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From vector spaces to hyperspaces, hypermaps and relations: Evolution of sets in metric spaces

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14. ABSTRACT
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During the four years of the project, together with Jean-Pierre Aubin, we were completing extensions of convex and set-valued analysis to sets deprived of any vector structure. Around 400 pages are being prepared for publication. This will be completed in 2023. Some of these results are already announced in published/accepted papers.
All the four postdoctoral fellows involved in the project, for the total of 44,5 months, were investigating control systems and dynamical systems on metric spaces. In particular, Mira Bivas considered optimal control of non-local differential inclusions with solutions defined via reachable sets and studied the optimal feedback in this context. Zeinab Badreddine’s

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"From vector spaces to hyperspaces, hypermaps and relations: Evolution of sets in metric spaces"
July 1, 2018 – June 30, 2022

PI: Hélène Frankowska*

September 24, 2022

Abstract. Control theory and, more generally, analysis of evolutions of systems, were and still are developed mostly in vector spaces with various additional structures. However not all the objects can be seen as elements of a vector space and powerful tools were created in set-valued analysis dealing with subsets of vector spaces. Going beyond vector spaces leads us to hyperspaces (that is subsets of any set, not-necessarily vector space, endowed with the Boolean algebra). The evolution of vectors has then to be replaced by the one of subsets of a hyperspace whose dynamics are described by hypermaps from one hyperspace to another. To regulate not only vectors, but also subsets of a hyperspace it was important to develop analogous of the successful tools of convex and set-valued analysis in such a new framework. Furthermore, a differential calculus of hypermaps had to be designed. In the context of metric spaces, similar problems do arise and the extensions had to be done in harmony with already known, more standard developments, like in the case of the (metric) Wasserstein spaces of probability measures.

During the four years of the project, together with Jean-Pierre Aubin, we were completing extensions of *convex and set-valued* analysis to sets deprived of any vector structure. Around 400 pages are being prepared for publication. This will be completed in 2023. Some of these results are already announced in published/accepted papers.

All the four postdoctoral fellows involved in the project, for the total of 44,5 months, were investigating control systems and dynamical systems on metric spaces. In particular, Mira Bivas considered optimal control of non-local differential inclusions with solutions defined via reachable sets and studied the optimal feedback in this context. Zeinab Badreddine's work was devoted to optimal control of mutational and morphological control systems with solutions defined as in earlier works by J.-P. Aubin. Her task was to investigate the related Hamilton-Jacobi inequalities. She also studied viscosity type solutions to a Hamilton-Jacobi-Bellman equation on the 2-Wasserstein space. Benoît Bonnet was extending control and differential inclusions theoretic results to the Wasserstein metric spaces of Borel probability measures. Here appropriate formulations of the Pontryagin principle and sensitivity relations in optimal control were obtained. Furthermore, he also investigated refined properties of the value function and upper semicontinuity of the optimal feedback map. More recently he extended necessary and sufficient conditions for viability and the Lyapunov second method to Wasserstein spaces. Finally, Thomas Lorenz worked on evolutions driven by merely measurable in time mutational differential inclusions on

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general metric spaces and obtained very general versions of the Filippov theorem and of the invariance theorem. Results were applied to control systems operating on the space of signed Radon measures.

Beside these topics directly related to the project, we pursued the study of state constrained control systems in finite and infinite dimensional Banach spaces, as well as investigations of metric regularity of control systems.

Participants: Let us underline that in France we have 12 months salaries and for this reason Senior scientists could only benefit from the travel support, while postdoctoral fellows had salaries fixed by CNRS according to their working experience. For this reason we cannot indicate the number of months of Senior scientists' work devoted to the project.

Jean-Pierre Aubin, Participant, Professor, retired from U. Paris-Dauphine, France
Hélène Frankowska, PI, DR CNRS, IMJ-PRG and Sorbonne Université, Paris, France
Marc Quincampoix, Participant, Professor, Université de Bretagne Occidentale, France

Present situation of involved postdoctoral fellows:
Mira Bivas, Assistant Professor, Bulgarian Academy of Sciences, Sofia, Bulgaria.
She worked for 6 months on the project.
Benoît Bonnet, Chargé de Recherche CNRS, LASS, Toulouse, France.
He worked for 15 months on the project.
Thomas Lorenz, Professor (W3), University of Rostock, Germany.
He worked for 6 months on the project.
Zeinab Badreddine, Assistant Professor at "Prépa Epsilon", https://prepa-epsilon.fr/ , school training for highly competitive exams for medical studies, Paris, France.
She worked for 16,5 months on the project.

Objectives of the project:

(a) Building of mathematical tools to study *hyperspaces*, that is families of subsets $A \subset X$, when X is any set deprived of any structure, either algebraic or topological. In this approach algebraic operations are replaced by Boolean ones, the sum of vectors by union of subsets, for instance.

(b) Investigation of the evolutions of states of systems which are subsets of X. For this aim it is necessary to define a concept of "velocity" in non-vector spaces, in particular, in plain metric spaces and hyperspaces. Then, armed with this concept of velocity, the objective was to give a meaning to sorts of differential equations (called *morphological equations*) governing the evolution of sets, their movement, their deformation, and to provide examples of evolutions.

Let us underline already here that all the postdoctoral fellows were involved in research on metric spaces, see section 2 below, especially Wasserstein spaces, and not on general spaces lacking a vector structure. This was done purposely, because on one hand, for getting hired, young people need publications on at least partially established topics and the one of Wasserstein spaces enjoys the recent attention of the international control theory and calculus of variations communities. Nevertheless all the postdoctoral fellows contributed with an innovative point of view by extending tools of set-valued analysis to metric spaces and applying them to control problems. On the other hand, extensions of dynamical systems to general sets deprived of any structure has to be coherent with hot examples of interest and some ideas elaborated in particular metric spaces could be then successfully extended to a more general framework.

Accomplishments. We displayed the list of technical achievements into two sections: the first one concerns general non-vector spaces and the second one metric spaces and, in particular, Wasserstein spaces. Research activities described in Section 1 were devoted to analogues of ODEs and differential inclusions in general non-metric and non-vector spaces and a book is in preparation. In parallel we also developed mathematical tools extending convex analysis to a very general framework.

In the case of metric spaces there were two major directions. The first one concerned the mutational systems, where mutational analysis was enriched by localized definitions of transitions. Here, the investigated questions concerned the celebrated Filippov theorem for mutational differential inclusions and its applications to invariance. Moreover, for the usual definitions of transitions, both viability and invariance were studied in a very general setting and applications to Hamilton-Jacobi-Bellman inequalities on proper metric spaces were provided. The obtained results were illustrated on the so-called morphological control systems on the space of compact subsets of $\mathbb{R}^d$, controlled continuity equations on Wasserstein spaces and on the space of signed Radon measures.

Before getting to these fairy general results, we worked in the second direction. Namely we considered the metric spaces with additional structures and studied Hamilton-Jacobi-Bellman inequalities and optimal feedback there, as described in more details in Section 2 below. Since the dual space to a metric space, in general, is not defined, our results are stated by using the extensions to metric spaces of tangents and the directional derivatives only. However, let us underline that the Wasserstein metric spaces do enjoy some additional structures that allowed us to introduce notions of derivatives, sub/superdifferentials and normals. This has lead also to results that are in the spirit of viscosity solutions to Hamilton-Jacobi-Bellman equation, but defined on metric spaces. Among other important achievements let us cite many extensions of classical control theoretic and differential inclusions results to Wasserstein spaces: maximum principle, sensitivity relations, upper semicontinuity of the optimal feedback (set-valued map), propagation of "differentiability" of the value function along optimal trajectories, relaxation theorem, compactness of the solution sets, etc. These results were published in the top mathematical journals and illustrated, as well, the hidden capabilities of earlier proofs based on set-valued analysis to be extended beyond vector spaces.

1 Analysis in the Absence of Topological and or Vector Structures

In the book on in preparation "Extremal Analysis" [44] of approximately 400 pages, we substantially progressed on this topic, omitting even the framework of metric spaces. In 2022 we tried to extend the project by one year to complete this book. However, because of technical failure of the CNRS side to activate the cage code in SAM before June 30, 2022, this was not possible. It looks like there are some new incompatibilities in the system of French Ministry of Armed Forces and the one of CNRS, that are also not compatible with SAM. This code was finally activated on September 1st, 2022 only.

Below we report only on published/accepted papers on this topic and we can transmit the preliminary version of the manuscript, in a pdf format, if requested.

1.1 Clustering Analysis

Clustering analysis is one of hot topics in Machine Learning, in the style of the prototypical one for vectors, known as k-means techniques. In [9], instead of analyzing time series of vectors or the problem of an allocation of vectors to clusters in vectors spaces, the same issues are investigated for time series and clusters of subsets ranging over the hyperset $\mathcal{P}(X)$ (composed of subsets of a "plain set" X deprived of any mathematical structure, let it be vectorial or topological). Then the Boolean operations take over arithmetic operations of vector spaces. In particular, the following tools were introduced to perform the clustering:

1. *dispersion gaps*$[[A_1, A_2]]$ between two disjoint subsets A_1 and A_2 : all the subsets K such that $A_1 \subset K \subset \complement A_2$ (instead of dispersion intervals $[v_1, v_2] \subset \mathbb{R}$);

2. *magnitudes* which are increasing hyperfunctions $\mu : K \in \mathcal{P}(X) \mapsto \mu(K) \in \mathbb{R}_+$ vanishing at the empty set (encompassing measure, capacities, etc.)

The main instrument of measure of a set $K \in [[A_1, A_2]]$ is its *echelon*

$$\mathbb{A}_\mu[[A_1, A_2]](K) \ := \ \frac{\mu(K \cap \complement A_1) - \mu(\complement K \cap \complement A_2)}{\mu(\complement A_1 \cap \complement A_2)} \ \in \ [-1, +1] \, .$$

Its (set-valued) inverse associates with any echelon

$$e \in [-1, +1] \rightsquigarrow \mathbb{A}_\mu[[A_1, A_2]]^{-1}(e) \ \in \ [[A_1, A_2]]$$

all the subsets in the dispersion gap sharing the same echelon. Echelons play the same role as the *quantiles* in statistics. Indeed, subsets with negative echelon are closer to A_1, while subsets with positive echelon are closer to A_2. The zero echelon corresponds to the median made of subsets "equidistant" from A_1 and A_2.

This clustering process is considered in [9] in the more general framework of clusters of a finite number of subsets and magnitudes and echelons of sets are used next to study time series of sets.

1.2 Extremal Analysis

In [35] we have announced some of results from [44] for which we first provide a short justification. The concept of value is polysemous, ranging from moral value (courage) to venal value (wealth), from the value of happiness to that of work, from the value of numbers to that of the shareholders, from the value of work to that of mathematics, and so many others, for no list being exhaustive. To the point where, more and more, one wishes to measure by numbers what can neither be enumerated nor measured by lack of units or measurement, unless using dubious palliatives such as "commodification" processes for assigning numerical values to assess symphonies, paintings, books, etc. Therefore we cannot use vector spaces to acclimate these entities in vector spaces to elaborate mathematical metaphors.

In this study, an *environment* X is an arbitrary set, *deprived of any mathematical structure*, made of elements which are regarded as *entities*. Actually, what matters are not only these entities $x \in X$ ranging over an arbitrary set, but also

1. "*baskets of entities*[1] $K \subset X$", which are subsets made of entities. These subsets are

[1]Instead of basket of goods preempted by economists to describe commodity vectors, which we also call *resources*.

ranging over what is called *hyperset*[2] $\mathcal{P}(X)$, the family of subsets K of X. More generally, we define recursively the hypersets $\mathcal{P}_r(X) := \mathcal{P}(\mathcal{P}_{r-1}(X))$ of order $r \in \mathbb{N}$ as the hyperset of the hyperset of order $r-1$;

2. *transforms* described on temporal windows[3] $[\Delta, \Omega]$ of duration $\Omega - \Delta > 0$, that allowed to distinguish an *input* $x_\Delta \in X$ at initial date Δ from the *output* $x_\Omega \in X$ at terminal date $\Omega > \Delta$ of the transform, summarized by the *input-output pair of entities* (x_Δ, x_Ω);

3. combinations of transforms.

Notwithstanding the set X may be deprived of any mathematical structures, its hyperset $\mathcal{P}(X)$ is always endowed with the *Boolean operations*: *unions*, *set-differences* and other Boolean operations on hypersets. They *play the same role as the addition and subtraction* of goods, granted with measurement units and commodity vectors.

Can we do more than just use the Boolean structures of hypersets?

Yes, since the field $\mathbb{R}$ of real numbers is equipped with the richest and prototypical mathematical structures, both algebraic, reticular, and topological, the temptation is great to convey its properties to any set X as well as its hypersets by associating with it the power spaces $X^{\mathbb{R}}$ and $\mathbb{R}^X$: that falls well since we recognize

1. the *plain set* $X^{\mathbb{R}}$ of *evolutions* associating with dates $t \in \mathbb{R} \mapsto X$ the elements $x(t) \in X$ (or *morphological evolutions* $t \in \mathbb{R} \mapsto K(t) \in \mathcal{P}(X)$) at date t;

2. the *vector space*[4] $\mathbb{R}^X$ of *valuators* $A : x \in X \mapsto A(x) \in \mathbb{R}$ *evaluating* elements $x \in X$ (and subsets $K \subset X$ by $K \in \mathcal{P}(X) \mapsto \sup_{x \in K} A(x) \in \mathbb{R} \cup \{+\infty\}$) as well,

which are the main themes of *extremal analysis*: some *mathematical structures of the set X and its hypersets $\mathcal{P}(X)$ can be derived from the choice of a vector space $\mathcal{S}(X) \subset \mathbb{R}^X$ of valuators* allowing us to transfer many results of functional analysis valid in vector spaces.

Our overall strategy is to transfer operations on intervals of values through valuators. Given a valuator $A \in \mathcal{S}(X)$ we associate with $x \in K$ its value $A(x) \in \mathbb{R}$ and with any subset $K \subset X$ its

$$\begin{cases} \textit{lower value}: & \Delta^\flat(K)(A) := \inf_{x \in K} A(x) \\ \textit{upper value}: & \Delta^\sharp(K)(A) := \sup_{x \in K} A(x) \\ \textit{interval value}: & [A](K) := [\Delta^\flat(K)(A), \Delta^\sharp(K)(A)] \\ \textit{confinement hull}: & K^{\bowtie} := \displaystyle\bigcap_{A \in \mathcal{S}(X)} A^{-1}([\Delta^\flat(K)(A), \Delta^\sharp(K)(A)]) \supset K. \end{cases}$$

The confinement hull of a subset $K \subset X$ depends on the choice of the vector space $\mathcal{S}(X)$ of valuators: the larger it is, the smaller the confinement hull $K_{\mathcal{S}}^{\bowtie}$. The set difference $K^{\bowtie} \setminus K$ is

[2] We make the choice to adopt the terminology "hyperset" to call the set of its subsets and not the term *"power" space* as it is often done, to avoid a *polysemic conflict* with the concept of power space $\mathbb{V}^{\mathbb{X}} := \prod_{x \in \mathbb{X}} \mathbb{V}_x$ where $\mathbb{V}_x := \mathbb{V}$ for all $x \in \mathbb{X}$, as well as the definition by *Bourbaki* of power of a set as its cardinal as well as $\mathcal{P}(X)$.

[3] Temporal windows are described by the pair of two numbers (Δ, Ω) of initial date Δ and terminal date $\Omega \geq \Delta$ or the pair $(\Omega, \Omega - \Delta)$ a terminal date Ω and the positive *duration* $\Omega - \Delta$ which offer an adequate mathematical description of *time*, a polysemous word meaning at least dates and duration.

[4] Even though we cannot add entities of $x \in X$, we can still add their values $A_1(x) \in \mathbb{R}$ and $A_2(x) \in \mathbb{R}$ and define the sum $A_1 + A_2 : x \in X \mapsto (A_1 + A_2)(x) := A_1(x) + A_2(x) \in \mathbb{R}$.
Generally speaking, any power space Y^X inherits properties of the set Y.

the set of elements which are evaluated on the same footing as elements of K without belonging to it. Only confined subsets $K = K^{\bowtie}$ can be strictly observed or perceived. Translating the individual "realities" of real numbers by $A^{-1}(\text{Property}([\Delta^\flat(K)(A), \Delta^\sharp(K)(A)]))$, the intersection $\text{Property}(K) := \bigcap_{A\in\mathcal{S}(X)} A^{-1}(\text{Property}([\Delta^\flat(K)(A), \Delta^\sharp(K)(A)]))$ is what *is "common" to all the individual evaluations* and is a way to transfer the "reality" of real numbers!

We proved, in particular, a general *Minsup Theorem* when the subset $K = K^{\bowtie}$ is a bounded confined subset (instead of bounded closed convex set) and also extended various results of convex analysis to this non standard framework. This approach also allowed

1. to extend notions of normals, tangents, derivatives and co-derivatives of set-valued maps;

2. to extend many results of optimization theory by introducing hyper derivatives of functions and characterizing extrema by a generalized Fermat rule and its dual;

3. to build on the evolutionary analysis by defining hyper-velocities of evolutions $x(\cdot) : t \in \mathbb{R} \mapsto x(t) \in X$, involving the coderivative $\partial x(t) := \partial x(t)(1) \subset \mathcal{S}(X)$ of an evolution $x(\cdot) : t \mapsto x(t) \in X$ at time t that is equal to the subset of valuators A satisfying

$$A \in \partial x(t) \text{ if and only if } 1 \leq \inf_{h>0} \frac{A(x(t)) - A(x(t-h))}{h}$$

The (retrospective) derivative of an evolution $x(\cdot)$ at t is the (possibly empty) subset of *hypervelocities* $v(t) \in X$ satisfying

$$\forall A \in \partial x(t), \quad A(v(t)) \leq 1.$$

For a set-valued map $F : x \in X \rightsquigarrow F(x) \subset X$, an evolution $x(\cdot) : t \in [t_0, T] \mapsto x(t) \in X$ is said to be governed by F if for every $t \in (t_0, T]$ there exists a *hypervelocity* $v \in F(x(t))$ such that

$$\begin{cases} \forall A \in \partial x(t), \quad A(v) \leq 1 \leq \inf_{h>0} \frac{A(x(t)) - A(x(t-h))}{h} \\ \forall A \in \mathcal{S}(X), \quad A(v) \leq \Delta^\sharp(F(x(t)))(A). \end{cases}$$

1.3 An Application to Economy

Chapter 18 of [44] did also allow to make an application to economy in [38] to answer the question: *How to provide means of payment to govern a viable economy*? This mathematical metaphor assumes that a lender/insurer of last resort supplies means of payment

1. between their imposed minimal and maximal *amounts of means of payment*;

2. alternating maximal *accelerations* and *decelerations* of the emission of means of payment, instead of imposing bounds on velocities or *growth rates*, as it is usually advised.

Bounds on accelerations instead of velocities describe a second-order *hysteron* instead of first-order one, prescribed by *Preisach* for explaining hysteresis in magnetism, and already used in economics. They provide a four-stroke monetary engine already detected by *Schumpeter*, passing through prosperity, recession, depression and recovery, making up *cyclic evolutions* of means of payments already revealed successively by *de Sismondi*, *Juglar*, *Kondratiev* and many economists ever since. Unfortunately, the phases of recession and recovery happen to be *mandatory* to reach the maximal and minimal amounts of means of payment in order to avoid monetary crisis.

The article [38] describes a mathematical metaphor of monetary-economic evolutionary systems, in which the authors no longer attribute to a deified *market* the task to use an *invisible hand*[5] which *manipulates prices by inventing ad hoc demand functions (in the static case) or differential equation systems (in the evolutionary case)* that economic agents use through their demand and supply functions. They replace these by

1. a *lender/insurer of last resort* creating means of payments;

2. *economic agents*, *makers* and *takers* of commodities *and* prices.

The market gone from the assumptions, only their behavior is taken into account. The economic agents are described by independent differential inclusions[6] governing commodities (ranging over vector spaces like in [38] or over *elements deprived of units of measure*) and are subject to

1. scarcity and eudemonic constraints on *commodities*;

2. political constraints on growth rates of *prices* and *means of payment*;

3. financial constraints on commodities, prices and means of payment.

Since the dynamic behavior of the lender of last resort and of each economic agent are not assumed to be *a priori* consistent, the differential inclusions of economic agents have to be modified to include the means of payments in their arguments and the differential inclusion of the lender to include commodities and prices. This provides *new differential inclusions regulating commodities and prices and involving means of payment for agents which were not present at the beginning, and a new differential inclusion allowing the lender to take into account commodities and prices.*

The above mathematical metaphor dispenses from inventing *a priori* systems involving commodities or goods, prices and means of payment. On the contrary, it reverts the focus on the decision to pilot *a posteriori* both commodities and prices. In other words, the financial constraints involving the money created by the banker lead them to integrate the means of payment into the behavior of both the banker and the economic agents who choose their goods and prices. The problem is *to regulate the joint evolution of commodities, prices and means of payment while taking into account their cyclic evolution* for the economic-monetary system to be both "economically and monetarily viable". The confrontation of uncertain evolutionary systems and constraints constitutes a "viability problem", justifying the use of the mathematical techniques designed to solve them.

The issues of dealing with the economic process of producing and exchanging commodities on the one hand, and using of prices to evaluate them with means of payments, on the other hand, have been addressed by many economists. Most of their answers amounted to impose *a priori* bounds on the *velocities* (or growth rates) of the supply of means of payment in the vain hope of stifling "prosperity (or inflation)" or "depression (deflation)". The unbridled growth of financial value coupled with too few constraints was therefore incompatible with the slower pace of economic activities evolving to meet viable economic constraints. They inevitably lead to either inflation or an economic slowdown and counterproductive rigour, and, sometimes to the cost of debt restructuring, violent social conflicts or wars.

[5] With good reason, *invisible*, since it requires an act of faith in a "Market", playing the role of a nowadays "intelligent designer".

[6] They are the evolutionary counterparts of demand functions in standard static equilibrium models.

2 Evolutions of Sets in Metric Spaces

2.1 Control Systems and Differential Inclusions under Uncertain Initial Conditions

Consider the control system

$$x'(t) = f(x(t), u(t)), \;\; u(t) \in U, \tag{1}$$

with $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ and a compact metric space U, which can be equivalently modeled by a differential inclusion

$$x'(t) \in F(x(t)) \tag{2}$$

where $F(x) := f(x, U)$. In many practical situations the initial condition $x(0)$ may be not known precisely and only a set E_0 of "possible" initial conditions is available. Then for any fixed control $u(\cdot)$, the set of possible initial conditions "propagates" as a tube $t \rightsquigarrow E(t)$. One can view $E(t)$ as a new state-variable. With any control $u(\cdot)$ and any set of initial conditions E_0 at time t_0 one can associate the tube $t \rightsquigarrow E(t)$ and a cost to be minimized at a given terminal time T

$$\min g(E(T)),$$

where g is a given map which associates to any set a real number. Then the value $V(t_0, E_0)$ is defined as the infimum of $g(E(T))$ over all possible controls. The interest to study the evolution of sets associated to (1) or (2) is motivated by multiagent control systems. At the initial time t_0 the set E_0 is the set of starting positions of infinitely many agents with dynamics (1) or (2). For example, this is the case in the evolution of the flock of sheeps, the swarm of fishes and the evolution of a crowd. In these cases there is a microscopic evolution : each agent has his own dynamic (1) and a macroscopic evolution which is described by the evolution of the "tube" $t \rightsquigarrow E(t)$ representing the set of possible positions of the agents at time t. It is then very natural to suppose that at every time t the evolution of each agent depends also on the state of the crowd

$$x'(t) = f(x(t), u(t), E(t)) \tag{3}$$

which generalizes equation (1). The dynamics (3) can be also described via the differential inclusion

$$x'(t) \in F(x(t), E(t)), \tag{4}$$

where for any $x \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$, $F(x, E) = f(x, U, E)$.

Under mild assumptions on f, U, for any control $u(\cdot)$ and any initial condition $E_0 \subset \mathbb{R}^d$, there exists a tube $t \rightsquigarrow E(t)$ such that

$$E(0) = E_0, \qquad E(t) = \{x(t), \text{ where } x(\cdot) \text{ solves (3) and } x(0) \in E_0 \}.$$

M. Bivas and M. Quincampoix have introduced a new notion of solution to (4), compatible with the above equality, and have shown that with this definition the important existence and compactness results together with a Gronwall-Filippov type estimates hold true.

In multiagent systems it is natural to assume that the control $u(\cdot)$ depends not only on time and the position of the agent, but also on the position of the crowd. So it is a function $(t, x, E) \mapsto u(t, x, E)$. In this case, under suitable Lipschitz regularity of controls, an existence result was proved.

Furthermore, given a cost function g, with any $t_0 \in [t_0, T]$ and any $E_0 \subset \mathbb{R}^d$ one can associate the value

$$V(t_0, E_0) := \min g(E(T)),$$

over all tubes $t \rightsquigarrow E(t)$ solving (4) and such that $E(t_0) = E_0$. The Lipschitz regularity of V was proved whenever g is Lipschitz continuous with respect to the Hausdorff distance. Also sufficient conditions for the existence of an optimal tube were proposed. In [16] an optimal control problem involving (4) was investigated together with a characterization of the corresponding value function through Hamilton-Jacobi inequalities and in [20] optimal feedback laws were derived. Motivated by the above results on set dependent set-valued dynamics, new results on Filippov's regularization of set-valued maps were derived in [18].

2.2 Morphological Control System

Consider the space $\mathscr{K}(\mathbb{R}^d)$ of nonempty compact subsets of $\mathbb{R}^d$ supplied with the Hausdorff distance, a complete separable metric space U and the Lebesgue measurable controls $u(\cdot) : [0,T] \to U$. Denote by $Lip(\mathbb{R}^d, \mathbb{R}^d)$ the set of bounded Lipschitz maps from $\mathbb{R}^d$ into itself. Let $f : \mathscr{K}(\mathbb{R}^d) \times U \to Lip(\mathbb{R}^d, \mathbb{R}^d)$, $K_0 \subset \mathbb{R}^d$ and consider the system

$$x'(\tau) = f\big(K(\tau), u(\tau)\big)\big(x(\tau)\big) \qquad \text{for a.e. } \tau \in [0,T], \quad x(0) \in K_0, \tag{5}$$

associated to a control $u(\cdot)$. Under mild assumptions, there exists a tube $t \mapsto K(t) \in \mathscr{K}(\mathbb{R}^d)$ so that $K(t)$ coincides with the reachable set of (5) at time t for every $t \in [0,T]$. This $K(\cdot)$ is the unique solution to the "morphological equation"

$$\overset{\circ}{K}(\cdot) \ni f\big(K(\cdot), u(\cdot)\big),\ K(0) = K_0$$

in the sense of J.-P. Aubin. The above control system seems to be well adapted to describe the movement of the crowd of agents and to control it by using either open-loop controls, or closed loop controls. System (5) can be interpreted in the following way : given a control $u(\cdot)$, every agent (indexed by its initial condition $x(0) = x_0 \in K_0$) has its dynamic depending on the evolution of the whole crowd of agents $K(\cdot)$ and its own evolution $x(\cdot)$. In [24, 33, 32] it was shown that the value function associated to the Mayer type optimal control problem involving the above morphological control system satisfies two generalized contingent Hamilton-Jacobi inequalities and the uniqueness of continuous solutions to these inequalities was investigated. One of these inequalities led as well to an expression of optimal feedback. This differs from the previous subsection because only morphological control systems are considered and the cost function may be merely continuous.

2.3 Viability and invariance of sets under mutational control systems

In [32] we investigated viability and invariance of proper subsets of a metric space (E,d) under mutational control system

$$\begin{cases} \overset{\circ}{x}(s) \ni f(x(s),\ u(s)) \quad \text{a.e.,} \quad u(\cdot) \in \mathscr{U} \\ x(0) = x_0, \end{cases}$$

where $f : E \times U \to \Theta(E)$, $\Theta(E)$ is a given set of transitions, (U, d_U) is a complete separable metric space and $\mathscr{U} := \{u(\cdot) : [0,\infty) \to U \text{ is Lebesgue measurable}\}$. Given a subset $\mathscr{K} \subset E$, viability means that for every initial condition $x_0 \in \mathscr{K}$ we can find trajectories of control system starting at this condition and remaining in $\mathscr{K}$. Invariance means that every trajectory of control system starting in the set of constraints $\mathscr{K}$ never violates them. We expressed necessary and sufficient conditions for viability and invariance using mutational tangents and also provided sufficient conditions for the existence and uniqueness of contingent solutions to the Hamilton-Jacobi-Bellman equation on a proper metric space, cf. [32]. As examples of application we considered controlled continuity equations on the metric space of compactly supported probability measures, endowed with the Wasserstein distance, and controlled morphological systems on the space of nonempty compacts subsets of the Euclidean space endowed with the Hausdorff metric.

Let us recall here that for this type of results, when stated in vector spaces, both the tangential and the dual approach based on normals (dual to tangents) are valid. For general metric spaces however, the notions of normals are absent. A notable exception are Wasserstein (metric) spaces, which, though being just metric, still have some structural properties that allowed us to introduce proximal normals to sets and to express necessary and sufficient conditions for viability and invariance in both the tangential and the normal forms [30].

2.4 Mutational Differential Inclusions on Metric Spaces

The major aim was to extend theory of measurably time-dependent differential inclusions to general metric spaces. The following fundamental results were already obtained :

- Theorem of *Filippov*'s type about the existence of solutions satisfying additional estimates with respect to a given absolutely continuous curve [36],
- Invariance Theorem, i.e., sufficient and necessary conditions for the invariance of an absolutely continuous tube of state constraints [39],
- Robust evolution of solutions to morphological inclusions in the space of measures [43].

Filippov's theorem is a corner stone of the differential inclusions theory that helps to study relaxation and linearization of control systems, viability and invariance of sets and Hamilton-Jacobi-Bellman equations in vector spaces. For this reason it was very important to start by its generalization to general metric spaces and to apply it for investigation of invariance of subsets and robust evolutions. The proofs of the general theorems are analytically challenging for mainly two reasons: First we dispensed with any linear-like structure of the transition space representing substitutes of velocities. Due to local effects we used just a pseudo-metric there instead, which is not necessarily positive definite like a metric. Second, the coefficients are assumed to be just measurable in time, but in such a pseudo-metric space, we could not apply the standard tools established for measurable set-valued maps.

Both the Filippov and the Invariance Theorem were applied explicitly to signed Radon measures which are supplied with the Kantorovich-Rubinstein metric and evolve according to transport equations with control [36, 39]. In comparison with the special and more popular case of probability measures, the time-dependent total variation of Radon measures quantifies the growth (and decline respectively) of the modelled multi-agents.

Furthermore all these results can be essentially applied to examples of problem classes published elsewhere before like in [Lorenz, Mutational Analysis 2010, Chapter 2] due to very similar general assumptions. They contain structured population models with Radon measures and the evolution of compact subsets of $\mathbb{R}^n$ under a more general class of differential inclusions than in the original works of J.-P. Aubin (i.e., one-sided Lipschitz and upper semicontinuous right-hand sides with sublinear growth instead of bounded Lipschitz ones).

It still remains writing down the already obtained results on sufficient and necessary conditions for viability of an absolutely continuous tube of state constraints and on characterizations of the epigraph of the value function in terms of mutational Hamilton-Jacobi inequalities satisfied by its directional derivatives.

2.5 Evolutions of Probability Measures and Control Systems

An alternative description of uncertain systems can be also done as follows. Fix a probability measure μ_0 on $\mathbb{R}^d$ whose support is E_0 and consider the image $\mu(t)$ of μ_0 by the flow at time t of a system modelled through a differential inclusion:

$$\begin{cases} \partial_t \mu(t) + \mathrm{div}(v(t)\mu(t)) = 0, \\ v(t,x) \in F(\mu(t), x) \text{ for } \mu(t)\text{-almost all } x \in \mathbb{R}^d \text{ and for a.e. } t \geq 0. \end{cases} \tag{6}$$

The above dynamical systems could be roughly understood as follows. The support of the probability measure $\mu(t)$ can be seen as the set of positions $x(t)$ of all the agents. The microscopic evolution of each agent is given by the differential inclusion

$$x'(t) =: v(t, x(t)) \in F(\mu(t), x(t)),\ t \geq 0,$$

while the macroscopic evolution of the crowd is given by $t \mapsto \mu(t)$ that aggregates the evolutions of all agents. The equality in (6) means that the total mass of the measure $\mu(t)$ is preserved.

Moreover one can include an additional information on the initial measure μ_0. For instance, that the unknown initial condition is the most probably on the boundary of E_0 or equidistributed on E_0, etc. This description is very rough because, in general, $v(t,\cdot)$ may be only a time dependent Borel vector field on $\mathbb{R}^d$ and (6) must be understood in the sense of distributions.

As often happens in mathematics, tools developed for evolutions of probability measures have allowed to obtain unexpected results on the existence of a differential game in the Wasserstein space [2]. This has a particular interest for repeated games with signals and partial monitoring.

In [22] a characterization of viability properties for such systems was obtained and a deterministic evolution in the Euclidean space of a multiagent system with a large number of agents (possibly infinitely many) was investigated. At each instant, besides the time and its current position, the set of velocities available to each agent is influenced by the set described by the current position of all the other agents. The latter is in turn determined by the overall motion of the crowd of all the agents. The interplay of the microscopical point of view of each single agent, and the macroscopical one (of the set-evolution) yields a non-trivial dynamical system. This two-level multiagent system can be described either by the evolution of a probability measure - describing the instantaneous density of the crowd - or by the evolution of a set - describing the positions where there is at least one agent. [22] describes the links between the two descriptions, providing also some quantitative estimates on the macroscopical admissible evolutions.

2.6 Solutions to Hamilton-Jacobi Equation on a Wasserstein Space

Let $\mathscr{P}_c(\mathbb{R}^d)$ be the space of compactly supported Borel probability measures on $\mathbb{R}^d$ endowed with the W_2 Wasserstein metric. Denote by $Lip(\mathbb{R}^d, \mathbb{R}^d)$ the space of all bounded Lipschitz continuous functions from $\mathbb{R}^d$ into itself with the topology of local uniform convergence and let (U, d_U) be a compact metric space of control parameters. Given a cost function $g : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, a mapping describing the dynamics $f : \mathscr{P}_2(\mathbb{R}^d) \times U \to Lip(\mathbb{R}^d, \mathbb{R}^d)$ and $(t_0, \mu_0) \in [0,1] \times \mathscr{P}_c(\mathbb{R}^d)$, consider the Mayer type optimal control problem

$$V(t_0, \mu_0) := \inf g(\mu(1))$$

over all the solutions defined on the time interval $[t_0, 1]$ of the continuity equation

$$\partial_t \mu(t) + \mathrm{div}(f(\mu(t), u(t))\mu(t)) = 0, \quad u(\cdot) \in \mathscr{U}, \quad \mu(t_0) = \mu_0.$$

V is called the value function. Consider time dependent compact valued tubes of probability measures

$$\Delta_r := \{(t,\mu) \in [0,1] \times \mathscr{P}_2(\mathbb{R}^d) \, : \, \mathrm{supp}(\mu) \subset B(0, r + \rho t)\},$$

where ρ is a bound on f. Define the Hamiltonian $\mathscr{H} : \mathscr{P}_2(\mathbb{R}^d) \times L^2(\mu; \mathbb{R}^d) \to \mathbb{R}$ by

$$\mathscr{H}(\mu, p) = \sup_{u \in U} \int_{\mathbb{R}^d} \langle p(x), f(\mu, u)(x) \rangle \mathrm{d}\mu(x)$$

and consider the Hamilton-Jacobi equation

$$-\partial_t w(t,\mu) + \mathscr{H}(\mu, -\nabla_\mu w(t,\mu)) = 0, \qquad w(1,\cdot) = g(\cdot), \qquad (HJB)$$

where $\nabla_\mu w$ refers to the "gradient with respect to the measure". To define solutions to (HJB) we have introduced new notions of Hadamard type super/subdifferentials:

Define the contingent cone to K at $\mu \in K$:

$$\overset{\circ}{T}_K(\mu) := \left\{ F \in Lip(\mathbb{R}^d, \mathbb{R}^d) \mid \liminf_{h \to 0+} \frac{1}{h} \mathrm{dist}((Id + hF)_{\#}\mu, K) = 0 \right\}$$

and consider $w : [0,1] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{\pm\infty\}$ and $(t,\mu) \in \mathrm{dom}(w)$.

We say that $(p_t, p_\mu) \in \mathbb{R} \times L^2(\mu; \mathbb{R}^d)$ belongs to the Hadamard subdifferential $\partial_H^- w(t,\mu)$ of w at (t,μ) if $\forall\, F \in Lip(\mathbb{R}^d, \mathbb{R}^d)$ and any $\kappa \in \mathbb{R}$,

$$p_t \kappa + \int_{\mathbb{R}^d} \langle p_\mu(x), F(x) \rangle \mathrm{d}\mu(x) \leq D_\uparrow w(t,\mu)(\kappa, F),$$

where for $\kappa \in \mathbb{R}$, $F \in Lip(\mathbb{R}^d, \mathbb{R}^d)$ with $(\kappa, F) \in \overset{\circ}{T}_{\mathrm{dom}(w)}(t,\mu)$

$$D_\uparrow w(t,\mu)(\kappa, F) := \liminf_{\substack{h \to 0+, \kappa' \to \kappa \\ W_2(\mu', (Id + hF)_{\#}\mu) = o(h) \\ (t + h\kappa', \mu') \in \mathrm{dom}(w)}} \frac{w(t + h\kappa', \mu') - w(t,\mu)}{h}$$

If $(\kappa, F) \notin \overset{\circ}{T}_{\mathrm{dom}(w)}(t,\mu)$, then set $D_\uparrow w(t,\mu)(\kappa, F) = +\infty$.

Hadamard superdifferential is defined by $\partial_H^+ w(t,\mu) := -\partial_H^-(-w)(t,\mu)$.

A function $w : [0,1] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfying $w(1,\cdot) = g(\cdot)$ is called a viscosity supersolution to (HJB) equation on Δ_r if for all $(t,\mu) \in \Delta_r$ with $t < 1$

$$-p_t + \mathscr{H}(\mu, -p_\mu) \geq 0, \quad \forall\, (p_t, p_\mu) \in \partial_H^- w(t,\mu),$$

where $\partial_H^- w(t,\mu)$ denotes the (Hadamard) subdifferential of w at (t,μ). Further, w is called a viscosity subsolution to (HJB) equation on Δ_r if for all $(t,\mu) \in \Delta_r$ with $t < 1$,

$$-p_t + \mathscr{H}(\mu, -p_\mu) \leq 0, \quad \forall\, (p_t, p_\mu) \in \partial_H^+ w(t,\mu),$$

where $\partial_H^+ w(t,\mu)$ denotes the (Hadamard) superdifferential of w at (t,μ). If w is simultaneously a viscosity super and subsolution, then it is called a viscosity solution to (HJB) on Δ_r.

We have shown that, under mild assumptions, continuous viscosity solutions on Δ_r coincide with the restriction $V_{|\Delta_r}$ for every $r > 0$, cf. [30]. An alternative approach was proposed in [15], where various notions of viscosity solutions in Wasserstein spaces and in L^2 spaces are investigated.

2.7 Differential Inclusions in Wasserstein Spaces

The above subsection deals with a pointwise extension of differential inclusions to the space of measures. We also considered a functional extension based on selections from a set-valued map $(t, \mu) \rightsquigarrow F(t, \mu) \subset L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$. We call a curve of measures $\mu(\cdot)$ a solution of the *differential inclusion*

$$\partial_t \mu(t) \in -\mathrm{div}\Big(F(t, \mu(t))\mu(t)\Big), \tag{7}$$

if there exists a *measurable selection* $v(t) \in F(t, \mu(t))$ such that (6) is satisfied in the sense of distributions. This is a convenient extension of (6) to differential inclusions because, under very mild assumptions, the set of solutions of the controlled continuity equation

$$\partial_t \mu(t) + \mathrm{div}\Big(f\big(t, \mu(t), u(t)\big)\mu(t)\Big) = 0,\ u(t) \in U$$

is equal to the set of solutions of the inclusion (7) with $F(t, \mu) = \{f(t, \mu, u)(\cdot) \,|\, u \in U\}$. Thanks to it, both closed loop and open loop control systems can be investigated using general results on differential inclusions.

In this general setting, we proved in [23, 17] three fundamental results of the theory of differential inclusions: Filippov type estimates, the relaxation theorem, and the compactness of the solution set. These contributions – which are based on novel properties of solutions of continuity equations – were then applied to derive a new existence result for fully non-linear mean-field optimal control problems with closed-loop controls and the appropriateness of introduced notions was illustrated on an example of leader-follower evacuation problem with soft congestions.

2.8 Necessary Conditions and Sensitivity Relations in Wasserstein Spaces

Consider the Mayer type optimal control problem

$$(P) \quad \begin{cases} \displaystyle\min_{u(\cdot)\in\mathscr{U}} g(\mu(1)) \\ \text{s.t.} \quad \begin{cases} \partial_t \mu(t) + \mathrm{div}\Big(f(t, \mu(t), u(t))\mu(t)\Big) = 0, \\ \mu(0) \ = \mu^0, \quad \mu(1) \in Q, \end{cases} \end{cases}$$

where $g : \mathscr{P}_c(\mathbb{R}^d) \to \mathbb{R}$ is a given final cost, an initial datum $\mu^0 \in \mathscr{P}_c(\mathbb{R}^d)$ is fixed and $f : [0, 1] \times \mathscr{P}_c(\mathbb{R}^d) \times U \times \mathbb{R}^d \to \mathbb{R}^d$ is a controlled non-local velocity field. The minimisation in (P) is taken over the set of admissible open-loop controls

$$\mathscr{U} := \Big\{u : [0, 1] \to U \text{ s.t. } u(\cdot) \text{ is Lebesgue-measurable}\Big\},$$

where (U, d_U) is a compact metric space, and the set of *final-point constraints* Q is defined by functional inequalities of the form

$$Q := \Big\{\mu \in \mathscr{P}_c(\mathbb{R}^d) \text{ s.t. } \Psi_i(\mu) \le 0 \text{ for all } i \in \{1, \dots, n\}\Big\},$$

where $\Psi_i : \mathscr{P}_c(\mathbb{R}^d) \to \mathbb{R}$ for every i. To derive first-order necessary optimality conditions for this problem, we introduced a new notion of localised metric subdifferential for compactly supported probability measures, and investigated the intrinsic linearised Cauchy problems associated to non-local continuity equations. We then made use of these novel concepts to provide a synthetic and geometric proof of a counterpart to the celebrated Pontryagin Maximum Principle. In addition, we proposed sufficient conditions ensuring its normality, cf. [25].

To simplify we cite this maximum principle in the absence of end-point constraints.

Theorem 2.1 (PMP) *Let (μ^*, u^*) be optimal for (P). Then the unique solution ν^* of*

$$\begin{cases} \partial_t \nu^*(t) + \mathrm{div}\big(J_{2d} H_\zeta(\nu^*(t), u^*(t))\nu(t)\big) = 0, \ \pi^1_\# \nu^*(t) = \mu^*(t) \\ \nu^*(1) = (\mathrm{Id}, -\nabla g(\mu^*(1))_\# \mu^*(1) \end{cases}$$

satisfies for a.e. $t \in [0,1]$ the maximality condition

$$H(\nu^*(t), u^*(t)) = \max_{u\in U} H(\nu^*(t), u)$$

In the above $T_\sharp\mu$ stands for the pushforward of the measure μ by the Borel map $T : \mathbb{R}^d \to \mathbb{R}^d$, while the Hamiltonian and the symplectic matrix of $\mathbb{R}^{2d}$ are given by

$$H(\zeta, u) = \int_{\mathbb{R}^{2d}} \big\langle r, f(\pi^1_\#\zeta, u, x)\big\rangle\, d\zeta(x, r), \ J_{2d} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

for any $\zeta \in \mathscr{P}_c(\mathbb{R}^{2d})$, $u \in U$.

Next, we investigated some of the supplementary properties of the value function associated to the above optimal control problem in the absence of end point constraints. Building on new interpolation and linearization formulas for non-local flows, we proved semiconcavity estimates for the value function, and established several variants of the so-called sensitivity relations which provide a connection between its superdifferenitals and the adjoint curves stemming from the maximum principle.

Theorem 2.2 (sensitivity relation) *Let $(\mu^*(\cdot), u^*(\cdot))$ be optimal and $\nu^*(\cdot)$ be the corresponding state-costate curve. Then,*

$$\big(H(\nu^*(t), u^*(t)), -\bar{\nu}^*(t)\big) \in \partial^+ V(t, \mu^*(t))$$

for almost every $t \in [0,1]$, where $\bar{\nu}^*(t) \in L^2(\mathbb{R}^d; \mu^*(t))$ denotes the barycentric projection of $\nu^*(t)$ onto $\mu^*(t)$ and $\partial^+ V$ stands for the Hadamard superdifferential of the value function V.

This relation, moreover, completed the necessary optimality condition to the sufficient one:

Theorem 2.3 (sufficient optimality condition) *Let $(\mu^*(\cdot), u^*(\cdot))$ be a trajectory-control pair and $\nu^*(\cdot)$ be a state-costate curve satisfying the PMP and the sensitivity relation. Then $(\mu^*(\cdot), u^*(\cdot))$ is optimal for (P) .*

We subsequently made use of these results to study the propagation of regularity for the value function along optimal trajectories, as well as to investigate optimal feedbacks for mean-field optimal control problems, cf. [29], [28].

2.9 Carathéodory Theory and a Priori Estimates for Continuity Inclusions

In [41] we extended the foundations of the theory of differential inclusions in the space of probability measures with compact support, which were recently laid down in one of our previous work, to the setting of general Wasserstein spaces. Anchoring our analysis in a novel series of estimates for solutions of continuity equations, we proved new variants of the Filippov theorem, compactness of solution set and relaxation theorem for continuity inclusions studied in the Cauchy-Lipschitz framework. We also proposed an existence result “à la Peano” for this class of dynamics under Carathéodory type regularity assumptions, based on a set-valued generalisation of the semi-discrete Euler scheme originally proposed by Filippov to study ordinary differential equations.

2.10 Viability and Exponentially Stable Trajectories for Differential Inclusions in Wasserstein Spaces

In [34] we proved a general viability theorem for continuity inclusions in Wasserstein spaces, and provided an application thereof to the existence of exponentially stable trajectories obtained via the second method of Lyapunov. More precisely, we derived the following two results.

Theorem 2.4 (Viability for proper constraints) *Let $F : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ be a set-valued map with convex images satisfying assumptions of [34], and $\mathcal{Q} \subset \mathscr{P}_2(\mathbb{R}^d)$ be a proper set such that*

$$F(t, \nu) \cap \overline{\mathrm{co}}\, T_{\mathcal{Q}}(\nu) \neq \emptyset$$

for almost every $t \in [0,T]$ and each $\nu \in \mathcal{Q}$. Then for each $\mu^0 \in \mathcal{Q} \cap \mathscr{P}_c(\mathbb{R}^d)$, there exists a solution $\mu(\cdot)$ to (7) with $\mu(0) = \mu^0$ such that $\mu(t) \in \mathcal{Q}$ for all times $t \in [0,T]$.

The above theorem was applied to obtain the existence of trajectories for which a Lyapunov functional $\mathscr{W} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{+\infty\}$ with domain $\mathrm{dom}(\mathscr{W}) \subset \mathscr{P}_2(\mathbb{R}^d)$ decays exponentially.

A map $\mathscr{W} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{+\infty\}$ is called a *strict Lyapunov function* for $F : \mathbb{R}_+ \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ if the following holds.

(*i*) $\mathscr{W}(\cdot)$ has compact sublevels in $\mathscr{P}_2(\mathbb{R}^d)$.

(*ii*) For almost every $t \geq 0$ and all $\mu \in \mathrm{dom}(\mathscr{W})$, there exists a $v \in F(t, \mu)$ for which

$$\mathrm{D}_{\uparrow}\mathscr{W}(\mu)(v) \leq -\rho \mathscr{W}(\mu),$$

where $\rho > 0$ is a fixed constant.

Theorem 2.5 (Exponentially stable trajectories) *Let $F : \mathbb{R}_+ \times \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ be a set-valued map with convex images satisfying assumptions of [34] and $\mathscr{W} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{+\infty\}$ be a strict Lyapunov function for $F(\cdot, \cdot)$. Then for each $\mu^0 \in \mathscr{P}_c(\mathbb{R}^d)$, there exists a solution $\mu(\cdot)$ to (7) with $\mu(0) = \mu^0$ such that*

$$\mathscr{W}(\mu(t)) \leq \mathscr{W}(\mu^0) e^{-\rho t},$$

for all times $t \geq 0$.

3 Recruited Post-Doctoral Associates Z. Badreddine, M. Bivas, B. Bonnet and T. Lorenz

Mira Bivas was recruited for six months on January 15th 2019. She was involved in the part of project concerning optimal control with uncertain initial conditions and also the optimal evolutions of multiagent systems. Her research concerned mainly the existence of tubes associated to a multiagent system together with investigation of a topological structure of the solution sets. She also studied some distance estimates for tubes starting from various initial sets. For this aim a new notion of solution of the reachable set-dependent inclusions was introduced. This collaboration led to articles [16, 20] and continued even after her departure [18].

Zeinab Badreddine was recruited for 17 months starting on September 1st 2019. She was first investigating optimal control of morphological control systems, cf. [24]. An extended abstract was accepted for MTNS 2020, that was cancelled because of the pandemic. She next worked in the framework of Wasserstein spaces on Hamilton-Jacobi inequalities and viability theorems [30] and on viability and invariance of mutational control systems in general metric spaces [32, 33].

Benoît Bonnet was recruited for 15 months starting on November 1, 2019. His task was to develop a functional approach to differential inclusions in Wasserstein spaces and then to apply the obtained results to optimal control in these spaces. Six articles were already published [17, 23, 25, 28, 29, 34]. B. Bonnet continued his collaboration with the project on the topic of differential inclusions and viability theory in the Wasserstein spaces [41, 42]. We should underline that in 2021 B. Bonnet was recruited by CNRS on the permanent position of researcher on the topic of Optimal Control.

Thomas Lorenz was recruited for six months starting on October 1, 2019. He was a senior postdoctoral fellow. Thanks to his work important results on mutational equations on general metric spaces were derived. They were applied to extend viability theory and Hamilton-Jacobi equations to systems discontinuous in time. Two long manuscripts [36, 39] were completed. In addition, he also published [13] concerning viability of non-local hyperbolic differential inclusions of first order with memory and applied it to non-local population models and [31] on approximation of reachable sets. He continued his collaboration with the project after the departure. Recently he got an offer to become a full professor position (W3) at U. of Rostock and will start working there in April 2023.

4 Dissemination

Results obtained in this project were mostly published in high level mathematical journals and also in Proceedings of CDC IEEE conferences. Three papers are already accepted for publication and will appear in 2023. Four more manuscripts were submitted recently.

H. Frankowska and M. Quincampoix organized the Special Session “Optimal Control and Calculus of Variations on metric spaces” at *15th Viennese Conference on Optimal Control and Dynamic Games* at Vienna University of Technology https://orcos.tuwien.ac.at/events/vc2022/ during the invited Session “Optimal Control and Calculus of Variations on metric spaces”, where two of speakers were postdoctoral fellows of the project (B.Bonnet and T. Lorenz) and M. Quincampoix and H. Frankowska presented results obtained in collaboration with the postdoctoral fellows M. Bivas and Z. Badreddine.

In addition some results directly related to the project were/will be presented at the following international scientific events:

1. 59th IEEE Conference on Decision and Control, December 8-11, 2020 (visio)
2. 15-th International Workshop on Well-Posedness of Optimization Problems and Related Topics, Borovets, Bulgaria, June 28 – July 2, 2021
3. Analysis, Control, and Numerics for PDE Models of Interest to Physical and Life Sciences, Levico Terme, Italy, September 20 - 24, 2021
4. PGMO days 2021, EDF Lab Paris Saclay, France, November 30 - December 1, 2021
5. 60th IEEE Conference on Decision and Control, December 12-15, 2021 (visio)
6. 20th French-German-Portuguese Conference on Optimization, Portugal, May 3-6, 2022
7. Theoretical and Numerical Trends in Inverse Problems and Control for PDEs, and Hamilton-Jacobi Equation: a French-Italian-Japanese Conference”, CIRM Luminy, France, June 13-17, 2022
8. International Workshop on Control and Optimization, Imperial College, London, England, July 5-6, 2022.
9. Deterministic and Stochastic Control, Politecnico di Milano, Italy, September 6-7, 2022

10. International workshop on Analysis and Control of (bi)linear PDEs, U. Rome Tor Vergata, Italy, September 7-9, 2022
11. 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Bayreuth, Germany, September 12-16, 2022
12. 61st IEEE Conference on Decision and Control, Cancun, Mexico, December 6-9, 2022

Two Conferences that we were planning to attend: MTNS 2020 and 13th AIMS Conference on Dynamical Systems, Differential Equations and Applications 2020, were cancelled because of pandemic.

H. Frankowska gave talks related to the topic of the project (control, set-valued analysis, differential calculus of set-valued maps with applications to optimal control) at the following conferences/workshops:
1. 14th Viennese Conference on Optimal Control and Dynamic Games, Wien, Austria, July 3-6, 2018
2. International Workshop Variational Analysis and Applications, Erice, Italy, August 28 – September 5, 2018
3. International Conference Dynamics, Control and Geometry, Banach Center, Warsaw, Poland, September 12-15, 2018
4. International Workshop Analysis, Control and Inverse Problems for PDEs, LIA COPDESC, U. di Napoli Federico II, Italy, November 26-30, 2018 (plenary speaker)
5. International Workshop Nonsmooth and Variational Analysis, ESI - Wien, Austria, January 28 – February 1, 2019
6. SIAM Control Conference, Chengdu, June 19-21, 2019
7. Singular nonlinear problems in Calculus of Variations and PDE's, University of Naples "Federico II", Italy, June 24-26, 2019
8. ICIAM, International Congress on Industrial and Applied Mathematics, Valencia, Spain, July 15-19, 2019
9. Congress for 100 years of Polish Mathematical Society, Krakow, Poland, September 3-7, 2019 (plenary speaker)
10. The French-German optimization conference, FGX'2019, Nice, France, September 17-20, 2019, (plenary speaker)
11. International Workshop on Optimal Control and Mean Field Games, Rio de Janeiro, Brazil, October 14-18, 2019
12. 58nd IEEE Conference on Decision and Control, Nice, France, December 11-13, 2019
13. Calculus of Variations and Applications, SISSA, Trieste, Italy, January 27 - February 1st, 2020
14. High Dimensional Hamilton-Jacobi Methods in Control and Differential Games, IPAM, Los Angeles, USA, March 30 - April 3, 2020 (visio because of the covid)
15. 2020 European Control Conference, Saint Petersburg, Russia, May 12-15, 2020 (visio because of the covid)
16. Challenges in Optimization with Complex PDE-Systems, Oberwolfach, Allemagne, February 14 – 20, 2021 (visio because of the covid)

5 Impacts

The proposed approach allowed to look on models in a new way:

(a) To avoid a priori constraints, but having in mind that environment does evolve and constraints may appear and disappear dynamically.

(b) Models do not need to be stated in vector spaces and variables may be multivalued or range over any set endowed with an extremal structure.

(c) Valuator of a basket of goods may depend on various factors, not just on quantities and prices. This enriches the environment of decision makers.

(d) Mathematical tools developed in the project make possible to work in a general setting beyond vector spaces, assuming that the state spaces are sets deprived of a given a priori algebraic or topological structure and that a vector space of valuators evaluating their elements by reals is chosen: this is sufficient to adapt many statements of functional analysis on topological vector spaces.

(e) The obtained results contributed to developments of control theory in the Wasserstein spaces using tools of set-valued analysis. This brought a mathematical progress also to this hot topic.

An impact on other disciplines, where modelling issues are of a crucial importance, is expected in the future.

We are very grateful to AFOSR for funding this project - this truly helped to create a new dynamic in this area of mathematics.

6 Changes

There were some delays with completing the book. This happened because with the time the ambitions were increasing, while staying with the same aim: building mathematical theory of Evolution of Sets in metric spaces. Another perturbation came, of course, from the pandemic, that complicated communication and isolated postdoctoral fellows from the Lab for 3 months.

We did not spend all the travel money, because some conferences passed to visio and some of them were cancelled. In 2022 we tried to extend the project by one year to disseminate better our results during 15th Viennese Conference on Optimal Control and Dynamic Games (July 2022), MTNS 2022 (September 2022) and 61th IEEE Conference on Decision and Control (December 2022). However, because of technical failure of the CNRS side to activate the cage code in SAM before June 30, 2022, this was not possible. It looks like there are some new incompatibilities in the system of French Ministry of Armed Forces and the one of CNRS, that are also not compatible with SAM. This code was finally activated on September 1st, 2022 only. So the participation in these conferences was financed from a different source, while we had over 16 000 dollars available for travel.

7 Publications since July 1, 2019 (in chronological order) with acknowledgment of the support from AFOSR:

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