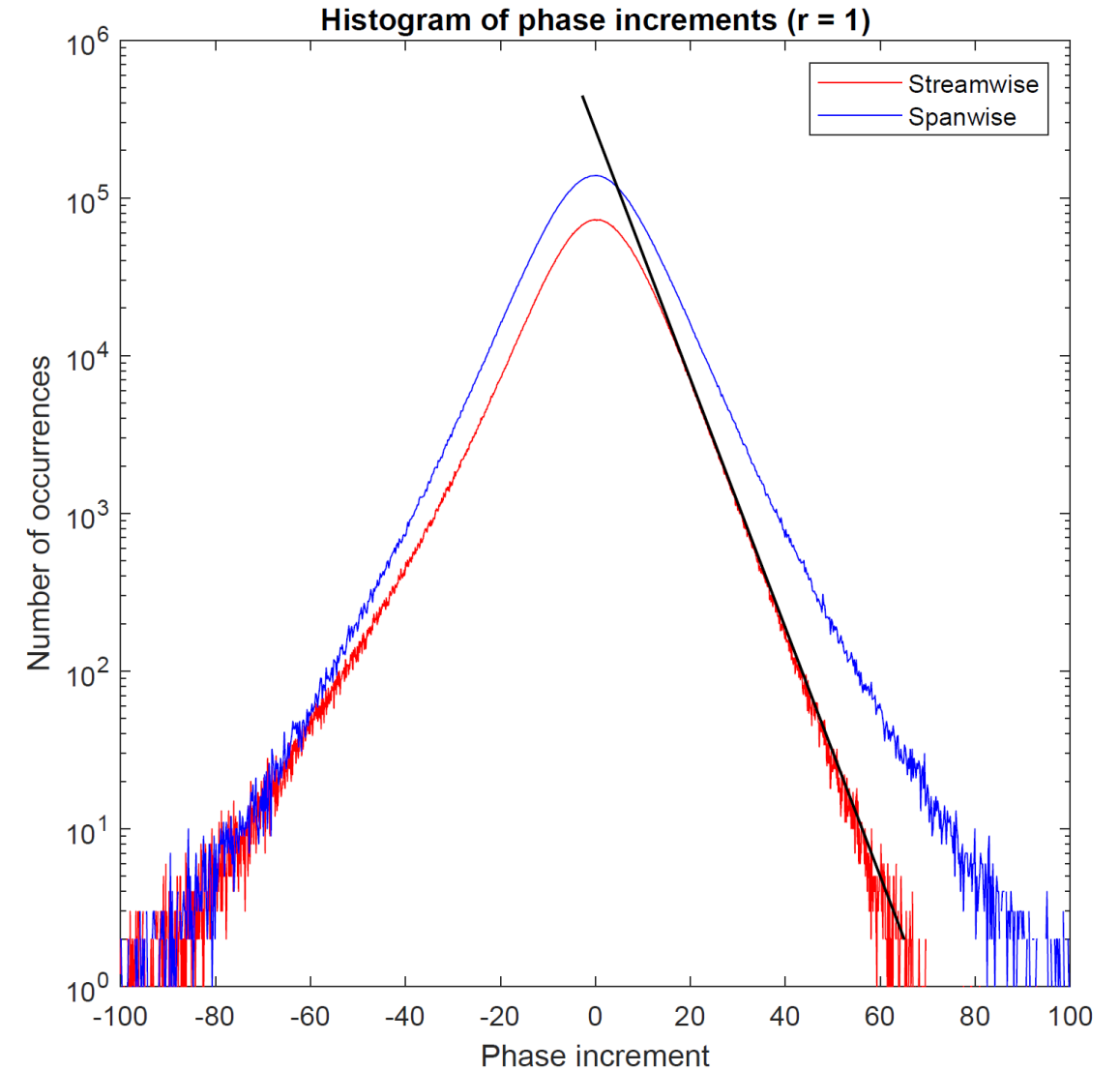
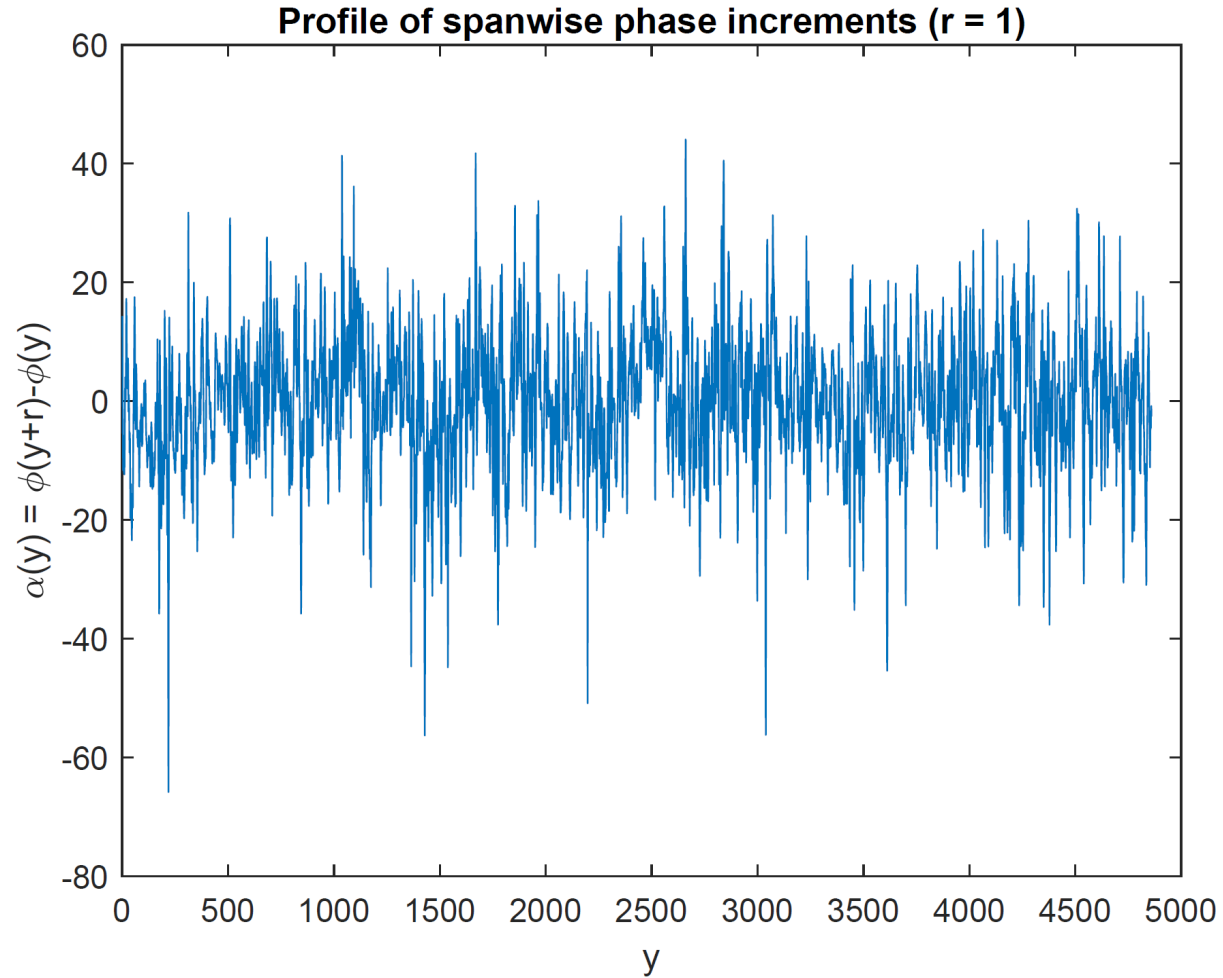
x: streamwise coordinate

y: spanwise coordinate

Here: Single realization of a 1D phase profile $\phi(x,y)$ for an arbitrarily chosen x

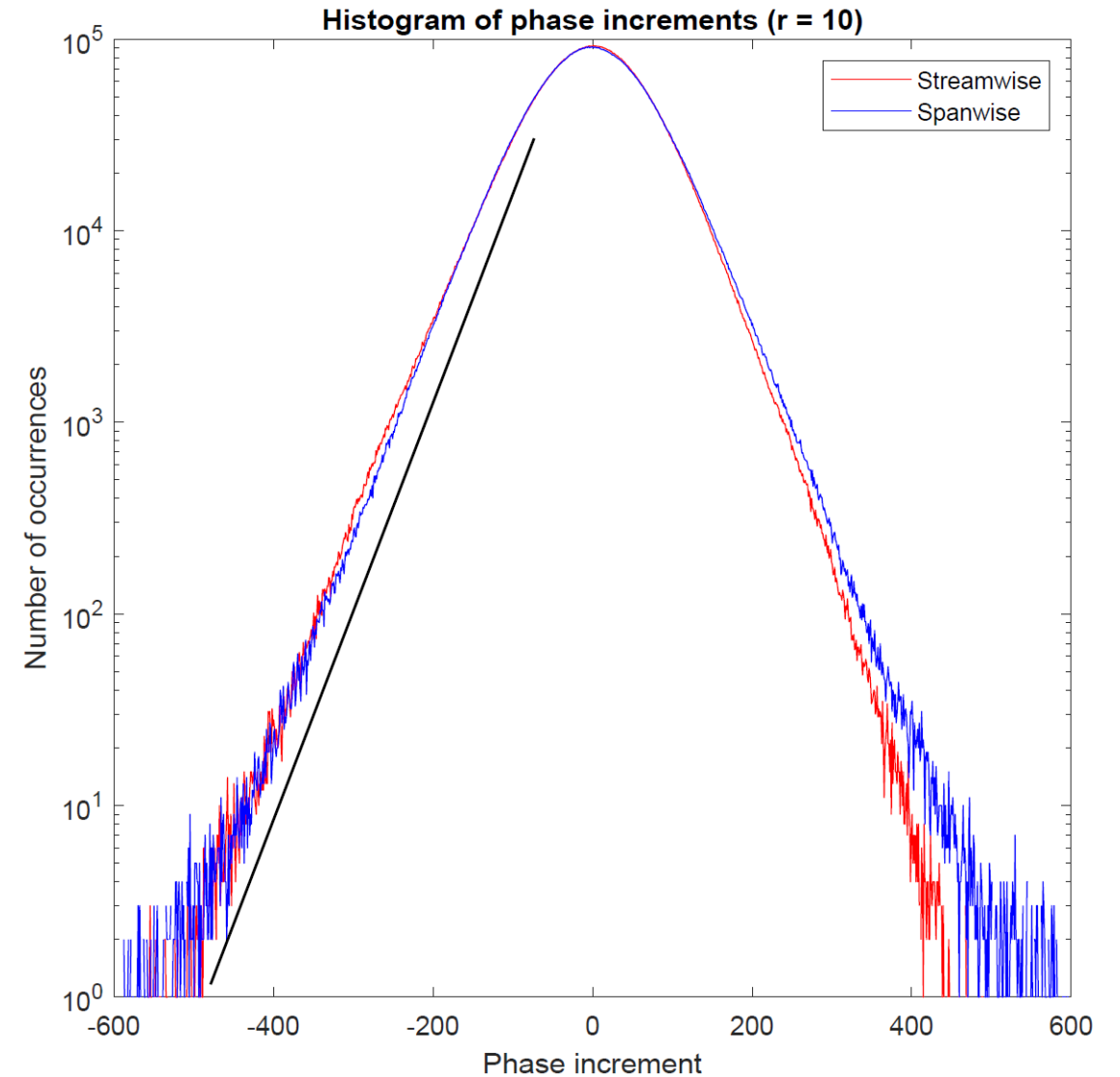
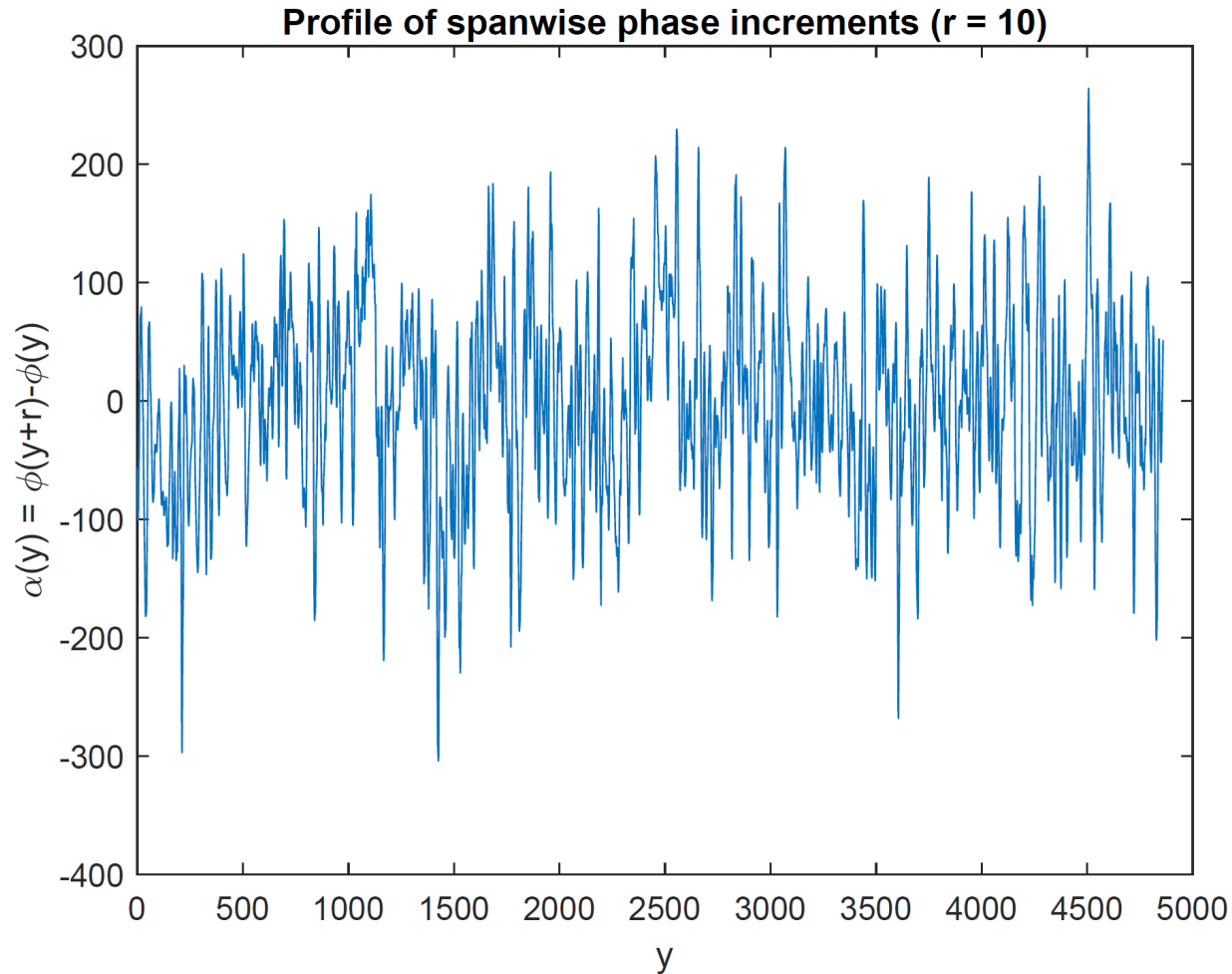
Phase screens from first-principle fluid dynamics

Phase increments for **separation $r = 1$ grid point**



Phase screens from first-principle fluid dynamics

Phase increments for **separation $r = 10$ grid points**



Second-order statistics of the **phase factor** of turbulent phase screens

Input-output relationship for Fresnel diffraction:

$$U(\mathbf{x}) = \iint U_0(\mathbf{x}')h(\mathbf{x} - \mathbf{x}')d\mathbf{x}', \quad (4)$$

where U_0 and U are the optical fields in the input and output planes, respectively, and $h(\mathbf{x})$ is a 2D impulse response function (the “Fresnel diffraction kernel”).

In the case of an unperturbed plane wave propagating through a single phase screen placed in the input plane, we have

$$U_0(\mathbf{x}) = A\psi(\mathbf{x}) \quad (5)$$

where A is a complex constant and $\psi(\mathbf{x})$ is the phase factor of the phase screen $\phi(\mathbf{x})$:

$$\psi(\mathbf{x}) = \exp[i\phi(\mathbf{x})]. \quad (6)$$

This leads to the 2D spectrum of U :

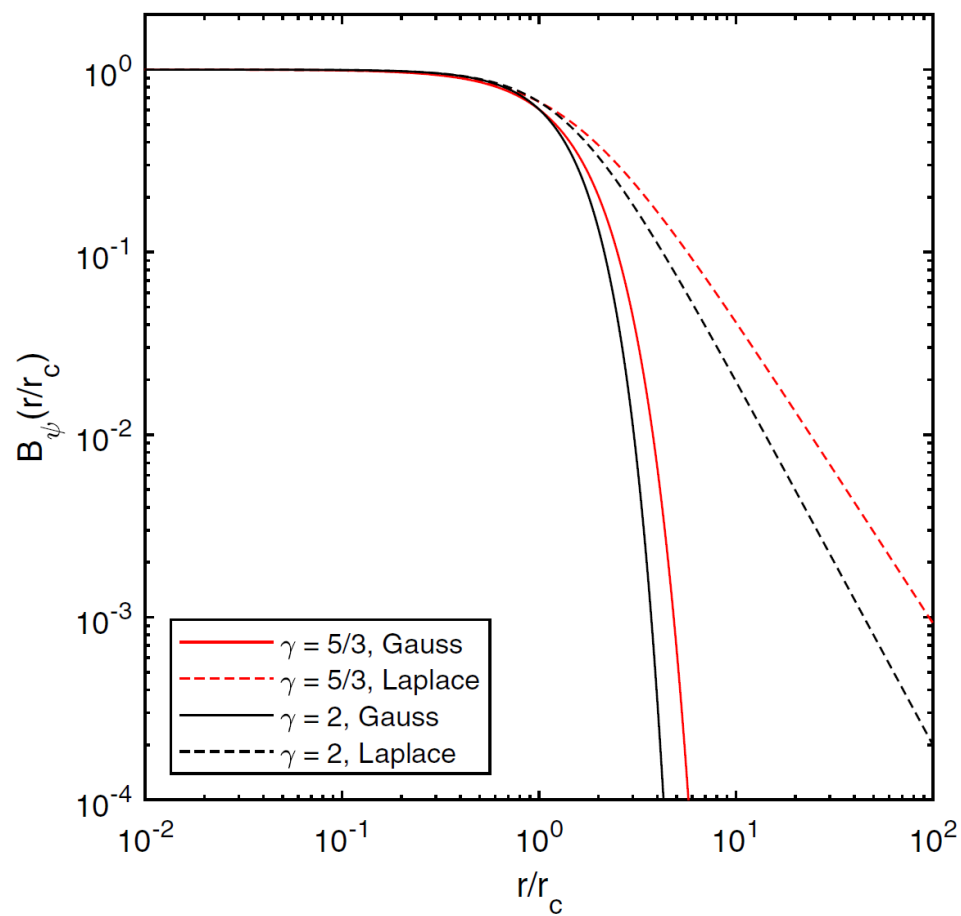
$$F_U(\boldsymbol{\kappa}) = F_\psi(\boldsymbol{\kappa})|AH(\boldsymbol{\kappa})|^2, \quad (7)$$

where

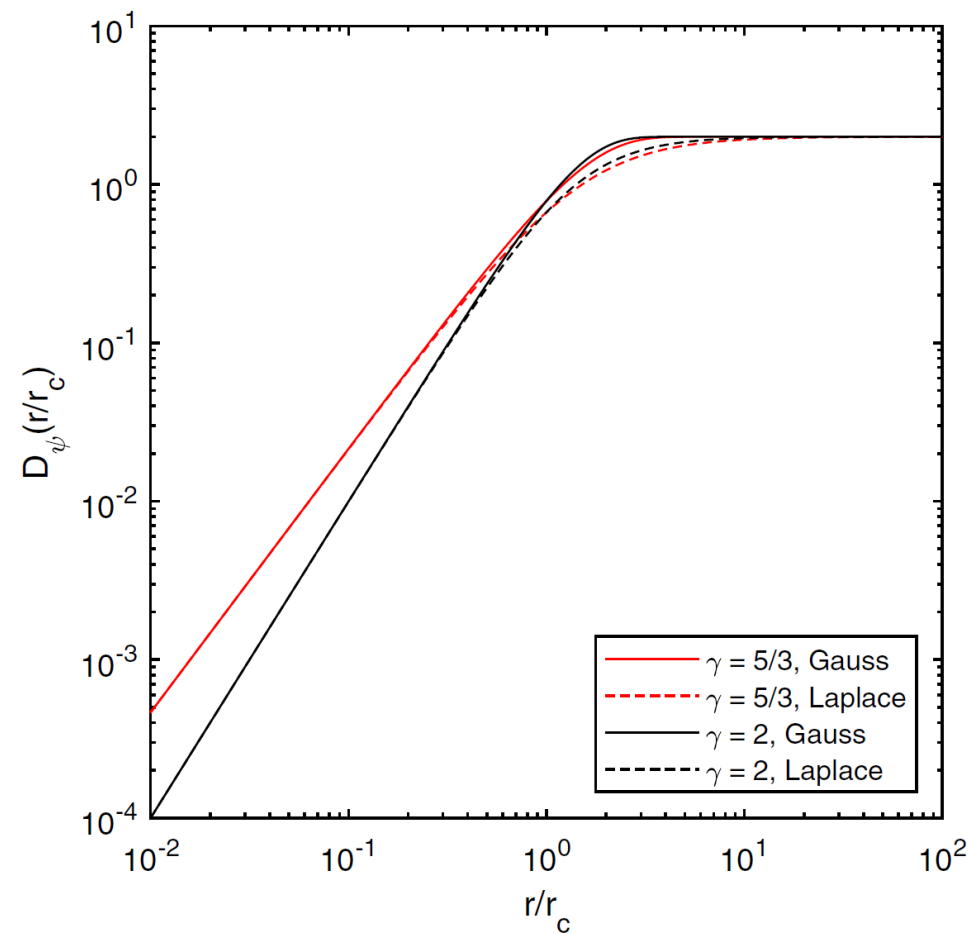
$$H(\boldsymbol{\kappa}) = \iint \exp(-i\boldsymbol{\kappa} \cdot \mathbf{x})h(\mathbf{x})d\mathbf{x} \quad (8)$$

is the 2D spatial transfer function and $F_\psi(\boldsymbol{\kappa})$ is the 2D spectrum of the phase factor.

Second-order statistics of the **phase factor** of turbulent phase screens

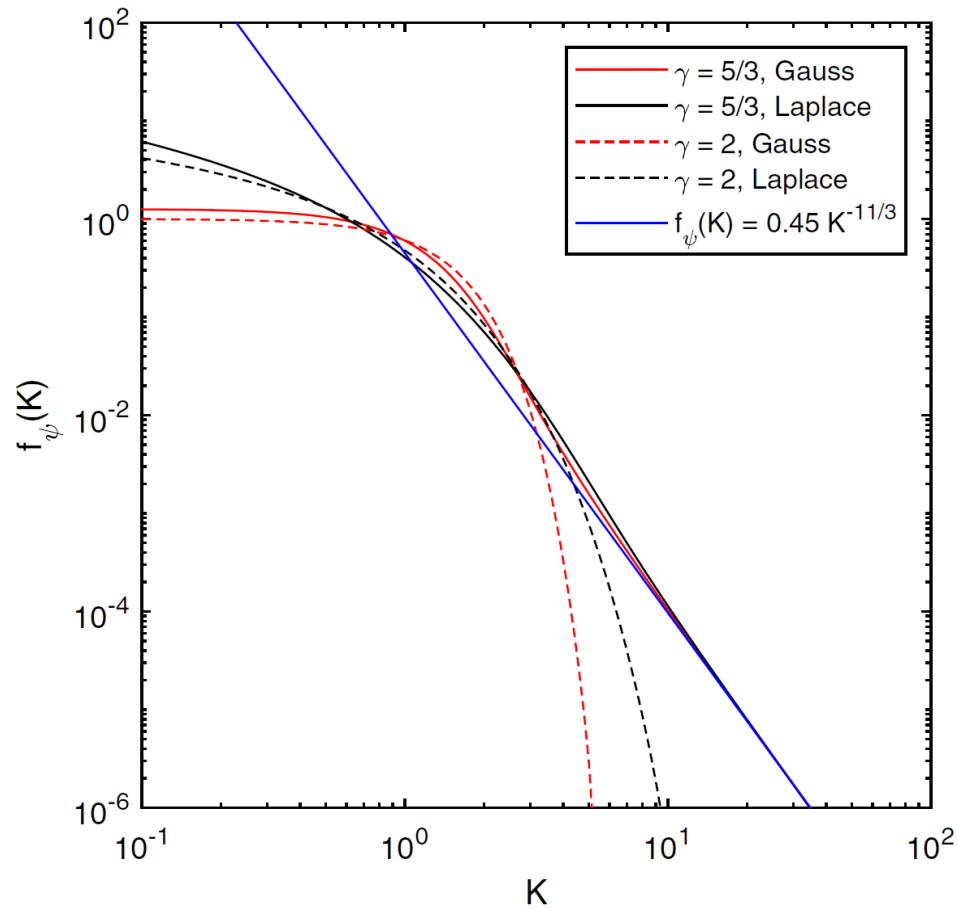


Covariance of ψ

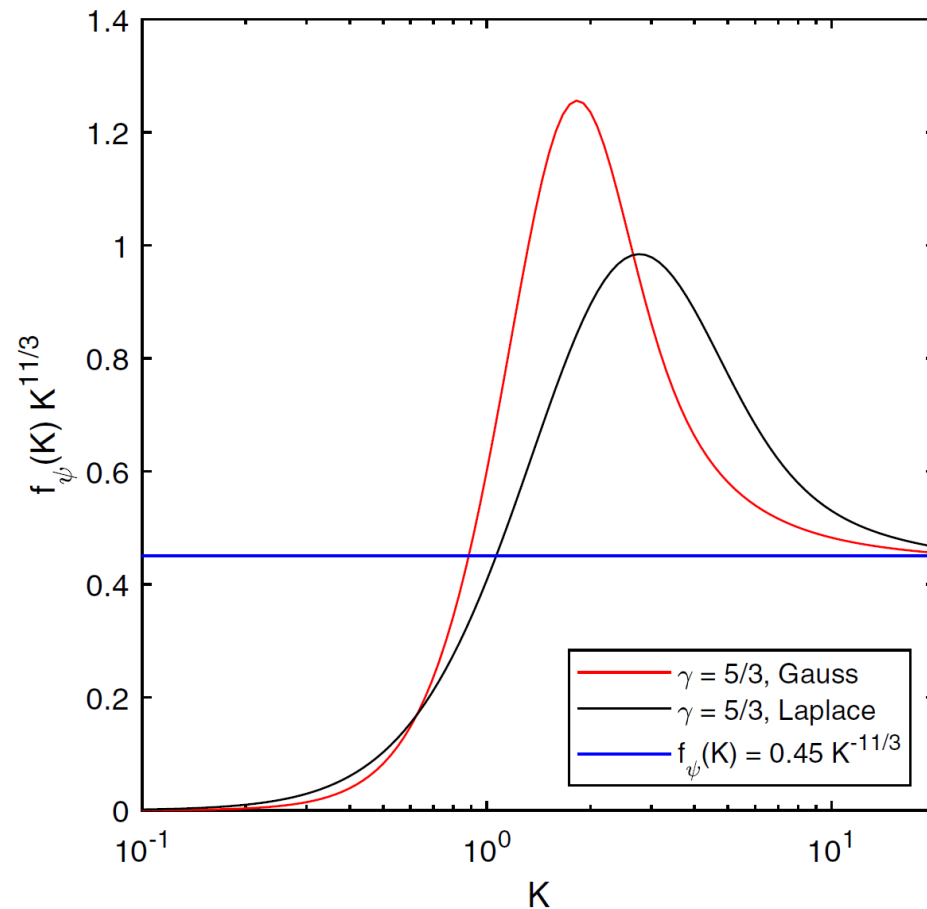


Structure function of ψ

Second-order statistics of the **phase factor** of turbulent phase screens



Normalized ψ spectrum



Compensated normalized ψ spectrum

Summary

- The **fine-wire thermometer** continues to be the instrument of choice for the in-situ measurement of turbulence at length scales **between 1 m and 1 mm** (inner-scale effects).
- Our fine-wire measurements of turbulence near the ground show the **Hill bump** and are consistent with Frehlich's (1992) model of the "**tail function**" $h(\kappa l_0)$
- **Turbulent temperature increments are non-Gaussian.** It is unclear to what extent non-Gaussian temperature increments lead to **non-Gaussian phase increments.**

First field measurements of optical turbulence near the ground with a newly developed fine-wire thermometer

Andreas Muschinski ^{a,b} and Eric L. Wagner^a

^aNorthWest Research Associates, Inc., Boulder, CO, USA; ^bAnn and H. J. Smead Aerospace Engineering Sciences, University of Colorado Boulder, Boulder, CO, USA

ABSTRACT

First results of optical turbulence field measurements collected with a newly developed fine-wire temperature sensing system are presented and discussed. The centerpiece of the sensing system is an array of fine-wire platinum resistance thermometers. The active fine wire in each sensor element has a diameter of $0.64\text{ }\mu\text{m}$ and a length between 0.5 and 1 mm. The sampling rate is 44.1 kHz, and the noise level is 1 mK for a bandwidth of 10 kHz. Data were recorded while the car onto which the sensors were mounted was traveling at a speed of about 40 mph, or 18 m s^{-1} . Estimates of the temperature structure function $D_{\theta\theta}(r)$ are compared against the classical Obukhov-Corrsin theory, which predicts r^α power-law asymptotes with $\alpha = 2$ in the viscous-diffusive range and $\alpha = 2/3$ in the inertial-convective range. For the pair of separations $r_1 = 5\text{ cm}$ and $r_2 = 10\text{ cm}$, we observed $\alpha = 0.63 \pm 0.06$. The frequency spectrum $S_{\theta\theta}(f)$ follows the theoretically predicted $f^{-5/3}$ power law in the inertial-convective sub-range. The ‘Hill bump’ in the transition regime between the inertial-convective and viscous-diffusive subranges is visible.

ARTICLE HISTORY

Received 17 December 2021
Accepted 8 May 2022

KEYWORDS

Optical turbulence; fine-wire thermometry; wave propagation; viscous-diffusive subrange; Hill bump

1. Introduction

The modern physics of wave propagation through the turbulent atmosphere was pioneered by Obukhov, Yaglom, Tatarskii and coworkers in the former Soviet Union in the 1950s and 1960s, as is documented in Tatarskii’s two seminal monographs [1,2]. Tatarskii’s main achievement was the integration of Obukhov’s theory of scalar turbulence [3], Yaglom’s mathematical theory of stochastic processes and random fields [4,5], and Maxwell’s electromagnetic field theory into a comprehensive theory of wave propagation through turbulent media. As pointed out by Monin and Yaglom [6, p. 496], however, Tatarskii also conducted ground-breaking observational research:

The most detailed measurements of the spatial structure function $D_{\theta\theta}(r)$ in the surface layer of the atmosphere (at heights of 1.5, 16, and 22 m, and r varying between 3 and 100 cm) were carried out by Tatarskii (1956) using two resistance thermometers and a special squaring device which automatically gives the mean square of the temperature difference at two points. Using the approximation $D_{\theta\theta}(r) \sim r^\alpha$, Tatarskii found a mean value of 0.81 for α (with relatively large spread of the individual values) which is close to the theoretical result $\alpha = 2/3 \approx 0.67$.

CONTACT Andreas Muschinski  andreas@nwra.com

© 2023 The Author(s). Published by Informa UK Limited, trading as Taylor & Francis Group
This is an Open Access article distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives License (<http://creativecommons.org/licenses/by-nc-nd/4.0/>), which permits non-commercial re-use, distribution, and reproduction in any medium, provided the original work is properly cited, and is not altered, transformed, or built upon in any way.

During the last 70 years since the publication of Obukhov's [3] and Corrsin's [7] theoretical papers on the spatial statistics of temperature fluctuations in fully developed turbulence, steady and significant progress has been made on this topic with respect to theoretical analysis [6,8–10], sensor development [11,12], field measurement [9,11,13], and computer simulation [14,15].

The purpose of the present paper is twofold: first, to demonstrate the usefulness of a newly developed fine-wire thermometer system for high-resolution turbulence measurements in the open atmosphere; second, to compare the measurements against theoretical predictions.

The paper is organized as follows. Section 2 summarizes the main results of the Obukhov-Corrsin theory. Section 3 gives a brief description of our newly developed fine-wire thermometer system. Section 4 outlines a field experiment that we conducted in the evening of 15 September 2020 in the Colorado Front Range near Longmont, CO. First results of the field observations are presented in Section 5, and summary and conclusions are given in Section 6.

2. Theory

The Obukhov-Corrsin theory [3,7] is the classical theory of fully developed, locally homogeneous and isotropic, turbulent temperature fluctuations, θ . Obukhov [3] found universal similarity relationships for the second-order structure function,

$$D_{\theta\theta}(\mathbf{r}) = \langle [\theta(\mathbf{x} + \mathbf{r}) - \theta(\mathbf{x})]^2 \rangle. \quad (1)$$

Independently of Obukhov's work, Corrsin [7] found the $-5/3$ power law for the wavenumber spectrum in the inertial-convective subrange and, by analogy with Heisenberg's [16] theory of fully developed turbulence in the velocity field, derived a -7 power law for the temperature spectrum in the viscous-diffusive range.

Obukhov predicted the existence of two different ranges for r : the inertial-convective subrange, where r is sufficiently large such that the molecular kinematic viscosity, ν , and the molecular diffusivity of heat, κ , have no effect on the spatial statistics of θ . This range is referred to as the inertial-convective subrange, within which $D_{\theta\theta}(\mathbf{r})$ must be a universal function of the separation $r = |\mathbf{r}|$, the scalar variance dissipation rate χ , and the dissipation rate ε of the turbulent kinetic energy. Obukhov used dimensional analysis to derive $D_{\theta\theta}(r)$ for the inertial-convective subrange:

$$D_{\theta\theta}(r) = \beta \varepsilon^{-1/3} \chi r^{2/3}, \quad (2)$$

where β is a universal, dimensionless constant known as the Obukhov constant, or Obukhov-Corrsin constant. It is often useful to restate Equation (2) as

$$D_{\theta\theta}(r) = C_\theta^2 r^{2/3} \quad (3)$$

[1, p. 46, Eq. 3.18], where

$$C_\theta^2 = \beta \varepsilon^{-1/3} \chi \quad (4)$$

[1, p. 47, Eq. 3.19] is referred to as the temperature structure parameter.

At very small scales, $D_{\theta\theta}(r)$ is predicted to depend only on χ , α , and r , and direct analysis, based on the scalar transport equation, leads to [3, Eq. 19]

$$D_{\theta\theta}(r) = \frac{1}{6} \frac{\chi}{\kappa} r^2. \quad (5)$$

The value of r at which the viscous-diffusive and inertial-convective asymptotes intersect is referred to as the inner scale,

$$\ell_0 = (6\beta\kappa)^{3/4} \varepsilon^{-1/4}. \quad (6)$$

The inner scale plays an important role for modeling the transition of the 3D wavenumber spectrum, $\Phi_{\theta\theta}(\kappa)$, from the inertial-convective subrange into the viscous-diffusive range [1,17,18].

3. A new fine-wire thermometer system

Here we give a brief description of our newly developed fine-wire thermometer system.

Each platinum resistance temperature detector (PRTD) was constructed using Wollaston wire (manufactured by Sigmund Cohn Corp.), comprised of silver-clad 0.64 μm platinum fine wire. A short length of the wire was affixed to a pair of pins, spaced 0.1 inches, and etched using a dilute nitric acid solution to expose a short (less than 1 mm) length of fine platinum wire.

The recording system ultimately provides accurate samples of PRTD resistance, as it varies in time. A circuit was built to provide a small excitation current to the sensors through comparatively-large-valued excitation resistors, via a battery supply, and to provide a differential signal interface to the recorder for digitization. 4-wire and 2-wire resistance temperature detector topologies were explored, and accuracies are comparable for the wire lengths involved, where calibration accounts for losses due to cable resistance and recorder input load resistance. High-quality shielded twisted pair wire was used to connect each fine wire sensor to the excitation circuit, which was in turn connected to the microphone inputs of a commercial multi-channel digital audio recorder, Tascam DR-680 (with 24-bit quantization and various sampling rates starting at 44.1 kHz). Recorder sensitivity was determined by splitting a 1 kHz sine wave of known voltage amplitude across the input channels, and referencing that to the corresponding unitless data values of the recorded signal, for each channel, at the gain settings used in the experiments. This measured sensitivity is included in the calibration to convert voltage samples to the corresponding PRTD resistance values.

4. Field measurements

In the evening of 15 September 2020, we collected temperature measurements by means of an array of three fine-wire sensors (labeled FW5, FW7, and FW8), which we attached to a microphone stand. The stand was mounted in a car and extended through the sun-roof 1.0 m above the roof line, angled forward somewhat, while the sensor mounting was adjusted to be horizontal, with sensors FW7 and FW8 spaced 5 cm above and below the mount, respectively, in front, facing into the wind. Sensor FW5 was placed at the same height as FW7, 10 cm directly behind FW7. The car's roof line was 1.5 m above the ground, such that the sensors were approximately 2.5 m above ground level (AGL).

Recording began at 2238 MDT (Mountain Daylight Time) and lasted less than 1 h. We divided the data into 13 ‘runs’, referred to as Run 408 through Run 420. In this paper, we present and analyze data only from Run 411. Run 411 began at 2247 MDT and lasted 3 min 12 s. During Run 411, the car was traveling southward on North 75th Street between Hygiene, CO and Niwot, CO. During the first minute of Run 411, the car traveled at a nominal speed of 40 mph (17.9 m s^{-1}), and later it was accelerated to 45 mph (20.1 m s^{-1}).

5. Results

5.1. Time series of temperature fluctuations

Figure 1 shows, as a representative example, 5-s long temperature signals measured with sensors FW5 and FW7. At a speed of 18 m s^{-1} , a 5-s interval corresponds to a path length of 90 m. The two signals track each other well, as expected. Closer inspection shows that the FW5 signal slightly lags behind the FW7 signal.

The delay of the FW5 signal with respect to the FW7 signal can be seen in Figure 2, which depicts a 0.2-s long subsection of Figure 1. The time interval of 0.2 s corresponds to a path length of 3.6 m. If the true air speed (the speed of the air relative to the sensors)

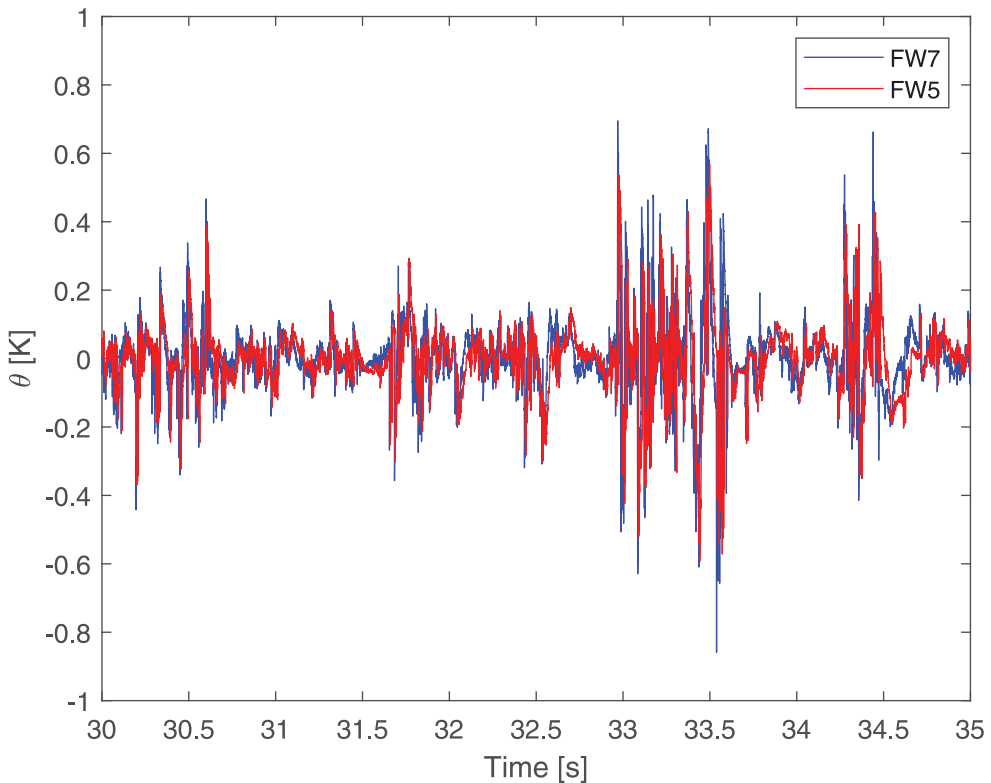


Figure 1. A 5-s-long segment of the temperature signals measured with the fine-wire sensors FW7 (blue) and FW5 (red) during Run 411. The measurements were made out of a car traveling at 40 mph (18 m s^{-1}), such that 5 s correspond to a path length 90 m.

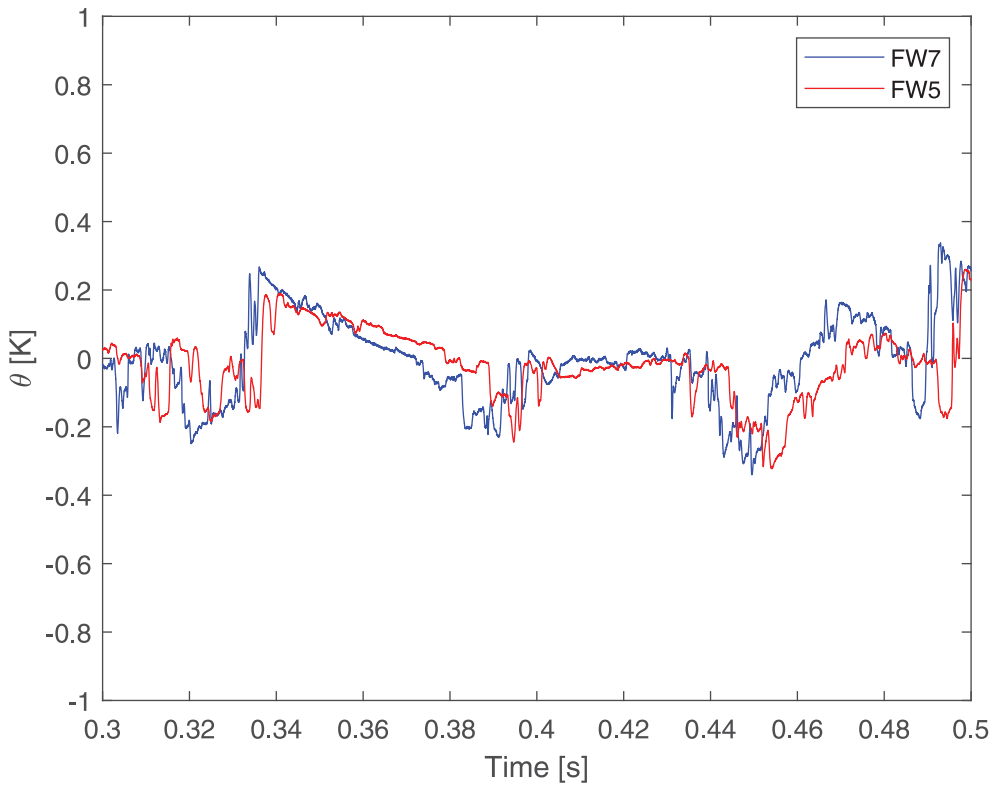


Figure 2. A 0.2-s long segment of the signals shown in Figure 1, corresponding to a path segment of 3.6 m. It can be clearly seen that the FW5 signal lags the FW7 signal by about 5 ms, consistent with the 10-cm spacing between the two sensors.

were 18 m s^{-1} , that is, if the wind speed relative to the ground were small compared to the speed of the car, then one would expect a delay of $0.1 \text{ m} / (18 \text{ m s}^{-1}) = 5.6 \text{ ms}$, which is in qualitative agreement with the delay that can be seen in Figure 2.

Figure 3 shows the cross-covariance function,

$$B_{12}(\tau) = \langle \theta_1(t + \tau) \theta_2(t) \rangle, \quad (7)$$

estimated from the two 5-s long time series measured with FW5 and FW7, respectively. The maximum of $B_{12}(\tau)$ is at $\tau_m = 5.3 \text{ ms}$, which for the sensor spacing $d = 10 \text{ cm}$ gives a true air speed

$$V = \frac{d}{\tau_m} \quad (8)$$

of 18.8 m s^{-1} , or 42 mph. That is, the wind speed relative to the ground was small compared to the nominal speed of the car. (We did not measure the wind speed relative to the ground directly.)

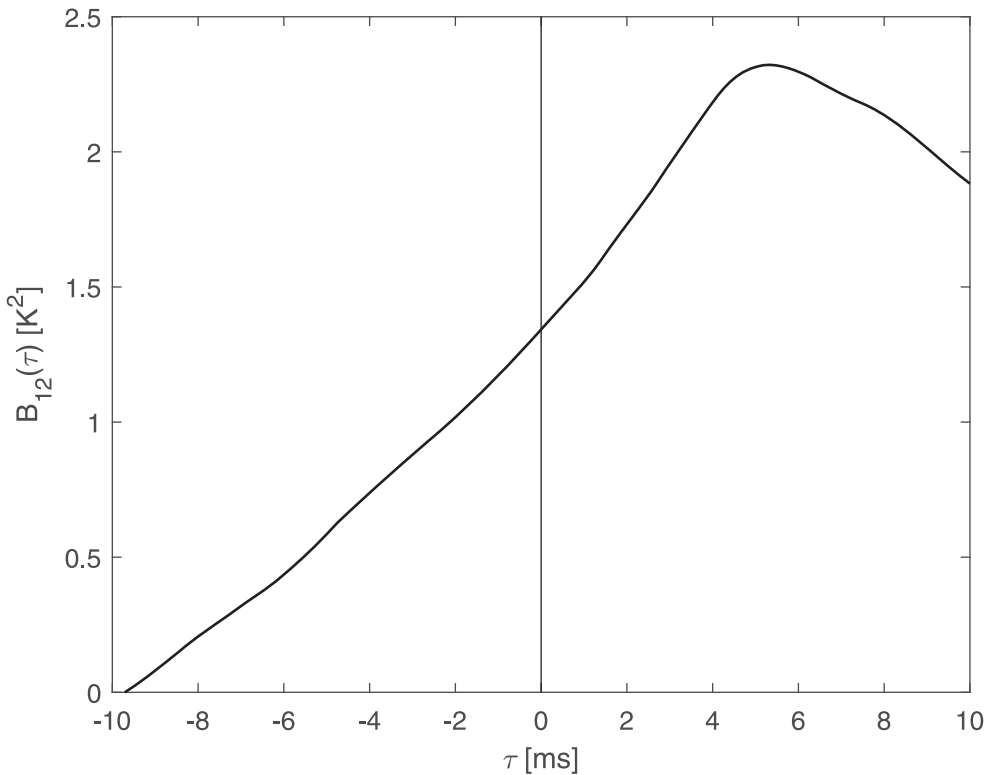


Figure 3. Cross-covariance function $B_{12}(\tau)$ estimated from the 5-s-long FW5 and FW7 signals shown in Figure 1. The peak of $B_{12}(\tau)$ occurs at $\tau = 5.3$ ms, corresponding to a true air speed of 18.8 m s^{-1} .

5.2. Temperature structure function

Figure 4 shows the structure function $D_{\theta\theta}(r)$ estimated from the 5-s long FW7 signal in Figure 1. The separation r was obtained from

$$r = V\tau, \quad (9)$$

where τ is the time lag and V is the true air speed obtained from Equation (8).

Figure 4 demonstrates that the observed $D_{\theta\theta}(r)$ follows an r^2 law for small r and an $r^{2/3}$ law for intermediate r , as predicted in Equations (5) and (3) for the viscous-diffusive subrange and inertial-convective subrange, respectively. Using Equation (3), we can calculate C_θ^2 from the $r^{2/3}$ asymptote and find $C_\theta^2 = 0.058 \text{ K}^2 \text{ m}^{-2/3}$. The inner scale ℓ_0 is the value of r at which the two asymptotes intersect, and we find $\ell_0 = 8.6 \text{ mm}$.

Moreover, Figure 4 shows that for r comparable to and larger than about 1 m, $D_{\theta\theta}(r)$ becomes flat. This is due to the fact that the recording system has a built-in high-pass filter with a cut-off frequency of order 10 Hz, which corresponds to a temporal lag τ of order 0.1 s and a spatial separation $r = V\tau$ of order 2 m. In addition to, and independently of, the high-pass filter effect, a flattening of $D_{\theta\theta}(r)$ occurs also due to the fact that in the lower part of the atmospheric surface layer, the outer scale of turbulence is comparable to the measurement height above ground level, which in our case is 2.5 m. In the current study, we make no attempt to disentangle the two effects from each other.

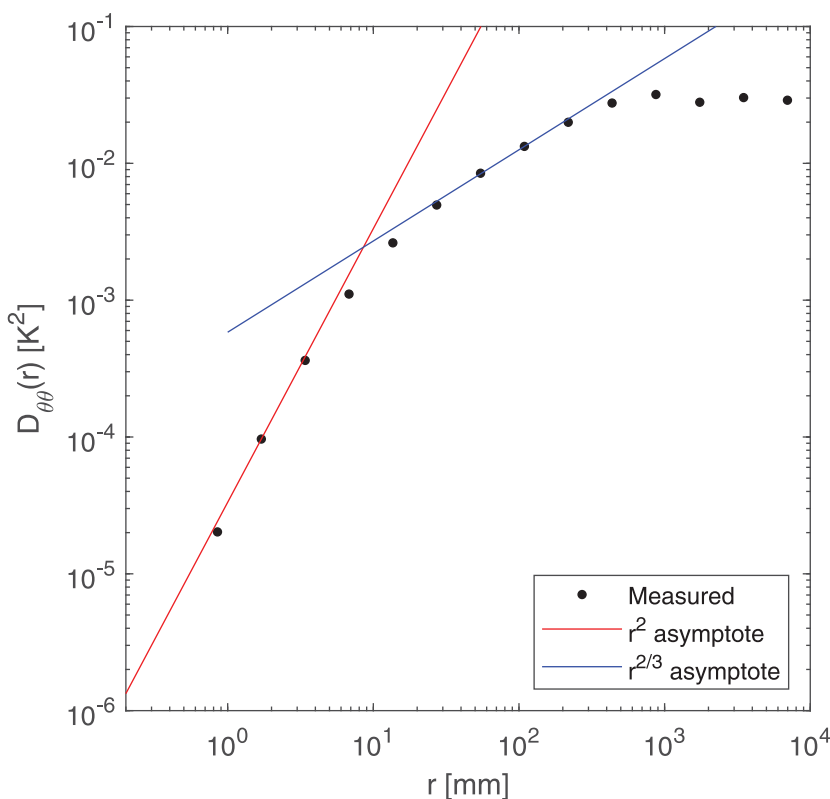


Figure 4. Temperature structure function $D_{\theta\theta}(r)$ estimated from the 5-s long FW7 time series shown in Figure 1. The measured structure function follows an r^2 power for small r and an $r^{2/3}$ power law at intermediate scales, as theoretically predicted by the Obukhov-Corrsin theory; see Equations (5) and (2), respectively.

5.3. Time series of true air speed, temperature structure parameter, inner scale, and power-law exponent

The analysis that we have shown above exemplarily for a single 5-s long interval we have performed for each 5-s interval within the 192-s long dataset collected during Run 411. Figure 5 shows the time series of the true air speed, V , which varies between 17 and 21.5 m s^{-1} . Our records indicate that the nominal driving speed was 40 mph (17.9 m s^{-1}) during the first minute of Run 411, and later the car was accelerated to 45 mph (20.1 m s^{-1}). The retrieved V values fluctuate around the nominal car speed by 2 m s^{-1} at most. Due to a lack of more accurate data of the car's speed, it is impossible to ascertain to what extent these fluctuations represent car-speed fluctuations, wind-speed fluctuations, or a combination of both. The magnitude of the fluctuations, however, is consistent with the wind speed of 2 m s^{-1} reported for 2248 MDT by the Boulder Municipal Airport.

We used the V estimates shown in Figure 5 to estimate C_θ^2 and ℓ_0 for each of these 5-s intervals, as shown in Figure 6. C_θ^2 varies between 0.011 and $0.087 \text{ K}^2 \text{ m}^{-2/3}$, and ℓ_0 varies between 5 and 13 mm. Closer inspection reveals that C_θ^2 and ℓ_0 are practically uncorrelated.

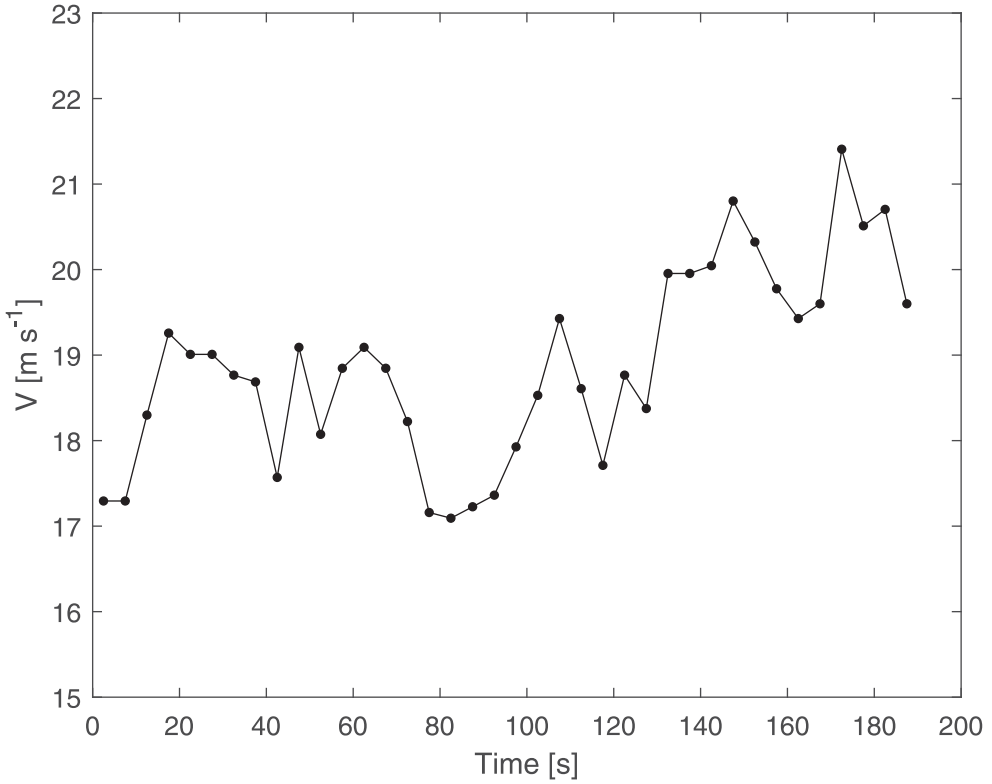


Figure 5. True air speed V determined for each 5-s interval for the entire 192-s long Run 411.

Of particular interest is the question about the robustness of the $r^{2/3}$ power law predicted for the inertial-convective range. In Figure 4 we have seen that the observed $D_{\theta\theta}(r)$ follows the $r^{2/3}$ power law most closely for r comparable to 10 cm, which is large compared to ℓ_0 and small compared to $h = 2.5$ m, the height above ground level. In fact, 10 cm is comparable to $\sqrt{\ell_0 h}$, the geometrical mean of ℓ_0 and h . If we take 10 mm as the typical value of ℓ_0 , then $\sqrt{\ell_0 h} = 16$ cm.

We analyzed $D_{\theta\theta}(r)$ for a set of r values that varied by a factor of 2, that is, $r = 2^n VT_s$, where $n = 0 \dots 15$ and $T_s = 1/44100 \text{ s} = 23 \mu\text{s}$ is the sampling period of our recording system. In order to estimate the power-law exponent α , we chose the pair of lags $\tau_1 = 128T_s$ and $\tau_2 = 256T_s$, which corresponds to the pair of separations $r_1 = 128VT_s$ and $r_2 = 256VT_s$. For $V = 18 \text{ m s}^{-1}$, this gives $r_1 = 5.2$ cm and $r_2 = 10.4$ cm. Assuming a power law, $D_{\theta\theta}(r) \propto r^\alpha$, we have

$$\frac{D_{\theta\theta}(r_2)}{D_{\theta\theta}(r_1)} = 2^\alpha, \quad (10)$$

such that

$$\alpha = \frac{\ln[D_{\theta\theta}(r_2)/D_{\theta\theta}(r_1)]}{\ln 2}. \quad (11)$$

Figure 7 shows the estimates of the power-law exponent α for each of the 5-s intervals during Run 411. The 38 α estimates have a mean value of 0.63, which is slightly smaller than

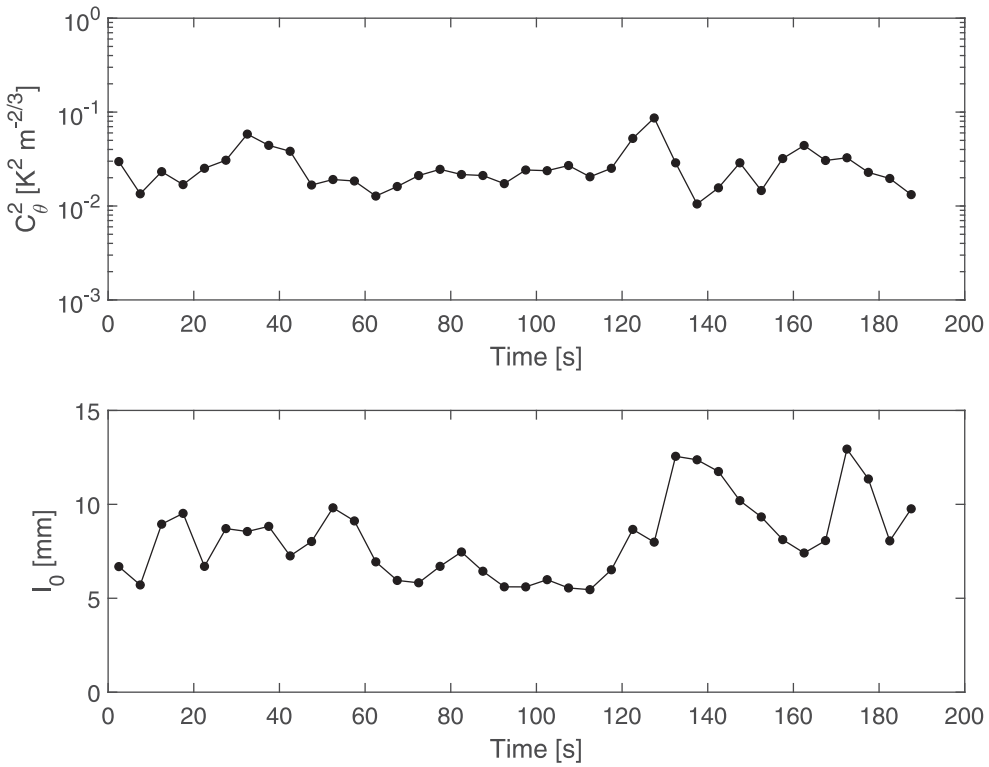


Figure 6. Temperature structure parameter C_θ^2 (upper panel) and inner scale ℓ_0 (lower panel) determined for each 5-s interval for the entire 192-s long Run 411.

the theoretically predicted value $2/3 \approx 0.67$. The standard deviation of the α estimates is 0.06, such that $2/3$ lies well within the interval 0.63 ± 0.06 .

5.4. Refractive-index structure parameter

If humidity effects are negligible, which is the case here (the relative humidity reported for 2248 MDT by the Boulder Municipal Airport was only 18%), the optical refractive index n of air is a function of the wavelength λ , the (absolute) temperature $T + \theta$ and the pressure p ,

$$n = 1 + a \left(1 + \frac{b}{\lambda^2} \right) \frac{p}{T + \theta}, \quad (12)$$

where T is the (absolute) mean temperature, θ is the turbulent temperature fluctuation, and $a = 7.76 \times 10^{-7} \text{ K Pa}^{-1}$ and $b = 7.52 \times 10^{-15} \text{ m}^2$ are constants [19, p. 10]. Turbulent pressure fluctuations are negligible if the Mach number is low, which is the case here, and we obtain the following relationship between the refractive-index structure parameter, C_n^2 , and the temperature structure parameter, C_θ^2 :

$$C_n^2 = a^2 \left(1 + \frac{b}{\lambda^2} \right)^2 \frac{p^2}{T^4} C_\theta^2. \quad (13)$$

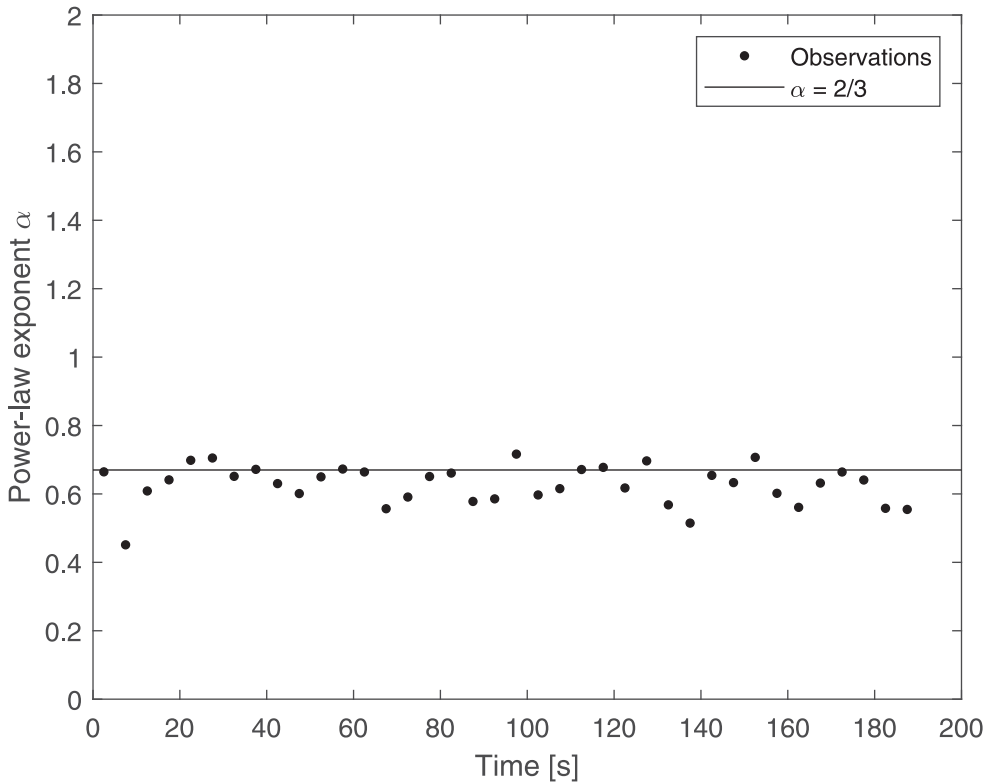


Figure 7. Power-law exponent in the inertial-convective range determined for each 5-s interval for the entire 192-s long Run 411.

We did not measure T and p as part of our fine-wire experiment, but we can use the observations reported by the Boulder Municipal Airport for 2248 MDT on 15 September 2020 ($T = 293$ K, $p = 842$ hPa) in order to estimate the constant of proportionality between C_n^2 and C_θ^2 as stated in Equation (13). For $\lambda = 500$ nm (green light), we obtain $C_n^2/C_\theta^2 = 6.2 \times 10^{-13} \text{ K}^{-2}$. That is, $C_n^2 = 6.2 \times 10^{-13} \text{ m}^{-2/3}$ if $C_\theta^2 = 1 \text{ K}^2 \text{ m}^{-2/3}$.

Figure 8 shows the C_n^2 time series computed from the C_θ^2 time series shown in Figure 6. The C_n^2 values range from $6 \times 10^{-15} \text{ m}^{-2/3}$ to $5 \times 10^{-14} \text{ m}^{-2/3}$, and the median of C_n^2 is $1.4 \times 10^{-14} \text{ m}^{-2/3}$. Nighttime C_n^2 values of order $1 \times 10^{-14} \text{ m}^{-2/3}$ in the atmospheric surface layer over land are consistent with earlier reports [20,21].

5.5. Temperature spectrum

Figure 9 shows the one-sided frequency power spectrum $S_{\theta\theta}(f)$ estimated from the entire 192-s long temperature time series recorded during Run 411. We estimated the spectrum by means of Welch's method [22] of the 'modified periodogram'. We used an overlap of 50% between subsequent 'Welch segments' and applied a Blackman window to each segment. The length of each Welch segment was 1 s, which gives $192 \times 2 - 1 = 383$ Welch segments, each of which was spectrally analyzed prior to incoherent averaging. Subsequently, we applied frequency-domain averaging to the averaged modified periodogram,

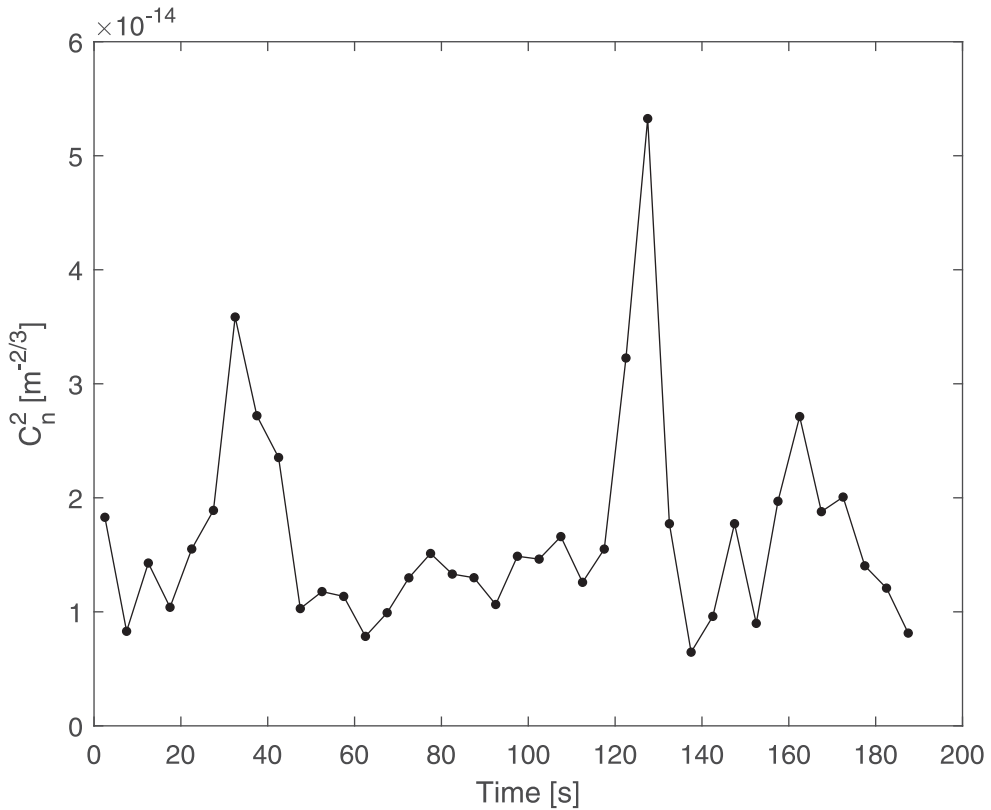


Figure 8. Time series of C_n^2 , computed from the C_θ^2 time series shown in Figure 6.

such that the final spectrum contains 30 spectral estimates per decade in the frequency range from 22.05 Hz to the Nyquist frequency of 22.05 kHz.

The spectrum follows the theoretically predicted $f^{-5/3}$ law at frequencies between about 10 Hz and several 100 Hz. We estimated the spectral level of the $f^{-5/3}$ asymptote, shown as the black line in Figure 9, from the observed $S_{\theta\theta}(f)$ in the interval from 20 to 100 Hz. The observed spectral density slightly exceeds the $f^{-5/3}$ asymptote in the frequency range from 100 to 600 Hz. This phenomenon is known as the ‘Hill bump’ [23,24]. Above 600 Hz, the observed $S_{\theta\theta}(f)$ drops rapidly, marking the transition into the viscous-diffusive range.

At very low and very high frequencies, the observed spectrum shows artifacts. The roll-off of $S_{\theta\theta}(f)$ to below the $f^{-5/3}$ asymptote at frequencies below about 10 Hz is caused by the recording system’s built-in high-pass filter. The flattening of the spectrum above about 4 kHz indicates noise. The noise spectral density is about $1 \times 10^{-10} \text{ K}^2\text{Hz}^{-1}$. If we consider 10 kHz as the useful bandwidth for turbulence measurements, $1 \times 10^{-10} \text{ K}^2\text{Hz}^{-1}$ corresponds to a noise r.m.s. value of 1 mK.

5.6. Collocated fine-wire and sonic measurements

The calibration of fine-wire thermometers relies on assumptions about the relationship between the fluctuation of the wire resistance, which is the primary measureable, and the

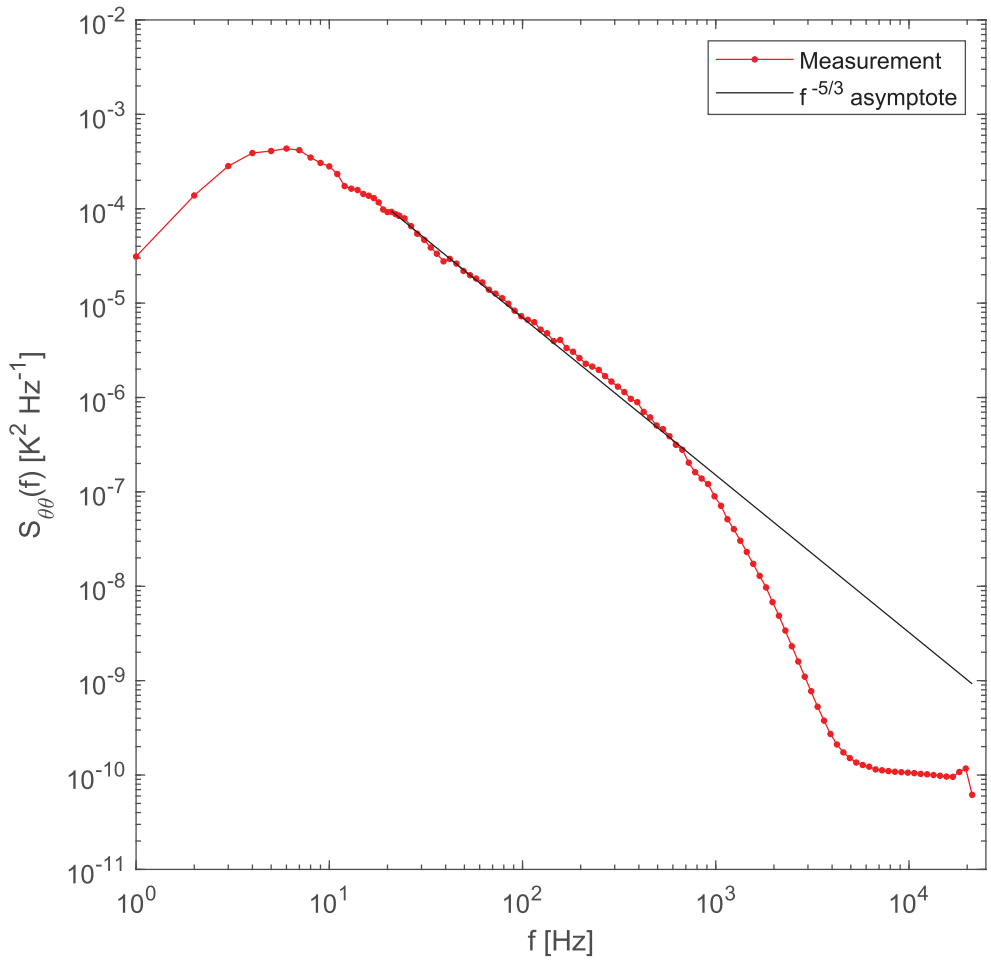


Figure 9. Temperature spectrum estimated from the entire FW7 temperature signal recorded during the 192-s long Run 411.

air-temperature fluctuation θ , which is the observable that we are interested in. The centerpiece of this relationship is an empirical model for the heat transfer between a heated cylindrical body (the electrically heated fine wire in our case) immersed in the cross-flow of a fluid (in our case the ambient air). We have reviewed the classical literature [12,25–30] and have developed a calibration procedure based on Churchill and Bernstein’s model [30, p.302] for the Nusselt number, Nu , as a function of the molecular Prandtl number, Pr , and the Reynolds number:

$$Nu = 0.3 + 0.62 \frac{Pr^{1/3}}{[1 + (0.4/Pr)^{2/3}]^{1/4}} Re^{1/2}. \quad (14)$$

For air, Pr can be treated as a constant, which leads to

$$Nu = A + B Re^{1/2}, \quad (15)$$

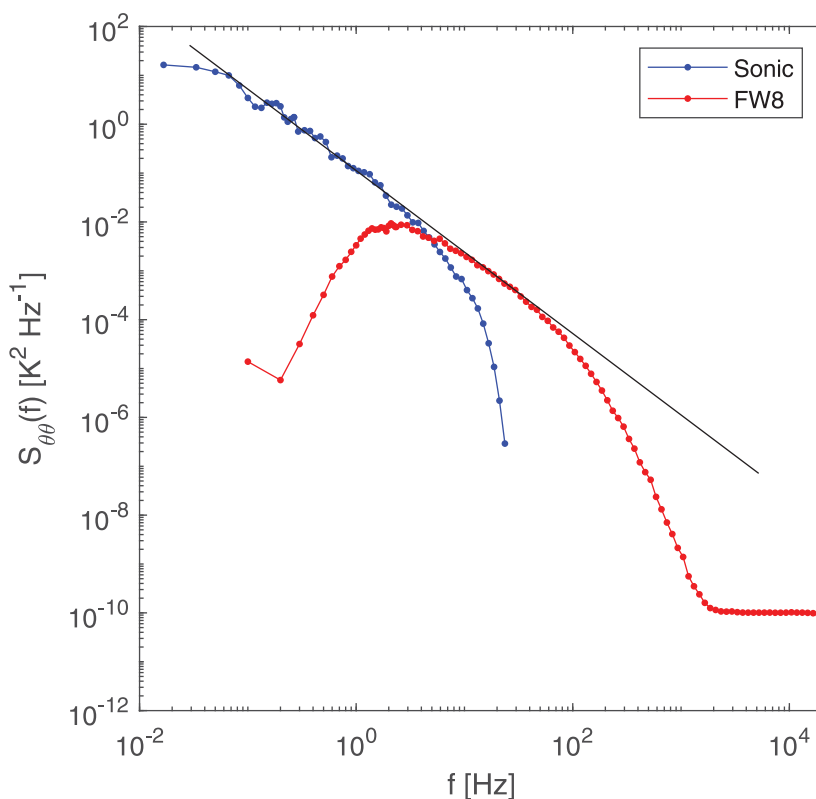


Figure 10. Temperature spectra estimated from 10-min long temperature time series simultaneously measured with two collocated sensors: an ultrasonic anemometer-thermometer (blue) and a fine-wire thermometer (red). The black line indicates an $f^{-5/3}$ power law.

a relationship known as ‘King’s law’. A more detailed account of our calibration procedure will be given elsewhere.

As an empirical test of the integrity of our calibration procedure, we measured turbulent air-temperature fluctuations in the atmospheric surface layer with our fine-wire thermometer system and with a collocated ultrasonic anemometer-thermometer (‘sonic’). The experiment was conducted on 5 October 2021 in a residential area in the Rocky Mountain foothills several miles west of Boulder, CO. The sonic (Model 81000, manufactured by R. M. Young Company, of Traverse City, MI) was mounted on a portable tower at 1.6 m AGL, and the fine-wire was attached to a microphone stand at the same height.

Figure 10 shows the temperature spectra estimated from the temperature fluctuations measured with fine-wire sensor FW8 and the sonic, respectively, during the 10-min period beginning at 1136 MDT. The mean wind speed was $U = 1.2 \text{ m s}^{-1}$ during this 10-min period. In the inertial-convective subrange, the observed spectra follow the same $f^{-5/3}$ power law, except where the fine-wire measurements are affected by the built-in highpass filter (at frequencies below about 10 Hz) and the sonic measurements are affected by lowpass filtering caused by the finite size of the sonic’s observation volume. The sonic spectrum begins to drop significantly between the $f^{-5/3}$ asymptote at a frequency of a few hertz, which for $U = 1.2 \text{ m s}^{-1}$ corresponds to a length scale $l = U/f$ of a few decimeters. The roll-off of

the fine-wire spectrum from the inertial-convective range into the viscous-diffusive range occurs at about 100 Hz, indicating an inner scale ℓ_0 of order 1 cm.

6. Summary and conclusions

We have shown that our fine-wire thermometer system is capable of collecting high-quality measurements of turbulent temperature fluctuations in the open atmosphere. Even though the 0.64- μm -diameter platinum fine wires are extremely delicate, our three sensors did not break when exposed to a true air speed of about 20 m s^{-1} over the course of several tens of minutes.

The active length of the wires is less than 1 mm, which is small compared to 5 mm, the smallest value of the inner scale ℓ_0 observed during Run 411. Therefore, the measured temperature fluctuations are not affected by spatial filtering due to the finite size of the sensors. The wires' response time is so short that temporal filtering plays no role either. The noise level of the entire system is so low that the spectrum can be measured well into the viscous-diffusive range.

We have shown that the delay between signals measured with a pair of longitudinally spaced sensors (we used a spacing of 10 cm) can be used for reliably measuring the true air speed (the speed of the air relative to the sensor) if the platform moves fast compared to the wind speed relative to the ground.

The temperature structure function $D_{\theta\theta}(r)$ and the frequency spectrum $S_{\theta\theta}(f)$ estimated from the measured temperature fluctuations are consistent with the predictions of the classical Obukhov-Corrsin theory. We determined the power-law exponent α of $D_{\theta\theta}(r)$ from observed values of the ratio $D_{\theta\theta}(r_2)/D_{\theta\theta}(r_1)$, where we chose $r_1 = 5$ cm and $r_2 = 10$ cm, and we found $\alpha = 0.63 \pm 0.06$, which is consistent with the theoretically predicted value $2/3 \approx 0.67$.

In the future, we intend to deploy our fine-wire sensing system on portable meteorological towers as well as on moving platforms such as cars, boats, balloons, and unmanned arial systems (UAVs). Arrays of fine-wire thermometers continue to be valuable tools for in-depth investigations of various aspects of optical turbulence, including intermittency [31], anisotropy and temperature 'sheets' [32–34], and the possible relevance of turbulent phase screens with non-Gaussian increments [35].

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by Air Force Office of Scientific Research [grant numbers FA9550-17-C-0021, FA9550-18-C-0011, FA9550-18-1-0009].

ORCID

Andreas Muschinski  <http://orcid.org/0000-0003-4648-2888>

References

- [1] Tatarskii VI. Wave propagation in a turbulent medium. New York: McGraw-Hill; 1961.
- [2] Tatarskii VI. The effects of the turbulent atmosphere on wave propagation. Jerusalem (Israel): Israel Program for Scientific Translation; 1971.
- [3] Obukhov AM. The structure of the temperature field in a turbulent flow. *Izv Akad Nauk SSSR, Ser Geogr i Geofiz.* 1949;13:58–69.
- [4] Yaglom AM. Introduction to the theory of stationary random functions. *Uspekhi Mat Nauk.* 1952;7:3–168.
- [5] Yaglom AM. An introduction to the theory of stationary random functions. Englewood Cliffs (NJ): Prentice-Hall; 1962.
- [6] Monin AS, Yaglom AM. Statistical fluid mechanics. Vol. 2. Cambridge (MA): The MIT Press; 1975.
- [7] Corrsin S. On the spectrum of isotropic temperature fluctuations in an isotropic turbulence. *J Appl Phys.* 1951;22:469–473.
- [8] Monin AS, Yaglom AM. Statistical fluid mechanics. Vol. 1. Cambridge (MA): The MIT Press; 1971.
- [9] Haugen DA, editor. Workshop on micrometeorology. Boston: American Meteorological Society; 1973.
- [10] Frisch U. Turbulence: the legacy of A. N. Kolmogorov. Cambridge: Cambridge University Press; 1995.
- [11] Kaimal JC, Finnegan JJ. Atmospheric boundary layer flows: their structure and measurement. Oxford (UK): Oxford University Press; 1994.
- [12] Bruun HH. Hot-wire anemometry. Oxford (UK): Oxford University Press; 1995.
- [13] Foken T. Micrometeorology. 2nd ed. Berlin (Germany): Springer; 2017.
- [14] Pope SB. Turbulent flows. Cambridge (UK): Cambridge University Press; 2000.
- [15] Sagaut P. Large eddy simulation for incompressible flow. 3rd ed. Berlin (Germany): Springer-Verlag; 2006.
- [16] Heisenberg W. Zur statistischen Theorie der Turbulenz. *Z Phys.* 1948;124:628–657.
- [17] Hill RJ, Clifford SF. Modified spectrum of atmospheric temperature fluctuations and its application to optical propagation. *J Opt Soc Am.* 1978;68:892–899.
- [18] Muschinski A. Temperature variance dissipation equation and its relevance for optical turbulence modeling. *J Opt Soc Am A.* 2015;32:2195–2200.
- [19] Clifford SF. The classical theory of wave propagation in a turbulent medium. In: Strohbehn JW, editor. Laser beam propagation in the atmosphere. Berlin: Springer; 1978. p. 9–43.
- [20] Lawrence RS, Ochs GR, Clifford SF. Measurements of atmospheric turbulence relevant to optical propagation. *J Opt Soc Am.* 1970;60:826–830.
- [21] Weseley ML, Alcaraz EC. Diurnal cycles of the refractive index structure function coefficient. *J Geophys Res.* 1973;78:6224–6232.
- [22] Welch PD. The use of fast Fourier transform for the estimation of power spectra: a method based on time averaging over short, modified periodograms. *IEEE Trans Audio Electroacoust.* 1967;AU-15:70–73.
- [23] Hill RJ. Models of the scalar spectrum for turbulent advection. *J Fluid Mech.* 1978;88:541–562.
- [24] Muschinski A, de Bruyn Kops SM. Investigation of Hill's optical turbulence model by means of direct numerical simulation. *J Opt Soc Am A.* 2015;32:2423–2430.
- [25] King LV. On the convection of heat from small cylinders in a stream of fluid: determination of the convection constants of small platinum wires with applications to hot-wire anemometry. *Phil Trans R Soc Lond A.* 1914;214:373–432.
- [26] Nusselt W. Das Grundgesetz des Wärmeüberganges. *Gesundheits-Ing.* 1915;38:477–482, 490–496.
- [27] Collis DC, Williams MJ. Two-dimensional convection from heated wires at low Reynolds numbers. *J Fluid Mech.* 1959;6:357–384.
- [28] Wyngaard JC. The effect of velocity sensitivity on temperature derivative statistics in isotropic turbulence. *J Fluid Mech.* 1971;48:763–769.
- [29] Williams RM, Paulson CA. Microscale temperature and velocity spectra in the atmospheric boundary layer. *J Fluid Mech.* 1977;83:547–567.

- [30] Churchill SW, Bernstein M. A correlating equation for forced convection from gases and liquids to a circular cylinder in crossflow. *J Heat Transfer*. 1977;99:300–306.
- [31] Muschinski A, Frehlich RG, Balsley BB. Small-scale and large-scale intermittency in the nocturnal boundary layer and the residual layer. *J Fluid Mech*. 2004;515:319–351.
- [32] Muschinski A, Wode C. First in situ evidence for coexisting submeter temperature and humidity sheets in the lower free troposphere. *J Atmos Sci*. 1998;55:2893–2906.
- [33] Balsley BB, Jensen ML, Frehlich RG, et al. Extreme gradients in the nocturnal boundary layer; structure, evolution, and potential causes. *J Atmos Sci*. 2003;60:2496–2508.
- [34] Muschinski A. Optical propagation through non-overturning, undulating temperature sheets in the atmosphere. *J Opt Soc Am A*. 2016;33:793–800.
- [35] Muschinski A. Phase-factor spectra of turbulent phase screens. *J Opt Soc Am A*. 2021;38:1339–1348.

Low-wavenumber compensation with Zernike tilt for non-Kolmogorov turbulence phase screens

ERANDI WIJERATHNA,^{1,*} HANYU ZHAN,¹ DAVID VOELZ,¹ AND ANDREAS MUSCHINSKI^{2,3}

¹Klipsch School of Electrical and Computer Engineering, New Mexico State University, Las Cruces, New Mexico 88001, USA

²NorthWest Research Associates, 3380 Mitchell Lane, Boulder, Colorado 80301, USA

³Ann and H. J. Smead Aerospace Engineering Sciences, University of Colorado Boulder, 3775 Discovery Drive, Boulder, Colorado 80303, USA

*Corresponding author: erandi@nmsu.edu

Received 26 September 2022; revised 21 December 2022; accepted 22 December 2022; posted 22 December 2022; published 6 February 2023

The fast-Fourier-transform-based filtering method for phase screen generation remains popular for numerical simulation of optical propagation through turbulence; however, these screens inherently underrepresent the spectral density at low wavenumbers. Here, the “Z-tilt” approach is explored to augment the spectral density at low wavenumbers by adding a random phase tilt, which is derived from the wavefront phase statistics of a Zernike polynomial basis. This approach is computationally efficient and can be applied to any statistically homogeneous and isotropic refractive index field. An analytic result is provided for the von Kármán spectrum with finite outer scale. In a quantitative comparison with phase screens compensated for using a common subharmonic approach, the Z-tilt method shows the best agreement with the analytical structure function when the outer scale is greater than about three times the screen dimension. For outer scales of the order of the screen dimension, the subharmonic and a modified Z-tilt method give the most accurate results. A propagation simulation demonstrates that the aperture-averaged angle-of-arrival variance is accurately predicted using the Z-tilt method. © 2023 Optica Publishing Group

<https://doi.org/10.1364/AO.475825>

1. INTRODUCTION

The split-step wave optics method is a powerful approach for numerical simulation of optical propagation through atmospheric turbulence and is widely applied for free space applications such as laser beam propagation, optical communication, adaptive optics, and imaging [1–7]. Propagation is based on a numerical implementation of scalar diffraction theory, and the propagation path is divided into segments where a phase screen that models the turbulence is placed in each segment. The existing techniques for creating phase screens mainly include the fast Fourier transform (FFT)-based method, also known as the spectral method [8,9], Zernike-polynomial-based method [10–13], phase-covariance-based method [14–16], and the sparse spectrum method [17]. Among these, the FFT-based method remains popular due to its computational efficiency and implementational convenience.

The FFT-based method starts with a two-dimensional array of uncorrelated, circularly symmetric, complex Gaussian random variables, and this array is filtered in the Fourier domain to impose a turbulence refractive index power spectral density such as the Kolmogorov, von Kármán, or Hill models. An inverse FFT produces a realization of the random phase screen in physical space. A well-known shortcoming of the approach is that the lowest wavenumber represented is $2\pi/L$, where L is the physical grid side length of the phase screen [3,8,9,18].

Consequently, the phase screens cannot correctly represent low-wavenumber turbulence effects (e.g., random tilt or beam wander) when L is smaller than the turbulence outer scale L_0 . To accurately represent these effects, wavenumbers down to $2\pi/L_0$ are generally required.

Much investigation has been devoted to the proper augmentation of the low wavenumbers in the FFT method. One approach involves mid-point displacement; for example, a bilinearly interpolated point between four existing points on the phase screen is randomly displaced [8,14]. Another recent method is based on the frequency shift property of the Fourier transform and a random spectral sampling of the grid [18]. Perhaps the most widely known method is the “addition of subharmonics,” where the missing low wavenumbers are incorporated through the creation and addition of screens that represent subharmonics of $2\pi/L$ [3,8,9,13,14]. These methods improve the statistical accuracy of the screen but can significantly increase the computational expense [18,19].

Other hybrid approaches have been devised where the FFT-generated phase screen is combined with a screen generated by another method. One such approach is to generate the compensating screen by the so-called covariance method where the phase correlation matrix is applied to build the low-frequency phase screen [20]. However, this approach is computationally inefficient due to the need to manipulate large covariance

matrices. An attractive hybrid approach aims to create the compensating screen by weighting low-wavenumber Zernike polynomial terms with appropriate random Gaussian amplitudes (coefficients). For instance, in [13], the compensation of an FFT-generated phase screen by means of a 37-order Zernike screen is described, although the methodology for parameter choices and the potential issue of “excess compensation” of the low frequencies were not addressed.

Apart from the hybrid approach, we note that the Zernike-polynomial-based method can also be used to generate the complete screen, but this again is computationally expensive because a large number of terms are required to depict the high-frequency terms accurately [17,20].

In this paper, our concern is with an extremely simplified form of the FFT and Zernike polynomial hybrid approach where only the two orthogonal tilt modes are added to the FFT screen. These tilt modes are commonly referred to as “Zernike tilt” or “Z-tilt.” This augmentation is approximate because low wavenumbers with orders higher than tilt are not explicitly modeled; however, we show that this approach often generates phase screen with accurate spatial structure functions.

The tilt modes, described in Section 2.A, require very little computation so the phase screen creation is very efficient. The combined screen, $\phi_{FZ}(x, y)$, is described by

$$\phi_{FZ}(x, y) = \phi_F(x, y) + \phi_Z(x, y), \quad (1)$$

where $\phi_F(x, y)$ is the FFT-based phase screen, and $\phi_Z(x, y)$ is the screen corresponding to the Z-tilt. We note that the Z-tilt method is not new and likely was first implemented around the time of Noll’s 1976 paper [10] that describes the Zernike mode contributions associated with the Kolmogorov spectrum, but literature detailing the approach is scarce. In [20], the authors discuss the addition of a tilt-only screen to the FFT screen, although their method does not take advantage of the Zernike modal analysis to determine the tilt contribution. Otherwise, the Z-tilt method is only briefly mentioned in several other publications [17,21,22] and was previously limited to screens with the Kolmogorov spectrum [7,9,23].

This paper makes the following contributions: (1) we describe the approach of computing realizations of the Z-tilt screen for any turbulence phase spectrum and present the result for the Kolmogorov spectrum as an example; (2) we derive the Z-tilt screen coefficients for the von Kármán spectrum with finite outer scale; (3) we examine the structure functions of the phase screens derived with the Z-tilt approach and compare the results with the subharmonic method and a modified form of the Z-tilt approach for various scenarios; and (4) we present aperture-averaged angle-of-arrival (AOA) propagation simulation results that demonstrate the performance differences when using the Z-tilt or subharmonic compensation methods. We find that for larger outer scales, the Z-tilt method produces screens with spatial structure functions that typically show higher accuracy at less computational expense than the results obtained with the subharmonic method. Such quantitative comparisons over a wide range of outer scale values have not been presented before.

2. THEORY AND PHASE SCREEN CONSTRUCTION

A. Zernike Tilt Definition

The phase of a waveform after propagation through turbulence can be represented by a series of orthonormal polynomials [10,24], where the Zernike polynomial basis is the most commonly used. Depending on the definition of the basis, the set of polynomials has several differing forms of normalizations [10,25,26]. Here, we choose the modified Zernike basis with Noll’s normalization [10] and follow the presentation by Conan [27]. The polynomial expansion of a zero-mean phase screen $\phi(r, \theta)$ over a circular aperture of radius R is given by

$$\phi(r, \theta) = \sum_{j=1}^{\infty} a_j Z_j \left(\frac{r}{R}, \theta \right), \quad (2)$$

where the polar coordinates of radial distance r and angle θ are defined over the ranges $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$, respectively. Z_j are Zernike polynomials defined over a unit circle and are separable in r and θ . The particular form of Z_j expanded over the circular aperture and the respective orthonormal conditions associated with the polynomials can be found in [27]. a_j is the Zernike coefficient, and its relationship with Z_j is defined in Appendix A, Eq. (A1). The mode ordering number j is a function of radial degree n and azimuth frequency m , where n and m are assigned such that $m \leq n$, and $n - |m|$ is even.

Of interest here are the x - and y -tilt Zernike polynomial modes given as

$$\text{when } j = 2, n = 1, m = 1, \quad Z_2 = 2(r/R) \cos \theta = 2x/R, \quad (3a)$$

$$\text{when } j = 3, n = 1, m = -1, \quad Z_3 = 2(r/R) \sin \theta = \frac{2y}{R}, \quad (3b)$$

where x and y are Cartesian coordinates, and $-R \leq x \leq R$, $-R \leq y \leq R$. The combined tilt contribution is given by $a_2 Z_2 + a_3 Z_3$. In practice, phase screens are commonly constructed in a square format rather than a circle. However, the definitions for the tilt terms in Eq. (3) also apply to a square when the side length of the square is equal to the circle diameter ($2R$) [28].

Construction of the Z-tilt phase screen requires knowing the variance of coefficients a_2 and a_3 , which depend on the phase power spectrum. Using Eqs. (A1)–(A5) of Appendix A and applying j , n , and m as listed in Eq. (3), the variance of the tilt coefficients is found to be

$$\langle |a_2|^2 \rangle = \langle |a_3|^2 \rangle = 8\pi \frac{1}{R^2} 2 \int_0^\infty d\kappa \Phi_\phi(\kappa) \frac{[J_2(R\kappa)]^2}{\kappa}, \quad (4)$$

where κ is the radial spatial wavenumber in units of rad/m, J_2 is the second order Bessel function of first kind, and $\Phi_\phi(\kappa)$ is the phase power spectrum.

1. Z-Tilt Phase Screen Construction

As an abundance of literature can be found on the generation of the primary FFT phase screen $\phi_F(x, y)$ (see [9] and references therein), it is not discussed here for brevity. To generate the Z-tilt

phase screen, we use the following prescription based on the sum of the two orthogonal Zernike tilt coefficients:

$$\phi_Z(x, y) = (A_x 2x/R + A_y 2y/R) \sqrt{\langle |a_2|^2 \rangle}, \quad (5)$$

where A_x and A_y are random number draws from a zero mean, unit variance Gaussian distribution, and $\langle |a_2|^2 \rangle$ is the variance of the tilt coefficients [Eq. (4)]. To generate a realization of the screen, a random value is drawn for A_x and another for A_y , and Eq. (5) is evaluated over the numerical grid. The screen corresponds to a physical size of $L \times L$, where $L = 2R$ is the screen side length. The complete turbulence phase screen is the sum of the FFT and Z-tilt screens as illustrated in Eq. (1).

3. ZERNIKE TILT VARIANCE EXAMPLES

A. Kolmogorov Spectrum

For illustration purposes, we first present the tilt variance for the Kolmogorov refractive index power spectrum. The 3D spectrum, which is also known as the Obukhov–Corrsin spectrum [29,30], is given by [31]

$$\Phi_n^{Kol}(K) = 0.033 C_n^2 K^{-11/3}, \quad (6)$$

where K is the radial spatial wavenumber, which is understood to be in 3D space, and C_n^2 is the refractive index structure constant. The associated phase power spectrum for a homogeneous and isotropic slab of turbulence of thickness Δz is found with the following relationship [27,31]:

$$\Phi_\phi(\kappa) = 2\pi k^2 \Delta z \Phi_n(\kappa), \quad (7)$$

where $k = 2\pi/\lambda$ is the optical wavenumber, λ is the wavelength, and the argument is now the radial spatial wavenumber κ , which is understood to be in a 2D plane. Equation (7) is generally applicable if the perturbations to a wavefront passing through the slab can be modeled as phase only. Utilizing Eqs. (6) and (7), the phase power spectrum is given by [9,10]

$$\Phi_\phi^{Kol}(\kappa) = (0.49/r_0^{5/3}) \kappa^{-11/3}, \quad (8)$$

where r_0 is the Fried parameter assuming a plane wave and the Kolmogorov spectrum [9,24]:

$$r_0 = (0.423 k^2 C_n^2 \Delta z)^{-3/5}. \quad (9)$$

The derivation of the covariance for the Zernike coefficients for the Kolmogorov spectrum is detailed in Eqs. (A6)–(A8) in Appendix A. The evaluation of Eq. (A8) for the tilt terms in Eq. (3) yields [32,33]

$$\langle |a_2|^2 \rangle^{Kol} = 0.449 \left(\frac{D}{r_0} \right)^{5/3}, \quad (10)$$

where $D = 2R$. This variance result is then used in Eq. (5) for a Kolmogorov spectrum screen where the screen side length is $L = D$.

B. von Kármán Spectrum

The so-called von Kármán spectrum includes a term to describe the effect of the finite outer scale L_0 [9,27,34]:

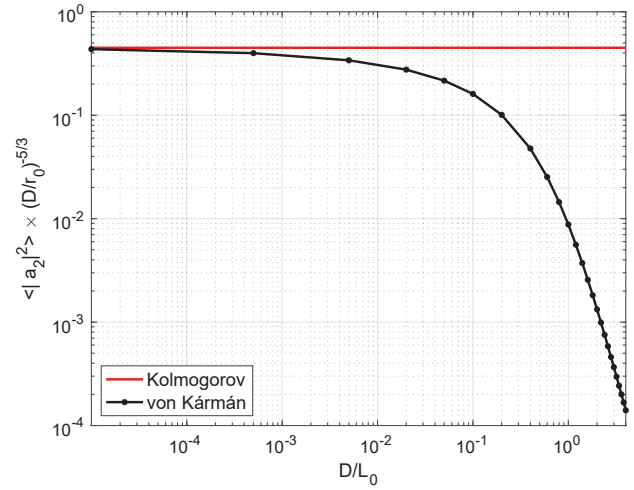


Fig. 1. Behavior of relative tilt variance with D/L_0 . For smaller D/L_0 , the curve asymptotes to the Kolmogorov constant 0.449 in Eq. (10).

$$\Phi_n^K(K) = 0.033 C_n^2 (K^2 + \kappa_0^2)^{-11/6}, \quad (11)$$

where $\kappa_0 = 2\pi/L_0$. Applying Eqs. (7) and (9), the phase power spectrum is

$$\Phi_\phi^{vK}(\kappa) = (0.49/r_0^{5/3}) (\kappa^2 + \kappa_0^2)^{-11/6}. \quad (12)$$

Following Eqs. (A9) and (A10), the closed form solution of the variance for tilt coefficients with a von Kármán turbulence spectrum can be written as [35–37]

$$\begin{aligned} \langle |a_2|^2 \rangle^{vK} = & \frac{0.49 \times 8\pi}{2^{5/3}} \left(\frac{D}{r_0} \right)^{5/3} 2 \left\{ \frac{(D\kappa_0/2)^{1/3}}{2^5} \frac{\Gamma(-\frac{1}{6})}{2\Gamma(3)\Gamma(\frac{11}{6})} \right. \\ & \times {}_2F_3 \left[2, \frac{5}{2}; 3, 5, \frac{7}{6}; \left(\frac{D\kappa_0}{2} \right)^2 \right] \\ & + \frac{1}{2^{14/3}} \frac{\Gamma(\frac{14}{3}) \Gamma(\frac{1}{6})}{\Gamma(\frac{29}{6}) [\Gamma(\frac{17}{6})]^2} \\ & \left. \times {}_2F_3 \left[\frac{11}{6}, \frac{7}{3}; \frac{17}{6}, \frac{29}{6}, \frac{5}{6}; \left(\frac{D\kappa_0}{2} \right)^2 \right] \right\}, \end{aligned} \quad (13)$$

where ${}_2F_3$ is the generalized hypergeometric function. To examine the behavior of Eq. (13), in Fig. 1, we plot a relative form of the variance, $\langle |a_2|^2 \rangle^{vK} \times (D/r_0)^{-5/3}$, as a function of D/L_0 . It is evident that as $L_0 \rightarrow \infty$ ($D/L_0 \rightarrow 0$), the Kolmogorov result is produced. As L_0 decreases, the tilt variance decreases exponentially. When $D/L_0 = 1$, the variance is about a factor of 50 smaller than the Kolmogorov result.

C. von Kármán–Tatarskii Spectrum

A more complete model of the turbulence power spectrum also includes the inner scale (l_0), and the question arises of whether it is necessary to include the inner scale effect when computing the

Z-tilt variance for such spectra. Since the phase spectral power generally decreases rapidly with κ , then an inner scale cutoff should have little effect on the integral result in Eq. (4). To test this, we considered the turbulence phase spectrum originally proposed by Tatarskii [38] that includes both the von Kármán outer scale and an inner scale contribution:

$$\Phi_{\phi}^{vKT}(\kappa) = (0.49/r_0^{5/3})(\kappa^2 + \kappa_0^2)^{-11/6} \exp\left(-\frac{\kappa^2}{\kappa_m^2}\right), \quad (14)$$

where $\kappa_m = 5.91/l_0$ [39]. A closed-form analytic solution for the tilt variance with this spectrum is not available, so Eq. (4) was evaluated using numerical integration. The results showed that the tilt variance is essentially equivalent to the von Kármán result with less than a 1% difference even when l_0 is as large as 10 mm and L_0 is as small as 250 mm.

4. PHASE SCREEN EVALUATION

A. Phase Screen Methods for Comparison

For comparison purposes, we analyzed the performance of the following four phase screens: the FFT-based screen with no low frequency compensation (results denoted by ϕ_F), the Zernike tilt compensated screen presented in this paper (ϕ_{FZ}), a subharmonic compensated screen (ϕ_{FSH}), and a Zernike tilt compensated screen where the overall tilt is removed from the primary FFT screen ($\phi_{F'Z}$). The results assumed the von Kármán spectrum (Section 3.B). Details about the last two screens follow.

We applied the method described by Lane *et al.* [8] for the subharmonic compensation screens because it appears to be the most common implementation in the literature [3,9]. The subharmonic method is widely used although it is known to underestimate the lowest frequencies for the Kolmogorov spectrum. More discussion is provided on this point in Section 5.B. The choice of the number of subharmonic modes (N_{SH}) used for this screen is important because a smaller number of modes improves the computational efficiency, whereas a larger number improves the modeling accuracy of the low wavenumbers. A practical value of $N_{SH} \approx 10$ has been proposed for modeling the Kolmogorov spectrum [19,40], and for the results presented here, we used $N_{SH} = 10$ for all outer scale cases considered.

When the outer scale L_0 is of the order of the grid side length L , the primary FFT screen already contains most of the required low frequencies, and therefore, adding the Z-tilt phase represents an excess compensation of the low-wavenumber contributions. To explore this issue, we generated a screen where the inherent tilt in the primary FFT phase screen is removed before adding the Z-tilt component. This phase screen is denoted by $\phi_{F'Z}$ where the prime indicates the tilt removal. The inherent tilt in the FFT screen was removed using a 2D polynomial fit based on the QR factorization provided in MATLAB.

B. Structure Function Analysis

Statistical accuracy of the turbulence phase screens was evaluated in terms of the phase structure function $D_{\phi}(r)$, where r is the spatial separation increment. Equations for the analytical structure functions $D_{\phi}^A(r)$ for the Kolmogorov and von Kármán

phase spectra are provided for reference in Appendix B. For numerical calculation of the phase screen structure function $D_{\phi}^S(r)$, the FFT was utilized. To reduce effects due to the finite size of the phase screen, which can act as an aperture (window), we employed the method in [9]. Specifically, the phase screen was embedded in a grid of twice its size (zero padded), and a circular window with a diameter corresponding to the screen width was applied. A structure function was computed for each individual screen realization, and an average over 10^4 independent realizations was performed. Finally, a radial average was applied to produce the average structure function with radial parameter r . We found that 10^4 realizations provided a repeatable result.

The structure function results were evaluated by computing the normalized root mean square error (NRMSE) between the analytical and radially averaged phase screens, as given by

$$\text{NRMSE} = \sqrt{\frac{\sum_{i=1}^N (D_{\phi}^A(r_i) - D_{\phi}^S(r_i))^2}{\sum_{i=1}^N (D_{\phi}^A(r_i))^2}}, \quad (15)$$

where N is the number of increment samples, and i is the index for spatial separations.

5. RESULTS

A. Structure Function Comparisons

Table 1 shows the parameters used for creating the random draws of phase screens in this study. Weak turbulent fluctuations were assumed ($\sigma_R^2 \ll 1$, where σ_R^2 is the Rytov variance) to ensure the applicability of the analytical expressions for $D_{\phi}(r)$. Note that the screen side length is $L = 1$ m.

Figure 2 shows the radially averaged structure function $D_{\phi}(r)$ results. The plots represent outer scale values L_0 ranging from “infinite” (Kolmogorov spectrum) to 0.5 m with the largest outer scale results shown in Fig. 2(a). The results include curves for the analytic solution, the FFT phase screen without compensation, and the three compensated screens described in Section 4.A.

For $L_0 \gg L$ [Fig. 2(a)], the structure function for the ϕ_{FZ} screen essentially overlays the analytic results. The other screens underestimate the structure function for large r . The result for $\phi_{F'Z}$ is the next closest to the analytic solution, and this is followed by the result for ϕ_{FSH} . As expected, the performance of the uncompensated for FFT screen ϕ_F is poor for large separations. In general, underestimation of the structure function for large r implies that the lowest mode (tilt) is underrepresented in the screen.

Table 1. Phase Screen Parameter Values for $D_{\phi}(r)$ Computation

Parameter	Notation	Values
Grid physical size	$L \times L$	1 m $\times$ 1 m
Grid sample spacing	$\Delta x, \Delta y$	1 mm
Fried parameter	r_0	0.139 m
Outer scale	L_0	∞ to 0.5 m
Number of subharmonics	N_{SH}	10
Number of realizations		10^4

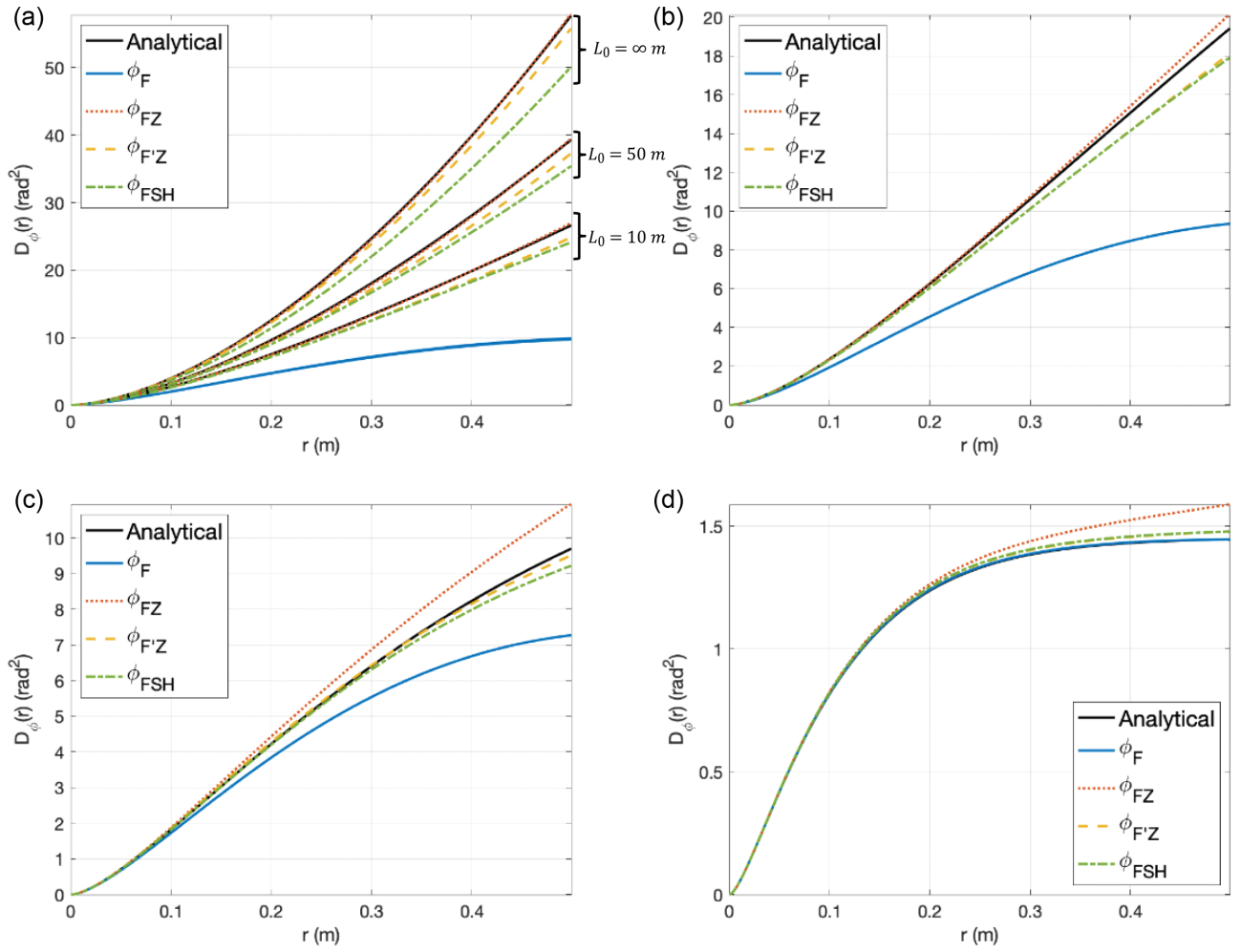


Fig. 2. Comparison of analytical phase structure function with average phase screen structure functions for (a) $L_0 = \infty$, 50 and 10 m, (b) $L_0 = 5$ m, (c) $L_0 = 2$ m, and (d) $L_0 = 0.5$ m. The parameters of the phase screens are listed in Table 1. The screen side length is $L = 1$ m.

For L_0 near the size of L [Figs. 2(b)–2(d)], the Z-tilt compensated for screen overestimates the structure function at large r due to the issue of excess compensation of the low wavenumbers. The structure functions for ϕ_{FSH} and $\phi_{F'Z}$ move closer to the analytic solution as L_0 decreases.

The differences apparent in Fig. 2 are quantified in Fig. 3, which shows the structure function normalized MSE [Eq. (15)] for the various screens as a function of L_0 . The ϕ_{FZ} screen shows the lowest structure function error for large L_0 (right side of Fig. 3). As the outer scale becomes shorter, there is a reversal near $L_0 \approx 3$ m ($L_0 \approx 3L$), where $\phi_{F'Z}$ and ϕ_{FSH} give the lowest error. For $L_0 \approx L$, all methods give similar performance except ϕ_{FZ} , due to the excess compensation. The uncompensated for FFT screen demonstrates the least error when $L_0 < L$.

As mentioned in Section 4.A, other researchers have found that the subharmonic method by Lane *et al.* underestimates the structure function for the Kolmogorov spectrum at large r [9,40,41]. Increasing the number of modes N_{SH} can improve the structure function comparison for the subharmonic screen; however, in our experiments, we found that the amount of improvement for $N_{SH} > 10$ was minimal. In fact, for $N_{SH} > 30$, the results can be inconsistent due to computing errors. These results are similar to the findings of Johansson and

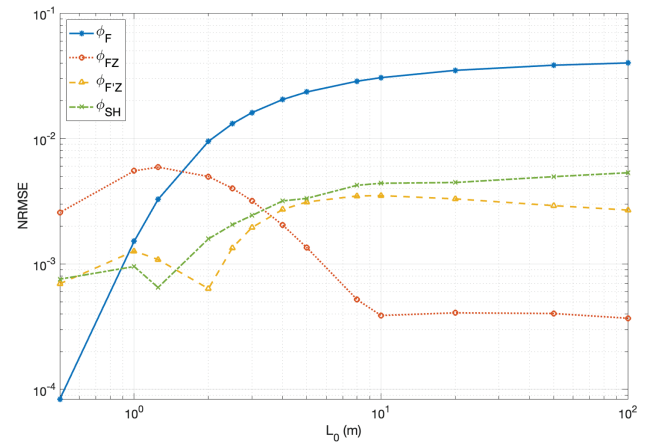


Fig. 3. Normalized root mean square error (NRMSE) of the radially averaged $D_\phi(r)$ for the four phase screens as a function of L_0 . The screen side length is $L = 1$ m.

Gavel [41]. These authors also proposed a hybrid subharmonic method that provides better accuracy in the Kolmogorov case, although the approach involves more complexity and can be computationally expensive.

B. Propagation Example for Angle of Arrival

The accuracy required for modeling the low wavenumbers for phase screens used in a propagation simulation depends on the quantity of interest. For example, the intensity scintillation index for a plane wave is a weak function of the overall wavefront tilt and outer scale. On the other hand, AOA fluctuations or the value of the phase coherence diameter (Fried parameter) can be more sensitive to tilt.

Propagation simulation studies that explore low-wavenumber compensation include Frehlich [3], who examined Gaussian beam and plane wave propagation through turbulence using the split-step simulation approach with FFT-based screens. He compared the results of no compensation with subharmonic compensation and demonstrated differences in intensity profiles and normalized intensity variance (scintillation index). He also demonstrated that low-wavenumber compensation has little effect on the scintillation index for a plane wave.

In contrast, our work focuses on the performance of different compensation methods, and here we chose to illustrate differences in simulated aperture-averaged AOA variance ($\langle \bar{\theta}^2 \rangle$) results for a plane wave propagated through turbulence using screens with the Z-tilt and subharmonic methods.

The split-step simulation and $\langle \bar{\theta}^2 \rangle$ calculations were conducted as described in [7,42]. The simulation parameters are listed in Table 2. The parameters correspond to the weak turbulent fluctuation regime ($\sigma_R^2 \ll 1$). Six phase screens were used to model the propagation path, and 10 subharmonics were used for the subharmonic screens. Briefly, to obtain one realization of the aperture-averaged AOA, a plane wave is propagated a distance z to a circular aperture of radius r . The maximum aperture radius was limited to one fourth of the screen size ($r = L/4$) to reduce grid edge effects. A focus function (lens) is applied to the aperture field, and the result is propagated to the focal plane where the centroid of the focal spot is calculated. The aperture-averaged AOA for a given realization is the respective centroid coordinate divided by the focal length. The simulation process is repeated 10^4 times, and the variance $\langle \bar{\theta}^2 \rangle$ is calculated.

The simulation results were also compared to an analytical result given by [43]

$$\langle \bar{\theta}^2 \rangle^{Kol} = \gamma_P(q) C_n^2 z (2r)^{-\frac{1}{3}} = 0.81 \gamma_P(q) \sigma_R^2 k^{-\frac{7}{6}} z^{-\frac{5}{6}} (2r)^{-\frac{1}{3}}, \quad (16)$$

Table 2. Simulation Parameter Values for $\langle \bar{\theta}^2 \rangle$ Computation

Parameter	Notation	Values
Grid physical size	$L \times L$	1 m $\times$ 1 m
Grid sample spacing	$\Delta x, \Delta y$	1 mm
Wavelength	λ	500 nm
Full path length	z	2000 m
Structure parameter	C_n^2	$10^{-16} \text{ m}^{-2/3}$
Fried parameter	r_0	0.139 m
Structure parameter		6
Outer scale	L_0	$\infty, 50 \text{ m}$ and 5 m
Number of subharmonics	N_{SH}	10
Number of realizations		10^4

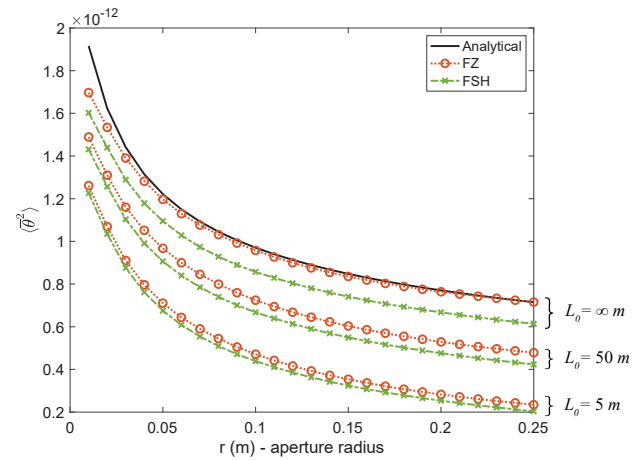


Fig. 4. Comparison of variances of the aperture-averaged AOA as a function of aperture radius r for $L_0 = \infty, 50$ and 5 m in weak turbulence $C_n^2 = 10^{-16} \text{ m}^{-2/3}$. The parameters of the phase screens are listed in Table 1. The screen side length is $L = 1 \text{ m}$.

where

$$\gamma_P(q) = \frac{\sqrt{3}}{16} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{8}{3}\right) \left(\frac{\beta}{2}\right)^{-\frac{1}{3}} \left\{ 1 + \frac{6}{5} \left(\frac{\pi}{2}\right)^{\frac{1}{6}} \beta^{\frac{1}{3}} q^{\frac{1}{3}} \right. \\ \left. \times \left[1 + \frac{\pi^2}{4} \beta^4 q^4 \right]^{\frac{5}{12}} \sin \left[\frac{5}{6} \arctan \left(\frac{2}{\pi \beta^2 q^2} \right) \right] \right\},$$

and

$$\beta = 0.5216, \quad q = \frac{2r}{\sqrt{\lambda z}}.$$

Equation (16) is based on weak turbulence fluctuation theory (Rytov approximation) and assumes a circular aperture and the Kolmogorov spectrum (infinite outer scale).

Figure 4 shows the simulation results for $\langle \bar{\theta}^2 \rangle$ as a function of aperture radius (r) and for L_0 values ranging from 5 m to infinity. The curve for infinite outer scale with the Z-tilt compensated screens (FZ) is in good agreement with the analytical curve. The slight departure of the FZ curve from the analytical result for small r can be attributed to a Fresnel filtering effect [7] where diffractive smoothing is modeled in the simulation but is not accounted for in the theory.

For all outer scales, the results involving the subharmonic screens (FSH) fall below the FZ curves, indicating the AOA values are underpredicted using the subharmonic screens. As expected, the difference between curves is the largest (up to $\sim 20\%$) for infinite L_0 , whereas the difference for $L_0 = 5 \text{ m}$ is relatively small. It is also apparent that the FZ and FSH differences tend to increase slightly with aperture radius.

C. Elapsed Time Comparison

Timing tests were carried out for generating the compensation phase screens for the Z-tilt and subharmonic methods. The results presented here do not include the time required for the primary FFT screen, as that is common to both. The machine

used for the comparisons was a Windows 10 HP Omen laptop with a 2.3–2.3–4.8GHz Intel processor with a GPU of NVIDIA GeForce RTX 2080 Turing GPU with 2944 CUDA and compute capability of 7.5. The code was implemented in MATLAB and timed with the tic/toc function. We found on average that the elapsed time to generate the Z-tilt screen is about 0.21 ms. This is 90 times shorter than for the subharmonic method with $N_{\text{SH}} = 2$, where the average elapsed time is 18.4 ms, more than 170 times shorter when $N_{\text{SH}} = 4$, where the average time is about 35.5 ms, and 400 times shorter when $N_{\text{SH}} = 10$, where the average time is 88.9 ms. For reference, the elapsed time to generate the primary FFT screen is on average about 31.1 ms, which is about 15 times longer than the Z-tilt screen. Other hardware and software language implementations will produce somewhat different execution time results, but we expect that the relative differences in speed between the Z-tilt and subharmonic methods will remain similar. These differences can translate to hours more execution time for subharmonic screens when statistics are computed with a simulation that involves multiple screens and tens of thousands of trials.

6. CONCLUSION

The additive Z-tilt method is a simple, efficient, and accurate approach to compensate for FFT-based turbulence phase

APPENDIX A: DERIVATIONS OF ZERNIKE COEFFICIENTS VARIANCE

The primary steps in the derivations of the Zernike coefficient covariances are summarized here for reference. The derivation follows the work of Conan [27] and Noll [10]. In accordance with Eq. (2) where $\phi(r, \theta)$ is defined over a circular aperture of radius R , the Zernike coefficients are defined to satisfy the relationship

$$a_j = \int d^2r W\left(\frac{r}{R}\right) \phi(r, \theta) Z_j\left(\frac{r}{R}, \theta\right), \quad (\text{A1})$$

where $W(r/R)$ is the aperture weight function (i.e., pupil function). The covariance of the coefficients is given by

$$\begin{aligned} \langle a_j^* a_{j'} \rangle &= \int d^2r \int d^2r' W\left(\frac{r}{R}\right) Z_j^*\left(\frac{r}{R}, \theta\right) \\ &\times \langle \phi^*(r, \theta) \phi(r', \theta') \rangle \times W\left(\frac{r'}{R}\right) Z_{j'}\left(\frac{r'}{R}, \theta'\right). \end{aligned} \quad (\text{A2})$$

The quantity $\langle \phi^*(r, \theta) \phi(r', \theta') \rangle$ is the phase covariance, and by virtue of the Wiener–Khinchine theorem, the Fourier transform of this is the phase power spectrum $\Phi_\phi(\kappa)$. The Fourier transform of the quantity $W(r/R) Z_j(r/R, \theta)$ is given by

$$\begin{aligned} Q_{\text{even } j}(\kappa, \varphi) &= \sqrt{n+1} \times \frac{1}{(2\pi)^2} \times \frac{J_{n+1}(R\kappa)}{R\kappa/2} (-1)^{(n-m)/2} i^m \sqrt{2} \cos m\varphi \\ Q_{\text{odd } j}(\kappa, \varphi) &= \sqrt{n+1} \times \frac{1}{(2\pi)^2} \times \frac{J_{n+1}(R\kappa)}{R\kappa/2} (-1)^{(n-m)/2} i^m \sqrt{2} \sin m\varphi \\ Q_j(\kappa, \varphi) &= \sqrt{n+1} \times \frac{1}{(2\pi)^2} \times \frac{J_{n+1}(R\kappa)}{R\kappa/2} (-1)^{n/2}, \end{aligned} \quad \begin{matrix} m \neq 0, \\ \\ m = 0. \end{matrix} \quad (\text{A3})$$

screens for the underprediction of spectral densities at wavenumbers that are small but still larger than the cutoff wavenumber $2\pi/L_0$. Here, the method was defined, and analytical results were derived for the von Kármán spectrum where the outer scale L_0 is finite. In a comparison with phase screens compensated for using a common subharmonic approach, the Z-tilt method shows the best agreement with the analytical structure function when L_0 is greater than about three times the grid side length L . When $L_0 \approx L$, the compensation methods of subharmonics, Z-tilt with the removal of the inherent tilt, and the uncompensated for FFT-based screen all show similar performances, whereas the basic Z-tilt method is less accurate because of excess compensation of the low frequencies. A propagation simulation example illustrates that aperture-averaged AOA variance results match analytical theory when using Z-tilt compensation, whereas subharmonic screens can result in underprediction. In addition to its accuracy for larger L_0 , a significant advantage of the Z-tilt approach is the fast calculation speed for the compensation screen.

Here, J_n is the n th order Bessel function of the first kind, and $i = \sqrt{-1}$. The 2D vector spatial frequency polar coordinates $\kappa = (\kappa, \varphi)$ are defined by the radial wavenumber κ in rad/m and the polar angle φ . Other works [10,27] use wavenumbers in 1/m, but our form is more common in the wave propagation community.

Rewriting $W(r/R) Z_j(r/R, \theta)$ in terms of its Fourier transform $Q_j(\kappa, \varphi)$, and applying the Wiener–Khinchine theorem, we can write Eq. (A2) as

$$\begin{aligned} \langle a_j^* a_{j'} \rangle &= \int d^2\kappa \int d^2\kappa' Q_j^*(R\kappa, \varphi) \\ &\times \left[(2\pi)^4 \Phi_\phi(\kappa) \delta(\kappa, \varphi; \kappa', \varphi') \right] Q_{j'}(R\kappa', \varphi'), \end{aligned} \quad (\text{A4})$$

where $\delta(\kappa, \varphi; \kappa', \varphi') = \frac{1}{\kappa} \delta(\kappa - \kappa') \delta(\varphi - \varphi')$ is the 2D Dirac delta function in polar coordinates, and Φ_ϕ is the phase power spectrum of the turbulence. Inserting Eq. (A3) into Eq. (A4), the covariance of the Zernike coefficients can be simplified to

$$\langle a_j^* a_{j'} \rangle = \begin{cases} 8\pi \sqrt{(n+1)(n'+1)} (-1)^{\frac{1}{2}(n+n'-(m+m'))} \delta_{mm'} & j, j' \text{ is even or odd,} \\ \times \int_0^\infty d\kappa \kappa \Phi_\phi(\kappa) \frac{J_{n+1}(\kappa R)}{\kappa R} \frac{J_{n'+1}(\kappa R)}{\kappa R} & \text{or } m = m' = 0 \\ 0, & \text{otherwise} \end{cases}, \quad (\text{A5})$$

where $\delta_{mm'}$ is the Kronecker delta function.

Kolmogorov Spectrum

Insert the Kolmogorov phase power spectrum [Eq. (8)] into Eq. (A5) to give the following:

$$\langle a_j^* a_{j'} \rangle^{Kol} = \frac{0.49 \times 8\pi}{R^2} \left(\frac{1}{r_0} \right)^{5/3} \sqrt{(n+1)(n'+1)} \\ \times (-1)^{\frac{1}{2}(n+n'-(m+m'))} \delta_{mm'} \int_0^\infty d\kappa \kappa^{-\frac{11}{3}} \frac{J_{n+1}(R\kappa) J_{n'+1}(R\kappa)}{\kappa} \quad (\text{A6})$$

The analytical form of the integration can be found in mathematical handbooks such as [44] and other literature [10,11,45], and the result is

$$\int_0^\infty d\kappa \kappa^{-\frac{11}{3}} \frac{J_{n+1}(R\kappa) J_{n'+1}(R\kappa)}{\kappa} \\ = \frac{R^{11/3}}{2^{14/3}} \frac{\Gamma\left(\frac{14}{3}\right) \Gamma\left(\frac{n+n'-5/3}{2}\right)}{\Gamma\left(\frac{n-n'+17/3}{2}\right) \Gamma\left(\frac{n'-n+17/3}{2}\right) \Gamma\left(\frac{n+n'+23/3}{2}\right)} \quad (\text{A7})$$

Combine Eqs. (A6) and (A7), set $j = j'$, and use the fact that $n - |m|$ is even by virtue of the Zernike polynomials definition to find the expression for the variance of the coefficients:

$$\langle |a_j|^2 \rangle^{Kol} = \frac{0.49 \times \pi}{2^{10/3}} \left(\frac{D}{r_0} \right)^{5/3} (n+1)(-1)^{(n-m)} \\ \times \frac{\Gamma\left(\frac{14}{3}\right) \Gamma(n-5/6)}{\left(\Gamma\left(\frac{17}{6}\right)\right)^2 \Gamma(n+23/6)}, \quad (\text{A8})$$

where $D = 2R$. A similar form can be found in the works of Noll [10], and Wang and Markey [46]. Roddier's result in [11] essentially produces the same result; however, it contains a scaling difference of $\pi/(2^{5/3})$.

von Kármán Spectrum

Insert the von Kármán phase power spectrum [Eq. (12)] into Eq. (A5) to get

$$\langle a_j^* a_{j'} \rangle^{vK} = \frac{0.49 \times 8\pi}{R^2} \left(\frac{1}{r_0} \right)^{5/3} \\ \times \sqrt{(n+1)(n'+1)} (-1)^{\frac{1}{2}(n+n'-(m+m'))} \delta_{mm'} \\ \times \int_0^\infty d\kappa \frac{J_{n+1}(R\kappa) J_{n'+1}(R\kappa)}{\kappa (\kappa^2 + \kappa_0^2)^{11/6}}, \quad (\text{A9})$$

where both j and j' are even or odd, or $m = m' = 0$. Otherwise, $\langle a_j^* a_{j'} \rangle = 0$. The integral term can be calculated numerically and also can be found in recent integral handbooks [47]. However, the analytical formulation can also be derived by representing the product of Bessel functions in integral form, then converting to a Mellin–Barnes' type integral and performing the contour integration [48]. The integral yields

$$\int_0^\infty d\kappa \frac{J_{n+1}(R\kappa) J_{n'+1}(R\kappa)}{\kappa (\kappa^2 + \kappa_0^2)^{11/6}} \\ = R^{11/3} \frac{(R\kappa_0)^{n+n'-5/3}}{2^{n+n'+3}} \frac{\Gamma\left(\frac{5}{6} - \frac{n+n'}{2}\right) \Gamma\left(\frac{n+n'}{2} + 1\right)}{\Gamma(n+2) \Gamma(n'+2) \Gamma\left(\frac{11}{6}\right)} \\ \times {}_3F_4 \left[\frac{n+n'}{2} + 1, \frac{n+n'}{2} + 2, \frac{n+n'+3}{2}; \right. \\ \left. n+2, n'+2, n+n'+3, \frac{1}{6} + \frac{n+n'}{2}; (R\kappa_0)^2 \right] \\ + R^{11/3} \frac{1}{2^{14/3}} \frac{\Gamma\left(\frac{14}{3}\right) \Gamma\left(\frac{n+n'}{2} - \frac{5}{6}\right)}{\Gamma\left(\frac{23}{6} + \frac{n+n'}{2}\right) \Gamma\left(\frac{17}{6} + \frac{n-n'}{2}\right) \Gamma\left(\frac{17}{6} + \frac{n'-n}{2}\right)} \\ \times {}_3F_4 \left[\frac{17}{6}, \frac{11}{6}, \frac{7}{3}; \frac{17}{6} + \frac{n-n'}{2}, \frac{17}{6} + \frac{n'-n}{2}, \frac{23}{6} \right. \\ \left. + \frac{n+n'}{2}, \frac{11}{6} - \frac{n+n'}{2}; (R\kappa_0)^2 \right],$$

where ${}_pF_q$ is the generalized hypergeometric function. Combining the above two equations, and setting $j = j'$ for the variance,

$$\langle |a_j|^{2vK} \rangle = \frac{0.49 \times 8\pi}{2^{5/3}} \left(\frac{D}{r_0} \right)^{5/3} (n+1)(-1)^{(n-m)} \\ \times \left\{ \frac{(D\kappa_0/2)^{2n-5/3}}{2^{2n+3}} \frac{\Gamma\left(\frac{5}{6} - n\right)}{(n+1) \Gamma(n+2) \Gamma\left(\frac{11}{6}\right)} \right. \\ \times {}_2F_3 \left[n+1, n+\frac{3}{2}; n+2, 2n+3, \frac{1}{6} + n; \left(\frac{D\kappa_0}{2} \right)^2 \right] \\ + \frac{1}{2^{14/3}} \frac{\Gamma\left(\frac{14}{3}\right) \Gamma\left(n - \frac{5}{6}\right)}{\Gamma\left(\frac{23}{6} + n\right) \left(\Gamma\left(\frac{17}{6}\right)\right)^2} \\ \times {}_2F_3 \left[\frac{11}{6}, \frac{7}{3}; \frac{17}{6}, \frac{23}{6} + n, \frac{11}{6} - n; \left(\frac{D\kappa_0}{2} \right)^2 \right] \Big\}. \quad (\text{A10})$$

Similar versions of Eq. (A10) can be found in the literature such as Conan [27] and Winker [36]. Sasiela in [35,37] derives the two-axis Z-tilt angle jitter variance, which differs from the single axis phase results in Eqs. (A10) and (13) by a multiplicative factor of $2[4/(kD)]^2$.

APPENDIX B: ANALYTICAL PHASE STRUCTURE FUNCTIONS

The analytical phase structure function $D_\phi^A(r)$ is given by [10,27,34]

$$D_\phi^A(r) = 2 [B_\phi(0) - B_\phi(r)] \\ = 4\pi \int_0^\infty \kappa \Phi_\phi(\kappa) [1 - J_0(\kappa r)] d\kappa. \quad (\text{B1})$$

Here, B_ϕ is phase covariance function. The analytical solutions for Kolmogorov and von Kármán spectra can be derived as [9,27]

$$\begin{aligned} \text{Kolmogorov, } D_\phi^A(r) &= 2 \left[\frac{24}{5} \Gamma \left(\frac{6}{5} \right) \right]^{5/6} \left(\frac{r}{r_0} \right)^{5/3} \\ &= 6.88 \left(\frac{r}{r_0} \right)^{5/3}, \end{aligned} \quad (\text{B2a})$$

$$\begin{aligned} \text{von Kármán, } D_\phi^A(r) &= 2 \frac{\Gamma(11/6)}{2^{5/6} \pi^{8/3}} \left[\frac{24}{5} \Gamma \left(\frac{6}{5} \right) \right]^{5/6} \left(\frac{2\pi}{r_0 \kappa_0} \right)^{5/3} \\ &\times \left[\frac{\Gamma(5/6)}{2^{1/6}} - (r \kappa_0)^{5/6} K_{5/6}(r \kappa_0) \right], \end{aligned} \quad (\text{B2b})$$

where K_n is the n th order modified Bessel function of the second kind.

Funding. Air Force Office of Scientific Research (FA9550-17-C-0021, FA9550-18-1-0009, FA9550-18-C-0011).

Disclosures. The authors declare no conflicts of interest.

Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

REFERENCES

- B. L. McGlamery, "Computer simulation studies of compensation of turbulence degraded images," *Proc. SPIE* **74**, 225–233 (1976).
- J. M. Martin and S. M. Flatté, "Intensity images and statistics from numerical simulation of wave propagation in 3-D random media," *Appl. Opt.* **27**, 2111–2126 (1988).
- R. Frehlich, "Simulation of laser propagation in a turbulent atmosphere," *Appl. Opt.* **39**, 393–397 (2000).
- D. K. Borah and D. G. Voelz, "Pointing error effects on free-space optical communication links in the presence of atmospheric turbulence," *J. Lightwave Technol.* **27**, 3965–3973 (2009).
- J. P. Bos and M. C. Roggemann, "Technique for simulating anisoplanatic image formation over long horizontal paths," *Opt. Eng.* **51**, 101704 (2012).
- R. C. Hardie, J. D. Power, D. A. LeMaster, D. R. Droege, S. Gladysz, and S. Bose-Pillai, "Simulation of anisoplanatic imaging through optical turbulence using numerical wave propagation with new validation analysis," *Opt. Eng.* **56**, 071502 (2017).
- D. Voelz, E. Wijerathna, A. Muschinski, and X. Xiao, "Computer simulations of optical turbulence in the weak- and strong-scattering regime: angle-of-arrival fluctuations obtained from ray optics and wave optics," *Opt. Eng.* **57**, 104102 (2018).
- R. G. Lane, A. Glindemann, and J. C. Dainty, "Simulation of a Kolmogorov phase screen," *Waves Random Media* **2**, 209–224 (1992).
- J. D. Schmidt, *Numerical Simulation of Optical Wave Propagation with Examples in MATLAB* (SPIE, 2010).
- R. J. Noll, "Zernike polynomials and atmospheric turbulence," *J. Opt. Soc. Am.* **66**, 207–211 (1976).
- N. A. Roddier, "Atmospheric wavefront simulation using Zernike polynomials," *Opt. Eng.* **29**, 1174–1180 (1990).
- B. Zhang, S. Qin, and X. Wang, "Accurate and fast simulation of Kolmogorov phase screen by combining spectral method with Zernike polynomials method," *Chin. Opt. Lett.* **8**, 3 (2010).
- L. Zhu, Y. Li, D. H. Zheng, and J. Wu, "Performance comparison of subharmonic and Zernike polynomials method for compensation of low-frequency components in FFT-based von Karman phase screen," in *International Conference on Wireless Communications & Signal Processing (WCSP)* (IEEE, 2015), pp. 1–5.
- C. M. Harding, R. A. Johnston, and R. G. Lane, "Fast simulation of a Kolmogorov phase screen," *Appl. Opt.* **38**, 2161–2170 (1999).
- F. Assémat, R. Wilson, and E. Gendron, "Method for simulating infinitely long and non stationary phase screens with optimized memory storage," *Opt. Express* **14**, 988–999 (2006).
- K. A. Winick, "Atmospheric turbulence-induced signal fades on optical heterodyne communication links," *Appl. Opt.* **25**, 1817–1825 (1986).
- M. Charnotskii, "Sparse spectrum model for a turbulent phase," *J. Opt. Soc. Am. A* **30**, 479–488 (2013).
- D. A. Paulson, C. Wu, and C. C. Davis, "Randomized spectral sampling for efficient simulation of laser propagation through optical turbulence," *J. Opt. Soc. Am. B* **36**, 3249–3262 (2019).
- M. Carbillet and A. Riccardi, "Numerical modeling of atmospherically perturbed phase screens: new solutions for classical fast Fourier transform and Zernike methods," *Appl. Opt.* **49**, G47–G52 (2010).
- J. Xiang, "Accurate compensation of the low-frequency components for the FFT-based turbulent phase screen," *Opt. Express* **20**, 681–687 (2012).
- G. Sedmak, "Performance analysis of and compensation for aspect-ratio effects of fast-Fourier-transform-based simulations of large atmospheric wave fronts," *Appl. Opt.* **37**, 4605–4613 (1998).
- H. Jakobsson, "Simulations of time series of atmospherically distorted wave fronts," *Appl. Opt.* **35**, 1561–1565 (1996).
- E. Wijerathna, D. Voelz, and H. Zhan, "Discrepancies between simulation and theory results for plane wave scintillation in atmospheric turbulence," in *Imaging and Applied Optics 2018 (3D, AO, AIO, COSI, DH, IS, LACSEA, LS&C, MATH, PcaOP)* (OSA, 2018), paper PTu21.4.
- D. L. Fried, "Statistics of a geometric representation of wavefront distortion," *J. Opt. Soc. Am.* **55**, 1427–1435 (1965).
- M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, 7th expanded ed. (Cambridge University, 2003).
- F. Zernike and F. J. M. Stratton, "Diffraction theory of the knife-edge test and its improved form, the phase-contrast method," *Mon. Not. R. Astron. Soc.* **94**, 377–384 (1934).
- R. Conan, "Mean-square residual error of a wavefront after propagation through atmospheric turbulence and after correction with Zernike polynomials," *J. Opt. Soc. Am. A* **25**, 526–536 (2008).
- M. Bray, "Orthogonal polynomials: a set for square areas," *Proc. SPIE* **5252**, 314–321 (2004).
- A. M. Obukhov, "The structure of the temperature field in a turbulent flow," *Izv. Akad. Nauk SSSR Ser. Geogr. i Geofiz.* **13**, 58–69 (1949).
- S. Corrsin, "On the spectrum of isotropic temperature fluctuations in an isotropic turbulence," *J. Appl. Phys.* **22**, 469–473 (1951).
- V. I. Tatarskii, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, 1961).
- G. A. Chanan, "Calculation of wave-front tilt correlations associated with atmospheric turbulence," *J. Opt. Soc. Am. A* **9**, 298–301 (1992).
- S. Thomas, "A simple turbulence simulator for adaptive optics," *Proc. SPIE* **5490**, 766–773 (2004).
- L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media* (SPIE, 2005).
- R. J. Sasiela and J. D. Shelton, "Transverse spectral filtering and Mellin transform techniques applied to the effect of outer scale on tilt and tilt anisoplanatism," *J. Opt. Soc. Am. A* **10**, 646–660 (1993).
- D. M. Winker, "Effect of a finite outer scale on the Zernike decomposition of atmospheric optical turbulence," *J. Opt. Soc. Am. A* **8**, 1568–1573 (1991).
- R. J. Sasiela, *Electromagnetic Wave Propagation in Turbulence*, 2nd ed. (SPIE, 2007).
- V. I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation* (Israel Program for Scientific Translations, 1971).
- A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, 1978), Vol. 2.
- T. Liu, J. Zhang, Y. Lei, R. Zhang, and J. Sun, "Optimal subharmonics selection for atmosphere turbulence phase screen simulation using the subharmonic method," *J. Mod. Opt.* **66**, 986–991 (2019).
- E. M. Johansson and D. T. Gavel, "Simulation of stellar speckle imaging," *Proc. SPIE* **2200**, 372–383 (1994).

42. E. Wijerathna, "Numerical wave optics simulations through atmospheric turbulence: simulation method improvements and a comparative study of optical scintillation," Ph. D. dissertation (New Mexico State University, 2021).
43. Y. Cheon and A. Muschinski, "Closed-form approximations for the angle-of-arrival variance of plane and spherical waves propagating through homogeneous and isotropic turbulence," *J. Opt. Soc. Am. A* **24**, 415–422 (2007).
44. F. W. J. Olver, D. W. Lozier, F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions Hardback and CD-ROM* (Cambridge University, 2010).
45. R. J. Mathar, "Karhunen-Loeve basis functions of Kolmogorov turbulence in the sphere," [arXiv](#), arXiv:0805.3979 (2008).
46. J. Y. Wang and J. K. Markey, "Modal compensation of atmospheric turbulence phase distortion," *J. Opt. Soc. Am.* **68**, 78–87 (1978).
47. I. S. Gradshteyn, I. M. Ryzhik, and A. Jeffrey, *Table of Integrals, Series, and Products*, 7th ed. (Academic, 2007).
48. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Mathematical Library (Cambridge University, 1966).

Scintillation Observed in an 84-km Propagation Experiment across White Sands Missile Range

Andreas Muschinski¹, David G. Voelz², Andrea Baca², and Robert Hull²

¹ NorthWest Research Associates, 3380 Mitchell Lane, Boulder, CO 80301, USA

² Klipsch School of Electrical and Computer Engineering, New Mexico State University,
P.O. Box 30001 MSC 3-O, Las Cruces, NM 88003, USA
andreas@nwra.com

Abstract: We present and discuss measurements collected during a long-range optical propagation experiment conducted in October, 2022 at the White Sands Missile Range, NM. The propagation path was 84 km long, and most of the path was between 600 m and 1500 m above ground level. © 2023 The Author(s)

1. Introduction

Turbulent fluctuations of the refractive index of air lead to phase fluctuations, phase-gradient fluctuations (i.e., angle-of-arrival fluctuations), and irradiance fluctuations (“scintillation”) of light propagating through the turbulent atmosphere. Tatarskii’s [1, p. 211] classical theory of wave propagation through the turbulent atmosphere predicts a scintillation index (the variance of $I/\langle I \rangle$, where I is the local and instantaneous irradiance) of

$$\sigma_I^2 = 1.23 C_n^2 k^{7/6} L^{11/6}, \quad (1)$$

where C_n^2 is the refractive-index structure parameter, $k = 2\pi/\lambda$ is the optical wavenumber, and L is the propagation path length. Eq. (1) relies on various assumptions, such as the validity of the Obukhov-Corrsin similarity theory [2, 3] of fully developed, locally homogeneous and isotropic turbulence; the negligibility of inner-scale effects; and the applicability of the Rytov approximations, which is valid only in the weak-fluctuation case, i.e., if $\sigma_I^2 \ll 1$. The scintillation index according to Eq. (1) is usually referred to as the “Rytov variance”. In contrast to Eq. (1), σ_I^2 approaches 1 in the strong-fluctuation case. This phenomenon is known as “saturation of scintillation” and has been corroborated by observations, theoretical analysis, and computer simulations [4, 5].

It is generally believed that angle-of-arrival statistics are less affected by strong-fluctuation effects than irradiance statistics. Wave-optics computer simulations involving multiple phase screens [6] have confirmed the validity of the geometrical-optics assumption for the angle-of-arrival variance well into the strong-fluctuation regime.

2. A long-range propagation experiment at White Sands Missile Range

Here we present and discuss observations of irradiance and angle-of-arrival statistics collected during a long-range optical propagation experiment across the Tularosa Basin and White Sands Missile Range in southern New Mexico. We conducted the experiment in October, 2022. An array of five test lights was deployed at the site of the Sunspot Observatory, NM, and we imaged these lights at nighttime with two telescopes, each equipped with a digital camera. The telescopes were located at the Aguirre Spring Campground on the east side of the Organ Mountains near Las Cruces, NM. The telescopes’ aperture diameters were 14 inch and 8 inch. Each of the five “sources” consisted of a red, a green, and a blue high-power, light-emitting diode (LED). Each LED emitted an average optical power of approximately 50 mW. The propagation path was 84 km long, and the light propagated for the most part through the residual layer [7] above the nocturnal boundary layer, between 600 m and 1500 m above the basin floor. More details of the experimental setup are presented in the companion paper by Baca *et al.* in this conference.

3. Statistics of aperture-averaged irradiance fluctuations

Fig. 1 shows some preliminary results from Run 8210, which consisted of a sequence of 60,554 images collected during a 17-min period in the evening of October 2, 2022, starting at 2215 MDT. The images were collected with a Canon EOS 5D Mark III digital color camera attached to the 14-inch telescope. The camera was operated in a video mode at 60 frames per second. The exposure time was 1 ms. The test lights were distributed laterally with non-equidistant spacing. The spacing between Source 1 and Source 5 was 70 m, which corresponds to 61/84,000 rad, or 0.7 milliradians. Sources 1, 2, and 5 could be easily separated in each of the frames but the images of Sources

3 and 4 (spaced by only 5 m, corresponding to $60\ \mu\text{rad}$) overlapped. In the initial phase of the post-processing, we decomposed each color frame into a red, a green, and a blue frame, and we retrieved the brightnesses for each individual source (except for sources 3 and 4, which we could not separate) and each color channel.

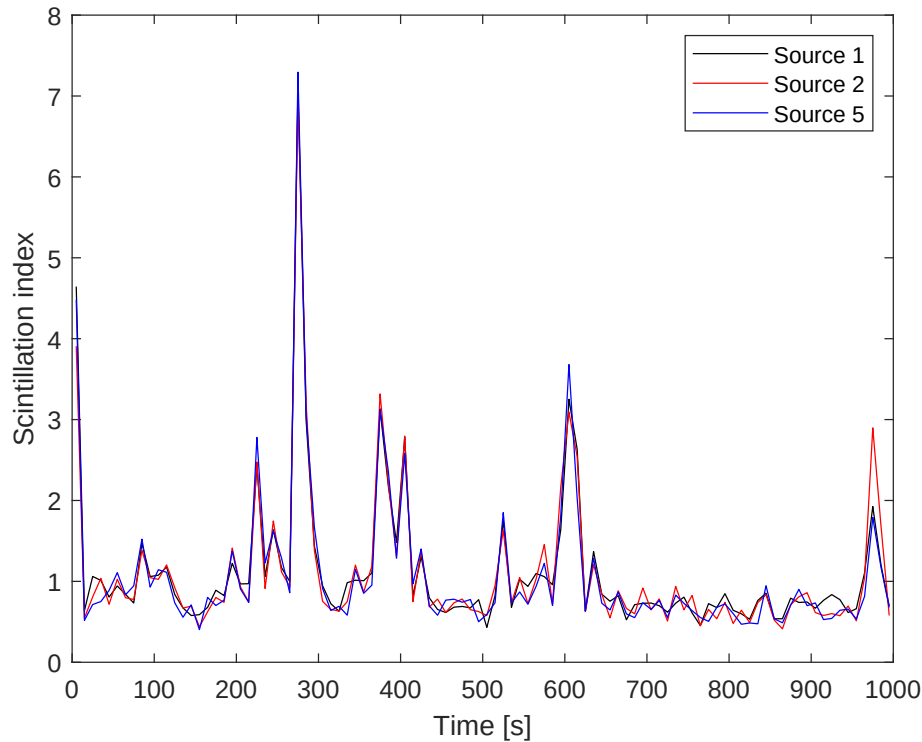


Fig. 1. Time series of 10-s estimates of the scintillation indices observed with three different test lights. The data were collected in October, 2022 at White Sands Missile Range, NM. The propagation path was 84 km long.

Fig. 1 shows the time series of 10-s estimates of the scintillation indices for sources 1, 2, and 5 observed through the green color channel. The scintillation indices for the three sources track each other well. Most of the time, the scintillation index ranged between 0.5 and 1.5, which is larger than what one would expect if the Fried length r_0 is small compared to the aperture diameter (a first rough estimate gave $r_0 \sim 2\text{ cm}$ based on the images' blur size), regardless of whether or not the weak-fluctuation assumption is fulfilled. Fig. 1 shows that the scintillation index increases occasionally to values even much larger than 1.5 (up to about 7 at $t = 275\text{ s}$).

Acknowledgments Air Force Office of Scientific Research (FA9550-17-C-0021, FA9550-18-1-0009, FA9550-18-C-0011).

References

1. V. I. Tatarskii, *Wave propagation in a turbulent medium* (McGraw-Hill, New York, 1961).
2. A. M. Obukhov, "The structure of the temperature field in a turbulent flow," *Izv. Akad. Nauk SSSR, Ser. Geogr. i Geofiz.* **13**, 58–69 (1949).
3. S. Corrsin, "On the spectrum of isotropic temperature fluctuations in an isotropic turbulence," *J. Appl. Phys.* **22**, 469–473 (1951).
4. E. L. Andreas, ed., *Turbulence in a refractive medium*, vol. MS 25 of *SPIE Milestone Series* (SPIE Optical Engineering Press, Bellingham, Washington, USA, 1990).
5. L. C. Andrews and R. L. Phillips, *Laser beam propagation through random media* (SPIE Press, Bellingham, Washington, 2005), 2nd ed.
6. D. G. Voelz, E. Wijerathna, A. Muschinski, and X. Xiao, "Computer simulations of optical turbulence in the weak- and strong-scattering regime: angle-of-arrival fluctuations obtained from ray optics and wave optics," *Opt. Eng.* **57**, 104102 (2018).
7. A. Muschinski, R. G. Frehlich, and B. B. Balsley, "Small-scale and large-scale intermittency in the nocturnal boundary layer and the residual layer," *J. Fluid Mech.* **515**, 319–351 (2004).