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Nonlocal PDEs: Modeling, Analysis, Control and Beyond

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The research proposal aims at making significant contributions to the broad area of analysis, dynamics, control and numerical approximations of models involving nonlocal Partial Differential Equations (PDEs) by addressing some key issues that remain poorly understood or unsolved and that appear systematically in real-life applications including but not limited to: collective behavior, wave propagation in heterogeneous media with high contrast, viscoelasticity, denoising algorithms and nonlocal image processing, anomalous transport and diffusion. This ambitious research program has been developed at the intersection of three areas that have evolved rapidly in the last decades: PDEs, Control and Numerics.

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Project Final Report

Part A: General Aspects

The research proposal aims at making significant contributions to the broad area of analysis, dynamics, control and numerical approximations of models involving nonlocal Partial Differential Equations (PDEs) by addressing some key issues that remain poorly understood or unsolved and that appear systematically in real-life applications including but not limited to: collective behavior, wave propagation in heterogeneous media with high contrast, viscoelasticity, denoising algorithms and nonlocal image processing, anomalous transport and diffusion. This ambitious research program has been developed at the intersection of three areas that have evolved rapidly in the last decades: PDEs, Control and Numerics.

During the period of the project, work has been done on essentially all aspects of the project as presented in the schedule of activities. These are evidenced by the list of articles written, most of them already published or accepted for publication, and some submitted for publication.

Part B: Specific Results Obtained

We list here some of the specific results obtained.

1. **A new notion of control for fractional PDEs.** The classical controllability properties for local PDEs consist of placing the control function inside the domain where the PDE is satisfied (interior control) or at the boundary of the domain (boundary control). We have shown that for nonlocal PDEs involving the fractional Laplace operator, this notion of boundary control

does not make sense. This is due to the fact that the corresponding PDE (stationary or time-dependent) is not well-posed if the control function is prescribed at the boundary. For such PDEs, we have introduced a new notion of control that replaces the above mentioned boundary control. More precisely, for fractional PDEs, we have shown that the control function must be localized outside the domain where the PDE is satisfied. We called it "exterior control". This notion of exterior controllability is consistent with many applications in real life phenomena. In a series of papers, we have studied the controllability properties from the exterior of space-time fractional diffusive equations, space-time fractional super-diffusive equations and fractional heat equations, where some null/exact controllability results, some lack of null/exact controllability results and some approximate controllability results have been proved.

2. **Controllability properties of fractional heat equations under constraints.**

The controllability properties of the fractional heat equation on open subsets of $\mathbb{R}^N$ ($N \geq 2$) are still not fully understood by the mathematical community. The classical tools like the Carleman estimates usually used to study the controllability for heat equations are still not available for the fractional Laplacian (except on the whole space $\mathbb{R}^N$). Even in the case $N = 1$, another difficulty for analyzing the fractional heat equation by using some spectral properties is that contrarily to the local case where the eigenvalues and eigenfunctions of the system are well known, for the fractional case, we just know an asymptotic for the eigenvalues and an explicit formula for the eigenfunctions is not accessible.

In the absence of constraints, the fractional heat equation on a bounded open interval $I \subset \mathbb{R}$ associated with the fractional Laplace operator $(-\Delta)^s$ $(0 < s < 1)$ is null-controllable (from the interior or from the exterior) in any positive time $T > 0$ if and only if $s > 1/2$. We have proved these results by using the gap condition on the eigenvalues, and we have also validated our results through numerical experiments. In space dimension $N \geq 2$, the best possible controllability result available for the fractional heat equation is the approximate controllability that we have obtained very recently and have mentioned above.

Regarding constrained controllability, we have proved the following results:

Firstly, we have shown that, if $s > 1/2$, then the fractional heat equation is controllable from any given initial datum in $L^2(-1, 1)$ to zero (and, by translation, to trajectories) in any positive time $T > 0$ by means of L^∞-controls. This extends our previous analysis where only L^2-controls were considered. The proof uses the canonical approach of reducing the question of controllability with an L^∞-control to a dual observability problem in L^1, and the use of Fourier series expansions to obtain a new result on the L^1-observation of linear combinations of real exponential functions.

Secondly, as a consequence of our first result, we prove the existence of a minimal (strictly positive) time $T_{\min}$ such that the fractional heat dynamics can be controlled to positive trajectories through the action of a positive control. Moreover, if the initial datum is supposed to be positive as well, then the maximum principle guarantees the positivity of the states too.

Thirdly, we have shown that in this minimal time, constrained controllability is achieved by means of a control that belongs to a certain space of Radon measures.

Fourthly, we have made some numerical simulations that confirm our theoretical results.

In many PDEs models some constraints need to be imposed when considering practical applications. This is for instance the case of diffusion processes (heat conduction, population dynamics, etc.), where realistic models have to take into account that the state represents some physical quantity which must necessarily remain positive. This topic is also related to some other relevant applications, like the optimal management of compressors in gas transportation networks, requiring the preservation of severe safety constraints. Finally, this issue is also important in other PDE problems based on scalar conservation laws, including the Lighthill-Whitham and Richards traffic flow models or the isentropic compressible Euler equation.

Most of the existing controllability theory for PDEs has been developed in the absence of constraints on the controls and/or the state. To the best of our knowledge, the literature on constrained controllability is currently very limited and the majority of the available results do not guarantee that controlled trajectories fulfill the physical restrictions of the processes under consideration.

The study of the controllability properties under positivity constraints is a very reasonable question for scalar-valued parabolic equations, which are canonical examples where positivity is preserved for the free dynamics. Therefore, the issue of whether the system can be controlled in between two states by means of positive controls, by possibly preserving also the positivity of the controlled solution, arises naturally.

The existence of a minimal time for constrained controllability is in counter-trend with respect to the unconstrained case, in which linear and semi-linear parabolic systems are known to be controllable at any positive time. Notwithstanding, often times, norm-optimal controls allowing to reach the target at the final time are restrictions of solutions of the adjoint system. Accordingly these controls experience large oscillations in the proximity of the final time, which are enhanced when the time horizon of control is small. This eventually leads to control trajectories that go beyond the physical thresholds and fail to fulfill the positivity constraint.

On the other hand, when the time interval is long, we expect the control property to be achieved with controls of small amplitude, thus ensuring small deformations of the state and, in particular, preserving its positivity. Roughly speaking, by imposing constraints to the control, we are somehow providing an impediment for the state to reach the target, unless the control time horizon is long enough. This behavior is then a warning that existing unconstrained controllability results, that are valid within arbitrarily short time, may be unsuitable in practical applications in which state-constraints need to be preserved along controlled trajectories.

3. **Controllabilty properties of fractional wave equations with memory terms.** Evolution equations involving memory terms are an effective tool for modeling a large spectrum of phenomena which apart from their current state are influenced also by their history. They appear in several different applications, including viscoelasticity, non-Fickian diffusion, and thermal processes with memory. Controllability problems for evolution equations with memory terms have been extensively studied in the past. Nevertheless, in the majority of these works the issue has been addressed focusing only on the steering of the state of the system to zero at time T,

without considering that the presence of the memory introduces additional effects that makes the classical controllability notion not suitable in this context. Indeed, driving the solution of the equation to zero is not sufficient to guarantee that the dynamics of the system reaches an equilibrium. If we were considering an equation without memory, once its solution is driven to rest at time T by a control, then it vanishes for all $t \geq T$ also in the absence of control. On the other hand, the introduction of a memory term may produce accumulation effects that affect the stability of the system. For these reasons, the classical notion of controllability for a wave equation, requiring the state and its velocity to vanish at time T, has been extended with the additional imposition that the control shall *shut down* also the memory effects. This special notion of controllability is generally called *memory-type null controllability.* We have completely analyzed the controllability properties of the wave equation with memory terms involving the fractional operators in the above mentioned framework. In more detail, the technique we use is based on a spectral analysis and an explicit construction of biorthogonal sequences. Although this approach limits our study to a one-dimensional case, it has the additional advantage of offering new insights on the behavior of this type of problems through the detailed study of the properties of the spectrum. Besides, as in other related previous works, we view the wave model with memory terms as the coupling of a wave-like PDE with an ODE. This approach enhances the necessity of a moving control strategy. Indeed, we show that the memory-type null controllability of the system fails if the support of the control is time-independent, unless of course in the trivial case. We mention that this strategy of a moving control has been successfully used in the past in the framework of viscoelasticity, the structurally damped wave equation and the Benjamin-Bona-Mahony equation.

4. **Lack of null-exact controllability for strong damping wave equations.** We have shown that strong damping wave equations associated with the Laplace operator or/and the fractional Laplace operator cannot be null or exact controllable. These results apply to interior or boundary controls for local PDEs and to interior or exterior controls for fractional PDEs. As a substitute, we have proved that they are indeed approximately controllable. This is the best possible result regarding the controllability properties of such PDEs.

5. **Controllability properties of fractional Moore-Gibson-Thompson equations.** For the first time, we have introduced the controllability properties from the exterior of a fractional version of the so called Moore-Gibson-Thompson (MGT) equation. The local MGT equation (which is originally a nonlinear equation), arises from modeling high amplitude sound waves. The classical nonlinear acoustics models include Kuznetsov's equation, the Westervelt equation and the Kokhlov-Zabolotskaya-Kuznetsov equation. A complete analysis concerning well-posedness, regularity, stability and asymptotic behavior of solutions has been done.

   Despite the wide range of applications of the local MGT equation, such as the medical and industrial use of high intensity ultrasound in lithotripsy, thermotherapy, ultra-sound cleaning, etc., there exists only one work about their controllability properties. In that work, the authors proved that the local MGT equation can be controlled using an interior control function supported on a moving subset of the domain, in a such way that it can visit all the domain. In other words, it is impossible to get an interior controllability to the local MGT equation when the control function is localized in a fixed subset of the domain. This poor control property is closely related to the fact that the damping term, in the local case, generates accumulation points in the spectrum. In particular, the boundary control problem will have the same issues. Consequently, and due to the nature of the applications, it is reasonable to ask if the dynamics of the model can be controlled by means of external forces.

We have shown that the fractional MGT equation is not exact or null controllable from the exterior or from the interior at time $T > 0$. However, we obtain that the system is indeed approximately controllable at any $T > 0$. We remark that this is the best possible conclusion that can be obtained regarding the controllability of the MGT equation.

6. **New class of source identification and optimal control of fractional elliptic and parabolic problems and numerical analysis.**

In many real life applications a source or a control is placed outside (disjoint from) the observation domain Ω where the PDE is satisfied. Some examples of inverse and optimal control problems where this situation may arise are: Acoustic testing, when the loudspeakers are placed far from the aerospace structures; Magnetotellurics (MT), which is a technique to infer earth's subsurface electrical conductivity from surface measurements; Magnetic drug targeting (MDT), where drugs with ferromagnetic particles in suspension are injected into the body and the external magnetic field is then used to steer the drug to relevant areas, for example, solid tumors; Electroencephalography (EEG) is used to record electrical activities in brain, in case one accounts for the neurons disjoint from the brain, one will obtain an external source problem. Having learned from the above mentioned exterior control, we have introduced a new class of source identification and optimal control problems where the source/control is located outside the observation domain where the PDE is satisfied.

The classical diffusion models lack this flexibility as they assume that the source/control is located either inside or on the boundary. This is essentially due to the locality property of the underlying operators.

We use the nonlocality of the fractional operator to create a framework that now allows placing a source/control outside the observation domain. We consider the Dirichlet, Robin and Neumann source identification or optimal control problems. These problems require dealing with the nonlocal normal derivative. We create a functional analytic framework and show well-posedness and derive the first order optimality conditions for these problems. We introduce a new approach to approximate, with convergence rate, the Dirichlet problem with nonzero exterior condition.

The numerical examples confirm our theoretical findings and illustrate the practicality of our approach.

Using our new notion of optimal control and source identification (inverse) problems mentioned above where we allow the control/source to be outside the domain where the fractional elliptic PDE is fulfilled, we have extended this work to the parabolic case. Several new mathematical tools have been developed to tackle the parabolic problem. We tackle the Dirichlet, the Neumann and the Robin cases. We introduce the notions of weak and very-weak solutions to the fractional parabolic Dirichlet problem.

We present an approach on how to approximate the fractional parabolic Dirichlet solutions by the fractional parabolic Robin solutions (with convergence rates). To the best of knowledge, such a result is not known even in the classical local case. A complete analysis for the Dirichlet and Robin optimal control problems has been discussed.

The numerical examples confirm our theoretical findings and further illustrate the potential benefits of nonlocal models over the local ones.

7. **Discrete dynamical systems.** We derive well-posedness results and an explicit representation of solutions in terms of special functions for semilinear space-time fractional evolution equations involving the discrete fractional Laplace operator. We prove the comparison principle. A special case of our equation is the discrete Fisher-KPP equation with or without delay. Our results include cubic nonlinearities and incorporates new results for the discrete Newell-Whitehead-Segel equation. We use Lévy stable processes as well as Mittag-Leffler, Wright and modified Bessel functions to describe the solutions of the linear lattice model, providing a useful framework for further study. For the nonlinear model, we use a generalization of the upper-lower solution method for reaction-diffusion equations in order the provide existence and uniqueness of solutions. We have also obtained an explicit formula of the fundamental solutions associated to the above mentioned systems.

8. **Fractional powers of sectorial operators.** Inspired by the characterization of the fractional Laplace operator by the so called Caffarelli-Silvestre extension, we have obtained a precise characterization (explicit description of their domain, minimal regularity needed for their existence) of fractional powers of sectorial operators given by a quadractic form (coercive or non-coercive) via a Dirichlet-to-Neumann map. These fractional powers operators turn out to be nonlocal. The dynamics (existence, regularity, continuous dependence and asymptotic behavior of solutions) of corresponding heat and wave equations have been examined.

9. **Semilinear fractional PDEs and applications.** In our monograph published this year in Springer Books Series, Mathématiques et Applications, we have considered fractional kinetic equations characterized by the presence of a nonlinear time-dependent source, generally of arbitrary growth in the unknown function, a time derivative in the sense of Caputo and the presence of a large class of diffusion operators. Beside classical examples involving the Laplacian, subject to standard (namely, Dirichlet, Neumann, Robin, dynamic/Wentzell and Steklov) boundary conditions, our framework includes also non-standard diffusion operators of fractional type subject to appropriate boundary or exterior conditions. We gave a unified scheme and analysis for the existence and uniqueness of strong and mild solutions, and then deal separately with the global regularity problem and the asymptotic behavior of solutions. Then we extended the analysis to systems of fractional kinetic equations that include prey-predator models of Volterra-Lotka type and chemical reactions models, all of them containing possibly some fractional kinetics.

10. **Controllability of multi-D fractional evolution equations.** In the context of PDEs involving the fractional Laplace operator $(-\Delta)^s$ $(0 < s < 1)$, the number of observability/controllability results is limited and have been essentially obtained by our team. They are essentially as follows:

    - **1-D problems**: In fact, 1-D problems can be handled by using Fourier series techniques as well as Ingham type inequalities and the corresponding parabolic versions. In this direction, we have proved several outstanding results on the interior and the exterior null controllability of the 1-D fractional heat equation.
    - **Multi-D problems**: We have shown the interior null controllability of multi-D fractional Schrödinger equations by using the Pohozaev identity for the fractional Laplace operator. Apart from that, we have proved the approximate controllability for the fractional heat equation, the strong damping fractional wave equation and the fractional Moore-Gibson-Thomson equation, in multi-D. These are the only multi-D results available before our works in the second year.

    In this second year of the project we made significant contributions by filling these gaps. We have proved that finite energy solutions of the fractional heat equation are null-controllable in any time $T > 0$ and in the range of exponents $s \in (1/2, 1)$. The methods that we developed are quite elaborated and employ many of the classical tools and techniques developed in the context of PDEs control theory, but carefully adapted to the fractional setting. The main results obtained in these topics can be summarized as follows:

    - **Partial observability/controllability of fractional wave equations through multiplier techniques.** Firstly, we have analyzed the observability and controllability properties of the fractional wave equation. More precisely, we have applied the Pohozaev identity for the fractional Laplace operator to develop the standard multiplier techniques in the fractional setting, and use it to prove the observability of the fractional wave equation from a certain neighborhood ω of the boundary $\partial\Omega$ (see Figure 1 below) for exponents s in the interval $[1/4, 1)$. The obtained observability results are only of partial nature, frequency-dependent. In fact, they are limited to solutions involving a finite number of Fourier components of the fractional Laplacian and they hold in a time horizon that grows as the number of involved eigenmodes increases. Let us notice that, this restriction comes naturally from the use of the Pohozaev identity. Furthermore, these results are also sharp in the sense that the lack of a uniform velocity of propagation for the fractional waves, when $0 < s < 1$, makes impossible to observe the high-frequency components in a uniform time. Note also that in this context, the assumption that $1/4 \leq s < 1$ seems to be necessary to accomplish the proof using the Pohozaev identity, although one could expect similar partial (frequency-dependent) results to be also true in the range $0 < s < 1/4$. Out of the partial observability inequality, by using some duality arguments, we established the null controllability of low-frequency projections of solutions to the fractional wave equation for every $1/4 \leq s < 1$.

- **Transmutation and controllability properties of fractional heat equations.** Using the observability for low-frequency solutions of the wave equation, we have transfered this result into the parabolic context. By duality, we have proved the null controllability for finite-dimensional frequency-dependent sub-spaces of solutions of the fractional heat equation with an explicit measure of the controllability cost. Getting this kind of explicit frequency-dependence estimates on the cost of control is essential to achieve our final goal. In the absence of such estimates the results can be achieved by combining unique continuation and finite-dimensional arguments, but then one losses control on the explicit dependence on the dimension of the finite-dimensional projections.
- **Controllability properties of finite energy solutions of fractional heat equations.** Finally, we discover a strategy to obtain the controllability of finite energy solutions of the fractional heat equation in the range of exponents $s \in (1/2, 1)$. Our methods work only if the observation domain ω is a neighborhood of the boundary $\partial\Omega^+ := \{x \in \partial\Omega : (x - x_0) \cdot \nu > 0\}$ as follows:

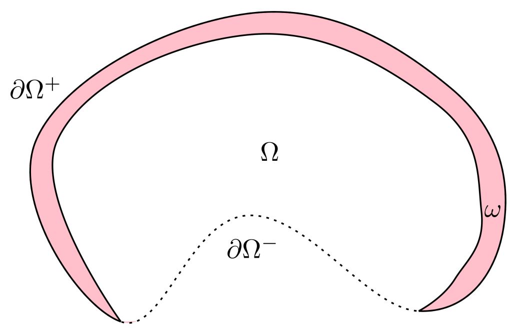

Figure 1: Domain Ω and control region ω.

11. **A new notion of controllabilty of fractional equations with memory terms.** Evolution equations involving memory terms are an effective tool for modeling a large spectrum of phenomena which apart from their current state are influenced also by their history. They appear in several different applications, including viscoelasticity, non-Fickian diffusion, and thermal processes with memory. Controllability problems for evolution equations with memory terms have been extensively studied in the past. Nevertheless, in the majority of these works the issue has been addressed focusing only on the steering of the state of the system to zero at time T, without considering that the presence of the memory introduces additional effects that makes the classical controllability notion not suitable in this context. Indeed, driving the solution of the equation to zero is not sufficient to guarantee that the dynamics of the system reaches an equilibrium. If we were considering an equation without memory, once its solution is driven to rest at time T by a control, then it vanishes for all $t \geq T$ also in the absence of control. On the other hand, the introduction of a memory term may produce accumulation effects that affect the stability of the system. For these reasons, for example, the classical notion of controllability for

a wave equation, requiring the state and its velocity to vanish at time T, has been extended with the additional imposition that the control shall *shut down* also the memory effects. This special notion of controllability is generally called *memory-type null controllability.* We have completely analyzed the controllability properties of the wave equation with memory terms involving the fractional operators in the above mentioned framework. In more detail, the technique we use is based on a spectral analysis and an explicit construction of biorthogonal sequences. Although this approach limits our study to a one-dimensional case, it has the additional advantage of offering new insights on the behavior of this type of problems through the detailed study of the properties of the spectrum. Besides, as in other related previous works, we view the wave model with memory terms as the coupling of a wave-like PDE with an ODE. This approach enhances the necessity of a moving control strategy. Indeed, we show that the memory-type null controllability of the system fails if the support of the control is time-independent, unless of course in the trivial case. We mention that this strategy of a moving control has been successfully used in the past in the framework of viscoelasticity, the structurally damped wave equation and the Benjamin-Bona-Mahony equation.

12. **Dynamics and asymptotic behavior of thermoelastic plate models.** In a bounded domain, we consider a thermoelastic plate with rotational forces. The rotational forces involve the spectral fractional Laplacian, with power parameter $0 \leq \theta \leq 1$. The model includes both the Euler-Bernoulli ($\theta = 0$) and Kirchhoff ($\theta = 1$) models for thermoelastic plate as special cases. First, we show that the underlying semigroup is of Gevrey class δ for every $\delta > (2-\theta)/(2-4\theta)$ for both the clamped and hinged boundary conditions when the parameter θ lies in the interval $(0, 1/2)$. Then, we show that the semigroup is exponentially stable for hinged boundary conditions, for all values of θ in $[0, 1]$. Finally, we prove, by constructing a counterexample, that, under hinged boundary conditions, the semigroup is not analytic, for all θ in the interval $(0, 1]$. The main features of our Gevrey class proof are: the frequency domain method, appropriate decompositions of the components of the system and the use of Lions' interpolation inequalities.

13. **Control of fractional elliptic PDEs with state and control constraints.:** In the elliptic case, several mathematical tools have been developed during the process to study these problems, for instance, the characterization of the dual of fractional order Sobolev spaces and the well-posedness of fractional elliptic equations with measure-valued data. These tools are widely applicable. We show well-posedness of the control problem and derive the first order optimality conditions. Notice that the adjoint equation is a fractional partial differential equation with a measure as the right-hand-side datum. We use the characterization of the fractional order dual spaces to study the regularity of solutions of the state and adjoint equations. As an application of the regularity result of the adjoint equation, we established the Sobolev regularity of the control. In addition, under this setup, even weaker controls can be used to obtain the same conclusion.

14. **Control of fractional parabolic PDEs with state and control constraints:** We introduced a new notion of optimal control and source identification (inverse) problems where we allow the control/source to be outside the domain where the fractional parabolic PDE is fulfilled.

Several new mathematical tools have been developed to handle the parabolic problem. We tackle the Dirichlet, the nonlocal Neumann and Robin cases. The need for these novel control concepts stems from the fact that classical PDEs models only allow placing the control/source either on the boundary or in the interior where the PDEs are satisfied. However, the nonlocal behavior of the fractional operator now allows placing the control/source in the exterior. We introduced the notions of weak and very-weak solutions to the fractional parabolic Dirichlet problem. We presented an approach on how to approximate the fractional parabolic Dirichlet solutions by the fractional parabolic Robin solutions (with convergence rates). A complete analysis for the Dirichlet and the nonlocal Robin control problems have been discussed. The obtained numerical examples confirm our theoretical findings and further illustrate the potential benefits of nonlocal models over the local ones.

15. **Control and numerical approximation of fractional diffusion equations.** This work has been submitted as a Chapter in the Handbook of Numerical Analysis. The aim of this chapter is to give a complete panorama of the control properties of fractional diffusive models. We have done this by surveying several research results we obtained in the last years, focusing in particular on aspects related with the numerical computation of the controls, though not forgetting to recall other relevant contributions which can be currently found in the literature of this prolific field. Our reference model is a non-local diffusive dynamics driven by the fractional Laplace operator on a bounded domain Ω. The starting point of our analysis is to conceive a Finite Element approximation for the associated elliptic model in one and two space-dimensions, for which we also provide error estimates and convergence rates in the L^2 and energy norm. In a second moment, we address two specific control scenarios: firstly, we consider the standard interior control problem, in which the control is acting from a small subset $\omega \subset \Omega$. Secondly, we move our attention to the exterior control problem, in which the control region $\mathcal{O} \subset \Omega^c$ is located outside the domain of definition. This exterior control notion is the extension of boundary control to the fractional diffusion framework, in which the non-local nature of the models does not allow for controls supported on the boundary of the domain. We conclude by discussing the interesting problem of simultaneous control, in which we consider families of parameter-dependent fractional heat equations and we aim at designing a unique control function capable of steering all the different realizations of the model to the same target configuration. In this framework, we see how the employment of stochastic optimization techniques may help in alleviating the computational burden for the approximation of simultaneous controls. Our discussion is complemented by several open problems related with fractional models which are currently unsolved and may be of interest for future investigation.

16. **Exterior controllability properties of a nonlocal Moore-Gibson-Thompson equation.** The three concepts of *exact, null, and approximate* controllabilities are analyzed from the exterior of the Moore–Gibson–Thompson equation associated with the fractional Laplace operator subject to the nonhomogeneous Dirichlet type exterior condition. We show that if $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with a Lipschitz continnuous boundary $\partial\Omega$, then there is no control function such that the fractional Moore-Gibson-Thompson equation with exterior controls data is exactly or null controllable in finite time $T > 0$. However, we prove that the

system is indeed approximately controllable for every exterior datum which is localized in an arbitrary non-empty open set $\mathcal{O} \subset \mathbb{R} \setminus \overline{\Omega}$.

17. **Approximate and mean approximate controllability properties for Hilfer time-fractional differential equations.** We study the approximate and mean approximate controllability properties of fractional partial differential equations associated with the so-called Hilfer type time-fractional derivative and a non-negative selfadjoint operator A with a compact resolvent on $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded open set, and the control function is localized in an open nonempty set $\omega \subset \Omega$. We show that the system is approximately controllable in any time $T > 0$ and any nonempty open set $\omega \subset \Omega$. In addition, if the operator A_B has the unique continuation property, then the system is also mean (memory) approximately controllable. The operator A_B can be the realization in $L^2(\Omega)$ of a symmetric, non-negative uniformly elliptic second order operator with Dirichlet or Robin boundary conditions, or the realization in $L^2(\Omega)$ of the fractional Laplace operator $(-\Delta)^s$ $(0 < s < 1)$ with the Dirichlet exterior condition, or nonlocal Robin type exterior conditions.

18. **Optimal control of fractional PDEs with state and control constraints.** In this work we consider optimal control of fractional (elliptic and parabolic) PDEs with both state and control constraints. The key challenge is how to handle the state constraints. Similarly to the elliptic case, we establish several new mathematical tools in the parabolic setting that are of wider interest. For example, existence of solution to the fractional parabolic equation with measure data on the right-hand-side. We employ the Moreau-Yosida regularization to handle the state constraints in both elliptic and parabolic cases. We establish convergence, with rate, of the regularized optimal control problems to the original ones. The spatial discretization is carried out using a finite element method and discretization error estimates are provided for the elliptic setting. Several illustrative numerical examples in both elliptic and parabolic setting have been provided.

19. **Optimal control of mixed local-nonlocal parabolic PDE with singular boundary-exterior data.** We consider parabolic equations on bounded smooth open sets $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ with mixed Dirichlet type boundary-exterior conditions associated with the elliptic operator $\mathcal{L} = -\Delta + (-\Delta)^s$ $(0 < s < 1)$. Firstly, we prove several well-posedness and regularity results of the associated elliptic and parabolic problems with smooth, and then with singular boundary-exterior data. Secondly, we show the existence of optimal solutions of associated optimal control problems, and we characterize the optimality conditions. This is the first time that such topics have been presented and studied in a unified fashion for mixed local-nonlocal PDEs with singular data.

20. **Optimal control problems of parabolic fractional Sturm-Liouville equations in a star graph.** In this work we deal with parabolic fractional initial-boundary value problems where the involved operator is a fractional version of the Sturm–Liouville operator in an interval and in a general star graph. We first give several existence, uniqueness and regularity results of weak and very-weak solutions. We prove the existence and uniqueness of solutions to a quadratic

boundary optimal control problem and provide a characterization of the optimal control via the Euler–Lagrange first order optimality conditions. We then investigate the analogous problems for a fractional Sturm–Liouville problem in a general star graph with mixed Dirichlet and Neumann boundary controls. The existence and uniqueness of minimizers, and the characterization of the first order optimality conditions are obtained in a general star graph by using the method of Lagrange multipliers.

21. **Turnpike and exponential turnpike property for fractional parabolic equations with non-zero data.** We consider averages convergence as the time-horizon goes to infinity of optimal solutions of time-dependent optimal control problems to optimal solutions of the corresponding stationary optimal control problems. Control problems play a key role in engineering, economics and sciences. To be more precise, in climate sciences, often times, relevant problems are formulated in long time scales, so that, the problem of possible asymptotic behaviors when the time-horizon goes to infinity becomes natural. Assuming that the controlled dynamics under consideration are stabilizable towards a stationary solution, the following natural question arises: Do time averages of optimal controls and trajectories converge to the stationary optimal controls and states as the time-horizon goes to infinity? This question is very closely related to the so-called turnpike property that shows that, often times, the optimal trajectory joining two points that are far apart, consists in, departing from the point of origin, rapidly getting close to the steady-state (the turnpike) to stay there most of the time, to quit it only very close to the final destination and time. In this work we deal with heat equations with non-zero exterior conditions (Dirichlet and nonlocal Robin) associated with the fractional Laplace operator $(-\Delta)^s$ $(0 < s < 1)$. We prove the turnpike property for the nonlocal Robin optimal control problem and the exponential turnpike property for both Dirichlet and nonlocal Robin optimal control problems.

22. **A unified framework for optimal control of fractional in time subdiffusive semilinear PDEs.** We consider optimal control of fractional in time (subdiffusive, i.e., for $0 < \gamma < 1$) semilinear parabolic PDEs associated with various notions of diffusion operators in an unifying fashion. Under general assumptions on the nonlinearity we first show the existence and regularity of solutions to the forward and the associated backward (adjoint) problems. In the second part, we prove existence of optimal controls and characterize the associated first order optimality conditions. Several examples involving fractional in time (and some fractional in space diffusion) equations are described in detail. The most challenging obstacle we overcome is the failure of the semigroup property for the semilinear problem in any scaling of (frequency-domain) Hilbert spaces.

23. **Control and numerical approximation of fractional diffusion equations.** The aim of this chapter is to give a complete panorama of the control properties of fractional diffusive models. We will do this by surveying several research results we obtained in the last years, focusing in particular on aspects related with the numerical computation of the controls, though not forgetting to recall other relevant contributions which can be currently found in the literature of this prolific field. Our reference model will be a non-local diffusive dynamics driven by the

fractional Laplace operator on a bounded domain Ω. The starting point of our analysis will be to conceive a Finite Element approximation for the associated elliptic model in one and two space-dimensions, for which we also provide error estimates and convergence rates in the L^2 and energy norm. In a second moment, we will address two specific control scenarios: firstly, we consider the standard interior control problem, in which the control is acting from a small subset $\omega \subset \Omega$. Secondly, we move our attention to the exterior control problem, in which the control region $\mathcal{O} \subset \Omega^c$ is located outside the domain of definition. This exterior control notion is the extension of boundary control to the fractional diffusion framework, in which the non-local nature of the models does not allow for controls supported on the boundary of the domain. We will conclude by discussing the interesting problem of simultaneous control, in which we consider families of parameter-dependent fractional heat equations and we aim at designing a unique control function capable of steering all the different realizations of the model to the same target configuration. In this framework, we will see how the employment of stochastic optimization techniques may help in alleviating the computational burden for the approximation of simultaneous controls. Our discussion is complemented by several open problems related with fractional models which are currently unsolved and may be of interest for future investigation.

24. **Optimal control, numerics, and applications of fractional PDEs.** This works provides a brief review of recent developments on two nonlocal operators: fractional Laplacian and fractional time derivative. We start by accounting for several applications of these operators in imaging science, geophysics, harmonic maps and deep (machine) learning. Various notions of solutions to linear fractional elliptic equations are provided and numerical schemes for fractional Laplacian and fractional time derivative are discussed. Special emphasis is given to exterior optimal control problems with a linear elliptic equation as constraints. In addition, optimal control problems with interior control and state constraints are considered. We also provide a discussion on fractional deep neural networks, which is shown to be a minimization problem with fractional in time ordinary differential equation as constraint. The paper concludes with a discussion on several open problems.

Part C: Travels and Presentations

1.
   - In June 2018, Dr. Mahamadi Warma, the project's PI has participated, and gave one of the main lecture, to the conference on " Fractional PDEs: Theory, Algorithms and Applications", organized by Brown University (Providence) at their ICERM center.
   - Warma spent one month (June 20 to July 20, 2018) in Chile where he did some research work with Dr. Carlos Lizama professor at the "Universidad de Santiago de Chile".
   - In September 2018, Warma has participated to the Dynamics and Control annual program review in Virginia where he has presented the most relevant results so far obtained.
   - In January 2019, Warma traveled to George Mason University in Fairfax (Virginia), where he did research work with Dr. Harbir Antil and Dr. Pablo Stinga from Iowa State University (Ames) on how to define fractional powers of elliptic operators with non-homogeneous boundary conditions.

- In March 2019, Warma has participated to the "34^{th}-Seminario Interuniversitario de Investigación en Ciencias Matemática" organized by the University of Puerto Rico, Humacao Campus, at Humacao.
- In March 2019, he also traveled to George Mason University in Fairfax (Virginia), to continue the research work started with Dr. Harbir Antil on optimal control for fractional PDEs. He also gave a presentation at the applied mathematics research seminar.
- In May 2019, Warma traveled to Germany to participate, and give one of the main lecture, to the workshop on "Parabolic Evolution Equations, Harmonic Analysis and Spectral Theory", organized by the Universities of Karlsruhe and Ulm (Germany).
- Warma traveled to Chile (May 20 to July 20, 2019) to do some research work with Dr. Carlos Lizama (Universidad de Santiago de Chile) and Dr. Rodrigo Ponce (Universidad de Talca).
- In September 19-22, Warma traveled to Florida for the D&C annual program review. He gave a talk on the progress and the main results obtained in the project.
- In March 1-5, 2021, Mahamadi Warma has participated to the virtual SIAM Conference on Computational Science and Engineering, and he gave a talk with the title: Controllability properties of fractional evolution equations.
- In February 26-27, 2021, Mahamadi Warma has participated to the virtual SIDIM conference of research in Mathematical Sciences in Puerto Rico. He gave one of the main lectures with the title: What are the classical boundary conditions for the fractional Laplace operator?
- March 10-16, 2022 Warma traveled to San Juan, Puerto Rico, to do some research work with Dr. Valentin Keyantuo and his team.
- March 21-23 Warma traveled to Carnegie Melon University in Pittsburgh to give a colloquium talk on the "turnpike property for fractional evolution equations."

2. In March 2019, Dr. Keyantuo CoPI of the project, has participated to the "34^{th}-Seminario Interuniversitario de Investigación en Ciencias Matemática" organized by the University of Puerto Rico, Humacao Campus, at Humacao.

3. In August 2018, Dr. Harbir Antil (George Mason University, Fairfax) and Dr. Pablo Stinga (Iowa State University, Ames) visited the University of Puerto, Rio Piedras Campus to do research work with Dr. Warma. They have investigated how to define fractional powers of elliptic operator with non-homogeneous boundary conditions. The team has met in January 2019 at George Mason University in Fairfax as mentioned above, and will meet in August 2019 at Iowa State University in Aimes to complete the research work on the above mentioned topic.

4. Dr. Umberto Biccari (DeustoTech, Bilbao, Spain) visited the UPRRP from November 19 to December 15, 2018 to do research work with Dr. Warma on controllability properties of fractional wave equations with memory terms.

5. Burhard Claus, PhD student at the Technical University of Dresden (Germany) has visited the UPRRP for one month in March 2019, to do research work with Dr. Warma. They have studied realizations of the fractional Laplace operator with nonlocal Neumann and Robin exterior conditions via forms method.

6. Jorge Gonzalez, PhD student at the "Universidad de Santiago de Chile" spent 5 months (March 01, 2018 to July 30, 2018) at the UPRRP to work with Dr. Valentin Keyantuo and Dr. Mahamadi Warma on fundamental solutions of space-time fractional diffusion equations involving the discrete fractional Laplace operator. He was financially supported by a Chilean scholarship.

7. Dr. Carole Louis-Rose (University of Guadeloupe, France) visited the UPRRP for two weeks in February 2019 to do research work with Warma on the controllability properties from the exterior of super diffusive equations.

8. Silvia Rueda, PhD student at the "Universidad de Santiago de Chile" spent 4 months (October 16, 2018 to February 15, 2019) at the UPRRP to do research work with Keyantuo and Warma on discrete fractional PDEs. She was financially supported by a Chilean scholarship.

9. In April 2019, Dr. Louis Tebou (Florida International University) spent one week at the UPRRP to work with Keyantuo and Warma on uniform analyticity and exponential decay of solutions of nonlocal thermoelastic plate equations.

10. In December 2018, Professor Enrique Zuazua (Universidad Autonoma de Madrid, Spain) spent two weeks in the Department of Mathematics of the University of Puerto Rico at Río Piedras were he carried out research work with Warma. During the visit, he delivered a lecture at the Departmental research seminar. The visit was also the opportunity to discuss various aspects of the project and make plans for future activities thereto related. We mention that Dr. Enrique Zuazua is the consultant of the research team of the project and he advises the team on all aspects of their research activities.

11. In December 2019, Professor Enrique Zuazua (University of Erlangen-Nürenberg, Germany) spent two weeks in the Department of Mathematics of the University of Puerto Rico at Río Piedras were he carried out research work with Warma. The visit was also the opportunity to discuss various aspects of the project and make plans for future activities thereto related.

12. In February 2020, Dr. Cirpian G. Gal (Florida International University) spent one week at George Mason University (Fairfax, Virginia) to work with Warma on control and transmission problems for fractional PDEs.

13. Due to the COVID-19 pandemic travels were and are still suspended at George Mason University (GMU) and also at the University of Puerto Rico, Rio Piedras Campus (UPRRP).

Part D: Training: Graduate Students

- Fabian Seoanes will complete his doctoral work this summer with a fellowship from the project. His Dissertation defense will take place at the end of August, 2019.

- Ernest Aragones will complete his doctoral work this summer with a fellowship from the project. His Dissertation defense will take place at the end of August, 2019.
- Dr. Thomas Brown, a postdoctoral fellow at GMU, working with Warma, is partially supported by the project.
- The PhD student Henrry Cortez Portillo was supported August-December 2020 with a fellowship from the project.
- Jesus Oliva Maza, a PhD student at the University of Zaragoza in Spain has spent three months (October-December, 2021) at GMU working with Dr. Warma. He was partially funded by the project.
- Andrew J. Bailey, a PhD student at GMU under the supervision of Dr. Warma has received a fellowship from the project.
- Madeline Horton, a PhD student at GMU under the supervision of Dr. Warma has received a fellowship from the project.

Part E: Workshop

On December 5-7, 2018, we have organized a workshop on "Dynamics, Control and Numerics for Fractional PDEs". The workshop was held in San Juan, Puerto Rico, at the Embassy Suites Hotel. There were a total of 45 participants including world class experts in the academia, researchers form the national laboratories, postdoctoral researchers and graduate students working in areas covered by the topics of the workshop. Additionally, two (2) program managers from the Air Force Office of Scientific Research were present for the entire duration of the workshop. There were five (5) keynote addresses of one hour each, covering all aspects of control theory, dynamics, numerical treatment and computational aspects of fractional PDEs. These ranged from the theoretical to the most applied aspects, and the related physical and engineering models. We had nine (9) invited talks of 45 minutes each given by junior researchers who have already obtained significant results in the field. For these talks, the speakers and the subject of their presentations covered all the above mentioned aspects. The program also included six (6) contributed talks of 30 minutes each given by postdoctoral researchers and graduate students working on the area of fractional PDEs and its applications. The workshop was sponsored by the AFOSR and the ARO.

Part F: Publications

Part I-Published:

1. Wolfgang Arendt, A.F.M. ter Elst and Mahamadi Warma. Fractional powers of sectorial operators via the Dirichlet-to-Neumann operator. **Communication in Partial Differential Equations 43 (2018), 1–24. DOI = 10.1080/03605302.2017.1363229**

2. Umberto Biccari, Mahamadi Warma and Enrique Zuazua. Local regularity for fractional heat equations. **Recent Advances in PDEs: Analysis, Numerics and Control. SEMA-SIMAI Springer Series (2018), 223–249**.

3. Mahamadi Warma. On the (s, p)-Dirichlet-to-Neumann operator on bounded Lipschitz domains. **Journal of Elliptic and Parabolic Equations 4 (2018), 223–269. DOI = 10.1007/s41808-018-0017-2**

4. Mahamadi Warma. Approximate controllabilty from the exterior of space-time fractional diffusive equations. **SIAM J. Control Optim. 57 (2019), 2037–2063. DOI = 10.1137/18M117145X**.

5. Edgardo Alvarez, Ciprian G. Gal, Valentin Keyantuo and Mahamadi Warma. Well-posedness results for a class of semi-linear super-diffusive equations. **Nonlinear Analysis 181 (2019), 24–61. DOI = 10.1016/j.na.2018.10.016**

6. Harbir Antil and Mahamadi Warma. Optimal control of the coefficient for the regional fractional p-Laplace equations: Approximation and convergence. **Mathematical Control and Related Fields 9 (2019), 1–38. DOI = 10.3934/mcrf.2019001**

7. Valentin Keyantuo, Carlos Lizama and Mahamadi Warma. Lattice dynamical systems associated to a fractional Laplacian. **Numerical Functional Analysis and Optimization 40 (2019), 1315–1343. DOI = 10.1080/01630563.2019.1602542**

8. Valentin Keyantuo, Fabian Seoanes and Mahamadi Warma. Fractional Gaussian estimates and holomorphy of semigroups. **Arch. Math. (Basel) 113 (2019), 629–647. DOI = 10.1007/s00013-019-01381-y**

9. Valentin Keyantuo, Carlos Lizama, Silvia Rueda and Mahamadi Warma. Asymptotic behavior of mild solutions for a class of abstract nonlinear difference equations of convolution type. **Adv. Difference Equ. 2019, Paper No. 251, 29 pp. DOI = 10.1186/s13662-019-2189-y**

10. Carlos Lizama, Jorge Gonzalez-Camus, Valentin Keyantuo and Mahamadi Warma. Fundamental solutions for discrete dynamical systems involving the fractional Laplacian. **Math. Methods Appl. Sci. 42 (2019), 4688–4711. https://doi.org/10.1002/mma.5685.**

11. Harbir Antil, Ratna Khatri and Mahamadi Warma. External optimal control of nonlocal PDEs. **Inverse Problems 35 (2019), no. 8, 084003, 35 pp. https://iopscience.iop.org/article/10.1088/1361-6420/ab1299/meta**.

12. Mahamadi Warma and Sebastian Zamorano. Null controllability from the exterior of a one-dimensional nonlocal heat equation. **Control & Cybernetics 48 (2019), 417–438.**

13. Ciprian G. Gal and Mahamadi Warma (Book). Fractional-in-time semilinear parabolic equations and applications. **Springer Books Series, Mathématiques et Applications Vol. 84, 2020. ISBN 978-3-030-45042-7. DOI = 10.1007/978-3-030-45043-4**

14. Umberto Biccari, Mahamadi Warma and Enrique Zuazua. Controllability of the one-dimensional fractional heat equation under positivity constraints. **Communications on Pure & Applied Analysis 19 (2020), 1949–1978. https://www.aimsciences.org/article/doi/10.3934/cpaa.2020086**

15. Umberto Biccari and Mahamadi Warma. Null-controllability properties of a fractional wave equation with a memory term. **Evolution Equations & Control Theory 9 (2020), 399–430. DOI: 10.3934/eect.2020011**.

16. Valentin Keyantuo, Louis Tebou and Mahamadi Warma. A Gevrey class semigroup for a Thermoelastic plate model with a fractional Laplacian: Between the Euler-Bernoulli and Kirchhoff models. **Discrete and Continuous Dynamical Systems (DCDS) Series A 40 (2020), 2875–2889. DOI: 10.3934/dcds.2020152**

17. Harbir Antil and Mahamadi Warma. Optimal control of fractional semilinear PDEs. **ESAIM Control Optim. Calc. Var. 26 (2020), Paper No. 5, 30 pp. https://doi.org/10.1051/cocv/2019003**.

18. Harbir Antil, Deepanshu Verma and Mahamadi Warma. External optimal control of fractional parabolic PDEs. **ESAIM Control Optim. Calc. Var. 26 (2020). DOI = 10.1051/cocv/2020005**

19. Mahamadi Warma and Sebastian Zamorano. Analysis of the controllability from the exterior of strong damping nonlocal wave equations. **ESAIM Control Optim. Calc. Var. 26 (2020), paper No. 42, 34 pp. DOI: 10.1051/cocv/2019028**.

20. Burkhard Claus and Mahamadi Warma. Realization of the fractional Laplacian with nonlocal exterior conditions via forms method. **J. Evolution Equations 20 (2020), 1597–1631. https://doi.org/10.1007/s00028-020-00567-0**

21. Harbir Antil, Deepanshu Verma, and Mahamadi Warma. Optimal control of fractional elliptic PDEs with state constraints and characterization of the dual of fractional order Sobolev spaces. **Journal of Optimization Theory and Applications 186 (2020), 1–23. DOI = 10.1007/s10957-020-01684-z**

22. Rafael Aparicio and Valentin Keyantuo. L^p-Maximal regularity for degenerate second order integro-differential equations on the real line. **Journal of Fourier Analysis and Applications 26 (2020), paper No. 34, 39p. https://doi.org/10.1007/s00041-020-09734-w**

23. Rafael Aparicio and Valentin Keyantuo. Besov Maximal regularity for degenerate second order integro-differential equations on the real line. **Math. Methods Appl. Sci. 43 (2020), 7239–7268. https://doi.org/10.1002/mma.6462**

24. Mahamadi Warma and Sebastian Zamorano. Exponential Turnpike property for fractional parabolic equations with non-zero exterior data. **ESAIM Control Optim. Calc. Var. 27 (2021), Paper No. 1, 35 pp. DOI = 10.1051/cocv/2020076**

25. Carole Louis-Rose and Mahamadi Warma. Approximate controllability from the exterior of space time fractional wave equations. **Applied Mathematics and Optimization 83 (2021), 207–250. DOI :10.1007/s00245-018-9530-9**.

26. Rodrigo Ponce and Mahamadi Warma. Asymptotic behavior and representation of solutions to a Volterra kind of equation with a singular kernel. **Semigroup Forum 102 (2021), 250–273. DOI: 10.1007/s00233-020-10157-8**

27. Ernest Aragones, Valentin Keyantuo and Mahamadi Warma. Approximate and mean approximate controllability properties for Hilfer time-fractional differential equations. **Vietnam J. Math. 49 (2021), 739–765. DOI = 10.1007/s10013-020-00453-9**

28. Carlos Lizama, Mahamadi Warma and Sebastian Zamorano. Exterior controllability properties of a nonlocal Moore-Gibson-Thompson equation. **Fract. Calc. Appl. Anal. 25 (2022), 887–923. DOI = 10.1007/s13540-022-00018-2**

29. Harbir Harbir, Ciprian G. Gal and Mahamadi Warma. A unified framework for optimal control of fractional in time subdiffusive semilinear PDEs. **Discrete Contin. Dyn. Syst. Ser. S 15 (2022), no. 8, 1883–1918. DOI = 10.3934/dcdss.2022012**

30. U. Biccari, Mahamadi Warma and E. Zuazua. Control and numerical approximation of fractional diffusion equations. **Handbook of Numerical Analysis 23 (2022), Pages 1–58. https://doi.org/10.1016/bs.hna.2021.12.001**

31. Hantil Antil, Thomas Brown, Ratna Khatri, Akwum Onwunta, Deepanshu Verma and Mahamadi Warma. Optimal control, numerics, and applications of fractional PDEs. **Handbook of Numerical Analysis 23 (2022), 87–114. https://doi.org/10.1016/bs.hna.2021.12.003**

Part II-Accepted for Publication:

1. Günter Leugering, Gisèle Mophou, Maryse Montamal, and Mahamadi Warma. Optimal control problems of parabolic fractional Sturm-Liouville equations in a star graph. **Mathematical Control and Related Fields, to appear. doi: 10.3934/mcrf.2022015**

2. Jean-Daniel Djida, Gisèle Mophou, and Mahamadi Warma. Optimal control of mixed local-nonlocal parabolic PDE with singular boundary-exterior data. **Evolution Equations and Control Theory, to appear. doi: 10.3934/eect.2022015**

3. Harbir Antil, Thomas Brown, Deepanshu Verma and Mahamadi Warma. Optimal control of fractional PDEs with state and conrol constraints. **Pure and Applied Functional Analysis, to appear**.

Part III-Submitted for Publication:

1. Jesús Oliva Maza and Mahamadi Warma. Introducing and solving generalized Black-Scholes PDEs through the use of functional calculus.

2. Gisèle Mophou and Mahamadi Warma. Quasi-reversibility methods of optimal control for ill-posed final value diffusion equations.

3. Umberto Biccari, Mahamadi Warma and Enrique Zuazua. Null control of multi-D fractional heat equations.

4. Harbir Antil, Umberto Biccari, Rodrigo Ponce, Mahamadi Warma and Sebastian Zamorano. Controllability properties from the exterior under positivity constraints for a 1-D fractional heat equation.