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Quantum information theory of observers in analogue and emergent gravity

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14. ABSTRACT Although I inherited this grant after completion, and the deliverables were very late, it would appear that the PI (Prof. Menicucci) has done excellent work. He has constructed a formalism for relativity using a sonic analogue. This is a very useful teaching tool, which we should investigate for possible inclusion in courses at the AF Academy. The grant produced 4 high quality papers detailing some interesting results in better understanding an observers interaction with the external relativistic universe. His paper on " Entanglement harvesting with coherently delocalized matter" was particularly intriguing possibilities for quantum communications - namely the idea of the teleportation of quantum states without the a priori requirement of entanglement. I note that he was subsequently given a follow-on award (FA2386-20-1-4015) to investigate some of these ideas further.					
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“Quantum information theory of observers in analogue and emergent gravity”

16 Feb 2024

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Summary of the project findings

Motivation

The behavior of quantum information in the presence of gravity is an important topic for quantum communication and other quantum technology in space. Laboratory analogues of curvature provide useful insight and pedagogical tools for use in an educational setting, as well as laboratory models that can be used to model relativistic and gravitational phenomena both theoretically and experimentally.

A critical gap in these models to date is any notion of how to model observers within such a “sonic universe.” Since general relativity is a relational theory, what observers see “from the inside” is crucial to understanding what the models can tell us.

Aim and long-term vision

This project aims to fill this gap with an explicit model for such observers and to study how these so-called *internal observers* interact with the analogue universe around them.

The long-term goal of this research direction is to study how the internal observers view quantum fluctuations in the medium and how they interpret interactions with external quantum particles. The project work made significant progress in this direction, documented in refereed journal articles commenced within the project period and, in many cases, completed and published in the following years.

Research outcomes

The main takeaways of this work are:

- 1) Einstein’s theory of relativity—which is critical for high-precision measurements, such as those required for accurate GPS timing—is hard to grasp, so intuition from sonic models of relativity is useful because we understand the physics of those systems intuitively.
- 2) There exist viable models of internal observers whose measurements will reflect the “sonic relativity” naturally present within acoustic materials. The means by which these observers come to establish beliefs about the nature of their world informs us about how we have come to understand our own [1].
- 3) The internal observers will believe their world to be entirely relativistic and free of any medium within which sound would propagate despite us being able to see otherwise from the outside [1].
- 4) Given the ability to scatter sound from external (non-sonic) particles—e.g., a piece of dust, an electron, a photon, etc.—the internal observers will see evidence of their motion with respect to the medium [2].

These findings are documented in the publications noted below and attached hereto.

Additional outcomes include two studies of vacuum entanglement extraction from a quantum field given two different deviations from standard physical assumptions: (1) a physical bandlimit on quantum field theory in attempts to extract entanglement from the quantum vacuum [3] and (2) a modification to the standard model of a quantum particle detectors to account for its quantum degrees of freedom [4].

Relativity curriculum development

A useful non-research outcome directly from this work has been new curriculum I developed to teach special relativity to first- or second-year undergraduate physics students by using sonic relativity as a bridge to establish intuition. The key focus of this pedagogy is an emphasis on the fact that the laws of physics are synthesized from measurements, and that making good measurements requires concepts we take for granted—e.g., the ability to synchronize distant clocks and the belief that yard sticks on one side of a lab are the same length as those on the other side of the lab. By going through the exercise to understand how internal observers would perceive their sonic world through the exchange of sound pulses, students gain insight into how we form the beliefs that we have about our own world.

A demo of the curriculum developed is available for review. If there is interest, I would be happy to work with instructors to craft a version of these ideas for possible future incorporation into Air Force or Space Force curriculum related to relativity.

List of Publications and Significant Collaborations that resulted from this AOARD supported project

Papers published in peer-reviewed journals

- [1] S. L. Todd and N. C. Menicucci, *Sound Clocks and Sonic Relativity*, Found. Phys. **47**, 1267 (2017).
- [2] S. L. Todd, G. Pantaleoni, V. Baccetti, and N. C. Menicucci, *Particle Scattering in a Sonic Analogue of Special Relativity*, Phys. Rev. D **104**, 064035 (2021).
- [3] L. J. Henderson and N. C. Menicucci, *Bandlimited Entanglement Harvesting*, Phys. Rev. D **102**, 125026 (2020).
- [4] N. Stritzelberger, L. J. Henderson, V. Baccetti, N. C. Menicucci, and A. Kempf, *Entanglement Harvesting with Coherently Delocalized Matter*, Phys. Rev. D **103**, 016007 (2021).

During the period of funding, I had several useful discussions with Dr Paul Alsing (AFRL), who has published significant works at the interface of quantum information and relativity theory.

This helped to establish a new collaboration with Prof Achim Kempf (U Waterloo), who works in multiple areas of quantum information theory, including those related to relativity, quantum field theory, and fundamental physics.

I was invited to give a talk on [1] at the University of Oxford Philosophy of Physics seminar series in Nov 2020, which led to a follow-up publication in 2023 [Bilson-Thompson et al., Phys. Rev. D **108**, 124020 (2023)], continuing the research that commenced with this project.

Attachments (below)

- Publications 1-4 above

Sent separately

- Completed DD882
- Completed SF425

Available upon request

- A demonstration of the introductory relativity curriculum developed as a result of this project.

Sound Clocks and Sonic Relativity

Scott L. Todd¹  · Nicolas C. Menicucci¹ 

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Abstract Sound propagation within certain non-relativistic condensed matter models obeys a relativistic wave equation despite such systems admitting entirely non-relativistic descriptions. A natural question that arises upon consideration of this is, “do devices exist that will experience the relativity in these systems?” We describe a thought experiment in which ‘acoustic observers’ possess devices called sound clocks that can be connected to form chains. Careful investigation shows that appropriately constructed chains of stationary and moving sound clocks are perceived by observers on the other chain as undergoing the relativistic phenomena of length contraction and time dilation by the Lorentz factor, γ , with c the speed of sound. Sound clocks within moving chains *actually* tick less frequently than stationary ones and must be separated by a shorter distance than when stationary to satisfy simultaneity conditions. Stationary sound clocks *appear* to be length contracted and time dilated to moving observers due to their misunderstanding of their own state of motion with respect to the laboratory. Observers restricted to using sound clocks describe a universe kinematically consistent with the theory of special relativity, despite the preferred frame of their universe in the laboratory. Such devices show promise in further probing analogue relativity models, for example in investigating phenomena that require careful consideration of the proper time elapsed for observers.

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1 Introduction

The repeated null results from experiments to detect the luminiferous aether towards the end of the nineteenth century—most notably the null result of the Michelson–Morley experiment [1]—culminated with many physicists, most notably George FitzGerald, Hendrik Lorentz, and Henri Poincaré, proposing mechanisms by which the aether was undetectable. FitzGerald [2] and Lorentz [3,4] independently (1889 and 1892 respectively) suggested that objects contract parallel to their direction of motion. Woldemar Voigt [4,5] suggested modification of the time coordinate to ensure that the wave equation for light worked in all reference frames, and Lorentz [4,6] also later introduced this same notion of ‘local time,’ though unlike length contraction he did not assign any physical importance to it. Poincaré [7], however, realised a physical significance of this notion of local time as suggested by Lorentz in that it would be the time recorded on clocks synchronised using light signals.

Eventually, aether theory fell victim to Ockham’s razor¹: Einstein’s theory of special relativity sufficed to explain all of the same phenomena with the added simplification of not requiring an undetectable aether. It is important to realise that while the theory of special relativity won out over any of the aether theories, aether-based models still produce the exact same kinematic results as the theory of special relativity when treated correctly due to both theories exhibiting the exact same mathematical formalism [8].

In an effort to describe black holes, Unruh [9] came up with an analogy in terms of sound waves propagating up a waterfall: if at some location along the waterfall the flow speed of the water exceeds the speed of sound in the water, it becomes impossible to sonically signal upstream anymore. This model is analogous to a black hole, except here sound takes the place of light. Such objects are called acoustic black holes or dumb holes (where ‘dumb’ is a synonym for mute). It is worth noting that such a model possesses a preferred reference frame: the reference frame in which the water is stationary.

Under several assumptions (no gravitational back-reaction, an unquantised gravitational field, and that at the Planck scale the wave equation for quantum fields is still applicable), Stephen Hawking famously demonstrated that black holes may be expected to evaporate by radiating at a characteristic temperature [10]. Whether the assumptions that Hawking made are valid is still unknown, and when Hawking’s assumptions were scrutinised by Unruh [11], the result of black hole evaporation was initially put into doubt. Acoustic black holes possess comparative problems: at sufficiently small length scales the continuum description breaks down (the expected equivalent of the Planck scale for spacetime), the field fluctuations (phonons) interact with the background that they propagate on (in analogy to gravitational back-reaction), and the analogy to the gravitational field itself (the fluid flow) is unquantised.

Unruh, following the same process as Hawking, theorised that acoustic black holes should emit an acoustic analogue of Hawking radiation [12]. Fortunately, in stark contrast to physics at the Planck scale, molecular physics, atomic physics, and fluid mechanics are well understood. While the validity of the underlying assumptions in

¹ Named in honour of William of Ockham, though often spelt “Occam’s razor”.

Hawking's derivation of black hole radiation could not be directly tested, the comparative assumptions in the acoustic black hole model could be. Unruh [13] and others (Brout *et al.* (analytically) [14], Corley (analytically) [15, 16], and Corley and Jacobson (numerically) [17, 18]) were able to show that acoustic black holes should radiate at a temperature as calculated using Hawking's approach for black holes even when the comparative assumptions in the acoustic black hole model were broken. This result demonstrates that the blackbody temperature of black holes as calculated by Hawking will not necessarily break down when the contributions of Planck scale physics are taken into account, providing us with some evidence that our current understanding of black hole thermodynamics could well be correct [9].

Models such as acoustic black holes are inherently nonrelativistic, yet nevertheless they appear to share some properties with relativistic systems [19]. This class of models are referred to as *analogue gravity models*. There are many examples of analogue gravity models. For example, quasiparticle production has been theorised [20] to occur in expanding Bose–Einstein condensates in analogy to particle production due to cosmic inflation. Bose–Einstein condensates have also been used to study analogue Hawking radiation, both theoretically [21, 22] and experimentally [23, 24]. An extensive list of the analogue gravity models known to exist up until 2011 can be found in the *Living Reviews in Relativity* article by Barceló *et al.* [19].

It is natural to wonder how far these analogies can be pushed before they break down. How many of the features of general relativity appear in analogous form within these models? If all of the features of general relativity emerge in some analogue form, then under what assumptions does this occur? Is there a microscopic mechanism behind the analogue form of gravity, and can we use this knowledge to infer anything about the origins of gravitation in our universe? If it is not possible to make all of the features of general relativity emerge in some analogous form in these systems, then why do we find the emergence of some aspects of general relativity and not others?

In order to answer these questions it is useful to understand the experience of observers within analogue relativity systems since observers were crucial in the understanding of general relativity, specifically in explaining the physical interpretation of general covariance [25]. The principle of general covariance states that the laws of physics should be independent of choice of coordinate system, as coordinates do not exist in nature *a priori*. It can be shown, however, that smoothly dragging the gravitational field around on a background spacetime manifold (a mathematical process called an active diffeomorphism) can result in a physical description of reality that is mathematically indistinguishable from a mere change in coordinates (a passive diffeomorphism). Therefore, if it is impossible to distinguish between an active diffeomorphism and a passive diffeomorphism, and if the principle of general covariance is taken to be true (that coordinates are indeed unphysical), then it follows that active diffeomorphisms must also be unphysical. Consequently, the spacetime manifold itself must be seen as being unphysical. It is only with respect to events or physical objects (in other words, observable quantities) that locations in spacetime have any meaning at all.

The interpretation here is that spacetime has no real, physical meaning independent of coincidences. What is meant by this is that points in spacetime are only defined by two or more events or objects coinciding, such as two particles passing through the

same point in space simultaneously. With this in mind, the existence of more than one solution to a covariant set of field equations is not seen to be problematic. If one metric tensor solves the field equations and yields spacetime paths for multiple observers that coincide at certain proper times for each observer, then *all* metric tensors consistent with the same initial conditions will yield spacetime paths for these observers that coincide at the same proper times. Einstein remarks on this, “... *the requirement of general covariance takes away from space and time the last remnant of physical objectivity*” [25].

The principle of general covariance and its interpretation on the meaning (or rather lack of meaning) of spacetime naïvely appears to be a problem that will be impossible to overcome in analogue gravity models because such models *do* possess an objective, physical analogue of spacetime; there is a preferred reference frame. In order to investigate if or how general covariance manifests in analogue gravity systems, we will need to consider the proper time elapsed by observers in analogue gravity models.

Barceló and Jannes [26] described the physics of ‘natural’ interferometers in analogue relativity models. Such devices would be constructed out of quasiparticles, themselves made up from the particles of the medium under consideration. Observers within these analogue universes, using these interferometers to perform Michelson–Morley type experiments, would find the exact same results that we found in our universe: that the aether is undetectable.

The reason for this is that the medium itself obeys a relativistic wave equation, and thus the resulting kinematics of the medium are subject to the symmetries of the Lorentz group. For example, it has been shown how electromagnetic-like theories based on models that admit a privileged reference frame can, in the low-energy limit, appear to obey Lorentz invariance to internal observers [27]. Consequently, any object constructed from the medium of the analogue universe will inherit the symmetries of the medium, *i.e.* the Lorentz group. Interferometers built this way will possess arms that will shrink in their direction of motion, and thus without any way to measure velocity relative to their aether, observers who only have access to these devices will come to believe the postulates of relativity via Ockham’s razor.

Additional discussions on the emergence of Lorentz symmetry and relativity in physical models that admit a rest frame can be found in the literature. For example, see Liberati et al. [28], Volovik [29], and Nandi [30].

While quasiparticle interferometers are *sufficient* in demonstrating the emergent relativity of these systems, we would like to ask, are such constructs *necessary* to demonstrate this relativity? An observer within such a system who is free to perform *any* experiment they would like using an interferometer made from a material other than the one used to construct the medium of their universe will be able to infer motion relative to their aether and will not come to believe the postulates of relativity. However, if certain restrictions are placed on which experiments such observers are allowed to perform with non-quasiparticle interferometers, can such observers come to believe the postulates of relativity through their observations?

Operationally speaking, constructing (or even describing how to construct) quasiparticles that would then in turn be used to build devices such as interferometers seems difficult, though such devices would most likely prove to be invaluable additions to the tool-kit of experimental physicists seeking to test analogue relativity systems. To

this end, we turn our attention towards devices that are inserted into analogue relativity systems from the laboratory in order to determine if, with certain constraints, such devices can appropriately act as relativistic observers in such analogue systems. With certain restrictions placed on what type of experiments can be conducted, we show here that *sound clocks*—the equivalent of light clocks in systems obeying sonic relativity—can be used as appropriate relativistic observers for a medium in which sound obeys a relativistic wave equation.

2 Approach

As a preface to what we will discuss here, it should be noted that the work carried out here is essentially equivalent to the work performed by Poincaré at the beginning of the twentieth century in his effort to incorporate his philosophy of relativity into the aether based model prevalent at the time. Here, our aether is that of a condensed matter system (such as a large slab of solid matter or a perfect liquid, either of which must be homogeneous and isotropic), and light is replaced by disturbances within our medium: sound. While Poincaré sought a description for the way in which mechanical bodies naturally behaved in order to obscure detection of the aether, we are, in some sense, attempting to do the reverse: how can we *manufacture* obfuscation of an aether from observers within it in such a way that the observers come to believe the postulates of relativity? What are the minimum constraints that we are required to put on experimental equipment in an aether-based system such that observers who only have access to that equipment have the existence of that aether hidden from them? We ask this questions in the context of analogue gravity models from which we aim to, if possible, make analogies to phenomena in our own universe. However, let it be made abundantly clear that we are not attempting to revive aether-based models as a basis for how our own universe works.

We go through a detailed analysis of the experience of observers who are both stationary and moving with a constant velocity with respect to the medium that they are confined to (which is at rest within the laboratory). The results obtained for stationary observers are in no way surprising, and calculations done from the laboratory frame without considering the technicalities of stationary observers' experience will yield the same results. However, going through the more painstaking analysis is instructive in how to approach the problem of analysing the experience of moving observers. The results found for the experience of moving observers *are* indeed surprising at first glance and are not what is expected from the naïve, simple calculation done from the laboratory (which just happens to work for the stationary observers because they share the reference frame of the laboratory, even if they are unaware of it). We will discuss the specifics of the analysis and the restrictions that are necessary as they become important.

3 Simple Sound Clocks

Here we describe devices called *sound clocks* for systems in which sound obeys a relativistic wave equation, *i.e.* systems that exhibit sonic relativity. A sound clock is a

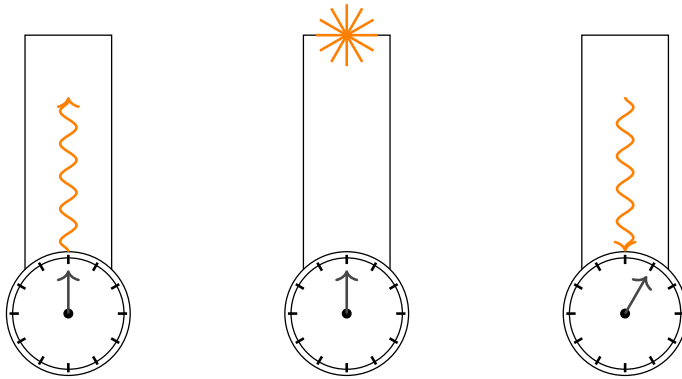


Fig. 1 The operation of a sound clock is, in principle, straightforward: a sound pulse is emitted by the clock when an observer wishes to begin recording time, and after travelling to the end of the sound clock arm it is detected, whereupon a new sound pulse (the ‘echo’) is then emitted back towards the central clock mechanism. Upon reception of the echo, the clock reading is advanced forward by one tick. Note that this figure depicts one sound clock at three distinct moments in time as opposed to three sound clocks at one moment in time

device that is analogous to the light clock used in thought experiments to show time dilation in special relativity. Sound clocks consist of a clock mechanism out from which an arm, called the *timing arm*, is extended. At both ends of the timing arm, sound can be detected and emitted. To record time, a sound pulse is emitted from the end of the timing arm in contact with the clock mechanism, and at some later point in time, part of the wavefront will intersect the far end of the timing arm and be detected. Immediately following detection, a sound pulse is then emitted back in the same fashion. Upon reception of this second sound pulse—what we will herein refer to as the ‘echo’—the clock mechanism advances its reading forward by one tick.

A diagrammatic outline of this process can be seen in Fig. 1. We could also imagine additional shorter timing arms being present to increment the sound clock by fractions of a tick, but for the sake of clarity in figures, we shall not include these. Observers thus have some sense of *local time* by counting ticks of their clock. However, if an observer wishes to know when an event occurred at some other location, then they must have access to the reading on the clock positioned at that location (for example, by sending a message to the observer at that location and requesting the reading for when a specified event happened).

Let us first consider a single sound clock as observed from the laboratory that is limited in the possible trajectories it can take: it is only allowed to travel in the direction perpendicular to its timing arm. If in the laboratory the length of the sound clock’s timing arm is known, and the velocity of the sound clock with respect to the medium is known, then the distance that n sound pulses have travelled can be determined via simple geometric arguments.

Furthermore, if one knows the speed of sound in the medium, then the time it took for these n sound pulses to trace out their paths can be determined. From Fig. 2, it can be determined that, in the laboratory frame, the total distance travelled by n sound pulses that are emitted and received by a sound clock travelling at velocity v for time

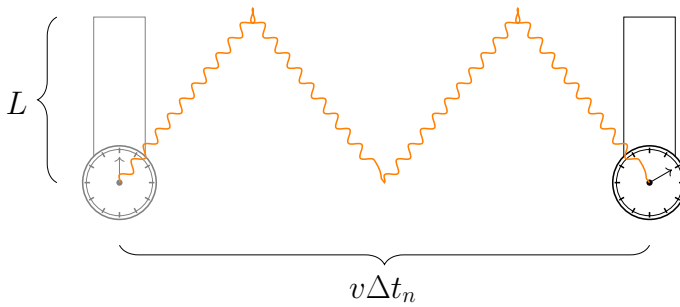


Fig. 2 A sound clock travelling with velocity v for some time Δt_n in which time n sound pulses are emitted and returned to the clock

Δt_n (Δt_n for the duration of time that is required for n sound pulses to be emitted and received) in the direction perpendicular to its timing arm is given by

$$s = \sqrt{(2Ln)^2 + (v\Delta t_n)^2}. \quad (1)$$

Dividing the distance that the n sound pulses have travelled by their speed, the speed of sound, c , yields the time, Δt_s (Δt_s for the time it took to trace out the path of length s), that it has taken for these n sound pulses to trace out their paths through space,

$$\Delta t_s = \frac{s}{c} = \sqrt{\left(\frac{2L}{c}n\right)^2 + \left(\frac{v}{c}\Delta t_n\right)^2}. \quad (2)$$

Note that Δt_s and Δt_n are equal, which is easiest to see for integer values of n : when n is integer, a sound pulse has just been detected; in order to detect a sound pulse, the sound clock and the sound pulse that is being detected must be at the same place at the same time. Defining some new variable $\Delta t := \Delta t_n = \Delta t_s$ (how long the clock has been recording ticks for), and also defining $\beta := v/c$, we can determine exactly how long it takes for a sound clock travelling at any velocity less than c to record n ticks of the clock,

$$\Delta t = \frac{2L}{c} \frac{n}{\sqrt{1 - \beta^2}}. \quad (3)$$

The tick frequency, or the period, of any sound clock is therefore given by,

$$T = \frac{\Delta t}{n} = \frac{2L}{c} \frac{1}{\sqrt{1 - \beta^2}}. \quad (4)$$

The Lorentz factor, $\gamma = \sqrt{1 - \beta^2}^{-1}$, has appeared for the tick frequency of a moving sound clock.

Note that for a sound clock at rest with respect to the medium in the laboratory frame we have $v = 0$, and thus $\beta = 0$, from which we obtain

$$\Delta t = \frac{2L}{c}n, \quad (5)$$

which corresponds to a period of

$$T = \frac{2L}{c}. \quad (6)$$

This is exactly the time we expect it to take for n sound pulses to bounce along and back an arm of length L at rest.

4 Sound Clock Chains

Consider now multiple sound clocks at different locations in space. A chain of regularly spaced sound clocks is the easiest such example of multiple sound clocks to consider. The sound clocks that form a chain are connected by arms of tunable length to their neighbours and are synchronised with the use of a sound pulse from some agreed-upon clock (call it the origin clock).

Consider the sound clocks to be labelled with integer values corresponding to how many steps away from the origin clock they are, with the origin clock itself being labelled clock 0. In general, there can be sound clocks to either side of the origin clock, with clocks on one side possessing positive integer labels, and clocks on the other side possessing negative integer labels. We shall adopt the following convention: when labelled from the laboratory, the sound clock with the largest positive-integer label is at the front of the chain if the chain is in motion; if the chain is stationary, we can freely label either side positive or negative.

Relative to a given sound clock, we call clocks closer to the front of a moving chain ‘upchain’ and clocks closer to the back of a moving chain ‘downchain’: from the definition of the labelling scheme outlined above, this means that the direction ‘upchain’ is parallel to the sound clock chain’s velocity vector, and the direction ‘downchain’ is anti-parallel to the sound clock chain’s velocity vector. The labelling convention is shown in Fig. 3.

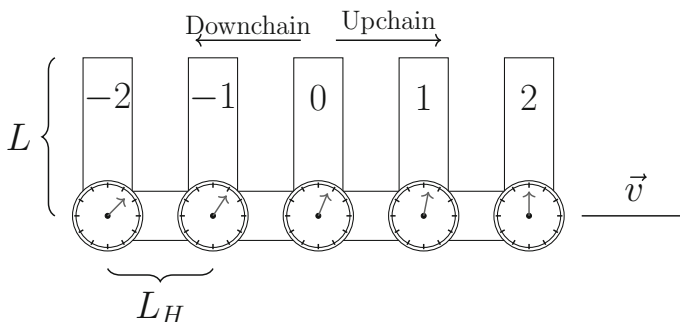


Fig. 3 A chain of sound clocks whose clocks have been separated and synchronised under the assumption that they are stationary. The length of the vertical timing arms (L) of all sound clocks in the chain is equal, as is the length of the horizontal spacing arms (L_H), though the lengths L_H and L are not necessarily equal (only at rest is this the case)

We wish for all of the sound clocks within a given chain to share a common coordinate system, so sound clocks within a chain must tick at the same frequency. The timing arms of adjacent sound clocks are assumed to be exactly parallel, and so to fulfil the requirement that they tick synchronously, the timing arms must be of equal length. This requirement and others will be discussed further in what follows.

4.1 Calibrating Clock Separation

A chain of sound clocks can be seen in Fig. 3, where L refers to the length of the ‘vertical’ arms (*i.e.* the timing arms), and L_H is the length of the horizontal arm separating sound clocks, which we also call the *spacing arms*. Note that for what follows the terms ‘vertical arms’ and ‘horizontal arms’ are interchangeable with ‘timing arms’ and ‘spacing arms’, respectively. The vertical arm of each sound clock is used to measure time directly in the same method as described for the single sound clock in Sect. 3, while the horizontal arm is used to space the sound clocks in an appropriate manner. For the purposes of our discussions here, the timing arms and spacing arms are considered to always be perpendicular to one another. We focus on the case where chains of sound clocks are *only* allowed to travel in the direction of the axis in which they are connected.

Observers who possess sound clocks and who are limited in their ability to make measurements as previously detailed have no ability to detect motion with respect to their medium. Any inertial motion within an analogue relativity system should therefore be indistinguishable from rest since observers who only possess sound clocks have no way to tell the difference between zero velocity with respect to the medium and constant non-zero velocity with respect to the medium.

Observers travelling with constant velocity separate sound clocks within a chain by simultaneously sending sound pulses along their timing arms and spacing arms. Believing themselves to be at rest, when the spacing arms are tuned to such a length that the two sound pulses return simultaneously, observers believe that the separation of their sound clocks (*i.e.* the spacing-arm length) is exactly the same length, L , of their timing arms. This belief happens to be true when a chain of sound clocks is actually at rest, but this is not the case when a chain of sound clocks is moving with a constant velocity.

By what distance should sound clocks travelling at a constant velocity be separated in order for this simultaneity condition to hold? Consider a simple chain of two sound clocks as seen in Fig. 4. The observer located at one of the sound clocks is able to adjust the separation distance of the two sound clocks by extending the arm that connects them or by reeling it back in.

Within a chain of sound clocks, the path taken by the sound pulse within the timing arms on its outbound journey and its inbound journey is symmetric for any motion perpendicular to the timing arms. This, however, is not the case for a sound pulse in the spacing arms. For sound pulses propagating between clocks in the spacing arms, there are two distinct paths taken when the clocks are travelling with non-zero velocity. As can be seen in Fig. 4, there is a downchain journey for which the sound pulse is travelling in the opposite direction to the sound clocks, and there is an upchain journey for which the sound pulse is travelling in the same direction as the sound clocks. The

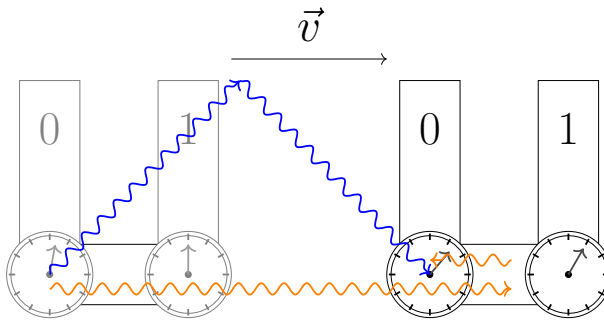


Fig. 4 The spacing arms between adjacent sound clocks in moving sound clock chains must be shorter than the timing arms by a factor of γ , to ensure that sound pulses that are emitted simultaneously along both arms also return simultaneously. Note that the readings of both clocks are advanced by exactly one tick when the sound pulses return (this is obscured partially for clock 0)

downchain journey, as seen in Fig. 4, takes less time to complete as v grows, while the upchain journey takes longer. The time it takes for a sound pulse to travel between two adjacent clocks downchain (Δt_d , which is greater than zero) and the time it takes to travel between two adjacent clocks upchain (Δt_u , which is greater than zero), as shown in Fig. 4, can be seen to obey the following relationships for a separation length of L_H ,

$$L_H - v\Delta t_d = c\Delta t_d, \quad (7)$$

$$\therefore \Delta t_d = \frac{L_H}{c + v}, \quad (8)$$

and,

$$L_H + v\Delta t_u = c\Delta t_u, \quad (9)$$

$$\therefore \Delta t_u = \frac{L_H}{c - v}. \quad (10)$$

For simultaneously emitted timing and synchronisation sound pulses to be detected simultaneously, we don't need to know exactly how long it takes the sound pulse to make either the downchain or upchain journey alone. We know that, by construction, the sum of the upchain and downchain times must be the same as the time for which one tick of the clock occurs as dictated by the timing arm, and one tick of the clock is given by (3) with $n = 1$. With the length of the vertical arm labelled L we have the relationship

$$\Delta t = \Delta t_d + \Delta t_u = \frac{2L}{c} \frac{1}{\sqrt{1 - \beta^2}}. \quad (11)$$

Substituting in the relationships for Δt_d and Δt_u allows us to solve for the separation distance, L_H , which we find to be

$$L_H = \frac{L}{\gamma}. \quad (12)$$

4.2 Synchronisation of Clocks

When the origin clock (clock 0) in a sound clock chain begins to record time, it simultaneously sends sound pulses along every arm connected to it. The sound pulse sent down its own timing arm is used to advance its own clock, whereas the pulses sent along the sound clock chain via the spacing arms are used to synchronise directly adjacent clocks (these are the synchronisation pulses). Upon receiving the synchronisation pulse, a given clock will simultaneously begin to record its own time and propagate the synchronisation pulse further along the chain. By this manner, all of the clocks in the chain can be synchronised with respect to the origin clock.

We can construct expressions for the time taken for the synchronisation pulse emitted from the origin clock to travel to another clock: $\Delta t_s^+ = |k| \Delta t_u$ is the time it takes for the synchronisation pulse to reach clock k (k steps in the upchain direction), while $\Delta t_s^- = |k| \Delta t_d$ is the time it takes for the synchronisation pulse to reach clock $-k$ (k steps in the downchain direction). When at rest, Δt_s^+ and Δt_s^- are equal. The relationships for the downchain and upchain synchronisation times can be combined into a single expression, and substituting in the expression for L_H from (12) we find

$$\Delta t_s^\pm = \frac{L}{c} \sqrt{\frac{1 \pm \beta}{1 \mp \beta}} |k|. \quad (13)$$

Despite the fact that the sound clocks are travelling in a medium with a preferred reference frame, the relativistic Doppler factor,

$$D = \sqrt{\frac{1 + \beta}{1 - \beta}}, \quad (14)$$

has appeared instead of the non-relativistic one, where β is the fractional speed of the sound clock chain with respect to sound. This is a result of observers within the sound clock chain setting the separation between adjacent clocks in such a way that the simultaneity of returned sound pulses occurs. In setting their separation in such a way, they have not only made their chain appear to exhibit the relativistic phenomenon of length contraction to observers in the laboratory, but they have also made their system appear to display the relativistic Doppler shift in regards to how long it takes sound to propagate between adjacent clocks to observers within the laboratory.

We now know how long it takes for any given clock, k , to tick n times (n_k) once it begins its clock: this is given by (3). We also know how long it takes for clock k to start its clock with respect to some initial clock (*i.e.* the time until it receives the synchronisation pulse): this is given by the appropriate choice of synchronisation time from (13). For n ticks of the k^{th} clock (n_k), the total amount of time that has transpired since the lead clock first began recording time is simply given by the sum of these two times.

For a clock a in the upchain direction we have the expression

$$t = \frac{L}{c} \sqrt{\frac{1+\beta}{1-\beta}} |a| + \frac{2L}{c} \frac{n_a}{\sqrt{1-\beta^2}}, \quad (15)$$

and for a clock b in the downchain direction we have the expression

$$t = \frac{L}{c} \sqrt{\frac{1-\beta}{1+\beta}} |b| + \frac{2L}{c} \frac{n_b}{\sqrt{1-\beta^2}}. \quad (16)$$

Let us also rearrange these equations for the number of ticks recorded by any clock post synchronisation. Upchain, the number of ticks recorded by clock a after time t is

$$n_a = \frac{ct\sqrt{1-\beta^2}}{2L} - \frac{|a|}{2}(1+\beta), \quad (17)$$

while downchain the number of ticks recorded by clock b after time t is

$$n_b = \frac{ct\sqrt{1-\beta^2}}{2L} - \frac{|b|}{2}(1-\beta). \quad (18)$$

Adding to (17) and (18) the offsets that the observers located at clocks a and b will add to their clocks to account for the believed synchronisation time ($|a|/2$ and $|b|/2$, respectively) yields the proper clock reading, denoted v , for clocks that are upchain and downchain, respectively. Note that, by the labelling scheme, we can express the proper clock reading of *any* clock in the chain, k (where k is any integer), at an instant in time with a single formula,

$$v_k := n_k + \frac{|k|}{2} = \frac{ct\sqrt{1-\beta^2}}{2L} - \frac{k}{2}\beta. \quad (19)$$

From this equation we can rearrange for t again, yielding

$$t = \frac{L}{c} (|k| + k\beta) \gamma + \frac{2L}{c} n_k \gamma = \frac{2L}{c} \left(v_k + \frac{k}{2}\beta \right) \gamma. \quad (20)$$

(15) and (16) are limiting cases of (20) and are now superfluous.

From (19) or (20), one can obtain the useful relationships for the difference in time as a function of the difference of proper clock reading of any clock or clocks,

$$\Delta t (\Delta v) = t(v_l) - t(v_k) = \frac{2L}{c} \gamma \Delta v + \frac{L}{c} \gamma \beta (l - k), \quad (21)$$

and for the difference in proper clock reading for any clock or clocks as a function of a difference in time,

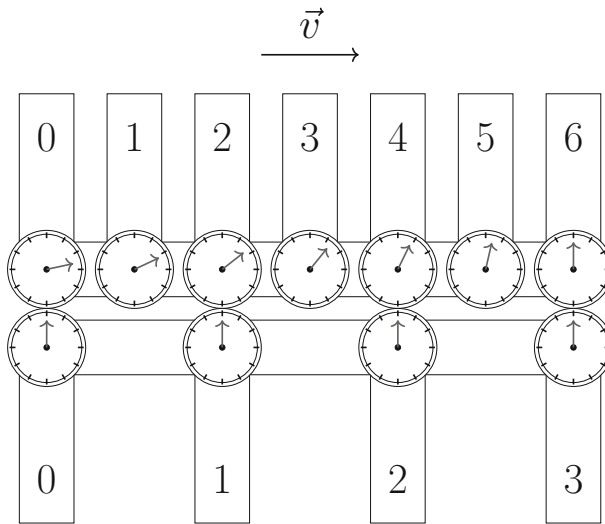


Fig. 5 Two chains of sound clocks instantaneously adjacent to one another after having carried out their calibration and synchronisation procedures. The sound clock chain at the *bottom* is stationary, while the sound clock chain at the *top* is travelling with a fractional velocity of $\beta = \sqrt{3}/2$, corresponding to a Lorentz factor of $\gamma = 2$. Simultaneity requirements lead to the moving chain possessing asynchronous clocks that are separated by a distance of L/γ , where L is the length of the *vertical* timing arms

$$\Delta v(\Delta t) = v_l(t_1) - v_k(t_0) = \frac{c}{2L\gamma} \Delta t + \frac{1}{2} \beta (k - l). \quad (22)$$

These two relationships, (22) and (21), are of crucial importance in Sects. 5 and 6. It is important to note that the quantity Δv present in (21) and (22) is entirely general and can correspond to the difference in proper clock reading as recorded by a single clock ($k = l$), which must of course occur at two different instances in time ($\Delta t \neq 0$), or to the difference in proper clock reading as recorded by different clocks ($k \neq l$), which can be calculated for any difference in time.

When observers—moving or not—request information from another observer on their chain for a believed simultaneous point in time, they will request information recorded at the same proper clock reading (*i.e.* $\Delta v = 0$). This is because all observers in inertial motion believe themselves to be at rest and so believe that all synchronised clocks possess the same proper clock readings at a given point in time. In reality, a moment in time is given by $\Delta t = 0$ for which, in general, different clocks will not read a difference in proper clock reading of $\Delta v = 0$ (which only happens to be true for chains of sound clocks at rest).

Figure 5 shows two chains of sound clocks, one at rest and one travelling with a velocity of v parallel to the stationary chain, at an instant in time. It can be seen that, as per (22), the stationary chain ($\beta = 0$) possesses clocks that all have the same reading at an instant in time, whereas the moving chain possesses clocks whose readings differ as a function of their separation from the origin clock.

5 Relativistic Effects Observed by Stationary Sound Clocks

Consider the following scenario: a moving chain of sound clocks passes by a stationary chain of sound clocks with a velocity of v , which corresponds to a fractional speed of β , with respect to sound. The moving chain travels in the direction parallel to its own spacing arms, and passes by the stationary chain parallel to it and close enough to it that the travel time for sound between clock faces on adjacent chains is small with respect to the time it takes clocks on either chain to tick. Observers located at each clock can only record the information accessible from their immediate surroundings: they can read their own clock face, they can count how many clock faces they have passed by on the adjacent chain, and they can read the value recorded on clocks in the adjacent chain when they are sufficiently close (*i.e.* next to them). Observers within a chain then have to talk to one another and exchange their own measurements in order to come to some understanding of whatever experiment they conducted.

As we shall demonstrate, observers located on the stationary chain of sound clocks determine that the sound clocks within the moving chain appear to be both separated by a shorter distance and tick less frequently than their own sound clocks. This is in keeping with what is seen in the laboratory. Later, when we consider measurements made by observers on a moving sound clock chain of a stationary chain we find that, contrary to what occurs in the laboratory, moving observers also determine that the clocks in the stationary chain are separated by a shorter distance than their own and ticks less frequently than their own. This only happens when both chains are treated equally in that observers are only allowed to use their own clocks as time references, observers can only signal with sound pulses, and observers have no means by which to detect motion with respect to the medium that they are embedded within.

5.1 Time Dilation as Seen by Stationary Observers

Imagine that observers within the stationary chain of sound clocks decide to focus on the lead sound clock of the moving chain as it passes by, as seen in Fig. 6. The first experiment they wish to conduct is:

“How many times do moving clocks appear to tick for every tick of stationary ones?”

To determine how many times sound clocks in a measuring chain believe themselves to tick for every one tick of a sound clock in a different chain, we use the following procedure:

1. Determine the separation in time, Δt , for some clock, z , in the chain that is being measured to increment its clock reading once. This is obtained using (21) with $z = l = k$ and $\Delta v_z = 1$.
2. Determine which clocks, k and l , in the chain performing measurements clock z is next to at two points in time separated by Δt . This is done by determining how far clock z has moved in Δt as calculated in Step 1.
3. Determine the proper clock reading on clock k when the clock that is being measured, z , is next to it, and determine the proper clock reading on clock l when the clock that is being measured, z , is next to it. The difference in these two proper

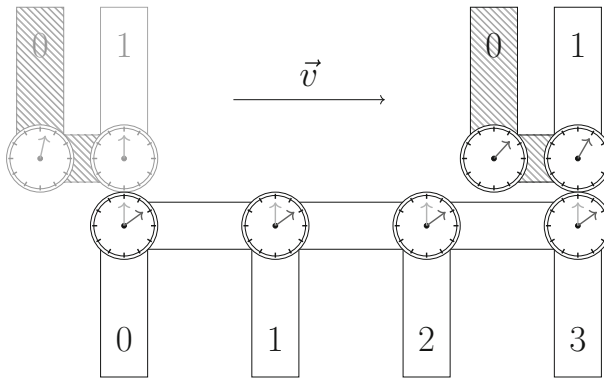


Fig. 6 The observers in the stationary chain focus only on the lead sound clock of the moving chain. The moving chain has a fractional velocity of $\beta = 3/\sqrt{13}$, with respect to sound, corresponding to a Lorentz factor of $\gamma = \sqrt{13}/2 \approx 1.8$. With this velocity, the moving chain of sound clocks travels a distance of $3L$ in the laboratory for every tick of its clocks. In this example, the clocks labelled k and l in the stationary chain are clocks 0 and 3, respectively

clock times is how many ticks observers within a measuring chain believe to have occurred for them for one tick of the clock they were measuring. In other words, evaluate (22) for clocks k and l as determined in Step 2 using time difference obtained in Step 1. The difference in these proper clock readings gives the perceived number of ticks that have transpired in the chain performing measurements for one tick of the clock being measured.

Consider that some clock, k , in the stationary chain is next to a clock in the moving chain, z , when the moving clock has a proper clock reading v_z^M (where the superscript ‘M’ denotes that the quantity pertains to the moving chain). At some later point in time, clock z in the moving chain has moved next to some other clock in the stationary chain, l , at the moment that clock z advances its proper clock reading forward one tick to $v_z^M + 1$. From (21), the time it takes for a given clock in the moving chain, z , to tick once ($\Delta v_z^M = 1$) is $2L\gamma/c$ where γ is the sonically relativistic Lorentz factor of the moving clock. The distance covered by a moving clock ticking once is then given by

$$x^M = v\Delta t = v \frac{2L}{c} \gamma = 2L\gamma\beta. \quad (23)$$

Stationary clocks are separated by a length of L , so the number of stationary-clock spacing-arm lengths that the moving clock has travelled in this time is given by bL , where b is just some number. Equating these two distances and solving for b , we find,

$$b = 2\gamma\beta. \quad (24)$$

We have then that $l = k + b$. What is the difference in the proper clock reading, $\Delta v^S = v_l^S - v_k^S$ (the superscript ‘S’ denotes that the quantity pertains to the stationary chain), corresponding to the difference in time, Δt , that it takes for clock z in the moving chain to travel between clocks k and l in the stationary chain? From (22) (with

$\beta = 0$ as we are considering the proper clock reading difference of the stationary chain), we find the difference in proper clock readings to be,

$$\Delta v^S(\Delta t) = v_l^S(t_1) - v_k^S(t_0) = \gamma. \quad (25)$$

Observers in the stationary chain determine that their clocks have all ticked γ times for one tick of the moving clock. This is, in fact, true. In the laboratory, all clocks within the stationary chain possess the same proper clock reading at the same instant in time, and all stationary clocks have indeed ticked γ times for one tick of the moving clock.

5.2 Length Contraction as Seen by Stationary Observers

Now imagine that observers located in the stationary chain decide to focus on more than just the first sound clock of the moving chain. The next experiment they wish to perform is:

“How are clocks in the moving chain spaced, relative to clocks in the stationary chain?”

When we want to determine the length of an object in a laboratory (at least for any experiment that takes place over reasonable distances and times) we will typically find the positions of both ends of the object in question at an instant in time and then determine the separation of these points in space. This is done with respect to some fixed coordinate system parallel to the object (*e.g.* a ruler). The definition of ‘an instant in time’, a notion that is crucial in operationally determining lengths, is a velocity-dependant quantity for observers who only have access to sound clocks. The only measure of time that observers with sound clocks have access to is their proper clock reading as given by (19). Within a given chain, observers at different clocks will possess a different proper clock reading at a given point in time if their chain is travelling with respect to the medium (*i.e.* $\beta \neq 0$ in (22)). To determine how observers with sound clocks measure distance, we must ask what measurements they make when *their* clocks have the same readings. The procedure we follow to determine what distance one chain of sound clocks measures another chain’s spacing arms to be is:

1. Determine which two clocks, k and l (where $k \neq l$), in a given chain are going to be used to perform measurements on lengths in another chain.
2. Determine the difference in time that is required for these two recording clocks to have the same proper clock reading, *i.e.* determine which two times correspond to $v_k = v_l$ using (19). The difference in time between these two clock readings is given by (21) for $\Delta v = 0$ and $k \neq l$.
3. Determine which clock, k' , in the chain that is being measured is next to clock k in the chain performing measurements at the time value corresponding to v_k , and which clock, l' , in the chain that is being measured is next to clock l in the chain that is performing measurements at the time value corresponding to v_l .
4. Determine how many spacing arms, b , separate the clocks in the chain performing measurements ($|b| = |l - k|$). Determine how many spacing arms, b' , separate

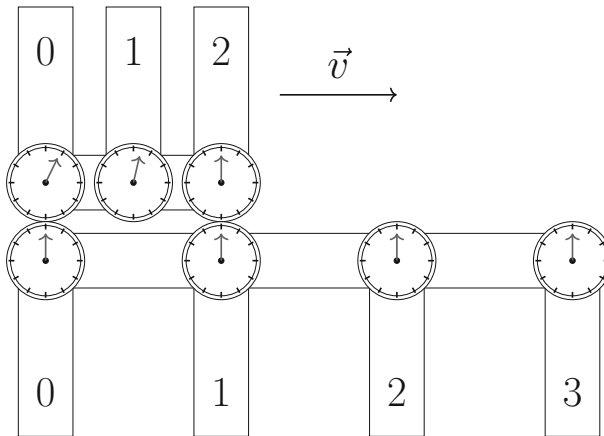


Fig. 7 The observers in the stationary chain of sound clocks determine the separation of sound clocks in the moving chain by determining which of their own clocks are simultaneously next to a given pair of clocks in the moving chain. From this information, they are able to determine how many spacing arms in the stationary chain are simultaneously parallel to a given number of spacing arms in their own chain, and with the knowledge that their own spacing arms are of length L , they can determine how long the spacing arms in the moving chain are. The moving chain is travelling with a fractional velocity of $\beta = \sqrt{3}/2$, corresponding to a Lorentz factor of $\gamma = 2$. In this example, clocks k and l in the stationary chain are clocks 0 and 1, respectively

the clocks in the chain being measured ($|b'| = |l' - k'|$). Using this information, determine how many spacing arms in the chain being measured simultaneously appear to be parallel to one spacing arm in the chain performing the measurements.

Consider that, at some time, one of the clocks in a moving chain, k' , is next to one of the clocks (call it k) in the stationary chain: we shall label this time $t(v_k^S)$. Which clock in the moving chain, l' , is next to some other clock in the stationary chain, call it $l = k + b$, when clock l has the same proper clock reading as clock k at time $t(v_k^S)$? From (21), with $\beta = 0$, the difference in time between the moments when clocks k and l have the same reading is 0. That is to say,

$$t(v_k^S) = t(v_l^S). \quad (26)$$

As expected, the clocks in the stationary chain possess the same proper clock reading at the same instant in time. This means that the moving chain has not moved relative to the stationary chain when clocks k and l perform their measurements at equal clock readings, as can be seen in Fig. 7, for $b = 1$.

Clock l is b spacing arms away from clock k , and the spacing arms in the stationary chain have length L ; therefore clocks k and l are separated by a distance of bL . Clock k' in the moving chain is next to clock k in the stationary chain, while simultaneously clock l' in the moving chain is next to clock l in the stationary chain; the number of spacing arms separating clocks k' and l' in the moving chain is labelled b' . Clocks in the moving chain are separated by spacing arms of length L/γ , so we have the equality

$$bL = b' \frac{L}{\gamma}. \quad (27)$$

This leads to the relationship

$$b' = \gamma b. \quad (28)$$

Directly adjacent clocks in the stationary chain (when $b = 1$ in (28)) will (correctly) conclude that, parallel to the single spacing arm that separates them, there are γ spacing arms in the moving chain.

Furthermore, if one were to extend the formalism and imagine that there were clocks at every point in space along a chain of sound clocks, the hypothetical clock with label $k + 1/\gamma$ would be next to some clock in the moving chain, whose neighbour would be next to clock k in the stationary chain.

6 Relativistic Effects Observed by Moving Sound Clocks

Consider that observers located on a moving chain of sound clocks wish to perform the same experiments as the stationary chain did in Sect. 5.

To reiterate: a moving chain of sound clocks passes by a stationary chain of sound clocks with a velocity of v , which corresponds to a fractional speed of β , with respect to sound. The moving chain travels in the direction parallel to its own spacing arms and passes by the stationary chain parallel to it and close enough to it that the time it takes sound to propagate between the two chains is small with respect to the time it takes clocks on either chain to tick. Observers located at each clock only have at hand the information accessible from their immediate surroundings; they can count how many clock faces they pass by on the adjacent chain, and they can read the value off of adjacent clock faces when they are sufficiently close. Observers within the chain then have to talk to one another and exchange their own measurements in order to come to some understanding of whatever experiment they conducted.

As foreshadowed in Sect. 5, we find that observers located on moving chains of sound clocks find that sound clocks within a stationary chain appear to be both separated by a shorter distance and tick less frequently than their own sound clocks. It is not obvious that this result should appear, and in fact it only does so when the observers are constrained to use their own clocks as time references, can only signal with sound pulses, and cannot detect motion with respect to the medium that they are embedded within. These constraints lead observers to ask for measurements made at the wrong laboratory time when they wish to aggregate and compare data recorded ‘simultaneously’ because the clocks in the moving chain do not actually possess the same reading at an instant in time.

6.1 Time Dilation as Seen by Moving Observers

Imagine that observers in the moving chain decide to focus on the lead sound clock of the stationary chain as they pass it, as seen in Fig. 8. The first experiment they wish to conduct is:

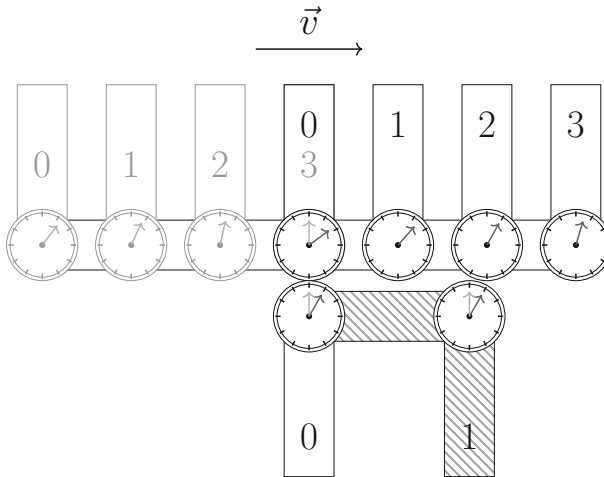


Fig. 8 The observers in the moving chain focus only on the lead sound clock of the stationary chain in order to determine how many times their own clocks tick for every one tick of a clock in the moving chain. Due to the asynchronicity of clocks in the moving chain, the observers within the moving chain come to the incorrect conclusion that their clocks tick γ times for every 1 tick of a clock in the stationary chain. The moving chain is travelling with a fractional velocity of $\beta = 3/\sqrt{13}$, with respect to sound, corresponding to a Lorentz factor of $\gamma = \sqrt{13}/2 \approx 1.8$. In this example, the clocks labelled k and l in the moving chain are clocks 3 and 0, respectively

“How many times do stationary clocks appear to tick for every tick of moving ones?”

The method used here is exactly the same as was outlined in Sect. 5.1. Consider that some clock in the moving chain, k , is next to a clock in the stationary chain, z , when the stationary clock has a proper clock reading v_z^S . At some later point in time, the moving chain has moved such that some other clock in the moving chain, l , is next to clock z at the moment that clock z advances its proper clock reading forwards one tick to $v_z^S + 1$. The difference in time, Δt , that it takes for clock z to tick once, $\Delta v_z^S = 1$, can be obtained using (21) (with $\beta = 0$ because we are considering the stationary clock). The time taken for a stationary sound clock to advance its proper clock reading by one tick is found to be $2L/c$ (as expected from (6)). The distance that the *moving* chain has travelled in this time is given by

$$x^M = v\Delta t = 2L\beta. \quad (29)$$

Sound clocks within the moving chain are separated by a distance of L/γ . The distance that the moving sound clock chain has travelled, x^M , is equal to some multiple, b , of its own sound clocks’ separation length,

$$b \frac{L}{\gamma} = 2L\beta. \quad (30)$$

From this, we can easily solve for b :

$$b = 2\gamma\beta \quad (31)$$

Note that the value of b determined here is exactly the same as b in (24).

The moving sound clock chain has travelled a distance of b multiples of its own spacing arm length in the time that it has taken the stationary sound clock, z , to tick once. Therefore, clock $l = k - b$ is next to the stationary clock, z , when it advances its time forward by one tick. The difference between the proper clock readings of clock k at some time t_0 and clock l at some later time $t_1 = t_0 + 2L/c$ can be obtained from (22):

$$\Delta v^M(\Delta t) = v_l^M(t_1) - v_k^M(t_0) = \gamma. \quad (32)$$

Where, again, the superscript M indicates that these are quantities pertaining to the moving chain. Even though the chain of moving sound clocks actually ticks less frequently than the chain of stationary sound clocks, observers travelling along with the moving chain believe that the stationary chain ticks less frequently than their own due to their incorrect belief that they are at rest, which results in their clocks being asynchronous.

6.2 Length Contraction as Seen by Moving Observers

The observers in the moving chain now decide to focus on more than just the first sound clock of the stationary chain in order to perform their next experiment:

“How are clocks in the stationary chain spaced, relative to clocks in the moving chain?”

We take the same approach as with the stationary chain’s experiment, except with the roles of the stationary and moving chain reversed: which two clocks in the moving chain are two adjacent clocks in the stationary chain simultaneously next to (where simultaneity is defined as equal proper clock readings)?

The method used here is exactly the same as was outlined in Sect. 5.2. Consider the scenario in which at some time, t_0 , some clock in the moving chain, k , has proper clock reading v_k^M while next to a clock in the stationary chain. After what period of time does another clock l , that is b spacing arms away ($l = k + b$), have the same proper clock reading as clock k ($v_l^M = v_k^M$)? With use of (21) we can find the difference in time, Δt , for $\Delta v^M = 0$ for clocks k and $l = k + b$:

$$\Delta t \left(\Delta v^M \right) = t(v_l^M) - t(v_k^M) = b \frac{L\gamma\beta}{c}. \quad (33)$$

In this time the moving chain has moved a total distance of,

$$x^M = v\Delta t = bL\gamma\beta^2. \quad (34)$$

Clocks in the moving chain are separated by a distance of L/γ , so clock $l = k + b$ is b multiples of L/γ away from clock k . Clock l has also travelled a total distance of x^M (from (34)) after the moment in time when clock k made its measurement, so the location of clock l at time t_1 as compared the location of clock k at time t_0 is given by

$$x_l^M(t_1) = b \frac{L}{\gamma} + bL\gamma\beta^2. \quad (35)$$

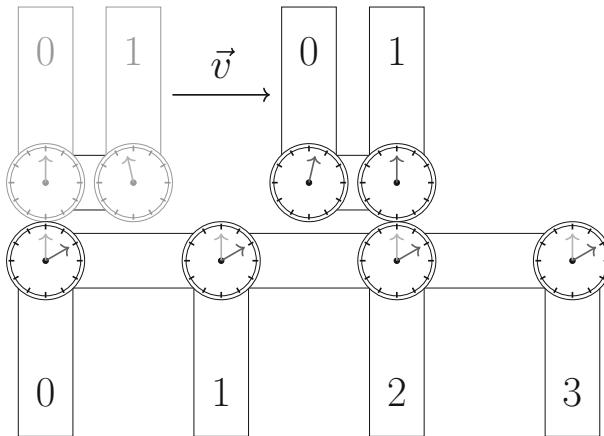


Fig. 9 The observers in the moving chain determine what they believe to be the separation of sound clocks in the stationary chain by determining how many sound clocks are between two of their own sound clocks at an instant in time (as they understand it). The moving chain has a fractional velocity of $\beta = \sqrt{3}/2$, with respect to sound, corresponding to a Lorentz factor of $\gamma = 2$. At this velocity, adjacent clocks in the moving chain only have the same clock reading when separated by a period of time that corresponds to having travelled a distance of $\gamma L = 2L$ in the laboratory. In this example, clocks k and l in the moving chain are clocks 0 and 1, respectively

This distance corresponds to some multiple, b' , of the *stationary* sound clocks' spacing arm length. Equating $b'L$ with $x_l^M(t_1)$ yields

$$b' = \gamma b. \quad (36)$$

Clocks k and $l = k + b$ in the moving chain register the same proper clock reading at two different instants in time. At these instants in time, clock k is situated over some clock in the stationary chain, and clock $l = k + b$ is situated over some other clock that is γb spacing arms away from the clock that k was situated over, as can be seen in Fig. 9, for $b = 1$. Directly adjacent neighbours in the moving chain (*i.e.* $b = 1$ in (36)) will (incorrectly) conclude that there are exactly γ clock arms simultaneously parallel to their single clock arm, a result that arises due to the asynchronicity of their clocks.

7 Sonic Relativity

We have now seen what look like relativistic effects from the perspective of internal observers: time dilatation appears to be given by (25) and (32), while apparent length contraction appears to be described by equations (28) and (36). However, none of these relationships explicitly deal with lengths or times. Ticks and numbers of clocks are both unitless, and furthermore, while γ as it appears in these equations is a quantity that we can calculate in the laboratory, internal observers as we have currently described them can only *measure* it (by counting how many clocks they pass in a given period of time or by comparing clock readings on their own chain to another chain). We shall

now cast all previous formulae in terms of quantities that internal observers themselves can measure and demonstrate the appearance of sonic relativity. Let us start with some defined quantities based on beliefs held by internal observers.

Assume that observers operationally define a unit of length called an ‘*arm*’ and a unit of time called a ‘*tic*’. A *timing arm* of length L (as measured in the laboratory) is operationally defined by internal observers to be 1 arm long, and 1 tic is operationally defined to be the time it takes for a sound clock with a timing arm of length 1 arm to advance its clock reading forward once. All observers in inertial motion believe themselves to be at rest, and so they believe 1 tic of time to be the time it takes for a sound pulse to travel 2 arm (to the end of the timing arm and back again). The speed of sound to any internal observer is then *defined* to be

$$\tilde{c} := 2 \frac{\text{arm}}{\text{tic}}. \quad (37)$$

With the belief that they are at rest, all observers co-moving with a chain of sound clocks that are calibrated and synchronised as per the procedures outlined in Sects. 4.1 and 4.2 believe that their *spacing arms* are exactly 1 arm in length each, as it takes exactly 1 tic of time for the echo of a sound pulse propagated between adjacent clocks in a chain to return. This requirement itself formed the basis of the calibration procedure as outlined in Sect. 4.1. With these definitions, we can determine at what velocity—in units of arm/tic—observers in a given chain believe another chain to be travelling.

In 1 tic of time for a stationary clock, $2\gamma\beta$ clock arms in the moving chain pass by (as per (31)). Recalling from (28) that stationary observers believe that γ spacing arms in the moving chain are simultaneously next to one of their own, observers on the stationary chain come to conclude that clocks in the moving chain are separated by γ^{-1} arm per clock. Stationary observers therefore (correctly) believe that $2\gamma\beta$ clocks in the moving chain have a length of 2β arm. These $2\gamma\beta$ sound clocks of length 2β arm take 1 tic to pass by. Thus, the perceived velocity of the moving chain, $\tilde{v}$, in units of arm/tic (up to a sign) is given by

$$|\tilde{v}| = \frac{2\beta \text{ arm}}{1 \text{ tic}} = 2 \frac{v}{c} \frac{\text{arm}}{\text{tic}} = \frac{\tilde{c}}{c} v = \tilde{c}\beta. \quad (38)$$

The perceived fractional velocity, $\tilde{\beta} := \tilde{v}/\tilde{c}$, of the moving chain with respect to sound using only measurements of quantities available to internal observers is related to the *actual* fractional velocity with respect to sound by

$$|\tilde{\beta}| = \beta. \quad (39)$$

Moving observers determine the same relationships. A moving clock passes by $2\gamma\beta$ stationary clocks in 1 tic of time (as per (24)), and from (36) moving observers deduce that γ spacing arms in the stationary chain lay simultaneously parallel to one of their own (*i.e.* moving clocks appear to be separated by γ^{-1} arm per clock). Because the moving observers think that they are the ones who are stationary and that the stationary

observers are moving, they believe that $2\gamma\beta$ clocks have passed by them, with a total length of 2β arm in a time of 1 tic. This leads to the (believed) velocity of the moving chain, $\tilde{v}$, in units of arm/tic (up to a sign) of

$$|\tilde{v}| = \frac{2\beta \text{ arm}}{1 \text{ tic}} = 2 \frac{v}{c} \frac{\text{arm}}{\text{tic}} = \frac{\tilde{c}}{c} v = \tilde{c}\beta. \quad (40)$$

The believed fractional velocity, $\tilde{\beta} := \tilde{v}/\tilde{c}$, of the stationary chain with respect to sound using only measurements of quantities available to internal observers is given by

$$|\tilde{\beta}| = \beta. \quad (41)$$

Both stationary and moving observers believe that the other chain of sound clocks is moving with a fractional velocity with respect to sound that is equal in magnitude to the value of the moving chain's fractional velocity with respect to sound in the laboratory frame. Utilising an agreed-upon coordinate system, observers in the moving chain will report the value of $\tilde{\beta}$ that they measure to be different than that as measured by stationary observers by a sign. Both stationary and moving observers define

$$\tilde{\gamma} := \frac{1}{\sqrt{1 - \tilde{\beta}^2}}. \quad (42)$$

A given observer measures length by counting the number of spacing arms between simultaneous measurements of the endpoints of an object: 1 arm is exactly the length of one spacing arm. We define $\tilde{\ell}'$ to be the length, as measured by a stationary observer, of an object whose length is measured to be $\tilde{\ell}$ by an observer at rest with respect to the object. Both stationary *and* moving observers believe that γb clocks in the other chain lay simultaneously next to b of their own, as per (28) and (36), respectively. Thus, observers in each frame will state that the total length spanned by some number of their own spacing arms is equal to the length spanned by γ times as many spacing arms of the other chain, leading to the relationship

$$\tilde{\ell}' = \frac{\tilde{\ell}}{\tilde{\gamma}}, \quad (43)$$

where γ has been replaced by the internal observer defined $\tilde{\gamma}$.

A given observer measures elapsed time by counting ticks of their own clock: 1 tic is exactly the time it takes for one's own clock to tick once. We define $\tilde{\tau}'$ to be the duration, as measured by a stationary observer, of a localised process whose duration is measured to be $\tilde{\tau}$ by an observer at rest with respect to the process. As per (25) and (32), both stationary *and* moving observers believe their own clock readings to have advanced $\tilde{\gamma}$ times in the time it takes a given clock in the other chain to advance its clock reading once. Therefore, both stationary and moving observers believe that a clock in the other chain takes $\tilde{\gamma}$ as much time to tick once as their own clock does, leading to the relationship

$$\tilde{\tau}' = \tilde{\gamma} \tilde{\tau}, \quad (44)$$

where, as before, the internal-observer-defined Lorentz factor $\tilde{\gamma}$ is used due to its equality with γ .

8 Discussion

Through an operational approach, it has been shown that it is possible for a certain class of inertial observers to deduce the existence of two key phenomena from special relativity—length contraction and time dilation—in condensed-matter systems for which the speed of sound plays an analogous role to the speed of light within our universe. The observers we discuss are significantly restricted in their ability to assign temporal and spatial values to events. They are only able to claim when events occur relative to their own clock, can only make claims about events that are sufficiently local such that the time it takes for the signal to reach them is negligible, and must confer with one another after taking local measurements to come to an understanding of the events that transpired.

The observers travelling at a constant velocity have no way to tell if they are stationary or moving. With no way to tell who is in motion, this question would become a philosophical one for internal observers, and some internal observers may even come to the same conclusions that we in our universe have: that all motion is relative. We have seen that when internal observers assume the former state of motion—that they are indeed stationary—then those who are in motion incorrectly set the separation of their clocks, a process that was achieved by fulfilling the requirement that sound simultaneously emitted along two objects of equal length will result in the simultaneous reception of echoes. This simultaneity condition results not only in the incorrect separation of clocks within a moving chain; it also results in the asynchronicity of those clocks.

In constructing their chains of sound clocks in such a way that local simultaneity conditions hold, observers who are stationary see the moving chain to be length contracted exactly as one would expect from a naïve application of relativistic formulae, with c being the speed of sound instead of the speed of light. The clocks within the moving chain also appear to be time dilated as would be expected from special relativity. This is due to the use of sound pulses to advance clock readings, and moving clocks increase the path length, thus increasing the time it takes for a sound pulse to return to the clock mechanism by exactly the Lorentz factor again.

More curiously, the observers in moving chains also witness stationary sound clock chains as being length contracted: this is not actually the case and is again a result of making use of simultaneity arguments. Moving observers *think* that the clocks within their own chain are synchronous, and thus when they wish to know what happened at some distant clock simultaneous with their own clock, they ask the observer located at that distant clock to provide them with information recorded when the distant clock's reading was the same as their own. These clocks are not actually synchronous, however, so the observers in the moving chain are actually comparing information from two separate instances in time. This happens to work out in exactly the right way to make the observations of moving observers and stationary observers symmetric: moving

observers also perceive stationary sound clock chains to be length contracted and time dilated exactly as one would expect from a naïve application of relativistic formulae, with c being the speed of sound instead of the speed of light.

We see that the ‘in-universe’ experience—the internal observers’ description of their universe—can be described by the mathematical formalism of special relativity provided that such observers believe the postulates of relativity, a conclusion that they would reasonably come to when given no ability to detect their aether. It is merely a misunderstanding of the Newtonian mechanics at play that results in the appearance of relativistic effects to these internal observers. Given the ability to detect their own state of motion relative to their aether, moving observers would quickly come to understand that they have incorrectly separated and calibrated their clocks, and fixing this problem would result in the disappearance of the apparent relativistic effects that are witnessed by moving observers.

We have intentionally remained within the realm of discussing devices that are operationally controlled by observers, and the relativity within the system described appears as a result of the belief that observers have about their state of motion. Nevertheless, it is worth noting that if we had access to devices that were built from quasiparticles made up from the medium itself, then sonic relativity would emerge naturally. As described by Barceló and Jannes [26], a device constructed from these quasiparticles (such as a sound clock chain) would shrink naturally as v approaches c , just as physical objects held together by the electromagnetic force do when travelling close to the speed of light [31]. This would entirely remove the role of the observers in tuning the separation of neighbouring clocks, and therefore the belief held by the observers on their state of motion would become inconsequential.

Furthermore, while we have restricted our analysis to chains of sound clocks that must keep their timing and spacing arms perpendicular, and that are only able to move in the direction perpendicular to their timing arms, devices built from quasiparticles would not be limited in these ways. If the sound clock chains that we describe here were permitted to travel with a velocity possessing a non-zero component in the direction parallel with the timing arms, then any change in the direction of motion mid-journey to include such a component of velocity would lead to the asynchronous return of echoes in the timing and spacing arms. Additionally, if these devices could alter the angle between their arms, this would again lead to the asynchronous return of sound pulses in the timing and spacing arms, and if the angle between the arms was closed to 0° , observers could quite easily determine that the timing and spacing arms are different lengths. These effects would be naturally taken care of in quasiparticle devices, however: the entire device’s dimensions would change accordingly with its velocity, conspiring to make the effects of length contraction impossible to detect to the observers located on the chain.

9 Conclusion

It is perhaps not too surprising that stationary observers in these systems infer that moving observers undergo relativistic time dilation and length contraction, but given the presence of a preferred reference frame it may not be immediately obvious that mov-

ing observers should see stationary observers subject to these same effects (although if one has a sufficient understanding of the history of special relativity, especially in the context of aether theory, then this may not actually be too surprising). Given that relativity *is* seen in both directions just as in our universe, it can be seen that the existence of a preferred reference frame is not immediately prohibitive in the emergence of a self-consistent description of relativity by internal observers in analogue gravity systems. While some aspects of relativity can be made to appear, it is not clear to what extent relativistic physics can be made to manifest in such systems.

If the role of observers in analogue gravity systems is taken more seriously, investigating the observations made by such observers might give us some insight into how many of the phenomena described by general relativity can be seen to arise in analogue gravity models in a self-consistent manner. Are there any analogue gravity models that appear to possess all of the phenomena of general relativity (in analogous forms) to internal observers? If this is not the case, then why not? Why should some of the phenomena of relativity emerge in a self-consistent manner in such models, but not others? Considering the experience of observers may help to answer these questions.

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Particle scattering in a sonic analogue of special relativity

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We investigate a simple toy model of particle scattering in the flat spacetime limit of an analogue-gravity model. The analogue-gravity medium is treated as a scalar field of phonons that obeys the Klein–Gordon equation and thus admits a Lorentz symmetry with respect to c_s , the speed of sound in the medium. The particle from which the phonons are scattered is external to the system and does not obey the sonic Lorentz symmetry that the phonon field obeys. In-universe observers who use the exchange of sound to operationally measure distance and duration find that the external particle appears to be a sonically Lorentz-violating particle. By performing a sonic analogue to Compton scattering, in-universe observers can determine if they are in motion with respect to their medium. If in-universe observers were then to correctly postulate the dispersion relation of the external particle, their velocity with respect to the medium could be found.

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I. INTRODUCTION

Analogue-gravity models provide an indirect way to probe the physics of gravitational systems for which the actual (i.e., nonanalogue) experiments are currently inaccessible. Perhaps most well known are the acoustic analogues to black holes (also called *dumb holes*¹), originally proposed by Unruh [1] to provide a theoretical and experimental testbed for the study of Hawking radiation [2,3] in a system where the microscopic physics is understood. Subsequently, analogue models for a variety of general relativistic and semiclassical gravitational phenomena have been discovered, and many such models are now being experimentally realized—notably, acoustic analogues for Hawking radiation [4–7], optical-media analogues for Hawking radiation [8], and analogues for cosmological expansion and particle production [9] have all successfully seen experimental realization to date. A comprehensive list of analogue-gravity models and the research into them as of 2011 can be found in the extensive review article by Barceló, Liberati, and Visser and the references therein [10], while references to some noteworthy experimental results in the interim can be found in a short article by Jacquet, Weinfurter, and König [11], which discusses both the history and future of experiments within the analogue gravity research endeavor. The aforementioned article by Jacquet *et al.* brings attention to experimental work on superradiant scattering by Torres *et al.* [12,13], as well as to developments on modeling

gravitational phenomena within superconducting circuits [14] (see Tian *et al.* [15,16] and Lang and Schützhold [17] for additional work on superconducting circuits within the context of analogue gravity). While by no means extensive, a selection of noteworthy theoretical results since 2011 include [18–23].

Studying relativistic phenomena with analogue models is, of course, only a valid approach provided that the analogue system can be faithfully mapped back to the actual physical system of interest. One obvious—and seemingly detrimental—way in which the mapping between analogue models and real physical systems of interest seems to fail is that analogue models are not truly relativistic. In the most obvious case, there exists a preferred reference frame: the rest frame of the analogue-gravity medium itself. Previous work by Barceló and Jannes [24] has highlighted that the natural way to consider analogue-gravity systems as genuine relativistic analogues is to use devices and observers that are *internal* to the analogue medium (e.g., they can be constructed from quasiparticle excitations of the medium itself) to make internal measurements of phenomena within the analogue universe. Phenomena like the Lorentz–FitzGerald contraction can be operationally shown to appear in a relativistically reciprocal manner by utilizing this notion of internal devices and observers: internal observers who are at rest with respect to the medium will *measure* moving observers’ devices to be length contracted (they are), and internal observers who are moving with respect to the medium will *measure* stationary observers’ devices to be length contracted (even though they are not) [25]. We call such observers *in-universe observers*.

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¹“Dumb,” in this case, being a synonym for “mute.”

One way to understand the operational emergence of relativity from an analogue-gravity model is to consider the flat-spacetime limit: for an acoustic system, the flat-spacetime limit is exactly a sonic analogue to Lorentz ether theory,² and it is known that Lorentz ether theory and special relativity are observationally equivalent [26,27]. Therefore, from the operational viewpoint that considers only the measurements made by in-universe observers [25], the sonic analogue to Lorentz ether theory can be seen to faithfully map to a sonic analogue of special relativity, which we call *sonic relativity*. For the purposes of this paper, our considerations will be restricted to such sonically relativistic analogues of special relativity.

In attempting to utilize analogue-gravity models as a means to study phenomena within the overlap of relativistic physics and quantum physics, it would prove useful to understand how operational measurements of quantum mechanical phenomena can be performed in such systems. In the real world, quantum field theory (QFT) is the most well-developed theory incorporating the effects of special relativity and quantum mechanics, and within the context of QFT, scattering experiments are one of our most valuable experimental tools for probing the dynamical interactions of physical phenomena.³ To this end—and as the title of this paper suggests—our focus within this paper shall be on scattering experiments rather than on other types of experiments that can be conducted within the framework of QFT.

If we are to attempt to utilize analogue-gravity models to their full potential (i.e., as a method by which to carry out actual experiments), then we should—as a first step—seek to describe and understand those detector models that can conceivably be experimentally realized and that can be used in conjunction with analogue-gravity systems. This leads to several questions. For example, what constitutes an appropriate model of a particle detector within an analogue-gravity model? How much of our understanding of quantum detectors from actually relativistic theories can we apply to analogue-gravity systems? If we consider deviations away from the idealized case in which we only consider that which is *internal* to an analogue-gravity model, how is our understanding of detector models impacted?⁴ In particular, this last question provides the basis of the work presented in this paper.

We will herein consider scattering experiments within analogue-gravity models. We do so in keeping with the philosophy of a previous paper that was published by two of the present authors [25]: we consider the measurements

made by in-universe observers within an analogue-gravity system, and we restrict these observers to measuring duration and distance solely through the exchange of sound pulses. This ensures their measuring devices obey sonic Lorentz symmetry. We assume that in-universe observers have access to the apparatus required to do scattering experiments with phonons. That is, they can produce phonons, and they can detect recoiling phonons through a “click” of a particular detector within an array of such detectors. We neglect the detailed questions of how such items are constructed, understanding that our assumptions are enough to meaningfully discuss scattering experiments in this setting.⁵ We seek to determine the results that in-universe observers measure of phonon scattering experiments, specifically in the case that phonons are scattered from an external particle—that is, one whose dispersion relation is nonrelativistic.

The contents of this paper are as follows: in Sec. II we review the notion of in-universe observers [25]. In Sec. III we elucidate further on our aims, provide a sketch of the approach that is used throughout this paper, state the conventions that we shall use, and give a schematic of the scattering events experiments that we consider. In Sec. IV we determine the kinematic expressions of a toy model of phonon scattering from two types of particles: one type of particle is sonically Lorentz obeying (internal particle), and the other is sonically Lorentz violating (external particle). In both cases, we first obtain the kinematic description of scattering in the laboratory frame and then in the frame of an in-universe observer who is comoving with the particle from which the phonons scatter. (This is the approach laid out in Sec. III.) In Sec. V we consider phonon scattering from a first-quantized particle with a Newtonian dispersion relation (an external particle) and present the results that demonstrate that these scattering experiments reveal information about the existence of a preferred rest frame to in-universe observers. In Sec. VI we explicitly explain in what way the results presented in the previous section reveal information about the existence of a preferred rest frame. In Sec. VII we summarize our results and demonstrate that the Standard-Model extension might provide a way to model phonon scattering from external particles as described by in-universe observers.

II. IN-UNIVERSE OBSERVERS

Assume that there are in-universe observers in an analogue-gravity universe who believe in the principle of

²Provided that one only considers the motion of sound waves belonging to the linear part of the medium’s dispersion relation.

³See almost any textbook on QFT for a discussion on scattering and its relevance to experiments, e.g., [28–31]. For a more in-depth discussion on the nature of detectors themselves, see [32,33].

⁴Where the notion of *internal* is as per the discussions of Barceló and Jannes [24].

⁵For the reader who is nevertheless curious about the possibility of actually realizing in-universe detectors, a sensible-seeming starting point would be to consider devices whose operation utilizes optomechanical interactions. To this end, recent work involving Brillouin scattering [34–37] and recent work involving surface acoustic waves (SAWs) coupled to qubits (to yield qubit-SAW devices) [38,39] may provide useful food for thought. Graphene and topological semimetals are yet another possibility [20,40].

special relativity, with the exception that sound takes the place of light [25]. So defined, in-universe observers can be constructed using classical (nonquantum) rulers and clocks, as in [25].⁶ Taking the coordinates x , y , z , and t to be measured operationally by some in-universe observer via the exchange of sound, and defining $\mathbf{x} = (x, y, z)$ as a notational shorthand, we can construct the following four-vector

$$X^\mu := \begin{pmatrix} c_s t \\ \mathbf{x} \end{pmatrix}. \quad (2.1)$$

This four-vector is a sonically Lorentz covariant object, and in keeping with special relativity we denote this the sonic four-position. Of course, there is nothing special about Cartesian coordinates: the sonically Lorentz covariant nature of the sonic four-position is true for any set of orthogonal coordinates. We have merely made the choice to pick Cartesian coordinates for the purposes of demonstration.

Consider now two sets of in-universe observers who are initially traveling in the same direction, which we shall call the z direction, by convention, with respect to their sound-carrying medium. The first set of observers are traveling with velocity $\mathbf{v}_1 = (0, 0, v_1)$ with respect to the medium, and the second set of observers are traveling with a velocity of $\mathbf{v}_2 = (0, 0, v_2)$ with respect to the medium. Note that the operational measurements of distance and duration allow for in-universe observers to operationally calculate velocities [25]. In the particular case that we are considering, the operationally determined sonic fractional velocity of the second set of observers as measured by the first set of observers will be given (in terms of the laboratory-frame values of quantities) by the following expression:

$$\beta = \frac{\beta_2 - \beta_1}{1 - \beta_2 \beta_1}, \quad (2.2)$$

where $\beta_1 = v_1/c_s$ and $\beta_2 = v_2/c_s$. The quantity β is the *boost parameter* of the second frame with respect to the first frame.

Each set of observers can describe the sonic four-position from their own operational measurements of distance and duration, as discussed above. Denote X^μ to be the components of the sonic four-position as operationally determined by the first set of observers, and X'^ν to be the components of the sonic four-position as operationally

determined by the second set of observers. The first set of observers can relate the components X'^ν to X^μ with the following familiar equation:

$$X'^\nu = \Lambda^\nu_\mu X^\mu. \quad (2.3)$$

For the particular case that we have been discussing Λ^ν_μ is given by⁷

$$\Lambda^\nu_\mu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \quad (2.4)$$

where β is the operationally determined sonic fractional velocity of the second set of observers with respect to the first set of observers, given by Eq. (2.2), and $\gamma := 1/\sqrt{1 - \beta^2}$.

Another way to see that measurements of duration and distance made using only the exchange of sound signals can be grouped together into a Lorentz covariant object, X^μ , is that the object being used to define spatial and temporal measurements (the phonon) obeys, in our simplified toy model, the Klein–Gordon equation [25]. The Klein–Gordon equation admits a Lorentz symmetry, and as a consequence of this, any measurements that are made using phonons inherit the Lorentz symmetry that the phonons themselves obey.

A consequence of the existence of the sonic four-position vector and its associated Lorentz transformation is that all physical quantities whose values can be determined geometrically—that is to say, from the components of the sonic four-position—transform exactly as expected from special relativity. For our purposes, we note that measurements of spatial and/or temporal coordinates (i.e., the components X^μ or X'^ν) can be used by in-universe observers to operationally determine the wavelength and frequency of sound waves, and also as a way to operationally determine angles: as a result, between any pair of in-universe observer reference frames the wavelengths and frequencies of sound waves obey the relativistic Doppler shift formula, and the angle of propagation of an acoustic ray will change via the relativistic aberration formula (where, in both cases, the speed of sound replaces the speed of light). The relativistic Doppler shift formula and the relativistic aberration formula are given, respectively, by

⁶In-universe observers could also be constructed from quasiparticle excitations of the medium itself, as proposed in [24]. However, by their nature, quasiparticle excitations are not, in general, well localized in energy, momentum, or space. This implies that considering a classical rest frame for in-universe observers may not be correct. If one were to consider in-universe observers made from quasiparticle excitations, the correct procedure would be to use the framework of quantum reference frames, as formulated in [41]. In order to avoid this additional layer of complexity we choose to work with classical in-universe observers as defined in [25].

⁷Note that this definition is *not* simply some vacuous one that we merely assert to be true axiomatically. While we do not offer a derivation within this paper, it is indeed possible to take the content of [25] and formulate a proper four-vector description of coordinates from the operational measurements made by in-universe observers. In doing so, the Lorentz transformation matrix naturally arises as the transformation relating the four-position between different in-universe observer frames.

$$\omega = \gamma(1 + \beta \cos \theta')\omega', \quad (2.5)$$

$$\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}, \quad (2.6)$$

where the last relation leads to the following useful expression,

$$(1 - \cos \theta) = \frac{1 - \beta}{1 + \beta \cos \theta'}(1 - \cos \theta'), \quad (2.7)$$

and where in all cases β is the boost parameter of the second frame with respect to the first [given by Eq. (2.2)], an *unprimed* symbol denotes the in-universe observer measured value of some physical quantity in the first frame, and a *primed* symbol denotes the value of that same physical quantity as measured by in-universe observers in the second frame. Note that in the specific case that we are considering, the angle θ is measured relative to the z axis and the angle θ' is measured relative to the z' axis.

For notational brevity in what follows, define the *Doppler factor* to be

$$D := \sqrt{\frac{1 + \beta}{1 - \beta}}. \quad (2.8)$$

In the particular case that $\theta' = 0$, the Doppler factor can be used to simplify Eq. (2.5) to $\omega = D\omega'$.

III. AIM AND APPROACH

A. Aim

Our purpose is to investigate a particular type of scattering experiment within the context of analogue-gravity models. In particular, we choose to investigate a scattering process that is in some sense analogous to Compton scattering. In true Compton scattering [28,42], a photon is scattered from a charged particle. By analogy we choose to analyze the scattering of phonons within an analogue-gravity system. We consider phonon scattering from two different types of particle, which we label as either *sonically Lorentz-obeying* or *sonically Lorentz-violating* based on their dispersion relation.

Our aim is the following: characterize the in-universe observer perspective of these scattering experiments, with the specific goal of determining what in-universe observers can learn from the results of scattering experiments involving sonically Lorentz-violating particles, including what they can learn about their state of motion with respect to the medium.

B. Sketch of our approach

In a typical derivation of scattering within quantum field theory, one often makes use of relativistic arguments to simplify the derivation by, for example, moving into the

center-of-mass frame of the system [28]. While such an approach indeed simplifies the derivation for phonon scattering from sonically Lorentz-obeying particles [24], this is certainly not the case for phonon scattering from sonically Lorentz-violating particles. In this paper we take the following approach to our derivations:

- (1) We anchor ourselves to the laboratory frame, which is the frame in which the analogue-gravity medium is assumed to be at rest. In this particular frame, we know the actual dynamics, and we know that energy and momentum must be conserved.
- (2) We identify, in the laboratory frame, the dispersion relations that apply to the phonon and to the particle from which the phonon will scatter.
- (3) Utilizing the relevant dispersion relations, we obtain the laboratory-frame kinematic expression for scattering by insisting on energy and momentum conservation.
- (4) With the laboratory-frame kinematics determined, we utilize the appropriate transformation rules to obtain the kinematic description of scattering from the perspective of the in-universe observer frame that is comoving with respect to the particle prior to phonon scattering.

This procedure grounds our calculation in the frame for which we know the dynamics: the laboratory frame. We do all of our dynamical calculations from this frame. The final step then determines how this behavior would appear to an in-universe observer who perceives the particle to be initially at rest in their own frame (even though both the observer and the particle may in fact be moving with respect to the medium). This allows us to tease out the differences between experiments that look the same to in-universe observers—particle initially at rest in the observer's frame—but that *actually* differ in their initial velocity with respect to the analogue-gravity medium—i.e., with respect to the *sonic ether*.

As a sanity check, we first utilize this approach to demonstrate that the kinematic description of phonon scattering from a sonically Lorentz-obeying particle shows no dependence on the initial motion with respect to the medium, as would be the case for ordinary scattering in a fully Lorentz-obeying model [24] and in ordinary QFT [28]. We then proceed to use the same approach to arrive at a kinematic description for phonon scattering from a sonically Lorentz-violating particle.

A mathematical sketch of this approach is the following. The energy and momentum conservation in the laboratory frame is simply⁸:

$$E_i + \hbar\omega_i = E_f + \hbar\omega_f, \quad (3.1)$$

⁸Note that we have restricted our considerations to the linear part of the medium's dispersion relation through our particular choices for the phonon's energy ($\hbar\omega$) and momentum ($\hbar\mathbf{k}$).

TABLE I. Notation for momenta of the phonon and particle. We restrict our initial setup to the case where $\mathbf{k}_i$ and $\mathbf{p}_i$ are in the positive- z direction.

Phonon	Particle	Meaning
$\hbar\mathbf{k}_i$	$\mathbf{p}_i$	Initial three-momentum (lab frame)
$\hbar\mathbf{k}_f$	$\mathbf{p}_f$	Final three-momentum (lab frame)
$k = \mathbf{k} $	$p = \mathbf{p} $	three-vector magnitude (lab frame)

$$\mathbf{p}_i + \hbar\mathbf{k}_i = \mathbf{p}_f + \hbar\mathbf{k}_f, \quad (3.2)$$

where the notation, which is used throughout, is defined in Table I. The dispersion (energy-momentum) relations allow us to express energies as functions of momenta:

$$E = E(\mathbf{p}), \quad \hbar\omega = \hbar\omega(\mathbf{k}), \quad (3.3)$$

from which we can rewrite the conservation of energy and momentum like

$$E_i(\mathbf{p}_i) + \hbar\omega_i(\mathbf{k}_i) = E_f(\mathbf{p}_f) + \hbar\omega_f(\mathbf{k}_f), \quad (3.4)$$

$$\mathbf{p}_f = \mathbf{p}_i - \hbar(\mathbf{k}_f - \mathbf{k}_i). \quad (3.5)$$

From here, we explicitly substitute our dispersion relations into Eq. (3.4), using the conservation of momentum [Eq. (3.5)] in the explicit functional form of $E_f(\mathbf{p}_f)$. (The specific functional forms that we will use will be given in subsequent sections.) Finally, we perform any valid algebraic steps that are required to obtain the desired kinematic expression for the Compton-like scattering of phonons. The resulting expression, by construction, obeys energy and momentum conservation, and is written entirely in terms of the initial parameters of our system—which, in principle, we are free to control—and the final state of the phonon—which we can envisage experimentally detecting using some appropriate apparatus, e.g., a detector array comprised of some sonic analogue to photomultiplier tubes.

Our final step is to utilize the laboratory-frame kinematic description of scattering—along with our understanding of how the values of physical quantities as measured by in-universe observers change between different reference frames—to determine the in-universe observer description of the kinematics. Specifically, we choose to obtain the kinematic description of scattering in the frame of an in-universe observer who is initially comoving with the particle from which the phonon will scatter—i.e., we are interested in the kinematic description as seen by the observer who initially believes the particle to be at rest. Before we can explicitly list the transformation rules for the values of physical quantities, it is important to recall a few key facts and to have a schematic of the scattering experiment in mind.

C. Schematic of our scattering experiment

Recall that the specific form of the transformation equations that we established in Sec. II were predicated on the assumption that pairs of in-universe observer reference frames had coordinate axes that were aligned, and that the relative motion between pairs of in-universe observer reference frames was constrained to be in the z direction. For the purposes of this paper, we choose only to consider experiments for which these assumption hold true. Also note that—as explicitly demonstrated in [25]—operational measurements of distance and duration that are made by in-universe observers who are stationary with respect to the analogue-gravity medium coincide with the equivalent measurements as made in the laboratory frame (up to unit conversions). That is to say, the laboratory frame *is* an in-universe observer reference frame; it simply happens to be the in-universe observer reference frame corresponding to observers who are stationary with respect to the medium.

With all of this in mind, we present in Fig. 1 a schematic of the scattering experiments that we will consider throughout the remainder of this paper. We note that our scattering experiments are initialized such that the trajectories of our particles are parallel with respect to the incoming phonons; we *define* these trajectories to point in the positive z direction. Denoting velocities $\mathbf{v} = (v_x, v_y, v_z)$, we can write the initial velocity of our particle $\mathbf{v}_i = (0, 0, v_i)$, where $v_i \geq 0$ by fiat of our assumptions; consequently, we have $|\mathbf{v}_i| = v_i$. Denoting sonic fractional velocities to be

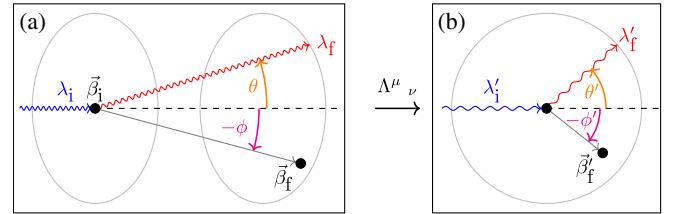


FIG. 1. A scattering event as seen in (a) the laboratory frame and (b) the comoving in-universe observer frame. All quantities labeled in the figure are measured operationally with sound-based clocks and rulers [25] and thus their values as measured by in-universe observers transform with respect to a sonic Lorentz transformation. We envisage the particle from which scattering occurs to be centered with respect to a hollow shell of detectors that can detect phonons incident upon them. To in-universe observers in the comoving frame (b), the detector array appears to form a spherical shell defined by constant spatial coordinates; in the laboratory frame (a), the array of detectors appears to be Lorentz contracted in the direction of motion and the spatial coordinates defining this shell change between the scattering event and the detection event. The relativistic Doppler formula and the relativistic aberration formula relate the operationally measured values of wavelengths and angles between reference frames, respectively. The operationally measured value of the final velocity of the particle in different reference frames is related by the general form of the relativistic velocity composition formula [43] (see [44] for an English translation).

$\beta := \mathbf{v}/c_s$, we can write $\beta_i := |\beta_i| = v_i/c_s$. Restricting ourselves to cases in which the particle's initial and final speeds are slower than c_s , we always have that $|\beta| < 1$; taken in combination with $v_i \geq 0$, we have then that $0 \leq \beta_i < 1$. The final velocity of the particle β_f is allowed to have components in any direction, provided that $|\beta_f| < 1$. As it turns out, though, we will ultimately be able to remove any reference to the final state of the particle (and thus to β_f) from the kinematic expressions that we obtain.

We previously provided equations for the relativistic Doppler shift of sound waves [Eq. (2.5)], and for the relativistic aberration formula [Eq. (2.6)]. As we have emphasized, in our particular scattering experiments we wish to consider the operational measurements made in the comoving in-universe observer frame. In the laboratory frame, we therefore have that $\beta = \beta_i$ —that is, the boost parameter of the comoving in-universe observer frame is numerically equal to the initial velocity of the particle, and thus the values of quantities in the laboratory frame (unprimed symbols) are related to the values of quantities in the comoving in-universe observer frame (primed symbols) in the following specific ways:

$$\omega_i = \gamma_i(1 + \beta_i)\omega'_i = D_i\omega'_i, \quad (3.6)$$

$$\omega_f = \gamma_i(1 + \beta_i \cos \theta')\omega'_f, \quad (3.7)$$

$$\cos \theta = \frac{\cos \theta' + \beta_i}{1 + \beta_i \cos \theta'}, \quad (3.8)$$

$$(1 - \cos \theta) = \frac{1 - \beta_i}{1 + \beta_i \cos \theta'}(1 - \cos \theta'), \quad (3.9)$$

where Eqs. (3.7)–(3.9) follow directly from Eqs. (2.5)–(2.7), respectively, and Eq. (3.6) follows from Eq. (2.5) with $\theta = \theta' = 0$ because the phonon is always initially traveling in the positive z direction.

IV. PHONON SCATTERING KINEMATICS

We now proceed to derive the kinematic expressions of Compton-like phonon scattering within an analogue-gravity model. As we stated in Sec. III, we first obtain the kinematic expression for phonon scattering for the case of *sonically Lorentz-obeying* particles (internal particles). Once we have obtained both the laboratory frame and comoving in-universe observer frame kinematics for phonon scattering from internal particles, we will then proceed to do the same for *sonically Lorentz-violating* particles (external particles).

A. Phonon scattering from internal particles

1. The laboratory frame kinematics of phonon scattering from internal particles

To provide some context, we imagine that, following Ref. [24], internal particles are collective-excitation quasiparticles whose dynamical description is covariant with respect to the same sonic Lorentz symmetry that in-universe observer reference frames transform under. In other words, these quasiparticles are relativistic with respect to the speed of sound c_s of the analogue-gravity medium. Most obviously, one might imagine these particles to arise from the analogue-gravity medium itself (hence the use of the word “internal”). However, there is no particular reason that these particles could not also arise in a separate medium with the same speed of sound, and to which the analogue-gravity medium is somehow coupled.

For the purposes of this paper, we choose the following simple and familiar energy-momentum relations for our internal particles:

$$E = \gamma m c_s^2, \quad \mathbf{p} = \gamma m \mathbf{v}, \quad (4.1)$$

where here $\mathbf{v}$ is the velocity of the particle as measured in the laboratory frame, $\gamma = 1/\sqrt{1 - \beta^2}$, $\beta = \mathbf{v}/c_s$, and $\beta^2 := \beta \cdot \beta$. The parameter m is the mass of the particle, and since the expressions for E and $\mathbf{p}$ can be collected into the Lorentz-covariant object P^μ , the quantity m behaves as a Lorentz scalar from the perspective of in-universe observers.

In order to be able to cleanly substitute all of our quantities into Eq. (3.4), we desire to be able to write $E = E(\mathbf{p})$. Noting that both $\mathbf{p}$ and γ are functions of $\mathbf{v}$, we can perform some simple algebra to obtain the following familiar expression for $\gamma = \gamma(\mathbf{p})$:

$$\gamma(\mathbf{p}) = \sqrt{1 + \frac{p^2}{m^2 c_s^2}}, \quad (4.2)$$

where $p^2 = \mathbf{p} \cdot \mathbf{p}$. Substituting this expression for γ into our expression for energy in Eq. (4.1), we obtain $E = E(\mathbf{p})$. It proves more useful for our purposes to consider $E^2(\mathbf{p})$, however (the form of which should also be quite familiar):

$$E^2(\mathbf{p}) = p^2 c_s^2 + m^2 c_s^4. \quad (4.3)$$

Now that we have $E = E(\mathbf{p})$, we can proceed precisely as we described in Sec. III. Rather than performing all of our substitutions and then algebraically simplifying afterwards, we choose to perform the reverse process. Square the energy and momentum conservation equations [Eqs. (3.4) and (3.5)] to obtain the following:

$$\begin{aligned}
 E_f^2(\mathbf{p}_f) &= \{E_i(\mathbf{p}_i) - \hbar[\omega_f(\mathbf{k}_f) - \omega_i(\mathbf{k}_i)]\}^2, \\
 &= E_i^2 - 2\hbar E_i(\omega_f - \omega_i) + \hbar^2(\omega_f^2 - 2\omega_i\omega_f + \omega_i^2);
 \end{aligned} \tag{4.4}$$

$$\begin{aligned}
 p_f^2 &= [\mathbf{p}_i - \hbar(\mathbf{k}_f - \mathbf{k}_i)]^2, \\
 &= p_i^2 - 2\hbar p_i k_f \cos \theta + 2\hbar p_i k_i \\
 &\quad + \hbar^2(k_f^2 + k_i^2) - 2\hbar^2 k_f k_i \cos \theta.
 \end{aligned} \tag{4.5}$$

Note that the $\cos \theta$ terms in Eq. (4.5) arise as a result of the experimental scenario that we are considering as per Fig. 1. In Eq. (4.4) we have suppressed the explicit functional dependency on momentum for notational brevity: this dependency still applies, of course. From here, we can directly substitute Eq. (4.3) into Eq. (4.4) wherever the square of the particle's energy appears. Doing so, and utilizing the square of the conservation of momentum [Eq. (4.5)], we obtain the following expression:

$$\begin{aligned}
 c_s p_i \hbar \omega_i - \hbar E_i \omega_i \\
 = \hbar \omega_f (c_s p_i + \hbar \omega_i) \cos \theta - E_i \hbar \omega_f - \hbar^2 \omega_i \omega_f,
 \end{aligned} \tag{4.6}$$

where we have made use of the fact that $\omega = c_s |\mathbf{k}|$. From here, it is a simple matter to isolate k_f . Doing so yields the following expression:

$$k_f = \frac{(E_i - c_s p_i) k_i}{E_i - c_s p_i + (1 - \cos \theta)(\hbar \omega_i + c_s p_i)}. \tag{4.7}$$

Some simple algebraic manipulations also allow us to phrase this in the following way:

$$\frac{\omega_f}{\omega_i} = \left[1 + \left(\frac{\hbar \omega_i + c_s p_i}{E_i - c_s p_i} \right) (1 - \cos \theta) \right]^{-1}. \tag{4.8}$$

Using the explicit forms of E and $\mathbf{p}$ as given in Eq. (4.1), and recalling that our experiment is initialized such that $|\beta_i| = \beta_i$ (with $\beta_i > 0$), we can further rewrite Eq. (4.8) entirely in terms of dimensionless quantities:

$$\frac{\omega_f}{\omega_i} = \left[1 + \left(\frac{\hbar \omega_i}{\gamma_i m c_s^2} + \beta_i \right) \left(\frac{1 - \cos \theta}{1 - \beta_i} \right) \right]^{-1}. \tag{4.9}$$

This is just the kinematic relationship for ordinary Compton scattering [42] as viewed from a boosted frame, except the boost is with respect to the speed of sound c_s rather than the speed of light. In other words, this is the kinematic description of a *sonic analogue to Compton scattering*.

2. The comoving in-universe observer frame kinematics of phonon scattering from internal particles

We now possess an unambiguous kinematic description of phonon scattering from an internal particle in the laboratory frame. Here we reexpress this from the perspective of an in-universe observer who is in the frame that is comoving with the particle prior to scattering. We use Eqs. (3.6) and (3.7) to write

$$\frac{\omega'_f}{\omega'_i} = \left(\frac{1 + \beta_i}{1 + \beta_i \cos \theta'} \right) \frac{\omega_f}{\omega_i}. \tag{4.10}$$

Plugging in Eq. (4.9) and using Eqs. (3.6)–(3.9) to rewrite all quantities from the comoving frame, this becomes

$$\begin{aligned}
 \frac{\omega'_f}{\omega'_i} &= \left(\frac{1 + \beta_i}{1 + \beta_i \cos \theta'} \right) \\
 &\times \left[1 + \left(\frac{\hbar \omega'_i}{m c_s^2} (1 + \beta_i) + \beta_i \right) \left(\frac{1 - \cos \theta'}{1 + \beta_i \cos \theta'} \right) \right]^{-1}.
 \end{aligned} \tag{4.11}$$

Straightforward algebra simplifies this to

$$\frac{\omega'_f}{\omega'_i} = \left[1 + \left(\frac{\hbar \omega'_i}{m c_s^2} \right) (1 - \cos \theta') \right]^{-1}. \tag{4.12}$$

Notice that all dependence on β_i has disappeared in the final form, as it must since both types of particle respect the sonic Lorentz symmetry.

While Eq. (4.12) may not be in its most familiar form, this is precisely the kinematic description of Compton scattering in the rest frame of a particle prior to scattering (with, of course, the understanding that all references to c are replaced by c_s). Recalling that our dispersion relation for phonons is taken to be linear, this expression can be easily cast into perhaps its most well known form:

$$\lambda'_f - \lambda'_i = \frac{h}{m c_s} (1 - \cos \theta'). \tag{4.13}$$

Note that the comoving in-universe observer frame's kinematic description of phonon scattering from internal particles makes no explicit reference to any velocity. This should not be too surprising, given the relativistic form of the energy-momentum relation that we chose for internal particles. It is important to note that we did not manually set $\beta_i = 0$ to obtain this result, though: all instances of β_i and γ_i were removed from the kinematic expression purely via the use of valid algebraic manipulations. The resulting expression, however, is entirely equivalent to taking Eq. (4.9), setting $\beta_i = 0$ (and hence, $\gamma_i = 1$), and then appending primes to the remaining operationally determined quantities—a method that should be familiar to any student of special relativity.

At this point, readers may find themselves wondering why we seem to be rederiving results that are well known in QFT [28]. While the naïve method of setting $\beta_i = 0$ in Eq. (4.9) does indeed provide a shortcut to the comoving in-universe description of scattering [i.e., without explicitly needing to consider the transformation equations Eqs. (3.6)–(3.9)], this only works for scattering from particles that are sonically Lorentz obeying in nature. Specifically, the equivalence of these two methods is a result of the fact that the energy-momentum relation for internal particles transforms covariantly under the same sonic Lorentz transformation that applies to in-universe observer reference frames.⁹ External particles, however, are *not* sonically Lorentz obeying in nature—their energy-momentum relations do not transform covariantly under the sonic Lorentz transformation—and so the shortcut that happened to work with internal particles *does not* apply when we are considering external particles.

Reobtaining the familiar expression for Compton scattering by utilizing the simple approach that we first detailed in Sec. III provides a sanity check for our approach, and it provides a familiar example through which to elucidate some important details regarding the derivation.

B. Phonon scattering from external particles

1. The laboratory frame kinematics of phonon scattering from external particles

As we have stated before, external particles are particles that are *not* sonically Lorentz obeying. That is to say, the dynamical description of external particles is not covariant with respect to the sonic Lorentz symmetry of the analogue-gravity medium. This, of course, leaves a wide range of possibilities in terms of selecting energy-momentum relations for external particles. For our purposes here, we choose specifically to consider our external particle to be an ordinary quantum-mechanical particle with the usual Newtonian energy-momentum relation:

$$E = \frac{p^2}{2m}, \quad \mathbf{p} = m\mathbf{v}. \quad (4.14)$$

Following the approach described in Sec. III, we directly substitute our expression for energy into Eq. (3.4) and utilize the conservation of momentum [Eq. (3.5)] to eliminate any reference to $\mathbf{k}_f$. This leads to an equation that is quadratic in k_f :

$$\begin{aligned} \frac{\hbar^2 k_f^2}{2m} - \hbar k_f \left[\frac{(\hbar k_i + p_i) \cos \theta}{m} - c_s \right] \\ + \hbar k_i \left(\frac{\hbar k_i + 2p_i}{2m} - c_s \right) = 0. \end{aligned} \quad (4.15)$$

⁹As noted in passing earlier, the internal particle's energy and momentum can be collected into the four-momentum P^μ . With $P^\mu := (E/c_s, \mathbf{p})$, one can use Eq. (4.1) ($E = \gamma mc_s^2$ and $\mathbf{p} = \gamma m\mathbf{v}$) to verify that P^μ is indeed a Lorentz-covariant object.

In the fully sonically Lorentz obeying case (that is, phonon scattering from internal particles) the conservation of energy and momentum lead to a linear equation in k_f , and thus for a given set of initial experimental parameters and for a given scattering angle θ there existed only one possible value of k_f . In this case, however, we have a quadratic equation, and so for a given set of initial experimental parameters and for a fixed scattering angle of θ , there are two possible values that k_f can take. We can solve Eq. (4.15) for $\hbar k_f$ and multiply the solutions by c_s to obtain solutions of the following form for the final energy of the phonon:

$$c_s \hbar k_{f,(1,2)} = \hbar \omega_{f,(1,2)} = c_s \bar{B} \pm c_s \sqrt{\bar{B}^2 - \bar{C}}, \quad (4.16)$$

where $c_s \hbar k_{f,(1)}$ (or equivalently, $\hbar \omega_{f,(1)}$) is taken to be the solution with the positive sign, $c_s \hbar k_{f,(2)}$ (or equivalently, $\hbar \omega_{f,(2)}$) is taken to be the solution with the negative sign, and

$$\bar{B} = (\hbar k_i + p_i) \cos \theta - mc_s, \quad (4.17a)$$

$$\bar{C} = \hbar k_i (\hbar k_i + 2p_i - 2mc_s). \quad (4.17b)$$

Note that the bar ($\bar{}$) serves only as a notational label; it has no particular physical or mathematical meaning. The quantity $\bar{B}$ has units of a three-momentum, whereas the quantity $\bar{C}$ has units of the square of a three-momentum.

The term $c_s \bar{B}$ in Eq. (4.16) expands to produce terms of the form $c_s \hbar k_i$, mc_s^2 , and $c_s p_i$. The first of these three terms is precisely the initial energy of the phonon, $\hbar \omega_i$. The remaining two terms have the structure of relativistic terms: by analogy to actual relativity, the second term appears to have the structure of a sonic analogue to rest mass energy, and the third term has the structure of the initial (relativistic) kinetic energy of the particle. With that said, these final two terms are *not* actually descriptions of the external particle's energy because the external particle is *not* sonically relativistic. Nonetheless, it is interesting to note that terms of a relativistic form appear naturally in the description of our nonrelativistic external particle.

As for the case of phonon scattering from internal particles, we also express the kinematic description of phonon scattering from external particles in terms of the ratio ω_f/ω_i :

$$\frac{\omega_{f,(1,2)}}{\omega_i} = B \pm \sqrt{B^2 - C}, \quad (4.18)$$

where, as per above, $\omega_{f,(1)}$ is taken to be the positive solution and $\omega_{f,(2)}$ is taken to be the negative solution. The quantities B and C are related to their barred versions and are given as follows:

$$B = \frac{c_s \bar{B}}{\hbar \omega_i} = \left(1 + \frac{c_s p_i}{\hbar \omega_i}\right) \cos \theta - \frac{mc_s^2}{\hbar \omega_i}, \quad (4.19a)$$

$$C = \frac{c_s^2 \bar{C}}{(\hbar \omega_i)^2} = 1 + \frac{2c_s p_i}{\hbar \omega_i} - \frac{2mc_s^2}{\hbar \omega_i}. \quad (4.19b)$$

The expressions B and C will prove useful when we move into the comoving in-universe observer reference frame. Before proceeding to determine the description of kinematics from that frame, we shall take a moment to address the nature of the two solutions for the kinematic description of phonon scattering from external particles.

2. Physical meaning of the two solutions

We will now focus on understanding the physical meaning of the two solutions given by Eq. (4.16). Scattering from either type of particle should always yield a real and positive value of the ratio of the final-to-initial phonon energies. In the case of scattering from the external particle, the requirement that the ratio of phonon energies be real puts restrictions on the allowed values of θ for given initial p_i and $\hbar \omega_i = c_s \hbar k_i$ since the discriminant of the square-root in Eq. (4.16) must be non-negative. From Eq. (4.18) we see that this leads to two conditions:

Condition 1.—Solutions 1 and 2 are *both* guaranteed to be real and positive when the following conditions are both satisfied:

$$\bar{C} \geq 0 \quad \text{and} \quad \bar{B} \geq \sqrt{\bar{C}}. \quad (4.20)$$

This condition directly translates into an inequality in $\cos \theta$, restricting the scattering angle range as follows:

$$1 \geq \cos \theta \geq \frac{mc_s + \sqrt{\hbar k_i (\hbar k_i + 2p_i - 2mc_s)}}{(\hbar k_i + p_i)}, \quad (4.21)$$

with

$$\hbar k_i (\hbar k_i + 2p_i - 2mc_s) \geq 0 \Rightarrow mc_s \leq \frac{\hbar k_i + 2p_i}{2}. \quad (4.22)$$

Condition 2.—If $\bar{C} \leq 0$ then solution 1 is always real and positive (note that $\bar{B}^2$ is always positive). This means that there aren't any restrictions on the scattering angle, and so $\cos \theta$ can take the usual range of values:

$$1 \geq \cos \theta \geq -1. \quad (4.23)$$

Restrictions are however placed on the absolute values of the three-momenta $\hbar k_i$ and p_i such that

$$\hbar k_i (\hbar k_i + 2p_i - 2mc_s) \leq 0 \Rightarrow mc_s \geq \frac{\hbar k_i + 2p_i}{2}. \quad (4.24)$$

These conditions will play a very important role in the calculation of the scattering cross section in Sec. V.

3. The comoving in-universe observer frame kinematics of phonon scattering from external particles

As we have discussed previously, the kinematic description of phonon scattering from external particles in the comoving in-universe observer frame is *not* simply given by taking the laboratory frame description and setting $\beta_i = 0$. The observers' geometric measurements constitute a Lorentz-covariant object (X^μ), but the energy and momentum of the external particle *cannot* be collected into a sonically Lorentz-covariant object. As a result of this mismatch in symmetry groups between the dynamically relevant physical quantities (energy and momentum) and the reference frames of observers, the kinematic description of phonon scattering from external particles is *not* Lorentz covariant. We take then the approach that we discussed in Sec. III (and which we previously applied to the kinematics for phonon scattering from internal particles in Sec. IV A 2) and use the relations given by Eqs. (3.6)–(3.9) to replace the value of all laboratory frame quantities with their comoving in-universe observer frame values.

Substituting these formulas into Eq. (4.16), some basic algebraic manipulations lead to the following description of phonon scattering from external particles in the comoving in-universe observer frame:

$$\begin{aligned} c_s \hbar k'_{f,(1,2)} &= \hbar \omega'_{f,(1,2)}, \\ &= \frac{c_s}{\gamma_i (1 + \beta_i \cos \theta')} (\bar{B}' \pm \sqrt{\bar{B}'^2 - \bar{C}'}), \end{aligned} \quad (4.25)$$

where

$$\bar{B}' = (D \hbar k'_i + mc_s \beta_i) \left(\frac{\cos \theta' + \beta_i}{1 + \beta_i \cos \theta'} \right) - mc_s, \quad (4.26a)$$

$$\bar{C}' = D \hbar k'_i (D \hbar k'_i + 2mc_s \beta_i - 2mc_s). \quad (4.26b)$$

$\bar{B}'$ and $\bar{C}'$ are just $\bar{B}$ and $\bar{C}$, respectively, written in the comoving ('') coordinates using Eqs. (3.6)–(3.9). Note that we have used the Doppler factor D as defined in Eq. (2.8), with $\beta = \beta_i$, for notational brevity in these expressions.

In the in-universe observer frame, the dimensionless ratio of the final to initial frequencies of the phonon can be given by the following relation:

$$\frac{\omega'_{f,(1,2)}}{\omega'_i} = \frac{(1 + \beta_i)}{(1 + \beta_i \cos \theta')} (B' \pm \sqrt{B'^2 - C'}), \quad (4.27)$$

where here B' and C' are

$$B' = \frac{\cos \theta' + \beta_i}{1 + \beta_i \cos \theta'} - \frac{mc_s^2 (1 - \beta_i^2)}{D \hbar \omega'_i (1 + \beta_i \cos \theta')}, \quad (4.28a)$$

$$C' = 1 - \frac{2mc_s^2}{D\hbar\omega'_i}(1 - \beta_i). \quad (4.28b)$$

In fact, the reader can verify, using Eqs. (3.6)–(3.9), that B' and C' are just B and C expressed in comoving ($'$) coordinates. The kinematic description of phonon scattering from an internal particle and the comoving in-universe observer frame [Eq. (4.12)] made no reference to the velocity of the particle (and hence the comoving observer), whereas the kinematic description of phonon scattering from an external particle [Eq. (4.27)] does retain reference to the velocity of the particle (and hence the comoving observer). The presence of velocity dependency in the comoving in-universe observer's description of phonon scattering from external particles means that, in principle, an in-universe observer could use such scattering experiments to identify their state of motion with respect to their analogue universe. We shall come back to this point in Sec. V E 5.

The forbidden scattering angles and three-momenta that arise in the laboratory frame description of phonon scattering from external particles take on the following forms in the comoving in-universe observer frame:

Condition 1.—Solutions 1 and 2 are *both* guaranteed to be real and positive when the following conditions are both satisfied:

$$\bar{C}' \geq 0 \quad \text{and} \quad \bar{B}' \geq \sqrt{\bar{C}'}. \quad (4.29)$$

This condition translates into the following restrictions on the scattering angle $\cos \theta'$:

$$1 \geq \cos \theta' \geq \frac{mc_s(1 - \beta_i^2) - \beta_i D\hbar k'_i + \sqrt{\bar{C}'}}{D\hbar k'_i - \beta_i \sqrt{\bar{C}'}} \quad (4.30)$$

with

$$mc_s^2 \leq \frac{D\hbar\omega'_i}{2(1 - \beta_i)}. \quad (4.31)$$

Condition 2.—If $\bar{C}' \leq 0$ then solution 1 is always real and positive [note that $(\bar{B}')^2$ is always positive]. In this case, there are no restrictions on the scattering angles, but the absolute values of the three-momenta are constrained:

$$1 \geq \cos \theta' \geq -1, \quad (4.32)$$

with

$$mc_s^2 \geq \frac{D\hbar\omega'_i}{2(1 - \beta_i)}. \quad (4.33)$$

V. SCATTERING CROSS SECTIONS

We now have an understanding of the kinematics of scattering for both the case of a phonon scattering from an internal particle (obeying sonic Lorentz symmetry), and the case of a phonon scattering from an external particle (not obeying sonic Lorentz symmetry). Our goal now is to understand whether, by analyzing the cross sections for phonons scattering on external particles, in-universe observers can infer anything about their velocity with respect to the laboratory reference frame.

In pursuit of this, we need to devise some appropriate quantum toy model for sonic particle scattering that captures the important features of actual particle scattering, such as Compton scattering [28,42]. The physics in the laboratory frame is unambiguous when we have such a model, and the method by which we determine the comoving in-universe observer description of events is by transforming observable quantities appropriately. We also continue to define all measurements operationally: we imagine that in-universe observers take an operational approach to detecting scattered phonons by using arrays of detectors that are, in their own frame, seen to be spherical and seen to be centered on the scattering event. Figure 2 highlights what this looks like in the laboratory frame. By performing several scattering events to create a statistical distribution and counting how many times any single detector clicks, we can obtain the *reaction rate* $\mathcal{R}$ —i.e., the number of scattering events per unit of volume per unit of time in a particular direction—to obtain the differential cross section.

In the following, we will start by carefully deriving the scattering cross section equations for external particles. Our system is a “hybrid” one, composed of three parts: the reference frames that we use to operationally perform measurements, the phonon, and the external particles. Only two of the three—the phonon and the reference frames—are always fully sonically relativistic, hence we cannot simply apply the QFT definition of the cross section since Lorentz covariance is deeply embedded in the derivation [28]. On the other hand, we have to upgrade the ordinary (nonrelativistic) quantum-mechanical scattering theory as this is usually derived only in the laboratory reference frame [45], while we want to be able to transform between Lorentz-obeying reference frames.¹⁰

A. General cross section definition

In quantum physics, the cross section measures the transition probability for a specific process to happen in

¹⁰The operational method of using arrays of detectors for the derivation of scattering cross section is a useful tool in our hybrid system, as we can easily transform the position of the detectors (with the transformation rules derived in Sec. IV) and use them to easily redefine a reaction rate in the new reference frame.

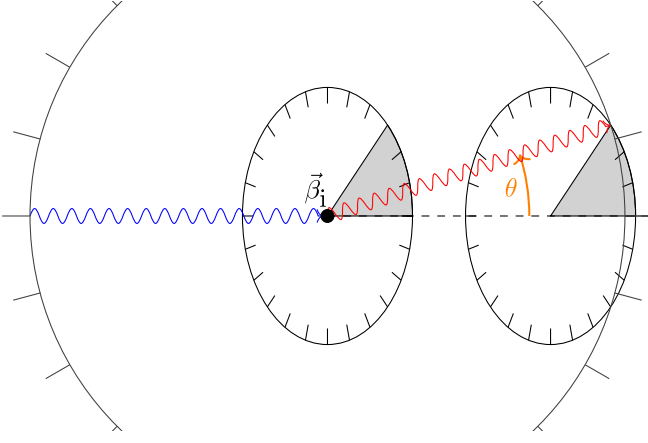


FIG. 2. Phonon scattering from an external particle as viewed in the laboratory frame. The large circle centered on the scattering event (clipped at the top and bottom) represents a 2D slice of a spherical array of detectors that is stationary with respect to the laboratory frame. The two ellipses represent 2D slices of a *single array of detectors* (comoving with the particle prior to scattering) at two different moments in time. Ticks on the circumference of both detector arrays are separated by 15° in the frame of that detector array. The scattering event occurs at the moment in time for which the geometric centers of both detector arrays are coincident. At some later point in time, the scattered phonon is again coincident with both detector arrays: the laboratory frame detector array detects the phonon at the detector located at an angle θ above the z axis (here $\sim 17.8^\circ$), which in the comoving frame corresponds to the detector located at the angle θ' above the z' axis (here 45° , indicated by the shaded wedge in the moving detector array). In this particular example $\beta_i = 0.75$, corresponding to $\gamma_i = 4/\sqrt{7} \approx 1.5$ (which is the factor by which the moving detector array is contracted in its direction of motion). Figure 1(a) shows the same scattering event indicated here, whereas Fig. 1(b) shows this scattering event in the frame of the moving detector array.

the scattering of two or more particles [46]. The total cross section is defined as

$$\sigma = \frac{\mathcal{R}}{\Phi}, \quad (5.1)$$

where $\mathcal{R}$ is the *reaction rate* (transition probability per unit time per unit volume), and Φ is the incident flux of particles. The cross section σ has units of area and qualitatively represents the effective cross sectional area (hence the name) that the particle presents to the incoming beam of particles to be scattered, and it will depend on the momentum of the incoming particles.

The cross section as defined in Eq. (5.1) can be expressed more explicitly as

$$\sigma[S|i] = \frac{1}{\Phi} \sum_{f \in S} \frac{w_{i \rightarrow f}}{L^3} = \frac{1}{\Phi} \sum_{\mathbf{k}_f} \sum_{\mathbf{p}_f} \frac{w_{i \rightarrow f}}{L^3}, \quad (5.2)$$

where $|i\rangle$ and $|f\rangle$ represent the initial and final states, respectively (often shown with ket symbols omitted), and where S is the subset of all final states that are different from i (in order to exclude the case of no scattering). We use box normalization (volume L^3) for our particles, so that the reaction rate $\mathcal{R}$, which is a transition rate per unit volume, becomes $w_{i \rightarrow f}/L^3$, with

$$w_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle f | \hat{H}_{\text{int}} | i \rangle|^2 \delta(E_{fi}), \quad (5.3)$$

being the transition rate from an initial state $|i\rangle$ at $t = -\infty$ to a final state $|f\rangle$ at time $t = T$, given some interaction Hamiltonian $\hat{H}_{\text{int}}$. The symbol E_{fi} represents the change in total energy of the system (as measured in the lab), and the states and the interaction Hamiltonian are all defined in a Schrödinger picture. For more details on how to derive Eq. (5.3), please see Ref. [45].

The notation $\sigma[S|i]$ represents the cross section for scattering to any final state $|f\rangle \in S$ (different from the initial one) given the particular initial state $|i\rangle$. The last form of Eq. (5.2) is the one we will use for our calculation.

B. Cross section in the in-universe observers' reference frame

Before going into the details of the cross section derivation, we want to clarify the behavior of the cross section σ when viewed from the frame of an in-universe observer in motion with respect to the laboratory frame.

Since our model lacks full (sonic) Lorentz invariance, we must reexamine some of the basic assumptions about cross sections—most especially, how it behaves when viewed from a moving frame. In ordinary QFT, which has full Lorentz invariance, the cross section behaves exactly like a cross sectional area: it is invariant under boosts in the direction of motion of the incoming particle to be scattered (assuming a stationary target particle) but not generally in other directions [28]. We cannot rely on this being true in our model. To examine how the cross section behaves in this hybrid model in which some pieces obey sonic Lorentz symmetry, we return to its basic form, Eq. (5.1), which is a ratio of the reaction rate $\mathcal{R}$ to the incoming flux Φ . We will examine how each of these would be perceived by an in-universe observer comoving with the external particle.

1. Reaction rate

The reaction rate $\mathcal{R}$ is a transition rate per unit volume. This can be reexpressed as the total transition *probability* per unit *spacetime volume*.

A spacetime volume element is invariant under a Lorentz transformation, and thus it remains invariant when measured by in-universe observers. Thus, the denominator of $\mathcal{R}$ is invariant.

Now let us examine the numerator. The probability of some particular event happening also remains invariant,

even though the different observers may disagree on the particular descriptors used to specify the event (e.g., direction, duration, etc.). If a given detector clicks, then all observers will agree that the detector clicked. At the end of a given set of experiments, all observers—regardless of state of motion—will therefore agree on the list of detectors that clicked, and how many times each detector clicked. Thus, the probability of a physical detector clicking must also be independent of an observer's state of motion.

This means that the transition probability for any given event $i \rightarrow f$ must be invariant. A moving observer would interpret this as the probability for the transition $i' \rightarrow f'$, where i' and f' are the initial and final states as interpreted by the moving observer, but the numerical value of this probability would be the same for both observers. The total transition probability is just the sum of all relevant individual transition probabilities of this form. Since this sum includes all possible final states (except the initial one), all observers must agree on this value. Thus the total transition probability (numerator of $\mathcal{R}$) is invariant.

Combining the above two results, we see that the reaction rate $\mathcal{R}$ is itself invariant, being the ratio of two invariant quantities: total transition probability and amount of spacetime volume.

2. Flux

The behavior of the flux Φ under a Lorentz transformation is the subject of much discussion, even in a fully relativistic theory such as QFT. A recent article [46] explains clearly the subtleties of this topic and provides the definitive resolution. The key results of that work also apply to our setup, and we repeat them here, along with a discussion of how our particular setup modifies them (or not).

The approach will be to write the flux in a manifestly Lorentz-invariant form based on the sonic analogue to the four-current, which we denote as

$$J^\mu := \begin{pmatrix} nc_s \\ n\mathbf{v} \end{pmatrix}, \quad (5.4)$$

with n being the number density of particles (in some frame) and $\mathbf{v}$ the three-velocity of the particle (as measured from the same frame). J^μ has units of speed times number density (or speed per unit volume), which has the intuitive interpretation of number of particles passing through a unit of area per unit of time. J^μ is a sonic Lorentz-covariant four-vector and will transform as such when changing from one in-universe observer's frame to that of another. Crucially, notice that this is a kinematical quantity, not a dynamical one. It is based on a description of the motion of the particle and makes no reference to its particular dispersion relation.

The crucial observation from Ref. [46] is that the flux for a two-particle scattering experiment can be written in an

invariant form using the four-currents J_1^μ and J_2^μ for the two particles¹¹:

$$\Phi = c_s^{-1} \sqrt{(J_1 \cdot J_2)^2 - (J_1)^2 (J_2)^2}, \quad (5.5)$$

where 1 and 2 are the two scattering particles, the dot $(\cdot)$ is the four-vector dot product (in our case, using c_s instead of c), and $(J)^2 = J \cdot J$. Importantly, every in-universe observer will agree on the numerical value of this quantity, so we only need to calculate it in one frame: the laboratory frame.

In the laboratory frame, the four-currents for the phonon and the external particle, respectively, are

$$J_s^\mu = n_s \begin{pmatrix} c_s \\ 0 \\ 0 \\ c_s \end{pmatrix}, \quad J_p^\mu = n_p \begin{pmatrix} c_s \\ 0 \\ 0 \\ v_i \end{pmatrix}, \quad (5.6)$$

where n_s and n_p are the initial number density of the phonon and of the particle. Then,

$$J_s \cdot J_p = n_s n_p c_s^2 (1 - \beta_i), \quad (5.7)$$

$$(J_s)^2 = 0, \quad (5.8)$$

$$(J_p)^2 = n_p^2 c_s^2 (1 - \beta_i^2). \quad (5.9)$$

Plugging these into Eq. (5.5) gives the formula for the flux in the laboratory frame:

$$\Phi = n_s n_p |c_s - v_i|, \quad (5.10)$$

For our choice of normalization,¹² the number densities are $n_p = 1/L^3$ and $n_s = 1/L^3$, so the flux in the laboratory frame is

$$\Phi = \frac{|c_s - v_i|}{L^6} = c_s \frac{|1 - \beta_i|}{L^6}. \quad (5.11)$$

This is the value of Φ in all frames.

Nevertheless, a quick confirmation shows that this is indeed what in-universe observers comoving with the particle would calculate. Simply apply the appropriate

¹¹Reference [46] uses natural units. We have included the prefactor c_s^{-1} to get the units right in our case.

¹²There is much discussion about choosing a Lorentz-invariant normalization in QFT textbooks (e.g., Ref. [28]), but this is merely a convention that has some calculational and conceptual utility in a fully relativistic setting. Importantly, it is not required for obtaining physically valid results—even in a fully relativistic setting. Any normalization will do as long as it is treated consistently throughout the calculation. And that is what we do.

Lorentz transformation to the currents, giving $J_s'' = D^{-1}n_s(c_s, 0, 0, c_s)^T$ and $J_p'' = \gamma_i^{-1}n_p(c_s, 0, 0, 0)^T$, and plug into Eq. (5.5). The result for Φ is unchanged.

3. Total cross section

The total cross section σ is the ratio of two sonic Lorentz scalars: $\mathcal{R}$ and Φ . The numerical value of each of these quantities is therefore agreed upon by all in-universe observers, and so, therefore, is their ratio $\sigma = \mathcal{R}/\Phi$. The total cross section is a sonic Lorentz scalar with respect to all in-universe observer frames moving in the z direction.

Importantly, however, note that we have taken care to only assert that these quantities are invariant with respect to different states of motion of the *observers*. The fact that the external particle violates the sonic Lorentz symmetry of the system means that the quantities in question are not necessarily invariant with respect to simultaneous boosts of all objects *with respect to the medium*. In fact, as we will discover in the coming pages, experiments that appear to in-universe observers to have the same initial conditions—assuming the observers in each case are comoving with the particle—will nevertheless lead to different outcomes depending on their initial velocity with respect to the medium. In the language of Lorentz-violating extensions to the Standard Model [47], the cross section is invariant with respect to *observer boosts* but not necessarily so with respect to *particle boosts*.

4. Differential cross section

While we have shown above that the numerical value of the total cross section σ is agreed upon by all observers, we are actually interested in the *differential* cross section $d\sigma/d\cos\theta$ with respect to the scattering angle θ in the laboratory frame and $d\sigma/d\cos\theta'$ with respect to scattering angle θ' in the comoving frame.

We can illustrate this situation as follows. We imagine a large spherical array of detectors that is stationary with respect to the laboratory frame and centered on a phonon scattering event, as illustrated in Fig. 2. Each detector subtends a small solid angle $d\Omega(\theta, \phi)$ at a given orientation (θ, ϕ) as measured in the laboratory frame. Whether a particular detector has clicked or not is manifestly invariant, as is the probability of any physical detector clicking, as discussed above. So too would this reasoning apply to a spherical array of detectors comoving with the particle, as illustrated in Fig. 2. Each of these detectors would subtend a solid angle $d\Omega'(\theta', \phi')$ at a given orientation (θ', ϕ') as measured in the comoving frame.

The total cross section can be split up into infinitesimal pieces $d\sigma$, corresponding to scattering (in the laboratory frame) at angle θ into a narrow ring of detectors for all azimuthal angles ϕ . We can reassemble these pieces into the total cross section:

$$\sigma = \int d\sigma = \int d\cos\theta \frac{d\sigma[\theta|i]}{d\cos\theta}, \quad (5.12)$$

where we make explicit the final scattering angle θ and initial state i . Since the probability that some detector in that ring clicks is agreed upon by all observers, it is merely a question of kinematics as to how that ring appears to in-universe observers in a different state of motion. To find this, we first find θ' and i' using Eqs. (3.8) and (3.6), and then we note that

$$d\sigma[\theta|i] = d\sigma[\theta'|i'] \quad (5.13)$$

by the scalar nature of σ and the argument above about the invariance of probabilities.

The comoving frame description of the scattering event is therefore obtained solely by appropriately transforming the laboratory-frame kinematic quantities into the comoving frame of the particle. From Eq. (5.13) we get the differential cross section in the comoving reference frame from the laboratory one [48]:

$$\frac{d\sigma[\theta'|i']}{d\cos\theta'} = \frac{d\sigma[\theta|i]}{d\cos\theta} J[\theta, \theta'], \quad (5.14)$$

where $J[\theta, \theta']$ is the Jacobian of the coordinate transformations between reference frames,

$$J[\theta, \theta'] = \frac{d\cos\theta}{d\cos\theta'} = \frac{1 - \beta_i^2}{(1 + \beta_i \cos\theta')^2}, \quad (5.15)$$

obtained using the transformation Eq. (3.8). With these tools in hand, our approach is to calculate the differential cross section in the laboratory frame (since that is the frame in which we know the dynamics) and then use Eq. (5.14) to express it from the comoving observers' perspective.

C. Phonons, quantized external particles, and the interaction Hamiltonian in the laboratory frame

In the laboratory frame, we assume that phonons are excitations of the analogue-gravity medium; for simplicity, we treat the phonon field as a scalar field.¹³

We use standard quantum mechanics to describe external particles, as this suffices to capture all of their relevant quantum mechanical degrees of freedom. Note that the only degree of freedom that we endowed our external particle with (see Sec. IV B) is its center-of-mass energy/momentum—we have not endowed the external particle with any additional degrees of freedom such as angular momentum.

¹³Treating phonons as a scalar field is appropriate in systems that are isotropic. For systems that exhibit anisotropies, e.g., any system with a crystal structure, a vector field description must be used to fully capture all appropriate degrees of freedom [49].

So in summary, our recipe for a quantum-mechanical interaction Hamiltonian includes the following:

- (1) First-quantized matter that describes the external particle.
- (2) A single excitation of a second-quantized phonon field that describes the scattering phonon.
- (3) An interaction Hamiltonian depending on the phonon's field amplitude at the (quantized) position of the external particle.

We define the phonon field as

$$\hat{\phi}(\mathbf{x}) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \sqrt{\frac{\hbar c_s}{2k}} (\hat{a}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}}). \quad (5.16)$$

Its units are $\sqrt{(\text{energy})/(\text{length})}$.

1. Interaction Hamiltonian

We want to build a toy model for phononic interaction with external particles. The Hamiltonian we will consider is

$$\hat{H}_{\text{int}} = \frac{g}{2} \hat{\phi}^2(\hat{x}), \quad (5.17)$$

where g is the “charge” of the interaction with units of length that we can, with foresight, use to define a dimensionless coupling constant

$$\alpha_s := \frac{g}{\tilde{\lambda}_p}, \quad (5.18)$$

in terms of a *reduced sonic Compton wavelength* of the external particle¹⁴

$$\tilde{\lambda}_p := \frac{\hbar}{mc_s}. \quad (5.19)$$

The cross section is evaluated by standard perturbation theory with the quantized interaction term

$$\begin{aligned} \hat{H}_{\text{int}} = & \frac{g}{2L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{\hbar c_s}{2\sqrt{k k'}} (\hat{a}_{\mathbf{k}} e^{i\mathbf{k} \cdot \hat{\mathbf{x}}} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \hat{\mathbf{x}}}) \\ & \times (\hat{a}_{\mathbf{k}'} e^{i\mathbf{k}' \cdot \hat{\mathbf{x}}} + \hat{a}_{\mathbf{k}'}^\dagger e^{-i\mathbf{k}' \cdot \hat{\mathbf{x}}}). \end{aligned} \quad (5.20)$$

The Hamiltonian is discrete as we are considering quantization in a finite “sonic universe” of volume L^3 .

Important to notice is that the interaction Hamiltonian in Eq. (5.20) is not Lorentz invariant—neither with respect to light, nor sound—as the operator $\hat{x}$ cannot be applied in a relativistic scenario. Hence, it is valid only for external particles. We will elaborate more in the next subsection, Sec. V D.

¹⁴Note that we are free to define the numerical quantity $\tilde{\lambda}_p$ despite the fact that the external particle is not a field excitation.

D. Internal particle cross section

We are now equipped with the relevant mathematical machinery and physical understanding to compute scattering cross sections for phonon scattering experiments involving external particles. Before we proceed to do so, we will discuss the nature of the scattering cross section for phonon scattering experiments involving internal particles, as we will make qualitative comparisons between these two types of scattering experiments in what follows.

In Sec. IV A we demonstrated that the kinematic description of phonon scattering from internal particles takes on the same mathematical form in both the laboratory frame and the comoving in-universe observer frame. In other words, we can refer to the kinematic description of phonon scattering from internal particles as being *sonically Lorentz covariant*. The types of particles that we would expect to behave in this manner would be collective excitation quasiparticles [50,51] belonging either to the analogue gravity medium itself (in addition to the phonons, which are also collective excitation quasiparticles) or to some other analogue gravity medium with the same characteristic speed of sound to which the phonon bearing medium is coupled.

Condensed-matter quantum field theory [31] is the appropriate way to study collective excitation quasiparticles, and so it is a reasonable hypothesis that the appropriate quantum description of sonically relativistic particles would therefore be given by a sonically relativistic condensed matter field theory, analogous to quantum field theory but with the speed of sound taking the place of the speed of light. There is in fact evidence to support this hypothesis: for example, both Volovik [50] and Barceló *et al.* [51]¹⁵ have demonstrated the emergence of quantum electrodynamics as an effective dynamical description of certain collective excitation quasiparticles within condensed matter systems. We will therefore assume that this hypothesis is correct, and so—in analogy to quantum field theory—the dynamical equations of motion governing the scattering of phonons from internal particles will be sonically Lorentz covariant. In-universe observers’ measurements of kinematics are *also* sonically Lorentz covariant, and so the differential cross section for phonon scattering from internal particles will trivially be sonically Lorentz invariant from the perspective of in-universe observers, akin to how the differential cross section for actual Compton scattering (described by the *Klein–Nishina formula* [28,52]) is actually Lorentz invariant.

When considering the differential cross section of phonon scattering from external particles we are not afforded the same convenience. We imagine our external particle to be some regular particle free of any inherent association to the analogue gravity medium: its classical description is merely that of a Newtonian particle, and its

¹⁵The latter work being based, in part, on the former.

TABLE II. Representation of single-particle states of the phonon and external particle. The phonon is treated as single excitations (second quantized) of a sonically relativistic field—i.e., one with a dispersion relation $\omega = c_s k$. The external particle is treated as an ordinary quantum-mechanical particle (first quantized). Our choice of normalization allows all types of particle to have the same form of the inner product and resolution of the (one-particle) identity operator. Our choice of normalization for the phonon differs from that of the usual one in quantum field theory [28]. This has no effect on observable quantities, including cross sections—see footnote 12.

Particle type	One-Particle state	Commutation relations	Inner product	One-Particle identity
Phonon	$ \hbar\mathbf{k}\rangle = \hat{a}_{\mathbf{k}}^\dagger 0\rangle_s$	$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'}$	$\langle\hbar\mathbf{k} \hbar\mathbf{k}'\rangle = \delta_{\mathbf{k},\mathbf{k}'}$	$\mathbb{I} = \sum_{\mathbf{k}} \hbar\mathbf{k}\rangle\langle\hbar\mathbf{k} $
External	$ \mathbf{p}\rangle = \frac{1}{L^{3/2}} \int_{\mathcal{D}_L} d^3x e^{i\mathbf{p}\cdot\mathbf{x}} \mathbf{x}\rangle$	N/A	$\langle\mathbf{p} \mathbf{p}'\rangle = \delta_{\mathbf{p},\mathbf{p}'}$	$\mathbb{I} = \sum_{\mathbf{p}} \mathbf{p}\rangle\langle\mathbf{p} $

corresponding quantum description is given by regular quantum mechanics. We must therefore take a careful and considered approach in calculating the differential cross section of phonon scattering from external particles, and to this end we apply the same general procedure that we applied in calculating the comoving in-universe observer frame description of the kinematics.

E. External particle cross section

To compute the differential cross section of phonon scattering from external particles, we apply Eqs. (5.2) and (5.3) to the specific experimental scenario that we have in mind, which is discussed in Sec. III and shown in Fig. 1.

1. Initial and final states

The quantum toy model we are considering is composed of two independent systems: the phonon and the external particle. Hence the initial and final states of the system can be written as follows:

$$|i\rangle = |\mathbf{p}_i\rangle \otimes |\hbar\mathbf{k}_i\rangle, \quad |f\rangle = |\mathbf{p}_f\rangle \otimes |\hbar\mathbf{k}_f\rangle. \quad (5.21)$$

$|\mathbf{p}_{i/f}\rangle$ are the initial/final states of the external particle, and $|\hbar\mathbf{k}_{i/f}\rangle$ are the initial/final states of the phonon. In Table II we illustrate the details of the states, where we have use the subscript $j = i, f$ for the final and initial states.

In line with the interaction Hamiltonian Eq. (5.20), states are discrete as they are defined in a finite “universe” of volume L^3 , and the delta functions are Kronecker deltas. Note that we are using a different normalization than what is usually used in literature, for example in Peskin [28]. However, this does not have any effects on measured quantities (see footnote 12.)

2. Cross section derivation

We now have all the general tools we need to calculate the differential and the total cross section in the laboratory frame and in the in-universe observers’ reference frame. We will start with calculating the transition rate for Hamiltonian Eq. (5.20), and the initial and final states in Table II:

$$\langle f | \hat{H}_{\text{int}} | i \rangle. \quad (5.22)$$

If we plug in the expressions for the interaction Hamiltonian Eq. (5.20), and initial and final states we obtain

$$\begin{aligned} \langle f | \hat{H}_{\text{int}} | i \rangle &= \frac{g}{2} \langle \mathbf{p}_f | \otimes_s \langle 0 | \hat{a}_{\mathbf{k}_f} \\ &\times \frac{1}{L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{\hbar c_s}{2\sqrt{k k'}} (\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}}) \\ &\times (\hat{a}_{\mathbf{k}'} e^{i\mathbf{k}'\cdot\hat{\mathbf{x}}} + \hat{a}_{\mathbf{k}'}^\dagger e^{-i\mathbf{k}'\cdot\hat{\mathbf{x}}}) | \mathbf{p}_i \rangle \otimes \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s. \end{aligned} \quad (5.23)$$

By considering the commutation relations in Table II, the orthogonality of the particle states, and well-known quantum mechanics relations,¹⁶ the transition amplitude can be rewritten as

$$\langle f | \hat{H} | i \rangle = \frac{g \hbar c_s}{2 L^3} \frac{1}{\sqrt{k_i k_f}} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar\mathbf{k}_i - \hbar\mathbf{k}_f}. \quad (5.24)$$

Given the expression Eq. (5.3) we can rewrite the transition rate as

$$\begin{aligned} w_{i \rightarrow f} &= \frac{2\pi}{\hbar} \left| \frac{g \hbar c_s}{2 L^3} \frac{1}{\sqrt{k_i k_f}} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar\mathbf{k}_i - \hbar\mathbf{k}_f} \right|^2 \delta(E_{fi}), \\ &= \frac{2\pi}{\hbar} \frac{g^2 (\hbar c_s)^2}{4 L^6} \frac{1}{E_{k_i} E_{k_f}} (\delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar\mathbf{k}_i - \hbar\mathbf{k}_f})^2 \delta(E_{fi}), \\ &= \frac{2\pi}{\hbar} \frac{g^2 (\hbar c_s)^2}{4 L^6} \frac{1}{k_i k_f} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar\mathbf{k}_i - \hbar\mathbf{k}_f} \delta(E_{fi}), \end{aligned} \quad (5.25)$$

where

$$\begin{aligned} E_{fi} &= E_f + \hbar\omega_f - E_i - \hbar\omega_i, \\ &= E_f(k_f, k_i, p_i) + \hbar c_s k_f - E_i - \hbar c_s k_i. \end{aligned} \quad (5.26)$$

If we want to reexpress the cross section Eq. (5.2) in terms of continuous variables for the final state of the phonon, we can use the relation

¹⁶For the full derivation see the Appendix.

$$\begin{aligned} \sum_{\mathbf{k}_f} &= \frac{1}{(\hbar \Delta k_f)^3} \sum_{\mathbf{k}_f} (\hbar \Delta k_f)^3, \\ &\approx \frac{1}{(\Delta k_f)^3} \int_{\mathbb{R}^3} d^3 k_f = \frac{L^3}{(2\pi)^3} \int_{\mathbb{R}^3} d^3 k_f, \end{aligned} \quad (5.27)$$

so that the total cross section is

$$\sigma[\mathcal{S}|i] = \frac{1}{\Phi} \frac{L^3}{(2\pi)^3} \int_{\mathbb{R}^3} d^3 k_f \sum_{\mathbf{p}_f} \frac{w_{i \rightarrow f}}{L^3}. \quad (5.28)$$

Equation (5.28) is very useful to deduce the differential cross section with respect to the scattering angle $\cos \theta$

$$\frac{d\sigma}{d\cos \theta} = \frac{1}{\Phi} \frac{L^3}{(2\pi)^3} \int_0^{+\infty} dk_f k_f^2 \int_0^{2\pi} d\phi \sum_{\mathbf{p}_f} \frac{w_{i \rightarrow f}}{L^3}. \quad (5.29)$$

Since we have assumed that the scattering processing we are considering is coplanar and colinear, we know that the quantity $w_{i \rightarrow f}$ does not depend on the angle ϕ . Hence we can solve the integral in ϕ straightaway

$$\frac{d\sigma}{d\cos \theta} = \frac{2\pi}{\Phi} \frac{L^3}{(2\pi)^3} \int_0^{+\infty} dk_f k_f^2 \sum_{\mathbf{p}_f} \frac{w_{i \rightarrow f}}{L^3}. \quad (5.30)$$

Therefore the differential cross section Eq. (5.30) becomes

$$\begin{aligned} \frac{d\sigma[\mathcal{S}|i]}{d\cos \theta} &= \frac{2\pi}{\Phi} \frac{L^3}{(2\pi)^3} \int_0^{+\infty} dk_f k_f^2 \\ &\times \sum_{\mathbf{p}_f} \frac{1}{L^3} \frac{2\pi}{\hbar} \frac{g^2}{4} \frac{(\hbar c_s)^2}{L^6} \frac{1}{k_i k_f} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f} \delta(E_{fi}), \\ &= 2\pi \frac{L^6}{|c_s - v_i|} \frac{L^3}{(2\pi)^3} \int_0^{+\infty} dk_f k_f^2 \\ &\times \sum_{\mathbf{p}_f} \frac{1}{L^3} \frac{2\pi}{\hbar} \frac{g^2}{4} \frac{(\hbar c_s)^2}{L^6} \frac{1}{k_i k_f} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f} \delta(E_{fi}), \\ &= \frac{1}{|1 - \beta_i|} \frac{\hbar c_s}{2\pi} \frac{\alpha_s^2 \lambda_p^2}{4} \\ &\times \int_0^{+\infty} dk_f k_f^2 \frac{1}{k_i k_f} \delta(E_{fi}) \Big|_{\mathbf{p}_f = \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f}. \end{aligned} \quad (5.31)$$

Equation (5.31) can be further simplified by rewriting the Dirac delta as

$$\delta(E_{fi}) = \sum_{k_0 \in \ker(E_{fi})} \frac{\delta(k_f - k_0)}{|(dE_{fi}/dk_f)_{k_f=k_0}|}, \quad (5.32)$$

where $\ker(E_{fi}) := \{k_0 : E_{fi}(k_0) = 0\}$, and, as before, E_{fi} is the analytic expression for the change in the total energy of the system (i.e., the difference between the final and the

initial energies of the system). The differential cross section Eq. (5.30) is

$$\begin{aligned} \frac{d\sigma}{d\cos \theta} &= \frac{1}{|1 - \beta_i|} \frac{\hbar c_s}{(2\pi)} \frac{\alpha_s^2 \lambda_p^2}{4} \int_0^{+\infty} dk_f k_f^2 \\ &\times \frac{1}{k_i k_f} \sum_{k_0 \in \ker(E_{fi})} \frac{\delta(k_f - k_0)}{|(dE_{fi}/dk_f)_{k_f=k_0}|} \Big|_{\mathbf{p}_f = \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f}. \end{aligned} \quad (5.33)$$

3. Scattering cross sections in the laboratory frame

We can now proceed to derive the differential cross section for phonon scattering from external particles in the laboratory frame. From the kinematic derivation in Sec. IV B 1 we know that we have two solutions $\hbar k_{f,1}$ and $\hbar k_{f,2}$, see Eq. (4.16), that satisfy the condition $E_{fi} = E_{fi}^{\text{tot}} - E_i^{\text{tot}} = 0$, see Eq. (4.15). The differential cross section Eq. (5.33) becomes

$$\begin{aligned} \frac{d\sigma[\mathcal{S}|i]}{d\cos \theta} &= \frac{1}{|1 - \beta_i|} \frac{\hbar c_s}{(2\pi)} \frac{\alpha_s^2 \lambda_p^2}{4} \frac{1}{k_i} \\ &\times \int_0^{+\infty} dk_f k_f \sum_{k_0 \in \{k_{f,1}, k_{f,2}\}} \frac{\delta(k_f - k_0)}{|(dE_{fi}/dk_f)_{k_f=k_0}|}. \end{aligned} \quad (5.34)$$

The denominator is easily calculated from Eq. (4.15)

$$\frac{dE_{fi}}{dk_f} = \frac{\hbar}{m} [\hbar k_f - (\hbar k_i + p_i) \cos \theta + mc_s]. \quad (5.35)$$

By expanding the sum and considering the two solutions of $\ker(E_{fi})$ the differential cross section becomes

$$\begin{aligned} \frac{d\sigma[\mathcal{S}|i]}{d\cos \theta} &= \frac{1}{|1 - \beta_i|} \frac{\hbar c_s}{(2\pi)} \frac{\alpha_s^2 \lambda_p^2}{4} \frac{1}{k_i} \int_0^{+\infty} dk_f k_f \\ &\times \left[\frac{\delta(k_f - k_{f,1})}{\left| \frac{1}{m} \hbar [\hbar k_{f,1} - (\hbar k_i + p_i) \cos \theta + mc_s] \right|} \right. \\ &\left. + \frac{\delta(k_f - k_{f,2})}{\left| \frac{1}{m} \hbar [\hbar k_{f,2} - (\hbar k_i + p_i) \cos \theta + mc_s] \right|} \right]. \end{aligned} \quad (5.36)$$

When we perform the integral in k_f we find

$$\begin{aligned} \frac{d\sigma[\mathcal{S}|i]}{d\cos \theta} &= \frac{1}{|1 - \beta_i|} \frac{\hbar c_s}{(2\pi)} \frac{\alpha_s^2 \lambda_p^2}{4} \frac{1}{k_i} \frac{m}{\hbar} \\ &\times \frac{(k_{f,1} H[k_{f,1}] + k_{f,2} H[k_{f,2}])}{\sqrt{\bar{B}^2 - \bar{C}}}, \end{aligned} \quad (5.37)$$

where $\bar{B}$ and $\bar{C}$ are defined in Eq. (4.17). The functions $H[k_{f,1}]$ and $H[k_{f,2}]$ are the Heaviside step functions that ensure that the two solutions $k_{f,1}$ and $k_{f,2}$ are real and

positive, as we have already seen in the kinematic Sec. IV B 1. By inserting the two solutions from Eq. (4.16) into Eq. (5.37), the explicit expression for the differential cross section is

$$\frac{d\sigma[S|i]}{d\cos\theta} = \frac{1}{|1 - \beta_i|} \frac{\hbar c_s}{(2\pi)} \frac{g^2}{4} \frac{1}{k_i} \frac{m}{\hbar^2} \times [2AH[\bar{B} - \sqrt{\bar{C}}]H[\bar{C}] + (A + 1)H(-\bar{C})], \quad (5.38)$$

where we have defined

$$A := \frac{\bar{B}}{\sqrt{\bar{B}^2 - \bar{C}}}. \quad (5.39)$$

The three conditions specified by the three Heaviside step functions $H(\bar{B} - \sqrt{\bar{C}})$, $H(\bar{C})$, and $H(-\bar{C})$ are the same as those in Eqs. (4.20). The differential cross section has units of area, as it must, since the quantity $\frac{\hbar c_s}{k_i} \frac{m}{\hbar^2}$ is dimensionless.

4. Scattering cross sections in the comoving in-universe observer frame

As we have seen multiple times throughout this paper, the comoving in-universe observer frame description of the scattering event is obtained by appropriately transforming the laboratory frame kinematic quantities into the comoving frame of the particle. We have already performed all of the heavy lifting required to obtain the differential cross section in the frame of the comoving in-universe observer. Our final task is to transform the kinematic quantities in Eq. (5.37) [or, alternatively, Eq. (5.38)] according to the transformations specified in Eqs. (3.6)–(3.8). Substituting $k_i = Dk'_i$, where D is the Doppler factor [Eq. (2.8)], and applying Eq. (5.14), the differential cross section becomes

$$\frac{d\sigma[S'|i']}{d\cos\theta'} = \frac{1 + \beta_i}{(1 + \beta_i \cos\theta')^2} \frac{g^2 \hbar c_s}{4(2\pi)} \frac{m}{\hbar^2} \frac{1}{Dk'_i} \times [2A'H(\bar{B}' - \sqrt{\bar{C}'})H(\bar{C}') + (A' + 1)H(-\bar{C}')], \quad (5.40)$$

where $\bar{B}'$ and $\bar{C}'$ are just $\bar{B}$ and $\bar{C}$ reexpressed in the comoving frame, as shown in Eq. (4.26), and A' is

$$A' = \frac{\bar{B}'}{\sqrt{\bar{B}'^2 - \bar{C}'}}. \quad (5.41)$$

5. Using Lorentz-violating sonic Compton scattering to determine absolute motion

The differential cross section written in terms of the values of quantities as measured by comoving in-universe

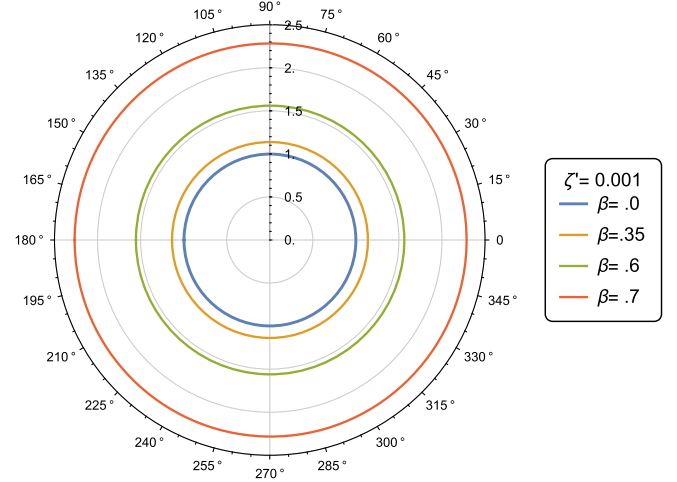


FIG. 3. Polar plot of the differential cross section for an external particle in the in-universe comoving frame for $\zeta' = 0.001$ and various values of β .

observers is given by Eq. (5.40). In principle, in-universe observers could use this expression to make qualitative statements regarding their state of motion by performing several scattering experiments on external particles with different initial velocities $v_i = \beta_i c_s$, using phonons with different initial energies $\hbar\omega'_i$ (or equivalently $\hbar c_s k'_i$). To facilitate the comparison between different energy and velocity regimes, we introduce the dimensionless quantity

$$\zeta' = \frac{\hbar\omega'_i}{mc_s^2} = \frac{\hbar k'_i}{mc_s} = \lambda_p k'_i, \quad (5.42)$$

that represents the ratio of the initial energy of the phonon to the “sonic rest-mass energy” that in-universe observers would think to associate to the particle.

The ratio ζ' is directly analogous to a ratio that appears in the description of actual Compton scattering, which we shall here explicitly denote¹⁷ ξ , the value of which has particular physical implications. The limit $\xi \ll 1$ corresponds to Thomson scattering [53], which describes the classical and nonrelativistic scattering of electromagnetic waves from charged particles, whereas the limit $\xi \gtrsim 1$ corresponds to scenarios in which both relativistic and quantum theoretic effects are prominent: as a result, a full quantum field theoretic description is necessary in the limit $\xi \gtrsim 1$ [54].

In the following we demonstrate what in-universe observers will measure if they were to perform several scattering experiments for the same value of ζ' and several values of β_i and vice versa. We provide the differential

¹⁷The expression for ξ is identical in form to that of ζ' except all references to sound are replaced by their corresponding references to light. That is to say, c takes the place of c_s , and the frequencies ω'_i and ω'_f correspond to frequencies of light, not sound.

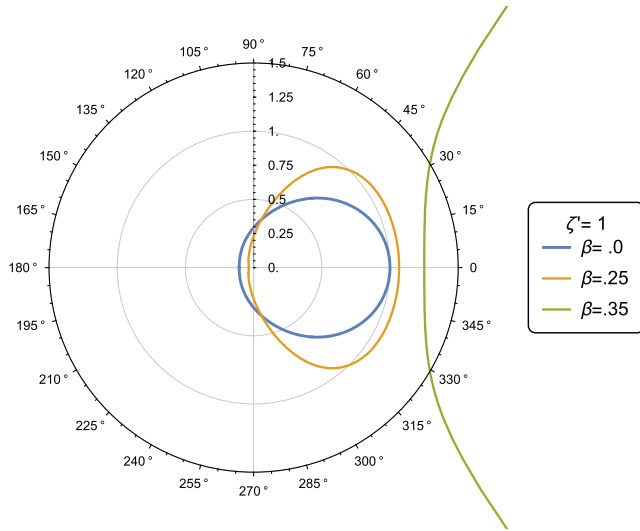


FIG. 4. Polar plot of the differential cross section for an external particle in the in-universe comoving frame for $\zeta' = 1$ and various values of β .

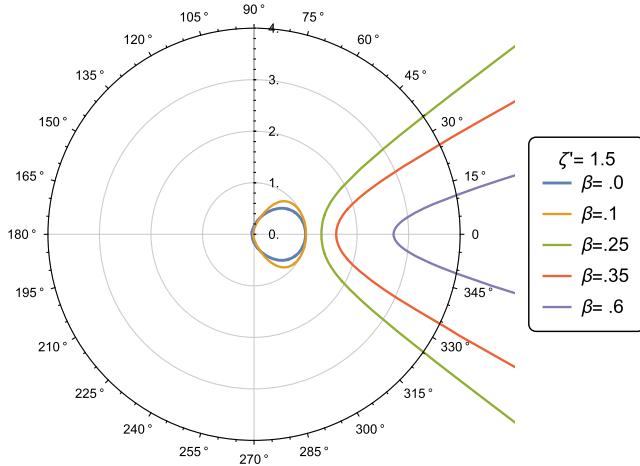


FIG. 5. Polar plot of the differential cross section for an external particle in the in-universe comoving frame for $\zeta' = 1.5$ and various values of β .

cross sections as viewed from the comoving frame for two types of experiments: in one type of experiment, the value of ζ' is held constant while β_i is varied; in the other type of experiment, β_i is held constant while ζ' is varied.

Fixed ζ' , varying β_i .—For this case we present the scattering cross sections for $\zeta' = \{0.001, 1, 1.5\}$. In Fig. 3 we can already see that, even for a very small value of ζ' , the cross section shows a marked dependence on different values of β_i . We can qualitatively compare this result with the Klein–Nishima differential cross section formula [52,55] for unpolarized photons, where it is easy—and expected—to see that there is no dependency of the cross section on the state of motion of the rest frame of the particle (for in actual

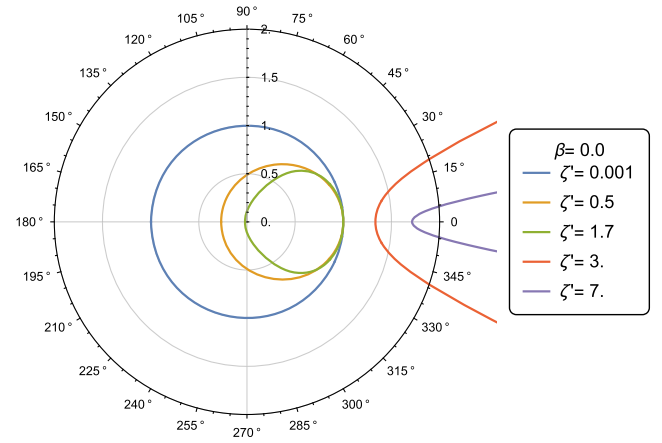


FIG. 6. Polar plot of the differential cross section for an external particle in the in-universe comoving frame for $\beta_i = 0$ and various values of ζ .

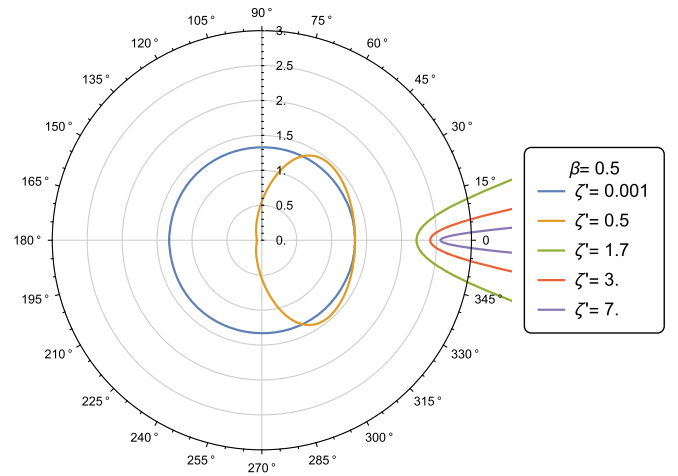


FIG. 7. Polar plot of the differential cross section for an external particle in the in-universe comoving frame for $\beta = 0.5$ and various values of ζ' .

relativity, there is no meaningful notion in which different inertial states of motion differ). For higher values of ζ' , another noteworthy feature becomes apparent: in Figs. 4 and 5 we can see that, as the initial velocity of the particle increases, the range of the scattering angle becomes smaller, eventually leading to a scenario in which the particle is prohibited from scattering outside of some given angular window. This effect is due to the conditions in Eq. (4.29), which place restrictions on the allowed angles of scattering as per Eq. (4.30).

Fixed β_i , varying ζ' .—The dependency of the scattering angle on the initial energy of the phonon ζ' becomes clearer when we consider the in-universe differential cross section for two values of $\beta_i = \{0, 0.5\}$ and several values of ζ' . In Figs. 6 and 7 we can see that as ζ' increases, the scattering angle becomes more forward, tending towards the direction

of motion of the particle in the laboratory frame. This effect becomes more pronounced with increasing β_i , as one can see by comparing Figs. 6 and 7.

VI. DISCUSSION

In considering phonon scattering from *external particles* we have restricted our considerations to that of nonrelativistic quantum mechanics for all values of ζ' because, by construction, our external particle is a non-relativistic quantum mechanical object. By analogy to true Compton scattering, phonon scattering from *internal particles* requires a full quantum field theoretic description¹⁸ because relativistic effects (with respect to sound) are important, and relativistic quantum mechanics is inapplicable because it is a fundamentally inconsistent theory [54]. We can however make qualitative comparisons between phonon scattering from internal particles and external particles in the limit $\zeta' \ll 1$, as in this limit relativistic effects become unimportant for internal particles. Strictly speaking, any comparisons of this type should be made in the specific case for which both types of particle are initially traveling very slowly in the laboratory frame ($\beta_i \ll 1$) as it is in this limit that the energy-momentum relations for internal and external particles coincide and thus the limit in which a quantum field theoretic description should coincide with an ordinary quantum mechanical description. With this in mind, Fig. 3 shows the differential cross sections for phonon scattering from external particles characterized by $\zeta' = 10^{-3}$: the limit $\zeta' \ll 1$ is respected here, and while the total amount of scattering is a function of β_i , the overall form of the differential scattering cross section (i.e., its angular dependency) is not. That the angular dependency of the differential scattering cross section is insensitive to β_i in this case is what we expect from the expected equivalence between phonon scattering from internal and external particles for low ζ' .

Any qualitative similarities between phonon scattering from internal and external particles vanishes for higher values of ζ' . In Figs. 4 and 5 the differential scattering cross sections for various values of β_i are plotted for $\zeta' = 1$ and $\zeta' = 1.5$, respectively. In both cases not only does the amount of scattering vary with initial β_i , but the overall form of the differential scattering cross sections is sensitive to changes in β_i too: this is in stark contrast to what would occur for scattering from internal particles which, again, must be insensitive to the value of β_i due to sonic Lorentz covariance that is inbuilt into internal particles.

The effect of varying ζ' for constant values of β_i *should* be expected to alter the angular distribution of scattering, even for internal particles. In real Compton scattering, forward scattering becomes preferentially favored as the initial photon energy is increased (see, for example, [56]).

The fact that there exists angular dependency in the differential scattering cross section of phonons from external particles for fixed β_i in Figs. 6 and 7 is therefore not in and of itself surprising or unexpected. With that said, for phonon scattering from external particles, increasing ζ' for fixed β_i eventually reveals the presence of forbidden scattering angles: this has *no* qualitative similarity to Compton scattering, and thus no qualitative similarity to phonon scattering from internal particles.

VII. CONCLUSION

Provided that in-universe observers in an analogue-gravity universe are allowed to interact with Newtonian particles external to their own medium, then the sonic analogue to Compton scattering—in which phonons scatter from these external particles—can be used by in-universe observers to infer that there must exist some preferred rest frame. In all but the most restrictive cases (i.e., unless $\beta \approx 0$ and $\zeta' \ll 1$), external particles result in qualitatively different scattering profiles than occur in fully relativistic scattering, such as Compton scattering. In the most dramatic cases, scattering from external particles results in differential scattering cross sections with forbidden angles, and as the energy of the phonon increases relative to the “sonic rest-mass energy” of the particle (ζ' increases) the window of allowed scattering angles becomes more tightly concentrated in the direction of the trajectory of the particle prior to scattering. Even in the cases for which scattering occurs at all angles, scattering for fixed values of ζ' shows a preference towards forward scattering for increasing β_i .

In principle, in-universe observers could conceivably utilize phonon scattering experiments from external particles to identify not only that a preferred rest frame must exist, but specifically which frame is the rest frame of their analogue universe. The ability to resolve which frame is actually the laboratory frame is fundamentally constrained by the mass of external particles and the energies of phonons that in-universe observers have access to. If in-universe observers are only able to probe the parts of parameter space corresponding to $\zeta' \ll 1$ then the angular distribution of scattering will not be considerably affected by their state of velocity (see Fig. 3) and thus they will only be able to detect their state of motion provided that they correctly deduce the relationship between β_i and the magnitude of the differential scattering cross section (that is, lower magnitudes correspond to lower β_i , as per Fig. 3). If, on the contrary, in-universe observers are able to probe regions of parameter space corresponding to $\zeta' \gtrsim 1$, then the presence of forbidden scattering angles could be utilized to locate the medium’s rest frame. In order for in-universe observers to utilize forbidden scattering angles to their advantage, they would either have to reverse engineer the energy-momentum relation for external particles, or postulate the correct energy-momentum relation

¹⁸As previously discussed, a condensed matter quantum field theory.

and confirm it experimentally. Given that the correct energy-momentum relation for external particles corresponds to the $\beta_i \ll 1$ limit of internal particles, it is not unreasonable to think that they would eventually postulate the correct relation.

Scattering experiments performed from internal particles must ultimately be equivalent to scattering experiments performed within a truly relativistic theory due to the fact that a Lorentz symmetry (with respect to c_s) is inbuilt into both internal particles themselves and the reference frames of in-universe observers. The interpretation here is that when internal particles are used, every part of the system is sonically Lorentz covariant, and when this is true the whole analogue-gravity system can be treated as being a sonic analogue to something like Lorentz ether theory, which is operationally indistinguishable from special relativity. Thus, when in-universe observers are only allowed to interact with internal particles, they cannot determine the presence of a preferred rest frame. It is only when the symmetry groups obeyed by the particle and the phonons are different that the analogy between our model and a Lorentz ether theory (and hence a relativistic theory) breaks down and the presence of the medium can be detected.

As a final note, to in-universe observers the sonic analogue to Compton scattering from external particles constitutes a breaking of the sonic Lorentz symmetry that they would otherwise believe in without access to external objects. From this point of view, the Standard Model extension [47] might provide a natural way to further investigate such scenarios from the perspective of in-universe observers who want to believe that the sonic Lorentz symmetry of their universe is fundamental. To highlight this point, consider that an excitation of a truly relativistic field (that is, relativistic with respect to the speed of light) would also constitute an example of an external particle from the perspective of in-universe observers. The Lagrangian density $\mathcal{L}_l$ describing a real scalar field that is actually relativistic (the subscript l denotes that the Lagrangian is invariant under Lorentz transformations with respect to c the speed of light) can be written in terms of some sonically relativistic Lagrangian $\mathcal{L}_s$ (the subscript s denotes that the Lagrangian is invariant under Lorentz transformations with respect to c_s the speed of sound) with some additional term $\mathcal{K}$ to account for the difference:

$$\mathcal{L}_l = \mathcal{L}_s + \mathcal{K}. \quad (7.1)$$

Viewed this way, in-universe observers might be able to describe certain external particles as though they were external particles that were coupled to some background vector field defining a preferred frame (the rest frame of the

medium). This is precisely the type of scenario that the standard model extension deals with, though in the standard model extension the Lorentz symmetry is taken to be fundamental at high energies and spontaneously broken at low energies, whereas the sonic Lorentz symmetry is emergent rather than fundamental.

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APPENDIX: GENERAL TRANSITION PROBABILITY DERIVATION

From the definition of cross section in Eq. (5.2), or equivalently from Eq. (5.28), we see that we need to evaluate the amplitude

$$\langle f | \hat{H}_{\text{int}} | i \rangle, \quad (A1)$$

where the initial and final states are, respectively,

$$|i\rangle = |\mathbf{p}_i\rangle \otimes |\hbar\mathbf{k}_i\rangle, \quad |f\rangle = |\mathbf{p}_f\rangle \otimes |\hbar\mathbf{k}_f\rangle, \quad (A2)$$

and where the states for external particles are defined in Table II. We consider the states represented in momentum space, so the phonon initial and final states can be written as

$$|\hbar\mathbf{k}_i\rangle = \hat{a}_{\mathbf{k}_i}^\dagger |0\rangle_s, \quad |\hbar\mathbf{k}_f\rangle = \hat{a}_{\mathbf{k}_f}^\dagger |0\rangle_s, \quad (A3)$$

where $|0\rangle_s$ is the phonon ground state. The amplitude Eq. (A1) then becomes

$$\begin{aligned}
 \langle f | \hat{H}_{\text{int}} | i \rangle &= \frac{g}{2} \frac{1}{L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{\hbar c_s}{2\sqrt{kk'}} \langle \mathbf{p}_f | \otimes_s \langle 0 | \hat{a}_{\mathbf{k}_f} (\hat{a}_{\mathbf{k}} e^{i\mathbf{k} \cdot \hat{\mathbf{x}}} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \hat{\mathbf{x}}}) (\hat{a}_{\mathbf{k}'} e^{i\mathbf{k}' \cdot \hat{\mathbf{x}}} + \hat{a}_{\mathbf{k}'}^\dagger e^{-i\mathbf{k}' \cdot \hat{\mathbf{x}}}) | \mathbf{p}_i \rangle \otimes \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s, \\
 &= \frac{g}{2} \frac{1}{L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{\hbar c_s}{2\sqrt{kk'}} \langle \mathbf{p}_f | \otimes_s \langle 0 | \hat{a}_{\mathbf{k}_f} (\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'}^\dagger e^{i(\mathbf{k}-\mathbf{k}') \cdot \hat{\mathbf{x}}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}'} e^{-i(\mathbf{k}-\mathbf{k}') \cdot \hat{\mathbf{x}}}) | \mathbf{p}_i \rangle \otimes \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s, \\
 &= \frac{g \hbar c_s}{2} \frac{1}{L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{1}{2\sqrt{kk'}} \langle \mathbf{p}_f | \otimes_s \langle 0 | \hat{a}_{\mathbf{k}_f} (\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'}^\dagger e^{i(\mathbf{k}-\mathbf{k}') \cdot \hat{\mathbf{x}}} + \underbrace{\hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}} e^{-i(\mathbf{k}'-\mathbf{k}) \cdot \hat{\mathbf{x}}}}_{(*)} | \mathbf{p}_i \rangle \otimes \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s.
 \end{aligned} \tag{A4}$$

In the term $(*)$ we have swapped $\mathbf{k}$ and $\mathbf{k}'$ as these are dummies variables. Using the commutation relations mentioned in Table II we obtain

$$\begin{aligned}
 \langle f | \hat{H}_{\text{int}} | i \rangle &= \frac{g \hbar c_s}{2} \frac{1}{L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{1}{2\sqrt{kk'}} \langle \mathbf{p}_f | \otimes_s \langle 0 | \hat{a}_{\mathbf{k}_f} (2\hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}} + \delta_{\mathbf{k}, \mathbf{k}'}) e^{i(\mathbf{k}-\mathbf{k}') \cdot \hat{\mathbf{x}}} | \mathbf{p}_i \rangle \otimes \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s, \\
 &= \frac{g \hbar c_s}{2} \frac{1}{L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{1}{2\sqrt{kk'}} \left(\underbrace{\langle \mathbf{p}_f | \otimes_s \langle 0 | \hat{a}_{\mathbf{k}_f} 2\hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}} e^{i(\mathbf{k}-\mathbf{k}') \cdot \hat{\mathbf{x}}} | \mathbf{p}_i \rangle \otimes \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s}_{(**)} + \underbrace{\langle \mathbf{p}_f | \otimes_s \langle 0 | \hat{a}_{\mathbf{k}_f} \delta_{\mathbf{k}, \mathbf{k}'} e^{i(\mathbf{k}-\mathbf{k}') \cdot \hat{\mathbf{x}}} | \mathbf{p}_i \rangle \otimes \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s}_{(***)} \right).
 \end{aligned} \tag{A5}$$

We evaluate $(**)$ first:

$$(***) = 2 \langle \mathbf{p}_f | e^{i(\mathbf{k}-\mathbf{k}') \cdot \hat{\mathbf{x}}} | \mathbf{p}_i \rangle_s \langle 0 | \hat{a}_{\mathbf{k}_f} \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s. \tag{A6}$$

Considering that $\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'}^\dagger = \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}} + \delta_{\mathbf{k}, \mathbf{k}'}$, and that $e^{i\mathbf{k} \cdot \hat{\mathbf{x}}} | \mathbf{p} \rangle = | \mathbf{p} + \hbar \mathbf{k} \rangle$, $(**)$ becomes

$$\begin{aligned}
 (**) &= 2 \langle \mathbf{p}_f | \mathbf{p}_i + \hbar \mathbf{k} - \hbar \mathbf{k}' \rangle_s \langle 0 | (\hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}_f} + \delta_{\mathbf{k}_f, \mathbf{k}'}) (\hat{a}_{\mathbf{k}_i}^\dagger \hat{a}_{\mathbf{k}} + \delta_{\mathbf{k}, \mathbf{k}_i}) | 0 \rangle_s, \\
 &= 2 \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k} - \hbar \mathbf{k}'} \delta_{\mathbf{k}_f, \mathbf{k}'} \delta_{\mathbf{k}, \mathbf{k}_i},
 \end{aligned} \tag{A7}$$

and $(***)$ becomes

$$\begin{aligned}
 (***) &= \langle \mathbf{p}_f | \mathbf{p}_i \rangle_s \langle 0 | \hat{a}_{\mathbf{k}_f} \hat{a}_{\mathbf{k}_i}^\dagger | 0 \rangle_s \delta_{\mathbf{k}, \mathbf{k}'}, \\
 &= \delta_{\mathbf{p}_f, \mathbf{p}_i} \delta_{\mathbf{k}_f, \mathbf{k}_i} \delta_{\mathbf{k}, \mathbf{k}'}.
 \end{aligned} \tag{A8}$$

The amplitude Eq. (A1) is therefore

$$\langle f | \hat{H}_{\text{int}} | i \rangle = \frac{g \hbar c_s}{2} \frac{1}{L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{1}{2\sqrt{kk'}} \left[\underbrace{2 \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k} - \hbar \mathbf{k}'} \delta_{\mathbf{k}_f, \mathbf{k}'} \delta_{\mathbf{k}, \mathbf{k}_i}}_{(1)} + \underbrace{\delta_{\mathbf{p}_f, \mathbf{p}_i} \delta_{\mathbf{k}_f, \mathbf{k}_i} \delta_{\mathbf{k}, \mathbf{k}'}}_{(2)} \right]. \tag{A9}$$

Evaluating (2) first we obtain

$$\begin{aligned}
 (2) &= \frac{g \hbar c_s}{2} \frac{1}{L^3} \delta_{\mathbf{p}_f, \mathbf{p}_i} \delta_{\mathbf{k}_f, \mathbf{k}_i} \sum_{\mathbf{k}, \mathbf{k}'} \frac{1}{2\sqrt{kk'}} \delta_{\mathbf{k}, \mathbf{k}'}, \\
 &= \frac{g \hbar c_s}{2} \frac{1}{L^3} \delta_{\mathbf{p}_f, \mathbf{p}_i} \delta_{\mathbf{k}_f, \mathbf{k}_i} \sum_{\mathbf{k}} \frac{1}{2k}.
 \end{aligned} \tag{A10}$$

This divergent term represents the case of no scattering. Since we are only interested in the set $\mathcal{S}$ of final states that *do not* contain the initial states (i.e., we are excluding the case of no scattering), we will not consider this term. Term (1) instead is

$$\begin{aligned}
(1) &= \frac{g \hbar c_s}{2 L^3} \sum_{\mathbf{k}, \mathbf{k}'} \frac{1}{2 \sqrt{k k'}} 2 \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k} - \hbar \mathbf{k}'} \delta_{\mathbf{k}_f, \mathbf{k}'} \delta_{\mathbf{k}, \mathbf{k}_i}, \\
&= \frac{g \hbar c_s}{2 L^3} \frac{1}{\sqrt{k_i k_f}} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f}.
\end{aligned} \tag{A11}$$

The amplitude is therefore

$$\langle f | \hat{H}_{\text{int}} | i \rangle = \frac{g \hbar c_s}{2 L^3} \frac{1}{\sqrt{k_i k_f}} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f}. \tag{A12}$$

The transition rate [45] for an initial state $|i\rangle$ at time $-\infty$ and a final state $|f\rangle$ at time T is

$$w_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle f | \hat{H}_{\text{int}} | i \rangle|^2 \delta(E_{fi}), \tag{A13}$$

where the states and the interaction Hamiltonian are defined in Schrödinger picture. Substituting Eq. (A12) into Eq. (A13) gives the explicit form of the transition rate:

$$\begin{aligned}
w_{i \rightarrow f} &= \frac{2\pi}{\hbar} \left| \frac{g \hbar c_s}{2 L^3} \frac{1}{\sqrt{k_i k_f}} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f} \right|^2 \delta(E_{fi}), \\
&= \frac{2\pi g^2 (\hbar c_s)^2}{\hbar 4 L^6} \frac{1}{k_i k_f} (\delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f})^2 \delta(E_{fi}), \\
&= \frac{2\pi g^2 (\hbar c_s)^2}{\hbar 4 L^6} \frac{1}{k_i k_f} \delta_{\mathbf{p}_f, \mathbf{p}_i + \hbar \mathbf{k}_i - \hbar \mathbf{k}_f} \delta(E_{fi}).
\end{aligned} \tag{A14}$$

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Bandlimited entanglement harvesting

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There are many reasons to believe that there is a fundamental minimum length scale below which distances cannot be reliably resolved. One method of constructing a quantum field with a finite minimum length scale is to use bandlimited quantum field theory, where the spacetime is mathematically both continuous and discrete. This is a modification to the field, which has been shown to have many consequences at the level of the field. We consider an operational approach and use a pair of particle detectors (two-level qubits) as a local probe of the field, which are coupled to the vacuum of the bandlimited massless scalar field in a time-dependent way through a switching function. We show that, mathematically, the bandlimit modifies the spatial profile of the detectors so that they are only quasilocal. We explore two different types of switching functions, Gaussian and Dirac delta. We find that, with Gaussian switching, the bandlimit exponentially suppresses the deexcitation of the detectors when the energy gap between the two levels is larger than the bandlimit. If the detectors are prepared in ground state, in certain regions of the parameter space they are able to extract more entanglement from the field than if there was no bandlimit. When the detectors couple with Dirac-delta switching, we show that a particle detector is most sensitive to the bandlimit when it couples to a small but finite region of spacetime. We find that the effects of a bandlimit are detectable using local probes. This work is important because it illustrates the possible observable consequences of a fundamental bandlimit in a quantum field.

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I. INTRODUCTION

One of the most important open problems of modern physics is the question of what is happening at the highest energy scales where quantum field theory and general relativity are expected to be incorporated into a larger theory of physics. These high-energy scales correspond to the shortest length scales, where it is widely thought that, due to quantum fluctuations of the metric, the notions of space and time break down [1]. Currently, there are many approaches to consistent theories of quantum gravity that describe the physics at these length scales, including string theory, loop quantum gravity and many others [2,3].

It has been shown that if there is a finite, minimum-length uncertainty in a quantum field, the field will obey the Shannon sampling theorem, meaning the continuous field can be reconstructed from a discrete set of sampling points [4]. This also implies the field will be bandlimited. The Shannon sampling theorem [5] is a theory in classical information that provides an equivalence between a continuous and a discrete representation of information. If a

signal $f(t)$ is bandlimited, so that it has no frequencies higher than some Λ , then knowing the value of the signal $f(t_n)$ on a lattice of points $\{t_n\}$ is enough to reconstruct the continuous signal for all t provided the average spacing of the lattice is less than or equal to the Nyquist spacing, π/Λ . Unlike naively putting a quantum field theory (QFT) on a lattice, the resultant bandlimited QFT preserves local Euclidian symmetries but is not Lorentz invariant. However, the conventional momentum cutoff generalizes to a covariant cutoff, where modes with a wavelength smaller than the cutoff have very small bandwidth and are effectively frozen out [6,7].

It was shown by Pye, Donnelly and Kempf [8] that applying a conventional bandlimit to a $(1+1)$ -dimensional scalar QFT results in field degrees of freedom that occupy an incompressible spatial volume. As a result, the two-point correlations of the field and entanglement entropy are modified from the case of no cutoff, with the least modification occurring at distances of the Nyquist spacing.

In that paper, the authors raised the idea of studying the interaction of Unruh-DeWitt (UDW) detectors [9]—two-level quantum systems—with bandlimited quantum fields as a natural next step. The UDW detectors serve as a local probe of the field. This could be used as a stepping stone

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toward the quantization of sampling theory. More specifically, in the entanglement harvesting protocol, the spatial profile of the detector—which quantifies where the detector couples to the field—and the field degrees of freedom enter the model at the same level. Exploring how they interact may provide a more operational understanding of the finite spatial volume of the discrete degrees of freedom.

In general, the study of UDW detectors as local particle detectors provides an operational method of probing quantum field fluctuations and correlations by providing a method of directly sampling the field in different spacetime regions through local interactions. It has been shown by Valentini [10] and Reznik *et al.* [11,12] that entanglement present in a quantum field can be extracted by a pair of initially separable UDW detectors that couple locally to the field in a process known as entanglement harvesting [13].

In most implementations of entanglement harvesting the setup is very simple: The detectors are generally coupled linearly to the vacuum of a scalar field. However, it has been shown that this simple model is a good approximation to the light-matter interaction, under the assumption that no angular momentum is transferred [14–16]. Entanglement harvesting has gained interest in both its applicational and foundational potential [13,17–24]. It may also have some implications for quantum gravity and the black hole information paradox [25–32].

In particular, this protocol has been shown to be incredibly sensitive to the properties of field to which they are coupled. In particular, it has been shown to be able to distinguish between a thermal bath and a de Sitter spacetime at the same temperature [33–35], and it is sensitive to the topology of spacetime [36,37], the presence of horizons [25] and the boundary conditions [38] of the field. UDW detectors have also been shown to be very sensitive to their state of motion [13,18]. There are generally two different assumptions made about the interaction between the detectors and the field. First, and more commonly, it is assumed that the detectors couple only weakly with the field. This allows for use of perturbation theory when calculating the final state of the system. In this regime, entanglement harvesting is possible, but multiphonon interactions are not considered. Additionally, this model may lead to divergences in limits where the detector response becomes large but finite. The second assumes that the detectors have Dirac-delta switching—i.e., they couple to the field at single moment in time. Here, the time-evolution operator takes a much simpler form, and in some cases the final state of the system can be known exactly. However, entanglement harvesting is not possible with delta switching [39,40].

In this paper we implement a conventional bandlimit on the scalar QFT by applying a hard cutoff and only allow modes with frequencies $|\mathbf{k}| < \Lambda$ to propagate. Such a cutoff is not Lorentz invariant, but we expect that a similar result

will hold in the case of a covariant cutoff, since our UDW detectors only couple to the field for a finite time, and modes with a small bandwidth would be further suppressed.

The rest of the paper is as follows. In Sec. II, we review the UDW model and use perturbation theory to find the final state of the two detectors to lowest order in the coupling strength. Before specifying the dimensionality, the spatial profile or switching function, we introduce the notion of the effective spatial profile, where we have absorbed the momentum cutoff of the field into a nonlocal modified spatial profile, allowing for a nonbandlimited field to model the same physics (if the modified profile is used instead of the original). Next, in Sec. III, we specify our model and take the detectors to be pointlike and couple to the field with a Gaussian function switching. We perturbatively calculate the excitation and deexcitation probability of a single detector in Sec. IV and the entanglement harvested by a pair of detectors in Sec. V. We use Dirac-delta switching to nonperturbatively calculate the transition probability of two detectors with Gaussian spatial profiles in Sec. VI. Finally, in Sec. VII, we conclude and discuss potential future work.

II. THE UNRUH-DEWITT MODEL

We will use UDW detectors to model the two particle detectors, A and B , which capture most of the features of the light-matter interaction when no angular momentum is exchanged. In this model, the detectors are described by two-level quantum systems, with ground and excited states given by $|0\rangle_D$ and $|1\rangle_D$, respectively, and separated by an energy gap of Ω_D , which couple locally to a quantum scalar field $\hat{\phi}(\mathbf{x}, t)$, with $D \in \{A, B\}$. The interaction of each detector is described, in the interaction picture, by the Hamiltonian

$$\hat{H}_D(\tau_D) = \lambda \chi_D(\tau_D) (e^{i\Omega_D \tau_D} \hat{\sigma}_D^+ + e^{-i\Omega_D \tau_D} \hat{\sigma}_D^-) \otimes \int d^n \mathbf{x} F_D[\mathbf{x} - \mathbf{x}_D(\tau_D)] \hat{\phi}(\mathbf{x}, t), \quad (1)$$

where λ is the coupling strength of the interaction, $\chi_D(\tau_D)$ is the switching function, which controls the interaction time, $\hat{\sigma}_D^+ := |1\rangle_D \langle 0|$ and $\hat{\sigma}_D^- := |0\rangle_D \langle 1|$ are the SU(2) ladder operators acting on the Hilbert space of detector D , $F_D(\mathbf{x})$ describes the spatial profile of the detector, and $\mathbf{x}_D(\tau_D)$ is the detector's spacetime trajectory parameterized by its proper time τ_D .

The time evolution of the detector-field system with respect to the time t is generated by the unitary operator

$$\hat{U} := \mathcal{T} \exp \left[-i \int dt \left(\frac{d\tau_A}{dt} \hat{H}_A[\tau_A(t)] + \frac{d\tau_B}{dt} \hat{H}_B[\tau_B(t)] \right) \right], \quad (2)$$

where $\mathcal{T}$ is the time-ordering operator.

Following the entanglement harvesting protocol, we consider the detectors to be initially ($t \rightarrow -\infty$) in the ground state and the field initially in the vacuum, so the joint state of the system is

$$|\Psi_0\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_\phi, \quad (3)$$

and, after the interaction ($t \rightarrow \infty$), the system is in the state

$$|\Psi_f\rangle = \hat{U}|\Psi_0\rangle = \sum_n \hat{U}^{(n)}|\Psi_0\rangle, \quad (4)$$

where $\hat{U}^{(n)}$ is the n th term in the Dyson expansion of the time-evolution operator (2)

$$\hat{U}^{(n)} = (-i)^n \int_{t_n < \dots < t_1} dt_1 \dots dt_n \hat{H}(t_1) \dots \hat{H}(t_n) \quad (5)$$

and $\hat{H}(t) = \sum_{D \in \{A, B\}} \frac{d\tau_D}{dt} \hat{H}_D[\tau_D(t)]$.

We are only interested in the partial state of the two detectors, so the Hilbert space of the field is traced out:

$$\begin{aligned} \hat{\rho}_{AB} &:= \text{Tr}_\phi[|\Psi_f\rangle\langle\Psi_f|] \\ &= \sum_{m,n} \text{Tr}_\phi[\hat{U}^{(m)}|\Psi_0\rangle\langle\Psi_0|(\hat{U}^{(n)})^\dagger]. \end{aligned} \quad (6)$$

It can be seen from Eqs. (1) and (5) that, in the expression $\hat{U}^{(n)}|\Psi_0\rangle$, the field operator $\hat{\phi}(\mathbf{x}, t)$ will be applied to the vacuum n times, and so the only terms that survive the partial trace and contribute to the partial state are terms where m and n have the same parity. Additionally, in the final state, $\hat{U}^{(n)}|\Psi_0\rangle$, an even n results in both detectors being excited or both remaining in the ground state, and an odd n results in only one detector being excited. Since m and n must be both even or both odd, the reduced density matrix of the two detectors will be of the form

$$\hat{\rho}_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{23} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix} \quad (7)$$

in the basis $\{|0\rangle_A \otimes |0\rangle_B, |0\rangle_A \otimes |1\rangle_B, |1\rangle_A \otimes |0\rangle_B, |1\rangle_A \otimes |1\rangle_B\}$.

The bandlimit Λ is a hard cutoff and is implemented by expanding the field operator in plane-wave modes and cutting off momenta where $|\mathbf{k}| > \Lambda$:

$$\hat{\phi}_\Lambda(\mathbf{x}, t) = \frac{1}{(2\pi)^{n/2}} \int_{|\mathbf{k}| < \Lambda} \frac{d^n \mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} (e^{i(\omega_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} \hat{a}_{\mathbf{k}}^\dagger + \text{H.c.}), \quad (8)$$

where, for a massless field the dispersion relation is $\omega_{\mathbf{k}} = |\mathbf{k}|$. This implementation of the cutoff is not covariant but is chosen this way for ease of calculation and to follow on from the framework of [8]. Reproducing these

calculations with a fully covariant cutoff [6] is left to future work. With this field operator, the interaction Hamiltonian of a single detector becomes

$$\begin{aligned} \hat{H}_D(\tau_D) &= \lambda \chi(\tau_D) (e^{i\Omega_D \tau_D} \hat{\sigma}_D^+ + e^{-i\Omega_D \tau_D} \hat{\sigma}_D^-) \\ &\otimes \int_{|\mathbf{k}| < \Lambda} \frac{d^n \mathbf{k}}{\sqrt{2|\mathbf{k}|}} \\ &\times (\tilde{F}_D^*(\mathbf{k}) e^{i[|\mathbf{k}|t(\tau_D) - \mathbf{k} \cdot \mathbf{x}_D(\tau_D)]} \hat{a}_{\mathbf{k}}^\dagger + \text{H.c.}), \end{aligned} \quad (9)$$

where

$$\tilde{F}_D(\mathbf{k}) = \frac{1}{(2\pi)^{n/2}} \int d^n \mathbf{x} F_D(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}} \quad (10)$$

is the Fourier transform of the spatial profile.

Mathematically, as was shown in [41], the hard momentum cutoff in the interaction Hamiltonian (9) can be absorbed into the definition of the spatial profile:

$$\begin{aligned} \hat{H}_D(\tau_D) &= \lambda \chi(\tau_D) (e^{i\Omega_D \tau_D} \hat{\sigma}_D^+ + e^{-i\Omega_D \tau_D} \hat{\sigma}_D^-) \\ &\otimes \int \frac{d^n \mathbf{k}}{\sqrt{2|\mathbf{k}|}} \Pi\left(\frac{|\mathbf{k}|}{2\Lambda}\right) \\ &\times (\tilde{F}_D^*(\mathbf{k}) e^{i[|\mathbf{k}|t(\tau_D) - \mathbf{k} \cdot \mathbf{x}_D(\tau_D)]} \hat{a}_{\mathbf{k}}^\dagger + \text{H.c.}) \\ &= \lambda \chi(\tau_D) (e^{i\Omega_D \tau_D} \hat{\sigma}_D^+ + e^{-i\Omega_D \tau_D} \hat{\sigma}_D^-) \\ &\otimes \int \frac{d^n \mathbf{k}}{\sqrt{2|\mathbf{k}|}} (\tilde{G}_D^*(\mathbf{k}) e^{i[|\mathbf{k}|t(\tau_D) - \mathbf{k} \cdot \mathbf{x}(\tau_D)]} \hat{a}_{\mathbf{k}}^\dagger + \text{H.c.}), \end{aligned} \quad (11)$$

where the new spatial profile is

$$G(\mathbf{x}) = F(\mathbf{x}) * \frac{1}{(2\pi)^{n/2}} \int d^n \mathbf{k} \Pi\left(\frac{|\mathbf{k}|}{2\Lambda}\right) e^{-i\mathbf{k} \cdot \mathbf{x}} \quad (12)$$

and $\Pi(k)$ is the rectangle function

$$\Pi(k) := \begin{cases} 0, & |k| > \frac{1}{2}, \\ 1/2, & |k| = \frac{1}{2}, \\ 1, & |k| < \frac{1}{2}. \end{cases} \quad (13)$$

In three dimensions, its inverse Fourier transform is

$$\mathcal{F}^{-1}\left[\Pi\left(\frac{|\mathbf{k}|}{2\Lambda}\right)\right] = \sqrt{\frac{2}{\pi}} \Lambda^2 \frac{j_1(\Lambda|\mathbf{x}|)}{|\mathbf{x}|}, \quad (14)$$

where $j_1(r)$ is the order-1 spherical Bessel function of the first kind.

By writing the interaction Hamiltonian in this way, it is made clear that an alternative interpretation of the hard bandlimit $|\mathbf{k}| < \Lambda$ is the pair of detectors that interact with a nonbandlimited ($\Lambda \rightarrow \infty$) massless scalar field have the

spatial profile given by Eq. (14). Since the rectangle function is compact in k space, the modified spatial profile is not compact—even if the initial profile $F(\mathbf{x})$ was compact or even pointlike.

Substituting the interaction Hamiltonian given by Eq. (9) into Eq. (6) and expanding to the lowest order in the interaction strength gives the reduced density matrix of the two detectors:

$$\hat{\rho}_{AB} = \begin{pmatrix} 1 - P_A - P_B & 0 & 0 & X^* \\ 0 & P_B & C^* & 0 \\ 0 & C & P_A & 0 \\ X & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4), \quad (15)$$

where

$$P_D := \frac{\lambda^2}{(2\pi)^n} \int \frac{d^n \mathbf{k}}{2|\mathbf{k}|} \left| \int_{-\infty}^{\infty} d\tau_D \tilde{F}_D^*(\mathbf{k}) \chi_D(\tau_D) e^{i[\Omega_D \tau_D + |\mathbf{k}|t(\tau_D) - \mathbf{k} \cdot \mathbf{x}_D(\tau_D)]} \right|^2, \quad (16)$$

$$C := \frac{\lambda^2}{(2\pi)^n} \int \frac{d^n \mathbf{k}}{2|\mathbf{k}|} \left[\left(\int_{-\infty}^{\infty} d\tau_A \tilde{F}_A^*(\mathbf{k}) \chi_A(\tau_A) e^{i[\Omega_A \tau_A + |\mathbf{k}|t(\tau_A) - \mathbf{k} \cdot \mathbf{x}_A(\tau_A)]} \right) \times \left(\int_{-\infty}^{\infty} d\tau_B \tilde{F}_B^*(\mathbf{k}) \chi_B(\tau_B) e^{i[\Omega_B \tau_B + |\mathbf{k}|t(\tau_B) - \mathbf{k} \cdot \mathbf{x}_B(\tau_B)]} \right)^* \right], \quad (17)$$

$$X := \frac{\lambda^2}{(2\pi)^n} \int \frac{d^n \mathbf{k}}{2|\mathbf{k}|} \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' e^{-i|\mathbf{k}|(t-t')} \times \left[\left(\frac{d\tau_A}{dt} \frac{d\tau_B}{dt'} \tilde{F}_A(\mathbf{k}) \tilde{F}_B^*(\mathbf{k}) \chi_A(\tau_A(t)) \chi_B(\tau_B(t')) e^{i[\Omega_A \tau_A(t) + \Omega_B \tau_B(t')] e^{i\mathbf{k} \cdot [\mathbf{x}_A(t) - \mathbf{x}_B(t')]} \right) + \left(\frac{d\tau_B}{dt} \frac{d\tau_A}{dt'} \tilde{F}_B(\mathbf{k}) \tilde{F}_A^*(\mathbf{k}) \chi_B(\tau_B(t)) \chi_A(\tau_A(t')) e^{i[\Omega_B \tau_B(t) + \Omega_A \tau_A(t')] e^{i\mathbf{k} \cdot [\mathbf{x}_B(t) - \mathbf{x}_A(t')]} \right) \right], \quad (18)$$

and $D, D' \in \{A, B\}$. Tracing out either detector B or detector A gives the reduced density matrix for the remaining detector:

$$\begin{aligned} \hat{\rho}_A &= \text{Tr}_B[\hat{\rho}_{AB}] = \begin{pmatrix} 1 - P_A & 0 \\ 0 & P_A \end{pmatrix}, \\ \hat{\rho}_B &= \text{Tr}_A[\hat{\rho}_{AB}] = \begin{pmatrix} 1 - P_B & 0 \\ 0 & P_B \end{pmatrix}, \end{aligned} \quad (19)$$

respectively, so we can interpret P_D as the transition probability of detector D . To leading order, the term X encodes the nonlocal correlations between the two detectors, and the term C encodes the total correlations [30].

In addition to quantifying effect of the bandlimit on the transition probability of the detectors, we also wish to quantify its effect on the amount of entanglement the pair extracts from the vacuum. We will use negativity [42] as our measure of entanglement. The negativity is an entanglement monotone and is defined as

$$N(\hat{\rho}_{AB}) := \frac{\|\hat{\rho}_{AB}^{\Gamma_A}\| - 1}{2} = \sum_{\lambda_i < 0} |\lambda_i|, \quad (20)$$

where Γ_A denotes the partial transpose with respect to A , $\|\cdot\|$ is the trace norm and the sum is over the negative eigenvalues of $\hat{\rho}_{AB}^{\Gamma_A}$. For a two-qubit system like ours (and also for a qubit-qutrit system), the negativity is zero if and only if the state is separable.

For a density matrix of the form of Eq. (7), the eigenvalues of $\hat{\rho}_{AB}^{\Gamma_A}$ are

$$\begin{aligned} \lambda_1 &= \frac{1}{2} \left(\rho_{22} + \rho_{33} - \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{14}|^2} \right) \\ &= \frac{1}{2} \left(P_A + P_B - \sqrt{(P_A - P_B)^2 + 4|X|^2} \right) + \mathcal{O}(\lambda^4), \\ \lambda_2 &= \frac{1}{2} \left(\rho_{22} + \rho_{33} + \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{14}|^2} \right) \\ &= \frac{1}{2} \left(P_A + P_B + \sqrt{(P_A - P_B)^2 + 4|X|^2} \right) + \mathcal{O}(\lambda^4), \\ \lambda_3 &= \frac{1}{2} \left(\rho_{11} + \rho_{44} - \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{23}|^2} \right) \\ &= 0 + \mathcal{O}(\lambda^4), \\ \lambda_4 &= \frac{1}{2} \left(\rho_{11} + \rho_{44} + \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{23}|^2} \right) \\ &= 1 - P_A - P_B + \mathcal{O}(\lambda^4). \end{aligned} \quad (21)$$

The only eigenvalue that can be zero to lowest order in the coupling strength is λ_1 , so we calculate the negativity of the reduced density matrix [Eq. (15)] as

$$\mathcal{N}(\hat{\rho}_{AB}) = \max \left[0, -\frac{1}{2} \left(P_A + P_B - \sqrt{(P_A - P_B)^2 + 4|X|^2} \right) \right], \quad (22)$$

which reduces to

$$\mathcal{N}(\hat{\rho}_{AB}) = \max[0, |X| - P_D] \quad (23)$$

when the transition probabilities of the two detectors are equal ($P_A = P_B = P_D$).

III. OUR SETUP

We will now consider (3 + 1)-dimensional Minkowski space and take the two UDW detectors to be separated by a distance S at rest in a common reference frame. We will consider the detectors to be pointlike:

$$F_D(\mathbf{x} - \mathbf{x}_D) = \delta^{(3)}(\mathbf{x} - \mathbf{x}_D) \quad (24)$$

and have identical energy gaps $\Omega_A = \Omega_B = \Omega$, and we will take the switching function to be a Gaussian with characteristic width σ :

$$\chi_D(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right). \quad (25)$$

With these assumptions the matrix elements in Eqs. (16) and (18) reduce to

$$P_{A,\Lambda} = P_{B,\Lambda} = P_{D,\Lambda} = \frac{\lambda^2}{4\pi} \left\{ e^{-\sigma^2\Omega^2} - e^{-\sigma^2(\Omega+\Lambda)^2} + \sqrt{\pi}\sigma\Omega[\text{erf}(\sigma\Omega) - \text{erf}(\sigma(\Omega+\Lambda))]\right\}, \quad (26)$$

$$X_\Lambda = \frac{\lambda^2\sigma}{4\sqrt{\pi}S} e^{-\sigma^2\Omega^2} e^{-S^2/(2\sigma^2)} \times \left[\text{erfi}\left(\frac{S}{2\sigma}\right) - \text{Re}\left(\text{erfi}\left(\frac{S}{2\sigma} + i\sigma\Lambda\right)\right) \right] + i \frac{\lambda^2\sigma^2}{2\pi S} e^{-\sigma^2\Omega^2} \int_0^\Lambda dk e^{-\sigma^2 k^2} \text{erfi}(\sigma k) \sin(Sk), \quad (27)$$

where the subscript Λ is used as a reminder that the scalar field is bandlimited, $\text{erfi}(x) := -i \text{erf}(ix)$, and erf is the error function.

Using the large- x expansion of

$$\text{erfi}(x) = \frac{2}{\sqrt{\pi}} \frac{e^{x^2}}{x} + \mathcal{O}\left(\frac{1}{x^3}\right), \quad (28)$$

the imaginary part of X_Λ can be approximated as

$$\begin{aligned} \text{Im}(X_\Lambda) &\approx \frac{\lambda^2\sigma^2}{2\pi S} e^{-\sigma^2\Omega^2} \left(\int_0^\infty dk e^{-\sigma^2 k^2} \text{erfi}(\sigma k) \sin(Sk) \right. \\ &\quad \left. - \frac{1}{\sqrt{\pi}} \int_\Lambda^\infty dk \frac{\sin(Sk)}{\sigma k} \right) \\ &= \frac{\lambda^2\sigma^2}{4\pi} \frac{e^{-\sigma^2\Omega^2}}{|S|} \left(e^{-S^2/(2\sigma^2)} - 1 + \frac{2}{\pi} \text{Si}(|S|\Lambda) \right), \end{aligned} \quad (29)$$

where $\text{Si}(x)$ is the sine integral

$$\text{Si}(x) := \int_0^x dt \frac{\sin t}{t}, \quad (30)$$

provided $\Lambda\sigma \gg 1$. Numeric testing has shown good agreement when $\Lambda\sigma > 5$.

IV. TRANSITION PROBABILITY

First we consider the dependence of the transition probability of a single detector on its energy gap, Ω , and the bandlimit of the quantum field, Λ , which is shown in Fig. 1. When the detector is initially in the ground state, the transition probability falls off exponentially with increasing energy gap. Additionally, as shown in the inset of Fig. 1(a), a small bandlimit lowers the transition probability, and this

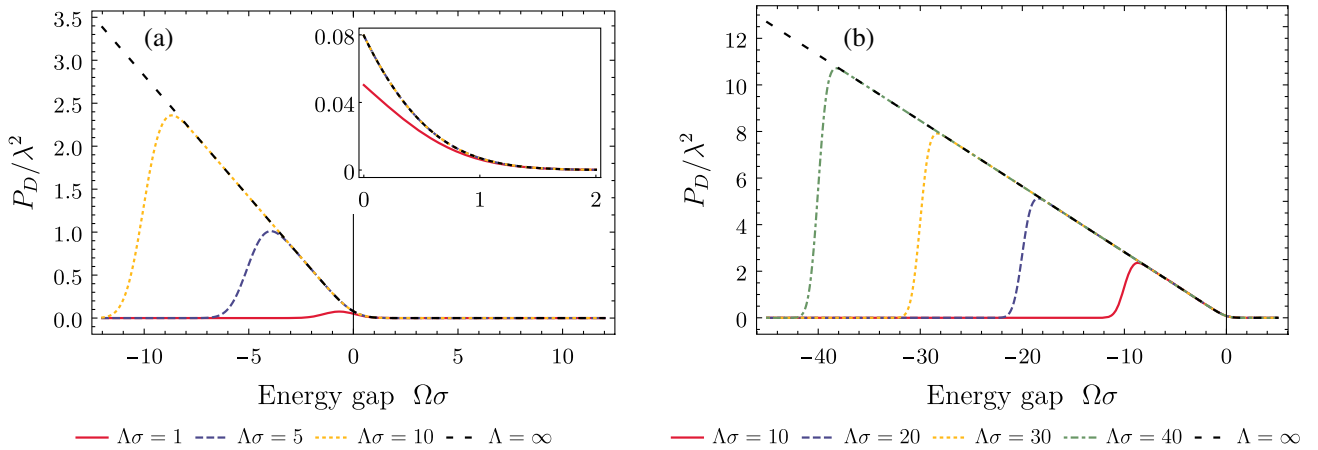


FIG. 1. The transition probability P_D of a single pointlike detector as a function of its energy gap Ω at various values of the bandlimit Λ of the field. A negative energy gap corresponds to an initially excited detector. When the energy gap of the detector is larger than the bandlimit, the transition probability of an initially excited detector rapidly falls off. When there is no bandlimit, the transition probability grows with the energy gap.

effect is strongest for small values of the energy gap due to the exponential suppression. We also find that for values of the bandlimit larger than $\Lambda\sigma \approx 2$, the transition probability (from the ground to the excited state) is nearly indistinguishable from the nonbandlimited case regardless of the energy gap.

In order to better understand this behavior, it is useful to write the transition probability in terms of the nonbandlimited case as

$$P_{D,\Lambda}(\Omega) = P_{D,\infty}(\Omega) - P_{D,\infty}(\Omega + \Lambda) - \frac{\lambda^2 \sigma \Lambda}{4\sqrt{\pi}} \text{erfc}(\sigma(\Omega + \Lambda)), \quad (31)$$

where $\text{erfc}(x) := 1 - \text{erf}(x)$. Since the bandlimit enters the expression like a modified energy gap, it is easy to see that for large values, both $P_{D,\infty}(\Omega + \Lambda)$ and $\text{erfc}(\sigma(\Omega + \Lambda))$ are exponentially suppressed and $P_{D,\Lambda}(\Omega) \approx P_{D,\infty}(\Omega)$.

If the detector was initially excited, which is mathematically equivalent to taking $\Omega < 0$, the transition probability increases linearly with increasing energy gap up to a maximum value after which it exponentially falls off, which is seen in the negative- Ω regions of Fig. 1. As the bandlimit is increased, we find that the maximum value of deexcitation probability occurs at larger (more-negative) values of energy gap and that the falloff after is sharper.

We define Ω_{crit} to be the value of energy gap associated with the maximum deexcitation probability. Since the transition probability falls off so rapidly when $|\Omega| < |\Omega_{\text{crit}}|$, this energy gap can be interpreted as the largest energy gap for which excited detectors are able to decay into the bandlimited field (any larger and the deexcitation probability is exponentially suppressed). This is not surprising, since bandlimiting the field also puts limits on the frequency of modes that exist in the theory, and since we have only

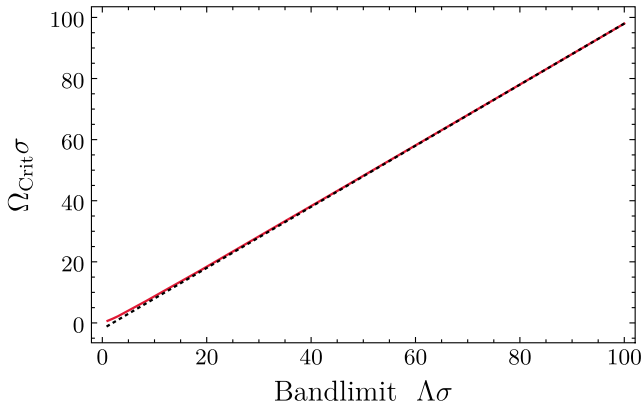


FIG. 2. The energy gap of the detector corresponding to the maximum deexcitation probability as a function of the bandlimit (red solid line). The deexcitation probability rapidly falls off for detectors with energy gaps greater than this critical value. The black dotted line corresponds to a line of best fit $\Omega_{\text{crit}} = \Lambda - 2/\sigma$.

calculated the reduced density matrix to lowest order in the coupling constant, we only consider single-mode deexcitations. In other words there is no room in the field for the detector to decay into.

The dependence of the Ω_{crit} on the bandlimit of the scalar field is plotted in Fig. 2, where the detector is assumed to be initially excited. Ω_{crit} grows with nearly the same rate as the bandlimit of the field. The most discrepancy occurs at small values of $\Omega\sigma$ where there is a high degree of quantum uncertainty in the effective energy gap of the detector. At large values of Ω_{crit} and Λ , the relationship approaches the linear relationship $\Omega_{\text{crit}} = \Lambda - 2/\sigma$.

V. NEGATIVITY

Now we consider the entanglement between two identical detectors A and B , which are initialized in the ground state and separated by a distance S . We first note that the qualitative dependence of negativity on the energy gap of the detectors does not change significantly as the bandlimit is changed. The actual value of the negativity at given value of Ω is much lower than the nonbandlimited case for a small enough value of Λ , and as Λ increases, the negativity oscillates around the nonbandlimited value. For this reason we only consider detectors with an energy gap of $\Omega\sigma = 0.01$ throughout this section.

In Fig. 3 we plot the dependence of the entanglement harvested on the bandlimit and the separation of the detectors. There are striking oscillations in the contours when considering a fixed value of separation, which we plot in Fig. 4. The frequency of these oscillations decreases

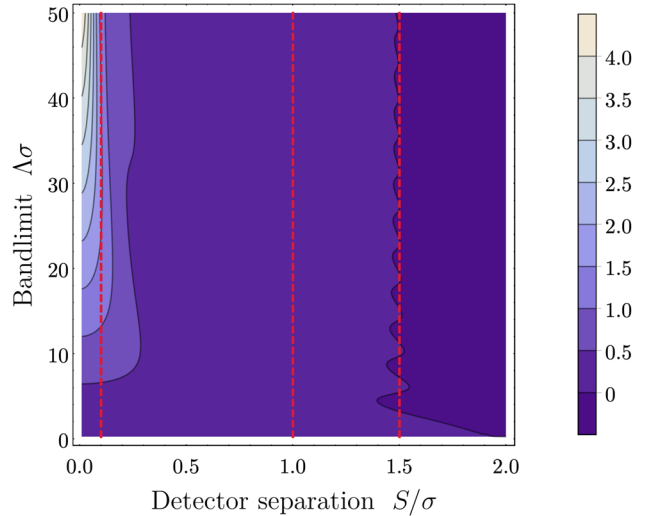


FIG. 3. The negativity of the reduced density matrix for the detectors' internal states, $\mathcal{N}(\hat{\rho}_{AB})/\lambda^2$, as a function of detector separation S and the bandlimit Λ of the field when the detectors have an energy gap of $\Omega\sigma = 0.01$. The darkest region corresponds to a value of zero negativity when the two detectors remain in a separable state. The red dashed line mark the constant separation slices shown in Fig. 4.

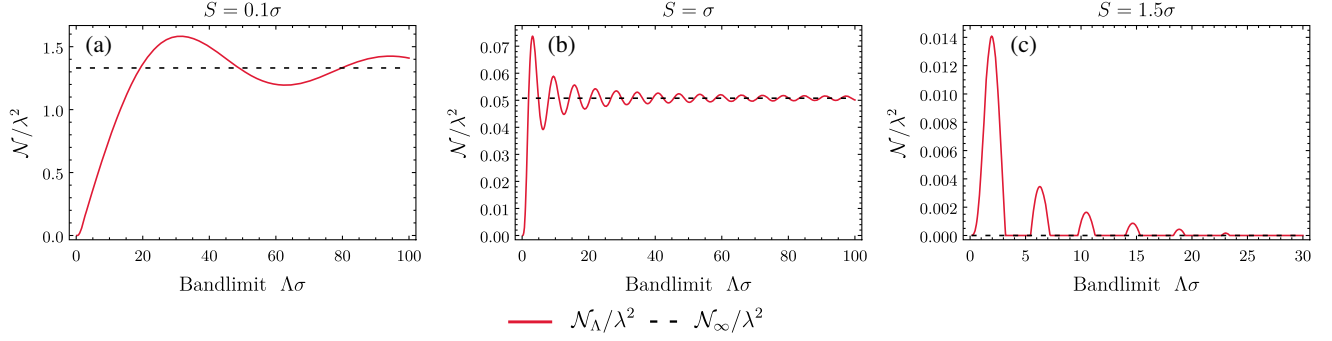


FIG. 4. Red solid line: The negativity of the reduced density matrix, $\mathcal{N}(\hat{\rho}_{AB})/\lambda^2$, as a function of the bandlimit Λ . Black dashed line: The value of the negativity when there is no bandlimit ($\Lambda \rightarrow \infty$). The detector energy gap is set to $\Omega\sigma = 0.01$.

when the detector separation is decreased. We also note the negativity decreases with increasing detector separation at fixed values of the bandlimit.

We take a closer look at the oscillations in negativity with respect to the bandlimit at fixed values of detector separation in Fig. 4. For values of separation where entanglement harvesting is possible in the nonbandlimited case [Figs. 4(a) and 4(b)], we find the negativity increases with increasing values of the bandlimit up to some maximum value. As the bandlimit increases further, the negativity will exhibit damped oscillations around the nonbandlimited value. The frequency of these oscillations increases as the detector separation is increased.

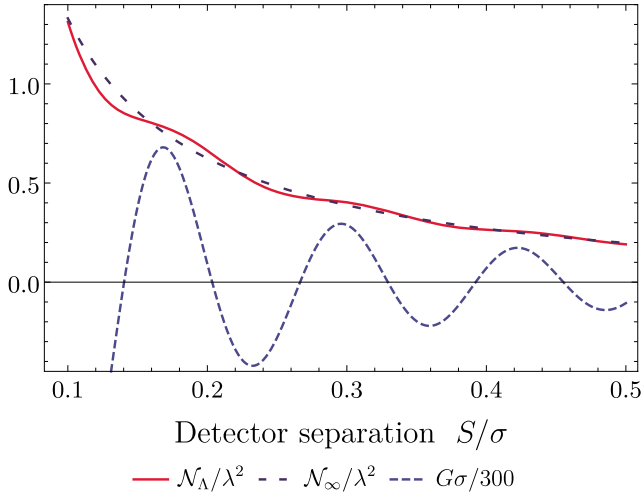


FIG. 5. Red solid line: The negativity of the reduced density matrix, $\mathcal{N}(\hat{\rho}_{AB})/\lambda^2$, as a function of the detector separation S when the bandlimit is set to $\Lambda\sigma = 50$. Black dashed line: The negativity as a function of detector separation when there is no bandlimit ($\Lambda \rightarrow \infty$). Blue dashed line: A plot of the effective spatial profile $G = \sqrt{\frac{2}{\pi}} \Lambda^2 \frac{j_1(\Lambda S)}{S}$ for $\Lambda\sigma = 50$ scaled by a factor of $1/300$ for ease of plotting. The energy gap is set to $\Omega\sigma = 0.01$. The overall decay of the bandlimited plot matches the decay of the nonbandlimited one, and oscillations in the bandlimited plot match up with the oscillations in the effective spatial profile.

These oscillations can also be present when separation of the detectors is too large for entanglement harvesting to be possible in the nonbandlimited case, which we show in Fig. 4(c). In this case, the damped oscillations that are still present in the function $|X| - P_D$ are cut off for negative values but remain when $|X| > P_D$. This results in values of the bandlimit where entanglement harvesting is possible, even if it not possible when there is no bandlimit.

In Fig. 5, we plot the negativity as a function of detector separation at a fixed bandlimit of $\Lambda = 50\sigma$ and $\Lambda \rightarrow \infty$. Similar to the plots in Fig. 4, we find the negativity exhibits damped oscillations around the nonbandlimited curve. Interestingly, we also find that these oscillations match up with the oscillations on the effective spatial profile [Eq. (14)], which implies that the oscillations in negativity may be a result of increased or decreased overlap of the detectors' effective spatial profiles.

Since neither the transition probability of the detectors nor the real part of the matrix element X_Λ depend significantly on the bandlimit when it is large, the oscillatory behavior in both Figs. 4 and 5 must be due the imaginary part of X and can be approximated as resulting from the $\text{Si}(S\Lambda)$ term in Eq. (29). Therefore, the oscillations with respect to Λ in the imaginary part of X can be shown to be upper bounded by

$$\frac{\lambda^2 \sigma}{2\pi^{3/2}} \frac{e^{-\sigma^2 \Omega^2}}{S^2 \Lambda} + \text{Im}(X_\infty). \quad (32)$$

Using this envelope, it is possible to tune the separation of pair of detectors of fixed energy gap so that the entanglement between them is zero for all values of the bandlimit greater than some chosen threshold and nonzero for some values of the bandlimit less than that threshold. In other words, for a chosen value of $\Lambda_{\text{threshold}}$ and a given value of Ω (or S), find a value of S (or Ω) so that

$$P_{D, \Lambda_{\text{threshold}}} = \left| \text{Re}(X_{\Lambda_{\text{threshold}}}) + i \left(\frac{\lambda^2 \sigma}{2\pi^{3/2}} \frac{e^{-\sigma^2 \Omega^2}}{S^2 \Lambda_{\text{threshold}}} + \text{Im}(X_\infty) \right) \right| \quad (33)$$

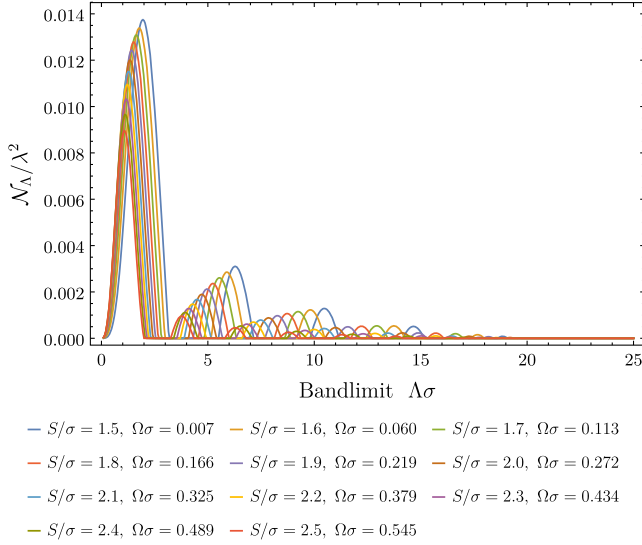


FIG. 6. An array of detectors with properly tuned energy gaps are set at different separation distances so that Eq. (33) is satisfied at $\Lambda_{\text{threshold}}\sigma = 20$. If the negativity of any one of the pairs of detectors (each corresponding to a different experiment) has nonzero negativity, then $\Lambda < \Lambda_{\text{threshold}}$.

is satisfied. It is important to note that if only one pair is used, then there will be regions of zero negativity at some values of $\Lambda < \Lambda_{\text{threshold}}$. For example, consider a value of $\Lambda\sigma = 4$ in Fig. 4(c). However, it is possible to build up an array of such pairs of detectors, each at different values of separation, so that the frequency of oscillations in negativity is different. This can be used to fill in the gaps of each individual curve and in principle test if a quantum field has a bandlimit $\Lambda \leq \Lambda_{\text{threshold}}$. An example of such an array is shown in Fig. 6.

VI. GOING BEYOND PERTURBATIVE: δ SWITCHING

An alternative approach to using perturbation theory to compute the reduced density matrix of the two detectors is to implement δ switching:

$$\chi_D(t) = \sqrt{2\pi\sigma}\delta(t - T_D), \quad (34)$$

which can be related to the switching used in the previous section by defining the Dirac delta as the weak limit of a normalized Gaussian:

$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{e^{-x^2/(2\sigma^2)}}{\sqrt{2\pi\sigma}}. \quad (35)$$

This choice of switching allows for the calculation of the time-evolution operator [Eq. (2)] exactly. Following the work of Simidzija and Martín-Martínez [39], we define detector A to be the one that switches first ($T_A \leq T_B$) and calculate the time-evolution operator as

$$\begin{aligned} \hat{U}_\delta = & [\mathbb{1}_A \otimes \mathbb{1}_B \otimes \cosh(\hat{Y}_B) + \mathbb{1}_A \otimes \hat{\mu}_B(T_B) \otimes \sinh(\hat{Y}_B)] \\ & \times [\mathbb{1}_A \otimes \mathbb{1}_B \otimes \cosh(\hat{Y}_A) + \hat{\mu}_A(T_A) \otimes \mathbb{1}_B \otimes \sinh(\hat{Y}_A)], \end{aligned} \quad (36)$$

where

$$\hat{\mu}_D(t) = e^{i\Omega_D\tau_D(t)}\hat{\sigma}_D^+ + e^{-i\Omega_D\tau_D(t)}\hat{\sigma}_D^- \quad (37)$$

and

$$\hat{Y}_D := -i\lambda\sqrt{2\pi\sigma}\frac{d\tau_D}{dt}\bigg|_{t=T_D} \int d^n\mathbf{x} F_D(\mathbf{x} - \mathbf{x}_D) \hat{\phi}(\mathbf{x}, T_D). \quad (38)$$

The detector-field system is initially in the state $|\Psi_0\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_\phi$ and the final state of the two-detector subsystem is

$$\hat{\rho}_{AB} = \text{Tr}_\phi[\hat{U}_\delta|\Psi_0\rangle\langle\Psi_0|\hat{U}_\delta^\dagger] = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix}, \quad (39)$$

where

$$\rho_{11} = \frac{1}{4}[1 + f_A + f_B \cos(2\theta) + f_A f_B \cosh(\omega)], \quad (40a)$$

$$\rho_{14} = \frac{1}{4}e^{-i(\Omega_A T_A + \Omega_B T_B)} f_B [i \sin(2\theta) + f_A \sinh(\omega)], \quad (40b)$$

$$\rho_{22} = \frac{1}{4}[1 + f_A - f_B \cos(2\theta) - f_A f_B \cosh(\omega)], \quad (40c)$$

$$\rho_{23} = -\frac{1}{4}e^{-i(\Omega_A T_A - \Omega_B T_B)} f_B [i \sin(2\theta) + f_A \sinh(\omega)], \quad (40d)$$

$$\rho_{33} = \frac{1}{4}[1 - f_A + f_B \cos(2\theta) - f_A f_B \cosh(\omega)], \quad (40e)$$

$$\rho_{44} = \frac{1}{4}[1 - f_A - f_B \cos(2\theta) + f_A f_B \cosh(\omega)], \quad (40f)$$

and we have defined

$$f_D := \exp\left(-2\pi\lambda^2\sigma^2 \int_{|\mathbf{k}| < \Lambda} \frac{d^n\mathbf{k}}{|\mathbf{k}|} |\tilde{F}_D(\mathbf{k})|^2\right), \quad (41a)$$

$$\begin{aligned} \theta := & -i[\hat{Y}_A, \hat{Y}_B] = -i\pi\lambda^2\sigma^2 \int_{|\mathbf{k}| < \Lambda} \frac{d^n\mathbf{k}}{|\mathbf{k}|} \\ & \times [\tilde{F}_A^*(\mathbf{k})\tilde{F}_B(\mathbf{k})e^{i|\mathbf{k}|(T_A - T_B)}e^{-i\mathbf{k}\cdot(\mathbf{x}_A - \mathbf{x}_B)} - \text{H.c.}], \end{aligned} \quad (41b)$$

$$\omega := 2\pi\lambda^2\sigma^2 \int_{|\mathbf{k}| < \Lambda} \frac{d^n \mathbf{k}}{|\mathbf{k}|} \times [\tilde{F}_A^*(\mathbf{k})\tilde{F}_B(\mathbf{k})e^{i|\mathbf{k}|(T_A-T_B)}e^{-i\mathbf{k}\cdot(\mathbf{x}_A-\mathbf{x}_B)} + \text{H.c.}], \quad (41c)$$

In order to compare to our perturbative results, we again consider $(3+1)$ -dimensional Minkowski space and take the UDW detectors to be identical ($\Omega_A = \Omega_B = \Omega$) and at rest in a common reference frame and separated by a

distance S . To avoid divergences in θ and ω in the nonbandlimited limit, we set the spatial profile of the detectors to be Gaussian:

$$F_D(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}a^3} \exp\left(-\frac{|\mathbf{x}|^2}{2a^2}\right) \quad (42)$$

rather than pointlike. With these simplifications, the matrix-element functions become

$$f_A = f_B = f = \exp\left(-\frac{\lambda^2\sigma^2}{2\pi a^2}(1 - e^{-a^2\Lambda^2})\right), \quad (43a)$$

$$\theta = \frac{\lambda^2\sigma^2}{4\sqrt{\pi}aS} \sum_{j \in \{0,1\}} (-1)^j \exp\left(-\frac{(S+(-1)^j T)^2}{4a^2}\right) \text{Im}\left[\text{erfi}\left(\frac{S+(-1)^j T}{2a} + ia\Lambda\right)\right], \quad (43b)$$

$$\omega = -\frac{\lambda^2\sigma^2}{2\sqrt{\pi}aS} \sum_{j \in \{0,1\}} \exp\left(-\frac{(S+(-1)^j T)^2}{4a^2}\right) \left[\text{erfi}\left(\frac{S+(-1)^j T}{2a}\right) - \text{Re}\left(\text{erfi}\left(\frac{S+(-1)^j T}{2a} + ia\Lambda\right)\right)\right], \quad (43c)$$

where $T := T_B - T_A$.

Finally, we calculate the transition probability of the detectors to be

$$P_{A,\Lambda} = \frac{1}{2}(1-f), \quad P_{B,\Lambda} = \frac{1}{2}(1-f\cos(2\theta)), \quad (44)$$

which are only equal when the field commutator vanishes. Since the time-evolution operator in Eq. (36) is calculated exactly rather than through a Dyson series expansion, this model allows for the backreaction of the detectors on the field. If detector A is in causal contact with detector B , then B will be interacting with the field that was modified by A . It is also worth noting that the transition probabilities are independent of the energy gap of the detector, and so they can be interpreted as both excitation and deexcitation probabilities.

A. One detector

First, we consider the delta-switching interaction of a single detector with a bandlimited scalar field. Since the bandlimit acts as an effective spatial profile, we can safely take the $a \rightarrow 0$ limit of Eq. (42) and again consider a pointlike detector. In this limit, the transition probability is

$$\lim_{a \rightarrow 0} P_{A,\Lambda} = \frac{1}{2} \left[1 - \exp\left(-\frac{\lambda^2\sigma^2\Lambda^2}{2\pi}\right) \right]. \quad (45)$$

Due to the exponential decay of f with increasing Λ , the transition probability of a single pointlike detector interacting with a nonbandlimited field becomes

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{2} \left[1 - \exp\left(-\frac{\lambda^2\sigma^2\Lambda^2}{2\pi}\right) \right] = \frac{1}{2}. \quad (46)$$

Additionally, this limit is well defined since taking the limits in the other order (first $\Lambda \rightarrow \infty$ then $a \rightarrow 0$) produces the same result.

One does not need to take limits of Eq. (42) to calculate the transition probability of a pointlike ($\tilde{F}(\mathbf{k}) = (2\pi)^{-n/2}$) detector in a nonbandlimited field. Instead, one can directly consider Eq. (41a), which leads to

$$P_{A,\infty} = \frac{1}{2} \left[1 - \exp\left(-\frac{2\pi\lambda^2\sigma^2}{(2\pi)^n} \int \frac{d^n \mathbf{k}}{|\mathbf{k}|}\right) \right] = \frac{1}{2} \quad (47)$$

provided $n \geq 2$. In other words, if a two-level detector is initialized in the ground (or excited) state and is coupled to the field at a single point in space and time, the result would be the completely mixed state.

This is still consistent with the conventional idea that the transition probability of a pointlike detector with delta switching will diverge [26] if one considers the perturbative calculation, since to lowest order in the coupling strength

$$P_{A,\infty} \approx \frac{2\pi\lambda^2\sigma^2}{(2\pi)^n} \int \frac{d^n \mathbf{k}}{|\mathbf{k}|} \rightarrow \infty. \quad (48)$$

This divergence is solely a mathematical artifact of using perturbation theory, however. The physical behavior in this case (interaction at a single spacetime point) is in fact well defined, finite, and given by Eq. (47).

In Fig. 7, we plot the transition probability of a single detector—with various widths of its Gaussian spatial

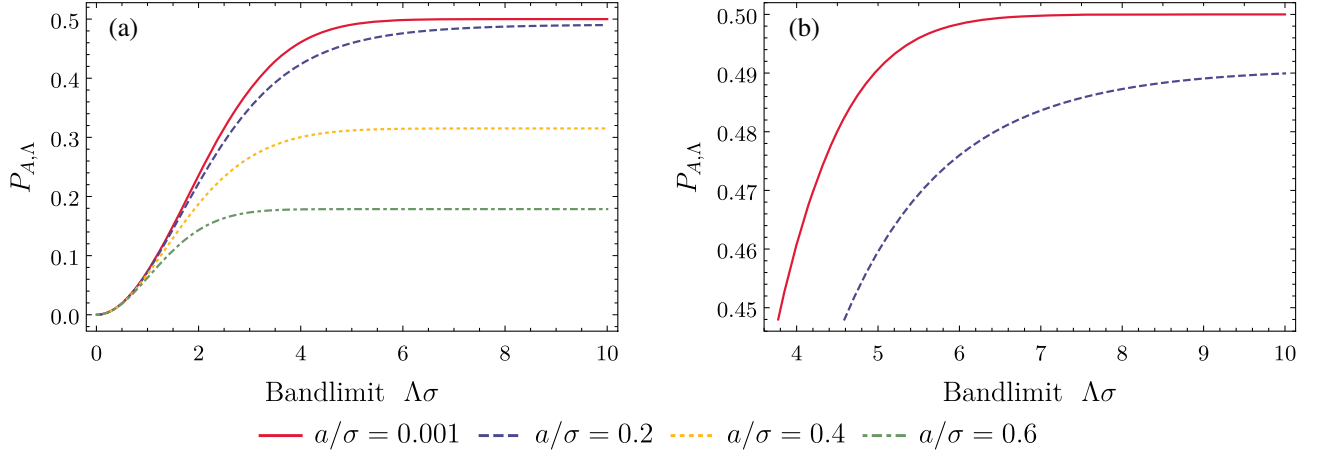


FIG. 7. The transition probability of a single detector interacting with a bandlimited scalar field with delta switching as a function of the bandlimit Λ for various widths of the spatial profile. As the bandlimit increases, the transition probability quickly asymptotes to the nonbandlimited ($\Lambda \rightarrow \infty$) value. The right-hand figure details the asymptotic behavior of the narrow ($a = 0.001\sigma$ and $a = 0.2\sigma$) detectors. The interaction strength is set to $\lambda = 1$.

profile, interacting via delta switching—versus the bandlimit of the scalar field. Since there is no dependence on the energy gap of the detector, this can be interpreted as either the probability of excitation or deexcitation. Regardless of the width of the detector, the transition probability increases with increasing value of the bandlimit and asymptotes to the nonbandlimited value.

We make comparisons to Sec. IV by considering the small- σ limit where the unitless energy gap $\Omega\sigma$ will also be close to zero. In this regime, the dependence of the transition probability on the bandlimit can be best seen in the inset of Fig. 1(a). Here, the same behavior is seen: When the bandlimit is small, the transition probability of the detector is increased when the bandlimit is increased. But for larger values of the bandlimit the transition probability is indistinguishable from the nonbandlimited ($\Lambda \rightarrow \infty$) case.

The width of the detector determines how quickly the transition probability approaches its corresponding nonbandlimited value, which can be seen in Fig. 7. When the spatial profile of the detector is relatively wide and the width is increased, the transition probability flattens out and approaches its asymptotic value at lower values of the bandlimit. However, when the spatial profile is very narrow, as seen in Fig. 7(b), the transition probability flattens for the detector with a width of $a = 0.001\sigma$ at a smaller value of the bandlimit than the detector with a width of $a = 0.2\sigma$.

This observation is made more precise in Fig. 8, where we plot the value of the bandlimit, denoted $\Lambda_{\max}$, for which the absolute difference between the transition probability and its corresponding nonbandlimited value is equal to a chosen tolerance as a function of the width of the spatial profile. This quantity was chosen for two reasons. First, it provides a concrete measure of determining that the transition probability is “close” to its asymptotic value, and

second, from an operational perspective the tolerance can be interpreted as the resolution of some experiment where UDW detectors are used to determine the bandlimit of the field.

We find that, for all three values of the chosen tolerance, the behavior of the absolute difference is the same. When the spatial profile is very narrow, increasing the width of the profile actually increases the value of $\Lambda_{\max}$ up to a maximum value, which occurs at $a \approx 0.2\sigma$. As the width is increased further, the value of $\Lambda_{\max}$ rapidly decreases.

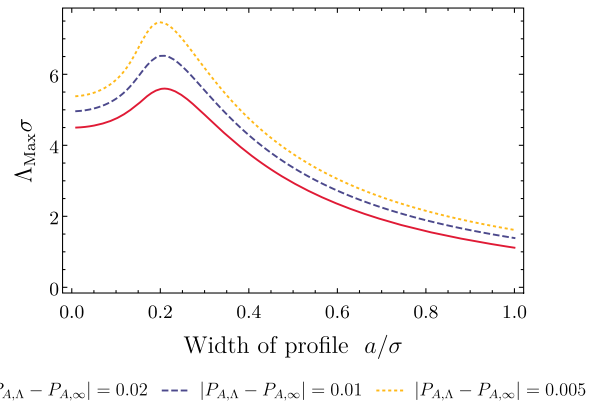


FIG. 8. The value of the maximum bandlimit $\Lambda_{\max}$ such that the difference between the transition probability in the bandlimited field and nonbandlimited field ($|P_{A,\Lambda_{\max}} - P_{A,\infty}|$) is equal to a specified (arbitrary) tolerance (0.02, 0.01, 0.005) as a function of the width of the spatial profile of the detector. When the bandlimit is larger than $\Lambda_{\max}$, the difference will be less than the tolerance. For all three values, a maximum occurs near at width of $a \approx 0.2\sigma$, indicating the transition probability of a detector with this width is more sensitive to the bandlimit than a wider or narrower detector. The interaction strength is set to $\lambda = 1$.

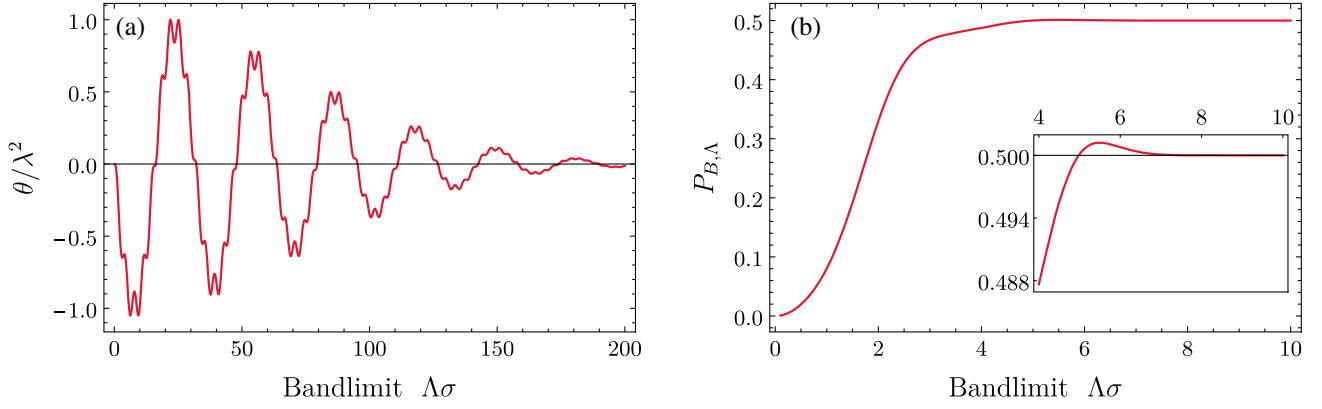


FIG. 9. A plot of (a) the parameter θ , which is proportional to the effective field commutator between detectors A and B , and (b) the transition probability of detector B as a function of the bandlimit Λ . The negative values in the parameter θ result in values of $P_B > \frac{1}{2}$. Although θ shows dependency on the bandlimit over a large range, we find that $P_{B,\Lambda}$ is only sensitive to the bandlimit when it is small ($\Lambda < 7\sigma$). The width of the spatial profile is set to $a = 0.01\sigma$, the detectors have a time delay of $T = \sigma$ and a separation of $S = 0.8\sigma$, and the interaction strength is set to $\lambda = 1$.

The region where increasing the width decreases $\Lambda_{\max}$ is easily explained by naively looking at Eq. (12). When the bandlimit is large enough such that $\frac{1}{\Lambda} \ll a$, the effective profile

$$G(\mathbf{x}) = \frac{e^{-a^2\Lambda^2/2}}{(2\pi)^{3/2}a^3} \left[e^{(a^4\Lambda^2 - |\mathbf{x}|^2)/(2a^2)} \text{Re} \left(\text{erf} \left(\frac{a^2\Lambda + i|\mathbf{x}|}{\sqrt{2}a} \right) \right) - \frac{\sqrt{2}a\Lambda}{\sqrt{\pi}} \text{sinc}(\Lambda|\mathbf{x}|) \right] \quad (49)$$

will be dominated by the spatial profile of the detector, and the transition probability will be nearly the same as the nonbandlimited case. However, when the width of the spatial profile is very small, the complicated relationship between the width and the bandlimit in the effective spatial profile leads to the increase in sensitivity with increasing width. The takeaway message is this: When comparing detectors with a Gaussian profile and delta switching, the detector most sensitive to the bandlimit is not the one with the narrowest profile, as one may naively think, but rather one with a width of approximately $a \approx 0.2\sigma$.

B. Two detectors

The no-go theorem described in [39,40] states that “a pair of UDW detectors with Dirac-delta switching functions and arbitrary spatial profiles and coupling strengths cannot harvest entanglement from a coherent state of a massless scalar field.” Therefore, the bandlimit—which is mathematically equivalent to a spatial profile—will not make a difference on the entanglement harvested (none).

One can still look at the transition probability of the second detector, detector B , which depends on the effective field commutator through the parameter θ . In Fig. 9(a), it is shown that θ , when plotted as a function of the bandlimit, exhibits damped oscillations with frequencies at $(S \pm T)$. By studying the parameter space, we have found that when

the spatial profile of the detector is made smaller, the oscillations decay more slowly, and in the limit of pointlike detectors, $a \rightarrow 0$, the oscillations do not decay. In this limit, θ reduces to

$$\lim_{a \rightarrow 0} \theta = \frac{\lambda^2 \sigma^2 \Lambda}{2\pi S} [\text{sinc}(\Lambda(T+S)) - \text{sinc}(\Lambda(T-S))], \quad (50)$$

which is simply the difference of the one-dimensional Fourier transforms of the rectangle function $\Pi(k/(2\Lambda))$ evaluated at $(T+S)$ and $(T-S)$.

The oscillations in θ are observable in the transition probability of the second detector when the bandlimit is small, and as shown in the inset of Fig. 9(b), these can result in a transition probability greater than $1/2$, the largest transition probability of a single detector. However, when the bandlimit is increased, the transition probability exponentially approaches its nonbandlimited quantity at the same rate as a single detector, and the oscillations resulting from θ are exponentially suppressed. If the bandlimit is small, then a second detector with instantaneous switching can be used to give some insight into the field commutator between the locations of the two detectors, but if the bandlimit is large, then a second detector will not provide any advantage.

VII. CONCLUSION

We have studied the entanglement harvesting protocol and response of two UDW detectors interacting with a conventionally bandlimited $(3+1)$ -dimensional scalar field, where modes with $|\mathbf{k}| \geq \Lambda$ do not exist. We find that the application of this cutoff is equivalent to modifying the spatial profile of the UDW detector by convolving it with the dimensionally appropriate Fourier transform of the rectangle function. This interpretation results in a nonlocal detector interacting with a nonbandlimited scalar field.

On the other hand, if the detectors were interpreted as local, then the degrees of freedom of the field are nonlocal. The two perspectives are entirely equivalent.

When a pointlike detector couples to the bandlimited field with Gaussian switching, we find to lowest order in the coupling strength that the probability of excitation is reduced compared to the nonbandlimited value. The effect only becomes pronounced, however, when the bandlimit is very small. We find, for all values of the bandlimit, when the detector is prepared in the excited state, the deexcitation probability increases linearly with increasing values of the energy gap of the detector, until the energy gap is slightly smaller than the bandlimit of the field. For larger energy gaps, the deexcitation probability exponentially falls off to nearly zero. The interpretation of this is that, to lowest order, the detector must emit a single phonon in order to decay, but if the energy gap is larger than the bandlimit, the emitted photon cannot propagate in the field.

We also note that it is possible for a pair of pointlike UDW detectors, coupling with Gaussian switching, to harvest entanglement from the vacuum of the bandlimited scalar field at larger separation distances than in the ordinary case of a field without a bandlimit. The enhancement results from oscillations in the imaginary part of the coherence term of the two detectors, which can be interpreted as resulting from the overlap of the nonlocal effective spatial profile of the detectors. By taking advantage of the enhancement in entanglement harvesting, one can in principle take an array of pairs of detectors, with specific energy gaps and separations, and put bounds on the bandlimit of a quantum field.

Finally, we use a Dirac-delta switching function to couple two detectors with Gaussian spatial profiles to the field at a single instance in time, which allows for nonperturbative solutions. Again, we find that the transition probability of the first detector is lower when the field is bandlimited, and the effect is much more apparent when the bandlimit is small. Surprisingly, the transition probability of a detector is most sensitive to the bandlimit when the detector's width is small—but not too small. We also find

that the transition probability of the second detector has a similar dependence on the bandlimit: Any changes are due to the effective field commutator between the two detectors.

Although it has a very small effect on the transition probability of the second detector, the value of the field commutator has significant dependence on the bandlimit, even when the latter is large. We expect that other setups that are highly dependent on the field commutator between two detectors will also be highly sensitive to the value of the bandlimit. One such application is “quantum collect calling” [43] where it is possible to signal using a massless scalar field such that no energy is transmitted from the sender to the receiver. Since this scheme significantly depends on the commutator of the field between the sender and receiver, it would be interesting further work to apply quantum collect calling to bandlimited QFT.

This work may also have applications in superconducting circuits where it was shown [44] that, in the ultrastrong coupling regime with nonadiabatic switching, it is in principle possible to gain knowledge about the ultraviolet cutoff behavior of the open transmission line by studying the probabilities of spontaneous vacuum excitation and spontaneous emission of a superconducting qubit.

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Entanglement harvesting with coherently delocalized matter

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We study entanglement harvesting for matter systems such as atoms, ions or molecules whose center of mass degrees of freedom are quantum delocalized and which couple to a relativistic quantum field. We employ a generalized Unruh-deWitt detector model for the light-matter interaction, and we investigate how the coherent spreading of the quantum center of mass wave function of two delocalized detector systems impacts their ability to become entangled with one another, via their respective interaction with a quantum field. For very massive detectors with initially highly localized centers of mass, we recover the results of entanglement harvesting for pointlike Unruh-deWitt detectors with classical center of mass degrees of freedom. We find that entanglement harvesting is Gaussian suppressed in the initial center of mass delocalization of the detectors. We further find that spatial smearing profiles, which are commonly employed to model the finite size of atoms due to their atomic orbitals, are not suited to model center of mass delocalization. Finally, for coherently delocalized detectors, we compare entanglement harvesting in the vacuum to entanglement harvesting in media. We find that entanglement harvesting is significantly suppressed in media in which the wave propagation speed is much smaller than the vacuum speed of light.

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I. INTRODUCTION

As is well known, first-quantized, localizable matter systems can become entangled with one another via their respective interaction with the electromagnetic vacuum [1–8]. This is possible due to the fact that the vacuum state of a quantum field is entangled [9–13]: apart from functional analytic subtleties, quantum fields, such as the Klein Gordon field, can be viewed as consisting of a harmonic oscillator at each point in space, with these oscillators being dynamically coupled through the Laplace operator in the equations of motion. In any circumstance, the ground state of coupled harmonic oscillators is generally an entangled state. This means that, for quantum fields, the ground state of the coupled local field oscillators (which in Minkowski space is the vacuum state) is a spatially entangled state. Early references are, e.g., [1,9]. The associated vacuum entanglement entropy obeys an area law in $3 + 1$ dimensions [2], which is of great interest, for example, in the context of the black hole information paradox. This is one of several

reasons why the entangling power of quantum fields in the vacuum has been of much interest in the field of relativistic quantum information [3–8].

Studies such as [9,10] showed that by coupling a quantum field to local first quantized systems via monopole or dipole interactions, entanglement can be swapped from the field to these so-called detectors.

Historically, the question of whether or not a pair of detectors can extract preexisting entanglement from the vacuum even at spacelike separation has been settled, positively, in [10]. With this question settled, and as detector couplings are generically neither completely spacelike or nonspacelike, the term “entanglement harvesting” has been established with a broader meaning in the literature [14–17]. Entanglement harvesting refers to the entangling power of quantum fields [7], both due to the extraction of preexisting vacuum entanglement and due to the additional production of entanglement at nonspacelike detector separations through field-mediated interactions. In the present paper, we investigate the impact of the detectors’ quantum delocalization on entanglement harvesting.

Entanglement harvesting has been studied extensively within the Unruh-deWitt (UdW) detector model. Despite its

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simplicity, the UdW detector model qualitatively captures many aspects of the light-matter interaction [18–20]. On the one hand, it models matter systems as two-level detector systems with classical center of mass degrees of freedom. On the other hand, it replaces the electromagnetic quantum field by a simpler scalar quantum field. Entanglement harvesting has been studied for UdW detectors not only interacting with the vacuum, but also for instance with general coherent field states [14,21], with quantum fields in curved spacetimes [3,22–25] and with quantum fields in spacetimes with nontrivial topology [15,26].

The process of entanglement harvesting depends very sensitively on the detector details [4,5,16,27–29]. For instance, it depends on whether or not the detectors are assumed to have a finite spatial extent, which is oftentimes incorporated into the UdW detector model by equipping the detectors with classical smearing functions [28]. Thus, for the purpose of studying entanglement harvesting in the light-matter interaction, it is crucial to let the details of the detector model reflect the physics of realistic matter systems, such as atoms, molecules or ions. The interaction of such matter systems and light is affected by two main aspects. First, matter systems such as atoms, molecules or ions can have a finite spatial extent due to their internal degrees of freedom, that is, due to their orbital wave functions. Second, their center of mass degrees of freedom can be quantum delocalized. While classical smearing functions have been shown to be appropriate for the purpose of modeling atomic orbitals [4], they are not suited to model center of mass delocalization [30].

The aim of this paper is to investigate how the quantum nature of the center of mass degrees of freedom of matter affects the process of entanglement harvesting. In the first section of this paper, we review the process of entanglement harvesting for both pointlike and spatially smeared UdW detectors, which interact with a scalar quantum field. In the second section, we then employ a generalized UdW detector model [30], such as to describe the interaction between quantum delocalized, first quantized matter systems and a second quantized field. We consider two quantum delocalized detectors in their respective ground states, and we ask how their ability to become entangled with each other is affected by their mass and their initial center of mass delocalization. We recover the results of vacuum entanglement harvesting for two pointlike UdW detectors, in the limit of very large detector masses and very sharply localized center of mass degrees of freedom. We find that there is however no limit in which one recovers the results of vacuum entanglement harvesting for two spatially smeared UdW detectors. Further, we find that vacuum entanglement harvesting is suppressed in the initial delocalization, and that very delocalized detectors can not harvest any entanglement from the vacuum.

Finally, in the third section of this paper, we briefly discuss entanglement harvesting for delocalized detectors in media—which might be of interest not only for the purpose of experimentally observing entanglement harvesting, but also to potentially make use of entanglement harvesting in quantum technologies. For the sake of simplicity, we model a medium as a scalar quantum field, whose wave propagation speed differs from the vacuum propagation speed of light. As the wave propagation speed decreases, we find that less entanglement can be harvested from the phononic ground state. Intuitively, this is because the phononic ground state transforms nontrivially under a quantum reference frame transformation into the rest frame of the coherently delocalized detectors, subjecting the detectors to noisy excitations, which make it more difficult for the detectors to harvest entanglement. We conjecture that matter systems are less likely to become entangled with each other by interacting with a medium than they are by interacting with the electromagnetic field.

II. REVIEW: ENTANGLEMENT HARVESTED BY UDW DETECTORS FROM THE VACUUM

Before taking into account the quantum delocalization of the centers of mass of the detectors, we briefly review the process of vacuum entanglement harvesting for detectors whose center of mass degrees of freedom are described classically. We consider two UdW detectors, labeled by $J = A, B$, which interact with a scalar quantum field. We assume that the two detectors have the same energy gap Ω and that their classical centers of mass are located, respectively, at the positions $\mathbf{x}_J$. We let $S := |\mathbf{x}_A - \mathbf{x}_B|$ denote the center of mass separation of the two detectors. We work within the interaction picture, in which operators evolve according to the free Hamiltonian

$$\hat{H}_0 = \sum_{J=A,B} \Omega |e_J\rangle \langle e_J| + \int d^3k k c \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}. \quad (1)$$

We here let $|e_J\rangle$ and $|g_J\rangle$ denote the respective excited and ground energy eigenstates of the two detectors. The operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ are the annihilation and creation operators of scalar field modes of momentum $\mathbf{k}$. We classically model the extended spatial profile of the detectors (as is common practice in the literature, see, e.g., [28]) by introducing smearing profiles $\xi(\mathbf{x} - \mathbf{x}_J)$ for the two detectors. We now assume that the monopole moment operators of the two detectors, $\hat{\mu}_J = |e_J\rangle \langle g_J| + |g_J\rangle \langle e_J|$, couple to the field operators $\hat{\phi}(\mathbf{x})$ via the linear monopole moment operator coupling. In the interaction picture, the state of the system then evolves in time according to the interaction Hamiltonian

$$\hat{H}_{\text{int}}(t) = \sum_{J=A,B} \hat{H}_J(t), \quad (2)$$

where the interaction Hamiltonians $\hat{H}_J(t)$, respectively, capture the interaction of the two detectors with the quantum field:

$$\hat{H}_J(t) := \lambda \chi(t) \hat{\mu}_J(t) \int d^3x \xi(\mathbf{x} - \mathbf{x}_J) \hat{\phi}(\mathbf{x}, t). \quad (3)$$

We here let λ denote the interaction strength, and we introduced a switching function $\chi(t)$ via which the interaction between the detectors and the quantum field may be switched on and off. The monopole moment operators and the field operators evolve according to their free Hamiltonians as follows:

$$\hat{\mu}_J(t) = e^{i\Omega t} |e_J\rangle \langle g_J| + \text{H.c.}, \quad (4)$$

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{c^2}{2k}} [e^{-ickt + i\mathbf{k} \cdot \mathbf{x}} \hat{a}_{\mathbf{k}} + \text{H.c.}]. \quad (5)$$

Perturbatively in the interaction strength, we can now study how much the two detectors become entangled with each other, via their respective interaction with the quantum field (see, e.g., [4,5] for a derivation and in-depth discussion of what follows in this paragraph). To this end, we consider the initial state $\rho(0) = |g_A\rangle \langle g_A| \otimes |g_B\rangle \langle g_B| \otimes |0\rangle \langle 0|$, for which the two detectors are unentangled and the quantum field is in its vacuum state $|0\rangle$. Evolving the initial state in time and taking the partial trace over the field degrees of freedom, one obtains the partial state $\rho_{AB}(t)$ of the two detectors after their interaction with the quantum field. Employing the basis $\{|g_A\rangle |g_B\rangle, |g_A\rangle |e_B\rangle, |e_A\rangle |g_B\rangle, |e_A\rangle |e_B\rangle\}$, the partial state of the detectors becomes

$$\rho_{AB}(t) =: \begin{pmatrix} 1 - 2P^c & 0 & 0 & \mathcal{M}^{c*} \\ 0 & P^c & \mathcal{L}^c & 0 \\ 0 & \mathcal{L}^c & P^c & 0 \\ \mathcal{M}^c & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4) \quad (6)$$

to second perturbative order in the interaction strength. Here, P^c denotes the excitation probability of the two detectors, respectively, and $\mathcal{M}^c$ has traditionally been referred to as the entangling term. The excitation probabilities of the two detectors are equal, since we here consider the same switching function and the same smearing profile (up to a displacement in space) for the two detectors. We here let the superscript c remind us of the classical nature of the center of mass degrees of freedom of the two UdW detectors.

As shown in Refs. [10,31], both matrix elements can be understood through Feynman-type diagrams like those given in Fig. 1. The transition of a UdW detector from the ground to excited state is always accompanied by either absorption or emission of a field quantum. Since the ground state of the field has no quanta to be absorbed, the lowest-order transition probability of detector A or B is the probability of emission, shown in Figs. 1(a) or 1(b), respectively. This probability (equal for both detectors) is captured as P^c . The entangling term, $\mathcal{M}^c$, is the coherence term—an amplitude, not a probability—associated with the process shown in Fig. 1(c) [10,31]. This process involves the emission of a field quantum from one detector and its absorption by the other detector. Crucially, while the *probability* for this process to happen is strictly nonzero—that is, the process actually occurs in the model we consider—it is also $\mathcal{O}(\lambda^4)$, so it does not appear explicitly in Eq. (6). Its corresponding coherence term $\mathcal{M}^c$, however, is $\mathcal{O}(\lambda^2)$ and thus appears in Eq. (6). Note that the diagram [Fig. 1(c)] is the same whether the field quanta being exchanged through this process are on or off shell, and it applies regardless of the spatial configuration of the detectors.

To measure the entanglement between the internal degrees of freedom of the two detectors, we will here employ the entanglement negativity [32], which is an entanglement monotone [33,34]. The entanglement negativity is defined for a density matrix as the absolute value of the sum of the negative eigenvalues of the partially transposed density matrix. To second perturbative order in the interaction strength, the negativity for the partial state of the two detectors becomes (see, e.g., [4,5])

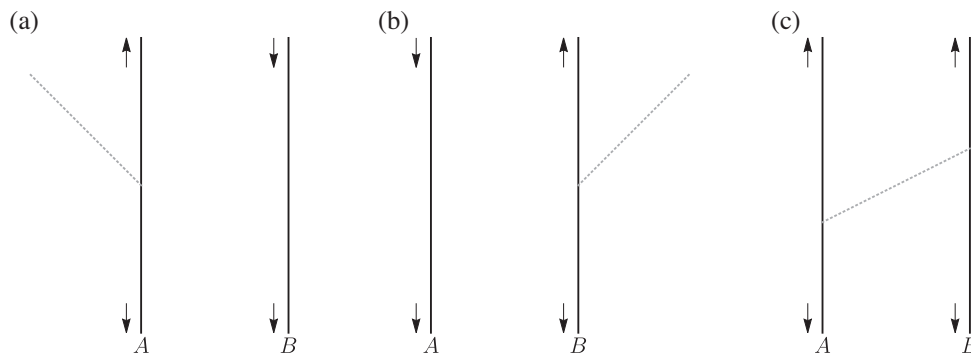


FIG. 1. Emission and absorption processes which contribute to the final state of the two UdW detectors (6). The arrows denote the state of the detector ($\downarrow$ represents $|g\rangle$ and $\uparrow$ represents $|e\rangle$) [10,31].

$$\mathcal{N}^c = \max\{0, -P^c + |\mathcal{M}^c|\}. \quad (7)$$

It is worth mentioning here that a rich variety of quantitative entanglement measures have been established in the literature (see, e.g., [35]). For instance, we could just as well use concurrence [36] as our measure for entanglement. For two identical detectors and to second perturbative order, concurrence and entanglement negativity have however been shown to be equivalent entanglement measures [26]. For the purpose of this paper, we will restrict our attention to entanglement negativity.

Let us now consider a switching function of compact support of the form

$$\chi(t) = \begin{cases} \sin(t/\sigma), & \text{for } 0 \leq t \leq \pi\sigma \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Examples for the use of compact switching functions can be found, e.g., in [5,37]. We employ a compact switching function in order to ensure that the interaction between the detectors and the field is switched on only during a compact time interval, $t \in [0, \pi\sigma]$. The importance of employing a compact switching function will become apparent in the next section. For the finite duration of time during which the coupling strength is nonzero, the negativity increases. After that time interval, that is, after time $\pi\sigma$, the detector is “off” again, and the joint state of the detectors, and its entanglement, stays constant. Throughout this paper, we will not discuss the details of the time evolution, but rather we will focus on the end result, i.e., on the amount of entanglement negativity that has accumulated after the end of the interaction and that then stays constant. We find the excitation probabilities and the entangling term after the end of the interaction to be given as follows,

$$P^c = \frac{\lambda^2 \sigma^2}{(2\pi)^3} \int \frac{d^3 k}{2ck} |\tilde{\xi}(\mathbf{k})|^2 A(k), \quad (9)$$

$$\mathcal{M}^c = -\frac{\lambda^2 \sigma^2 e^{i\pi\Omega\sigma}}{c} \int \frac{d^3 k}{k} \frac{e^{2i\mathbf{k} \cdot \mathbf{x}_0} \tilde{\xi}(\mathbf{k})^2 B(k)}{1 - \sigma^2(\Omega + ck)^2}, \quad (10)$$

where $\tilde{\xi}(\mathbf{k})$ denotes the Fourier transformation of the spatial smearing profile $\xi(\mathbf{x})$ and where we defined the following functions:

$$A(k) := \frac{1 + \cos(\pi\sigma(\Omega + ck))}{(\sigma^2(\Omega + ck)^2 - 1)^2}, \quad (11)$$

$$B(k) := \frac{i(2\Omega + ck) \sin(\pi\Omega\sigma)}{2\Omega(1 - \Omega^2\sigma^2)} + \frac{e^{-i\pi\Omega\sigma} + e^{-i\pi ck\sigma}}{1 - \sigma^2(\Omega - ck)^2}. \quad (12)$$

We can now specify a spatial smearing function according to which the detectors couple to the field. We could for instance model the spatial extent of the detectors via Gaussian smearing profiles of standard deviation $L/2$:

$$\xi(\mathbf{x}) = \left(\frac{2}{\pi L^2}\right)^{3/2} e^{-2\mathbf{x}^2/L^2}. \quad (13)$$

We will here refer to L as the “width” of the smearing profiles. Since the smearing profiles are normalized, we may interpret them as classical probability distributions, according to which the detectors couple to the quantum field. We find that the excitation probabilities and the entangling term depend on the width L of the smearing profiles, the energy gap Ω of the detectors and the total interaction time $\pi\sigma$, and the entangling term additionally depends on the separation S of the two detectors:

$$P_L^c = \frac{\lambda^2 \sigma^2}{2\pi^2 c} \int_0^\infty dk k e^{-\frac{L^2 k^2}{4}} A(k), \quad (14)$$

$$\mathcal{M}_L^c = -\frac{\lambda^2 \sigma^2 e^{i\pi\Omega\sigma}}{2\pi^2 c S} \int_0^\infty dk \frac{\sin(kS) e^{-\frac{L^2 k^2}{4}} B(k)}{1 - \sigma^2(\Omega + ck)^2}. \quad (15)$$

We here let the subscript L indicate that we employed Gaussian smearing profiles. In the limit of very sharply peaked smearing profiles ($L \rightarrow 0$), we recover the excitation probabilities and the entangling term for pointlike UdW detectors:

$$P_0^c = \frac{\lambda^2 \sigma^2}{2\pi^2 c} \int_0^\infty dk k A(k), \quad (16)$$

$$\mathcal{M}_0^c = -\frac{\lambda^2 \sigma^2 e^{i\pi\Omega\sigma}}{2\pi^2 c S} \int_0^\infty dk \frac{\sin(kS) B(k)}{1 - \sigma^2(\Omega + ck)^2}. \quad (17)$$

The upper plot in Fig. 2 shows the negativity after the end of the interaction for a pointlike detector, as a function of the energy gap Ω of the detectors and their separation S . The pointlike detector will be our reference setup for comparison below.

We observe in passing that these pointlike UdW detectors do not harvest entanglement when they are spacelike separated [37]. This is due to our choice of switching function in Eq. (8). Spacelike entanglement harvesting is possible, for example, when choosing a Gaussian switching function [4,5], and it is possible also in setups where the detectors are switched on only during strictly compact time intervals that are fully and unambiguously spacelike separated [10], thereby sidestepping any questions about the effects of long Gaussian tails.

The ability to harvest preexisting entanglement from the vacuum arises inseparably with the ability to harvest entanglement by letting two detectors interact via the field. This can be seen by considering the exchange of a field quantum as shown in Fig. 1(c), which involves the Feynman propagator. Notably, the tails that all Feynman propagators possess outside the light cone are expressing correlations that are solely due to vacuum entanglement since for spacelike separations there are no correlations that

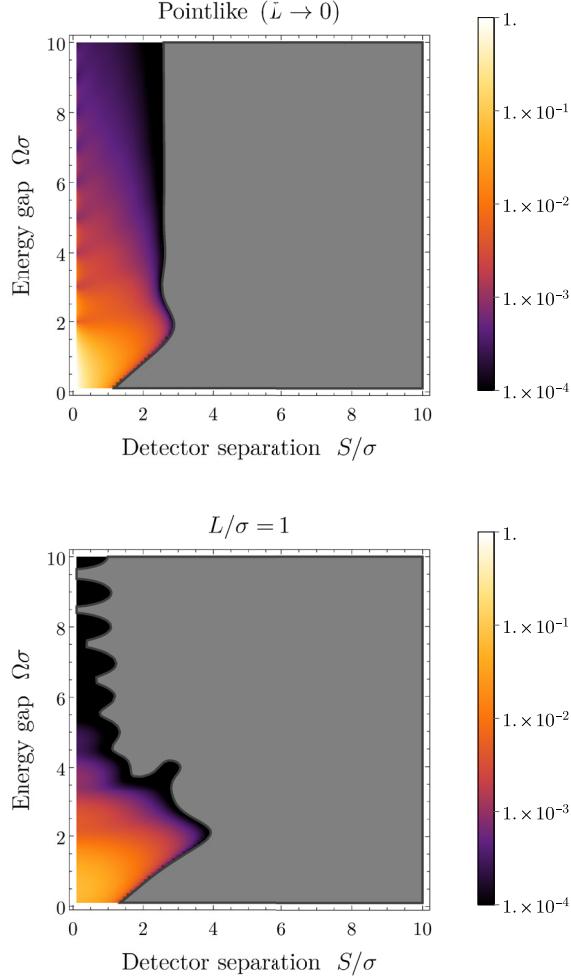


FIG. 2. The negativity $\mathcal{N}$ after the end of the interaction, as a function of the energy gap Ω and the separation S of the detectors, plotted (top) for pointlike UdW detectors and (bottom) for spatially smeared UdW detectors with $L = \sigma$. The regions of zero negativity are marked in gray.

are due to causal interaction. This can equally well be described as being due to the (acausal) exchange of virtual field quanta [38]. For other types of separation between extended events, the explanation for the origin of any generated entanglement is ambiguous. This is because the Feynman propagator, being a time-ordered two-point correlator, describes both (1) field correlations that are due to causal interaction involving the exchange of real (i.e., on-shell) field quanta and (2) those that are due to the exchange of virtual (i.e., off-shell) field quanta.

In fact, Fig. 1(c) makes no distinction at all between these explanations and applies regardless of the nature of the separation between the events. Previous literature has paid particular attention to the phenomenon of entanglement harvesting specifically through extraction of preexisting vacuum entanglement at spacelike separations [9–13] in order to unambiguously distinguish between explanations (1) and (2). We do not dwell on such concerns here.

Instead, we focus in this work on the question of how the ability of detectors to harvest entanglement in a given circumstance depends on the way the spatial extent of each individual detector is modelled. To this end, after considering (presently) the case of spatially smeared detectors, we will turn our attention in Sec. III to the novel situation of quantum-delocalized detectors.

Returning now to the plots above, we notice that the lower plot in Fig. 2 shows the negativity for Gaussian-smeared UdW detectors. The Gaussian smearing profile given in Eq. (13) provides a simple modification to the UdW detectors that is distinct from delocalization of their center of mass. We find that the smeared detectors can harvest entanglement for larger values of separation of the center of their distributions than is possible with pointlike detectors. Physically, this is expected since the smeared detectors have a spatial extent, so their near sides will be closer than their centers of mass. As expected, this extends the range for harvesting entanglement.

Additionally, we find that the negativity decreases with increasing widths, L , of the smearing profiles. Intuitively, we can understand this behavior as follows: we can interpret the spatially smeared UdW detectors as having a physical size, which introduces a soft momentum cutoff, and, as a result, they have a reduced probability of emitting very high-energy field quanta. This ultimately reduces the amplitude of the exchange of a field quantum between the detectors, Fig. 1(c), reducing the entanglement. From an alternative perspective, spatially smeared detectors average the quantum field fluctuations over extended spatial regions, and the larger these spatial regions are, the harder it is to swap correlations from the quantum field (since the small-scale correlations are inaccessible).

In the negativity plots in Fig. 2 we further observe a resonancelike behavior, for energy gaps that are multiples of the switching scale σ , which manifests itself in slight ripples in the negativity for pointlike UdW detectors, and more pronounced oscillations for spatially smeared UdW detectors.

III. ENTANGLEMENT HARVESTED BY COHERENTLY DELOCALIZED DETECTORS FROM THE VACUUM

The results for entanglement harvesting in the previous section relied on the assumption that the center of mass degrees of freedom of the matter systems under investigation are classical. In this section, we want to study how the process of entanglement harvesting is affected when the center of mass motion of the two detectors is quantum uncertain.

A possible setup we have in mind here is the following: let us consider two atoms, which are initially localized in a certain region of space via a center of mass position measurement, and which are then left to evolve freely. The center of mass wave functions of the two initially

localized atoms spread and the atoms dynamically delocalize in space. Let us now imagine that these two coherently delocalizing atoms interact with the electromagnetic vacuum. How much will the internal degrees of freedom of the atoms become entangled with each other? How will the result depend on the mass of the detectors, their initial localization, and their dynamical delocalization process? How will the results compare to the results for entanglement harvesting with classical center of mass degrees of freedom?

To study these questions, we employ a generalized UdW detector model, as described in [30]: we again replace the electromagnetic field by a simpler scalar field, and we model the atoms as two-level detector systems. However, in order to allow the detectors to coherently delocalize, we now consider the center of mass degrees of freedom of the two detectors to be quantized. We let $\hat{\mathbf{x}}_J$ and $\hat{\mathbf{p}}_J$ denote the center of mass position and momentum operators of the two detectors. We will here assume that the two detectors are of equal mass M . In the interaction picture, operators then evolve according to the free Hamiltonian

$$\hat{H}_0 = \sum_{J=A,B} \left[\frac{\hat{\mathbf{p}}_J^2}{2M} + \Omega |e_J\rangle\langle e_J| \right] + \int d^3k c k \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}. \quad (18)$$

We again couple the detectors to the quantum field via the monopole moment coupling. However, the field operators now take the center of mass position operators of the two detectors as their argument:

$$\hat{H}_{\text{int}}(t) = \lambda \chi(t) \sum_{J=A,B} \hat{\mu}_J(t) \hat{\phi}(\hat{\mathbf{x}}_J, t). \quad (19)$$

We note that the interaction Hamiltonian involves functions $\hat{\phi}(\hat{\mathbf{x}}_J, t)$ which are dependent on operators and which are operator valued themselves. An interaction of this form can be made sense of via the spectral theorem of functional calculus [30,39,40] as follows:

$$\hat{\phi}(\hat{\mathbf{x}}_J, t) := \int d^3x_J \hat{P}(\mathbf{x}_J, t) \hat{\phi}(\mathbf{x}_J, t), \quad (20)$$

where the operators $\hat{P}(\mathbf{x}_J, t)$ are given by

$$\hat{P}(\mathbf{x}_J, t) := |\mathbf{x}_J(t)\rangle\langle \mathbf{x}_J(t)| \quad (21)$$

$$= \int \frac{d^3p d^3q}{(2\pi)^3} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}_J + i t \frac{p^2 - q^2}{2M}} |\mathbf{p}\rangle\langle \mathbf{q}|. \quad (22)$$

We again assume that initially the two detectors are in their ground states and the field is in its vacuum state. We further let $|\varphi_J\rangle$ denote the initial center of mass states of the two detectors. The initial state of the system then reads $\rho(0) = |\varphi_A\rangle\langle \varphi_A| \otimes |g_A\rangle\langle g_A| \otimes |\varphi_B\rangle\langle \varphi_B| \otimes |g_B\rangle\langle g_B| \otimes |0\rangle\langle 0|$. We can express the initial center of mass states both in terms of

the initial center of mass wave functions in the position and in the momentum representation:

$$|\varphi_J\rangle = \int d^3x \varphi_J(\mathbf{x}) |\mathbf{x}\rangle = \int d^3p \tilde{\varphi}_J(\mathbf{p}) |\mathbf{p}\rangle. \quad (23)$$

We again want to investigate how entangled the internal degrees of freedom are with each other after the end of the interaction of the detectors with the quantum field. To this end, we evolve the initial state in time and trace over both the field and the center of mass degrees of freedom. To second perturbative order, we again obtain the partial state of the two detectors after the end of the interaction:

$$\rho_{AB} = \begin{pmatrix} 1 - P_A - P_B & 0 & 0 & \mathcal{M}^* \\ 0 & P_B & \mathcal{L} & 0 \\ 0 & \mathcal{L} & P_A & 0 \\ \mathcal{M} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4). \quad (24)$$

We again calculate the entanglement negativity for this density matrix, as a quantitative measure for how much entanglement the internal degrees of freedom of the two coherently delocalizing detectors can harvest from the vacuum:

$$\mathcal{N} = \max \left\{ 0, \sqrt{\frac{(P_A - P_B)^2}{4} + |\mathcal{M}|^2} - \frac{P_A + P_B}{2} \right\}. \quad (25)$$

We find the excitation probabilities P_J of the two detectors to be

$$P_J := \frac{\lambda^2}{(2\pi)^3} \int_0^{\pi\sigma} dt_1 \int_0^{\pi\sigma} dt_2 \chi(t_1) \chi(t_2) \int d^3p \\ \times \int \frac{d^3k}{2ck} |\tilde{\varphi}_J(\mathbf{p})|^2 e^{i(t_1 - t_2)(\Omega + ck + \frac{k^2 - 2\mathbf{p}\cdot\mathbf{k}}{2M})}, \quad (26)$$

and we note that the excitation probabilities of the two detectors depend on their center of mass states only via their respective momentum probability distributions $|\tilde{\varphi}_J(\mathbf{p})|^2$. Unlike the excitation probabilities, we find that the entangling term $\mathcal{M}$ also depends on the phases of the initial center of mass momentum wave functions:

$$\mathcal{M} := -\lambda^2 \int_0^{\pi\sigma} dt_1 \int_0^{t_1} dt_2 \chi(t_1) \chi(t_2) \int \frac{d^3p_1 d^3p_2}{(2\pi)^3} \\ \times \int \frac{d^3k}{2ck} \tilde{\varphi}_A(\mathbf{p}_1 + \mathbf{k}) \tilde{\varphi}_B(\mathbf{p}_2 - \mathbf{k}) \tilde{\varphi}_A^*(\mathbf{p}_1) \tilde{\varphi}_B^*(\mathbf{p}_2) \\ \times e^{i(t_1 + t_2)(\Omega - \frac{\mathbf{k}\cdot(\mathbf{p}_1 - \mathbf{p}_2)}{2M} - \frac{k^2}{2M})} [e^{-i(t_1 - t_2)(ck - \frac{\mathbf{k}\cdot(\mathbf{p}_1 + \mathbf{p}_2)}{2M})} \\ + e^{-i(t_1 - t_2)(ck + \frac{\mathbf{k}\cdot(\mathbf{p}_1 + \mathbf{p}_2)}{2M})}]. \quad (27)$$

The phases of the momentum wave functions carry the position information of the two wave functions, and, as

expected, the entangling term thus depends on the spatial locations of the two center of mass wave packets. While the excitation probabilities of the detectors, respectively, only depend on the properties of one detector, the entangling term thus depends on the properties of both detectors. As suggested, e.g., in [4], we can thus think of the excitation of the respective detectors according to P_J as local noise, while the nonlocal entangling term $\mathcal{M}$ describes entangling excitations which are shared by the two detectors.

We can now specify the initial center of mass wave functions for the two detectors. Let us here consider detectors whose center of mass position wave functions are Gaussian wave packets of initial width L , respectively, centered around $\mathbf{x}_J$, at a separation $S := |\mathbf{x}_A - \mathbf{x}_B|$:

$$\varphi_J(\mathbf{x}) = \left(\frac{2}{\pi L^2}\right)^{3/4} e^{-\frac{|\mathbf{x}-\mathbf{x}_J|^2}{L^2}}, \quad (28)$$

$$\tilde{\varphi}_J(\mathbf{p}) = \left(\frac{L^2}{2\pi}\right)^{3/4} e^{-\frac{p^2 L^2}{4} + i\mathbf{p} \cdot \mathbf{x}_J}. \quad (29)$$

The momentum probability distributions resulting from these momentum wave functions are the same for both detectors, $|\tilde{\varphi}_J(\mathbf{p})|^2 = L^3/(2\pi)^{3/2} e^{-p^2 L^2/2}$. We thus find that the excitation probabilities of the two detectors are equal, $P_A = P_B =: P$, and the negativity reduces to

$$\mathcal{N} = \max\{0, |\mathcal{M}| - P\}. \quad (30)$$

In order for the two detectors to harvest entanglement from the vacuum, the nonlocal entangling excitations thus need to dominate over the local excitation of the respective detectors [4].

We can now see why it is important to employ a switching function of compact support. Under the free quantum mechanical time evolution, the wave packets in Eq. (29) start out completely delocalized in space for $t \rightarrow -\infty$, then flow together to Gaussians of width L at time $t = 0$, and then spread again into completely delocalized states for $t \rightarrow \infty$. If we employed a switching function of noncompact support, such as a Gaussian switching function, we would need to consider the completely delocalized center of mass wave packets at time $t \rightarrow -\infty$ as the initial center of mass states. However, we want to consider the localized wave packets in Eq. (29) as the center of mass states at the initial time $t = 0$, since our aim here is to study how entanglement harvesting is affected by the center of mass spreading of initially localized detectors. We therefore need to ensure that the interaction with the quantum field is switched on precisely at time $t = 0$, which in turn is why we need to employ a switching function of compact support. Let us here again employ the compact sine switching function in Eq. (8). We obtain

$$P = \frac{\lambda^2 \sigma^2}{\pi c} \int_0^\infty dp p^2 |\tilde{\varphi}_J(\mathbf{p})|^2 \mathcal{U}(p) \quad (31)$$

for the excitation probabilities. We here defined the template function

$$\mathcal{U}(p) := \int_{-1}^1 dz \int_0^\infty dk k \left[1 + \cos\left(\pi\sigma\left(F - \frac{kpz}{M}\right)\right) \right] \times \left[\sigma^2 \left(F - \frac{kpz}{M}\right)^2 - 1 \right]^{-2}, \quad (32)$$

where p is the detector's recoil momentum and where we defined $F := \Omega + ck + k^2/(2M)$ for convenience of notation. We refer to $\mathcal{U}(p)$ as a “template function” due to the fact that the function is independent of the center of mass states of the detectors. For the entangling term we obtain

$$\mathcal{M} = \frac{\lambda^2 \sigma^2 L^2}{c(2\pi)^3} \int_{-\infty}^\infty dp_1 \int_{-\infty}^\infty dp_2 \int_0^\infty dk \frac{\sin(kS)}{S} \times e^{-L^2(p_1^2 + p_2^2)/2} e^{-L^2 k^2/4} \mathcal{V}(k, p_1, p_2). \quad (33)$$

We here again defined a template function,

$$\mathcal{V}(k, p_1, p_2) := \sum_{j=0}^1 \frac{e^{i\pi\alpha\sigma}}{1 - \sigma^2(\alpha + \beta_j)^2} \times \left(\frac{i(2\alpha + \beta_j) \sin(\pi\alpha\sigma)}{2\alpha(1 - \alpha^2\sigma^2)} + \frac{e^{-i\pi\alpha\sigma} + e^{-i\pi\beta_j\sigma}}{1 - \sigma^2(\alpha - \beta_j)^2} \right), \quad (34)$$

and we defined $\alpha := \Omega - k(p_1 - p_2)/(2M)$ as well as $\beta_j := ck + (-1)^j k(p_1 + p_2)/(2M)$ for convenience of notation.

Since we work within a framework in which the center of mass dynamics are described by the Schrödinger equation, we need to ensure that the virtual center of mass velocities are well within the nonrelativistic regime. That is, we need to ensure that the momentum probability distributions $|\tilde{\varphi}_J(\mathbf{p})|^2$ have contributions only for momenta corresponding to virtual velocities much smaller than the speed of light. Let us here restrict the virtual velocities to velocities no larger than 1% of the vacuum speed of light, $v := p/M \leq 0.01c$. The Gaussian momentum probability distributions of the detectors have a standard deviation of $1/L$. We can thus assume to a very good approximation (within 3.5 standard deviations away from the mean) that the center of mass momenta $\mathbf{p}$ in the probability distributions satisfy $pL \lesssim 3.5$. The nonrelativistic regime therefore corresponds to parameters L and M satisfying

$$LMc \gtrsim 3.5 \times 10^2. \quad (35)$$

The center of mass wave function of a coherently delocalized detector spreads faster for smaller L and M , that is,

for initially more sharply localized detectors and for smaller detector masses. Consequently, for a given detector mass, there is a minimal initial delocalization width we can consider while staying within the nonrelativistic regime for the virtual center of mass velocities.

Provided that we chose appropriate parameters M and L , we can now expand the template functions $\mathcal{U}$ and $\mathcal{V}$ for nonrelativistic virtual center of mass velocities. We Taylor expand $\mathcal{U}$ around $p/(Mc) = 0$, and we Taylor expand $\mathcal{V}$ around both $p_1/(Mc) = 0$ and $p_2/(Mc) = 0$. To second order, we then obtain the following simplified expressions for the excitation probabilities and the entangling term:

$$P = \frac{\lambda^2 \sigma^2}{4\pi^2 c} \left[\mathcal{U}(0) + \frac{3}{2L^2} \frac{\partial^2 \mathcal{U}}{\partial p^2}(0) + \mathcal{O}((LMc)^{-4}) \right], \quad (36)$$

$$\mathcal{M} = \frac{\lambda^2 \sigma^2}{4\pi^2 c} \int_0^\infty dk \frac{\sin(kS)}{S} e^{-L^2 k^2/4} \left[\mathcal{V}(k, 0, 0) + \frac{C(k)}{(LMc)^2} + \mathcal{O}((LMc)^{-4}) \right]. \quad (37)$$

For notational convenience, we here defined:

$$C(k) := \frac{M^2 c^2}{2} \left[\frac{\partial^2 \mathcal{V}}{\partial p_1^2}(k, 0, 0) + \frac{\partial^2 \mathcal{V}}{\partial p_2^2}(k, 0, 0) \right]. \quad (38)$$

As displayed in Fig. 3, we find that the excitation probabilities decrease both for increasing energy gaps Ω and decreasing detector masses M . Intuitively, this behavior can be explained as follows: switching the interaction on and off breaks time translation invariance and therefore provides energy for the excitation and the recoil of the detectors and the excitation of the field. The kinetic energy of the recoil becomes larger for smaller detector masses. Since the excitation of the detector is always accompanied

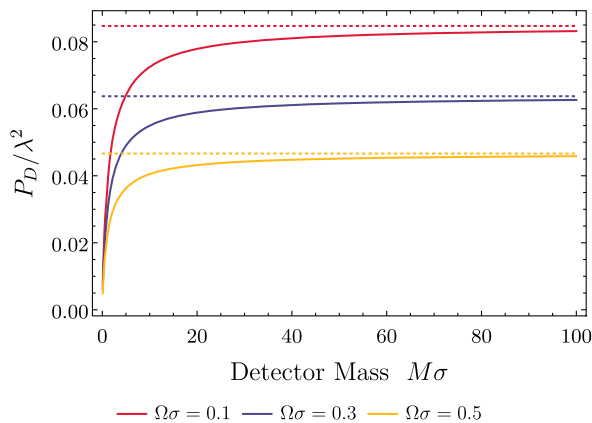


FIG. 3. The transition probability for a detector as a function of its mass, after the end of the interaction, for different energy gaps and with $L = 1000\sigma$. The dotted lines represent the excitation probabilities of pointlike UdW detectors with the same energy gaps as the respective massive detectors.

by the emission of a field quantum and the recoil of the detector, the excitation process becomes energetically more expensive for larger energy gaps and smaller detector masses. In the limit of infinitely small energy gaps, the detectors essentially turn into simple charges and all the switching energy can go into the recoil of the detectors and the excitation of the field. Similarly, in the limit of infinitely large detector masses, the kinetic recoil energy tends to zero and all the switching energy can go into the excitation of the field and the internal degrees of freedom. For the excitation probability of a very massive detector, it is therefore justifiable to neglect the recoil of the detector and to model the center of mass degrees of freedom classically. As can be seen in Fig. 3, the excitation probability in the limit of large detector masses indeed approaches the excitation probability of a pointlike UdW detector.

As displayed in Fig. 4, we further find that the entangling term is suppressed both in the separation of the two detectors and in the initial center of mass delocalization widths.

The intuition for this follows from the intuition given for the behavior seen in Fig. 2: the more delocalized the detectors are initially, the larger the length scale of the detectors and the smaller the entangling term. In fact, as we can see from Eq. (37), the entangling term for coherently delocalized detectors is Gaussian suppressed in the initial delocalization width—contrary to the excitation probabilities, whose leading order term does not depend on the initial delocalization width at all, as we can see from Eq. (36). Therefore, the ability of quantum delocalized detectors to harvest entanglement from the vacuum is

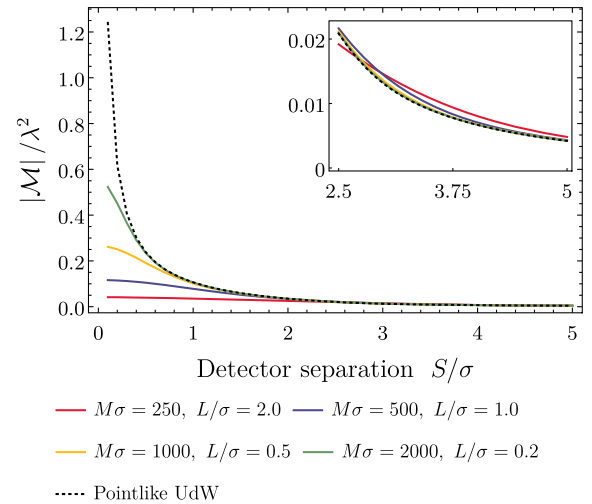


FIG. 4. The entangling term $\mathcal{M}$ for pointlike UdW detectors as well as for massive detectors, after the end of the interaction, as a function of the detector's separation S (with $\Omega\sigma = 0.1$). For the massive detectors, we chose different values for M and L such as to keep their product constant ($ML = 500$), which fixes the virtual velocities at which the detectors dynamically delocalize.

Gaussian suppressed in the initial center of mass delocalization.

Let us now see whether we can find a limit in which we recover the entanglement harvesting results for UdW detectors, that is, for detectors with classical center of mass degrees of freedom. We start by exploring the negativity in the limit of very large detector masses ($M \rightarrow \infty$), while keeping the initial delocalization width L fixed. One might expect to recover the classical behavior of UdW detectors in this limit, since the dynamical quantum center of mass delocalization process becomes very slow: the virtual center of mass velocities satisfy $v \lesssim 3.5/(LM)$ and thus tend to zero in this limit. Even though the detectors each have a finite initial delocalization width, their center of mass wave packets do not coherently spread any further. We indeed find that the excitation probabilities and the entangling term respectively reduce to the excitation probabilities and the entangling term for UdW detectors. However, there is a twist: The excitation probabilities reduce to the excitation probabilities for pointlike UdW detectors, $P \rightarrow P_0^c$, while the entangling term reduces to the entangling term for Gaussian smeared UdW detectors, $\mathcal{M} \rightarrow \mathcal{M}_L^c$. Thus, in the limit of very large detector masses and for finite initial delocalization widths L , the negativity neither reduces to the negativity for a pair of pointlike UdW detectors, nor to the negativity for a pair of Gaussian smeared UdW detectors.

Intuitively, we can understand this behavior as follows. In the infinite mass limit, the kinetic energy of the recoil of the detectors tends to zero. The center of mass degrees of freedom no longer play a role in the energy balance of the excitation process of the detectors, and the recoil of the detector becomes negligible. We can thus effectively interpret the center of mass probability distributions as classical probability distributions, of finite and constant width, for the positions of two pointlike UdW detectors. Since the excitation probability of a pointlike UdW detector is independent of the position of the detector, we recover the results for pointlike UdW detectors for the excitation probabilities. On the other hand, the nonlocal entangling excitations shared by two pointlike UdW detectors depend on the detector separation, and therefore they also depend on the classical position probability distributions for the two detectors. Consequently, the entangling term for incoherently delocalized detectors does not reduce to the entangling term for pointlike UdW detectors, but rather to the entangling term for spatially smeared UdW detectors.

Let us now recall that Gaussians of width L approach delta distributions in the limit $L \rightarrow 0$. Clearly, we should be able to recover the entanglement harvesting results for a pair of pointlike UdW detectors in the limit of very large detector masses and center of mass distributions, which are very sharply peaked (and thus essentially completely localized) at all times. However, we need to approach this limit in a way that ensures that the virtual center of mass

velocities stay within the nonrelativistic regime identified in Eq. (35). To this end, we define $M =: m/\gamma$ and $L =: l\gamma$, with γ as a regularization factor and with m and l as constants satisfying $lmc \gtrsim 3.5 \times 10^2$. Letting $\gamma \rightarrow 0$ then lets the initial center of mass localization become very sharp ($L \rightarrow 0$) and the detector masses become very large ($M \rightarrow \infty$), while keeping the virtual center of mass velocities fixed and therefore nonrelativistic. In the limit $\gamma \rightarrow 0$, the excitation probabilities and the entangling term reduce to

$$P \rightarrow P_0^c + \frac{\lambda^2 \sigma^2}{4\pi^2 c (lmc)^2} \int_0^\infty dk \frac{\sigma^2 c^2 k^3}{((\Omega + ck)^2 \sigma^2 - 1)^4} \\ \times (20(\Omega + ck)^2 \sigma^2 + 4 + \cos(\pi\sigma(\Omega + ck))) \\ \times [20(\Omega + ck)^2 \sigma^2 + 4 - \pi^2((\Omega + ck)^2 \sigma^2 - 1)^2] \\ + 8(\Omega + ck)\pi\sigma[(\Omega + ck)^2 \sigma^2 - 1] \sin(\pi\sigma(\Omega + ck))) \\ + \mathcal{O}((lmc)^{-4}), \quad (39)$$

$$\mathcal{M} \rightarrow \mathcal{M}_0^c + \frac{\lambda^2 \sigma^2}{4\pi^2 c (lmc)^2} \int_0^\infty dk \frac{\sin(kS)}{S} C(k) \\ + \mathcal{O}((lmc)^{-4}). \quad (40)$$

By letting the virtual center of mass velocities go to zero, we can then describe two detectors whose center of mass degrees of freedom are localized very sharply at all times. Indeed, taking the limit $\gamma \rightarrow 0$ first and then taking the limit $1/(lmc) \rightarrow 0$, we recover the excitation probabilities and the entangling term for two pointlike UdW detectors, $P \rightarrow P_0^c$ and $\mathcal{M} \rightarrow \mathcal{M}_0^c$. We hence identified the limit in which entanglement harvesting for a pair of coherently delocalized detectors reduces to entanglement harvesting for a pair of pointlike UdW detectors. On the other hand, we find that there is no limit in which the results reduce to entanglement harvesting for a pair of spatially smeared UdW detectors. This confirms what we mentioned before, namely that spatial smearing profiles are appropriate to model the finite spatial extent of atoms due to their electronic orbitals [4] but not to model the coherent center of mass delocalization of an atom [30].

In Fig. 5, we plotted the entanglement negativity for two coherently delocalizing detectors, as a function of the energy gap and the separation of the detectors. We can clearly see how the negativity reduces to the negativity for a pair of pointlike UdW detectors, when first letting $\gamma \rightarrow 0$ and then also letting $1/(lmc) \rightarrow 0$. We also observe that entanglement harvesting is indeed highly suppressed in the initial center of mass delocalization width.

Overall, we find that entanglement harvesting is suppressed for coherently delocalized detectors (and thus for actual physical matter systems such as atoms, ions or molecules), compared to entanglement harvesting for UdW detectors, whose center of mass degrees of freedom are assumed to be classical. An intuitive explanation for this

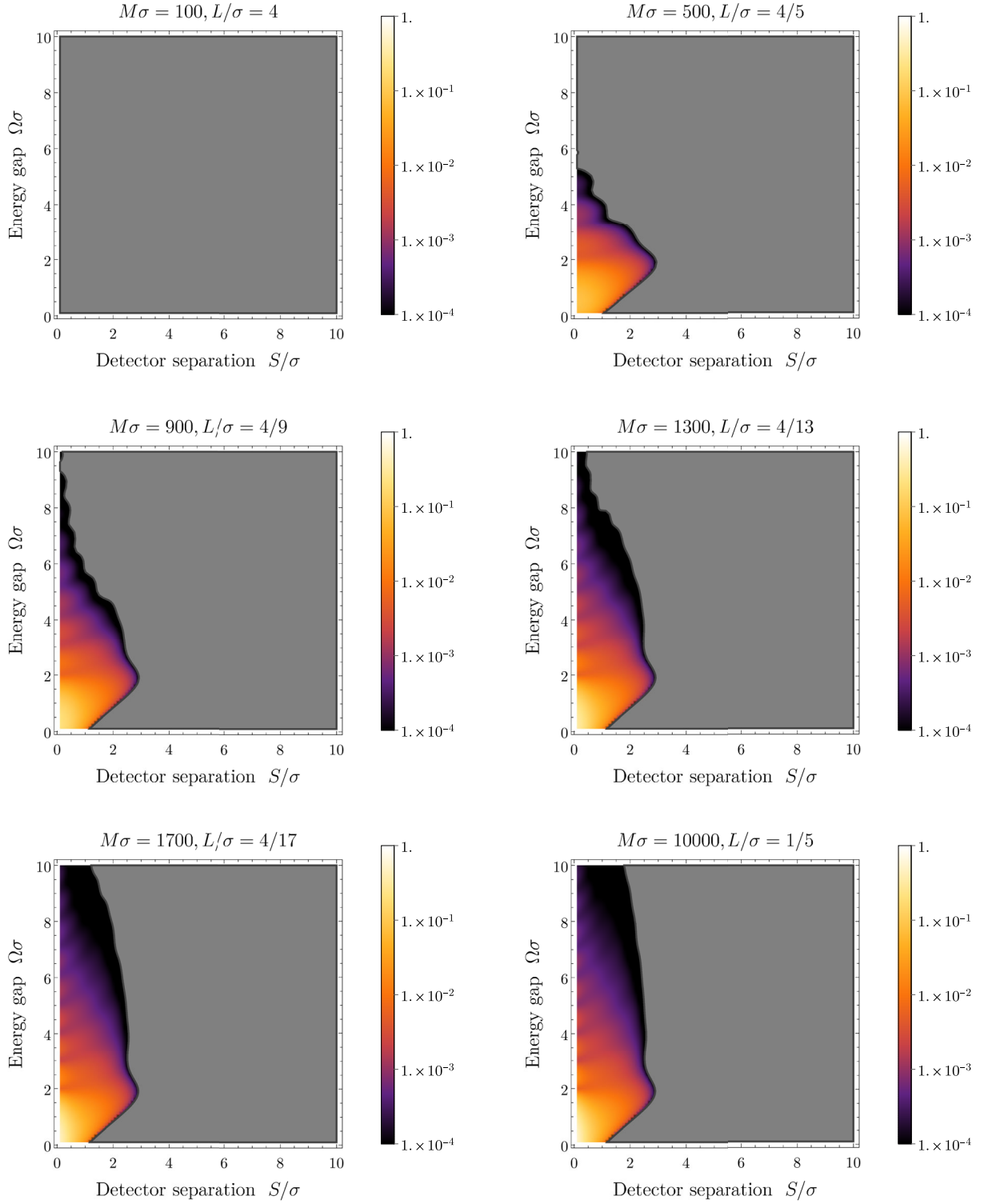


FIG. 5. The negativity $\mathcal{N}$ for two coherently delocalizing detectors, after the end of the interaction, plotted as a function of the energy gap Ω and the separation S of the two detectors. Regions of zero negativity are marked in gray. We chose the detector masses M and the initial center of mass localization widths L so that γ decreases from left to right and from top to bottom. In the first five plots we fixed $1/(lmc) = 2.5 \times 10^{-3}$, while in the sixth plot we chose parameters satisfying $1/(lmc) = 5 \times 10^{-4}$, such as to see what happens to the negativity as we further decrease $1/(lmc)$. As expected, as we approach the limit $\gamma \rightarrow 0$ and $1/(lmc) \rightarrow 0$, we find that the negativity resembles more and more the negativity displayed in Fig. 2 for two pointlike UdW detectors.

suppression might be the following. We here focused on the entanglement harvested by the internal degrees of freedom of the two detectors. However, further (bipartite as well as multipartite) entanglement could potentially build up between the respective internal and center of mass degrees of freedom of the two detectors. This entanglement, which remains to be calculated, might build up at the expense of entanglement between the internal degrees of freedom of the two coherently delocalized detectors.

IV. ENTANGLEMENT HARVESTED BY COHERENTLY DELOCALIZED DETECTORS FROM THE GROUND STATE OF A MEDIUM

Experimentally verifying entanglement harvesting from the vacuum is a difficult task [16,17,27,41]. It might be more feasible to experimentally observe entanglement harvesting from the ground state of a medium, e.g., by sending atoms through a thin foil or a Bose-Einstein condensate. We here want to shed some light on whether the internal degrees of freedom of quantum delocalized atoms might become entangled with each other, via their respective interaction with the entangled ground state of a medium.

In the previous sections, we modeled the electromagnetic field via a simple scalar quantum field with dispersion relation $\omega = ck$, where c stands for the vacuum propagation speed of light. We will here model a medium via a scalar quantum field with dispersion relation $\omega = c_s k$, with $c_s < c$ the wave propagation speed in the medium. The propagation of waves in the scalar field could then for instance model the propagation of light in a medium, or the propagation of sound in a phononic field, both of which are known to propagate slower than light in the vacuum. For concrete experimental setups, it will be very interesting to pursue analogous calculations with the there relevant realistic dispersion relation. Repeating the calculations we performed in the previous section, we obtain the excitation probabilities

$$P = \frac{\lambda^2 \sigma^2}{4\pi^2 c_s} \left[\mathcal{U}(0) + \frac{3}{2L^2} \frac{\partial^2 \mathcal{U}}{\partial p^2}(0) + \mathcal{O}((LMc)^{-4}) \right], \quad (41)$$

and the entangling term

$$\mathcal{M} = \frac{\lambda^2 \sigma^2}{4\pi^2 c_s} \int_0^\infty dk \frac{\sin(kS)}{S} e^{-L^2 k^2/4} \left[\mathcal{V}(k, 0, 0) + \frac{C(k)}{(LMc)^2} + \mathcal{O}((LMc)^{-4}) \right], \quad (42)$$

where $\mathcal{U}$, $\mathcal{V}$ and C are defined as in Eqs. (32), (34) and (38), with the exception that c is being replaced by c_s in the definitions of $\mathcal{U}$ and β_j .

In Fig. 6 we plotted the excitation probability, the entangling term and the negativity for a pair of coherently

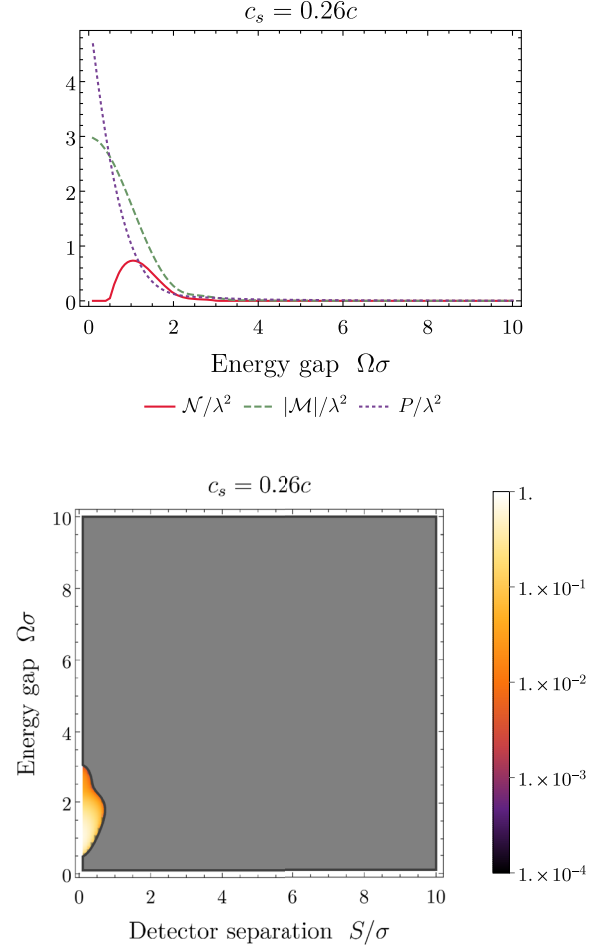


FIG. 6. We consider two detectors (with detector masses $M\sigma = 900$ and initial localization widths $L\sigma = 4/9$) in a medium with wave propagation speed $c_s = 0.26c$. We plot (top) the transition probability P , the entangling term $\mathcal{M}$ and the negativity $\mathcal{N}$, after the end of the interaction, as function of the energy gap Ω and for a detector separation $S = \sigma/10$, and (bottom) the negativity $\mathcal{N}$, after the end of the interaction, as a function of the energy gap and the detector separation. The region of zero negativity is marked in gray.

delocalized detectors in a medium with wave propagation speed $c_s = 0.26c$. We find that the negativity is significantly suppressed compared to the negativity in Fig. 5, in which the detectors were in the vacuum. Intuitively, this behavior can be explained as follows: transforming into the quantum uncertain rest frame of the delocalizing detectors, the phononic ground state transforms nontrivially into an excited field state which might be more entangled than the phononic ground state. The entangling excitations thus potentially increase, but at the same time, also the local “noisy” excitations increase. For the parameters we chose here, these two competing effects play out in such a way that the negativity decreases significantly. Both light and sound can be slowed down significantly in media (e.g., light in crystals or sound in Bose-Einstein condensates), to

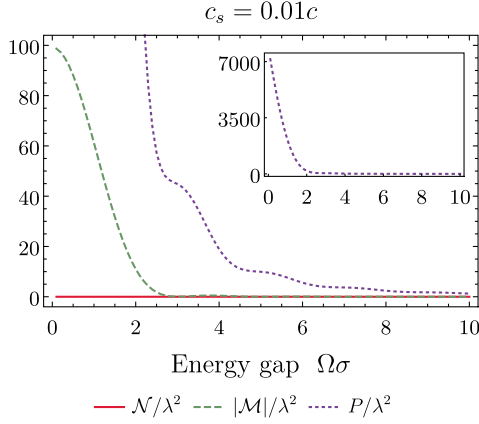


FIG. 7. We consider two detectors (with detector masses $M\sigma = 900$, initial localization widths $L\sigma = 4/9$ and detector separation $S = \sigma/10$) in a medium with wave propagation speed $c_s = 0.01c$. We plot the transition probability P and the entangling term $\mathcal{M}$, after the end of the interaction and as function of the energy gap, and we find that the negativity $\mathcal{N}$ vanishes for this choice of parameters.

the extreme of being stopped completely [42,43]. The detectors in such media could coherently delocalize with virtual velocities that are comparable to, or even larger than, the propagation speed in the medium, $v \gtrsim c_s$, while remaining well within the nonrelativistic regime, $v \leq 0.01c$. Gaussian center of mass wave packets with support for supersonic virtual center of mass velocities are ones for which $LMc_s \gtrsim 3.5$, while the nonrelativistic regime is characterized by $LMc \gtrsim 3.5 \times 10^2$. In Fig. 7 we plotted the excitation probabilities, the entangling term and the negativity for two detectors in a medium whose wave propagation speed is 1% of the vacuum speed of light. We chose the parameters so that the maximal virtual center of mass velocities in the Gaussian wave packet are close to the speed of sound in the medium ($LMc_s = 4$), while staying well within the nonrelativistic regime ($LMc = 400$). We find that both the entangling term and the excitation probabilities for detectors in the medium are significantly enhanced, and we find that overall the negativity vanishes. We thus find that if the wave propagation speed in the medium is too small, the internal degrees of freedom of a pair of coherently delocalizing detectors cannot become entangled with each other.

We conjecture that it is generally harder for detectors to harvest entanglement from a medium than from the vacuum. Entanglement harvesting experiments in media might however still be worth considering, given that they may be more easily conducted than the harvesting of entanglement from the vacuum.

V. CONCLUSIONS AND OUTLOOK

We here studied the ability of two quantum delocalized detectors to become entangled with one another, via their respective interaction with a scalar quantum field.

Specifically, we accounted for the quantum mechanical nature of the center of mass degrees of freedom of the detectors and calculated the entanglement negativity for the internal degrees of freedom of the two detectors. Our calculations show that the process of entanglement harvesting is affected by the coherent delocalization of matter and, in particular, that delocalized detectors harvest less entanglement than detectors whose center of mass degrees of freedom are assumed to behave classically.

We identified the limit in which the results for entanglement harvesting for coherently delocalized detectors reduce to the results for detectors with classical external degrees of freedom: for two detectors of very large mass, whose center of mass wave functions are initially very sharply peaked and which dynamically delocalize very slowly, we recover the negativity for two pointlike UdW detectors. This limit corresponds to detectors whose centers of mass are essentially completely localized at all times.

Further, we found that the delocalization of the centre of mass degree of freedom is fundamentally different from the finite extent of a detector's charge distribution that arises from the finite size of its electronic orbitals. If the description of the latter is desired, it can be modeled separately through the use of smearing functions. Finally, we discussed entanglement harvesting in media, where we found that entanglement harvesting for coherently delocalized detectors decreases with decreasing wave propagation speeds.

Looking forward, it could be useful to adapt and apply methods of quantum reference frames [44–48] to the study of entanglement harvesting. On one hand, these methods may allow one to extend our study to relativistic virtual center of mass velocities. On the other hand it might also be insightful to apply the quantum reference frame formalism to entanglement harvesting for coherently delocalized detectors in media. One could for instance consider the ground state of a medium and perform a quantum reference frame transformation into the quantum uncertain rest frame of the detectors. One might then explicitly see whether the ground state transforms into an excited field state, and whether this excited field state is more entangled than the ground state. One could then try to apply these insights towards finding quantum uncertain detector states of motion that are best suited for the purpose of optimally harvesting entanglement from the medium.

A deeper understanding of entanglement harvesting in the vacuum and in media may also be obtainable through the study of entanglement harvesting from excited field states. Intuitively, states with field quanta delocalized over large distances may contain harvestable entanglement over large distances. The intuition here is nontrivial, however, because while the entangling term might increase, the excitation probabilities might also increase significantly. Overall, their difference, the entanglement negativity, might therefore decrease, similarly to what we already observed in this paper for detectors in media.

Further, in this paper we focused on the entanglement harvested only by the internal degrees of freedom of quantum delocalized detectors. It will be very interesting to investigate to what extent the center of mass degrees of freedom of coherently delocalized detectors can harvest entanglement from the vacuum. In addition, the center of mass degrees of freedom can become entangled with the internal degrees of freedom in the harvesting process. We conjecture, for example, that, for faster virtual recoil velocities, the center of mass degree of freedom harvest larger amounts of entanglement while, possibly due to entanglement monogamy, the internal degrees of freedom then might harvest less.

It will be technically difficult, however, to calculate entanglement measures, such as the negativity, for the center of mass degrees of freedom, since they possess Hilbert spaces that are infinite dimensional. We anticipate that this can be addressed, for example, by either postselecting for specific recoil momenta, by discretizing the momentum space, e.g., by placing the detectors in a confining potential or cavity and placing an energy cutoff, or by binning momenta into a finite number of momentum regions.

Such methods could then allow one, for example, to study to what extent, e.g., a preexisting entanglement between the center of mass degrees of freedom can help or hinder the harvesting of entanglement. It should also be very interesting to explore how the delocalization of the center of mass interacts with models of natural ultraviolet cutoffs in quantum field theory, such as band limitation, see, e.g., [49–51].

We close by highlighting that, after the harvesting process, the extracted entanglement is available for

arbitrary purposes, including use in protocols that also involve two detectors interacting with a quantum field, such as quantum state teleportation and quantum energy teleportation. It should be interesting to explore if these protocols could be usefully merged with entanglement harvesting into one protocol for, e.g., quantum state or energy teleportation without the need of preexisting entanglement.

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