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Pixel Matrices and Other Compositional Analyses of Interconnected Systems

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14. ABSTRACT
The new world of interconnected-everything brings new challenges to those who wish to understand it and keep society safe from unintended and unimagined consequences. With constant communication and feedback loops being the norm, the space of behaviors is too large to analyze by simulation alone. Today's nearly unlimited computational power must be used more wisely, so that our knowledge of a system can evolve along with the system itself. New mathematical techniques are needed to provide the algebraic formulas for \emph{combining our insights}, just as we combine components, allowing us to anticipate the behavior of an assembled system. Category theory is the mathematics of combination and
compositionality, so it is well-suited as a foundation for such work. We propose to investigate compositional techniques for analyzing systems of all sorts. At the mathematical center of many disciplines, one needs to solve a system of simultaneous equations. As mundane, abstract, and worked-over as this may seem, a new elementary technique was recently discovered with the potential to change how we approach such problems. This technique is highly compositional---the solutions to subsystems can be combined to form a solution of the whole---and it emerged out of a similarly compositional approach to understanding the behavior of networked machines. Just as circuits can be combined to form computers, machines of all scales can be interconnected to form more complex machines. The common theme is compositionality: whether combining the constraints and requirements necessary to design a robot, or combining the equations that describe its function, our goal is to find analyses that are scalable and reusable, so that the knowledge we gain today is efficiently utilized in the networks of tomorrow.

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Final report for AFOSR grant FA9550-17-1-0058

Pixel matrices and other compositional analyses of interconnected systems

David I. Spivak

March 18, 2021

This report, submitted to Fred Leve at the Air Force Office of Scientific Research, summarizes my group's scientific progress on the research sponsored by AFOSR grant FA9550-17-1-0058, titled *Pixel matrices and other compositional analyses of interconnected systems*.

Category theory is a theory of relationships and composition. As such it has become a gateway to pure math, providing rigorous connections between algebra and number theory, logic and programming, geometry and topology, and probability and measure. My goal during this grant was to show that category theory is also broadly applicable outside of mathematics, from probability and programming, to dynamics and safety.

Below I will briefly summarize the work that this grant has enabled my team—consisting of myself, Brendan Fong, Rémy Tuyeras, and Paolo Perrone—to complete over the past three years.

1 Summary of scientific progress

Our team worked on a number of scientific problems related to this grant.

The pixel array method

We explored the pixel array method for solving systems of nonlinear equations, in particular those arising as the steady states of dynamical systems. It is a fully discretized approach that uses linear algebra to solve even nonlinear systems. While we have found it to be much faster than Newton-based methods when searching for *all solutions in a bounding box*, we have not been able to optimize it enough to be useful in many practical situations.

One particular case where it is useful is when the variable sharing formula is tree-shaped. This turned out to be the situation in a smart-grid application considered at NIST, where the pixel array method was put to use. The method was also used in considering several PDEs, and while the results were not as exciting as may have been wished, close cousins of this idea have been implemented in AlgebraicJulia by Evan Patterson and James Fairbanks in their pursuit of several DARPA awards, e.g. the GATEM project.

Temporal type theory

We continued work on temporal type theory, a new logical framework for dynamical systems. With researchers from Honeywell, we applied these methods to robotics problems. We also presented work at the Categorical Probability and Statistics Workshop showing how probability in this context generalizes the usual notion of stochastic process. Finally, together with researchers from LIDS at MIT, we used these ideas to propose a new approach to monitoring and diagnosing perception systems in autonomous vehicles.

Categorical probability theory

Our group worked significantly on the development of a new generalization of probability theory, called Markov categories. Here one can use many of the techniques and prove many of the usual theorems of probability and statistics, such as Bayesian inversion, Basu's theorem, etc., but all within a generic context that does not require measure theory. In particular, our group proved stochastic dominance and the Blackwell theorem, the De Finetti theorem, and several others.

Category theory for programmers

In keeping with the imperative to make our work computational, we wrote a book with famed programmer Bartosz Milewski. Through this, our group developed a working knowledge of programming in Haskell and Idris, a dependently typed language, as well as expertise in the mathematics behind them.

Polynomial functors

Finally, our work on the above topics led to a breakthrough in formalizing interacting dynamical systems that can reconfigure their connection pattern in real-time, namely using the theory of polynomial functors. This became the impetus for a new research grant from the AFOSR.

2 Publications and presentations

Below I will list some publications and presentations with which I have been involved—either as author or as postdoc supervisor (postdoc name in **bold**)—that were produced during the performance period of this AFOSR grant. Some are about the above topics, others are about tangential 'spin-off' topics.

2.1 Books

- Fong, B.; Milewski, B.; **Spivak, D.I.** (2020) *Categories for Programmers*. (*to appear*).
- Fong, B.; **Spivak, D.I.** (2018) *An Invitation to Applied Category Theory: Seven Sketches in Compositionality*. Cambridge University Press.

2.2 Journal papers, book chapters, conferences, and preprints

- Antonante, P.; **Spivak, D.I.**; Carlone L. (2020) "Monitoring and Diagnosability of Perception Systems". Available online: `http://arxiv.org/abs/2005.11816`.
- **Spivak, D.I.** (2020) "Poly: An abundant categorical setting for mode-dependent dynamics." Available online: `http://arxiv.org/abs/2005.01894`.
- Myers, D.J.; **Spivak, D.I.** (2020) "Dirichlet Functors are Contravariant Polynomial Functors". Available online: `https://arxiv.org/abs/2004.04183`.
- **Spivak, D.I.**; Myers D.J. (2020) "Dirichlet polynomials form a topos". Available online: `https://arxiv.org/abs/2003.04827`.
- Fritz, T.; Gonda, T.; **Perrone, P.**, Rischel, E. (2020) "Representable Markov Categories and Comparison of Statistical Experiments in Categorical Probability". `https://arxiv.org/abs/2010.07416`
- Fritz, T.; **Perrone, P.**; Rischel, E.; Gonda T. (2020) "Stochastic dominance and the Blackwell theorem in Markov categories." *To appear*
- Fritz, T.; **Perrone, P.**; Rischel, E.; Gonda T. (2020) "Distribution monads and the De Finetti theorem in Markov categories." *To appear*
- **Perrone, P.**; Tholen, W. (2020) "Kan extensions are partial colimits." `https://arxiv.org/abs/2101.04531`
- Constantin, C.; Fritz, T.; **Perrone, P.**; Shapiro B. (2020) "Partial Evaluations and the Compositional Structure of the Bar Construction." Available online: `https://arxiv.org/abs/2009.07302`.
- Fong, B.; Speranzon, A.; **Spivak, D.I.** (2019) "Temporal Landscapes: A graphical temporal logic for reasoning". Available online: `https://arxiv.org/abs/1904.01081`.
- Liu, C.T.; **Spivak, D.I.** (2018) "Evaluating the Pixel Array Method as Applied to Partial Differential Equations". Available online: `https://arxiv.org/abs/1808.01724`.
- Speranzon, A.; **Spivak, D.I.**; Varadarajan, S. (2018) "Abstraction, Composition and Contracts: A Sheaf Theoretic Approach". Available online: `http://arxiv.org/abs/1802.03080`.
- **Fong, B.**; Zanasi, F. (2018) "Universal Constructions for (Co)Relations: categories, monoidal categories, and props." *Logical Methods in Computer Science* Volume 14, Issue 3. Available online: `https://arxiv.org/abs/1710.03894`.
- Baez, J.C.; **Fong, B.** (2018) "A Compositional Framework for Passive Linear Networks." Available online: `https://arxiv.org/abs/1504.05625v5`.

2.3 Invited Presentations

Over the course of this performance period, I spoke in the following seminars and conferences:

- Tallinn University of Technology, Computer Science Theory Seminar, 2020/06/25
- Workshop on Categorical Probability and Statistics, 2020/06/07
- MIT Algebra, Statistics, and Optimization seminar, 2019/05/09
- Institute for Systems Research Seminar, University of Maryland, 2018/09/28
- Toposes in Como, 2018/06/27
- Special Seminar, Max Planck Institute for Mathematics in the Sciences, 2018/05/07
- Applied Category Theory workshop, National Institute of Standards and Technology, 2018/03/16