



AFRL-AFOSR-VA-TR-2024-0024

Onset and Prediction of Orbital Motions of Streamwise Vortices

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11/03/2023
Final Technical Report

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Air Force Research Laboratory
Air Force Office of Scientific Research
Arlington, Virginia 22203
Air Force Materiel Command

REPORT DOCUMENTATION PAGE

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1. REPORT DATE 20231103		2. REPORT TYPE Final		3. DATES COVERED	
				START DATE 20190601	END DATE 20230531
4. TITLE AND SUBTITLE Onset and Prediction of Orbital Motions of Streamwise Vortices					
5a. CONTRACT NUMBER		5b. GRANT NUMBER FA9550-19-1-0095		5c. PROGRAM ELEMENT NUMBER 61102F	
5d. PROJECT NUMBER		5e. TASK NUMBER		5f. WORK UNIT NUMBER	
6. AUTHOR(S) Justin Jaworski					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) LEHIGH UNIVERSITY 526 BROADHEAD AVE BETHLEHEM, PA 18015 USA					8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Air Force Office of Scientific Research 875 N. Randolph St. Room 3112 Arlington, VA 22203				10. SPONSOR/MONITOR'S ACRONYM(S) AFRL/AFOSR RTA1	11. SPONSOR/MONITOR'S REPORT NUMBER(S) AFRL-AFOSR-VA-TR-2024-0024
12. DISTRIBUTION/AVAILABILITY STATEMENT A Distribution Unlimited: PB Public Release					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT This research program investigates theoretically the evolution and stability of trailing vortices and their interactions with solid surfaces or flexible wings. Several analytical and numerical approaches are applied across five model scenarios with different vortex structures and configurations with respect to a planar surface and/or an aeroelastic lifting body. A nonmodal, temporal stability analysis of a three-dimensional trailing vortex pair approaching an inviscid ground plane indicates that the suppression of the dominant Crow instability near the ground plane can be described with inviscid fluid dynamics and does not require an appeal to dissipative viscous flow effects. To help protect your privacy, Micro so ft Office prevented automatic downlo ad o f this picture from the Internet. AFOSR word mark					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:				17. LIMITATION OF ABSTRACT	
a. REPORT U	b. ABSTRACT U	c. THIS PAGE U	UU		18. NUMBER OF PAGES 101
19a. NAME OF RESPONSIBLE PERSON GREGG ABATE				19b. PHONE NUMBER (Include area code) 425-1779	

Standard Form 298 (Rev.5/2020)
Prescribed by ANSI Std. Z39.18

Report Coversheet

Award Number

FA9550-19-0095

Report Type

Final

Reporting Period

Start 1 Jun 2019 - End 31 May 2023

Distribution Statement

Distribution A - Approved for Public Release

Program Officer Name

Dr. Gregg Abate

Principal Investigator Name

Dr. Justin Jaworski

Project Title

Onset and prediction of orbital motions of streamwise vortices

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This research program investigates theoretically the evolution and stability of trailing vortices and their interactions with solid surfaces or flexible wings. Several analytical and numerical approaches are applied across five model scenarios with different vortex structures and configurations with respect to a planar surface and/or an aeroelastic lifting body. A non-modal, temporal stability analysis of a three-dimensional trailing vortex pair approaching an inviscid ground plane indicates that the suppression of the dominant Crow instability near the ground plane can be described with inviscid fluid dynamics and does not require an appeal to dissipative viscous flow effects.

ONSET AND PREDICTION OF ORBITAL MOTIONS OF STREAMWISE VORTICES

AFOSR GRANT FA9550-19-1-0095

JUNE 1, 2019 TO MAY 31, 2023

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Part I

OUTCOMES AND PRODUCTS

1 ACCOMPLISHMENTS

1.1 Research objectives

The overall objective of this research effort is to develop theoretical models to better understand how finite-core trailing vortices are affected by the presence of a moving or elastic wing, which is part of a greater coordinated effort with completed experimental work directed by Prof. Donald Rockwell (Lehigh) and ongoing computational work under the direction of Drs. Miguel Visbal and Daniel Garmann (AFRL). The present theoretical effort is principally organized into the following five objectives.

1. Model the downstream evolution of trailing vortices in generalized flow fields
2. Develop a mathematical framework to evolve a time-dependent trailing vortex condition at a planar boundary into a semi-infinite domain
3. Predict onset conditions for vortex precession and helical saturation
4. Establish stability criteria of streamwise vortices in proximity to a solid planar surface
5. Enhance basic advances through collaboration with AFRL

1.2 Research accomplishments

This research program investigates theoretically the evolution and stability of trailing vortices and their interactions with solid surfaces or flexible wings. Several analytical and numerical approaches are employed across five model scenarios involving different vortex structures and configurations with respect to a planar surface and/or an aeroelastic lifting body. A non-modal, temporal stability analysis of three-dimensional trailing vortices is developed and carried out for a vortex pair approaching an inviscid ground plane, which indicates that the phenomenon of suppression of the dominant Crow instability near the ground plane can be described with inviscid fluid dynamics alone and does not require an appeal to viscous flow or dissipative effects. A separate numerical model for the helical wake created by a rotor with flexible blades investigates the aeroelastic coupling between the near-wake region and the self-induced blade motions. The stability of the helical tip vortices and the time history of the rotor wakes reveal that blade flexibility may generate tip vortices that are less sensitive to low wavenumber perturbations but more sensitive to higher wavenumber perturbations, and the introduction of blade pitch alters the local induced velocity field and alters tip vortex stability such that peak eigenvalue trends do not adhere to classical stability criteria. In addition to three-dimensional trailing vortex stability, several predictive modeling frameworks based on conformal mapping techniques are constructed for complex two-dimensional hydrodynamic vortex-structure interactions that also motivate complementary acoustic models for these aeroelastic interactions. An analytical model for an aeroelastic vortex gust encounter couples the incident, bound, and shed vorticity of the system, where a comparison with available experimental data for fixed airfoils shows that the boundary layer plays an important role in the trajectory

and strength of the incoming vortex. This inviscid model furnishes close predictions with experiment for trajectories that do not have a strong viscous interaction with the boundary layer and points to a current lack in experimental data for validation for the aeroelastic model. The interaction of a stationary vortex-airfoil interaction near a ground plane is separately considered, which requires the adoption of a sophisticated conformal mapping technique based on the Schottky-Klein prime function to represent the multiply-connection regions involved. An aeroelastic model for a turbulent vortical eddy passing over a flexible, wall-mounted fiber is constructed and carried out to investigate the coupled dynamics and resulting acoustic emission. In this configuration, the noise production is greatest for specific parameter combinations and initial conditions where the relative motion between the vortex and fiber is maximized at the instant that they are closest to one another.

1.3 Dissemination of results

The foregoing results have been disseminated to relevant technical communities through archival journal papers, papers and presentations and major at international conferences, NATO STO meetings, and doctoral dissertations that are detailed in §§4 and 5.

2 IMPACTS

The technical achievements of this program produced predictive tools for streamwise vortex evolution and stability in the presence of solid bodies, as well as incorporated new theoretical paradigms from the applied mathematics community to handle the complex and dynamically-coupled aeroelastic interactions of vortex gusts with flexible structures. The modeling outcomes invite companion experimental and/or computational work for future validation and model refinement.

This program has supported an international student exchange between Lehigh University (M. Herndon) and the University of Cambridge (S. Bradford) under ISEP, as well as several undergraduate student research experiences that were organized in coordination with the Lehigh STEM-SI program (S. Bapat, A. Peters, B. Pierce), which focuses on providing access to research experiences for underrepresented cohorts in engineering. The graduate student research preparation and experience of M. Herndon was augmented by his participation in AFRL SFFP (2019) and through continued interactions with Dr. Daniel Garmann (AFRL, WPAFB) over the course of his doctoral program. These various modes of support are gratefully acknowledged.

3 CHANGES

A 1-year no-cost extension was approved within the reporting period on 27 May 2022.

4 THESES AND DISSERTATIONS

4.1 PhD dissertations

Huansheng Chen (in progress, expected Fall 2023): Noise generation of graded poroelastic edges from vortex rings

Mark Herndon (in progress, expected Spring 2024): Analytical and computational analysis of perturbed finite-core trailing vortices

5 PUBLICATIONS

5.1 Journal papers

Herndon, M. A. & Jaworski, J. W. 2023 Linear stability of a counter-rotating vortex pair approaching an inviscid wall. *Theoretical and Computational Fluid Dynamics*, **37**, pp. 519-532.

Rodriguez, S. N., Jaworski, J. W. & Michopoulos, J. G. 2021 Stability of helical vortex structures shed from flexible rotors. *Journal of Fluids and Structures*, **104**, 103279.

Jaworski, J. W. & Peake, N. 2020 Aeroacoustics of silent owl flight. *Annual Review of Fluid Mechanics*, **52**, pp. 395-420.

Chen, H. & Jaworski, J. W. 2020 Aeroelastic interactions and trajectory selection of vortex gusts impinging upon Joukowski airfoils. *Journal of Fluids and Structures*, **96**, 103026.

Rodriguez, S. N. & Jaworski, J. W. 2020 Strongly-coupled aeroelastic free-vortex wake framework for floating offshore wind turbine rotors. Part 2: Application. *Renewable Energy*, **149**, pp. 1018-1031.

Rodriguez, S. N. & Jaworski, J. W. 2019 Strongly-coupled aeroelastic free-vortex wake framework for floating offshore wind turbine rotors. Part 1: Numerical framework. *Renewable Energy*, **141**, pp. 1127-1145.

Jaworski, J. W. 2019 Sound from aeroelastic vortex-fibre interactions. *Philosophical Transactions of the Royal Society A*, **377**, 20190071.

Herndon, M. A., Jaworski, J. W. & Garmann D. J. High fidelity simulation of a perturbed trailing vortex pair in ground effect. In preparation for *Journal of Fluid Mechanics*.

5.2 Conference papers

Herndon, M. A. & Jaworski, J. W. 2022 Linear stability and optimal perturbations of a trailing vortex pair in ground effect. *Proceedings of 2022 AIAA SciTech Forum*, 3-7 January. Paper AIAA-2022-1700.

Chen, H. & Jaworski, J. W. 2020 Aeroelastic encounters of spanwise vortex gusts and the self-rotation of trailing vortices. *Proceedings of 2020 AIAA SciTech Forum*, Orlando, FL, 6-10 January. Paper AIAA-2020-0555.

Chen, H. & Jaworski, J. W. 2019 Acoustic emission of aeroelastic vortex-gust interactions. *Proceedings of 23rd International Congress on Acoustics*, Aachen, Germany, 9-13 September, pp. 604-611.

Chen, H., Baddoo, P. J. & Jaworski, J. W. 2019 Modeling of Brown-Michael vortices in ground effect. *Proceedings of 2019 AIAA Aviation Forum*, Dallas, TX, 17-21 June. Paper AIAA-2019-3041.

Herndon, M. A. & Jaworski, J. W. Unsteady evolution of a trailing vortex subject to time-varying inflow boundary conditions. In preparation for 8th *International Symposium on Marine Propulsors*, 17-21 March 2024, Hurghada, Egypt.

5.3 Abstracts with presentations

Herndon, M. A. & Jaworski, J. W. Linear stability and optimal perturbations of a trailing vortex pair in ground effect. APS DFD 2021, Phoenix, AZ. Abstract #E24.00007.

5.4 Presentations

Jaworski, J. W. Discussion of a possible follow-on group to AVT-347. NATO STO Meeting, AVT-347, London, United Kingdom. 23 May 2023.

Tiomkin, S. & Jaworski, J. W. Unsteady aerodynamics of membrane wings. NATO STO Meeting, AVT-347, Varna, Bulgaria. 28 September 2022.

Baddoo, P., Hajian, R. & Jaworski, J. W. Unsteady aerodynamics of porous aerofoils: harmonic motions & gust response. NATO STO Meeting, AVT-347, virtual. 4 November 2020.

Chen, H. & Jaworski, J. W. Aeroelastic vortex-structure interactions with barriers and airfoils. NATO STO Meeting, AVT-282, Trondheim, Norway. 7 October 2019.

Part II

TECHNICAL UPDATES

6 LINEAR STABILITY OF VORTEX PAIRS APPROACHING AN INVISCID WALL

M. Herndon, J. W. Jaworski

6.1 Background

The dynamic properties of aircraft wakes are critical to understand due to the danger they pose to downstream aircraft (Rossow, 1999). Aircraft wakes may be idealized by a pair of equal-strength counter-rotating vortices (Spalart, 1998), which are subject to a range of complex behavior, including merging, rotation, instability, and breakdown. Perturbations of tip vortex formation due to, for example, wing pitching and heaving or atmospheric turbulence can significantly alter the trajectory and evolution of the trailing vortex downstream (Fishman & Rockwell, 2018; Garmann & Visbal, 2016). Multiple vortices interacting with one another will further complicate the dynamics of these vortices and their stability. Therefore, a clear understanding of the response of an isolated vortex or vortex pair system to self-induced or external perturbations is critical to the anticipation and prediction of the evolution of an aircraft wake and the wakes of control surfaces.

This research effort creates a model to determine the temporal evolution and stability characteristics of a counter-rotating trailing vortex pair in the presence of an inviscid planar surface. The mathematical model permits results for the limiting case of vortex pairs near a boundary where the viscous vortex-boundary-layer interaction is negligible. The inviscid interaction between the vortices and their image in the ground plane also clarifies the near-wall behavior observed by As-selin & Williamson (2017) and asserts that viscous flow effects are not essential to demonstrate perturbation growth inhibition near the wall.

6.2 Model formulation

A counter-rotating vortex pair in ground effect is modeled as a pair of vortex filaments of finite core radius a with their corresponding mirror image across a ground plane. The governing equations for a general n -vortex system are adapted from Crouch (1997) and Fabre *et al.* (2002), who employed a vortex filament method to derive the perturbation equations. Figure 1 illustrates the three-dimensional vortex system, where the centers of the physical vortices are separated by a distance b and lie at a height h above the ground plane. A sinusoidal perturbation of wavelength kb is prescribed along the entire infinite length of each vortex, where k is the perturbation wavenumber. In mathematical terms, the position vector of the n^{th} vortex element with respect to the global coordinate system is

$$\mathbf{r}_n(x, t) = x \hat{e}_x + (Y_n(t) + \eta_n(t)e^{ikx}) \hat{e}_y + (Z_n(t) + \zeta_n(t)e^{ikx}) \hat{e}_z, \quad (1)$$

where Y_n and Z_n are the coordinates the unperturbed vortex center position at time t , and η_n and ζ_n are the perturbation amplitude components with respect to the local coordinate system attached to vortex n . The unperturbed vortices are oriented along the $\hat{e}_x$ -direction such that x is the variable

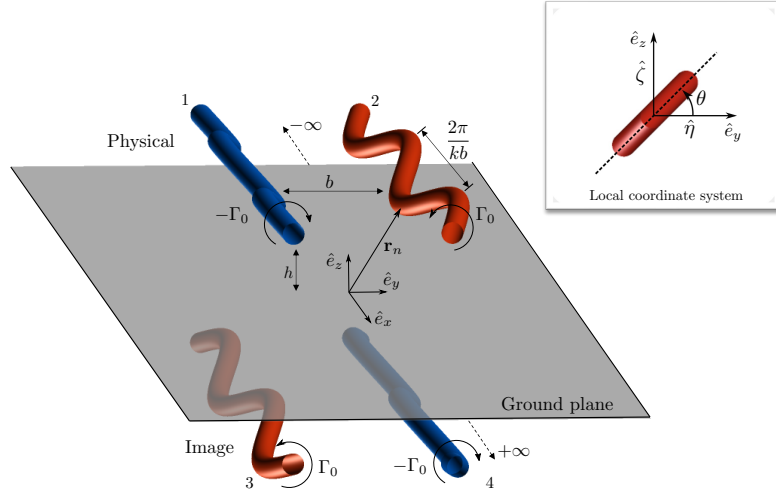


Figure 1: Three-dimensional representation of the model for a perturbed vortex pair in ground effect. Two wavelengths are shown of an infinite sinusoidal perturbation that extends to $\pm\infty$ in the $\hat{e}_x$ -direction. The ground effect constraints are held along the entire length of the vortices. The coordinate system local to an individual vortex is shown, where the perturbation directions η and ζ are shown with the respective angular orientation θ of the vortex.

of integration for the Biot-Savart law,

$$\mathbf{U}_n = \sum_{m \neq n} \frac{\Gamma_m}{4\pi} \int_{-\infty}^{\infty} \frac{\mathbf{r}_m - \mathbf{r}_n}{\|\mathbf{r}_m - \mathbf{r}_n\|^3} \frac{\partial \mathbf{r}_m}{\partial x_m} dx_m, \quad (2)$$

where n is understood to be the total number of vortices. The Biot-Savart law computes the induced velocity, $\mathbf{U}_n$, by each vortex onto the others, and here we use an alternative method to compute the effects of self induction for when $m = n$, which is described ahead.

The governing equations of the vortex system are determined by linearizing the Biot-Savart law to first order in the perturbation amplitudes η_n and ζ_n . The mean vortex center positions are calculated from the kinematics of the temporally-varying position vector and the leading-order expansion of the Biot-Savart law, and the perturbation amplitude equations are found at the next order. The characteristic length scale is the initial vortex separation b , the time is non-dimensionalized as

$$\tau = \frac{\Gamma_0}{2\pi b^2} t, \quad (3)$$

and all other non-dimensional quantities are denoted with a hat. Vortices of equal strength and core sizes are assumed in the present investigation, yet the method can permit more general vortex configurations. The resulting non-dimensional equations for the unperturbed vortex center positions are

$$\frac{d\hat{Y}_n}{d\tau} = \sum_{m \neq n} \frac{\hat{\Gamma}_m \hat{Z}_{mn}}{\hat{R}_{mn}^2}, \quad \frac{d\hat{Z}_n}{d\tau} = - \sum_{m \neq n} \frac{\hat{\Gamma}_m \hat{Y}_{mn}}{\hat{R}_{mn}^2}, \quad (4a,b)$$

and the non-dimensional equations for the perturbation amplitudes are organized in a similar fashion to Fabre *et al.* (2002):

$$\begin{aligned} \frac{d\hat{\eta}_n}{d\tau} = & \frac{\hat{\Gamma}_n}{\hat{a}^2} \bar{\omega}(ka) \hat{\zeta}_n + \sum_{m \neq n} \frac{2\hat{\Gamma}_m \hat{Y}_{mn} \hat{Z}_{mn}}{\hat{R}_{mn}^4} (\hat{\eta}_n - \phi(kb\hat{R}_{mn})\hat{\eta}_m) \\ & - \sum_{m \neq n} \frac{\hat{\Gamma}_m}{\hat{R}_{mn}^2} \left[\left(1 - \frac{2\hat{Z}_{mn}^2}{\hat{R}_{mn}^2} \right) \hat{\zeta}_n - \left(\psi(kb\hat{R}_{mn}) - \frac{2\hat{Z}_{mn}^2}{\hat{R}_{mn}^2} \phi(kb\hat{R}_{mn}) \right) \hat{\zeta}_m \right], \end{aligned} \quad (5a)$$

$$\begin{aligned} \frac{d\hat{\zeta}_n}{d\tau} = & -\frac{\hat{\Gamma}_n}{\hat{a}^2} \bar{\omega}(ka) \hat{\zeta}_n - \sum_{m \neq n} \frac{2\hat{\Gamma}_m \hat{Y}_{mn} \hat{Z}_{mn}}{\hat{R}_{mn}^4} (\hat{\zeta}_n - \phi(kb\hat{R}_{mn})\hat{\zeta}_m) \\ & + \sum_{m \neq n} \frac{\hat{\Gamma}_m}{\hat{R}_{mn}^2} \left[\left(1 - \frac{2\hat{Y}_{mn}^2}{\hat{R}_{mn}^2} \right) \hat{\eta}_n - \left(\psi(kb\hat{R}_{mn}) - \frac{2\hat{Y}_{mn}^2}{\hat{R}_{mn}^2} \phi(kb\hat{R}_{mn}) \right) \hat{\eta}_m \right]. \end{aligned} \quad (5b)$$

The right-hand-sides of Equations 5a and 5b involve modified mutual induction functions ϕ and ψ , originally determined by Crow (1970), whose definitions can be found in Eq. 2.4 in Fabre *et al.* (2002). The self-induced rotation, $\bar{\omega}$, is calculated in the same manner as Fabre & Jacquin (2000) which is inspired by the Saffman (1992) analysis of Kelvin waves on a vortex filament.

Saffman (1992, Chapter 12) analyzed oscillations of a vortex column to calculate the self-induced rotation frequency for small disturbances along a uniform rectilinear vortex filament by obtaining roots to the dispersion relation (Saffman, 1992, Eq. 11.3.4) for Kelvin waves on a vortex filament. The self-induced rotation frequency, $\bar{\omega}$, is calculated by obtaining the lowest retrograde root of this dispersion relation for the bending mode and evaluating the frequency from via Saffman (1992, Eq. 12.1.5). This method of calculating self-induction has been shown to not produce spurious instabilities at short wavelengths when compared to the original self-induction function of Crow (1970). Figure 2 illustrates the equivalence of the cutoff method and the results from the dispersion relation for low wavenumbers and small core sizes when substituted into Crow's original equations (Eqs. 10-11). It is understood that α in Fig. 2 is the highest positive symmetric root from (Crow, 1970, Eq. 11).

The equations for the perturbation amplitudes is a linear, non-autonomous system. To consider cases near a wall boundary, we impose the constraints $\hat{Y}_1 = \hat{Y}_3$, $\hat{Y}_2 = \hat{Y}_4$, and $\hat{Z}_1 = -\hat{Z}_3$, $\hat{Z}_2 = -\hat{Z}_4$ for the vortex center positions. Likewise, $\hat{\eta}_1 = \hat{\eta}_3$, $\hat{\eta}_2 = \hat{\eta}_4$, and $\hat{\zeta}_1 = -\hat{\zeta}_3$, $\hat{\zeta}_2 = -\hat{\zeta}_4$ for the perturbation amplitudes. Because these constraints must hold for all time to represent the ground effect problem, the non-autonomous system may be reduced from an 8×8 system in Fabre *et al.* (2002) to a modified 4×4 system. The model vortex problem may be compactly expressed as a system of ordinary differential equations, where the matrix $\mathbf{A}$ depends on the time-dependent vortex center positions:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x} = (\hat{\eta}_1, \dots, \hat{\eta}_n, \hat{\zeta}_1, \dots, \hat{\zeta}_n)^T. \quad (6)$$

The governing equations are integrated numerically via a fourth-order Runge-Kutta scheme over a range of dimensionless wavenumbers kb to observe how the perturbations grow with respect to time. The key parameters of interest in this problem are the initial dimensionless vortex height h/b and the fundamental time scale τ , which is the time it takes for an undisturbed vortex pair

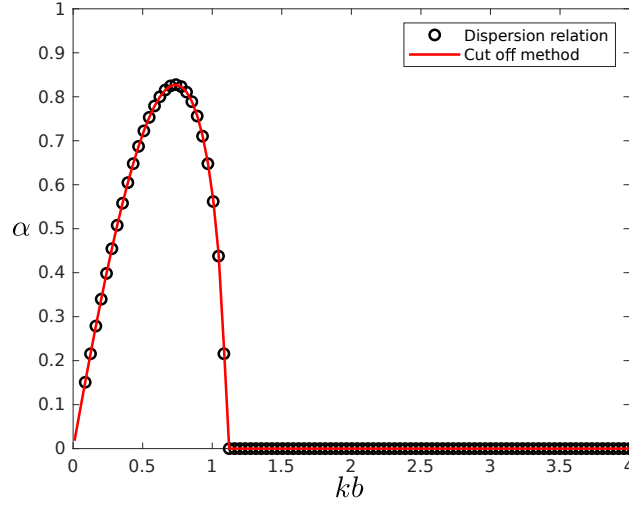


Figure 2: Spectrum of eigenvalues α with respect to wavenumber employing Crow's original theory with both a cutoff method and the dispersion relation for Kelvin waves on a Rankine vortex to calculate self-induction. There is agreement between both methods, as they have equivalent values in the low wavenumber limit.

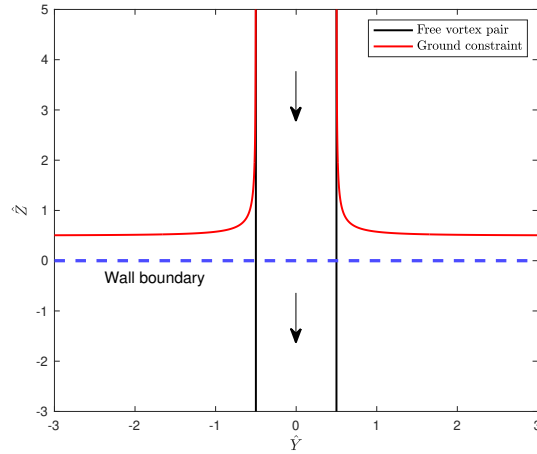


Figure 3: Induced trajectory of the vortex pair in free space and approaching an inviscid wall. The wall requires a hyperbolic trajectory of the vortex pair, in accordance with classical theory (Crow, 1970).

to translate a distance b in the $-\hat{e}_z$ direction. Note that Lamb (1932) has previously calculated the theoretical trajectory of a pair of point vortices, where the vortices descend on a hyperbolic trajectory away from each other along an inviscid wall. The true trajectory of the vortex pair descending towards a ground plane would rebound and persist above the wall due to the viscous effects, as explained by Harvey & Perry (1971). Figure 3 illustrates representative trajectories of vortex pairs with or without an inviscid ground constraint.

The small finite-core size $\hat{a} = a/b = 0.098$ is chosen to validate the present method with Crow's original theory for a free vortex pair. Initial starting heights are selected as $h/b = 5$ and $h/b = 10$ to observe the transient growth of perturbations with respect to distance away from the wall for the constrained cases, with a larger core size of $a/b = 0.23$, based on the direct numerical simulations of Dehtyriov *et al.* (2020a).

6.3 Time dependent stability and optimal perturbations

The stability of perturbations to time-dependent dynamical systems has been formalized by Farrell & Ioannou (1996) in their work on generalized stability theory. Generalized stability theory is a non-modal approach that can capture asymptotic and finite-time behaviors of perturbations to time-dependent systems, where the non-autonomous operator may also be non-normal. In fact, it has been shown in Farrell & Ioannou (1996) that non-normality of a system can lead to transient growth in both autonomous and non-autonomous systems such that the systems are asymptotically unstable. For the present work, the finite-time stability of infinitesimal perturbations is the primary focus, where we are not concerned with the asymptotic stability of the system.

6.4 Non-modal stability theory

Generalized stability theory seeks a fundamental solution matrix $\Phi_{[t,t_0]}$, or the *propagator* matrix, which maps the state of the system at $\mathbf{x}(t_0)$ to $\mathbf{x}(t)$, where $\Phi_{[t,t_0]}$ satisfies

$$\frac{d\Phi_{[t,t_0]}}{dt} = \mathbf{A}(t)\Phi_{[t,t_0]}, \quad \Phi_{[t_0,t_0]} = \mathbf{I}. \quad (7)$$

This solution is analogous to the familiar matrix exponential solution for autonomous systems. The fundamental solution matrix maps the system at some time t_0 to a future time t subject to an initial condition $\mathbf{x}_0$ and characterizes the system,

$$\mathbf{x} = \Phi_{[t,t_0]}\mathbf{x}_0. \quad (8)$$

The fundamental solution matrix is analogous to the matrix exponential for autonomous systems, where the solution would be expressed as

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0. \quad (9)$$

The asymptotic stability of an autonomous system are characterized by the eigenvalues of the matrix $\mathbf{A}$, where the amplification of an initial perturbation at some time t is

$$\sigma(t) = e^{\alpha t}, \quad (10)$$

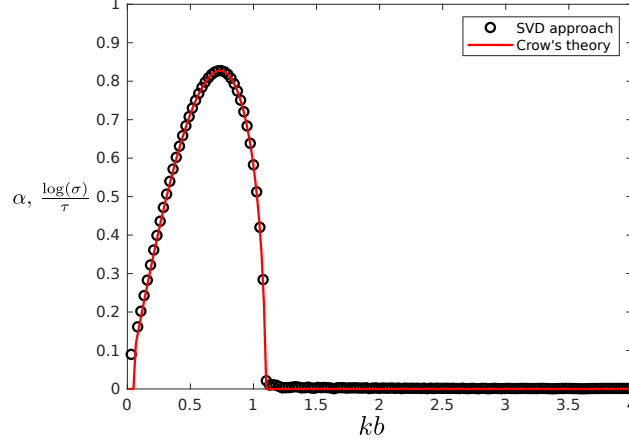


Figure 4: Agreement between methods with an $\alpha_{\max} = 0.83$ at $kb = 0.73$, where α is the real component of the complex eigenvalue defined by Crow. $\ln \sigma/\tau$ approximates α to make clear comparison between Crow's method and the present non-modal (SVD) approach. The vortex core size is selected as $\hat{a} = a/b = 0.098$.

where α is the largest real eigenvalue of $\mathbf{A}$. For a non-autonomous system, stability is *not* quantified by the eigenvalues of the time-dependent operator $\mathbf{A}$, as the eigenvalues may change at each instant in time. Instead, the optimal perturbation may be deduced by taking the singular value decomposition (SVD) of the fundamental solution matrix at some time instant of interest t , where the amplification σ is the highest singular value. The right singular vector of the SVD corresponds to the initial condition that would lead to the optimal amplification. Further details of the non-modal stability theory are also described by Rapaka *et al.* (2008).

6.5 Results and discussion

The case of a free vortex pair studied by Crow (1970) reduces to a linear autonomous system where the asymptotic behavior of initial perturbations are measured by largest real eigenvalue α of his system. To relate the optimal perturbation of a free vortex pair to the results from Crow, the maximum eigenvalue is approximated as

$$\alpha = \lim_{t \rightarrow \infty} \frac{\ln(\sigma(t))}{t}, \quad (11)$$

where σ is the highest singular value (amplification of an initial perturbation) obtained from the SVD of the fundamental solution matrix. This is the same definition from Farrell & Ioannou (1996) for the first Lyapunov exponent, which quantifies the asymptotic exponential rate of growth or decay of time-dependent systems.

To compare the eigenvalue approximation Eq. 11 from the SVD approach to Crow's original work, we select $a/b = 0.098$ to match his parameters for the trailing vortex core radius of a real aircraft wake vortex. Figure 4 illustrates the agreement between both methods, where there is agreement in the maximally growing mode at $kb = 0.73$. However, it was necessary to integrate forward in time substantially to match the full wavenumber spectrum, roughly 80 non-dimensional time steps. The reason for the difference at initial times would be in part due to the non-normality of the system, where transient growth may not correspond with the asymptotic behavior defined

by the eigenvalues. For finite-time stability analysis, this long time requirement is not a relevant concern, as we are interested in the maximally growing mode at a particular instant in time.

Two illustrative examples are provided to compare the evolution of the long-wavelength instability for a vortex pair in ground effect to an unconstrained vortex pair. The vortex core radius is chosen as $a/b = 0.23$ inline with a subset of the parameters from Dehtyriov *et al.* (2020b), but it should be noted that the choice of vortex core size is not the primary interest in this study. Modifying the vortex core size (for the equal strength, equal size configurations studied here) will modify at what wavenumber the optimal perturbations occur yet retains the overall behavior of the system. With the goal of comparing the unconstrained free vortex pair to the constrained ground effect case, it is only necessary to be consistent in a choice of core size. We consider an initial height of $h/b = 5$ and $h/b = 10$ and compare the evolution of the vortex systems with respect to an unconstrained free vortex pair.

Figure 5 shows the growth of an initial perturbation across all wavenumbers considered for an initial height of $h/b = 10$ above the wall compared to the free vortex pair case. Prior to wall interaction, the spectrum of perturbation growth for the ground constrained case coincides closely with the free vortex pair case. Normalizing the amplification σ with respect to the maximum amplification of the free vortex pair highlights the difference between the two configurations. As the trajectories diverge, the amplification of the ground constrained vortex pair perturbations are less than half of the free vortex pair. For a real viscous fluid, initial heights of $h/b = 10$ will have the Crow instability dominate such that the vortex pair converts into vortex ring before the wall interaction. This phenomenon does not occur in our model, as vortex ring creation is not an inviscid process. Despite the discrepancy for those initial heights, the dynamics of the system is still in agreement with experiments in the context of perturbation growth inhibition. For long time intervals, the perturbation amplitudes grow exponentially as if they were a free vortex pair if they were to follow along their inviscid idealized trajectory. Kliment & Rokhsaz (2008) investigated experimentally vortex pairs in ground effect and noted that viscous effects may be negligible for heights above the wall 0.5-1.5 times the vortex span b , and that the wall effect may be modeled with image vortices. We have limited our investigation to time intervals and spatial positions that are reasonably outside of the boundary layer, where secondary vorticity generation and adverse pressure gradients cannot be neglected.

Figure 5 shows the same qualitative behavior as Fig. 6 for a vortex pair that starts closer to the wall at an initial height $h/b = 5$. The amplification curves are in alignment at early times before the trajectories diverge followed by clear perturbation growth inhibition. Beyond the clear reduction of perturbation growth in the ground constrained case, it is evident that the wavenumber of the optimal perturbation shifts to higher values as the vortex pair approaches the wall. This observation indicates that wavenumbers higher than the Crow instability (optimal wavenumber for the unconstrained case) should be selected to maximize the growth of the vortex pair at the wall.

Direct integration of the system and examining the growth curves make this fact more evident. To see the time history of the optimal perturbations, the optimal initial condition and wavenumber given by the SVD approach can be directly applied to the system and integrated in time. The instantaneous energy gain of the system G is defined by

$$G = \frac{\|\mathbf{x}\|}{\|\mathbf{x}_0\|} \quad (12)$$

and represents the amplification of an initial perturbation. Note that this his definition of G is

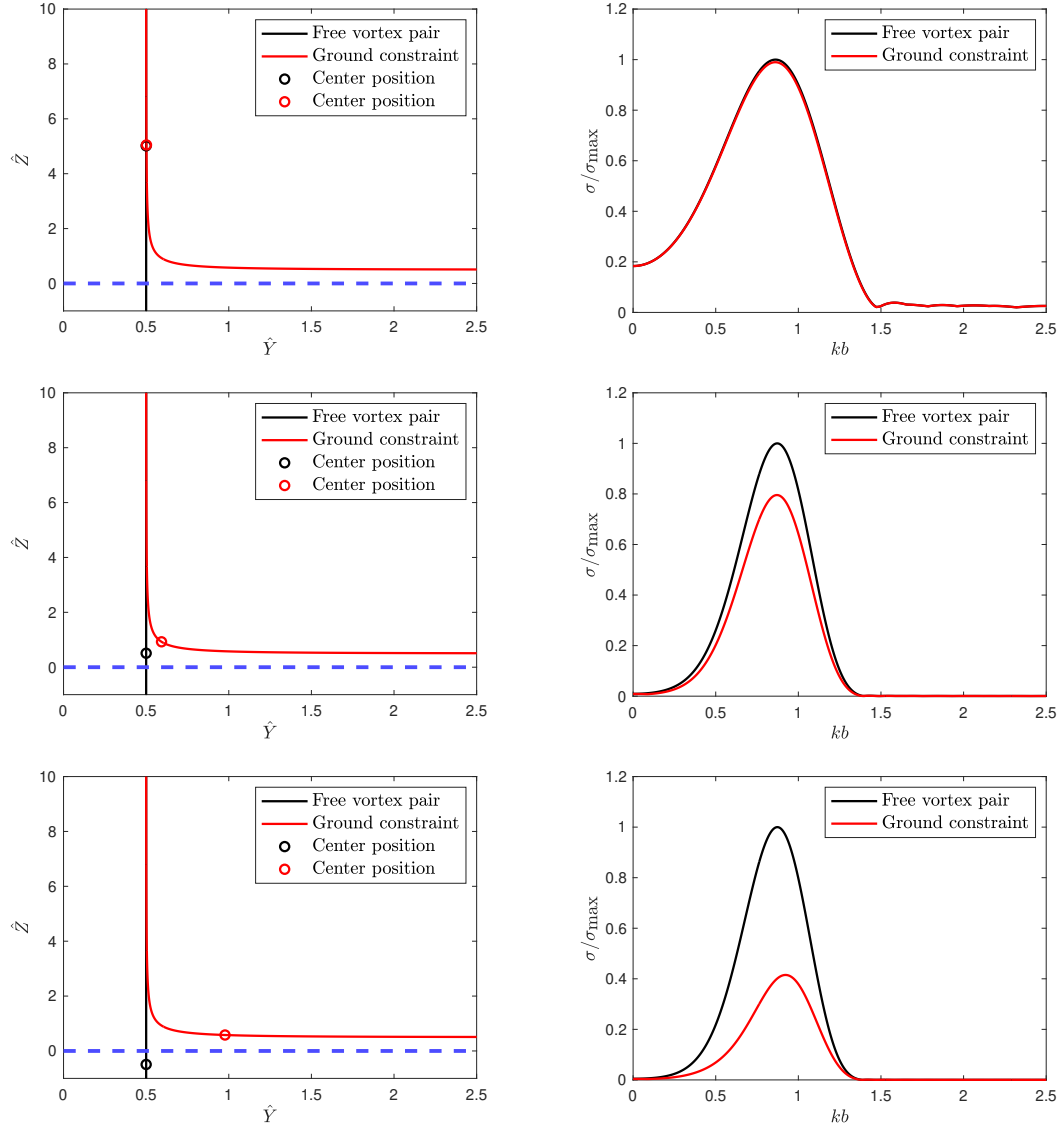


Figure 5: Vortex trajectories and optimal wavenumber spectrum for an initial height $h/b = 10$ at $\tau = 5.5$ (top row), $\tau = 9.5$ (middle row), and $\tau = 10.5$ (bottom row). The right column are the amplifications σ of an initial perturbations scaled by the corresponding maximum amplification of the unconstrained free vortex pair case $\sigma_{\max}$. The vortex core size is $\hat{a} = 0.23$.

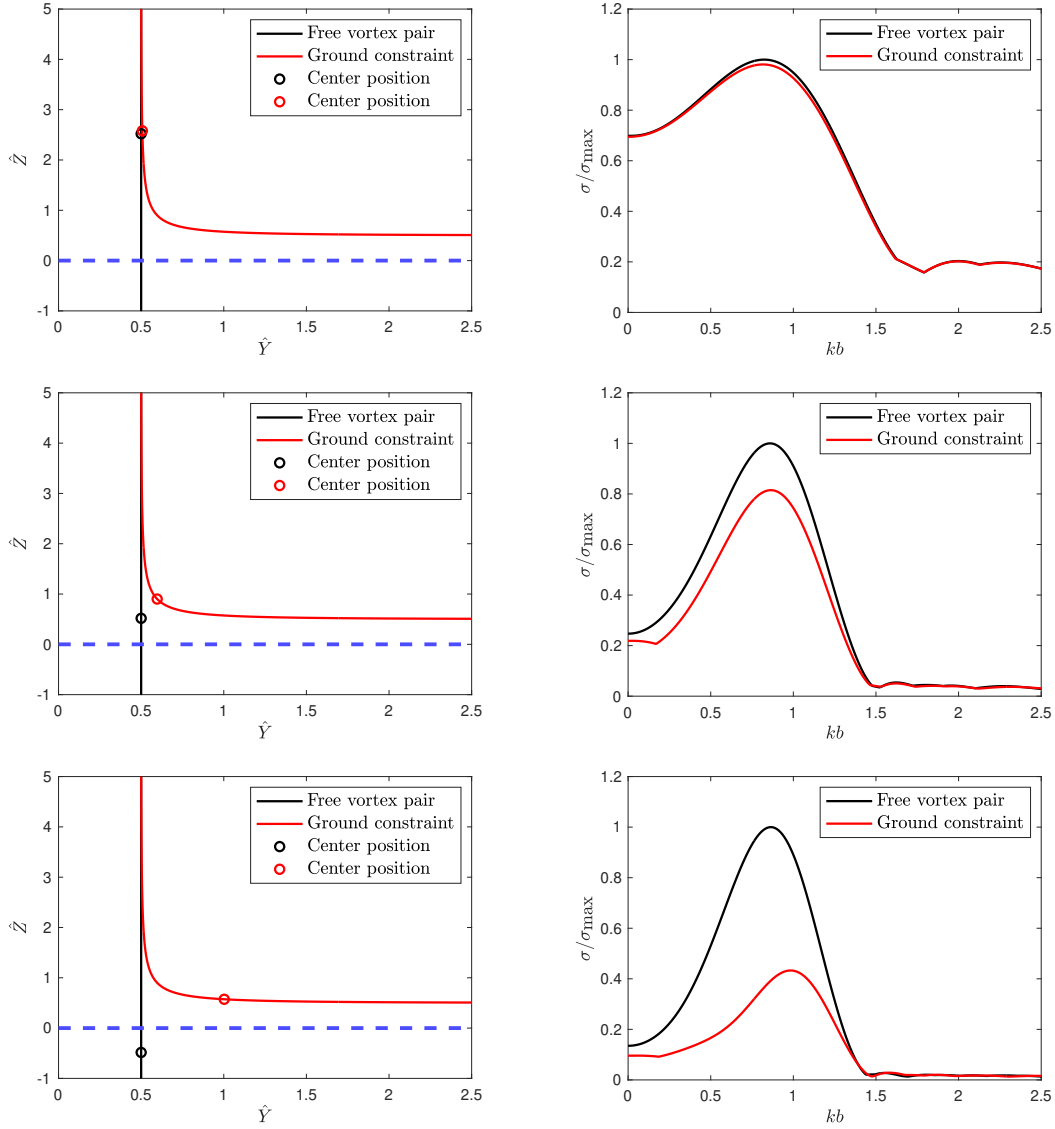


Figure 6: Vortex trajectories and optimal wavenumber spectrum for an initial height $h/b = 5$ at $\tau = 2.5$ (top row), $\tau = 5.0$ (middle row), and $\tau = 5.5$ (bottom row). The right column are the amplifications σ of an initial perturbations scaled by the corresponding maximum amplification of the unconstrained free vortex pair case $\sigma_{\max}$. The vortex core size is $\hat{a} = 0.23$.

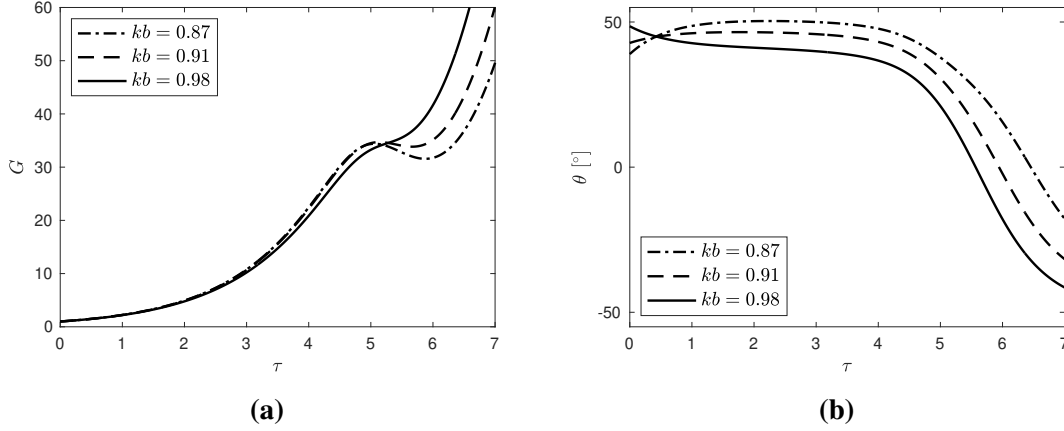


Figure 7: Energy gain (a) and instability plane angle (b) as a function of time for three wavenumbers and there optimal initial condition for select times of interest. $kb = 0.87$ maximizes the gain at $\tau = 2.5$, $kb = 0.93$ maximizes energy gain at $\tau = 5.0$, and $kb = 0.98$ maximizes energy gain at $\tau = 5.5$. Vortex core size is $\hat{a} = 0.23$ for each case.

different than that of σ , as σ is the amplification of a perturbation provided the optimal initial condition is considered for a particular wavenumber at some point in time. It is important to recall that the search for the optimal perturbation seeks the maximally amplified initial condition at each instant in time *per* wavenumber. This implies that the wavenumber of maximal perturbation growth can be different in time, as shown in Figures 5 and 6, which can be expected due to the non-autonomous nature of the governing equations.

Integration of the governing equations at particular wavenumbers and initial condition further reveal the time-dependent dynamics of the system. Figure 7a illustrates the growth curves for three different conditions on the optimal perturbation. Specifically, we plot the perturbation energy gain for the optimal perturbation at $\tau = 2.5$, $\tau = 5.0$, and $\tau = 5.5$. The wavenumbers of the associated perturbations are $kb = 0.87$, $kb = 0.91$, and $kb = 0.98$, respectively. The above times are selected to showcase how the optimal perturbation is time-dependent, i.e., optimizing growth for a particular instant in time will not guarantee maximal growth at a later time. Recall that the initial condition is obtained from the right singular vector of the singular value decomposition of the fundamental solution matrix at the time of interest τ corresponding to the maximally growing wavenumber. For optimal perturbations considered before $\tau = 5.5$, a decrease in the perturbation amplitudes after the wall interaction is observed, which has also been shown in the experiments of Asselin & Williamson (2017). This effect has been attributed to the rotation of the plane of which the instability exists as the vortices are swept away from each other along the wall. Recall that the Crow instability balances the self-induced rotation of the vortex, the induced strain field from the other vortex, and the mutual interaction between the perturbed vortices, allowing the perturbations to grow radially in a fixed plane Leweke *et al.* (2016). The rotations of the plane of instability are shown in Fig. 7b, which all illustrate the rotation induced by the wall. The optimal perturbation at $\tau = 5.5$ shows a decrease in the rate of amplification, yet is maximally excited compared to the other cases. Beyond choosing the optimal wavenumber at a particular instant in time, it is also necessary to consider the effect of the corresponding initial condition

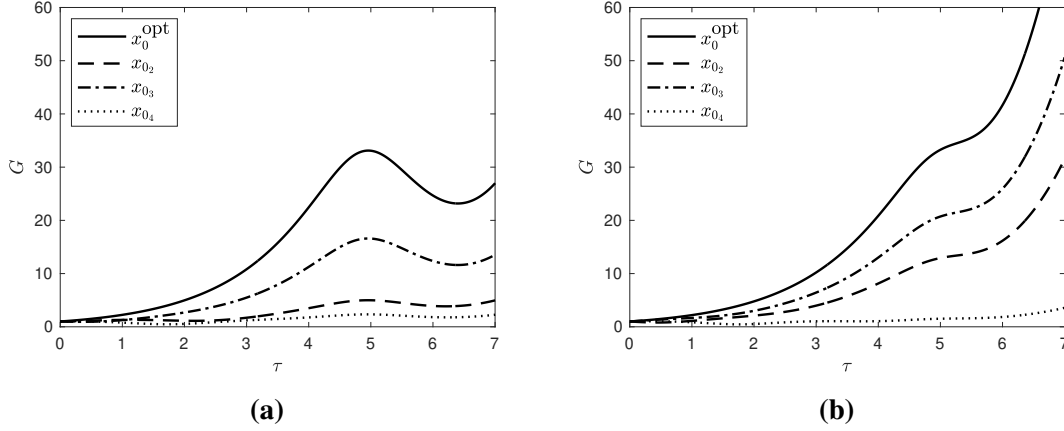


Figure 8: Comparison on the effect of initial condition for the wavenumbers $kb = 0.82$ (a) and $kb = 0.98$ (b). x_0^{opt} indicates the optimal initial condition of unit length from the right singular vector of the SVD of propagator matrix, where in this case the chosen times of optimization are $\tau = 2.5$ (s) and $\tau = 5.5$ (b). The initial conditions x_{0_2} , x_{0_3} , and x_{0_4} are randomly generated and scaled to be of unit length. The random initial conditions for the perturbations showcase how for the same wavenumber, optimal energy gain is highly dependent on the initial conditions. The initial height above the wall is $h/b = 5$ with a vortex core size of $\hat{a} = 0.23$ for all each case.

$\mathbf{x}_0$. The optimal initial condition for a given wavenumber is denoted $\mathbf{x}_0^{\text{opt}}$, which corresponds to the right singular vector of the singular value decomposition of the fundamental solution matrix at an instant in time τ . The right singular vector is of unit magnitude and describes the geometric orientation of the initial perturbation. To compare $\mathbf{x}_0^{\text{opt}}$ to other initial conditions, we randomly generate three other initial conditions of unit magnitude and directly integrate the systems. A similar illustration of this idea is presented by Rapaka *et al.* (2008) in the context of non-modal growth of perturbations in density-driven convection in porous media. Figure 8b exemplifies that, for a given wavenumber, the geometric initial condition can have significant impact on subsequent perturbation growth. For the configurations studied in this work, all optimal perturbations were geometrically symmetric. This observation is not surprising, as the Crow instability (Crow, 1970) is a symmetric perturbation, where anti-symmetric modes are notably less amplified. For the 4-vortex systems studied by Crouch (1997) and Fabre *et al.* (2002), symmetric configurations were also determined to be maximally growing.

6.6 Conclusions

The time-dependent characteristics of a perturbed equal strength and equal size counter-rotating vortex pair interacting with an inviscid wall have been studied as a special case of the equations and methods of Fabre *et al.* (2002) and Crouch (1997). The non-modal approach of Farrell & Ioannou (1996) enables the finite-time stability of the temporally varying vortex system to be studied, where we draw connection to the limiting case of the time-independent and non-periodic free vortex pair system by approximating the eigenvalues at long times via Eq. 11. To elucidate the effect of the wall on perturbation growth, the general 4-vortex system is recast as a 2-vortex system with ground

constraints. The ground constrained system is compared against an unconstrained free vortex pair to further highlight the effect of the wall on the development of the long-wavelength instability.

There are important differences in the present analysis compared to the experimental work of Asselin & Williamson (2017) and the computational work by Dehtyriov *et al.* (2020a,b). In a real viscous fluid, the vortex pair will undergo viscous dissipation and lose circulation strength during its descent, diminishing the magnitude of the interaction between the vortices. Secondary vorticity generation at the wall boundary will emerge and lead to vortex rebound from the surface. It is understood in the present work that we emphasize the *inviscid* interaction between the vortex pair and the wall, and limit our analysis to where boundary layer formulation can be neglected.

Direct integration of the system allows for further investigation on the effect of initial conditions of the present configurations. The present non-modal approach yields both the optimal wavenumber for perturbation gain and the initial condition that leads to maximum growth for any wavenumber at any instant in time. The optimal initial condition for all cases considered corresponded to a symmetric initial condition, where the vortices are originally oriented at $40 - 50^\circ$ to the horizontal. The effect of choosing the optimal initial condition is illustrated in Figs. 8a and 8b, where random initial conditions of unit magnitude were generated for comparison. The initial orientation and symmetry of the vortices significantly influence the resulting magnitude of perturbation growth, yet do not alter the general trend of the energy gain curves. We observe that upon wall interaction, a decrease in perturbation growth rate is observed, with some parameters yielding a temporary decrease in total perturbation amplitude during wall interaction. Choosing the optimal wavenumber at a given instant in time also notably impacts perturbation evolution, where selecting an initial wavenumber higher than the free vortex pair Crow instability results in larger perturbation amplification at the wall. From a control strategy perspective, it would be critical to not only excite a particular wavenumber of interest, but also initialize the perturbation such that it best realizes the corresponding optimal initial condition. Despite the limitations of the present inviscid analysis, the above findings can better inform the study of more realistic configurations to study the time-dependent nature of a vortex pair in ground effect.

7 STABILITY OF HELICAL VORTEX STRUCTURES FROM FLEXIBLE BLADES

S. Rodriguez, J. W. Jaworski

7.1 Background

The trailing vortices produced by flexible wings encountering a gust are perturbed away from a straight line and produce a self-induced corkscrew motion downstream. This orbiting vortex motion is also found in the wake of rotating blade systems with rotors. However, unlike the vortices shed from fixed-wing aircraft, the wake of a rotor consists of a helical vortex structure that persists within proximity of the rotor plane. The helical vortex structure is a byproduct of the lift distribution across the span of the rotor blade and the mutual interaction of the vortex elements that roll up and generate strong tip and root vortices, as shown in Fig. 9. However, bluff bodies at the center of rotors, such as nacelles on wind turbines, have historically been associated with the destabilization and dissipation of root vortices (Sherry *et al.*, 2013). Therefore, tip vortices are commonly the dominant aerodynamic structure in the wake of rotors and rotorcraft (Bhagwat & Leishman, 2000b). These tip vortices persist as strong coherent structures that can impact rotor blade loading and their associated performance significantly (Bhagwat & Leishman, 2000b; Leishman, 2006).

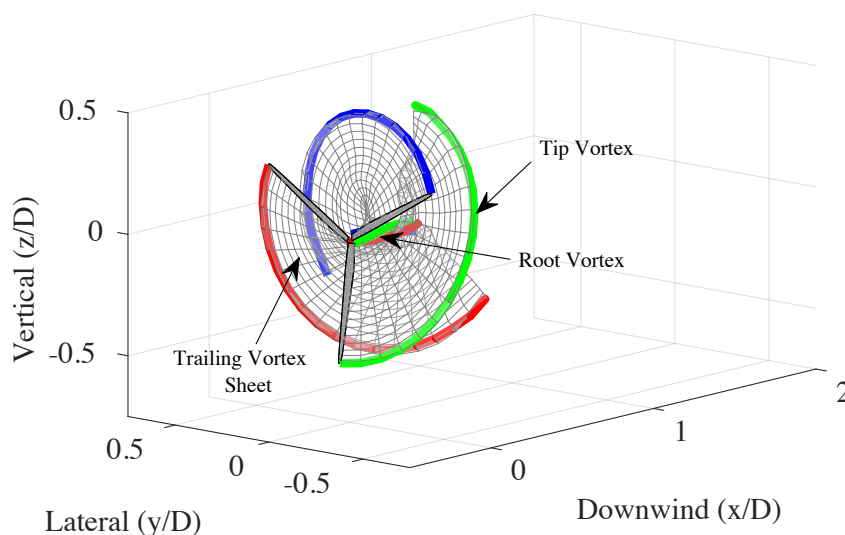


Figure 9: Rotor near-wake aerodynamic vortex structures

Most published research concerned with the near-wake is generally aimed toward understanding the impact of near-by vortex structures on rotor performance and the vortex interactions with rotor blades (Leishman, 2006; Bhagwat & Leishman, 2000a; Ramasamy *et al.*, 2010; Ramasamy, 2015; Yoon *et al.*, 2014; S. *et al.*, 2017; Cesnik *et al.*, 2004). These near-wake investigations are chiefly motivated by the need to understand, control, and minimize adverse operational impact, such as excessive blade-vortex interaction noise.

The present effort focuses on the fluid dynamic aspects of the kinematics and stability of shed helical vortices in the near-wake region. Investigations into the stability of coherent helical vortex structures generally study the influence of two types of disturbances: short-wave and long-wave perturbations (Quaranta *et al.*, 2015). Short-wave perturbations are able to disturb vortex cores and are generally a byproduct of external strain fields or the curvature and torsion of the vortex itself (Quaranta *et al.*, 2015). The present study is concerned with long-wave perturbations, which are generated by external disturbances with characteristic lengths much larger than the vortex core, such as gusts and atmospheric turbulence (Quaranta *et al.*, 2015).

The methods and outcomes of this research effort is structured as follows. Section 7.2 outlines the free-vortex wake method that is used to simulate rotor operation and to generate the helical wake. Section 7.3 introduces a linear eigenvalue stability analysis, which is used to quantify temporal stability characteristics of tip vortices. This section also introduces a canonical three-bladed rotor configuration of the National Renewable Energy Laboratory (NREL) 5MW reference wind turbine, which is a well-studied, flexible rotor in the literature. Section 7.4 applies the stability analysis to tip vortices generated by a stationary symmetric (i.e., rotor-plane perpendicular to inflow) configuration of this flexible rotor under a range of static rotor (non-rigid body rotor kinematics) conditions, such as variable tip speed ratio, inflow speed, and blade pitch. Finally, the conclusions of this research effort are discussed in Section 7.6.

7.2 Free-vortex wake method

The numerical model used to simulate rotor operation and generate the helical vortex system is based upon the free-vortex wake method (FVM) presented by Rodriguez (2018) and Rodriguez & Jaworski (2019, 2020) developed to emulate the NREL 5MW reference wind turbine and its aeroelastic performance. Their model coupled a linear kinematic beam theory for spinning structures to the free-vortex wake method code developed by Sebastian and Lackner, known as WInDS (Sebastian, 2012; Sebastian & Lackner, 2012b). To provide comprehensive context, the FVM framework employing both the Vatisas vortex model (Leishman, 2006; Sebastian & Lackner, 2012b) and vortex cut-off models (Sebastian & Lackner, 2012b) are now briefly outlined. Note that in the context of this paper all vectors are noted with a bold face, (i.e., $\mathbf{x}$) and matrices are noted as a bold faces inside square brackets (i.e., $[\mathbf{x}]$).

The flow physics modeled by the free-vortex wake method is governed by,

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}_\infty + \mathbf{V}_{\text{induced}} + \mathbf{V}_{\text{rbm}}, \quad (13)$$

where the left hand side tracks the velocity of the discrete filaments in the wake, and $\mathbf{V}_\infty$, $\mathbf{V}_{\text{induced}}$, and $\mathbf{V}_{\text{rbm}}$, are the freestream velocity, the induced velocity, and the velocity generated by rigid body motions (rbm) of the rotor (such as wave-induced motions of a floating offshore wind turbine), respectively. A uniform freestream velocity in the axial direction is assigned and prescribed in the present study. The rigid-body motion velocity may be superimposed onto the rotor kinematics, as has been done in Rodriguez (2018) and Rodriguez & Jaworski (2019, 2020) for wind turbine blades in an offshore environment. However, no such rotor motion is imposed in this investigation, i.e., $\mathbf{V}_{\text{rbm}} = \mathbf{0}$. The induced velocity is computed using the Biot-Savart law where the before-mentioned vortex core models are employed. Two standard forms of the Biot-Savart law

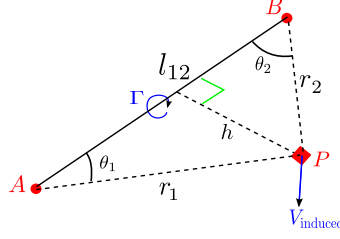


Figure 10: Induced velocity relationship of an i^{th} semi-infinite vortex filament

appear in the literature: the traditional form (Leishman, 2006) and another that is computationally more efficient (Phillips & Snyder, 2000). Both of these relations are provided below.

The traditional form of the Biot-Savart law for semi-infinite filaments is

$$\mathbf{V}_{\text{induced}} = \frac{\Gamma}{4\pi} \int_l \frac{d\mathbf{l} \times \mathbf{r}}{|\mathbf{r}|^3} \quad (14)$$

where $\mathbf{r}$ is the position along a semi-infinite vortex filament. For a straight-segment filament the Biot-Savart can be written as

$$\mathbf{V}_{\text{induced}, i} = \frac{\Gamma}{4\pi h} (\cos \theta_1 - \cos \theta_2) \mathbf{e}, \quad (15)$$

where i indicates the i^{th} filament, i.e. the induced velocity contribution of the i^{th} vortex filament on a point in space. Geometric relationships of the i^{th} filament configurations illustrated in Fig. 10 lead to the following expressions

$$h = \frac{|\mathbf{l}_{12} \times \mathbf{r}_1|}{l_{12}}, \quad \cos \theta_1 = \frac{\mathbf{l}_{12} \cdot \mathbf{r}_1}{l_{12}r_1}, \quad \cos \theta_2 = \frac{\mathbf{l}_{12} \cdot \mathbf{r}_2}{l_{12}r_2}, \quad \mathbf{e} = \frac{\mathbf{l}_{12} \times \mathbf{r}_1}{|\mathbf{l}_{12} \times \mathbf{r}_1|}, \quad (16)$$

where $\mathbf{e}$ is a column vector in $\mathbb{R}^3$. Substitution of these expression into Eq. 14 and algebraic manipulation then yields

$$\mathbf{V}_{\text{induced}, i} = \frac{\Gamma}{4\pi} \frac{\mathbf{l}_{12} \times \mathbf{r}_1}{|\mathbf{l}_{12} \times \mathbf{r}_1|^2} \mathbf{l}_{12} \cdot \left(\frac{\mathbf{r}_1}{r_1} - \frac{\mathbf{r}_2}{r_2} \right). \quad (17)$$

as presented in Phillips & Snyder (2000). Equation 17 can be rearranged to benefit numerical calculations by using the trigonometric relations

$$\mathbf{l}_{12} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos \theta_1, \quad |\mathbf{r}_1 \times \mathbf{r}_2| = r_1 r_2 \sin \theta_1. \quad (18)$$

Substitution of these relations into Eq. 17 yields

$$\begin{aligned} \mathbf{V}_{\text{induced}, i} &= \frac{\Gamma}{4\pi} \frac{\mathbf{l}_{12} \times \mathbf{r}_1}{|\mathbf{l}_{12} \times \mathbf{r}_1|^2} \left((\mathbf{r}_1 - \mathbf{r}_2) \cdot \left(\frac{\mathbf{r}_1}{r_1} - \frac{\mathbf{r}_2}{r_2} \right) \right) \\ &= \frac{\Gamma}{4\pi} \frac{(r_1 + r_2)(\mathbf{l}_{12} \times \mathbf{r}_1)}{l_{12}r_1(l_{12}r_1 + \mathbf{l}_{12} \cdot \mathbf{r}_1)}. \end{aligned} \quad (19)$$

The form of the Biot-Savart law presented by Phillips & Snyder (2000), Sebastian & Lackner (2012b), and Sebastian (2012) includes the vortex cutoff model (Sebastian & Lackner, 2012b) and the Vatisstas vortex model (Leishman, 2006; Sebastian & Lackner, 2012b):

$$\mathbf{V}_{\text{induced}, i} = \begin{cases} \frac{\Gamma}{4\pi} \frac{(r_1+r_2)(\mathbf{l}_{12} \times \mathbf{r}_1)}{l_{12}r_1(l_{12}r_1 + \mathbf{l}_{12} \cdot \mathbf{r}_1) + (\delta l_{12})^2}, & \text{if cutoff model,} \\ \frac{C_\nu \Gamma}{4\pi} \frac{(r_1+r_2)(\mathbf{l}_{12} \times \mathbf{r}_1)}{l_{12}r_1(l_{12}r_1 + \mathbf{l}_{12} \cdot \mathbf{r}_1)}, & \text{if Vatisstas model,} \end{cases} \quad (20)$$

where

$$C_\nu = \left(\frac{(l_{12}r_1)^2 - (\mathbf{l}_{12} \cdot \mathbf{r}_1)^2}{l_{12}^2} \right) \left(r_c^{2n} + \left(\frac{(l_{12}r_1)^2 - (\mathbf{l}_{12} \cdot \mathbf{r}_1)^2}{l_{12}^2} \right)^{2n} \right)^{-1/n}. \quad (21)$$

The geometric relations in Eq. 16 enable Eq. 20 to be expressed in the traditional form as

$$\mathbf{V}_{\text{induced}, i} = \begin{cases} \frac{\Gamma}{4\pi} \frac{1}{(h + (\delta l_{12})^2)} (\cos \theta_1 - \cos \theta_2) \mathbf{e}, & \text{if cutoff model,} \\ \frac{\Gamma}{4\pi} \frac{h}{(r_c^{2n} + h^{2n})^{1/n}} (\cos \theta_1 - \cos \theta_2) \mathbf{e}, & \text{if Vatisstas model.} \end{cases} \quad (22)$$

The stability analysis presented in this investigation is derived from Bhagwat & Leishman (2000a), who used the traditional form of the Biot-Savart law (Eq. 22 with the Vatisstas model), which for consistency will be used for the remainder of this paper.

7.3 Methods for stability and dynamics analysis

The current investigation considers long-wave perturbations on the rotor wake of a stationary wind turbine rotor (i.e., no rigid body motion due to offshore wave-induced forcing). The analysis used in this work was originally developed by Bhagwat & Leishman (2000a) to perturb the helicopter wake geometries harmonically in both space and time. Their investigation evaluated the tip-vortex stability of a hovering rotor, in which the tip-vortex geometry was generated by an FVM aerodynamic model. The current investigation also employs FVM. However, a major difference between the FVM employed by Bhagwat & Leishman (2000a) and the FVM used herein is that the current framework takes consideration of the impact that the trailing vortex sheet and its temporal changes have on the production of the tip vortex and roll-up effects. Hence, the wake geometry presented by this work is able produce additional physical insight into these effects on the stability analysis.

It is also important to note that Bhagwat and Leishman investigated perturbations of the tip vortices only, neglected blade elasticity, and modeled a simple canonical rectangular rotor blade geometry. The work presented here will look into the stability of wakes generated by the flexible, non-uniform, and tapered rotor-blade geometry of the NREL rotor blade. However, the current stability analysis of the wake, like Bhagwat and Leishman, will only consider perturbing the tip vortices in isolation, even though the wake geometry was generated by accounting for the presence of the trailing vortex sheet.

Due to the different modeling approach taken in this study to investigate tip-vortex stability, it can be anticipated that this study will recover similar trends reported by Bhagwat and Leishman, but will also bring to light any deviation of these stability trends due to geometric distortion of tip-vortices as a result of rotor-blade deformation. The derivation of the tip vortex stability analysis is now presented.

7.3.1 Perturbed induced velocity formulation

The free-vortex wake method depends on computing the local velocity of Lagrangian markers cast into the wake. Hence, the induced velocity field is perturbed by displacing the wake geometry by

a small quantity, $\delta \mathbf{r}$. To evaluate the stability of the system, the governing equation are perturbed as follows:

$$\frac{d(\mathbf{r} + \delta \mathbf{r})}{dt} = \mathbf{V}_{\text{induced},i}(\mathbf{r} + \delta \mathbf{r}) \rightarrow \frac{d\mathbf{r}}{dt} + \frac{d(\delta \mathbf{r})}{dt} = \mathbf{V}_{\text{induced},i}(\mathbf{r} + \delta \mathbf{r}) \quad (23)$$

$$\delta \dot{\mathbf{r}} = \mathbf{V}_{\text{induced},i}(\mathbf{r} + \delta \mathbf{r}) - \mathbf{V}_{\text{induced},i}(\mathbf{r}). \quad (24)$$

where the overdot in $\delta \dot{\mathbf{r}}$ represents the time derivative. The perturbed velocity can be expressed as an ordered series in $\delta \mathbf{r}$ as follows, where quadratic and higher order terms can be neglected:

$$\mathbf{V}_{\text{induced},i}(\mathbf{r} + \delta \mathbf{r}) = \mathbf{V}_{\text{induced},i}(\mathbf{r}) + \delta \mathbf{V}_{\text{induced},i}(\delta \mathbf{r}) + O((\delta \mathbf{r})^2). \quad (25)$$

Substituting Eq. 25 into Eq. 24, yields the perturbed governing equation:

$$\delta \dot{\mathbf{r}} = \delta \mathbf{V}_{\text{induced},i}(\delta \mathbf{r}). \quad (26)$$

The unperturbed velocity field is determined by the Biot-Savart law. Thus, substituting $\delta f = \frac{\partial f}{\partial x} \delta x$, where f is some function of interest being perturbed that is dependent on a variable x , into the Biot-Savart law using the Vatistas or cutoff model for semi-infinite straight filaments from Eq. 22 will yield the perturbed induced velocity field:

$$\begin{aligned} \mathbf{V}' = \bar{\Gamma} \left[(\cos \theta_1 - \cos \theta_2) \mathbf{e} + (\delta \cos \theta_1 - \delta \cos \theta_2) \mathbf{e} \right. \\ \left. + h_f \delta h (\cos \theta_1 - \cos \theta_2) \mathbf{e} + (\cos \theta_1 - \cos \theta_2) \delta \mathbf{e} \right] \end{aligned} \quad (27)$$

where for the Vatistas and cutoff models

$$h_f = \begin{cases} -\frac{1}{(h + (\delta l_{12})^2)}, & \text{if cutoff model,} \\ h^{-1} - \frac{2h^{2n-1}}{r_c^{2n} + h^{2n}}, & \text{if Vatistas model,} \end{cases} \quad (28)$$

and

$$\bar{\Gamma} = \begin{cases} \frac{\Gamma}{4\pi} \frac{1}{(h + (\delta l_{12})^2)}, & \text{if cutoff model,} \\ \frac{\Gamma}{4\pi} \frac{h}{(r_c^{2n} + h^{2n})^{1/n}}, & \text{if Vatistas model.} \end{cases} \quad (29)$$

Note that $\mathbf{V}' = \mathbf{V}_{\text{induced},i} + \delta \mathbf{V}_{\text{induced},i}$, and the spatial parameters, $\cos \theta_1$, $\cos \theta_2$, h , and $\mathbf{e}$ must all be perturbed to arrive at the governing perturbed induced velocity. The final expression for the perturbed induced velocity field requires further simplification of the perturbed parameters δh , $\delta (\cos \theta_1)$, $\delta (\cos \theta_2)$, and $\delta \mathbf{e}$, in terms of the position vectors, $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_P$. Applying this simplification yields the final form of the perturbed induced velocity on the point of interest, P ,

$$\delta \mathbf{V}_{\text{induced},i} = \bar{\Gamma} \{ [\mathbf{M}] \delta \mathbf{r}_A + [\mathbf{N}] \delta \mathbf{r}_B + [\mathbf{O}] \delta \mathbf{r}_P \}, \quad (30)$$

where

$$\delta \mathbf{r}_A^T = [\delta r_{Ax}, \delta r_{Ay}, \delta r_{Az}], \quad \delta \mathbf{r}_B^T = [\delta r_{Bx}, \delta r_{By}, \delta r_{Bz}], \quad \delta \mathbf{r}_P^T = [\delta r_{Px}, \delta r_{Py}, \delta r_{Pz}].$$

The coefficient matrices $[\mathbf{M}]$, $[\mathbf{N}]$, and $[\mathbf{O}]$ are now defined. First, consider the second term in Eq. 27 as

$$(\delta \cos \theta_1 - \delta \cos \theta_2) \mathbf{e} = \bar{\mathbf{A}}_1 \mathbf{e}^T \delta \mathbf{r}_A + \bar{\mathbf{B}}_1 \mathbf{e}^T \delta + \bar{\mathbf{P}}_1 \mathbf{e}^T \delta \mathbf{r}_P, \quad (31)$$

where

$$\bar{\mathbf{A}}_1^T = \left(\frac{\partial(\cos \theta_1)}{\partial r_{Ax}} - \frac{\partial(\cos \theta_2)}{\partial r_{Ax}}, \frac{\partial(\cos \theta_1)}{\partial r_{Ay}} - \frac{\partial(\cos \theta_2)}{\partial r_{Ay}}, \frac{\partial(\cos \theta_1)}{\partial r_{Az}} - \frac{\partial(\cos \theta_2)}{\partial r_{Az}} \right), \quad (32)$$

$$\bar{\mathbf{B}}_1^T = \left(\frac{\partial(\cos \theta_1)}{\partial r_{Bx}} - \frac{\partial(\cos \theta_2)}{\partial r_{Bx}}, \frac{\partial(\cos \theta_1)}{\partial r_{By}} - \frac{\partial(\cos \theta_2)}{\partial r_{By}}, \frac{\partial(\cos \theta_1)}{\partial r_{Bz}} - \frac{\partial(\cos \theta_2)}{\partial r_{Bz}} \right), \quad (33)$$

$$\bar{\mathbf{P}}_1^T = \left(\frac{\partial(\cos \theta_1)}{\partial r_{Px}} - \frac{\partial(\cos \theta_2)}{\partial r_{Px}}, \frac{\partial(\cos \theta_1)}{\partial r_{Py}} - \frac{\partial(\cos \theta_2)}{\partial r_{Py}}, \frac{\partial(\cos \theta_1)}{\partial r_{Pz}} - \frac{\partial(\cos \theta_2)}{\partial r_{Pz}} \right). \quad (34)$$

The coefficient matrices are collected as

$$\begin{aligned} [\mathbf{M}_1] &= \bar{\mathbf{A}}_1 \mathbf{e}^T, \\ [\mathbf{N}_1] &= \bar{\mathbf{B}}_1 \mathbf{e}^T, \\ [\mathbf{O}_1] &= \bar{\mathbf{P}}_1 \mathbf{e}^T. \end{aligned} \quad (35)$$

Next, the third term in Eq. 27 is defined as

$$h_f \delta h (\cos \theta_1 - \cos \theta_2) \mathbf{e} = h_f (\cos \theta_1 - \cos \theta_2) \left(\bar{\mathbf{A}}_2 \mathbf{e}^T \delta \mathbf{r}_A + \bar{\mathbf{B}}_2 \mathbf{e}^T \delta \mathbf{r}_B + \bar{\mathbf{P}}_2 \mathbf{e}^T \delta \mathbf{r}_P \right), \quad (36)$$

where the coefficient matrices are collected and defined as

$$\begin{aligned} [\mathbf{M}_2] &= h_f (\cos \theta_1 - \cos \theta_2) \bar{\mathbf{A}}_2 \mathbf{e}^T, \\ [\mathbf{N}_2] &= h_f (\cos \theta_1 - \cos \theta_2) \bar{\mathbf{B}}_2 \mathbf{e}^T, \\ [\mathbf{O}_2] &= h_f (\cos \theta_1 - \cos \theta_2) \bar{\mathbf{P}}_2 \mathbf{e}^T, \end{aligned} \quad (37)$$

using

$$\bar{\mathbf{A}}_2^T = \left(\frac{\partial h}{\partial r_{Ax}}, \frac{\partial h}{\partial r_{Ay}}, \frac{\partial h}{\partial r_{Az}} \right), \quad (38)$$

$$\bar{\mathbf{B}}_2^T = \left(\frac{\partial h}{\partial r_{Bx}}, \frac{\partial h}{\partial r_{By}}, \frac{\partial h}{\partial r_{Bz}} \right), \quad (39)$$

$$\bar{\mathbf{P}}_2^T = \left(\frac{\partial h}{\partial r_{Px}}, \frac{\partial h}{\partial r_{Py}}, \frac{\partial h}{\partial r_{Pz}} \right). \quad (40)$$

Finally, the fourth term in Eq. 27 is defined as

$$(\cos \theta_1 - \cos \theta_2) \delta \mathbf{e} = (\cos \theta_1 - \cos \theta_2) [\bar{\mathbf{A}}_3] \delta \mathbf{r}_A + (\cos \theta_1 - \cos \theta_2) [\bar{\mathbf{B}}_3] \delta \mathbf{r}_B + (\cos \theta_1 - \cos \theta_2) [\bar{\mathbf{P}}_3] \delta \mathbf{r}_P, \quad (41)$$

where

$$[\bar{\mathbf{A}}_3] = \begin{bmatrix} \frac{\partial e_x}{\partial r_{Ax}} & \frac{\partial e_x}{\partial r_{Ay}} & \frac{\partial e_x}{\partial r_{Az}} \\ \frac{\partial e_y}{\partial r_{Ax}} & \frac{\partial e_y}{\partial r_{Ay}} & \frac{\partial e_y}{\partial r_{Az}} \\ \frac{\partial e_z}{\partial r_{Ax}} & \frac{\partial e_z}{\partial r_{Ay}} & \frac{\partial e_z}{\partial r_{Az}} \end{bmatrix}, \quad [\bar{\mathbf{B}}_3] = \begin{bmatrix} \frac{\partial e_x}{\partial r_{Bx}} & \frac{\partial e_x}{\partial r_{By}} & \frac{\partial e_x}{\partial r_{Bz}} \\ \frac{\partial e_y}{\partial r_{Bx}} & \frac{\partial e_y}{\partial r_{By}} & \frac{\partial e_y}{\partial r_{Bz}} \\ \frac{\partial e_z}{\partial r_{Bx}} & \frac{\partial e_z}{\partial r_{By}} & \frac{\partial e_z}{\partial r_{Bz}} \end{bmatrix}, \quad [\bar{\mathbf{P}}_3] = \begin{bmatrix} \frac{\partial e_x}{\partial r_{Px}} & \frac{\partial e_x}{\partial r_{Py}} & \frac{\partial e_x}{\partial r_{Pz}} \\ \frac{\partial e_y}{\partial r_{Px}} & \frac{\partial e_y}{\partial r_{Py}} & \frac{\partial e_y}{\partial r_{Pz}} \\ \frac{\partial e_z}{\partial r_{Px}} & \frac{\partial e_z}{\partial r_{Py}} & \frac{\partial e_z}{\partial r_{Pz}} \end{bmatrix}. \quad (42)$$

the coefficient matrices are then defined as

$$\begin{aligned} [\mathbf{M}_3] &= (\cos \theta_1 - \cos \theta_2) [\bar{\mathbf{A}}_3], \\ [\mathbf{N}_3] &= (\cos \theta_1 - \cos \theta_2) [\bar{\mathbf{B}}_3], \\ [\mathbf{O}_3] &= (\cos \theta_1 - \cos \theta_2) [\bar{\mathbf{P}}_3]. \end{aligned} \quad (43)$$

Finally, the global coefficient matrices of Eq. 30 are obtained by summing the appropriate coefficient matrices:

$$[\mathbf{M}] = [\mathbf{M}_1] + [\mathbf{M}_2] + [\mathbf{M}_3], \quad [\mathbf{N}] = [\mathbf{N}_1] + [\mathbf{N}_2] + [\mathbf{N}_3], \quad [\mathbf{O}] = [\mathbf{O}_1] + [\mathbf{O}_2] + [\mathbf{O}_3]. \quad (44)$$

Substituting Eq. 44 into Eq. 30, defines the perturbed induced velocity field completely.

7.3.2 Long-wave harmonic perturbation

Now that the perturbed induced velocity is expressed by its independent parameters, the type of perturbation to be used on the Lagrangian marker velocity must be defined, i.e., the left hand side of Eq. 26. For this analysis the wake is perturbed in a harmonic fashion. In other words, the form of the perturbation is assumed to be a travelling wave such that a series of normal mode perturbations could describe any arbitrary disturbance (Bhagwat & Leishman, 2000a). These perturbations are set in cylindrical coordinates for convenience, and their amplitudes are denoted by δ_0 . The harmonic perturbation is defined by the following expression:

$$\delta \mathbf{p}_k = \begin{bmatrix} \delta x_0 \\ \delta r_0 \\ \delta \theta_0 \end{bmatrix} e^{\alpha t + i\omega \zeta_k} = \delta_0 e^{\alpha t + i\omega \zeta_k}, \quad (45)$$

where $k = A, B$, or P , which correspond to endpoints of the straight segment vortex filament and a point in space, as shown in Fig. 10. It is important to note here that $i = \sqrt{-1}$ and is not the filament index. Unidirectional perturbations are illustrated below in Figs. 11-13 for a helical vortex, at an arbitrary time, age, divergence rate, and a fixed frequency.

The cylindrical coordinate transformation matrix is defined by

$$[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -r \sin \theta \\ 0 & \sin \theta & r \cos \theta \end{bmatrix}, \quad (46)$$

where the harmonic perturbation Eq. (45) is transformed to Cartesian coordinates as follows:

$$\delta \mathbf{r}_A = [\mathbf{T}] \delta \mathbf{p}_A, \quad \delta \mathbf{r}_B = [\mathbf{T}] \delta \mathbf{p}_B, \quad \delta \mathbf{r}_P = [\mathbf{T}] \delta \mathbf{p}_P. \quad (47)$$

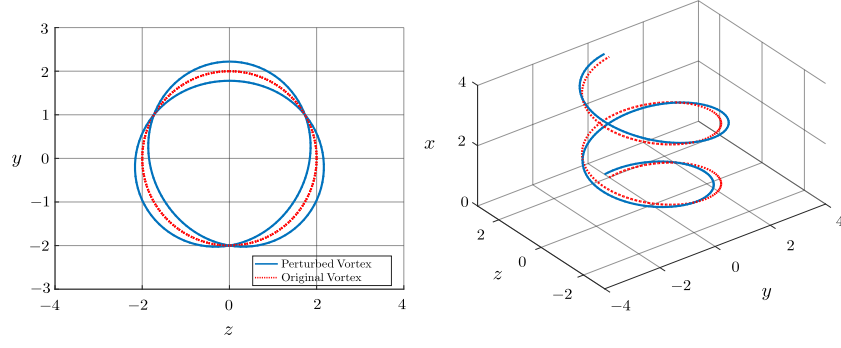


Figure 11: Illustration of a radial perturbation at $\omega = 1.5 \text{ rad}^{-1}$

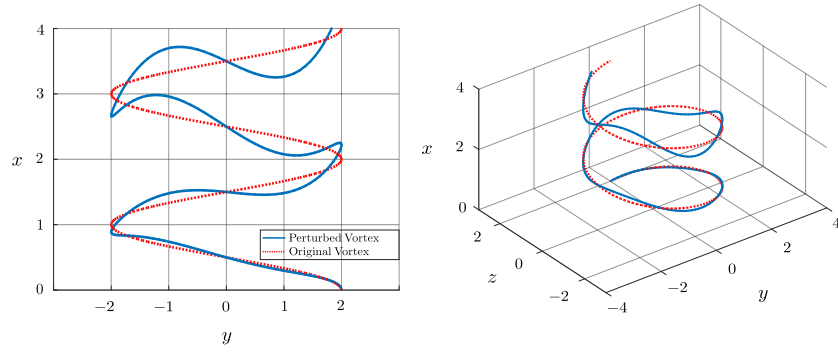


Figure 12: Illustration of an axial perturbation at $\omega = 3 \text{ rad}^{-1}$

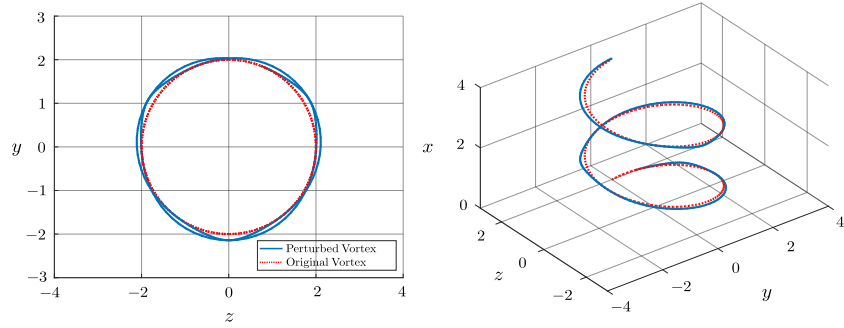


Figure 13: Illustration of an azimuthal perturbation at $\omega = 1.5 \text{ rad}^{-1}$

Also note the corresponding transformation matrix expressions:

$$\dot{\mathbf{r}}_k = [\mathbf{T}] \dot{\mathbf{p}}_k, \quad (48)$$

$$\delta \dot{\mathbf{r}}_k = [\mathbf{T}] \delta \dot{\mathbf{p}}_k + [\mathbf{T}_2] \delta \mathbf{p}_k, \quad (49)$$

$$[\mathbf{T}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\theta} \sin \theta & -\dot{z} \\ 0 & \dot{\theta} \cos \theta & \dot{y} \end{bmatrix}, \quad (50)$$

$$\cos \theta = \frac{y}{\sqrt{y^2 + z^2}}, \quad \sin \theta = \frac{z}{\sqrt{y^2 + z^2}}, \quad \dot{\theta} = \frac{y\dot{z} - z\dot{y}}{y^2 + z^2}. \quad (51)$$

7.3.3 Eigenvalue analysis

A complete set of equations has now been obtained (Eqs. 30, 49) that is necessary to formulate an eigenvalue stability analysis. First, substitute Eq. 49 into the left hand side of the governing perturbed equation, Eq. 26,

$$\delta \dot{\mathbf{r}}_k = [\mathbf{T}] \delta \dot{\mathbf{p}}_k + [\mathbf{T}_2] \delta \mathbf{p}_k = (\alpha [\mathbf{T}] + [\mathbf{T}_2]) \delta_0 e^{\alpha t + i\omega \zeta_k}. \quad (52)$$

Next, substituting Eq. 47 into Eq. 30 will give

$$\begin{aligned} \delta \mathbf{V}_{\text{induced},i} &= \bar{\Gamma} ([\mathbf{M}] \delta \mathbf{r}_A + [\mathbf{N}] \delta \mathbf{r}_B + [\mathbf{O}] \delta \mathbf{r}_P) \\ &= \bar{\Gamma} ([\mathbf{M}] [\mathbf{T}] \delta_0 e^{\alpha t + i\omega \zeta_A} + [\mathbf{N}] [\mathbf{T}] \delta_0 e^{\alpha t + i\omega \zeta_B} + [\mathbf{O}] [\mathbf{T}] \delta_0 e^{\alpha t + i\omega \zeta_P}) \\ &= \bar{\Gamma} ([\mathbf{M}] e^{i\omega(\zeta_A - \zeta_P)} + [\mathbf{N}] e^{i\omega(\zeta_B - \zeta_P)} + [\mathbf{O}]) [\mathbf{T}] \delta_0 e^{\alpha t + i\omega \zeta_P} \\ &= [\mathbf{V}_i] [\mathbf{T}] \delta \mathbf{p}_P, \end{aligned} \quad (53)$$

where by only considering the real contribution of $e^{i\omega(\zeta_A - \zeta_P)}$ and $e^{i\omega(\zeta_B - \zeta_P)}$:

$$[\mathbf{V}_i] = \bar{\Gamma} (\cos \omega(\zeta_A - \zeta_P) [\mathbf{M}] + \cos \omega(\zeta_B - \zeta_P) [\mathbf{N}] + [\mathbf{O}]). \quad (54)$$

The subscript i denotes the i^{th} filament (the segment between $\mathbf{r}_B$ and $\mathbf{r}_A$). The superposition of each individual filament's perturbed induced velocity field on a point of interest is expressed as $\delta \mathbf{V} = \sum_i \delta \mathbf{V}_{\text{induced},i}$. Likewise, the expression $[\mathbf{V}] = \sum_i \mathbf{V}_i$ is the total perturbed induced velocity field in cylindrical coordinates on the point of interest. Substituting Eqs. 52 and 53 into Eq. 26 yields, after some rearrangement,

$$\alpha \delta \mathbf{r}_P = ([\mathbf{V}] - [\mathbf{T}_2] [\mathbf{T}]^{-1}) \delta \mathbf{r}_P. \quad (55)$$

Setting $[\mathbf{W}] = ([\mathbf{V}] - [\mathbf{T}_2] [\mathbf{T}]^{-1})$ and rearranging Eq. 55 further gives the final form of the eigenvalue problem for each *point of interest* along the tip vortex:

$$([\mathbf{W}] - \alpha [\mathbf{I}]) \delta \mathbf{r}_P = \mathbf{0}, \quad (56)$$

where solving for the maximum α along the age of the tip vortex, at a specified radial wavenumber ω and instant in time t , produces the stability trend, ω v. α , of the tip vortices.

Stability of tip vortices shed from a canonical rotor: method comparison Three conditions must be met to validate the stability analysis as presented. The stability trends (divergence rate versus perturbation wavenumber, α v. ω) must obey the following classical results: 1) peak divergence rates occur at perturbation wavenumbers $\omega = N_b (k - \frac{1}{2})$, 2) divergence rates must converge to a constant value as the wavenumber goes to infinity, 3) because the canonical rotor configuration generates symmetric results, it is required that the stability trends of each individual tip vortex be identical. In addition to these requirements, it can also be anticipated that stability trends have larger divergence rates as the number of blades increase, which was observed by Bhagwat & Leishman (2000a).

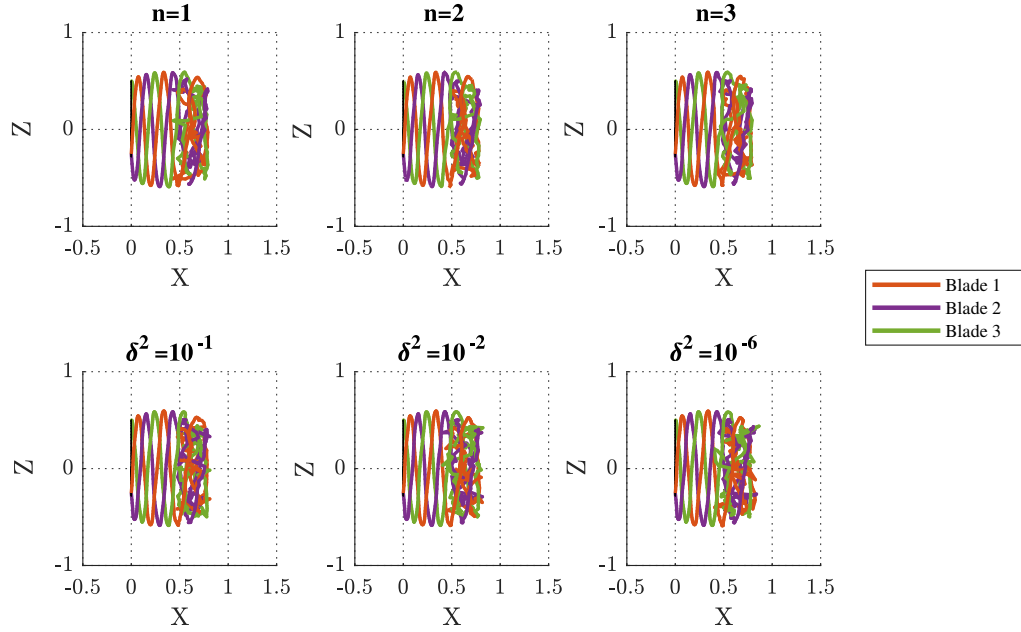
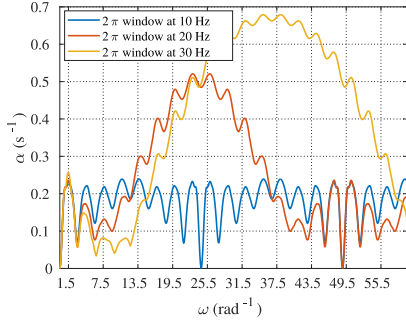


Figure 14: Three-bladed rotor wakes modeled by employing the Vatistas core model (first row) and the cutoff model (second row). Shedding frequency is $f = 20$ Hz.

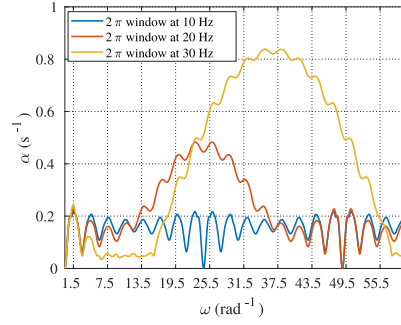
Stability results are now presented for the tip vortices produced by the canonical rotor. Figure 14 presents side views of the three-bladed rotor and its wake, where the Vatistas core modeling and cutoff core modeling have been used where indicated. The two-bladed and four-bladed rotor wakes are presented and discussed as auxiliary studies in Rodriguez *et al.* (2021). First, the stability trends (α v. ω) are presented in Figure 15 for the Vatistas model and cutoff model at variable vortex shedding frequencies. Perhaps the most notable feature of Fig. 15 is the periodic behavior in divergence rate oscillations as a function of wavenumber. Classical stability studies (Widnall, 1972; Gupta & Loewy, 1974; Bhagwat & Leishman, 2000a) have traditionally looked at low wavenumbers that are consistent with the underlying long-wave perturbation qualitative assumption, i.e., long-wave perturbations are defined as a perturbations much larger than the vortex core. However, because of the long-wave assumption, no investigation, to our knowledge, has presented stability trends for a high range of wavenumber perturbations to visualize the limits of long-wave perturbation analyses.

It is believed that this periodic behavior is caused by both the long-wave perturbation analysis and a numerical artifact from the time-marching scheme (second-order Runge-Kutta) of the free-vortex wake method currently used based on the following observations. Notice that for a relatively low vortex shedding frequency, such as $f = 10$ Hz (blue trends in Fig. 15), the periodic behavior has lower amplitude and lower period than at shedding frequencies of $f = 20$ and 30 Hz. Higher shedding frequencies of $f = 20$ and 30 Hz show a high bell-shaped divergence rate trend. It is also observed that the width of this “bell” increases with the shedding frequency. This result suggests that if the shedding frequency approaches infinity then the rotor-wakes simulated by the current free-vortex wake method would be most sensitive to high wavenumbers, i.e., short-wave

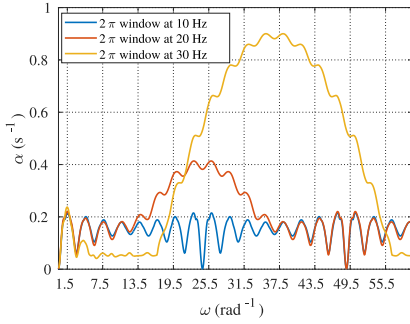
perturbations. Finally, the bell curve maxima occur at approximately the same location ($\omega \approx 25$ (1/rad) at $f = 20$ Hz; and $\omega \approx 37$ (1/rad) at $f = 30$ Hz) for rotor wake geometries employing the Vatistas model, which further reinforces the conjecture that the periodic bell-curve is an artifact of the long-wave perturbation analysis, the numerical integration, and their relationship to the Vatistas finite core modeling.



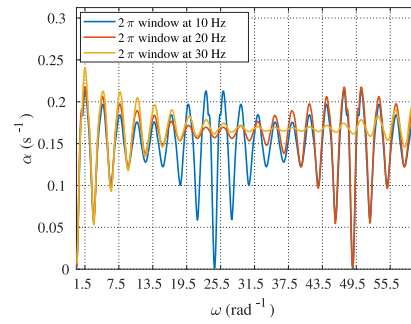
(a) Vatistas vortex core model, $n = 1$



(b) Vatistas vortex core model, $n = 2$



(c) Vatistas vortex core model, $n = 3$



(d) Cutoff vortex core model, $\delta^2 = 10^{-1}$

Figure 15: Stability trends for one tip vortex shed from a three-bladed rotor at variable vortex shedding frequencies (10, 20, 30 Hz) with a one-rotation (2π) tip vortex window. Vatistas models for $n = 1, 2$, and 3 are employed, and the cutoff model is presented with a radius of $\delta^2 = 0.1$.

At wavenumbers between $0 \leq \omega \leq 7.5 \text{ rad}^{-1}$, the stability trends generated by the Vatistas models ($n = 1, 2$, and 3) with shedding frequencies $f = 10$ and $f = 20$ Hz show very good agreement with reported classical stability trends, i.e., divergence-rate peaks occur at $\omega = N_b (k - 1/2)$, where k is any natural number. However, at $f = 20$ and 30 Hz, the stability trends do not exhibit the correct behavior beyond the second divergence rate peak. In fact, the stability trend becomes erratic as wavenumbers increase and approach the bell. This result may indicate that the shedding frequency has neighboring vortices spaced too closely to remain numerically stable to record the vortex stability trends reported by classical studies with the Vatistas model. Limitations of the Vatistas model with regard to capturing the classical stability results at higher vortex shedding fre-

quencies is further highlighted by the cutoff vortex model results in Fig. 15d. The cutoff model shows very consistent oscillations of the divergence rates for all vortex shedding frequencies. It appears that as the vortex frequency increases the period of the bell behavior tends to infinity. This dependence on vortex shedding frequency suggests that an infinite vortex shedding frequency is required to achieve classical stability trend results for the cutoff model. The absence of the bell as seen for the Vatisstas models also implies that the cutoff model is not susceptible to short-wave (high wavenumber) numerical instabilities. The differences between the Vatisstas and cutoff models motivated further investigation into reducing the cutoff radius, which becomes the Rankine vortex as $\delta^2 \rightarrow 0$. Similarity between stability trends for variable cutoff radii further reinforces the notion that the Vatisstas core model may be more sensitive to short-wave numerical instabilities.

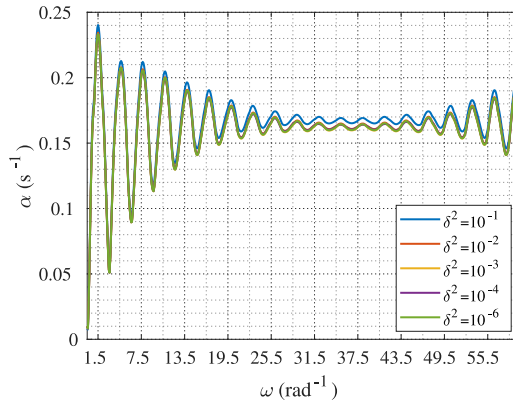
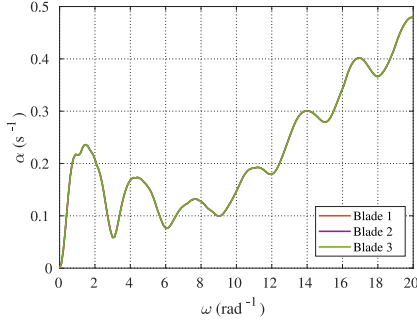


Figure 16: Stability trends for the three-bladed rotor wake with shedding frequency set at $f = 30$ Hz and reducing the cutoff radii to $\delta^2 \rightarrow 0$.

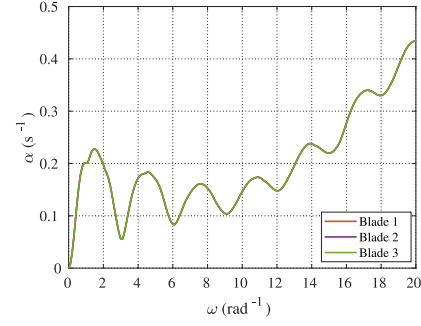
The above stability trends are for single tip vortex shed from one blade. Recall that the stability trends for each tip vortex on a multi-bladed rotor must be identical for the analysis to be valid. The stability trends for each individual tip vortex for a three-bladed rotor are presented in Fig. 17. Here it is demonstrated that the stability trends for the tip vortex from each blade are identical, as expected due to the symmetric rotor configuration. For both the Vatisstas and the cutoff models, the divergence rates peak at perturbation wavenumbers $\omega = N_b (k - 1/2)$, i.e., the classical stability trend criteria, for wavenumber perturbations up to $\omega = 13.5 \text{ rad}^{-1}$. After $\omega = 13.5 \text{ (rad}^{-1})$ peak divergence rates correspond to wavenumbers that are shifted forward by some value ϵ (i.e., peaks occur at some $\omega = N_b (k - 1/2) + \epsilon$), which deviates from the classical stability trend.

7.4 Stability and dynamics of tip vortices shed from flexible rotors

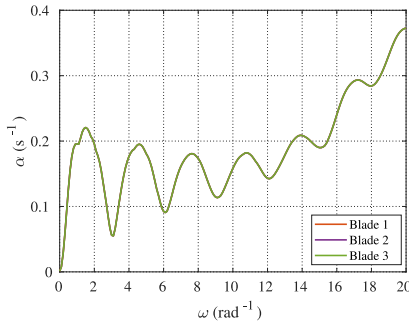
Tip vortices shed from the NREL 5MW Reference Wind Turbine rotor, with a rigid rotor, and a flexible rotor, are now investigated. Three cases spanning low-level and high-level operating conditions of the NREL 5MW reference wind turbine are considered to evaluate a range of rotor aeroelasticity and its impact on tip vortex dynamics and stability. The operational conditions considered in the current work were first presented in Sebastian & Lackner (2012a). The tip-speed ratio λ (defined as $\frac{\text{blade-tip speed}}{V_\infty}$), blade pitch θ_{bl} , inflow velocity V_∞ , and rotor diameter-based Reynolds



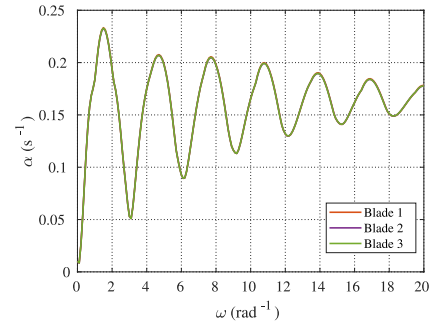
(a) Vatistas vortex core model, $n = 1$



(b) Vatistas vortex core model, $n = 2$



(c) Vatistas vortex core model, $n = 3$



(d) Cutoff vortex core model, $\delta^2 = 10^{-6}$

Figure 17: Stability trends for all the individual tip vortices shed from a three-bladed rotor from Vatistas and cutoff finite core models

number $Re_D = \rho D V_\infty / \mu$, where for all cases $\rho = 1.23 \text{ kg/m}^3$, $\mu = 1.20 \times 10^{-5} \text{ kg/(m}\cdot\text{s)}$, $D = 126 \text{ m}$ (Jonkman *et al.*, 2009), are used to describe case conditions.

Table 1 lists the parameters for all cases. In addition, tip vortices shed from a symmetric rotor configuration (rotor-plane is perpendicular to the horizontal uniform inflow) and the rotor operational configuration as specified by NREL (rotor-plane at 5° tilt) will be investigated. All simulations were run for 60 s with a vortex shedding frequency of $f = 12 \text{ Hz}$, using the Vatistas finite core model with $n = 2$. As in the canonical rotor case, simulation snapshots are presented with $X = x/D$ and $Z = z/D$ axis labels, where x , z , and D are distances in the downstream direction, distance in the vertical direction, and the rotor diameter, respectively. The stability analyses employed the window cutoff specified by Rodriguez (2018), such that Case 1 employs a 2π -window due to small tip vortex pitch rate, and Cases 2 and 3 employ a 4π window due to a larger tip vortex pitch rate. Thus, it is important to remember that by performing the stability analysis on a windowed portion of the wake, conclusions to be drawn apply only to early ages of tip vortices and not as a whole vortex structure that includes far-wake regions. To satisfy the window specifications, the stability analyses are conducted on tip vortices between simulation

instants $t = 20$ s and $t = 40$ s.

Table 1: Parametric conditions considered

	λ	V_∞ [m/s]	θ_{bl} [°]	Re_D
Case 1:	9.63	6	0	7.75×10^7
Case 2:	7	11.4	0	14.7×10^7
Case 3:	4.43	18	15	23.2×10^7

7.4.1 Case 1: Below-rated operating conditions

First, Case 1 corresponding to the below-rated, off-design condition of the NREL 5MW wind turbine is investigated. Snapshots of the symmetric and operational configurations at $t = 60$ s are shown in Fig. 18. The difference in rigid symmetric and operational configurations is highlighted in the location at which the numerical onset of qualitative wake breakdown occurs. The symmetric configuration appears coherent approximately between $0 \leq X \leq 1$, whereas the operational configuration maintains coherence approximately between $0 \leq X \leq 0.7$. Each of the flexible rotor configurations both exhibit earlier breakdown than the rigid configurations. The earlier wake breakdown in flexible rotors is likely due to the initial transient impact by the blade deflection and its subsequent influence on the formation of the tip vortices. However, for both flexible rotor configurations (symmetric and tilted) the location at which the numerical wake break-down begins is approximately the same, which indicates that rotor-configuration is not a major contribution to wake-breakdown at relatively low-inflow conditions. This result is likely due to the wake breakdown being largely dominated by the computational start-up transients of the blade deflection rather than being dominated by periodic aerodynamic loading, as discussed in Rodriguez & Jaworski (2019).

The qualitative evaluation of the tip vortices above is reinforced by the stability analysis performed on the snapshot $t = 40$ s in Fig. 19. As expected, via the classical stability trend criteria, the maximum eigenvalue for all configurations as shown in Fig. 19 corresponds to a perturbation wavenumber of $\omega = 1.5 \text{ rad}^{-1}$. The operational configurations exhibit slightly higher peaks than the symmetric rotor configuration counterparts, which bolsters the notion that operational configurations generate more unstable tip vortices for rigid rotors. However, it is interesting to note in Fig. 19 that wake breakdown appears to occur at similar locations downstream, despite the flexible rotor operational configuration having larger maximum eigenvalues than the flexible rotor symmetric configuration. It is important to note, though, that similar locations of wake breakdown may be a result from the initial transient perturbations from blade deformation that dominate the wake formation downstream. However, as time progresses the “windowed” stability analysis captures the inherent tip vortex stability of each corresponding rotor configuration, which results in higher maximum eigenvalues for operational rotor configurations in the 4π window.

To further highlight the impact blade flexibility has on tip vortex stability, eigenvalues corresponding to perturbation wavenumbers $\omega = 1.5, 4.5$ and 7.5 rad^{-1} , i.e. $N_b = 3$ and $k = 1, 2, 3$

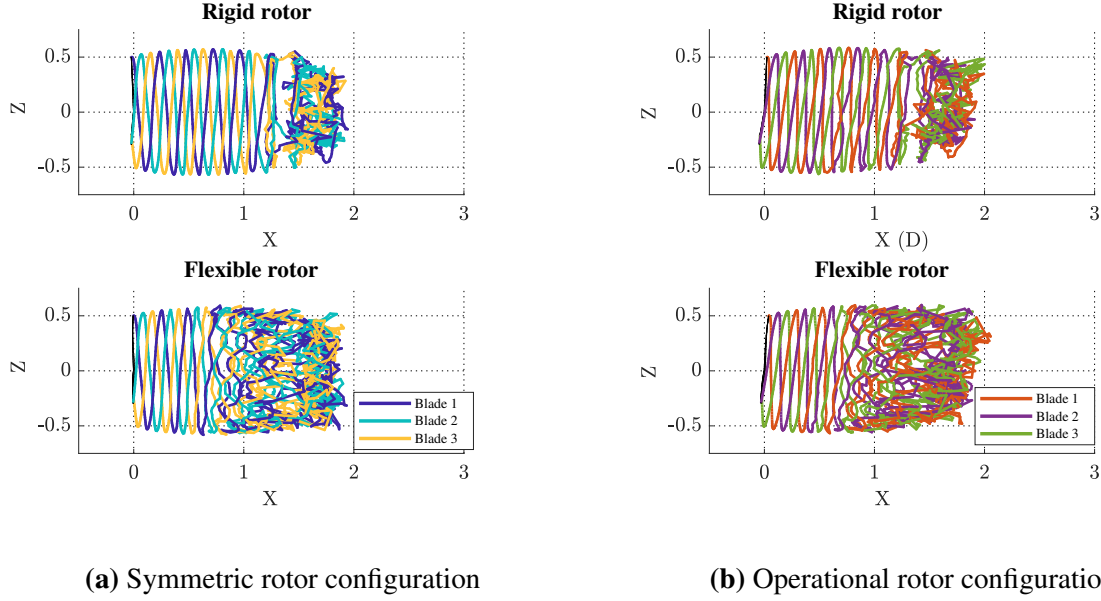


Figure 18: Case 1 tip-vortex snapshot at $t = 60$ s for rigid and flexible rotor operation

for $\omega = N_b(k - 1/2)$, are tracked in time for symmetric (sym.) and operational (op.) configurations and results are presented in Fig. 20. Despite the rotor configuration, it is shown in Fig. 27a that within the 2π window of tip vortices under consideration, blade flexibility initially generates less unstable (lower positive eigenvalues) tip vortices than rigid rotors. As time progresses, the eigenvalues generated from the stability of rigid rotor tip vortices tend to approach the temporal eigenvalue characteristics of tip vortices generated by flexible rotors. The impact of blade flexibility on the stability of tip vortices is likely an indication that aeroelastic interactions allow the wake to more quickly approach its equilibrium state.

The conjecture presented earlier that operational rotor configurations break down earlier than symmetric rotor configurations is supported by Fig. 27a, which shows that operational configurations periodically reach higher maximum eigenvalues (α corresponding to $\omega = 1.5 \text{ rad}^{-1}$) than symmetric operational configurations. The periodic behavior seen in Fig. 27a, and is not present in the symmetric configuration, corresponds to a change in the angle-of-attack of individual blades as they pass through a full rotation in the tilted rotor plane. Figure 27b further illustrates the out-of-phase behavior of the maximum eigenvalues (corresponding to perturbation wavenumber $\omega = 1.5 \text{ rad}^{-1}$) of each tip vortex shed off of individual blades. It is seen that each individual tip vortex contains the same stability trend but with a phase shift that corresponds to the periodic change in angle of attack of each rotor blade as it moves around the tilted rotor plane.

The impact that the operational configuration has on stability trends is highlighted by the fast-Fourier transform (FFT) of eigenvalues corresponding to $\omega = 1.5 \text{ rad}^{-1}$, in Fig. 21. Both symmetric (21a) and operational (21b) configurations are compared to evaluate rotor-plane tilt impact on tip vortex stability. The FFT for the rigid symmetric rotor configuration reflects a low-frequency contribution of $f \approx 0.02 \text{ Hz}$ that is not trivial to interpret. This contribution is seen in Fig. 27a where the time-history of the eigenvalue shows a slight decay and growth, which may likely be caused by the meandering of the initial transient effects of the simulation, i.e., the dynamics of the agglomeration of filaments downstream in Fig. 18. The flexible symmetric rotor configuration also

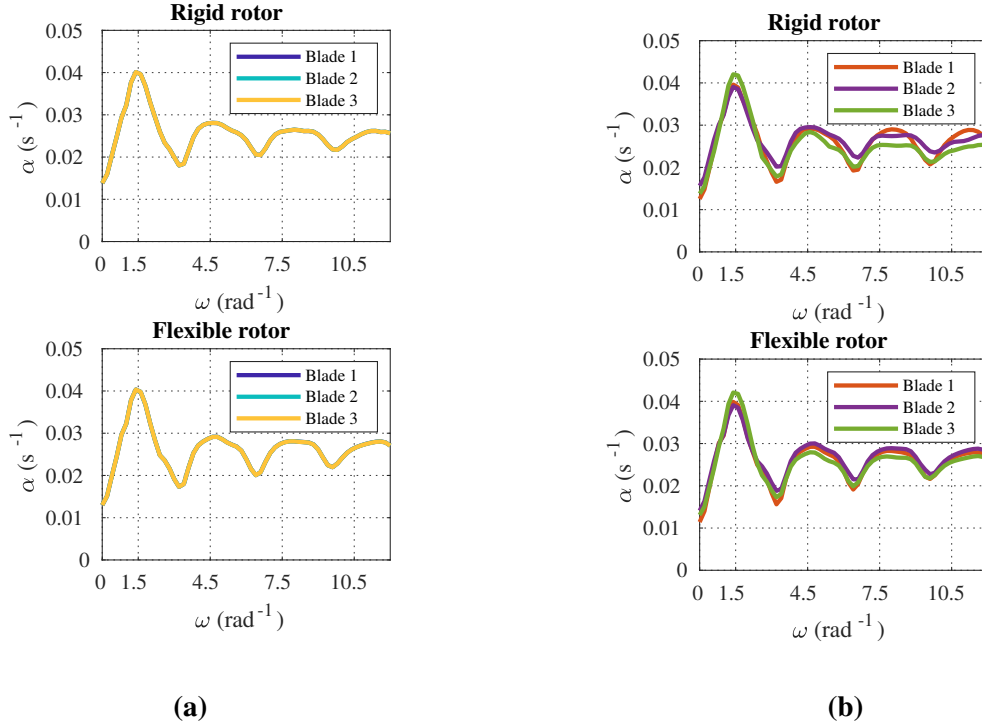


Figure 19: Case 1 stability trend snapshots at $t = 40$ s of tip vortices shed from rigid and flexible (a) symmetric rotor and (b) operational rotor configurations.

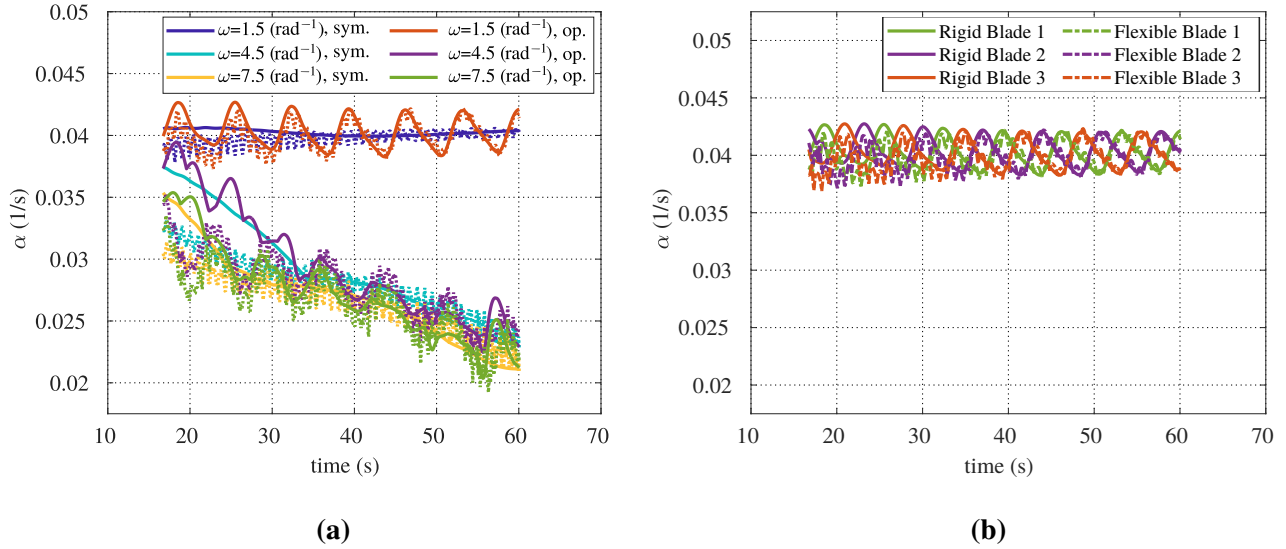


Figure 20: Case 1 time history growth rates: a) eigenvalues corresponding to perturbation wavenumbers $\omega=1.5, 4.5, 7.5 \text{ rad}^{-1}$ for symmetric and operational rotor configurations (legend provides color scheme of rotor configuration and dashed lines represent time-history of growth-rates from flexible blades); and b) operational configuration eigenvalues corresponding to $\omega=1.5 \text{ rad}^{-1}$ for all three blades (solid lines = rigid rotors, dashed lines = flexible rotors).

exhibits the low-frequency contribution of $f \approx 0.02 \text{ Hz}$ but also exhibits a significant contribution

of $f \approx 1.1$ Hz, which is near the first natural frequency of the NREL rotor blade, $f_1 = 1.2$ Hz. The FFT-derived frequency spectra of the operational configuration highlights a dominant contribution of $f = \Omega/2\pi \approx 0.14$ Hz. This frequency was first presented by Rodriguez & Jaworski (2017) and then by Rodriguez (2018), where it was determined that stability trends fluctuate at a period $\Lambda = 2\pi/\Omega$. However, Rodriguez & Jaworski (2017) and Rodriguez (2018) concluded the stability trend fluctuation was a byproduct of the blade passing frequency, and not the rotor plane tilt. Figure 21 shows that the fluctuating frequency, $f = \Omega/2\pi$, corresponds to the impact of the operational configuration (rotor plane tilt) on the rotor blade angle of attack and stability trends, and does not correspond to the rotation rate alone.

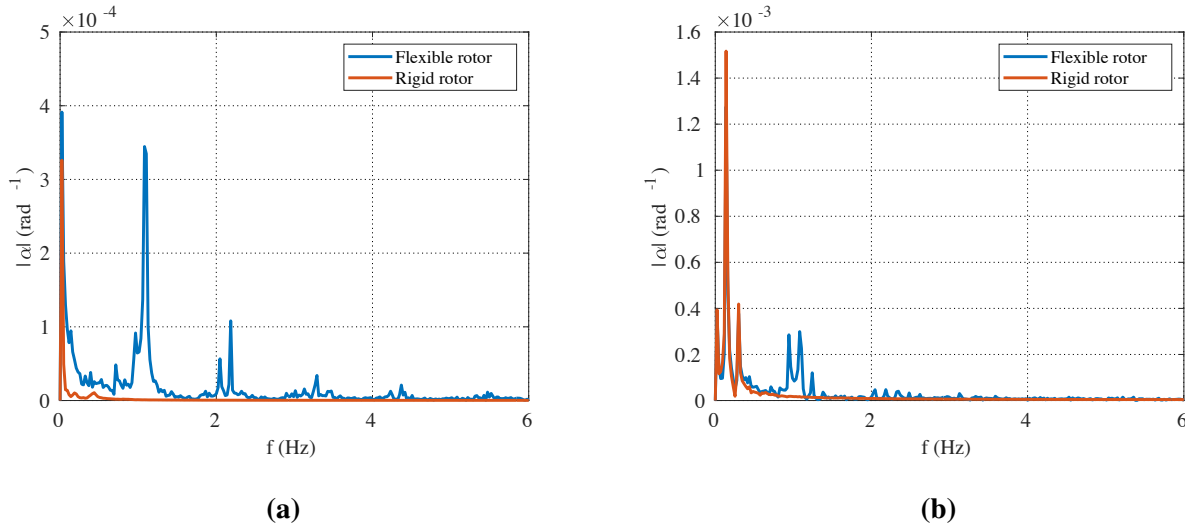


Figure 21: FFT-derived frequency spectra of eigenvalue time history signal for perturbation wavenumber $\omega = 1.5 \text{ rad}^{-1}$: a) symmetric rotor configuration; b) operational rotor configuration

7.4.2 Case 2: Rated operating conditions

Case 2 conditions are now investigated. Snapshots of the symmetric and operational configurations at $t = 60$ s are shown in Fig. 22. Differences in wake breakdown location between rigid symmetric and operational rotors are less visible qualitatively than case 1 rotor-wakes. Differences in wake breakdown location are difficult to identify likely because of the decrease in tip-speed ratio, which increases tip-vortex to tip-vortex spacing (pitch), as thereby generating less unstable vortices and mitigating the effect of perturbation propagation across tip vortices compared to case 1 operation. However, one major difference between symmetric and operational wakes is the transition into the agglomeration of tip vortices in the far wake. The symmetric configuration exhibits a smooth transition into the far wake as opposed to the operational configuration which exhibits distortion in tip vortices as early as $X \approx 1.8$ downstream. Similarly to Case 1, the flexible rotor configurations exhibit earlier breakdown than both the rigid configurations. These earlier occurrences of wake breakdown in flexible rotors is likely due to the initial transient impact that blade deflection has on the formation of the tip vortices. Rotor configuration (symmetric vs. operational) for flexible

rotors seems to not have a substantial impact with regard to the onset of wake breakdown location. The minimal influence of rotor configuration on wake breakdown in this case is likely due to the wake breakdown being largely dominated by the initial transients of the blade deflection rather than being dominated by periodic aerodynamic loading as discussed in Rodriguez & Jaworski (2019).

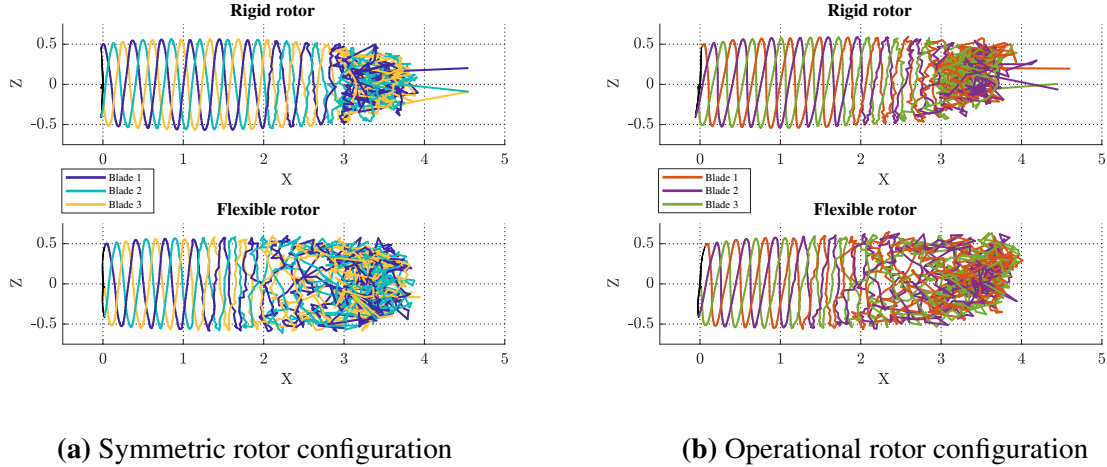


Figure 22: Case 2 tip-vortex snapshot at $t = 60$ s for rigid and flexible rotor operation

Snapshots at $t = 40$ s of the stability analysis performed on tip vortices for two rotations (4π window) are presented in Fig. 23. Maximum eigenvalues occur at the expected perturbation wavenumber of $\omega = 1.5 \text{ rad}^{-1}$. The rigid rotor operational configuration exhibits higher eigenvalue peaks than its symmetric rotor counterpart, which again supports the notion that operational configurations tend to further destabilize tip vortices. However, it is interesting to note in Fig. 23 that wake breakdown appears to occur at similar locations downstream in Fig. 22, despite the flexible rotor operational configuration having larger maximum eigenvalues than the flexible rotor symmetric configuration. It is important to note, though, that similar locations of wake breakdowns may be a result from the initial transient perturbations from blade deformation that dominate the wake formation downstream. However, as time progresses the “windowed” stability analysis captures the inherent tip vortex stability of each corresponding rotor configuration, which results in higher maximum eigenvalues for operational rotor configurations in the 4π -window. Furthermore, the stability analysis for flexible rotors result in distorted (non-smooth) stability trends as perturbation wavenumbers increase, which was not as prevalent in case 1 and is likely due to blade vibrations. The level may be an indicator that blade flexibility may reduce, shift, or distort tip vortex instability at specified perturbation frequencies.

The time histories of eigenvalues corresponding to perturbation wavenumbers $\omega = 1.5, 4.5$ and 7.5 rad^{-1} , i.e., $N_b = 3$ and $k = 1, 2, 3$ for $\omega = N_b (k - 1/2)$ are now tracked in time for symmetric (sym.) and operational (op.) configurations in Fig. 24. Independent of the configuration, Fig. 24a shows that within the 4π window of tip vortices under consideration, blade flexibility enables less unstable (lower positive eigenvalues) tip vortices at $\omega = 1.5 \text{ rad}^{-1}$ than rigid rotors. As time progresses, the presence of blade flexibility may generate tip vortices that are more sensitive to higher wavenumber perturbations as shown in the time histories for stability trends for $\omega = 4.5$ and 7.5 rad^{-1} . The same higher wavenumber sensitivity was seen in tip vortices for conditions in case 1, but for case 2 this sensitivity is more pronounced in Fig. 24.

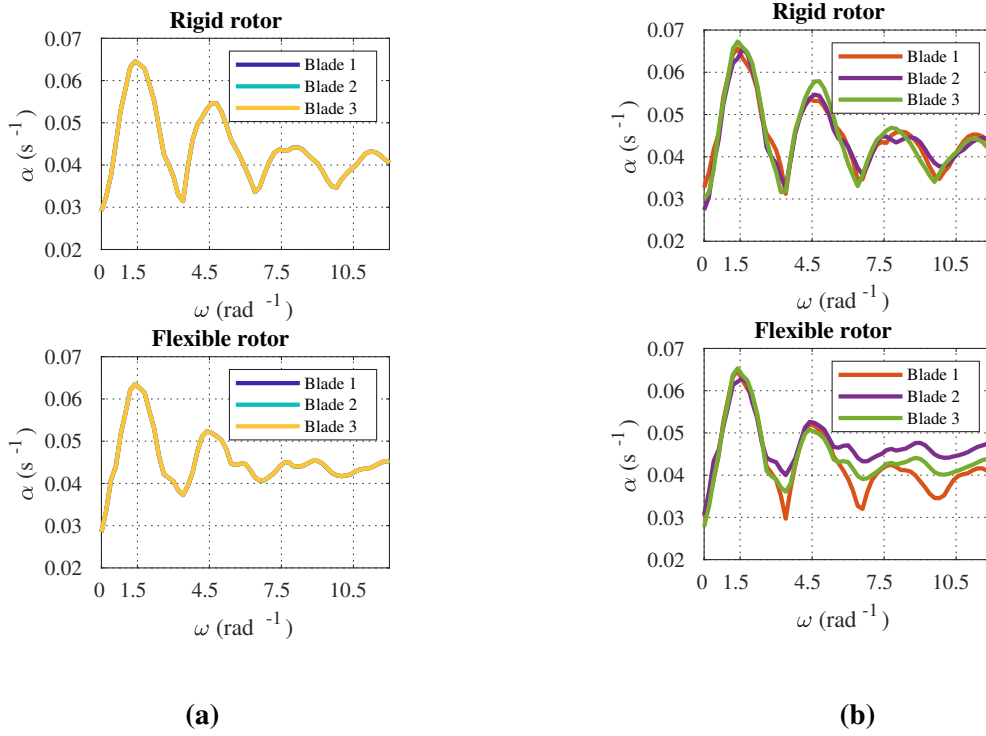


Figure 23: Stability trend snapshots at $t = 40$ s of tip vortices shed from rigid and flexible symmetric rotor (left column) and operational rotor configurations (right column).

Figure 24a quantitatively supports the qualitative observation stated earlier that operational rotor configurations breakdown earlier than symmetric rotor configurations. As in case 1, it is shown in Fig. 24a that operational configurations periodically reach higher maximum eigenvalues (α corresponding to $\omega = 1.5 \text{ rad}^{-1}$) than symmetric operational configurations. This periodic behavior seen in Fig. 24a, not present in the symmetric configuration, corresponds to a change in the angle-of-attack of individual blades as they pass through a full rotation in the tilted rotor plane, as was also observed in case 1 operation. Figure 24b further illustrates the out-of-phase behavior of the maximum eigenvalues (corresponding to perturbation wavenumber $\omega = 1.5 \text{ rad}^{-1}$) of each tip vortex shed off of individual blades.

The frequency spectra of the eigenvalue time-histories for Case 2 symmetric and operational rotor configurations is shown in Fig. 25. For the symmetric rotor configuration, the FFT shows a dominant low-frequency contribution that was also seen in case 1. As mentioned previously, identifying this low-frequency contribution is not trivial but seems to be related to the agglomeration of tip vortices downstream. This low frequency contribution is seen in Fig. 27a through a small and almost negligible eigenvalue fluctuation in time. For the symmetric flexible rotor configuration, the FFT is showing a secondary frequency of $f \approx 1.1 \text{ Hz}$ contribution near the first natural frequency of the NREL rotor blade ($f_1 \approx 1.2 \text{ Hz}$), which was also present in case 1. The operational configuration FFT is dominated by the frequency at which the angle-of-attack changes in the tilted rotor plane, which for case 2 is $f = \Omega/2\pi \approx 0.2 \text{ Hz}$.

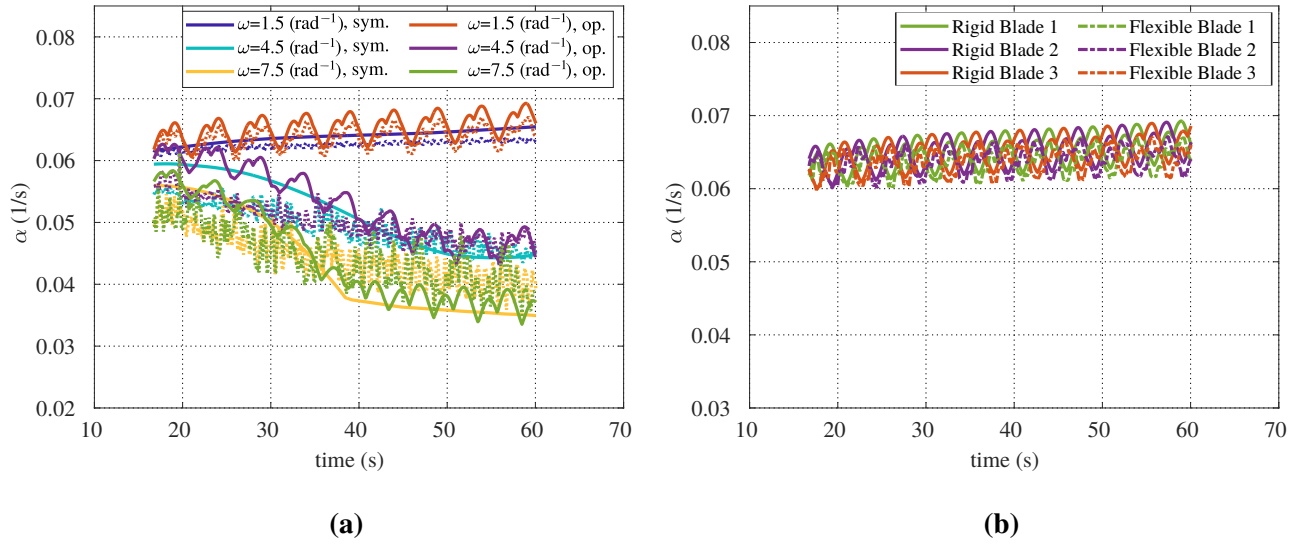


Figure 24: Case 2 time history operation growth rates: a) eigenvalues corresponding to perturbation wave numbers $\omega=1.5, 4.5, 7.5$ rad $^{-1}$ for symmetric and operational rotor configurations (legend provides color scheme of rotor configuration and dashed lines represent time-history of growth-rates from flexible blades); b) Operational configuration eigenvalues corresponding to $\omega=1.5$ rad $^{-1}$ for all three blades

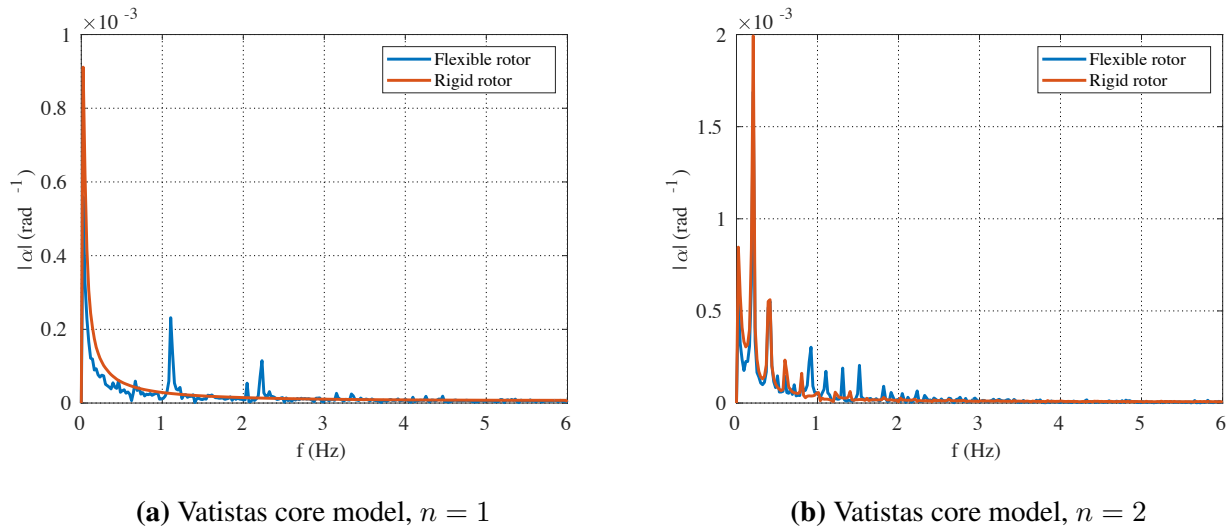


Figure 25: FFT-derived frequency spectra of eigenvalue time history signal for perturbation wave number $\omega = 1.5$ rad $^{-1}$: a) symmetric rotor configuration; b) operational rotor configuration

7.4.3 Case 3: Above-rated operating conditions

Case 3 of the NREL 5MW wind turbine is now investigated. Snapshots of the symmetric and operational configurations at $t = 60$ s are shown in Fig. 26. The generation of tip vortices presented in

Fig. 26 exhibit much more coherent and smooth helical vortex geometry than any of the other operational conditions (Cases 1 and 2) for both symmetric and tilted configurations. The coherence of the tip vortex geometry is attributed to the decrease in tip-speed ratio, which increases the distance between adjacent vortices (helical pitch) that mitigates the influence of perturbation propagation. Unlike prior Cases 1 and 2, it is difficult to qualitatively highlight in Case 3 the influence of rotor-plane tilt on the stability of tip vortices as both symmetric and tilted configurations maintain a very coherent tip vortex structure. Therefore, no clear qualitative conjecture can be made about the influence of operational configurations on numerical and qualitative wake break down. Similarly, the influence of blade flexibility on the stability of tip vortices generated in symmetric and operational configurations is not clearly observed qualitatively, as the 15° blade pitch has reduced aerodynamic loading of the blades and blade deformation. Thus, unlike prior rotor operational conditions, no clear qualitative hypothesis can be made on the influence of blade elasticity on the stability of tip vortices.

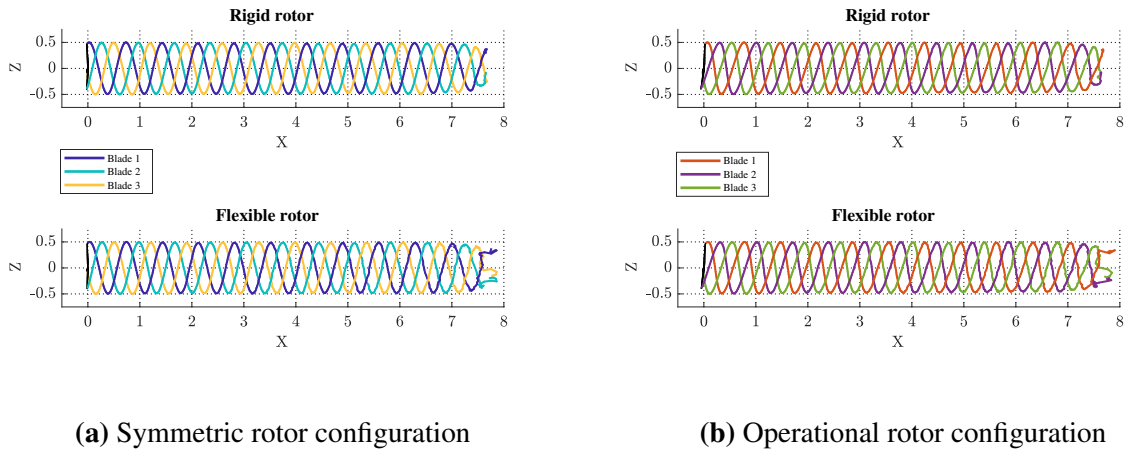


Figure 26: Case 3 tip-vortex snapshot at $t = 60$ s for rigid and flexible rotor operation

Snapshots of the stability analysis performed at $t = 40$ s for tip vortices in a two rotation window (4π window) are presented in Fig. 27. Snapshots of the stability analyses qualitatively show that eigenvalues show less variance across a range of perturbation wavenumbers than in other cases. In fact, the current case 3 appears not to adhere to the classical stability criteria, namely that eigenvalue peaks of stability trends do not correspond to wavenumber perturbations equal to $\omega = N_b (k - 1/2)$. In fact, the stability trend shifts forward some ϵ amount, i.e. $\omega = N_b (k - 1/2) + \epsilon$. Figure 27 shows that at $t = 40$ s the stability for operational rotor configurations reaches higher eigenvalues than symmetric rotor configurations.

The time histories of eigenvalues corresponding to perturbation wavenumbers $\omega = 1.5, 4.5$ and 7.5 rad^{-1} , i.e., $N_b = 3$ and $k = 1, 2, 3$ for $\omega = N_b (k - 1/2)$ are now tracked in time for symmetric (sym.) and operational (op.) configurations in Fig. 28. Unlike cases 1 and 2 shown in Figs. 20 and 24, respectively, the time histories of eigenvalues in case 3 for perturbation wavenumbers $\omega = 1.5, 4.5$ and 7.5 are nearly identical in both the symmetric and operational configurations. For both symmetric and operational configurations, it is seen that as time progresses blade flexibility generates more stable (lower positive eigenvalues) than rigid rotors. Finally, although qualitative

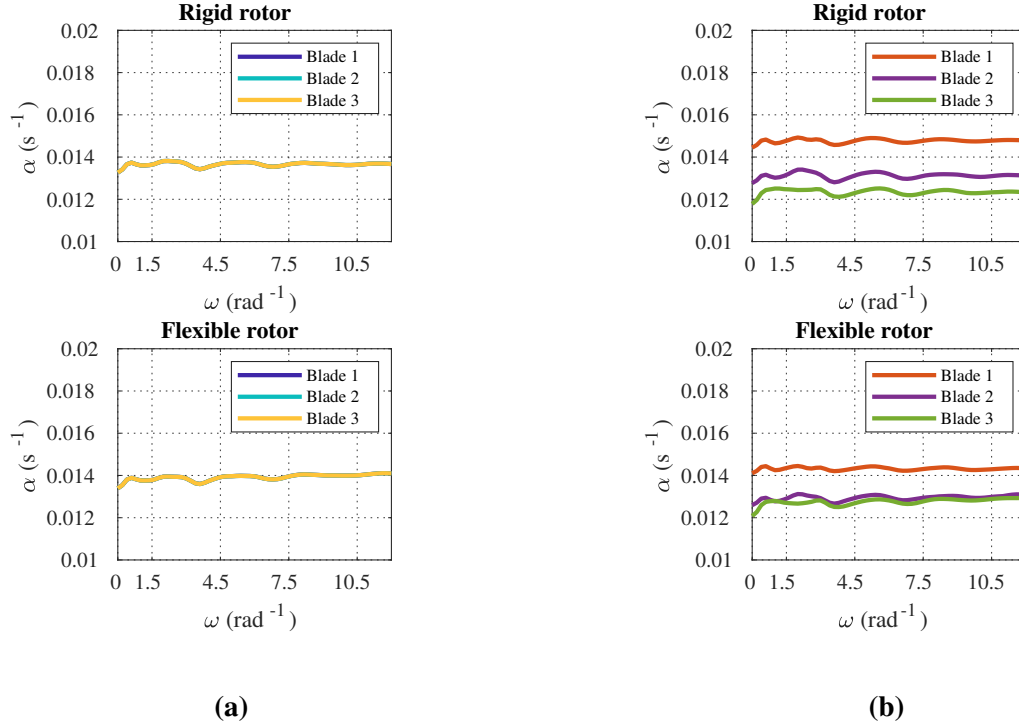


Figure 27: Stability trend snapshots at $t = 40$ s of tip vortices shed from rigid and flexible symmetric rotor (left column) and operational rotor configurations (right column).

differences between tip vortices generated by symmetric and operational rotor in Fig. 26 are minor, the underlying physical differences are highlighted by the temporal stability characteristics shown in Fig. 28, which demonstrates that the periodic change in angle-of-attack generated by the rotor-plane tilt results in higher eigenvalues of the operational rotor configuration, thereby generating more unstable tip vortices. Figure 28b also shows the out-of-phase behavior of tip vortex stability trends generated by the out-of-phase period change in angle-of-attack for individual rotor blades.

The frequency spectra of the eigenvalue time-histories for Case 3 symmetric and operational rotor configurations is shown in Fig. 29. For the symmetric rotor configuration, the FFT shows a dominant low frequency contribution that was also seen previously in cases 1 and 2. It is also interesting to note that for the tip vortices generated by symmetric flexible rotors, this low frequency contribution is much more pronounced than a rigid rotor. As mentioned previously, identification of this low-frequency contribution is not trivial but seems to be related to the agglomeration of tip vortices downstream, and it may be that blade flexibility amplifies this low-frequency contribution in the amplitude spectrum. For the flexible rotor symmetric configuration, the FFT shows a secondary frequency near the first natural frequency of the NREL rotor blade ($f_1 \approx 1.2$ Hz) and at exactly the first natural frequency of the NREL rotor blade. The operational configuration FFT is dominated by the frequency at which the angle-of-attack changes in the tilted rotor plane, which for case 3 is also $f = \Omega/2\pi \approx 0.2$ Hz, since the rotation rate is equal to the rotation rate of case 2 operational conditions.

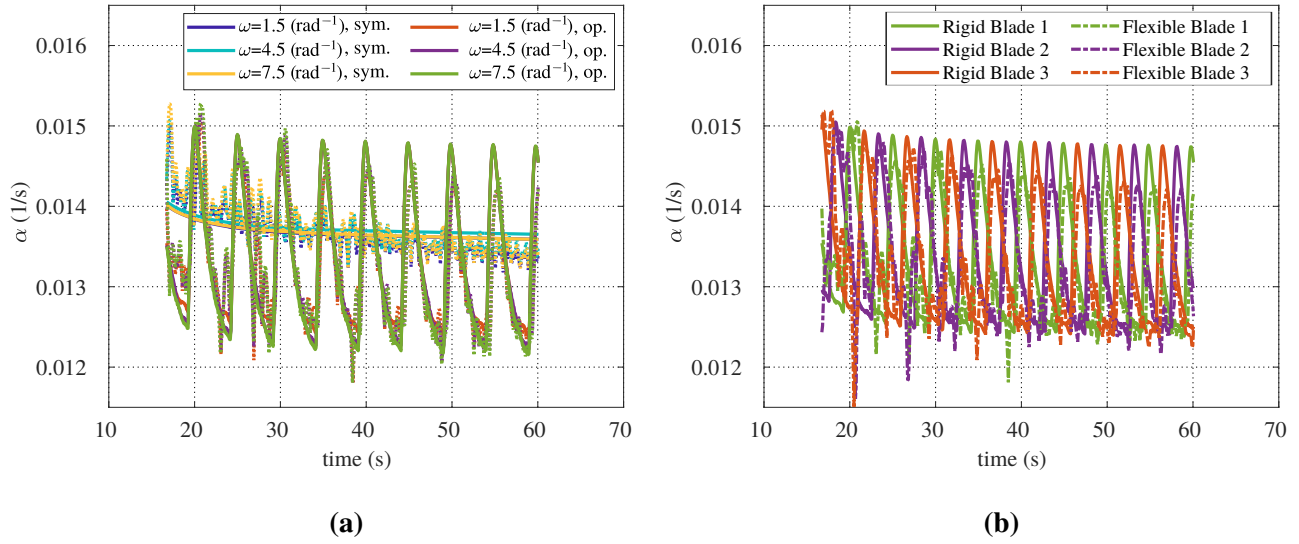


Figure 28: Case 3 time history operation growth rates tip vortex 1: a) eigenvalues corresponding to perturbation wavenumbers $\omega=1.5, 4.5, 7.5$ rad $^{-1}$ for symmetric and operational rotor configurations (legend provides color scheme of rotor configuration and dashed lines represent time-history of growth-rates from flexible blades); b) operational configuration eigenvalues corresponding to $\omega=1.5$ rad $^{-1}$ for all three blades.

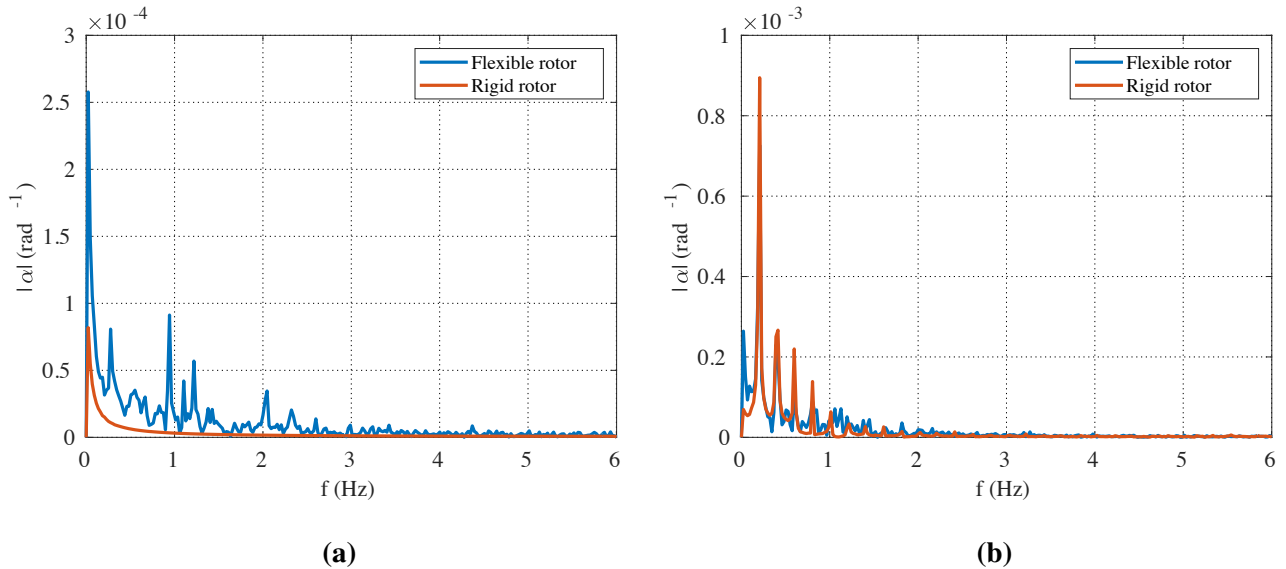


Figure 29: FFT-derived frequency spectra of eigenvalue time history signal for perturbation wave number $\omega = 1.5$ rad $^{-1}$: a) symmetric rotor configuration, and b) operational rotor configuration.

7.5 Discussion

The tip vortex analyses presented show that a rotor-plane tilt and blade flexibility alters the time-dependent content of tip vortex stability. By introducing the rotor-plane tilt, the time-history of eigenvalues corresponding to perturbation wave numbers $\omega = 1.5, 4.5$ and 7.5 (rad^{-1}) fluctuate at a frequency of $f = \Omega/2\pi$. Rodriguez & Jaworski (2017) advanced the claim that stability trends are dependent on the blade passing frequency corresponding to $f = \Omega/2\pi$. However, their investigation incorrectly identified time-varying stability trends for tip vortices shed from different blades despite their base simulation being a symmetric rotor configuration. Rodriguez (2018) also identified the same frequency content in time-history trends for Cases 1-3, but it was also assumed that the stability trend fluctuation was a byproduct of the blade passing frequency. The current investigation shows that the stability trend fluctuation is not a byproduct of the blade passing frequency, but a result of the periodic change in angle-of-attack as the rotor blade travels around the tilted rotor-plane. It was also found that blade flexibility adds higher frequency content to the stability trends equal to or about the first natural frequency of the rotor blade. The impact that both blade flexibility and rotor-plane tilt have had on the stability trends indicate that tip vortices are sensitive to the dominant dynamics of rotor operation.

The analyses presented herein have also shown the impact that blade flexibility has on tip vortex stability for an aeroelastic rotor based on the NREL 5MW reference wind turbine rotor. Rodriguez & Jaworski (2017) investigated initially the role of flexibility on the stability of tip vortices and concluded that blade deformation destabilizes tip vortices. However, their stability analyses were conducted on the entire tip vortex geometry, which included perturbing tip vortex locations that were already initially perturbed by artificial transients due to the impulse loading of the inflow conditions and numerical instabilities. To avoid performing stability analyses on tip vortices perturbed by numerical artifacts, Rodriguez (2018) performed a windowed analysis to identify regions where numerical instabilities would not corrupt stability analyses. Rodriguez (2018) employed the windowed stability analysis and found that by evaluating time-history of eigenvalues corresponding to $\omega = 1.5 \text{ rad}^{-1}$ wavenumbers, blade flexibility can generate less unstable (lower positive eigenvalues) tip vortices than rigid rotors. However, no time history analyses were performed on higher wavenumber perturbations. The present analyses align well with results presented in Rodriguez (2018) for wavenumbers $\omega = 1.5 \text{ rad}^{-1}$. Furthermore, it was also observed that for Cases 1 and 2, where no rotor blade pitch is present, blade flexibility can reduce sensitivity to low wavenumber perturbations and increase sensitivity to higher wavenumbers.

7.6 Conclusions

The stability of tip vortices shed from flexible rotors has been investigated numerically. An aeroelastic free-vortex wake method was used to generate tip vortex structures and a linear-eigenvalue stability analysis was employed to highlight the underlying tip vortex physics. Towards validation of the stability analysis, it was found that the Vatisstas model used to desingularize the Biot-Savart was susceptible to numerical artifacts not present in the cutoff modeling. Further investigation into the impact of numerical time integration with Vatisstas and cutoff finite core modeling are necessary to understand the numerical issues at hand. Nevertheless, it was found that by employing a windowed stability analysis, whereby only the early-aged segments of tip vortices are considered, classical stability trends were recovered on a three-bladed canonical rotor.

The validated stability analysis was then employed on the tip vortices generated by the NREL 5MW reference wind turbine for rigid and flexible blades under symmetric (no tilt) and operational (5° rotor-plane tilt) configurations. The tip vortex stability analyses and corresponding time-history analyses demonstrated three main findings: 1) tip vortex stability is sensitive to dominant dynamics of rotor operation; 2) blade flexibility may generate tip vortices less sensitive to low wavenumber perturbations but more sensitive to higher wavenumber perturbations; and 3) introducing blade pitch alters the local induced velocity field and alters tip vortex stability such that peak eigenvalue trends do not adhere to classical stability criteria.

8 MOTION AND SOUND OF AEROELASTIC VORTEX-AIRFOIL INTERACTIONS

H. Chen, J. W. Jaworski

8.1 Background

The encounter of a vortex gust with an aerodynamic body is a canonical fluid-structure interaction with implications for the prediction of transient loads on fliers and swimmers and their generation of vortex sound. Among the various orientations that the incident vortex can take with respect to the solid body (cf. Rockwell (1998)), the alignment of the spanwise vortex with the orientation of the bound vorticity on the body, the wake, and the boundary-layer vorticity allows for their dynamic coupling during a vortex-airfoil encounter. For this reason, a number of theoretical, numerical, and experimental studies have been carried out to understand and model two-dimensional vortex-structure interactions.

Many prior investigations of two-dimensional vortex-body interactions consider static solid structures, where the flow unsteadiness arises from the motion of one or many coherent vortices. The simplest analytical models of these scenarios represent the vortex structures as point vortices, where the solid boundaries are often transformed via conformal mapping techniques to simple geometries where the solution is known or is easily determined. Analytical vortex-structure modeling has historically focused on the effects of boundary geometry on the vortex path, which is of interest to gain insight into the evolution of coherent flows and is critical in the assessment of vortex sound generation (Crighton, 1972a; Howe, 1997; Conlisk & Veley, 1985; Jaworski, 2016, 2018). These analytical models can be generalized to handle multiply-connected domains (Baddoo & Ayton, 2018; Crowdy, 2010), as well as vortex clouds (Panaras, 1987) or finite-sized vortices (Newton, 2013). The close encounter of real vortices with surface nonuniformities (e.g., an edge, corner, or surface asperity) and the large surface pressure fluctuations they can impart are sensitive to the incoming vortex strength (Conlisk & Rockwell, 1981), the geometrical details of the boundary (Rockwell & Knisely, 1979), and the alignment of the vortex with the solid body (Conlisk & Rockwell, 1981; Ziada & Rockwell, 1982; Gursul & Rockwell, 1990). To date, analytical investigations of direct vortex impingement have not been developed for structures under aeroelastic motions, which is a key contribution of the present study. Furthermore, the vortex sound produced by this interaction has also not been heretofore investigated.

The local vortex-body interaction of near or direct impingement can lead to significant distortion of the incoming vortex (Gursul & Rockwell, 1990) and its possible severing by the body into two parts (Rockwell & Knisely, 1979; Lee & Smith, 1991; Barnes & Visbal, 2018a,b). Strong viscous and nonlinear effects in the neighborhood of the solid boundary can also lead to transient separation and turbulent transition of the surface boundary layer (Barnes & Visbal, 2018a,b), the eruption of vorticity from this boundary layer (Doligalski *et al.*, 1994), the creation of additional vortices (Ziada & Rockwell, 1982), and a time-dependent change in the circulation of the incident vortex gust (Peng & Gregory, 2015). It is argued in the present work (along the lines of Rogler (1978)) that regimes with significant viscous interaction between the incident vortex and boundary layer indicate the practical limit of point-vortex models for aeroelastic vortex gust simulations.

Dynamic structural motions affect the paths of the incident vortex and the vorticity shed into the wake, which is the product of unsteady loading by the vortex gust on the aerodynamic structure or airfoil. These structural motions may be either prescribed (Manela, 2013) or the result of aeroe-

lastic coupling with the vortex gust, where in the latter case the compliance of either the airfoil or its support provides the means for a dynamic two-way energy transfer between the fluid and the solid body. The introduction of structural elasticity into the physical system leads to the possibility of resonance effects, which have been studied in the context of flow noise amplification and suppression by Howe (1994) for a turbulent inflow and by Manela (2011) for a point-vortex gust. Aeroelastic point-vortex interactions have lacked attention in the literature in comparison to cases with static structures or prescribed motion. Manela and Huang Manela & Huang (2013) investigated the interaction of an incident point vortex with a trailing edge flap mounted to a torsional spring, and Riso *et al.* (2016) investigated impulsive and transient gust loads using a discrete free wake with vortices shed at every instant in time. These studies were carried out for flat plates and did not consider the effects of airfoil geometry on the vortex-structure interaction. The present work is motivated by the need to assess this geometrical effect in addition to the dynamically-coupled interactions of the incident vortex, the aeroelastic airfoil motion, and the vortex wake.

The production and evolution of vorticity shed into the wake determines the unsteady aerodynamic load on an airfoil. Accordingly, researchers have sought to develop low-order models for unsteady vortex shedding that are accurate with respect to load prediction and can also provide physical insight into the flow field at a modest computational cost relative to a full numerical simulation of the Navier-Stokes equations. Early wake models for transient airfoil motions (von Kármán & Sears, 1938; Bisplinghoff *et al.*, 1996) supposed a flat, continuous vortex sheet that is appropriate for small-amplitude airfoil motions but becomes increasingly inaccurate for large-amplitude airfoil motions or severe gust responses (Riso *et al.*, 2016). Brown & Michael (1954, 1955) and Edwards (1954) developed independently a point-vortex model to describe the static vortices positioned over lifting delta wings due to leading-edge separation. This model was adapted by Cheng (1954) and Rott (1956) for unsteady two-dimensional flow with point vortices whose strengths are allowed to vary in time. The Brown and Michael model framework supposes that the vorticity shed from an edge rolls up via a connecting vortex sheet into a concentrated point vortex, whose strength is set by the Kutta condition and whose position is described by a first-order differential equation: the so-called Brown and Michael equation. The original model formulation brought about a spurious surface force on the airfoil that was identified by Peters & Hirschberg (1993) and was eliminated by Howe (1996) in his emended form of the Brown and Michael equation, which is used in the present work. Michelin & Llewellyn Smith (2009) reviewed and formalized the derivation of the Brown and Michael equation, which results from a linear momentum balance requiring zero net force on both the shed vortex and on the connecting vortex sheet. However, Michelin & Llewellyn Smith (2009) pointed out that the Brown and Michael framework has an insufficient number of free parameters to impose both linear and angular momentum balances on the vortex and the connecting sheet, in addition to the condition that the strength of a trailing-edge point vortex cannot decrease in time. Wang & Eldredge (2013) built upon the conservation of impulse argument by Tchieu & Leonard (2011) to develop a new evolution model for shed point vortices that agrees well with airfoil simulations at high pitch rates with shedding from both leading and trailing edges. More recently, Darakananda & Eldredge (2019) developed a hybrid modeling approach where the rate at which the vortex sheets roll up into point vortices may be adjusted by the analyst to gain the desired level of physical insight into the fluid dynamics. Eldredge & Jones (2019) provide a full contemporary review of low-order approaches to vortex modeling.

The objective of the present work is to model the dynamic vortex-gust interactions of non-flat airfoils that are mounted to a linear suspension. The theoretical model is used to investigate

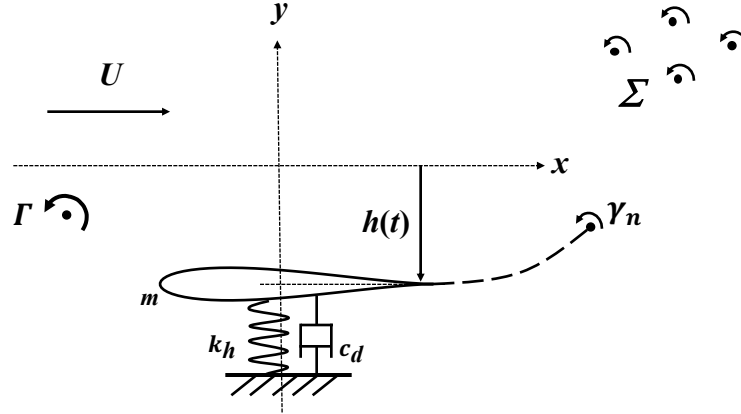


Figure 30: Schematic of the generalized model problem of an incident vortex interaction with a symmetric Joukowski airfoil on elastic translational support in a uniform flow, where U is the flow speed and $h(t)$ denotes the displacement of the airfoil. The airfoil can be considered as a damped harmonic oscillator with mass m , damping coefficient c_d , and spring stiffness k_h . Γ denotes the incident line vortex, and γ_n is the tethered trailing-edge vortex whose motion is determined by the emended Brown and Michael equation. Σ is the set of free vortices generated at the airfoil trailing edge due to unsteady airfoil loads in response to the incident line vortex or the airfoil motion. The airfoil has zero angle of attack.

the dynamic interplay between a vortex gust, unsteady vortex shedding, the airfoil shape, and the aeroelastic airfoil motion, as well as to predict the conditions for direct vortex impingement. To these ends, Section 8.2 develops the mathematical model and dynamic formulation for vortex-gust interactions with a Joukowski airfoil, as well as presents the framework to assess the vortex sound generation. Section 8.3 first studies the effect of airfoil thickness and compares the model results against previous experiments with fixed airfoils. The dimensionless aeroelastic parameters and the initial placement of the vortex gust to achieve direct airfoil impingement are then determined numerically. The acoustic implications of the aeroelastic vortex-airfoil interaction are also presented: Section 8.3.4 validates the analytical/numerical framework against previous acoustic results for a stationary flat plate, and new results for the acoustic emission of a (non-flat) Joukowski airfoil that is free to move aeroelastically under the influence of the vortical field is then presented and discussed. Section 8.4 summarizes the results from this study and presents concluding remarks.

8.2 Mathematical formulation

Figure 30 illustrates the model problem of a Joukowski airfoil on elastic supports in two-dimensional uniform flow with an incident line vortex Γ and the vorticity field Σ shed into the wake. All vortices are defined to have positive circulation in the counterclockwise direction. The strength of the trailing-edge vortex γ_n satisfies the Kutta condition, and its motion obeys the emended Brown and Michael equation Howe (1996). The mathematical formulation of the aeroelastic system is now described.

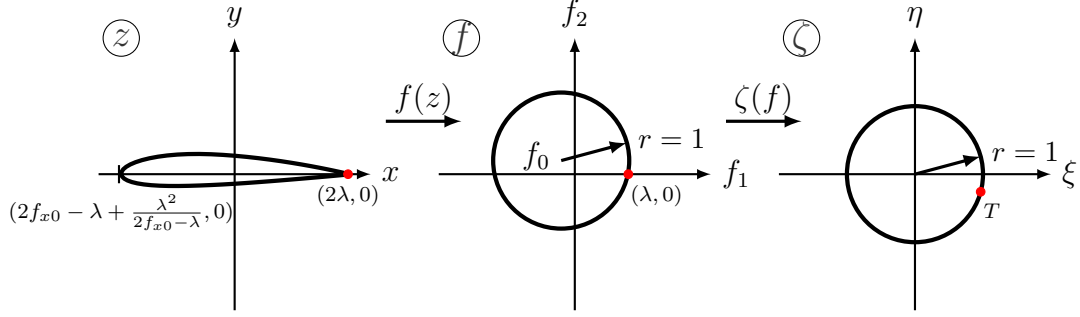


Figure 31: Successive mappings of a generalized Joukowski airfoil in the physical z -plane to a unit circle centered at the origin in the ζ -plane. The red dot indicates the position of the trailing edge through successive mappings and is located at the complex-valued position T in the ζ -plane.

8.2.1 Mapping

The conformal mapping of the Joukowski airfoil between the physical z -plane and the mapped ζ -plane is described by

$$\zeta(z) = \frac{1}{2} \left(z + \sqrt{z^2 - 4\lambda^2} \right) - f_0. \quad (57)$$

where f_0 is the origin of the unit circle in the f -plane, as shown in Fig. 31, and λ is the transformation parameter of the conformal mapping defined by $\lambda = \sqrt{r^2 - f_{y0}^2} - f_{x0}$. Using Eq. (57), the Joukowski airfoil in the physical z -plane ($z = x + iy$) with its trailing edge located at $(2\lambda, 0)$ is mapped to a circle with radius $r = 1$ in the f -plane ($f = f_1 + if_2$), as shown in Fig. 31. Note the offset of the circle center at $f_0 = f_{x0} + if_{y0}$ and the corresponding trailing edge at $(\lambda, 0)$. For the symmetric Joukowski airfoils considered in this work, $f_{y0} = 0$. The unit circle in the f -plane is then shifted by an elementary mapping to the origin in the ζ -plane.

When time-dependent airfoil motions are considered, Eq. (57) becomes

$$\zeta(s) = \frac{1}{2} \left[s(z, t) + \sqrt{s^2 - 4\lambda^2} \right] - f_0, \quad (58)$$

where $s(z, t) = z - ih(t)$. The airfoil displacement $h(t)$ may be either prescribed or be part of the solution in the case of aeroelastic airfoil motions.

8.2.2 Flow complex potential

The complex potential of the flow is

$$w(\zeta) = w_\gamma + w_\Gamma + w_h + U \left(\zeta + f_0 + \frac{\lambda^2}{\zeta + f_0} \right), \quad (59)$$

where $w_\gamma(\zeta)$, $w_\Gamma(\zeta)$, and $w_h(\zeta)$ are the complex potential contributions due to the shed and free vorticity field, the incident vortex, and the airfoil motion, respectively. The last term of Eq. (59) represents the uniform background flow.

The complex potential $w_\Gamma(\zeta)$ of the incident vortex located instantaneously at ζ_Γ is determined from the model of a vortex moving around a cylinder Howe (2003), which yields

$$w_\Gamma(\zeta) = -\frac{i\Gamma}{2\pi}\log(\zeta - \zeta_\Gamma) + \frac{i\Gamma}{2\pi}\log\left(\zeta - \frac{1}{\zeta_\Gamma^*}\right) - \frac{i\Gamma}{2\pi}\log\zeta, \quad (60)$$

where the last two terms guarantee that the total circulation inside the cylinder is initially zero, and the asterisk denotes the complex conjugate. Similarly, the complex potential of the n vortices shed into the wake and located at positions ζ_{γ_n} is

$$w_\gamma(\zeta) = \sum_{k=1}^n \left[-\frac{i\gamma_k}{2\pi}\log(\zeta - \zeta_{\gamma_k}) + \frac{i\gamma_k}{2\pi}\log\left(\zeta - \frac{1}{\zeta_{\gamma_k}^*}\right) \right]. \quad (61)$$

By appeal to Kelvin's theorem, the bound vorticity inside the cylinder at any time has total circulation $-\sum_{k=1}^n \gamma_k$, which is used to compute the aerodynamic lift force.

The complex potential of the airfoil motion is Batchelor (1967)

$$w_h(\zeta) = iV \left(\zeta - \frac{1}{\zeta} \right), \quad (62)$$

where $V = dh/dt$ is the instantaneous velocity of the airfoil in the downward direction.

8.2.3 Vortex shedding and the emended Brown and Michael equation

The motions of the shed vortices are described by the emended Brown and Michael equation (Howe, 1996),

$$\frac{d\mathbf{x}_{\gamma_n}}{dt} \cdot \nabla \Psi_j + \frac{\Psi_j}{\gamma_n} \frac{d\gamma_n}{dt} = \mathbf{v}_{\gamma_n} \cdot \nabla \Psi_j, \quad j = 1, 2, \quad (63)$$

where $\mathbf{x}_{\gamma_n}$ denotes the position of the n th shed vortex with circulation γ_n that is tethered to the trailing edge, and $\mathbf{v}_{\gamma_n}$ is the fluid velocity when the self-induced local velocity of γ_n is excluded. A Cartesian coordinate system is used, where $\mathbf{x} \equiv (x, y)$. $\Psi_j(\mathbf{x}, t)$ denotes the stream function of the complex potential of the flow in the j -direction. For a Joukowski airfoil mapped to the ζ -plane, the components of the stream function are Howe (2014)

$$\Psi_1 = \text{Im} \left\{ \zeta + \frac{1}{\zeta} \right\} \quad \text{and} \quad \Psi_2 = \text{Im} \left\{ -i \left(\zeta - \frac{1}{\zeta} \right) \right\}. \quad (64)$$

The instantaneous circulation of the tethered vortex $\gamma_n(t)$ is obtained by enforcing the Kutta condition at the trailing edge of each instant in time,

$$\gamma_n(t) = \frac{|T^* \zeta_{\gamma_n} - 1|^2}{|\zeta_{\gamma_n}|^2 - 1} \left(\frac{2\Gamma(1 - \text{Re}\{T^* \zeta_\Gamma\})}{|T^* \zeta_\Gamma - 1|^2} - \sum_{k=1}^{n-1} \gamma_k \frac{|\zeta_{\gamma_k}|^2 - 1}{|T^* \zeta_{\gamma_k} - 1|^2} - 2\pi V \text{Re}\{T^*\} \right), \quad (65)$$

in which T^* is the complex conjugate of the trailing edge T location ($T = \lambda - f_0$) in the ζ -plane. The tethered trailing-edge vortex is released and becomes a free vortex when $d\gamma_n/dt$ changes

sign, at which time another tethered vortex is placed at the airfoil trailing edge whose motion and instantaneous circulation are determined by

$$\frac{dz_{\gamma_n}^*}{dt} + (H_1 - iH_2) \frac{1}{\gamma_n} \frac{d\gamma_n}{dt} = v_{\gamma_n}^*, \quad (66)$$

and Eq. (65), respectively (Howe, 1996; Graham, 1977, 1980). As described by Michelin & Llewellyn Smith (2009), this criterion for vortex shedding is based on the physical argument that the intensity of the point vortex cannot decrease in value, else it would imply a negative diffusion in the point vortex.

In Eq. (66), $z_{\gamma_n}^* = x - iy$ and $v_{\gamma_n}^* = v_x - iv_y$. The functions H_1 and H_2 involve the stream functions Ψ_j and their derivatives. Equation (66) is the equivalent scalar form of the emended Brown and Michael equation, which is employed for the theoretical analysis in this work. Specific details related to the derivation of Eq. (66) and the expressions of H_1 and H_2 from Chen & Jaworski (2018) are presented in Chen & Jaworski (2020). Also,

$$v_{\gamma_n}^* = -\frac{i\gamma_n \zeta''(z_{\gamma_n})}{4\pi \zeta'(z_{\gamma_n})} + F'_{\gamma_n}(z_{\gamma_n}), \quad (67)$$

is the complex velocity of the shed vortex with self-potential velocity excluded (Howe, 2003). The first term on the right side of Eq. (67) is the so-called Routh correction (Clements, 1973), and the second term is the desingularized complex velocity at z_{γ_n} ,

$$F'_{\gamma_n}(z_{\gamma_n}) = \zeta' \left[\frac{dw}{d\zeta} + \frac{i\gamma_n}{2\pi} \frac{1}{\zeta - \zeta_{\gamma_n}} \right]. \quad (68)$$

8.2.4 Kinematics of the incident and free vortices

Similarly, the complex velocity of the incident line vortex at s_Γ is (Howe, 2003)

$$\frac{ds_\Gamma^*}{dt} = -\frac{i\Gamma \zeta''(s_\Gamma)}{4\pi \zeta'(s_\Gamma)} + F'(s_\Gamma), \quad (69)$$

where

$$F'_\Gamma(s_\Gamma) = \frac{i\Gamma}{2\pi} \frac{\zeta'_\Gamma}{(\zeta_\Gamma^2 - 1)\zeta_\Gamma} - \frac{i\zeta'_\Gamma}{2\pi} \sum_{k=1}^n \gamma_k \left(\frac{1}{\zeta_\Gamma - \zeta_{\gamma_k}} - \frac{1}{\zeta_\Gamma - 1/\zeta_{\gamma_k}^*} \right) + iV\zeta'_\Gamma \left(1 + \frac{1}{\zeta_\Gamma^2} \right) + U. \quad (70)$$

Note that $\zeta_\Gamma = \zeta_\Gamma(s)$ and $\zeta_{\gamma_k} = \zeta_{\gamma_k}(s)$.

Also, the equation of motion for each of the $n - 1$ free vortices is

$$\frac{ds_{\gamma_k}^*}{dt} = -\frac{i\gamma_k \zeta''(s_{\gamma_k})}{4\pi \zeta'(s_{\gamma_k})} + F'_{\gamma_k}(s_{\gamma_k}), \quad (71)$$

where

$$F'_{\gamma_k}(s_{\gamma_k}) = \frac{i\Gamma}{2\pi} \zeta'_{\gamma_k} \left(\frac{1}{\zeta_{\gamma_k} - \zeta_{\Gamma}} - \frac{1}{\zeta_{\gamma_k} - 1/\zeta_{\Gamma}^*} + \frac{1}{\zeta_{\gamma_k}} \right) + \frac{i\gamma_k}{2\pi} \frac{\zeta'_{\gamma_k}}{\zeta_{\gamma_k} - 1/\zeta_{\gamma_k}^*} - \frac{i\zeta'_{\gamma_k}}{2\pi} \sum_{m=1, m \neq k}^n \gamma_m \left(\frac{1}{\zeta_{\gamma_k} - \zeta_{\gamma_m}} - \frac{1}{\zeta_{\gamma_k} - 1/\zeta_{\gamma_m}^*} \right) + iV \zeta'_{\gamma_k} \left(1 + \frac{1}{\zeta_{\gamma_k}^2} \right) + U. \quad (72)$$

8.2.5 Airfoil motion and loads

The airfoil moves aeroelastically under its lift force and the equation of motion of the elastic mount,

$$m \frac{d^2 h}{dt^2} + c_d \frac{dh}{dt} + k_h h = -L', \quad (73)$$

where $h(t)$ denotes the plunge displacement of the airfoil with chord length c . The elastic suspension can be considered as a damped harmonic oscillator with airfoil mass m , damping coefficient c_d , and spring stiffness k_h . The unsteady sectional lift force L' on a thin airfoil is determined by (Katz & Plotkin, 2001)

$$L' = \rho U \Gamma_a(t) + \rho \int_0^c \frac{\partial}{\partial t} \Gamma_a(x, t) dx, \quad (74)$$

where the airfoil circulation $\Gamma_a(t)$ is the integrated vorticity bound to the airfoil, $\Gamma_a = \int_0^c \gamma(x, t) dx$. The first term is the quasi-steady Kutta-Joukowski lift due to the instantaneous circulation, and the second term includes the unsteady contributions to the lift due to the fluid loading on the airfoil (added mass) and the downwash effect of the wake.

The instantaneous circulation Γ_a is determined by Kelvin's theorem,

$$\Gamma_a(t) = - \sum_{k=1}^n \gamma_k(t), \quad (75)$$

where γ_k denotes the strength of the k th shed vortex. The airfoil vorticity distribution may be rewritten analytically as a trigonometric expansion of the following form (Glauert, 1983):

$$\gamma(\theta, t) = 2U \left[A_0(t) \frac{1 + \cos \theta}{\sin \theta} + \sum_{n=1}^{\infty} A_n(t) \sin n\theta \right], \quad (76)$$

where the first term in the bracket is the vorticity distribution for the symmetric airfoil, and the second term summation represents the camber distribution. Note that for symmetric airfoils with zero angle of attack, as considered in the present work, $A_1 = A_2 = A_3 = \dots = A_n = 0$, and only $A_0(t)$ is not identically zero at all times. Eq. (74) can be expanded as

$$L' = \pi \rho c U^2 A_0 + \pi \rho c^2 U \frac{3}{4} \frac{\partial A_0}{\partial t}, \quad (77)$$

where the first term is equivalent to the quasi-steady lift in Eq. (74), and the second term is equivalent to the lift contribution due to time dependency. Details of this expansion can be found in Katz & Plotkin (2001).

The system of dynamical equations is formed from Eqs. (65), (66), (69), (71), and (77), which consists of $2(n + 2)$ first-order ordinary differential equations for the position $(s_{x\Gamma}(t), s_{y\Gamma}(t))$ of the incident line vortex, the positions $(s_{x\gamma_n}(t), s_{y\gamma_n}(t))$ of n trailing-edge vortices, and the instantaneous airfoil displacement $h(t)$ and the plunging velocity $V(t)$. The system of equations is marched forward in time using ODE45 in MATLAB, where the relative and absolute solution tolerances are set to 10^{-8} . Once solved, the results are mapped to the physical z -plane using $z(t) = s(t) + ih(t)$.

8.2.6 Acoustic emission

The aerodynamic sound problem at low Mach numbers can be solved by the Powell-Howe acoustic analogy (Howe, 1998, 2003)

$$\left(\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) B = \text{div}(\omega \wedge \mathbf{v}) + \frac{dV}{dt} \delta(y), \quad (78)$$

where B is the stagnation enthalpy per unit mass of the fluid (Howe, 1975) and c_0 is the isentropic speed of sound. The first term on the right hand side denotes the vorticity contribution via the divergence of the Lamb vector, and the second term is the airfoil motion contribution. In the acoustic far field, the acoustic pressure and stagnation enthalpy are related by the linearized approximation

$$p(\mathbf{x}, t) \approx \rho B(\mathbf{x}, t). \quad (79)$$

The acoustic pressure can be written as

$$p(\mathbf{x}, t) = p_v(\mathbf{x}, t) + p_\omega(\mathbf{x}, t), \quad (80)$$

where $p_v(\mathbf{x}, t)$ is the acoustic pressure due to the airfoil motion,

$$p_v(\mathbf{x}, t) \approx \frac{2\rho\lambda^2 \cos \theta}{\sqrt{2c_0|\mathbf{x}|}} \frac{\partial^2}{\partial t^2} \int_{-\infty}^{[t]} \frac{dh/dt}{\sqrt{[t] - \tau}} d\tau, \quad (81)$$

and $p_\omega(\mathbf{x}, t)$ denotes the contribution from all vortices including the incident vortex and shedding vortices, which can be calculated using Howe's formula Howe (2003),

$$p_\omega(\mathbf{x}, t) \approx \frac{-\rho\omega x_j}{2\pi\sqrt{2c_0|\mathbf{x}|}^{(3/2)}} \frac{\partial}{\partial t} \int_{-\infty}^{[t]} \left(\frac{dx}{d\tau} \frac{\partial Y_j}{\partial y} - \frac{dy}{d\tau} \frac{\partial Y_j}{\partial x} \right)_{z(\tau)} \frac{d\tau}{\sqrt{[t] - \tau}}. \quad (82)$$

Here ω is the vortex strength, $x_j/|\mathbf{x}|$ denotes the observer direction θ , where $\sin \theta = x_1/|\mathbf{x}|$ and $\cos \theta = x_2/|\mathbf{x}|$, $\mathbf{x}$ is the observer location in the far field ($|\mathbf{x}| \rightarrow \infty$), $[t] = t - |\mathbf{x}|/c_0$ is the acoustic retarded time, t denotes the observer time, and τ is the source time. Y_j ($j = 1, 2$) are the components of the Kirchhoff vector that are established from the conformal mapping,

$$Y_1 = \text{Re}(\zeta + 1/\zeta) \quad \text{and} \quad Y_2 = \text{Re}\{-i(\zeta - 1/\zeta)\}. \quad (83)$$

The integrals in Eq. (82) can be evaluated numerically once the path of the vortex $z(\tau) = x(\tau) + iy(\tau)$ has been determined.

8.3 Results

The numerical framework developed in Section 8.2 is verified against Manela (2013) in the limiting cases of a flat plate that is either stationary or under prescribed harmonic plunging motions; further details of this verification are provided in Chen & Jaworski (2020). New results are now presented to examine the effect of airfoil thickness on gust interactions with stationary bodies. These results are compared against the gust experiments conducted by Peng & Gregory (2015, 2017) to assess the predictive limitations of the present theory. Aeroelastic gust-airfoil interactions are then explored, and the upstream vortex conditions to achieve direct impingement are determined numerically. Finally, the vortex sound produced by the interaction of the vortex gust coupled to the aeroelastic motions of the airfoil is evaluated.

The model problem is nondimensionalized by $\bar{x} = x/(2\lambda)$, $\bar{y} = y/(2\lambda)$, $\bar{t} = Ut/(2\lambda)$, and $\bar{\Gamma} = \Gamma/(4\pi U\lambda)$. All simulations are initialized with the initial trailing-edge vortex positioned at $\bar{z}_\gamma(0) = z_\gamma(0)/(2\lambda) = (1, 10^{-6})$. Unless otherwise stated, the initial horizontal position of the vortex gust is $\bar{x}_\Gamma(0) = \text{Re}\{\bar{z}_\Gamma(0)\} = -20$. Note that the dimensionless parameters in Section 8.3.2 follow those used by Peng & Gregory (2015, 2017) for the sake of comparison and are slightly different from the parameters in other sections. The circle centers of the symmetric airfoils considered herein require $f_{y0} = 0$.

8.3.1 Effect of airfoil thickness on vortex-gust interaction

The effect of airfoil thickness on a vortex gust interaction with a stationary, symmetric Joukowski airfoil is now examined. The time histories of the bound circulation and the trajectories of the incident and shed trailing-edge vortices are investigated as a function of airfoil thicknesses. Joukowski airfoils with thickness-to-chord ratios of 0%, 12%, 20%, and 25% corresponded to the mapped circle in the f -plane centered at horizontal positions $f_{x0} = \{0, -0.092, -0.155, -0.2\}$, respectively. The initial vertical vortex location is $\bar{y}_\Gamma(0) = \text{Im}\{\bar{z}_\Gamma(0)\} = 0.2$, and the vortex strength is $\bar{\Gamma} = 0.2$. This nondimensional vortex strength is selected to match the numerical simulations by Manela (2013) for the special case of a fixed flat plate airfoil. Details of this verification are presented in Chen & Jaworski (2020).

The numerical simulations produce three trailing-edge vortices as a result of gust encounter for all airfoil thickness considered, which also occurs in the case of a stationary flat plate Manela (2013). Figure 32 compares the trajectories of the incident vortex Γ and shed vortices γ_1 , γ_2 , and γ_3 for increasing values of airfoil thickness. The trajectories of γ_1 , γ_2 , γ_3 , and Γ all have a similar shape for different values of the airfoil thickness, where greater deviations are observed at long times for larger airfoil thicknesses. The comparison of airfoil circulation for each case is shown in Fig. 33. Despite minor differences in the airfoil circulation for different cases, larger airfoil thicknesses lead to larger peak-to-peak values of airfoil circulation and may result in larger sectional lift coefficients in the course of a vortex gust encounter.

8.3.2 Model comparison against experiments

The analytical-numerical model in this work permits comparisons of the present scheme against vortex gust experiments for non-flat airfoils that are available in the literature. Here, a comparison is made with an experimental study by Peng & Gregory (2015) on the interaction of an incident vortex with a stationary NACA0012 airfoil in a uniform flow. Their study tracked the change in

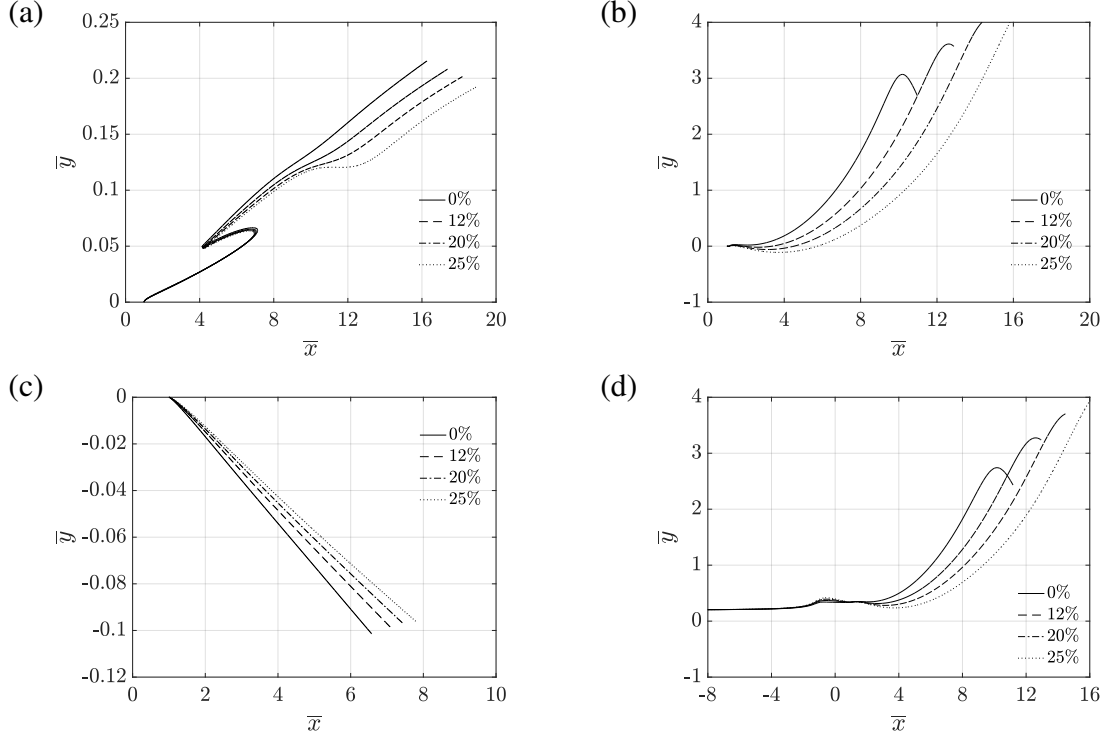


Figure 32: Effects of thickness-to-chord ratio on the trajectories of an incident line vortex Γ and the three shed vortices γ_1 , γ_2 , and γ_3 resulting from its encounter with a stationary Joukowski airfoil in uniform flow with zero angle of attack: (a) trajectories of trailing edge vortex γ_1 ; (b) trajectories of trailing edge vortex γ_2 ; (c) trajectories of trailing edge vortex γ_3 ; (d) trajectories of incident vortex Γ . Initial vertical vortex position $\bar{y}_\Gamma(0) = 0.2$ and vortex strength $\bar{\Gamma} = 0.2$.

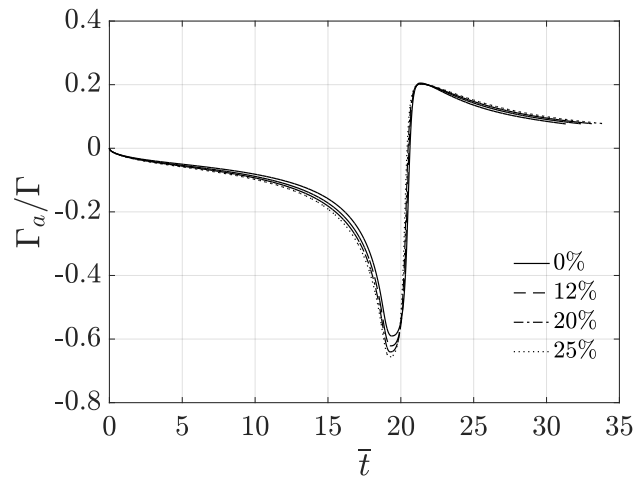


Figure 33: Time histories of scaled bound circulation of a vortex gust encounter with a stationary airfoil as a function of thickness-to-chord ratio. Initial vertical vortex position $\bar{y}_\Gamma(0) = 0.2$ and vortex strength $\bar{\Gamma} = 0.2$.

the incident vortex strength due to its encounter with the airfoil in addition to the trajectory of this vortex as a function of the initial vertical vortex positions.

To furnish a comparison, the present numerical simulations use a symmetric Joukowski airfoil with 12% thickness, where $f_{x0} = -0.092$. Since the present area of interest is the parabolic airfoil leading edge, small differences in airfoil shape are negligible in this comparison. The dimensionless initial vertical locations are $y_\Gamma(0)/c = \{0, -0.1, 1, -0.2, -0.33\}$, where the chord length of the airfoil is related to λ by the expression $c = 3\lambda - 2f_{x0} - \lambda^2/(2f_{x0} - \lambda)$. For a NACA0012 airfoil, $c/\lambda \approx 4.034$. The strength of the incoming vortex is matched with the experiments with $\Gamma/(Uc) = -0.48$ (clockwise). It is worth noting that the vortex strength does not change in the current analytic work due to its representation as a point vortex.

Figure 34 compares results from the current inviscid model simulations with available experimental measurements. The experimental results are denoted by asterisks whose different colors correspond to various vortex positions, and the associated simulation results are indicated by dashed lines. For initial vortex locations $y_\Gamma(0)/c = -0.2$ (blue), and -0.3 (black), the numerical and experimental results for trajectories agree when the vortex is far from the airfoil. Small deviations of the vortex trajectory results are observed once the vortex passes beyond $x/c > -0.3$, which is likely due to the displacement effect of the airfoil boundary layer. However, for initial vortex position $y_\Gamma(0)/c = \{-0.1, 0\}$, large deviations are observed between the predicted vortex trajectories of the numerical simulation and the experimental data when the vortex is close to the airfoil. As shown in Fig. 34(b), Peng & Gregory (2015) report a significant (30% to 40%) drop in vortex strength due to deceleration effects of the strong adverse pressure gradient near the leading-edge of the airfoil, where the vortex decay may not be sufficient to explain completely the large deviations of the vortex trajectories. A collision of the vortex with the airfoil leading-edge is observed in the experiment when the initial incident vortex location is $y_\Gamma(0)/c = 0$. In this case, the incident vortex with a finite core radius of $0.1c$ is split by the airfoil leading edge, as shown in Fig. 34(a). Clearly, the point vortex model in the inviscid numerical simulation cannot represent the physics associated with finite vortex cores nor any viscous-inviscid interactions between the vortex and airfoil. Thus, our inviscid simulation model cannot predict the experimental trends in vortex strength and trajectories when the vortex passes sufficiently near the airfoil to have significant viscous interactions with the airfoil. However, our model shows good agreement with the experimental results when there is sufficient distance between the vortex and the airfoil for the inviscid assumption of the model to hold.

A second comparison is made between the current analytic simulations and a separate experiment by Peng & Gregory (2017) for $\Gamma/(Uc) = -0.196$ (clockwise). Figure 35(a) compares the incident vortex trajectories from initial locations $y_\Gamma(0)/c = \{0.05, -0.05\}$. The large deviations of vortex trajectories are observed in Fig. 35(a) are similar to the results in Fig. 34(a) for small separation distances between the vortex and airfoil. We note that the higher vortex initial position $y_\Gamma(0)/c = 0.05$ allows the vortex to pass above the airfoil in the experimental measurements, whereas the vortex continues to take a path below the airfoil in the numerical simulation. The abrupt change in vortex path to ‘dive’ below the oncoming edge was observed experimentally by Ziada & Rockwell (1982) for shear layer impingement on an edge and was modeled analytically by Rogler (1978) for patch vortices encountering a flat plate. Figure 35(b) shows that the increasing of vortex initial location $\bar{y}_\Gamma(0)$ significantly affects the corresponding vortex trajectories. Adjustment of the initial vortex position in the numerical simulation indicates that above $y_\Gamma(0)/c = 0.111$, the vortex finally passes above the airfoil and the vortex trajectory offsets downward due to its

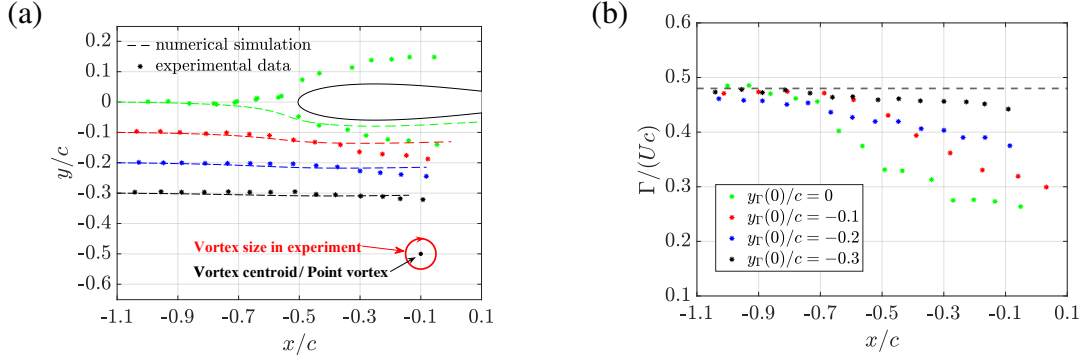


Figure 34: Comparison of incident vortex gust trajectory near a 12% thickness airfoil against the experiments data of Peng & Gregory (2015, Fig. 16) for $\Gamma/(Uc) = -0.48$ (clockwise). The nondimensional vortex strength in the numerical simulation matches the value in the experiments prior to the airfoil encounter. (a) Dependence of vortex trajectory on initial vortex placement. Splitting occurs for experiments with the vortex aligned with the airfoil. The radius of the vortex core in the experiment is $0.1c$. (b) Variation of the vortex strength as a function of initial position and downstream locations. The point vortex does not change its initial strength and is denoted by the dashed line. Asterisks denote averaged experimental data.

clockwise direction and its interaction with the airfoil.

8.3.3 Aeroelastic vortex-airfoil interactions

Attention is now turned to the aeroelastic interactions between the airfoil, the incident, wake and bound vorticity. The investigations are separated into two parts. First, the effects of different lift contributions to the aerodynamic model are examined numerically. Second, then the conditions to achieve direct vortex impingement on the airfoil are explored with respect to different aerodynamic parameters.

The linear airfoil suspension described by Eq. (73) introduces additional three dimensionless parameters to the simulations: the reduced natural frequency $\bar{\omega}_n = (2\omega_n \lambda)/U$, the mass ratio $\mu = m/m_f$, and the damping ratio $\xi = c_d/(2m\omega_n)$, where $m_f = \pi\rho c^2/4$ is the apparent mass of the fluid. The mass ratio is fixed to $\mu = 10$, which is representative of a typical aluminum airfoil in air. The airfoil thickness-to-chord ratio of 12% corresponds to $f_{x0} = -0.092$ in the conformal mapping, and the angle of attack is identically zero. Structural damping is neglected ($\xi = 0$) in all simulations presented.

Comparison of aerodynamic lift contributions in vortex-gust interaction Per the full unsteady lift expression (74), the unsteady aeroelastic motion of the airfoil is influenced by force contributions from quasi-steady circulatory lift, apparent mass effects, and the unsteady wake. Thus, the effects of the aerodynamic model are assessed for cases of increasing physical sophistication, from the quasi-steady case to the full unsteady case. In this assessment, the reduced natural frequency is set to $\bar{\omega}_n = 0.5$, the initial y -coordinate of the incident vortex is $\bar{y}_\Gamma(0) = 0.2$, and the strength of incident vortex is $\bar{\Gamma} = 0.2$. A more complete examination of aerodynamic model assumptions may be carried out over various ranges of $\bar{\omega}_n$, $\bar{y}_\Gamma(0)$, and $\bar{\Gamma}$, but such an examination

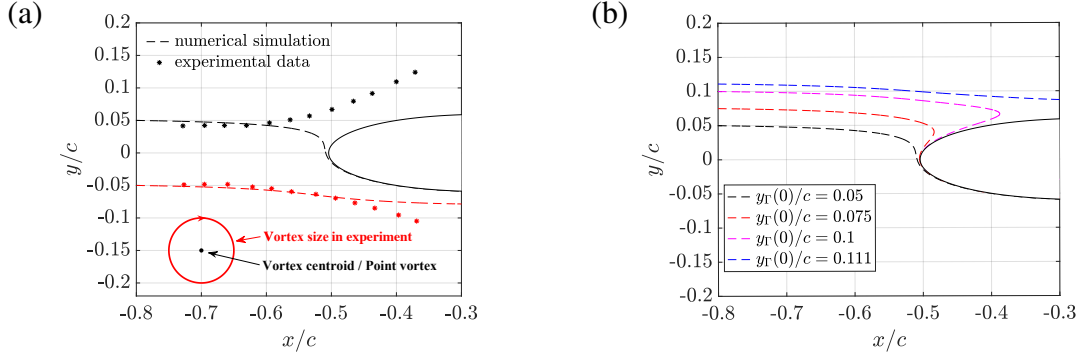


Figure 35: Comparison of incident vortex gust trajectory near a 12% thickness airfoil against the experiments data of Peng & Gregory (2017, Fig. 9) for $\Gamma/(Uc) = -0.196$ (clockwise). (a) Dependence of vortex trajectory on initial vortex placement. The radius of the vortex core in the experiment is $0.1c$. The point vortex does not change its initial strength and is denoted by the dashed line. Asterisks denote averaged experimental data. (b) Computed vortex trajectories from various initial vertical vortex locations.

is beyond the scope of this work and is not carried out here.

Figure 36(a) compares the time variations of the scaled airfoil circulation for all these aerodynamic models. The airfoil circulation history for each model increases with time from the beginning as the incident line vortex approaches to the airfoil and has a sharp change at time $\bar{t} \approx 19.7$. The sharp response results from the strong gust-airfoil interaction of the incident line vortex approaching and passing by the airfoil, after which there is a damped oscillation in the bound circulation (i.e., airfoil lift). In the quasi-steady case, the incident vortex passes by the leading-edge of the airfoil at $\bar{t} = 19.24$, and passes by the trailing edge at $\bar{t} = 20.03$. Similarly in the added mass case, the incident vortex passes by the leading-edge of the airfoil at $\bar{t} = 19.31$, and passes by the trailing edge at $\bar{t} = 20.19$, and in the full unsteady case, the incident vortex passes by the leading-edge of the airfoil at $\bar{t} = 19.24$, and passes by the trailing edge at $\bar{t} = 20.06$. Therefore, the total vortex passage times over the airfoil for the three aerodynamic model variations are $\Delta\bar{t} = 0.79, 0.89$, and 0.82 , respectively. Moreover, compared to the quasi-steady case, the added mass and full unsteady results have lower-frequency responses in airfoil circulation due to the inclusion of the inertia of the fluid about the airfoil in these cases. Note that the full unsteady case has a stronger initial response to the gust-airfoil interaction and engenders additional aerodynamic damping due to the downwash effect of the wake. Figure 36(b) shows the time histories of the airfoil displacement for the three aerodynamic model variations, which have a similar response to the airfoil circulation histories in Fig. 36(a).

Vortex impingement The present scenario fixes the values of the mass ratio $\mu = 10$ and the structural damping $\xi = 0$. Therefore, the mathematical model is controlled by two dimensionless parameters: the strength of the incident vortex $\bar{\Gamma}$, and the reduced natural frequency of the oscillator $\bar{\omega}_n$. Different initial locations of the incident vortex result in the passage of the incident vortex either above or below the airfoil. Here, the bisection method is used to infer the initial vortex position $\bar{y}_\Gamma(0)$ to achieve direct airfoil impingement given $\bar{\Gamma}$ and $\bar{\omega}_n$. The accuracy of $\bar{y}_\Gamma(0)$ is controlled by the tolerance of the bisection method, which is set to $\bar{\vartheta} = \vartheta/(2\lambda) = 0.01$ in the

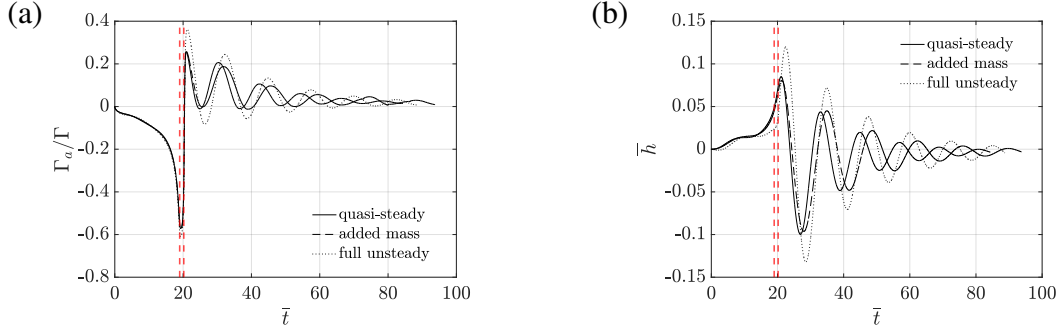


Figure 36: Comparisons of time histories of (a) scaled airfoil circulation and (b) airfoil displacement for different cases of lift contribution: quasi-steady case, added mass case, and full unsteady case. Vertical lines represent the intervals in time when the incident vortex passes by the airfoil leading and trailing edges. $\bar{\omega}_n = 0.5$, $\bar{y}_\Gamma(0) = 0.2$, and $\bar{\Gamma} = 0.2$.

present work. Figure 37 shows the incident vortex trajectories for different initial vortex positions and the corresponding time histories of the airfoil circulation. Figure 37(a) plots the trajectories of the incident vortex in the aeroelastic simulations with different initial locations, atop the steady streamlines of the flow field in the absence of vortices for visual reference. For $y_\Gamma(0) = -0.4$, the incident vortex follows initially the steady streamline from the left inflow locations and moves below the airfoil at large times. As the initial location moves up to the impingement location $y_\Gamma(0) = -0.31$, the incident vortex aligns with the steady streamline at early times. However, near the airfoil, the vortex path deviates backward under influence of the shed vorticity and airfoil motion and moves along a path near the stagnation streamline at the airfoil. As the initial location of the incident vortex moves upward, smaller deviation of the path of the incident vortex is observed in Fig. 37(a).

Figure 37(b) presents time variations of the airfoil circulation that occur over the initial time period where only the first two trailing-edge vortices are shed. For the cases when the initial vortex location is $y_\Gamma(0) = \{0, -0.1\}$, which cause the incident vortex to move above the airfoil, the corresponding magnitudes of the airfoil circulation time histories increase similarly at early times. These time histories have a sharp change when the incident vortex passes closest to the airfoil due to the shedding of the second vortex. However, for the case when the initial vortex moves below the airfoil, a smaller peak-to-peak value and an extended flat response in the airfoil circulation with respect to time is observed in Fig. 37(b), which may infer a weak interaction between the incident vortex and the airfoil. The initial vortex location $y_\Gamma(0) = -0.31$ that results in vortex impingement demonstrates a similar flat circulation response. However, the shedding of the second vortex is delayed when the incident vortex passes near the stagnation streamline at the airfoil.

The effects of different aeroelastic parameters such as $\bar{\Gamma}$ and $\bar{\omega}_n$ are now studied with respect to the selection of initial incident vortex locations to achieve vortex impingement. In the current numerical scheme, vortex impingement commonly occurs while only the first trailing-edge vortex exists and has convected downstream of the airfoil, where its downwash effect on the airfoil may be neglected. Figure 38 presents results for the vortex initial vertical position to achieve direct airfoil impingement as a function for $\bar{\omega}_n$ for $\bar{\Gamma} = 0.1$. In these simulations, the downwash effect of the airfoil wake has been neglected in the aerodynamic model. It is clear from Fig. 38 that there

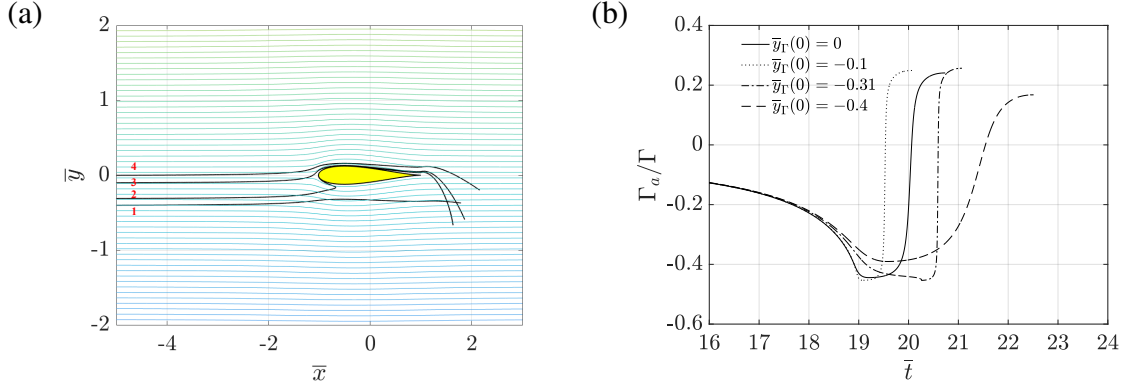


Figure 37: Time histories of incident vortex trajectories and the bound circulation for different initial vortex positions, with reduced natural frequency $\bar{\omega}_n = 0.5$ and incident vortex strength $\bar{\Gamma} = 0.1$: (a) time-varying trajectories of the incident line vortex Γ (denoted 1, 2, 3, 4) resulting from an incident vortex Γ with different initial positions $y_\Gamma(0) = \{-0.4, -0.3, -0.1, 0\}$ past a symmetrical Joukowski airfoil with 12% thickness. The streamlines in the flow field are for the case when all vortices are absent, and the motion of the incident vortex across these lines under the influence of the airfoil is noted. (b) Time variations of corresponding bound circulation on the airfoil during the shedding of the first two trailing-edge vortices.

are upper and lower asymptotic limits on the initial vertical vortex positions to have impingement, where $\bar{y}_\Gamma(0) \rightarrow -0.17$ for small values of $\bar{\omega}_n$ and $\bar{y}_\Gamma(0) \rightarrow -0.328$ for large values. Recalling that these simulations are performed at fixed mass ratio and without structural damping, the lower and upper limits correspond to negligible spring restoring force (free airfoil) and dominant spring (rigid support) limits. Importantly, the results demonstrate a monotonic change in initial vortex position $\bar{y}_\Gamma(0)$ with respect to the reduced natural frequency for a given value, and these values of $\bar{y}_\Gamma(0)$ are bounded by the low and high frequency limits of $\bar{\omega}_n$.

Lastly, Fig. 39 determines $\bar{y}_\Gamma(0)$ for impingement for fixed reduced frequency $\bar{\omega}_n = 0.5$ as a function of the incident vortex strength $\bar{\Gamma}$. The value the initial vertical position of the incident vortex decreases monotonically with increasing $\bar{\Gamma}$ and indicates the need for lower vortex positioning for direct impingement for stronger incident vortices.

Some additional commentary is warranted at this point with respect to the validity of this inviscid model to predict vortex impingement conclusions accurately for real fluids. A comparison of the present model against the airfoil experiments in Section 3.2 indicates clearly that the airfoil boundary layer has a strong influence on the trajectories of vortices passing closely above or below the airfoil. However, it remains unclear what influence if any the boundary layer has the conditions for direct impingement for fixed or aeroelastic airfoil. Companion experimental efforts to test the validity of the present model predictions would be valued but have not yet conducted to the authors' knowledge.

8.3.4 Vortex sound generation

Verification of simulations Consider the degenerate case of the generalized model in Fig. 30 of a fixed, flat airfoil of length 4λ immersed in a uniform flow with speed U in the x -direction. An incident line vortex of constant strength $\bar{\Gamma} = 0.2$ is released into the flow at the initial instant in

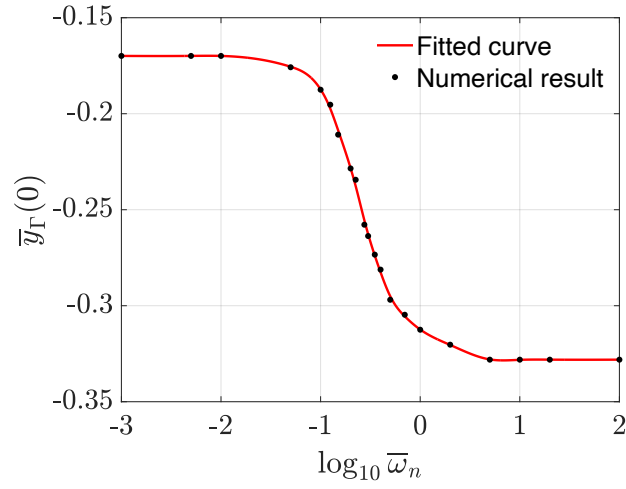


Figure 38: Effect of reduced natural frequency $\bar{\omega}_n$ on the initial incident vortex vertical position $\bar{y}_\Gamma(0)$ to achieve direct airfoil impingement. Incident vortex has a strength of $\bar{\Gamma} = 0.1$ and moves past a symmetric Joukowski airfoil with 12% thickness.

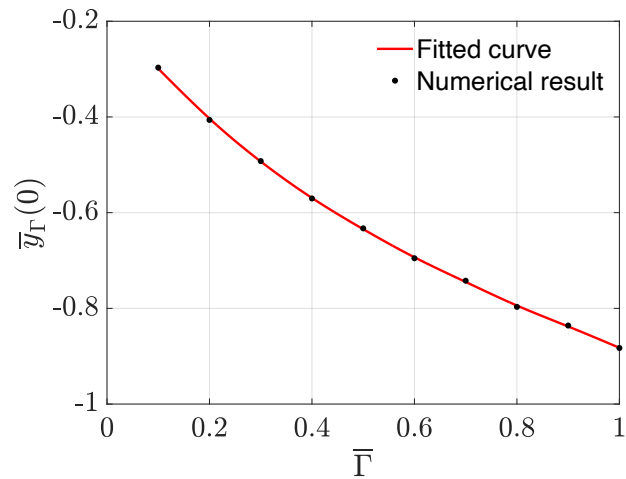


Figure 39: Variations of selection of initial incident vortex location $\bar{y}_\Gamma(0)$ with varying strength of incident vortex $\bar{\Gamma}$ when vortex impingement occurs, with reduced natural frequency $\bar{\omega}_n = 0.5$.

time ($\bar{t} = 0$) at position $(\bar{x}_0, \bar{y}_0) = (-20, 0.2)$ and advects freely over the airfoil. This scenario has been previously investigated by Manela (2013) and furnishes a verification case for the more general framework developed in this paper. Figure 40(a) illustrates the comparison of the numerical results for the nondimensional acoustic pressure measured directly above a stationary flat plate or a stationary Joukowski airfoil at $\theta = \pi/2$, where the solid curve agrees with (Manela, 2013, Fig. 3(c)).

Aeroelastic case Now consider the case of an incident vortex past by a Joukowski airfoil under aeroelastic motion, as illustrated in Figure 30. The Joukowski airfoil has 12% thickness and zero angle of attack. The initial strength and location of the incident vortex are the same as in Section 8.3.4, and the additional dimensionless parameters for the aeroelastic case are the damping ratio $\xi = 0$, reduced natural frequency $\bar{\omega}_n = 0.5$, and mass ratio $\mu = 10$. The sound produced by the unsteady suction ($\theta = \pi/2$) in different aspects is shown in Figure 40(b) and (c), respectively. In Figure 40(b), the solid curve denotes the incident vortex acoustic contribution, and the dashed curve denotes the wake contribution which includes the effect of 12 Brown-Michael vortices that are shed by the trailing edge over the course of the simulation. The reduced pressure of the incident vortex contribution decreases when the incident vortex passes by the leading edge, and the pressure of the wake contribution starts to fluctuate due to more vortices shedding from the airfoil trailing edge. After the incident vortex passes by the trailing edge, the wake contribution and the incident vortex contribution start to balance together with same strength in opposite signs and yield a net zero total acoustic pressure in long times, as illustrated in Figure 40(c). Compared with the total sound for a stationary flat and thick Joukowski airfoils in Figure 40(a), the result in Figure 40(c) shows that aeroelastic effects have a modest slight influence in the total sound for the parameters considered here. However, the comparison of flat airfoil and Joukowski airfoil in Figure 40(a) shows that the airfoil thickness controls significantly the peak noise due to the acceleration of the vortex in the leading-edge interaction. Figure 40(d) presents the sound observed along a different observation angle ($\theta = 0$) produced by the unsteady lift contributions of the incident vortex, the wake, and the airfoil motion.

8.4 Conclusions

An analytical system of equations is developed to simulate numerically the aeroelastic interactions of an incident line vortex with a Joukowski airfoil on a linear elastic suspension. The mathematical framework is modelled by using dynamic conformal mapping techniques. The dynamical problem for the incident vortex motion and trailing edge wake evolution is studied using potential flow theory and the Brown and Michael equation for vortices shed from the airfoil trailing edge.

Numerical simulations of the analytical model are verified in the limiting cases of stationary and harmonically-plunging flat plates, and the effect of airfoil thickness on the vortex gust interaction is systematically examined for stationary, symmetric Joukowski airfoils. Comparisons of the numerical solutions against vortex gust interaction data for fixed real airfoils indicate good agreement for vortex path prediction when the incident vortex is sufficiently far from the airfoil to have mild to negligible viscous interactions with the airfoil and its boundary layer. However, the numerical agreement of vortex path predictions is increasingly poor for incident vortex positions closer to the airfoil. The change in strength of the incident vortex due to viscous effects of the airfoil boundary layer that was identified by Peng & Gregory (2015) is shown to be a good indicator

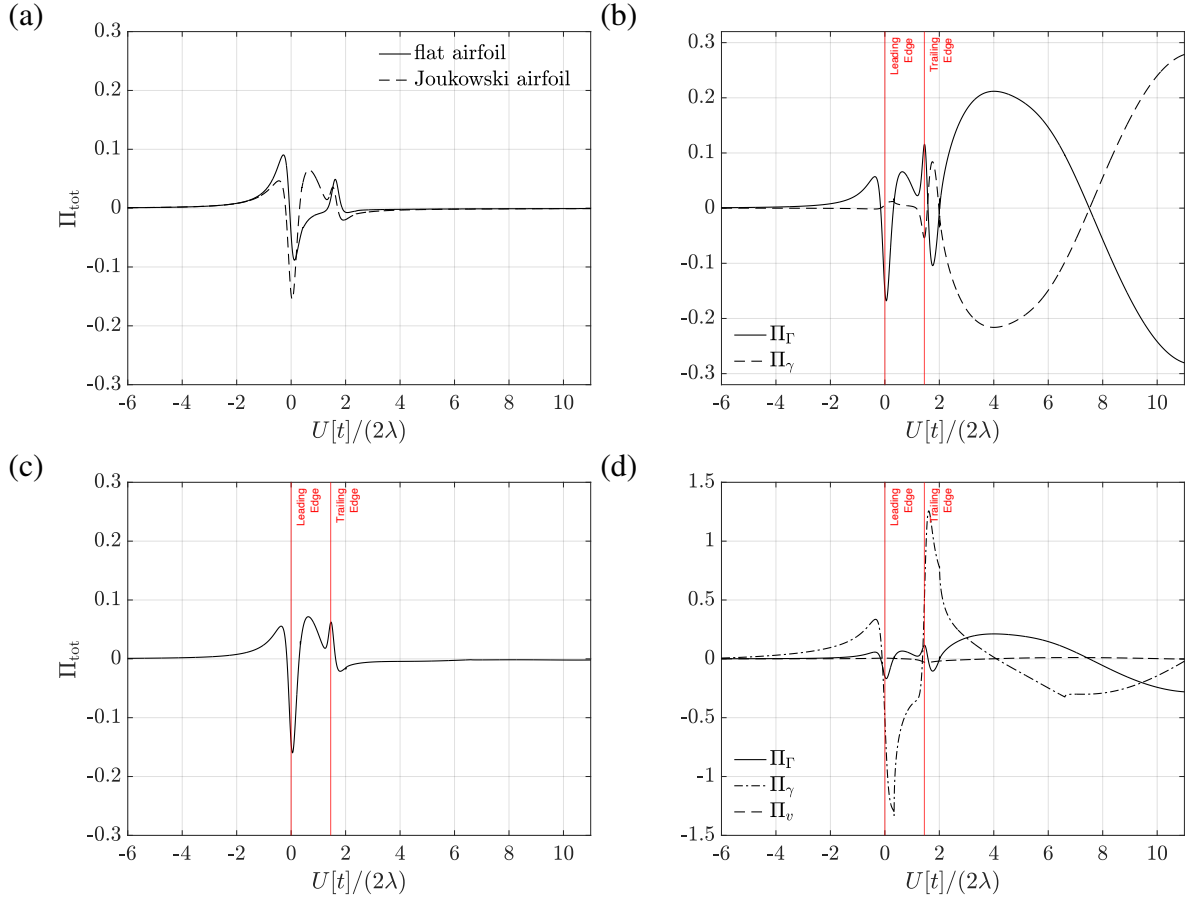


Figure 40: Acoustic radiation of a line vortex encountering a stationary flat-plate airfoil or a Joukowski airfoil with 12% thickness: (a) comparison of total acoustic radiation for the stationary flat and Joukowski airfoils along $\theta = \pi/2$; (b) acoustic contributions of incident vortex (Π_Γ) and wake ($\Pi_\gamma = \sum_{k=1}^{k=12} \Pi_{\gamma_k}$) for a Joukowski airfoil under aeroelastic motion along $\theta = \pi/2$; (c) total normalized acoustic pressure for a Joukowski airfoil under aeroelastic motions observed at angle $\theta = \pi/2$; (d) acoustic contributions of the incident vortex (Π_Γ), vortical wake (Π_γ), and the airfoil motion (Π_v) for a Joukowski airfoil under aeroelastic motion, as observed along $\theta = 0$. For the aeroelastic simulations in (b-d), $\xi = 0$, $\bar{\omega}_n = 0.5$, and $\mu = 10$.

of the degree of violation of the inviscid model assumptions and of the disagreement between the model and experimental observation.

The influence of lift contributions in the aerodynamic model are assessed in the context of the aeroelastic vortex gust-airfoil framework. The aeroelastic results of all three aeroelastic models of increasing sophistication (quasi-steady, added mass, and full unsteady) demonstrate an oscillatory damping of airfoil circulation and plunge displacement in long times in the absence of structural damping. The full unsteady case adds more aerodynamic damping to the aeroelastic model and imparts a larger impulse to the airfoil circulation and structural response from its gust encounter relative to other variations of the aerodynamic model that is due to the unsteady wake contribution to the lift.

The aeroelastic framework is then used to determine the conditions for achieving direct impingement of an incident line vortex with the elastically-mounted airfoil. The aeroelastic results of a symmetric Joukowski airfoil with 12% thickness-to-chord ratio demonstrate that the incident vortex follows initially the steady streamline in the uniform flow, but deviates backward and then moves along a path near the stagnation streamline at the airfoil near where the vortex impingement occurs. The initial vortex placement for airfoil impingement changes monotonically with the reduced natural frequency in a specific range that depends on the vortex strength. However, for fixed incident vortex strength, asymptotic limits on the vortex placement to achieve impingement are discussed numerically in the limits of high and low reduced natural frequency of the elastic suspension. Therefore, there is a finite range of vortex positions where vortex impingement occurs for aeroelastic vortex gust encounters that depends upon the strength of the incident vortex. Companion experimental comparisons of aeroelastic vortex-airfoil interactions and the associated fluid effects on impingement prediction have not yet been carried out to the authors' knowledge and are desired for validation and to identify any necessary improvements to the physics modeling.

In complement to the dynamical frame for the vortex-airfoil interactions, the aeroelastic vortex dynamics study by Chen & Jaworski (2018) is extended to study the corresponding acoustic emissions using the Powell-Howe acoustic analogy. The acoustic signature for the special case of a stationary flat-plate airfoil is verified against available results in the literature. New acoustic results show that the airfoil thickness controls significantly the total sound, while a modest difference in total sound is observed in the comparison of stationary and aeroelastic cases for the parameters considered. A complete parameter sweep of the aeroelastic cases is possible with the model developed to determine conditions for minimum noise production and the associated dynamics of the gust-airfoil interaction.

9 MODELING OF VORTEX-AIRFOIL ENCOUNTERS IN GROUND EFFECT

H. Chen, P. J. Baddoo, J. W. Jaworski

9.1 Background

Natural fliers commonly utilize a ground plane to enhance their lift-to-drag ratio (Rayner, 1991). Vehicle designers (NATO RTO, 1999; Holloran & O’Meara, 1999; Rozhdestvensky, 2000) also seek to exploit this “ground effect” phenomenon, and consequently the accurate mathematical modelling of the flow field is required. Such a model must be physically accurate and computationally efficient if it is to be used in any optimisation loop for vehicle design. However, current vortex models (Michelin & Llewellyn Smith, 2009; Darakananda & Eldredge, 2019) are restricted to single-body interactions and cannot take into account the effect of a ground plane. There are both numerical (Abramowski, 2007; Willis *et al.*, 2007; Quinn *et al.*, 2014) and asymptotic (Widnall & Barrows, 1970; Iosilevskii, 2008, 2012) investigations, but this paper proposes a vortex model that is swift to compute and is not restricted to an asymptotic regime where the body is either very close or far away from the planar wall.

Two-dimensional unsteady flow with low Mach number and high Reynolds number past an airfoil is a classical problem in fluid-structure interactions that has received considerable attention in the aeronautical literature over the past century. For theoretical analyses in particular, the representation of how vorticity is shed into the wake to satisfy the Kutta condition at the trailing edge plays a crucial role in gust-airfoil interactions and related unsteady airfoil problems. An early model to account for the effect of vortex shedding of a delta wing was formulated by Brown & Michael (1954). Their model supposes that the vorticity shed from an edge rolls up via a connecting vortex sheet into a point vortex with time-varying circulation. We refer to such vortices as Brown-Michael vortices, whose circulation is set instantaneously to satisfy the Kutta condition. The Brown and Michael equation describes the motion of the vortex tethered to the trailing edge before it is released as a free vortex into the flow. However, the original formulation does not guarantee the vanishing of a reaction force due to an unbalanced couple, which is of minor concern to the fluid dynamic problem but is important for aeroacoustic predictions based on the model. Howe (1996) provided an emended version of the Brown-Michael equation which does not result in an unbalanced couple.

In the present work, the analyses of Manela & Huang (2013) and Chen & Jaworski (2018) are extended to include the influence of a ground plane. Advances in conformal mapping theory (Crowdy & Marshall, 2006; Crowdy, 2013) allow the complex potential to be expressed in terms of the Schottky-Klein prime function, which may be rapidly computed using a recently developed algorithm (Crowdy *et al.*, 2016). Adapting the approach of Baddoo & Ayton (2018) allows the complex potential to be constructed and applied as a forcing for shed Brown-Michael vortices. The results of insight into the influence of a ground plane on lift production and aerodynamic performance.

9.2 Mathematical formulation

Our analysis follows the work of Baddoo & Ayton (2018) and is restricted to inviscid, irrotational, incompressible, two-dimensional flow. Therefore, there exists a complex potential function

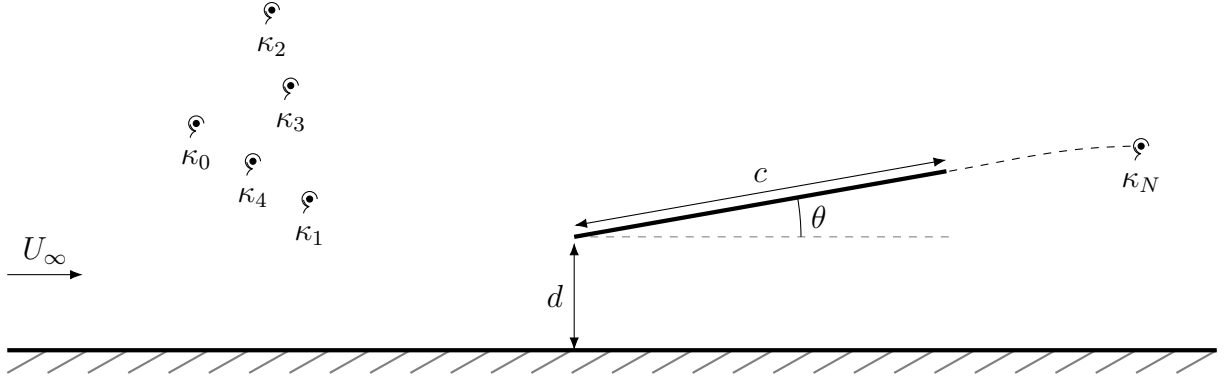


Figure 41: A flat plate in ground effect with labelled length-scales. The chord length is $c = 1$, pitching angle θ , and ground proximity d . The origin is placed on the ground in-line with the trailing edge, while x is the streamwise direction and y is the lateral direction. Point vortices of strengths κ_j are labelled with spirals, and the dashed line indicates a branch cut connecting the trailing edge to the Brown-Michael vortex.

$$w(z) = \phi(z) + i\psi(z),$$

where ϕ and ψ are the velocity potential and stream function in the physical z -plane, respectively. Similarly to Baddoo & Ayton (2018), we consider a plate of length c , inclined at angle θ to the ground, where the distance from the leading edge to the ground is d , as illustrated in figure 41. Far upstream and downstream, the flow is tangential to the ground and has speed U_∞ . The fluid domain may contain a number of free and attached point vortices, which represent incident vorticity on the flat plate, its vorticity shed into the wake, and the Brown-Michael vortex shed from the trailing edge. No-flux boundary conditions are imposed on the ground and plate. Additionally, we specify the Kutta condition of finite flow velocity at the trailing edge. We non-dimensionalize length by c and velocity by U_∞ .

9.2.1 Conformal map

Unlike the simple case of a single aerofoil (Chen & Jaworski, 2018), our domain is doubly-connected and therefore the traditional methods of conformal mappings to the unit disk or upper half-plane are not applicable. Consequently, we appeal to a similar conformal map as that used in the study of vortex equilibria in ground effect (Baddoo & Ayton, 2018).

Since our domain is doubly-connected, the canonical circular domain is an annulus. We label the exterior circle of the annulus C_0 and the interior circle C_1 , as illustrated in figure 42. The radius of the interior circle is given by q , which we vary to change the height from the ground, d . The conformal map we use is called the “radial slit map” (Crowdy, 2013), which depends on two parameters, and maps circular regions in the ζ -plane to slits pointing to a particular point in the z -plane. However, if one of the parameters is taken to lie C_0 , the radial slit map becomes a radial half-plane map and C_0 is mapped to an infinite line. Furthermore, we may choose the second parameter in the radial slit map such that the (finite) image of C_1 is inclined at angle θ to

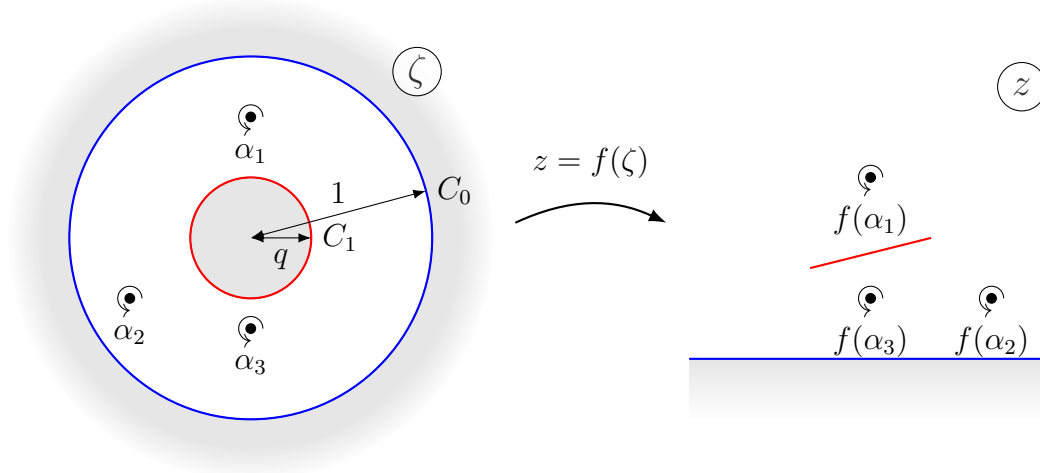


Figure 42: Illustration of the conformal map from the transform ζ -space to the physical z -space. The inner red circle is mapped to the flat plate and the outer blue circle is mapped to the ground plane. The gray areas indicate regions that are outside the domain of definition. The point vortices with strengths κ_j located at α_j in the ζ -plane map to the z -plane, where the vortex strengths are preserved under the conformal mapping.

the ground. The corresponding map is given as (Baddoo & Ayton, 2018)

$$z = f(\zeta) = A \frac{\omega(\zeta, e^{2i\theta})}{\omega(\zeta, 1)} + s, \quad (84)$$

where A is a constant required to scale the plate to unit length and s is a constant that moves the plate so that the leading edge is above the origin. Both of these may be determined numerically using a simple Newton method. The Schottky-Klein prime function is denoted by $\omega(\cdot, \cdot)$.

The Schottky-Klein prime function for the annulus can be expressed in two rapidly convergent forms Crowdy *et al.* (2016); Crowdy (2010, 2006):

$$\omega(\zeta, \gamma) = \frac{\zeta - \alpha}{C^2} \prod_{k=1}^{\infty} \left(1 - q^{2k} \frac{\zeta}{\alpha} \right) \cdot \left(1 - q^{2k} \frac{\alpha}{\zeta} \right) \quad (85)$$

$$= -\frac{\alpha}{C^2} A \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)} \left(\frac{\zeta}{\alpha} \right)^n, \quad (86)$$

where

$$C = \prod_{k=1}^{\infty} (1 - q^{2k}), \quad A = \frac{\prod_{n=1}^{\infty} (1 + q^{2n})^2}{\sum_{n=1}^{\infty} q^{n(n-1)}}.$$

9.2.2 Complex potential

Since we have a conformal map from the annular domain to the physical domain, it is sufficient to calculate the complex potential in the annular domain. We appeal to the new calculus of vortex

dynamics proposed by Crowdy (2010), which provides a framework for the calculation of instantaneous complex potentials associated with multiply-connected domains. An advantage of this approach which we will exploit is its ability to specify the circulations around individual bodies.

We follow the procedures of Chen & Jaworski (2018) and Baddoo & Ayton (2018), where the complex potential is split into three contributions: the contribution from the incident vortex W_Γ , the contribution from the shedding vortices W_κ , and the contribution from the uniform flow W_U .

Contribution from the incident vortex Due to Crowdy (2010), we know that the hydrodynamic Green's function,

$$G^{(0)}(\zeta, \alpha) = \frac{1}{2\pi i} \log \left(\frac{\omega(\zeta, \alpha)}{|\alpha| \omega(\zeta, 1/\bar{\alpha})} \right), \quad (87a)$$

is the complex potential for a unit circulation point vortex at α in any canonical circular domain. An overbar denotes the complex conjugate. The hydrodynamic Green's function produces circulation -1 around the object which is the image of C_0 and zero circulation around the other objects. To specify a non-zero circulation around the obstacle whose boundary is the image of C_1 , we use the Möbius map $\theta_1(\zeta) = q^2 \zeta$ and write the *modified* hydrodynamics Green's function

$$G^{(1)}(\zeta, \alpha) = \frac{1}{2\pi i} \log \left(\frac{\omega(\zeta, \alpha)}{|\alpha| \omega(\zeta, \theta_1(1/\bar{\alpha}))} \right). \quad (87b)$$

The above corresponds to a point vortex of unit circulation at α but now has circulation -1 around the obstacle whose boundary is the image of C_1 , i.e. the flat plate. Additionally, the modified hydrodynamics Green's function produces zero circulation around all other obstacles.

We are free to choose either Eq. (87a) or Eq. (87b) to represent the complex potential due to the vortices. However, since we wish to cancel the contributions to the circulation around any body, we are required to choose Eq. (87b). In order to cancel out the circulation around all bodies we must place a sufficiently strong vortex at the pre-image of infinity. The pre-image of infinity is $\zeta_\infty = 1$ and lies on the body C_0 . Substituting $\alpha = \zeta_\infty$ into Eq. (87a) results in a constant, so we choose Eq. (87b) to represent the complex potential induced by a point vortex in the flow.

Therefore, the complex potential induced by the incident vortex of strength Γ at α_Γ , and produces no circulation around the plate C_1 is

$$\begin{aligned} W_\Gamma(\zeta) &= \Gamma G^{(1)}(\zeta, \alpha_\Gamma) - \Gamma G^{(1)}(\zeta, 1) \\ &= \frac{\Gamma}{2\pi i} \log \left(\frac{\omega(\zeta, \alpha_\Gamma)}{|\alpha_\Gamma| \omega(\zeta, q^2/\bar{\alpha}_\Gamma)} \right) - \frac{\Gamma}{2\pi i} \log \left(\frac{\omega(\zeta, 1)}{\omega(\zeta, q^2)} \right). \end{aligned} \quad (88)$$

Contribution from vortex shedding Using Eq. (87b), the complex potential induced by N shed vortices of strength κ_n at α_n , which have $-\kappa$ circulation around the plate C_1 , is given by

$$\begin{aligned} W_\kappa(\zeta) &= \sum_{n=1}^N \kappa_n G^{(1)}(\zeta, \alpha_n) \\ &= \frac{1}{2\pi i} \sum_{n=1}^N \kappa_n \log \left(\frac{\omega(\zeta, \alpha_n)}{|\alpha_n| \omega(\zeta, q^2/\bar{\alpha}_n)} \right), \end{aligned} \quad (89)$$

where $\kappa = \sum_{n=1}^N \kappa_n$.

Contribution from uniform flow The complex potential induced by a uniform flow of magnitude 1 far upstream which produces no circulation around the plate is given by

$$W_U(\zeta) = -a \frac{\omega_\alpha(\zeta, 1)}{\omega(\zeta, 1)}. \quad (90)$$

where $\omega_\alpha(\zeta, 1)$ is the first derivative of $\omega(\zeta, 1)$ with respect to α and a is a constant such that

$$z \sim \frac{a}{\zeta - 1},$$

as $\zeta \rightarrow 1$. Expanding (84) in its Laurent series yields

$$a = A \omega(1, e^{2i\theta}).$$

Total complex potential In summary, the total complex potential for a flat plate in ground effect with N shed vortices of strength κ_n located at α_n and an incident vortex of strength Γ located at α_Γ around the flat plate is given by

$$\begin{aligned} W \equiv \phi + i\psi &= W_\Gamma(\zeta) + W_\kappa(\zeta) + W_U(\zeta) \\ &= \frac{\Gamma}{2\pi i} \log \left(\frac{\omega(\zeta, \alpha_\Gamma)}{|\alpha_\Gamma| \omega(\zeta, q^2/\bar{\alpha}_\Gamma)} \right) - \frac{\Gamma}{2\pi i} \log \left(\frac{\omega(\zeta, 1)}{\omega(\zeta, q^2)} \right) \\ &\quad + \frac{1}{2\pi i} \sum_{n=1}^N \kappa_n \log \left(\frac{\omega(\zeta, \alpha_n)}{|\alpha_n| \omega(\zeta, q^2/\bar{\alpha}_n)} \right) - a \frac{\omega_\alpha(\zeta, 1)}{\omega(\zeta, 1)}. \end{aligned}$$

The complex velocity field induced by this configuration of vortices may be calculated by differentiating the above, which yields

$$\begin{aligned} \frac{dW}{d\zeta} &= U(\zeta) - iV(\zeta) \\ &= \Gamma G_\zeta^{(1)}(\zeta, \alpha_\Gamma) - \Gamma G_\zeta^{(1)}(\zeta, 1) + \sum_{n=1}^{N-1} \left(\kappa_n G_\zeta^{(1)}(\zeta, \alpha_n) + \kappa_N G_\zeta^{(1)}(\zeta, \alpha_N) \right) + \frac{dW_U}{d\zeta}. \end{aligned} \quad (91)$$

In terms of the Schottky-Klein prime function (86), the complex velocity is

$$\begin{aligned} \frac{dW}{d\zeta}(\zeta) = U(\zeta) - iV(\zeta) &= \frac{\Gamma}{2\pi i} \left(\frac{\omega_\zeta(\zeta, \alpha_\Gamma)}{\omega(\zeta, \alpha_\Gamma)} - \frac{\omega_\zeta(\zeta, q^2/\bar{\alpha}_\Gamma)}{\omega(\zeta, q^2/\bar{\alpha}_\Gamma)} \right) \\ &\quad - \frac{\Gamma}{2\pi i} \left(\frac{\omega_\zeta(\zeta, 1)}{\omega(\zeta, 1)} - \frac{\omega_\zeta(\zeta, q^2)}{\omega(\zeta, q^2)} \right) \\ &\quad + \frac{1}{2\pi i} \sum_{n=0}^{N-1} \kappa_n \left(\frac{\omega_\zeta(\zeta, \alpha_n)}{\omega(\zeta, \alpha_n)} - \frac{\omega_\zeta(\zeta, q^2/\bar{\alpha}_n)}{\omega(\zeta, q^2/\bar{\alpha}_n)} \right) \\ &\quad + \frac{\kappa_N}{2\pi i} \left(\frac{\omega_\zeta(\zeta, \alpha_N)}{\omega(\zeta, \alpha_N)} - \frac{\omega_\zeta(\zeta, q^2/\bar{\alpha}_N)}{\omega(\zeta, q^2/\bar{\alpha}_N)} \right) \\ &\quad - a \left(\frac{\omega_{\alpha,\zeta}(\zeta, 1)}{\omega(\zeta, 1)} - \frac{\omega_\alpha(\zeta, 1)\omega_\zeta(\zeta, 1)}{\omega^2(\zeta, 1)} \right), \end{aligned}$$

where κ_N is the circulation of the shedding vortex, and κ_n ($n = 1, 2, \dots, N-1$) are the circulations of the free vortices which have been shed from the trailing edge of the plate.

The Kutta condition states that the velocity at the trailing edge of the plate is finite. Accordingly, W must vanish at $\zeta = \zeta_t$, where ζ_t is the location of the trailing edge of the plate in ζ -space. Therefore, the instantaneous circulation of the shedding vortex κ_N is

$$\kappa_N = - \left[\Gamma G_\zeta^{(1)}(\zeta_t, \alpha_\Gamma) - \Gamma G_\zeta^{(1)}(\zeta_t, 1) + \sum_{n=1}^{N-1} \kappa_n G_\zeta^{(1)\zeta}(\zeta_t, \alpha_n) + B \right] / G_\zeta^{(1)}(\zeta_t, \alpha_N), \quad (92)$$

where B is defined as

$$B = \frac{dW_U}{d\zeta} = -a \left(\frac{\omega_{\alpha, \zeta}(\zeta_t, 1)}{\omega(\zeta_t, 1)} - \frac{\omega_\alpha(\zeta_t, 1)\omega_\zeta(\zeta_t, 1)}{\omega^2(\zeta_t, 1)} \right). \quad (93)$$

We note that the derivatives of the hydrodynamic Green's function are expressible in terms of the Schottky-Klein prime function (86) as

$$\begin{aligned} G_\zeta^{(1)}(\zeta, \alpha) &= \frac{1}{2\pi i} \left(\frac{\omega_\zeta(\zeta, \alpha)}{\omega(\zeta, \alpha)} - \frac{\omega_\zeta(\zeta, q^2/\bar{\alpha})}{\omega(\zeta, q^2/\bar{\alpha})} \right), \\ G_{\zeta, \alpha}^{(1)}(\zeta, \alpha) &= \frac{1}{2\pi i} \left(\frac{\omega_{\alpha, \zeta}(\zeta, \alpha)\omega(\zeta, \alpha) - \omega_\zeta(\zeta, \alpha)\omega_\alpha(\zeta, \alpha)}{\omega^2(\zeta, \alpha)} \right). \end{aligned}$$

The derivative of κ_N with respect to time t is given by

$$\begin{aligned} \frac{d\kappa_N}{dt} &= - \frac{\Gamma G_{\zeta, \alpha}^{(1)}(\zeta_t, \alpha_\Gamma)}{G_\zeta^{(1)}(\zeta_t, \alpha_N)} \cdot \frac{d\alpha_\Gamma}{dt} - \frac{1}{G_\zeta^{(1)}(\zeta_t, \alpha_N)} \sum_{n=1}^{N-1} \kappa_n G_{\zeta, \alpha}^{(1)}(\zeta_t, \alpha_n) \cdot \frac{d\alpha_n}{dt} \\ &+ \frac{G_{\zeta, \alpha}^{(1)}(\zeta_t, \alpha_N) \left[\Gamma G_\zeta^{(1)}(\zeta_t, \alpha_\Gamma) - \Gamma G_\zeta^{(1)}(\zeta_t, 1) + \sum_{n=1}^{N-1} \kappa_n G_\zeta^{(1)}(\zeta_t, \alpha_n) + B \right]}{\left(G_\zeta^{(1)}(\zeta_t, \alpha_N) \right)^2} \cdot \frac{d\alpha_N}{dt}. \end{aligned}$$

9.2.3 Evolution of vortex shedding

The motion of the most recent vortex shed from the trailing edge is determined by the Brown and Michael equation (Howe, 1996),

$$\frac{d\mathbf{x}_N}{dt} + \frac{\mathbf{x}_N}{\kappa_N} \frac{d\kappa_N}{dt} = \mathbf{v}_N, \quad (94)$$

where $\mathbf{x}_N$ represents the location of a shed vortex tethered to the trailing edge with circulation κ_N in a vector form with respect to the rectangular coordinate system $\mathbf{x} \equiv (x, y)$, $\mathbf{v}_N$ is the fluid velocity at the location of $\mathbf{z}_N$ in a vector form when its local self-induced velocity contribution is excluded $\mathbf{v}_N \equiv (u, v)$. The Brown and Michael equation (94) governs the motion of the trailing-edge vortex until $\frac{d\kappa_N}{dt} = 0$. At this instant the circulation of the Brown-Michael vortex is fixed, the vortex is released into the wake as a free vortex, and a new Brown-Michael vortex is created at the trailing edge.

Equation (94) can be rewritten in a scalar form:

$$\frac{d\alpha_N}{dt} + \frac{1}{f'(\alpha_N)} \frac{f(\alpha_N) - f(\alpha_t)}{\kappa_N} \frac{d\kappa_N}{dt} = \frac{v_N^*}{f'(\alpha_N)}, \quad (95)$$

where α_N and α_t are the mapped vortex location of $z_N = x + iy$ and the trailing-edge location in ζ -plane, respectively and

$$v_N^* = \left[\frac{1}{f'(\alpha_N)} \left(\frac{i\kappa_N}{4\pi} \frac{f''(\alpha_N)}{f'(\alpha_N)} + F'(\alpha_N) \right) \right]^*, \quad (96)$$

The desingularized velocity field is represented by

$$F'(\alpha_N) = \lim_{\zeta \rightarrow \alpha_N} \left[\frac{dW}{d\zeta} - \frac{1}{2\pi i} \frac{\kappa_N}{\zeta - \alpha_N} \right]. \quad (97)$$

Similarly, the motion for the incident vortex is governed by

$$\frac{d\alpha_\Gamma}{dt} = \frac{\left[\left(\frac{i\Gamma}{4\pi} \frac{f''(\alpha_\Gamma)}{f'(\alpha_\Gamma)} + F'(\alpha_\Gamma) \right) \right]^*}{|f'(\alpha_\Gamma)|^2}, \quad (98)$$

where

$$F'(\alpha_\Gamma) = \lim_{\zeta \rightarrow \alpha_\Gamma} \left[\frac{dW}{d\zeta} - \frac{1}{2\pi i} \frac{\Gamma}{\zeta - \alpha_\Gamma} \right]. \quad (99)$$

Finally, the motions for the $N - 1$ "frozen" trailing edge vortices follow a similar form

$$\frac{d\alpha_n}{dt} = \frac{\left[\left(\frac{i\kappa_n}{4\pi} \frac{f''(\alpha_n)}{f'(\alpha_n)} + F'(\alpha_n) \right) \right]^*}{|f'(\alpha_n)|^2}, \quad (100)$$

with

$$F'(\alpha_n) = \lim_{\zeta \rightarrow \alpha_n} \left[\frac{dW}{d\zeta} - \frac{1}{2\pi i} \frac{\Gamma}{\zeta - \alpha_n} \right], \quad \text{for } n = 1, 2, \dots, N - 1. \quad (101)$$

9.3 Results

In this section we plot results to investigate the vortex shedding from the trailing edge with the ground effect. The numerical simulations are carried out and are parameterized uniformly as follows:

$$\bar{x} = 2x/c, \quad \bar{y} = 2y/c, \quad \bar{\Gamma} = \Gamma/(\pi U c), \quad \bar{t} = 2Ut/c.$$

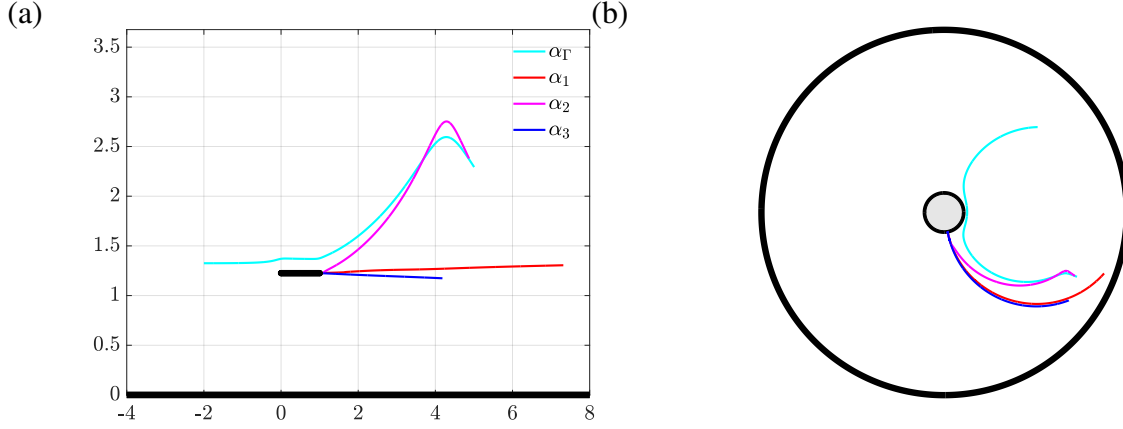


Figure 43: Trajectories of the incident vortex α_Γ , and three shed vortices α_1 , α_2 and α_3 in physical z -plane and mapped ζ -plane. $\theta = 0^\circ$.

9.3.1 Zero airfoil inclination angle ($\theta = 0^\circ$)

We first consider the case when an incident vortex α_Γ starts at the height $\Delta\bar{h} = 0.2$ above the airfoil with constant strength $\bar{\Gamma} = 0.2$ and passes by a flat plate airfoil with zero inclination angle ($\theta = 0$) in ground effect. The incident vortex induces fluid loading on the airfoil thereby causing the shedding of vorticity at the trailing edge. The trajectories of the vortices in physical z -plane and mapped ζ -plane are shown in Fig. 43, and the streamline at $\bar{t} = 14.97$ is shown in Fig. 44. Our simulations indicate that only three vortices are shed during this vortex-airfoil interaction, and the results are similar to previous work (Manela, 2013; Chen & Jaworski, 2018) in the absence of a ground plane. Notably, while the trajectories of α_1 and α_3 remain in the vicinity of the x -axis, the trajectory of α_2 , having an opposite sign of circulation to α_Γ , follows the trajectory of the incident vortex.

Figure 45 compares the corresponding time histories of airfoil circulation against that in the case of a free airfoil (without ground effect). The ground effect causes higher fluctuation of the airfoil circulation during the time when the incident vortex passes by the leading edge and the trailing edge of the airfoil. The approximate passing times are marked in red in Fig. 45. Notably, when the third shed vortex starts to slightly moving downward (see the blue curve in Fig. 43 (a)), the ground effect quickens approach of the airfoil circulation to its asymptotic long-time value.

9.3.2 Small airfoil inclination angle ($\theta = -10^\circ$)

We also consider the case when the airfoil has a pitched angle to the x -axis. The vortex trajectories in both z -plane and ζ -plane, and the streamlines at different points in time are given in Fig. 46, Fig. 47, Fig. 48, Fig. 49 and Fig. 50, respectively.

9.4 Conclusion

This paper has considered the shedding of vorticity from a plate interacting with an incident vortex in ground effect. The model framework is built using the Schottky-Klein prime function and is solved as a potential flow problem. The motion of the shedding vortices are determined by the

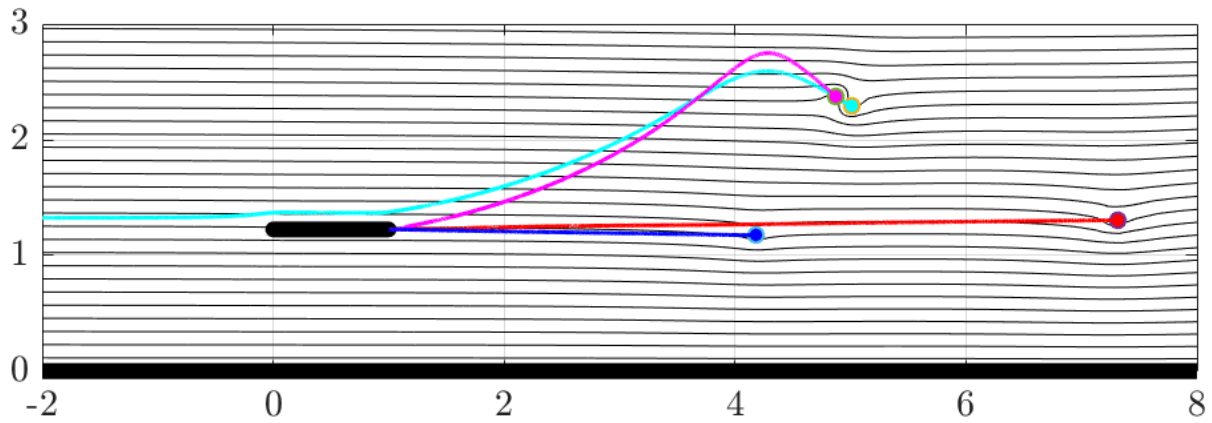


Figure 44: The steady, uniform flow past a flat plate with zero inclination angle ($\theta = 0^\circ$) in ground effect at $\bar{t} = 14.97$, with the Kutta condition specified. The instantaneous locations of the incident vortex and three vortices are marked as dots in cyan, red, magenta and blue. The corresponding previous vortex trajectories are marked in the same color as the dots.

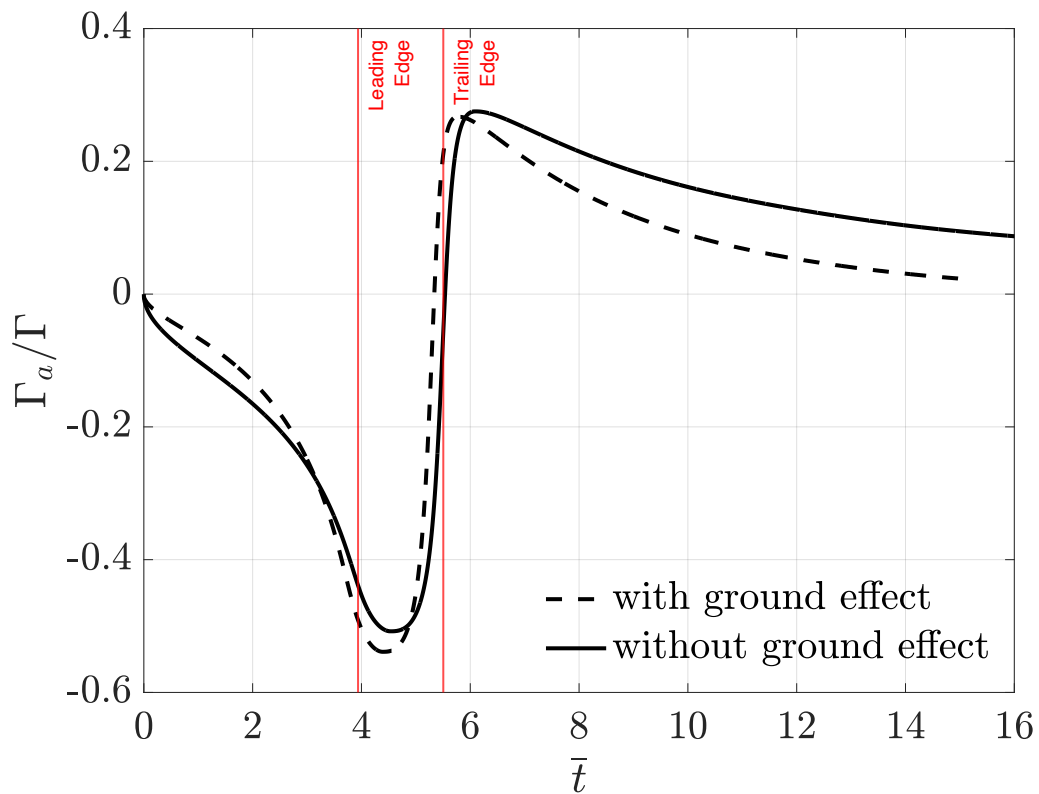


Figure 45: Comparisons of airfoil circulation histories with and without ground effect. $\theta = 0^\circ$.

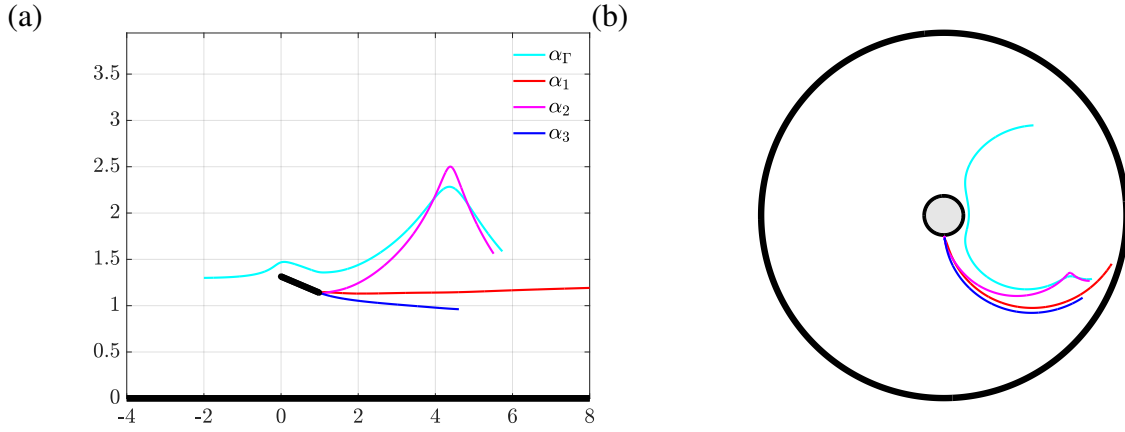


Figure 46: Trajectories of the incident vortex α_Γ , and three shed vortices α_1 , α_2 and α_3 in physical z -plane and mapped ζ -plane. $\theta = -10^\circ$.

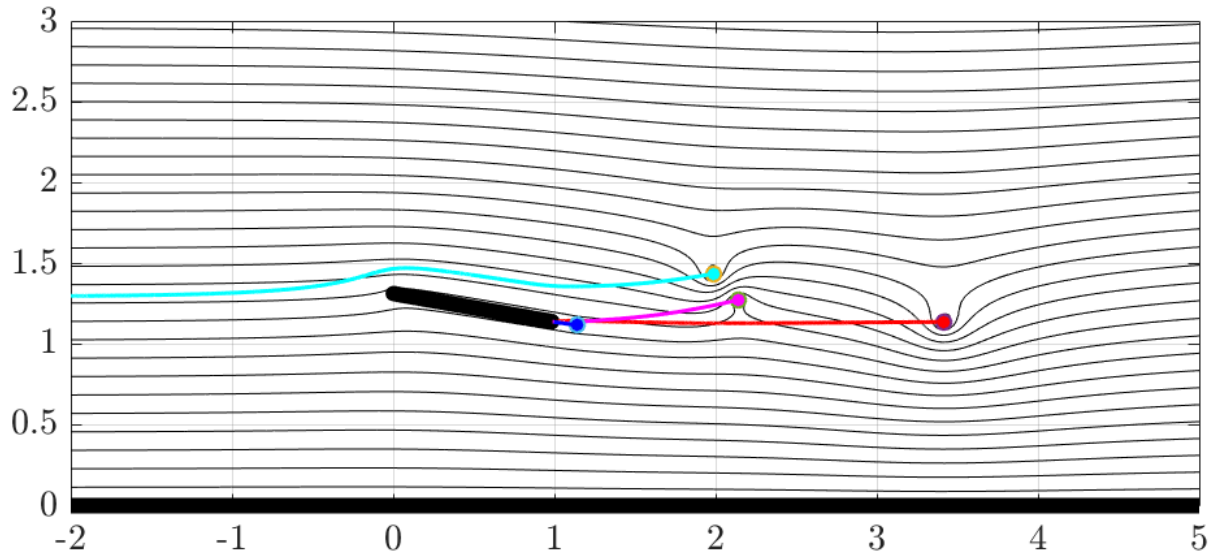


Figure 47: The steady, uniform flow past a flat plate with inclination angle $\theta = -10^\circ$ in ground effect at $\bar{t} = 6.37$, with the Kutta condition specified, for positive angle of attack. The instantaneous locations of the incident vortex and three vortices are marked as dots in cyan, red, magenta and blue. The corresponding previous vortex trajectories are marked in the same color as the dots.

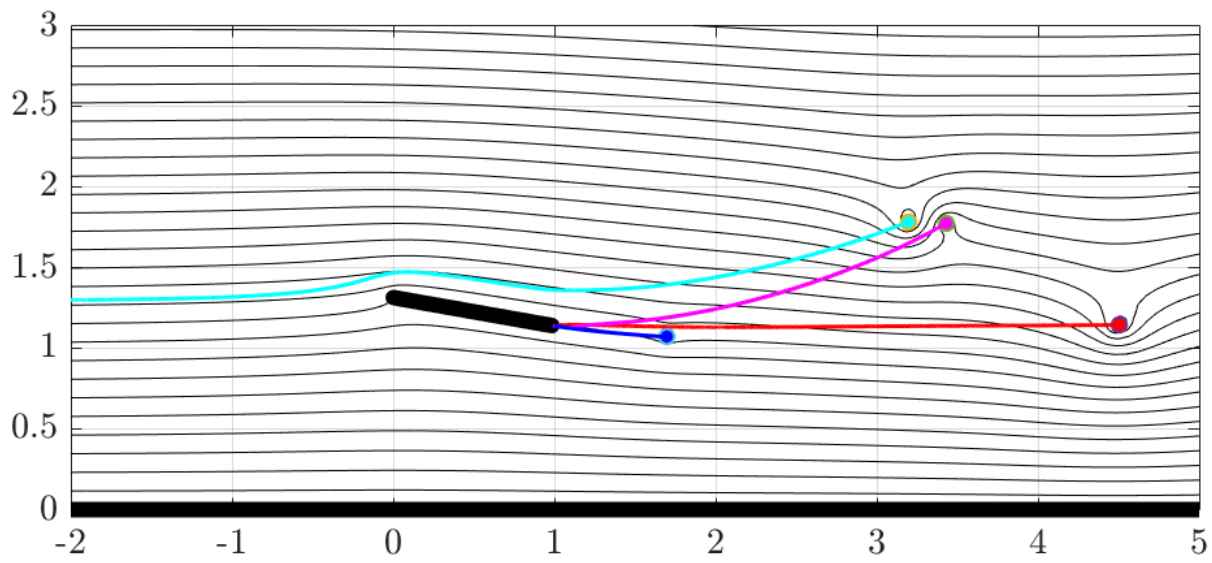


Figure 48: The steady, uniform flow past a flat plate with inclination angle $\theta = -10^\circ$ in ground effect at $\bar{t} = 8.37$, with the Kutta condition specified, for positive angle of attack. The instantaneous locations of the incident vortex and three vortices are marked as dots in cyan, red, magenta and blue. The corresponding previous vortex trajectories are marked in the same color as the dots.

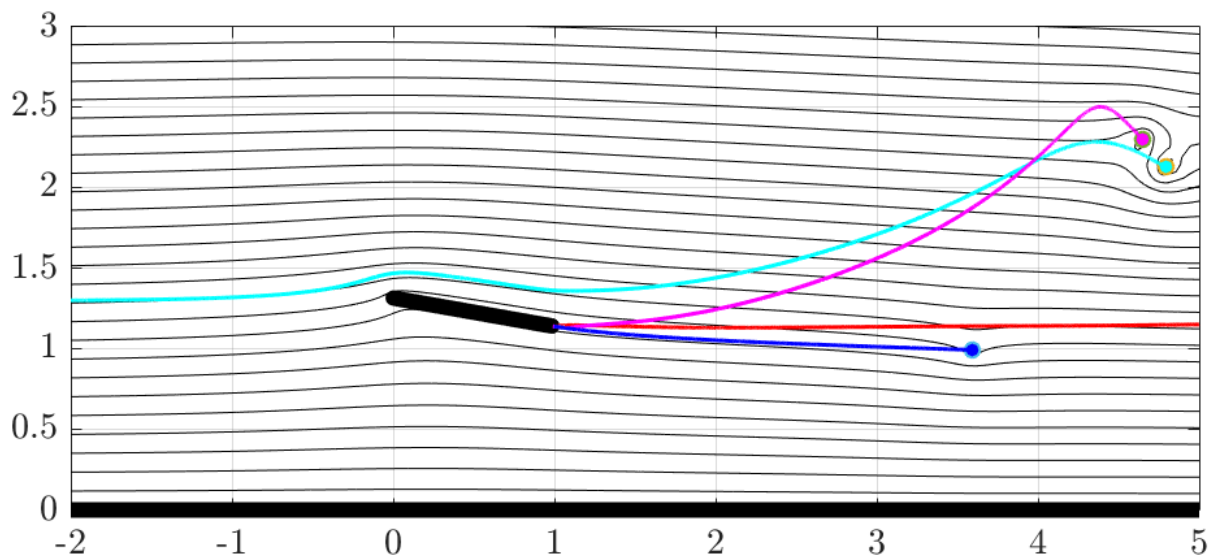


Figure 49: The steady, uniform flow past a flat plate with inclination angle $\theta = -10^\circ$ in ground effect at $\bar{t} = 13.37$, with the Kutta condition specified, for positive angle of attack. The instantaneous locations of the incident vortex and three vortices are marked as dots in cyan, red, magenta and blue. The corresponding previous vortex trajectories are marked in the same color as the dots.

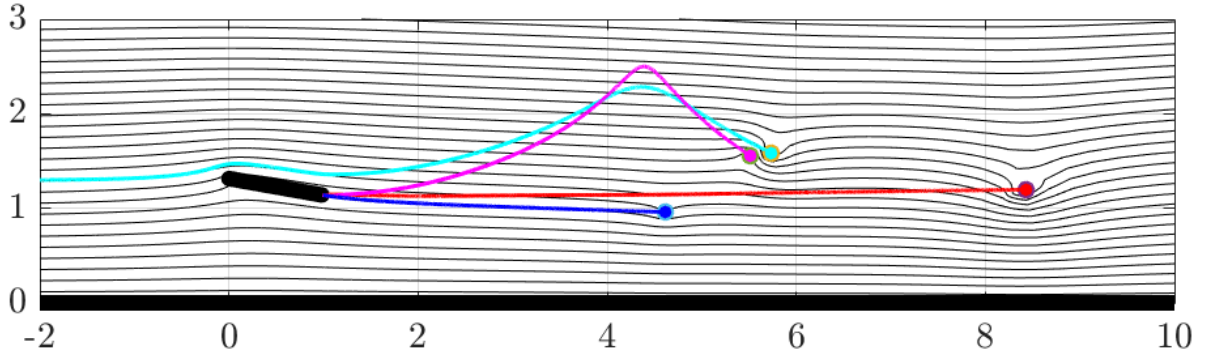


Figure 50: The steady, uniform flow past a flat plate with inclination angle $\theta = -10^\circ$ in ground effect at $\bar{t} = 15.62$, with the Kutta condition specified, for positive angle of attack. The instantaneous locations of the incident vortex and three vortices are marked as dots in cyan, red, magenta and blue. The corresponding previous vortex trajectories are marked in the same color as the dots.

Brown and Michael equation, and the Kutta condition is satisfied to guarantee the flow leaving the trailing edge smoothly. We compared the simulation results with ground effect against previous works by Manela (2013) and Chen & Jaworski (2018) in which the ground effect is not considered. We obtained similar results for the vortex trajectories, but observed the effect of ground in stabilizing the airfoil circulation. This study highlights the ability of the Schottky-Klein prime function mapping procedure to be adapted for complex, multibody vortex interactions, including but certainly not limited to an inviscid ground plane.

10 SOUND FROM AEROELASTIC VORTEX-FIBER INTERACTIONS

J. W. Jaworski

10.1 Background

The aeroelastic and acoustic analyses performed in previous sections motivated a related research effort involving the vortex (turbulence) interactions with a flexible surface element, which is inspired in part by a potential noise-reduction wing feature of an owl wing.

Owls are believed to fly in effective silence as a result of three distinctive and unique physical features Graham (1934): a comb of stiff feathers at the wing leading edge, a fringe of flexible filaments at the trailing edges of the feathers and wing, and a soft downy coating on the upper wing surface. In contrast to the amount of research literature devoted to leading- and trailing-edge modifications for turbulence noise mitigation (Hersh *et al.*, 1974; Mathews & Peake, 2018; Ffowcs Williams & Hall, 1970; Crighton & Leppington, 1970; Crighton, 1972b; Cannell, 1975, 1976; Howe, 1992, 1993; Jaworski & Peake, 2013; Cavalieri *et al.*, 2016; Ayton, 2016), the mechanisms for acoustic attenuation by the downy material on owl wings have received relatively little attention. Despite this lack of research activity, Lilley (1998) postulated that the down material was essential to the owl ‘hush kit,’ suggesting that the large bandwidth of noise suppression resulted from a mechanism of the down to eliminate turbulence noise at its source. Clark *et al.* (2014) discovered the unique structure of the down material, which is composed of several flexible barbules (fibers) that interlock to form a porous barrier that offsets the boundary-layer flow and its noise sources from the wing surface by a distance on the order of a millimeter. Their experiments with various porous canopies demonstrated that surface pressure fluctuation level reductions of up to 30dB could be achieved with this small offset, and complementary theoretical models developed in that investigation linked the low-frequency level reductions to the mixing-layer instability occurring at the suspended porous layer. This work led to the development of rigid, streamwise-aligned structures termed ‘finlets’ designed to modify the boundary layer upstream of the trailing edge before encountering the trailing edge, which yielded frequency-dependent far-field noise reductions of up to 10dB on a DU96-W180 airfoil section relative its untreated counterpart (Clark *et al.*, 2015). Jaworski & Peake (2020) provide a detailed review of this and other noise-reduction technologies inspired by the aeroacoustics of owls.

The present study extends the theoretical model of Clark *et al.* (2014) to include the effect of *hydroelastic fiber motions* on the motion of the incoming vortex and the resulting noise generation. A single fiber is modeled to represent the physical limit of a sparse distribution of down fibers, as illustrated in Fig. 51. The basic idea here is to determine what role the hydrodynamic movement of the flexible down fibers in the presence of an advecting vortex has on the strength of its acoustic field relative to its stationary-fiber counterpart. Previous works have addressed the vortex noise generation from rigid fibers perpendicular to a half-plane (Kasoev, 1976; Abou-Hussein *et al.*, 2002; Howe, 2003), and the Schwarz-Christoffel conformal mapping of Clark *et al.* (2014) and Lighthill (1973) for arbitrary fiber inclinations enable the analysis of coupled dynamic motions of the vortex and fiber. A generalized mathematical model for a vortex moving past a rigid wedge with a movable fiber extending from its tip is presented in §10.2, which includes the special case of a flexible fiber extending from a half plane investigated in this paper. A theoretical framework for the hydrodynamic coupling between the rotational motion of the fiber to the vortex path is also

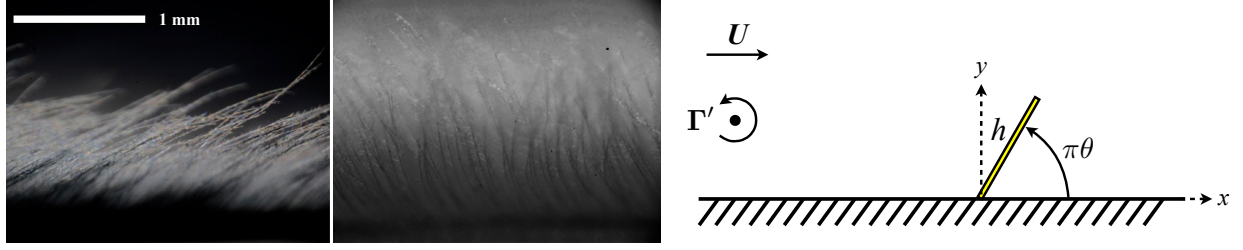


Figure 51: Representation of owl down barbules (great grey owl, left; snowy owl, middle) on the upper surface of flight feathers as a two-dimensional fiber (right) that is elastically restrained to rotate about the origin. A point vortex of strength Γ' represents a turbulent eddy immersed in a uniform fluid flow with speed U .

presented. Section 10.3 investigates the dynamic aeroelastic coupling between the angular motions of the elastically-restrained fiber and the vortex path in a steady mean flow, and the resulting noise generation is estimated from vortex sound theory (Howe, 2003). Conclusions from this investigation are presented in §10.4.

10.2 Mathematical model

10.2.1 Conformal mapping

The mathematical analysis for sound due to a vortex passing round an inclined fiber (cf. Fig. 52) follows from the procedure outlined by Howe (2003) for a vortex passing over a rigid, upright spoiler in a background mean flow, which was extended for arbitrary angles of inclination $\theta \in (0, 1)$ by Clark *et al.* (2014) and whose surface geometry appears in Lighthill (1973) in the context of the Weis-Fogh clap-and-fling lift mechanism. The present paper generalizes the half plane in these works to an arbitrary wedge, which may also be used for vortex sound analyses of trailing edges. The mapping between the physical z -plane and the ζ -plane is achieved using the Schwarz-Christoffel transformation Milne-Thompson (1968).

$$z = \frac{K}{2 - \alpha} (\zeta + 1)^{1-\theta} (\zeta - 1)^{1-\alpha+\theta} \quad (102)$$

The mapping constants b and K are determined by linking the z - and ζ -planes, yielding

$$b = \frac{\alpha - 2\theta}{2 - \alpha} \quad \text{and} \quad K = \frac{(2 - \alpha)h}{f(\alpha, \theta)}, \quad (103)$$

where

$$f(\alpha, \theta) = \frac{2^{2-\alpha}(1 - \theta)^{1-\theta}(1 - \alpha + \theta)^{1-\alpha+\theta}}{(2 - \alpha)^{2-\alpha}} e^{i\pi(1-\alpha)}. \quad (104)$$

The expression $f(\alpha, \theta) = \exp(i\pi(1 - \alpha))g(\alpha, \theta)$ defines real-valued $g(\alpha, \theta)$. The mapping between the z - and ζ -planes is completed by equating the background mean flows far from the fiber, which can be achieved for finite flow speeds only in the special case of the planar wall ($\alpha = 1$) (Batchelor, 1967, pp. 410-412) where $V = KU$.

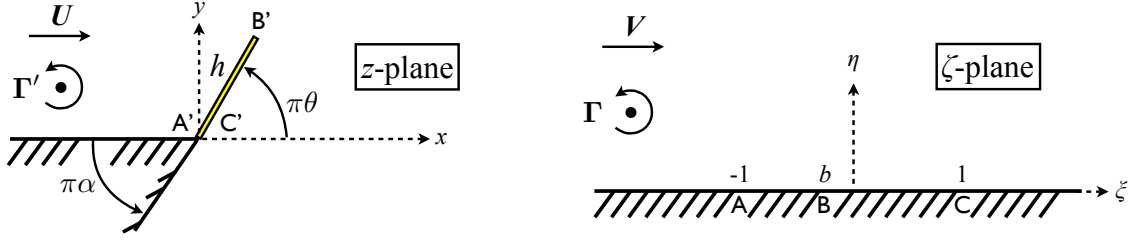


Figure 52: Generalized conformal mapping of a wedge with a finite extension from its tip. The Schwarz-Christoffel transformation relates the physical z -plane to the ζ -plane where the vortex motion problem is solved. The impermeable wall condition is satisfied automatically by adding an appropriate image vortex in the solid boundary that is complemented by a kinematic boundary condition on the fiber. For conformal maps, $\Gamma' = \Gamma$. Uniform background flows are considered only when $\alpha = 1$ for the case of a half plane.

10.2.2 Equations of vortex motion

The nonlinear equations for the motion of the vortex at $z_0(t) = x_0(t) + i y_0(t)$ are

$$\begin{aligned} \frac{d\bar{z}_0}{dt} &= \frac{dx_0}{dt} - i \frac{dy_0}{dt} \\ &= -\frac{i\Gamma}{4\pi} \frac{\zeta''(z_0)}{\zeta'(z_0)} + \left(\frac{i\Gamma}{2\pi} \frac{1}{\zeta(z_0) - \bar{\zeta}(z_0)} + UK - \frac{h^2}{4} \frac{d\theta}{dt} \int_{-1}^1 \frac{|Z_f(\chi, \alpha, \theta)|^2}{(\chi - \zeta(z_0))^2} d\chi \right) \zeta'(z_0), \end{aligned} \quad (105)$$

where

$$\frac{\zeta''(z_0)}{\zeta'(z_0)} = [\theta(\zeta + 1)^{-1} - (\zeta - b)^{-1} + (\alpha - \theta)(\zeta - 1)^{-1}] \zeta'(z_0). \quad (106)$$

Here the primes denote $()' = d/dz$, and the integral

$$I(\zeta, \alpha, \theta) = \int_{-1}^1 \frac{|Z_f(\chi, \alpha, \theta)|^2}{(\chi - \zeta)^2} d\chi \quad (107)$$

follows from the kinematic boundary condition imposed on the moving fiber, which is detailed in appendix A; (128) defines function $|Z_f|$. In general, (105) depends implicitly on the vortex position through $\zeta(z_0)$, which requires the solution of (102) for a given z_0 . The vortex equations of motion (105) can be expressed in dimensionless form,

$$\begin{aligned} \frac{d\bar{Z}}{dT} &= \left\{ -i \frac{f(\alpha, \theta)}{2 - \alpha} \left[\theta(\zeta + 1)^{-1} - (\zeta - b)^{-1} + (\alpha - \theta)(\zeta - 1)^{-1} - \frac{2}{\zeta - \bar{\zeta}} - i \frac{1}{4} \frac{d\theta}{dT} I(\zeta, \alpha, \theta) \right] + \epsilon \right\} \\ &\quad \times (\zeta + 1)^\theta (\zeta - b)^{-1} (\zeta - 1)^{\alpha - \theta}, \end{aligned} \quad (108)$$

where $\zeta = \zeta(Z)$, using the following dimensionless groups:

$$Z = \frac{z_0}{h}, \quad S = \frac{\Gamma}{4\pi h}, \quad T = \frac{tS}{h}, \quad \epsilon = \frac{U}{S}. \quad (109)$$

10.2.3 Dynamic fiber model

The vortex equations of motion (108) hold generally for all fiber angles, which in the dynamic case $\theta = \theta(t)$. The fiber motion may be either prescribed or be the solution of an oscillator model for the perturbation angle $\hat{\theta}$ about elastic equilibrium position θ_0 , such that $\theta = \theta_0 + \hat{\theta}$. The linear oscillator model considered here is

$$I_0 \frac{d^2 \hat{\theta}}{dt^2} + c_\theta \frac{d\hat{\theta}}{dt} + k_\theta \hat{\theta} = M_0, \quad (110)$$

which in nondimensional form may be written as

$$\frac{d^2 \hat{\theta}}{dT^2} + 2\Omega c_d \frac{d\hat{\theta}}{dT} + \Omega^2 \hat{\theta} = \mu^{-1} C_{M_0}. \quad (111)$$

Here $\omega_\theta^2 = k_\theta/I_\theta$, $\Omega = \omega_\theta h/S$, $\mu = I_0/(\rho h^4)$, $c_d = c_\theta/(2I_0\omega_\theta)$, $C_{M_0} = M_0/(\rho S^2 h^2)$, and ρ is the fluid density. The coefficient of moment C_{M_0} is computed by integrating the instantaneous pressure difference across the fiber times its moment arm from the elastic axis. Four integral terms arise that represent the individual contributions from the unsteady and dynamic head portions in the linearized Bernoulli equation evaluated on the fiber that sum to C_{M_0} . These integrals are first formulated in the z -plane and then recast as integrations along the ξ -axis in the ζ -plane.

The total unsteady moment coefficient in the anti-clockwise direction is

$$C_{M_0} = \frac{2 - \alpha}{g^2(\alpha, \theta)} \left[M_1 \frac{d^2 \theta}{dT^2} + M_2 \left(\frac{d\theta}{dT} \right)^2 + M_3 \left| \frac{dZ}{dT} \right| + M_4 \right]. \quad (112)$$

Time-dependent coefficients M_1 , M_2 , and M_3 are integrals resulting from the unsteady pressure contribution and are given appendix B. Coefficient M_4 is linked to the moment contribution associated with the dynamic head,

$$M_4 = -\frac{1}{2} \oint_{-1}^1 \frac{(A^2 + \epsilon^2 - 2A\epsilon \cos \pi\alpha)(1 - \xi^2)}{\xi - b} d\xi, \quad (113)$$

where the Cauchy principal value must be taken, and

$$A = \frac{g(\alpha, \theta)}{2 - \alpha} \left(\frac{4\eta_0(T)}{(\xi - \xi_0(T))^2 + \eta_0^2(T)} - \frac{1}{4} \frac{d\theta}{dT} I(\xi, \alpha, \theta) \right). \quad (114)$$

Moment integrals (113) and (129)-(131) determine the instantaneous hydrodynamic moment (112) in (111) acting on the flexible fiber, which can be evaluated using standard numerical integration routines. A fully hydroelastic numerical simulation of the vortex-fiber interaction round a generalized wedge is made possible with the present approach by integrating (108) and (111) forward in time. In addition to the owl down model problem considered in this paper, this numerical framework also enables the examination of vortex sound generation from link models of flexible trailing edges with finite or zero trailing-edge angles for cases where $\alpha < 1$, which is beyond the scope of this paper but will be pursued in future work. In addition, the singular flow velocity at the fiber tip in the present theoretical model may be regularized by vortex shedding, which could be pursued using point vortices or other low-order models (Howe, 1997; Michelin & Llewellyn Smith, 2009; Manela & Huang, 2013; Darakananda & Eldredge, 2019; Eldredge & Jones, 2019) to investigate this vorticity production on the acoustic signature.

10.2.4 Acoustic emission

In the special case of a fiber protruding from a half plane ($\alpha = 1$, cf. Fig. 51), the acoustic emission from the fiber can be expressed in terms of the vortex path, the fiber angular motion, and the streamwise component of the Kirchhoff vector, provided that the fiber is acoustically compact. Under this assumption, Howe (2003, p. 193) defines a function

$$\mathcal{W}(Z) = \frac{d}{dz}(K\zeta) = (\zeta + 1)^\theta (\zeta - b)^{-1} (\zeta - 1)^{\alpha-\theta}, \quad (115)$$

such that the reduced acoustic pressure in the far field due to the vortex motion is

$$P_T(\mathbf{x}, t) \equiv \frac{p_T(\mathbf{x}, t)}{\rho S^2 \sqrt{M} \sin \beta (h/|\mathbf{x}|)^{1/2}} \approx 2^{5/2} \frac{\partial}{\partial T} \int_0^\infty \text{Im} \left[\mathcal{W}(Z) \frac{dZ}{d\hat{T}} ([T] - \lambda^2) \right] d\lambda. \quad (116)$$

The second term of the integrand implies the functional dependence $dZ/d\hat{T}([T] - \lambda^2) = u([T] - \lambda^2) + iv([T] - \lambda^2)$, and $\hat{T} = [T] - \lambda^2$. Equation (116) measures the acoustic pressure at far-field location $\mathbf{x}$, which is inclined at an angle β measured clockwise from the y -direction. In (116), c_0 is the isentropic speed of sound, the Mach number is defined as $M = S/c_0$, and $[T]$ is the nondimensional retarded time (Clark *et al.*, 2014). It is worthwhile to note that the influence of the fiber position on the vortex sound is handled intrinsically by the time dependence of θ in $\mathcal{W}(Z)$.

The acceleration of the fiber generates an additional dipolar acoustic emission that must also be included in the total acoustic field (Manela & Huang, 2013). The pressure contribution from the fiber acceleration is (Howe, 2003, p. 127)

$$p_f(\mathbf{x}, t) = \rho \int_{-\infty}^\infty \oint_{S_f} \frac{\partial v_n}{\partial \tau}(\mathbf{y}, \tau) G(\mathbf{x}, \mathbf{y}, t - \tau) dS_f(\mathbf{y}) d\tau, \quad |\mathbf{x}| \rightarrow \infty, \quad (117)$$

where v_n is the velocity normal to the moving solid boundary S_f . The compact Green's function for a compact body at a plane wall is (Howe, 2003, p. 138)

$$G(\mathbf{x}, \mathbf{y}, t - \tau) \approx \frac{x Y_1}{\pi \sqrt{2c_0} |\mathbf{x}|^{3/2}} \frac{\partial}{\partial t} \left\{ \frac{H(t - \tau - |\mathbf{x}|/c_0)}{\sqrt{t - \tau - |\mathbf{x}|/c_0}} \right\}, \quad |\mathbf{x}| \rightarrow \infty, \quad (118)$$

which involves Heaviside function H . The term Y_1 denotes the first component of the Kirchhoff vector that corresponds to the ideal flow of unit speed in the x direction passing round the fiber, which in this configuration is $Y_1 = \text{Re}[K\zeta] = h\xi/g(\alpha = 1, \theta)$. Equation (117) may be reorganized using (102), (109), and (118) to identify the reduced acoustic pressure from fiber acceleration,

$$P_f(\mathbf{x}, t) \approx -\frac{2^{1/2}}{\pi} \frac{\partial}{\partial T} \int_0^\infty \left[\frac{I_f(\theta)}{g^3(\alpha = 1, \theta)} \frac{d^2 \theta}{d\hat{T}^2} \right] ([T] - \lambda^2) d\lambda, \quad (119)$$

using the same pressure scaling as in (116). The expression in the brackets of (119) is a function of $\hat{T} = [T] - \lambda^2$, and

$$I_f(\theta) = \int_{-1}^{1-2\theta} F(\xi, \theta) d\xi - \int_{1-2\theta}^1 F(\xi, \theta) d\xi, \quad (120)$$

where

$$F(\xi, \theta) = \xi[\xi - (1 - 2\theta)](1 + \xi)^{1-2\theta}(1 - \xi)^{2\theta-1}. \quad (121)$$

The total reduced pressure in the acoustic far field is $P(\mathbf{x}, t) = P_T(\mathbf{x}, t) + P_f(\mathbf{x}, t)$.

10.3 Results

Previous works have considered the sound generated by a vortex passing over an oscillating (Lighthill, 1973) or a stationary, upright (Kasoev, 1976; Abou-Hussein *et al.*, 2002; Howe, 2003) or leaning (Clark *et al.*, 2014) two-dimensional barrier emerging from a half plane. The mathematical model presented in §10.2 extends beyond these works in two ways. First, the unsteady fiber motion and resulting hydrodynamic moment due to an incident vortex are modeled. Second, the half plane is generalized as a solid wedge, which enables vortex motion prediction round edges of arbitrary wedge angle with an elastic extension. In this paper, the vortex motion in a uniform flow and the resulting acoustic emission for prescribed, harmonic fiber motions and aeroelastic motions due to hydrodynamic vortex-fiber coupling are evaluated to anticipate their effect on the far-field acoustic signature.

10.3.1 Prescribed fiber motion

To form a basis for comparison against prior works, all vortex paths investigated herein start far upstream of the fiber at $Z = -10 + 0.75i$. The fiber angular position is described by $\theta(T) = \theta_0 + \theta_1 \sin(\omega T + \psi)$, where $\theta_0 = 0.5$ and $\theta_1 = 0.1$ to produce a harmonic oscillation about the upright fiber position. The relative starting positions of the vortex and fiber are phased by $\psi = 0, \pi/2, \pi$, and $3\pi/2$, and the frequency is varied across $\omega = 0.1, 1$, and 10 . Vortex trajectories and acoustic emissions are investigated for scenarios without ($\epsilon = 0$) and with ($\epsilon = 1$) a background mean flow.

In the absence of a mean flow, Fig. 53 shows the influence of the fiber oscillation frequency and the initially phasing between the vortex and fiber on the reduced acoustic pressure, as compared to the limiting case of a stationary, upright fiber, whose results have been verified against the literature (Abou-Hussein *et al.*, 2002; Howe, 2003). To facilitate comparison, the retarded time is referenced with respect to when the vortex crosses $x = 0$. For $\omega = 0.1$ and $\omega = 1$, the effect of the change in phase angle ψ is predominantly to shift the acoustic signature left or right, which agrees with the shift in the vortex path in Figs. 54a and 54b. The extrema of the acoustic signals increase with the oscillating frequency ω , with the exception of $\omega = 1$ when $\psi = \pi/2$ or $3\pi/2$. In these cases, Fig. 54b indicates that the phase between the vortex and fiber motion increases the minimum distance between the two, leading to a weaker acoustic emission. For the largest oscillating frequency considered, $\omega = 10$, it is clear that prescribed fiber motion modulates the acoustic signal and leads to large increases in the acoustic noise level. These results are consistent with the fact that a larger relative velocity between the vortex and fiber when in close proximity leads to peaks in the acoustic signature. The high oscillation frequency also affects the vortex trajectory, bringing the vortex nearer the fiber and breaking strongly the left-right symmetry of the vortex path.

The qualitative parametric trends for $\epsilon = 0$ also hold for when a mean flow of magnitude $\epsilon = 1$ is introduced in Figs. 55 and 56. With mean flow, a case is again found for $\omega = 1$ when the oscillating fiber creates a weaker acoustic output than for a fixed fiber, occurring instead at $\psi = 0$, which indicates that the background flow also plays a substantial role in the phasing between the vortex and fiber motions. As anticipated, the acoustic pressure extrema occur at greater magnitudes due to the enhanced relative velocities between the vortex and fiber when in closest proximity.

These results for vortex sound generated by a fiber oscillating with prescribed motion suggest at least two strategies in which fiber unsteadiness could weaken the resulting acoustic emission.

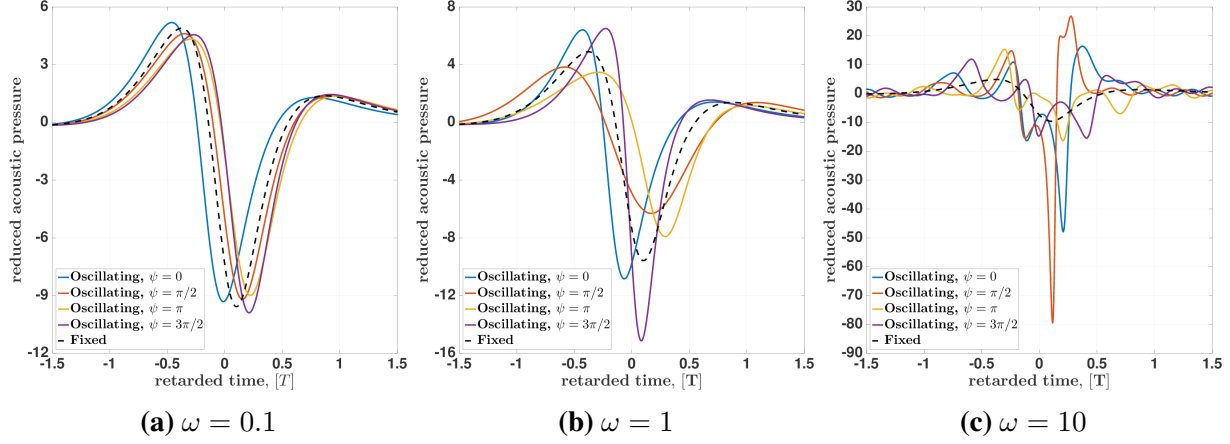


Figure 53: Dependence of the acoustic signature on oscillation frequency of the fiber ω and initial phase ψ relative to the fixed fiber case without a mean flow, $\epsilon = 0$: (a) $\omega = 0.1$; (b) $\omega = 1$; (c) $\omega = 10$. The retarded time $[T]$ is referenced to the instant the vortex passes $x = 0$.

First, if the phasing of the motion influences the vortex trajectory in a way that increases the minimum distance between the vortex and fiber, the peaks of the acoustic signature are reduced. Second, the acoustic emission is reduced when the speed of the vortex in close proximity to the fiber is reduced. These strategies based on the vortex-fiber interplay are supported by the fact that the reduced acoustic pressure P_r due to the vortex (116) dominates the total acoustic field. The radiation due to the angular acceleration of the fiber is largest (but still relatively insignificant) for $\omega = 10$ with an amplitude of $|P_f| \approx 1.04$; this acoustic pressure amplitude changes by a factor of ω^2 to render this contribution negligible at smaller fiber vibration frequencies.

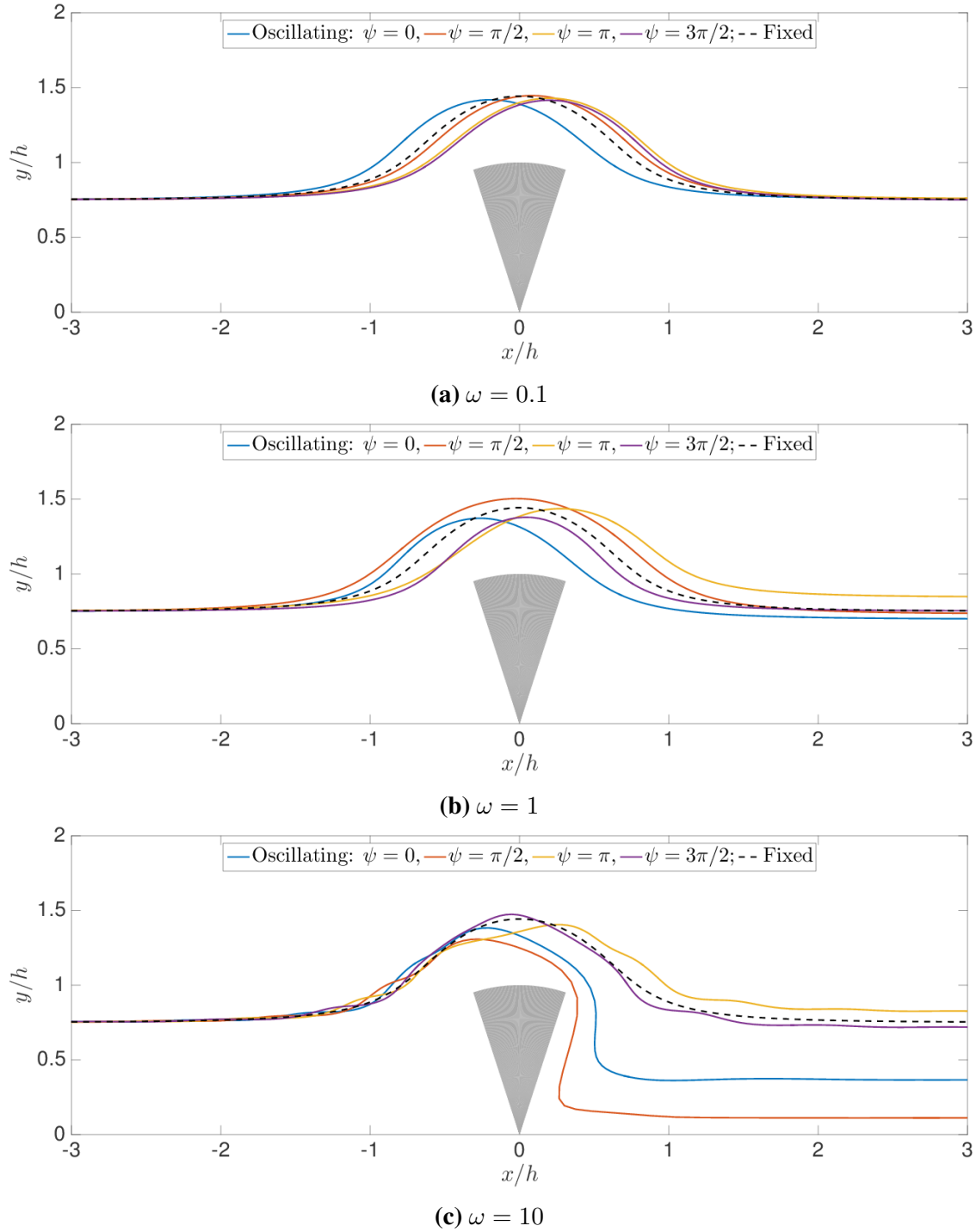


Figure 54: Vortex trajectories past an oscillating fiber of frequency ω with initial phase ψ without a mean flow, $\epsilon = 0$: (a) $\omega = 0.1$; (b) $\omega = 1$; (c) $\omega = 10$. The gray sector indicates the region in which the fiber oscillates.

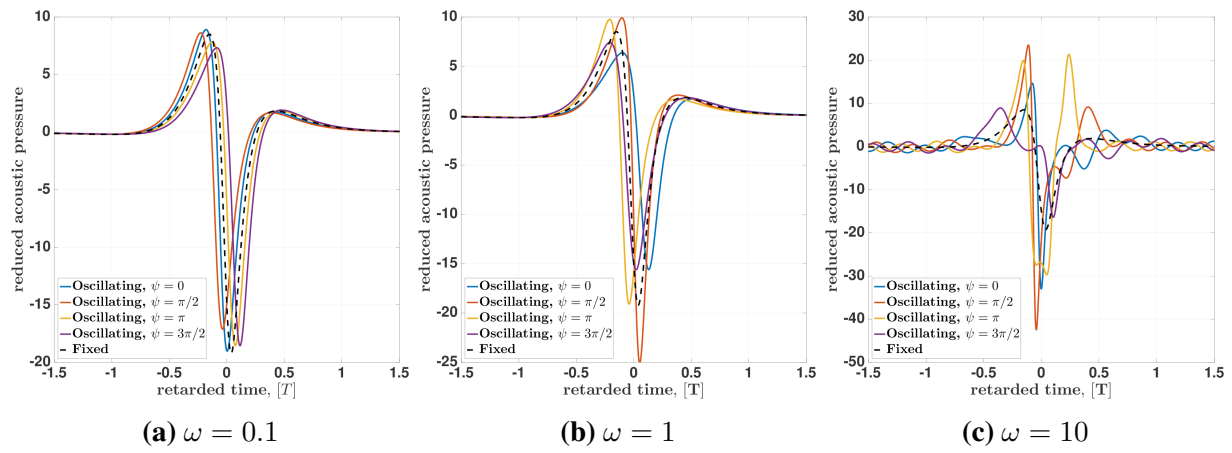


Figure 55: Dependence of the acoustic signature on oscillation frequency of the fiber ω and initial phase ψ relative to the fixed fiber case with a mean flow, $\epsilon = 1$: (a) $\omega = 0.1$; (b) $\omega = 1$; (c) $\omega = 10$. The retarded time $[T]$ is referenced to the instant the vortex passes $x = 0$.

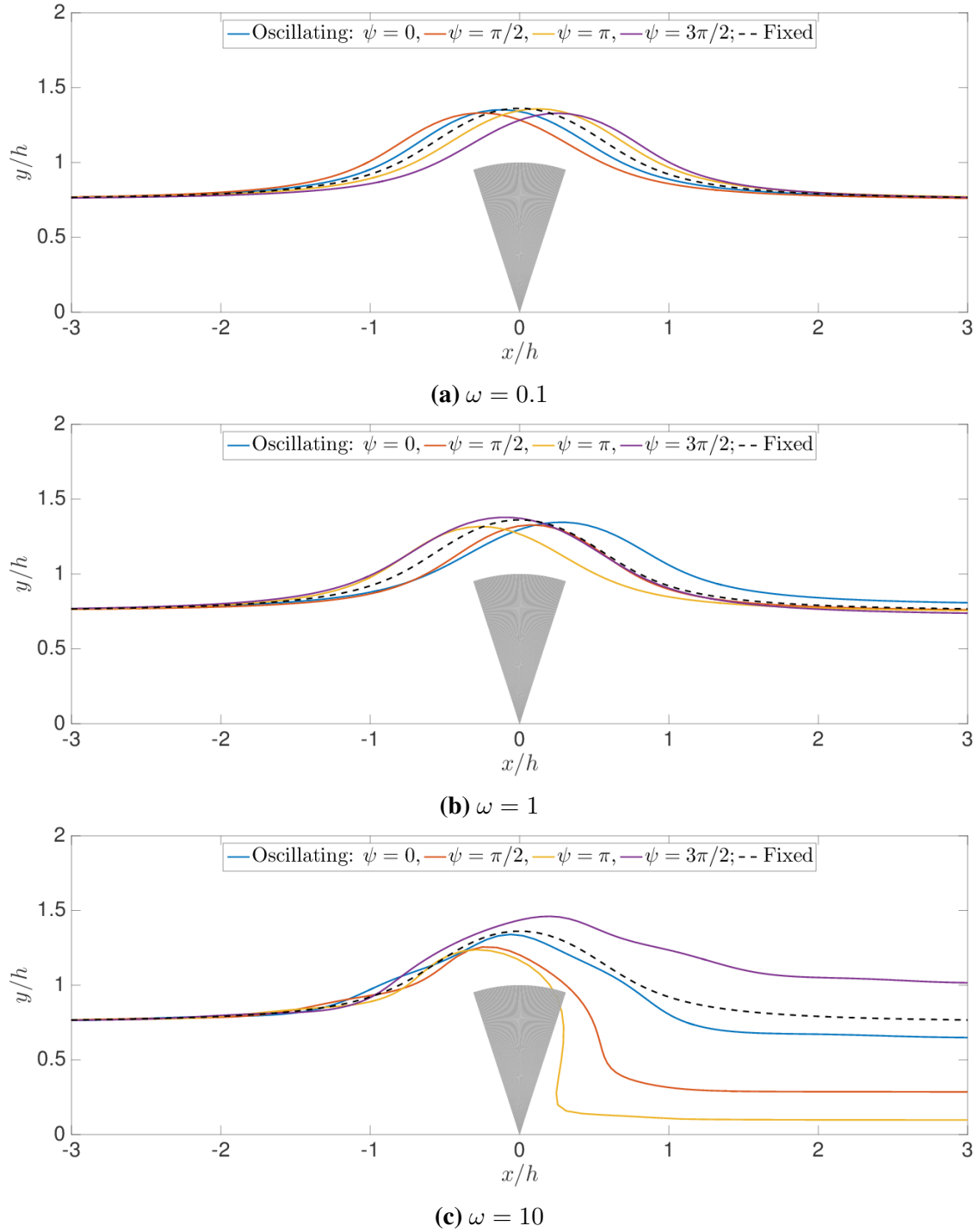


Figure 56: Vortex trajectories past an oscillating fiber of frequency ω with initial phase ψ with mean flow, $\epsilon = 1$: (a) $\omega = 0.1$; (b) $\omega = 1$; (c) $\omega = 10$. The gray sector indicates the region in which the fiber oscillates.

10.3.2 Aeroelastic vortex-fiber interactions

The modeling framework is now demonstrated for the aeroelastic case of a incident vortex with initial position $Z = -10 + 3i$ and a mean background flow with strength $\epsilon = 1$, recalling that ϵ is defined by (109) as the ratio of the mean flow speed to the characteristic vortex speed. The elastic equilibrium fiber position is set to $\theta_0 = 0.5$, and the aeroelastic parameters representative of an owl down fiber in air are $\Omega = 0.761$, $c_d = 1.85 \times 10^{-3}$, and $\mu = 0.130$, as detailed in appendix C. Three vortex-fiber interaction scenarios are explored and compared to one another with respect to the predicted vortex paths and acoustic signatures: (i) a rigid fiber; (ii) a flexible fiber that is initially still at the elastic equilibrium position $\theta_0 = 0.5$; and (iii) a flexible fiber that is initially still but is displaced away from elastic equilibrium at $\theta = 0.6$. In the latter case, the fiber is able to oscillate initially under free vibration before the aeroelastic coupling to the incident vortex affects its dynamical response. Lastly, some qualitative changes in the acoustic emission and fiber dynamics are investigated for various initial vertical locations of the incoming vortex.

Figure 57a presents and compares the far-field acoustic prediction for each vortex-fiber scenario. For the aeroelastic parameters considered here, the case of the fiber initially at the equilibrium position yields an acoustic signature that is indistinguishable from the rigid fiber case. The corresponding low-amplitude angular displacement history shown in Fig. 57b corroborates the minor influence of the fiber motion on the noise generation. However, the impact of aeroelastic vortex-fiber coupling on the acoustic emission is more substantial for the case when the fiber is initially displaced away from static equilibrium. In this case, Fig. 57b demonstrates that the frequency of fiber oscillation is modulated higher when under the influence of the incident vortex. The acoustic signal for this case is suggestive of a superposition of the rigid-fiber noise signature with higher-frequency oscillatory content that is due to the fiber oscillations. This high-frequency content is modulated in the same manner as the time history of θ in Fig. 57b and indicates that the changes in the acoustic signature arise primarily from the fiber oscillatory motions; the fact that the vortex paths of these three cases are virtually coincident lends support to this claim. The additional noise component due to fiber motions increases the peak acoustic pressure value in this case.

Figure 58 plots the paths of the incident vortex given the initial position $Z_0 = -6 + Y_0 i$ with $\epsilon = 1$. The coincident paths for the aeroelastic cases mentioned above fall on the line for $Y_0 = 0.3$, and the influence of the fiber on the upward motion of the incoming vortex increases with decreasing Y_0 . Figure 59 shows the corresponding acoustic signals and displacement histories of the fiber, where the ringing of the fiber at its natural frequency at large Y_0 in Fig. 59b becomes suppressed by stronger vortex-fiber interactions for lower initial vortex placements. This change in fiber motion is accompanied in Fig. 59a by stronger acoustic peaks and changes in the acoustic response waveform as Y_0 decreases. However, it remains unclear from these results alone whether the fiber could be aeroelastically tuned instead for vortex sound suppression due to the weak differences between the cases of the rigid and flexible cases of the fiber initially at the equilibrium position.

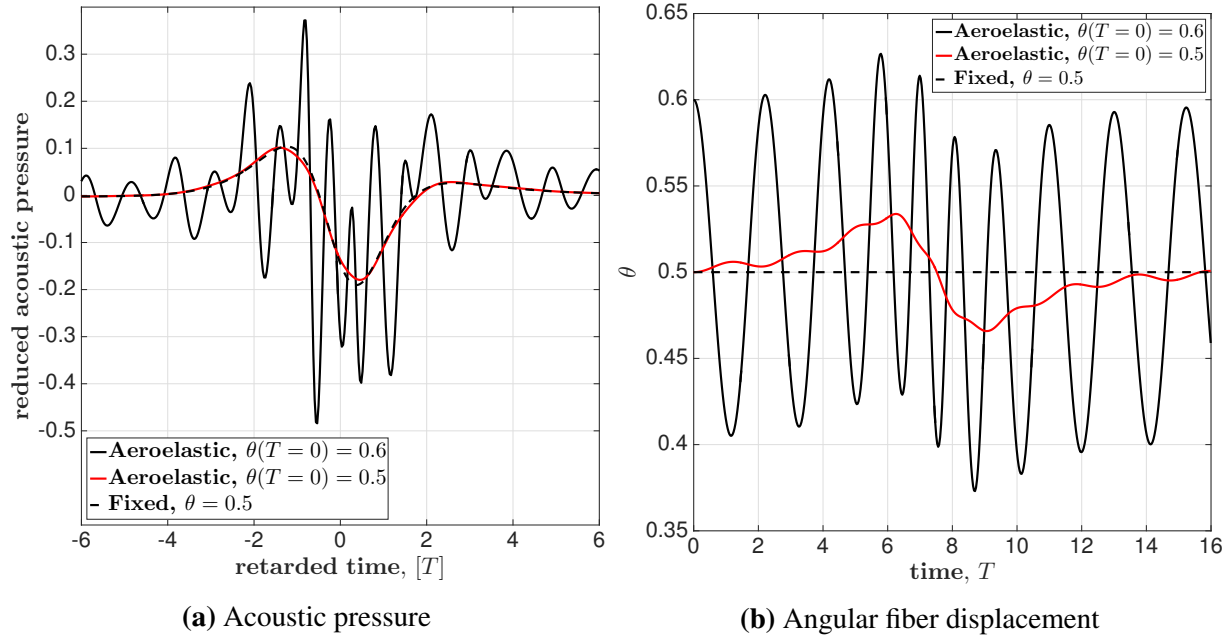


Figure 57: Dynamic and acoustic responses of a flexible fiber immersed in a uniform flow that is aeroelastically coupled to an incident point vortex, $\epsilon = 1$: (a) far-field acoustic pressure as a function of retarded time; (b) angular fiber displacement as a function of nondimensional time. Initial vortex placement is $Z_0 = -10 + 3i$.

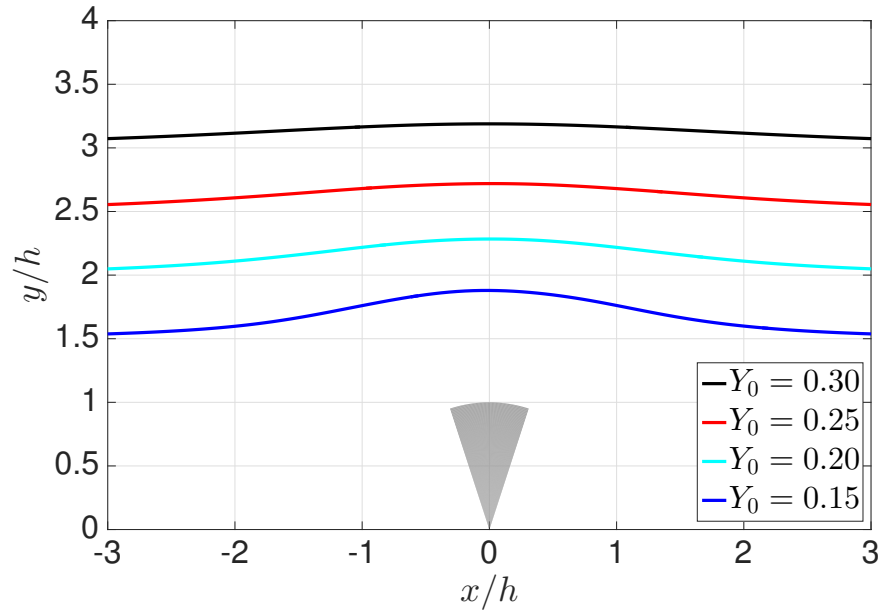


Figure 58: Trajectories of an incident point vortex passing a flexible fiber of frequency that rotates under aeroelastic influence. Initial vortex placement $Z_0 = -6 + Y_0 i$. The gray sector indicates the region in which the fiber oscillates during the strongest encounter when $Y_0 = 0.15$.

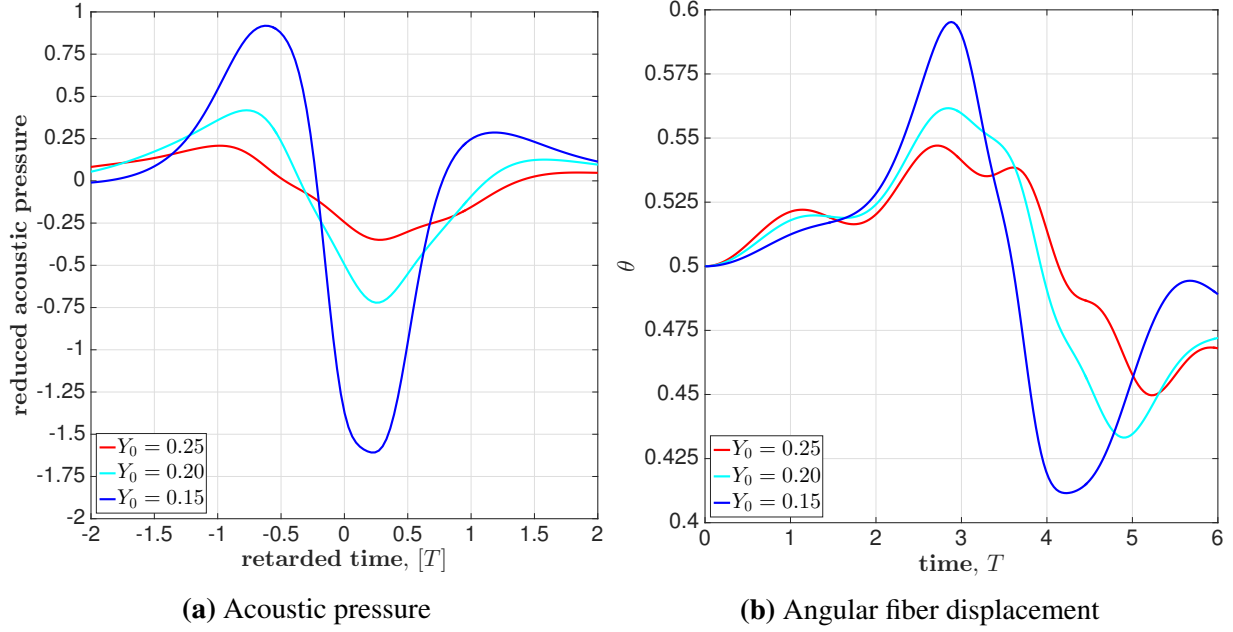


Figure 59: Dynamic and acoustic responses of a flexible fiber immersed in a uniform flow that is aeroelastically coupled to an incident point vortex as a function of initial vertical position, $\epsilon = 1$: (a) far-field acoustic pressure as a function of retarded time; (b) angular fiber displacement as a function of nondimensional time. Initial vortex placement $Z_0 = -6 + Y_0 i$.

10.4 Conclusion

A mathematical framework is developed to model the nonlinear trajectory of a vortex under self-induced motion moving past an oscillating two-dimensional barrier in a wedge. The wedge formulation enables subsequent analysis of roughness and trailing-edge noise problems with an elastic extension. In the limit of the wedge forming a half-plane, the problem of vortex-noise generation from a single two-dimensional barrier is carried out, corresponding to the limit of a sparse distribution of owl down barbules (pennula) on the upper wing surface. The prescribed harmonic motion of the barrier leads generally to an increase in the extrema in the acoustic signature as the barrier oscillation frequency and background mean flow velocity are increased. However, the phase between the vortex path and barrier motion can lead to noise reductions by altering the vortex trajectory to increase the minimum distance between the vortex and barrier, and by reducing the relative speed between the vortex and barrier when in close proximity. For the range of parameters considered, the acoustic emission of the vortex interaction with the solid surface dominates the acoustic radiation attributed solely to the fiber motion.

Aeroelastic parameters representative of an isolated owl down fiber in air furnish a preliminary assessment of vortex-fiber coupling effects on the motion of the incident vortex and the resulting acoustic emission. For the set of parameters considered, the acoustic signatures of a rigid or flexible fiber that is at static equilibrium at the initial instant are virtually identical. If the flexible fiber is initially displaced away from equilibrium, the aeroelastic coupling between the incident vortex and fiber modulates the fiber dynamic response, and the acoustic response appears as a superposition of the rigid fiber case with higher-frequency sound that depends upon the fiber motion. Preliminary

exploration of aeroelastic effects on the noise production and fiber dynamics indicate that at close proximity the coupled vortex-fiber interaction dominates the free vibration dynamics of the fiber, which is accompanied by a change in the waveform of the acoustic response. Additional work is needed to determine whether or not the hydroelastically-coupled motion of the vortex and barrier would lead to an increase or decrease in the acoustic level, and in case of the latter event, the parametric ranges where a reduction is possible. Further research into the vortex sound generated by flexible trailing edges is also desired and may be pursued using the present dynamic conformal mapping framework.

10.5 Appendix A: Complex potential of kinematic boundary condition

The physical requirement that the flow cannot penetrate the (moving) solid boundary produces an additional term that must be included in the total complex velocity of the vortex-fiber system. This kinematic boundary condition may be written in vector form as

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n} \quad \text{on } \partial D, \quad (122)$$

where $\mathbf{u}$ is the fluid velocity, $\mathbf{n}$ is the normal vector into the fluid on the fluid-solid boundary ∂D , and $\mathbf{U}$ is the velocity of the boundary, which is identically zero except on the fiber surface. Boundary condition (122) may be reexpressed in complex form by noting that along ∂D , which is parameterised here by the arc length s , the tangent vector $\mathbf{t}$ is dz/ds and the normal vector $\mathbf{n}$ becomes idz/ds . The complexification of the velocity vectors to $u(s)$ and $U(s)$ converts (122) into

$$\text{Re} \left[\bar{u}(s) \times i \frac{dz}{ds} \right] = \text{Re} \left[\bar{U}(s) \times i \frac{dz}{ds} \right]. \quad (123)$$

One can clearly write $\bar{u} = dw/dz$ by introducing complex velocity potential $w(z)$ that is associated with the kinematic boundary condition, which enables the bracketed term on the left-hand side to be expressed compactly as idw/ds via the chain rule. The angular velocity of the fiber is $i\pi z d\theta/dt$ for points z along the fiber, which permits (123) to be expressed as

$$\text{Re} \left[i \frac{dw}{ds} \right] = \text{Re} \left[\pi \frac{d\theta}{dt} \bar{z} \frac{dz}{ds} \right]. \quad (124)$$

The identity $\frac{d}{ds}|z|^2 = 2\text{Re} \left[\bar{z} \frac{dz}{ds} \right]$ enables (124) to be directly integrated in s , where the integration constant may be set to zero without loss of generality. Therefore, the complex velocity associated with kinematic boundary condition satisfies

$$\text{Re} [iw] = \frac{\pi}{2} \frac{d\theta}{dt} |z|^2 \quad \text{on } \partial D. \quad (125)$$

Now suppose a kinematic boundary condition mapped to the ζ -plane (cf. Fig. 52) where $W(\zeta) \equiv w(z(\zeta))$. The goal is to determine an analytic function $\widetilde{W}(\zeta) = iW(\zeta)$ such that the real part of $\widetilde{W}$ equals the right-hand side of (125). Straightforward application of the Cauchy integral formula for Dirichlet boundary value problems on a half plane yields

$$\widetilde{W}(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\frac{\pi}{2} \frac{d\theta}{dt} |z_t(\chi)|^2}{\chi - \zeta} d\chi, \quad (126)$$

where $z_f(\xi)$ denotes the values z on the fiber surface in the mapped ζ -plane. Therefore, the complex velocity potential of interest which satisfies the kinematic boundary condition simplifies to

$$W(\zeta) = -\frac{h^2}{4} \frac{d\theta}{dt} \int_{-1}^1 \frac{|Z_f(\chi)|^2}{\chi - \zeta} d\chi. \quad (127)$$

The integral in (127) has been put into nondimensional form using $z_f = hZ_f$, where

$$|Z_f(\xi)| = \frac{(1 + \xi)^{1-\theta}(1 - \xi)^{1-\alpha+\theta}}{g(\alpha, \theta)}. \quad (128)$$

Note that the dependence of (128) on θ makes $|Z_f|$ an implicit function of time.

10.6 Appendix B: Unsteady moment integrals

Unsteady contributions from the linearized Bernoulli equation to the total hydrodynamic moment (112) depend on the integrals

$$M_1 = \int_{-1}^1 I_{M_1}(\xi, \alpha, \theta) \kappa(\xi, \alpha, \theta) d\xi, \quad (129)$$

$$M_2 = \int_{-1}^1 I_{M_2}(\xi, \alpha, \theta) \kappa(\xi, \alpha, \theta) d\xi, \quad (130)$$

$$M_3 = \left(\frac{\partial \xi_0}{\partial s^*} + \frac{1}{\eta_0} \frac{\partial \eta_0}{\partial s^*} \right) \int_{-1}^1 \frac{4\eta_0}{(\xi - \xi_0)^2 + \eta_0^2} \kappa(\xi, \alpha, \theta) d\xi, \quad (131)$$

where

$$I_{M_1}(\xi, \alpha, \theta) = \frac{1}{4} \oint_{-1}^1 \frac{|Z_f(\chi, \alpha, \theta)|^2}{\chi - \xi} d\chi, \quad (132)$$

$$I_{M_2}(\xi, \alpha, \theta) = -\frac{1}{2} \oint_{-1}^1 \frac{m(\chi, \alpha, \theta)}{\chi - \xi} d\chi, \quad (133)$$

$$\kappa(\xi, \alpha, \theta) = (1 + \xi)^{1-2\theta} (\xi - b)(1 - \xi)^{1-2\alpha+2\theta}, \quad (134)$$

$$m(\chi, \alpha, \theta) = \left[\ln \frac{1 + \chi}{1 - \chi} + \ln \frac{1 - \alpha + \theta}{1 - \theta} \right] \frac{(1 + \chi)^{2(1-\theta)} (1 - \chi)^{2(1-\alpha+\theta)}}{g^2(\alpha, \theta)}. \quad (135)$$

Cauchy principal values must be taken where indicated by the dashed integral, and it is implicitly understood that $\theta = \theta(T)$ such that (129)-(131) are functions of time. Note that the vortex position Z maps to point $\zeta = \xi_0(T) + i\eta_0(T)$ in the ζ -plane. The partial derivatives $\partial \xi_0 / \partial s^*$ and $\partial \eta_0 / \partial s^*$ are readily found from differentiating (102) along a streamline with nondimensional arc length $s^* = s/h$ passing through the instantaneous vortex location and by taking the real or imaginary part.

10.7 Appendix C: Estimation of parameters based on owl pennula

Approximate order-of-magnitude estimates of the fiber dynamics parameters are determined in this appendix based on physical characteristics of flexible pennula on the upper surface of owl wings (cf. Fig. 51). In the sparsely-distributed limit, an isolated pennulum is modeled as a cantilevered elastic cylindrical rod of diameter d . Assuming a linear load profile along the length of the fiber, the equivalent torsional spring stiffness at the root of a rigid fiber that yields the same fiber tip deflection of an elastic rod with the same load profile is $k_\theta = \frac{5\pi}{22}Ed^4/h$, where E is the elastic modulus. The mass moment of inertia per unit depth into the page for the fiber of thickness d is $I_0 \approx \frac{\sqrt{\pi}}{6}\rho_f h^3 d$, where ρ_f is the fiber volumetric density.

The natural frequency of the rigid fiber mounted to a torsional spring is $\omega_\theta^2 = k_\theta/I_\theta$. The rotational total mass I_θ is the sum of the physical rotational inertia and the effective mass of the fluid surrounding the fiber when oscillating, $\frac{\pi}{4}\rho d^2$ (Dong, 1978). The square of the natural frequency can then be written as

$$\omega_\theta^2 = \frac{15}{22} \frac{E}{\rho_f d^2} \left(1 + \frac{\rho}{\rho_f}\right)^{-1} \left(\frac{d}{h}\right)^2. \quad (136)$$

An approximate Reynolds number can be formed based on $\omega_\theta h$, which in air at standard conditions is $R_{\omega_\theta} \approx 1.6$; this value is consistent with diameter-based Reynolds number estimates of $R = O(1)$ for owl pennula (Clark *et al.*, 2016). The damping coefficient c_θ is crudely modeled here on the basis of a viscous drag restoring moment, $c_\theta \approx \frac{1}{6}\rho\omega_\theta h^3 d C_D$. The drag coefficient in air for $R = 1.6$ is taken to be $C_D \approx 4.8$ using the drag model of Tomotika & Aoi (1950, figure 6).

The material properties of the feather keratin that forms the owl pennula are similar to polypropylene, which has handbook values of $\rho_f \approx 900 \text{ kg/m}^3$ and $E \approx 2 \text{ GPa}$ (Jaworski & Peake, 2013). The pennula have representative dimensions $d \approx 6 \text{ }\mu\text{m}$ and $h \approx 1 \text{ mm}$ (Klän *et al.*, 2012). The vortex speed parameter is assumed to be approximately at the convective speed of eddies near a wall, $S \approx 0.7U$ (Chase, 1980), where $U = 8 \text{ m/s}$ is a representative owl flight speed (Kroege *et al.*, 1972; Jaworski & Peake, 2020).

The dimensionless parameters in (111) can now be estimated from the above information: $\Omega = \omega_\theta h/S \approx 0.761$, $c_d = c_\theta/(2I_0\omega_\theta) \approx 1.85 \times 10^{-3}$, and $\mu = I_0/(\rho h^4) \approx 0.130$.

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