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Microscopic and Macroscopic Electromagnetic Polarization Forces

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14. ABSTRACT
Dipolar forces and hidden momenta are determined for the perfectly electrically conducting sphere. The original additional-boundary-condition method developed by the principal investigator is made applicable to a wider class of spatially dispersive materials and metamaterials. The speckle patterns and correlation functions are determined for the fields radiated by the incoherent sources of spherical "antennas" like the sun and other stars. It is proven that the most commonly used circuit representation of antennas, namely a frequency dependent resistor in series with a reactance, violates time-domain causality and passivity near antiresonances of the antenna. Using a convenient three-vector formulation of the Landau-Lifshitz equation of motion of charged particles, the velocity and powers as functions of time are derived for an electron moving in a counterpropagating laser beam.

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Final Performance Report for AFOSR Grant FA9550-19-1-0097

Microscopic and Macroscopic Electromagnetic Polarization Forces

Program Manager: Dr. Arje Nachman, Principal Investigator: Dr. Arthur D. Yaghjian

1 Introduction

The primary tasks that were undertaken and successfully completed on the AFOSR two-and-a-half-year Grant FA9550-19-1-0097 can be briefly summarized as follows:

1. Determine the dipolar force and "hidden momentum" for general electrically small perfect conductors of arbitrary shape and, in particular, for the perfectly electrically conducting sphere in order to verify the general results;
2. Improve upon the original method developed by the principal investigator for deriving additional boundary conditions (ABCs) to make the method applicable to a wider class of spatially dispersive materials and metamaterials;
3. Applying the combination of the theory of spherical wave expansions and time-domain fields to the electromagnetic fields radiated by incoherent sources, determine the speckle patterns and correlation functions of the fields from spherical radiators like the sun and other stars.
4. Formulate and rigorously prove the result discovered unexpectedly by the PI that the most commonly used circuit representation of antennas, namely a frequency dependent resistor in series with a reactance, violates time-domain causality and passivity near antiresonances of the antenna.
5. Using a convenient three-vector formulation of the Landau-Lifshitz equation of motion of charged particles, determine closed-form solutions of the velocity and powers as functions of time of an electron moving in a counterpropagating laser beam.

Selected highlights of these accomplishments, each of which have been published during the past three years in archival journals will be summarized in the remainder of this final report. An additional interesting discovery published during the past three years but not required by the grant proposal, namely Maxwell's definition and interpretation of electric polarization density as the displacement vector $\mathbf{D}$, rather than today's polarization density $\mathbf{P}$, and the ramifications of this discovery, is described in the last section of the report.

2 Time-Domain Force and Hidden Momentum for a Perfectly Conducting Sphere

As part of the previous grant, we proved that the Amperian magnetic dipoles induced on arbitrarily shaped, electrically small perfect electrical conductors (PECs) by arbitrarily time-varying externally applied fields contain a "hidden-momentum" electromagnetic force that makes the force on these time-varying Amperian magnetic dipoles equal to the force on magnetic-charge magnetic dipoles with the same magnetic dipole moment in the same externally applied fields [1]. Rigorous proof that this hidden-momentum force exists for arbitrarily time-varying dipoles and fields had not been found previously, even though as early as 1967 Shockley and James argued that a hidden momentum should exist for a static Amperian magnetic dipole in a static external electric field [2]. Under the present grant, the exact Mie solution to the perfectly conducting sphere under plane-wave illumination was used to prove that the expressions for the total force (which includes the hidden-momentum) on the arbitrarily shaped, electrically small PECs correctly predict the time-domain forces on perfectly conducting spheres.

Given the relatively simple, straightforward expressions for the fields in the Mie solution, it may seem somewhat surprising that this determination of the time-domain force on a PEC sphere had not been done previously. One reason for this is that to determine the time-domain force on the sphere of radius a in the plane-wave field, the real fields must be considered and not just the complex phasor fields. Another reason may be that usually only the dipole fields are kept in the Mie solution as $ka \to 0$ ($k = \omega\sqrt{\mu_0\epsilon_0}$ with ω the angular frequency) since the multipole moments and their far fields of higher order than dipoles are negligible compared with the electric and magnetic dipole moments and their far fields as $ka \to 0$. Remarkably, however, we find that although the ratios of the moments and far fields of the higher-order multipoles to those of the dipoles approach zero as $ka \to 0$, the near fields at the surface of the sphere of the electric and magnetic quadrupoles must be retained in the Mie solution to obtain the correct time-domain force and, in particular, the hidden-momentum force on the dipoles of the PEC sphere.

The total electromagnetic force on the PEC sphere is given from the Lorentz force expression as

$$\mathbf{F}(t) = \int_V [\rho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t) + \mu_0 \mathbf{J}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)]dV \tag{1}$$

with the illuminating external plane-wave fields given by

$$\mathbf{E}_e(\mathbf{r},t) = \hat{\mathbf{x}}\mathrm{Re}[E_0 e^{-i(\omega t - kz)}] = \hat{\mathbf{x}}E_0\cos(\omega t - kz) \tag{2a}$$

$$\mathbf{H}_e(\mathbf{r},t) = \hat{\mathbf{y}}\mathrm{Re}[H_0 e^{-i(\omega t - kz)}] = \hat{\mathbf{y}}H_0\cos(\omega t - kz). \tag{2b}$$

The rectangular coordinates (x, y, z) have associated spherical coordinates (r, θ, ϕ) so that $z = r\cos\theta$ and $E_0 = \sqrt{\mu_0/\epsilon_0}H_0$.

The force $\mathbf{F}^E(t)$ exerted by the total electric field $\mathbf{E}(\mathbf{r},t)$ on the induced charge density $\rho(\mathbf{r},t)$ of the sphere is

$$\mathbf{F}^E(t) = \int_V \rho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t)dV = \int_S \hat{\mathbf{r}}\cdot\overline{\mathbf{T}}^E(\mathbf{r},t)dS \tag{3}$$

where the electric stress dyadic is $\overline{\mathbf{T}}^E = \epsilon_0\left[\mathbf{EE} - \frac{1}{2}\bar{\mathbf{I}}E^2\right]$ and the surface S of V just encloses all the charge-current on the PEC sphere so that the volume integral normally added to the right-hand side of (3) vanishes because the total fields are zero inside the PEC sphere and finite throughout the infinitesimally thin surface layer of charge-current. The stress dyadic $\overline{\mathbf{T}}^E(\mathbf{r},t)$ is evaluated on S just outside the PEC sphere. Because $\mathbf{E}_t = 0$ on S, (3) reduces to

$$\mathbf{F}^E = \frac{\epsilon_0}{2}\int_S E_r^2\hat{\mathbf{r}}dS = \frac{\epsilon_0 a^2}{2}\int_0^{2\pi}\int_0^{\pi}(E_{er} + E_{sr})^2\hat{\mathbf{r}}\sin\theta d\theta d\phi \tag{4}$$

with the subscripts "r" and "t" denoting the radial (normal) and tangential vector components with respect to the spherical surface S. The subscript "e" refers to the external fields and "s" refers to the scattered fields produced by the charge and current induced on the sphere by the external fields.

To evaluate the double integral in (4), we can insert the fields from the Mie solution in [3, p. 564] with the coefficients in [3, eq.(13) on p. 565] for the PEC sphere. However, only the terms of order a^3 need be retained in (4) because the electric and magnetic dipole moments, which are the lowest order multipole moments, are of order a^3 and thus all higher order terms become negligible as the sphere radius approaches zero, in particular, for electrically small spheres. In other words, the force on the multipole moments of higher order than dipole moments can be ignored as ka approaches zero. This means that only the portion of the fields to order ka need be retained under the integral signs in (4) for electrically small spheres. Specifically, we have from (2a)

$$E_{er} \approx E_0\sin\theta\cos\phi(\cos\omega t + ka\cos\theta\sin\omega t) \tag{5}$$

and from [3, p. 564]

$$\begin{aligned} E_{sr} &\approx E_0\hat{\mathbf{r}}\cdot\mathrm{Re}\left[\left(\frac{3}{2}b_1\mathbf{n}_{\mathrm{e}1} + \frac{5}{6}ib_2\mathbf{n}_{\mathrm{e}2}\right)e^{-i\omega t}\right] \\ &\approx E_0\sin\theta\cos\phi\left(2\cos\omega t + \frac{3}{2}ka\cos\theta\sin\omega t\right) \end{aligned} \tag{6}$$

where $\mathbf{n}_{\mathrm{e}1}$ and $\mathbf{n}_{\mathrm{e}2}$ are the Stratton "even" electric dipole and electric quadrupole exterior electric fields, respectively, and b_1 and b_2 are their coefficients, which are functions of ka. Embedded in $\mathbf{n}_{\mathrm{e}1}$ and $\mathbf{n}_{\mathrm{e}2}$ are the spherical Hankel functions $h_1^{(1)}(ka)$ and $h_2^{(1)}(ka)$, respectively. To obtain the last approximate expression in (6), use has been made of the small ka approximations

$$\frac{b_1 h_1^{(1)}(ka)}{ka} \overset{ka\to 0}{\sim} \frac{2}{3}, \quad \frac{b_2 h_2^{(1)}(ka)}{ka} \overset{ka\to 0}{\sim} \frac{1}{10}ka \tag{7}$$

found from Stratton's Mie solution for the PEC sphere. In view of (5) and (6)

$$E_r^2 \approx E_0^2\sin^2\theta\cos^2\phi\,(9\cos\omega t + 15ka\cos\theta\sin\omega t)\cos\omega t. \tag{8}$$

Inserting (8) under the double integral sign of (4) and performing the integrations shows that only the ka term in (8) multiplied by the z component of $\hat{\mathbf{r}}$ survives to give

$$\mathbf{F}^E(t) = 2\pi\epsilon_0 ka^3 E_0^2 \sin\omega t \cos\omega t\,\hat{\mathbf{z}}\,. \tag{9}$$

For electrically small spheres ($ka \to 0$), the magnetic dipole moment of the PEC sphere approaches the value of

$$\mathbf{m}(t) = -2\pi a^3 H_0 \cos\omega t\,\hat{\mathbf{y}}, \quad \frac{d\mathbf{m}(t)}{dt} = 2\pi\omega a^3 H_0 \sin\omega t\,\hat{\mathbf{y}}. \tag{10}$$

The combination of (10) and (2a) reveals that $\mathbf{F}^E(t)$ in (9) can be rewritten as

$$\mathbf{F}^E(t) = -\mu_0\epsilon_0 \frac{d\mathbf{m}(t)}{dt} \times \mathbf{E}_e(0,t). \tag{11}$$

This has the form of the force on a magnetic-charge magnetic dipole rather than the $\mu_0\epsilon_0\mathbf{m}\times\partial\mathbf{E}/\partial t$ force on an Amperian magnetic dipole and, thus, it implicitly contains the hidden-momentum force, $-\mu_0\epsilon_0\partial(\mathbf{m}\times\mathbf{E})/\partial t$. It is also noteworthy that the force in (11) requires the electric quadrupolar electric field in (6) at the surface of the sphere, even though the electric quadrupole moment and far fields are negligible compared to the electric and magnetic dipole moments and far fields for electrically small spheres (that is, their ratios approach zero as $ka \to 0$).

The force $\mathbf{F}^H$ exerted by the total field $\mathbf{H}$ on the induced current density $\mathbf{J}$ of the sphere is determined from

$$\mathbf{F}^H(t) = \mu_0 \int\limits_{V_a} \mathbf{J}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)dV = \int\limits_{S} \hat{\mathbf{r}}\cdot\overline{\mathbf{T}}^H(\mathbf{r},t)dS \tag{12}$$

where the magnetic stress dyadic is $\overline{\mathbf{T}}^H = \mu_0\left[\mathbf{HH} - \frac{1}{2}\overline{\mathbf{I}}H^2\right]$. Because $H_r = 0$ on S, (12) reduces to

$$\mathbf{F}^H(t) == -\frac{\mu_0 a^2}{2}\int\limits_0^{2\pi}\int\limits_0^{\pi} |\mathbf{H}_{et}+\mathbf{H}_{st}|^2\hat{\mathbf{r}}\sin\theta d\theta d\phi. \tag{13}$$

Evaluating the double integral in (13), as we did for the electric-field force, we obtain (analogous to (11))

$$\mathbf{F}^H(t) = \mu_0\frac{d\mathbf{p}(t)}{dt}\times\mathbf{H}_e(0,t). \tag{14}$$

The resulting force in (14) requires the magnetic quadrupolar magnetic field at the surface of the sphere, even though the magnetic quadrupole moment and far fields are negligible compared to the electric and magnetic dipole moments and far fields for electrically small spheres (that is, their ratios approach zero as $ka \to 0$).

Adding $\mathbf{F}^E(t)$ in (11) to $\mathbf{F}^H(t)$ in (14) gives the total electromagnetic force exerted on the charge-current of the electrically small PEC sphere by the external plane-wave illumination in (2)

$$\mathbf{F}(t) = \mu_0\frac{d\mathbf{p}(t)}{dt}\times\mathbf{H}_e(0,t) - \mu_0\epsilon_0\frac{d\mathbf{m}(t)}{dt}\times\mathbf{E}_e(0,t) \tag{15}$$

which checks with the expression for the total force found in [**?**], [1] on the charge-current of an electrically small PEC of arbitrary shape centered at $\mathbf{r} = 0$ in an arbitrary time-varying external fields. The agreement between the two expressions in (15) and [**?**], [1] mutually confirms the analyses used to derive each of them and further substantiates that a microscopic Amperian magnetic dipole contains an internal ("hidden-momentum") force that makes the total force exerted on their charge-current in the external fields identical to the force on a microscopic magnetic-charge magnetic dipole with the same magnetic dipole moment in the same external fields.

3 Additional Boundary Conditions (ABC's) for Spatially Dispersive Continua

Electric quadrupolar continua satisfying a physically reasonable constitutive relation supports both an evanescent and a propagating eigenmode. Thus, three interface boundary conditions, two plus an "additional boundary condition" (ABC), are required to obtain a unique solution to a plane wave incident from free space upon an electric quadrupolar half space. By determining a constitutive relation for the electric quadrupolar continua that holds everywhere including the transition layer between the free space and the quadrupolar continuum, we derived under the previous grant these three boundary conditions directly from Maxwell's differential equations. During the present grant, we systematized and generalized the method used in the previous work [4] so that it could be used to derive the ABCs for other spatially dispersive continua (not just electric quadrupolar continua).

By letting the general electric quadrupolarizability density satisfy $\alpha_0(z) = \alpha_Q u(z)$ in [4, eq. (13)] with $u(z)$ the unit step function and α_Q the electric quadrupolarizability of the continuum, we are implicitly assuming that the thickness ℓ of the transition layer from free space to the electric quadrupolar continuum approaches zero (that is, $\ell \to 0$). This can occur only if the electric quadrupolar material is an ideal continuum whose average separation distance d between its electric quadrupoles approaches zero, that is, $d \to 0$ or, in terms of a dimensionless parameter, $k_0 d \to 0$ where $k_0 = \omega/c$ with ω the angular frequency and c the free-space speed of light. However, the separation distance d does not appear in the continuum formulation except implicitly in the parameter α_Q, which has dimensions of length squared, that is, dimensions of d^2. Thus, one can infer that [4, eq. (13)] is an approximation that holds with increasing accuracy as $k_0^2\alpha_Q \to 0$. In other words, [4, eq. (13)] should actually be expressed as an asymptotic approximation with respect to the parameter $k_0^2\alpha_Q$, namely

$$\alpha_0 = \alpha_Q u(z)\left[1 + z_o(k_0^2\alpha_Q)\right] \tag{16}$$

where $z_o(x)$ denotes a function that approaches zero as x approaches zero; that is, $z_o(x) = o(1)$ as $x \to 0$. As a consequence, many of the other equations should also be expressed as asymptotic approximations. In particular, the electric quadrupolar constitutive relation [4, eq. (14)] and the associated Maxwell second equation [4, eq. (16)] can be rewritten as

$$\overline{\mathbf{Q}}_0 = \alpha_Q\epsilon_0 u(z)\left[\frac{1}{2}(\nabla\mathbf{E} + \mathbf{E}\nabla) - \frac{1}{3}(\nabla\cdot\mathbf{E})\overline{\mathbf{I}}\right]\left[1 + z_o(k_0^2\alpha_Q)\right] \tag{17}$$

$$\frac{1}{\mu_0}\nabla\times\mathbf{B} + i\omega\epsilon_0\mathbf{E} - \frac{1}{2}i\omega\nabla\cdot\overline{\mathbf{Q}}_0\left[1 + z_o(k_0^2\alpha_Q)\right] = 0. \tag{18}$$

As a result of these modifications, the boundary conditions in [4, eqs. (10)–(12)] can also be rewritten in asymptotic form as

$$\mathbf{E}_s^{(2)} - \mathbf{E}_s^{(1)} = z_o(k_0^2\alpha_Q) \tag{19}$$

$$\mathbf{B}_s^{(2)} - \mathbf{B}_s^{(1)} = -\frac{i\omega\mu_0}{2}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}^{(2)})\left[1 + z_o(k_0^2\alpha_Q)\right] \tag{20}$$

$$\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}^{(2)}\cdot\hat{\mathbf{n}} = z_o(k_0^2\alpha_Q) \tag{21}$$

which reduce to [4, eqs. (10)–(12)] as $k_0^2\alpha_0 \to 0$. These asymptotic approximations enable the improved derivation of the boundary condition in [4, eq. (50)] given next.

In the paper [4], we state that for $k_0^2\alpha_Q \ll 1$ the $k_0^2\alpha_Q/12$ term in [4, eq. (47)] can be neglected in order to obtain the boundary condition in [4, eq. (50)]. A closer look at [4, eq. (47)], modified according to the asymptotic approximations in (16)–(18) above, reveals that the neglect of the $k_0^2\alpha_Q/12$ term is not necessarily valid unless

$$k_0^2\alpha_Q\left(E_z^{(2)} - E_z^{(1)}\right) = z_o(k_0^2\alpha_Q). \tag{22}$$

To prove that (22) holds, we first note that in the paper [4] the boundary conditions (modified as given in (19)–(21) above) are derived by integrating the modified [4, eq. (19)], z times the modified [4, eq. (19)], and the modified [4, eq. (20)] over the transition layer of thickness ℓ. If, in addition, we integrate z times the modified [4, eq. (20)] over the transition layer, we get the result

$$\frac{1}{6}k_0^2\alpha_Q\left(E_z^{(2)} - E_z^{(1)}\right) = z_o(k_0^2\alpha_Q) \tag{23}$$

which is equivalent to (22) and thus leads to the boundary condition in [4, eq. (50)] as $k_0^2\alpha_Q \to 0$. (Integration over the transition layer of [4, eqs. (19) and (21)] times z^n with $n \geq 2$ to obtain the second and higher order moments of these equations gives no additional information.) Note that (22) or (23) does not necessarily imply that $E_z^{(2)} - E_z^{(1)} = 0$ as $k_0^2\alpha_Q \to 0$.

In summary, the accuracy of the electric quadrupolarizability [4, eq. (13)] and electric quadrupolar constitutive relation [4, eq. (14)] is determined along with its implications for some of the other important equations in the paper [4]. Specifically, the derivation of the boundary condition in [4, eq. (50)] is made rigorous by means of an improvement enabled by the modifications to the equations from the inserted limits of accuracy. Moreover, and quite significantly, the improved derivation of the boundary conditions given here ensures that the method applied in [4] to derive the boundary conditions for electric quadrupolar continua directly from Maxwell's differential equations generalizes to the derivation of deterministic boundary conditions for higher order multipolar continua (and any other spatially dispersive continua). It can be argued that this direct method should also reveal in the asymptotic limit the required number of additional boundary conditions (ABC's) because the method uses all the relevant physics, namely Maxwell's equations and the asymptotically valid constitutive relations, but nothing more.

4 Speckle Patterns and Correlation Functions for the Partially Coherent Fields of Spherical Radiators

Because this landmarkian work with the partially coherent electromagnetic fields radiated by spherical incoherent sources like the sun and other stars is fully documented in the 16-page article in the IEEE Transactions on Antennas and Propagation [5], a paper that won the 2020 Schelkunoff Prize Paper award for the best paper in the APS Transactions, just a summary of the major results will be given here.

Using a realistic model of the sun and other stars at optical wavelengths as a spherical antenna composed of statistically independent volume sources, rather than the usual planar aperture-field, circular-disc model of the sun and other stars, a self-contained, straightforward, detailed derivation is given for the narrow-bandwidth received fields and speckle patterns radiated by the star and for the first- and second-order correlation functions satisfied by the fields and measured in Michelson phase stellar interferometry and Hanbury Brown–Twiss intensity stellar interferometry, respectively. The derivation hinges on the use of a newly derived spherical wave expansion that involves a Fourier series in both the spherical angles θ and ϕ and on determining the time averages of the associated spherical-wave coefficients as required by the assumed Lambertian radiation of the sun and other stars within the visible spectrum. It is shown that the ka bandlimit that holds for the order of the spherical Hankel functions in the spherical wave expansions of electrically large, nonsuper-reactive, coherent sources also applies to the incoherent stellar sources, and that the $\pi(k_0a)^2$ quasi-monochromatic spherical mode coefficients are uncorrelated (their temporal cross correlation is zero).

Working directly with the real-valued time-domain fields and their correlations, and without having to invoke van Cittert-Zernicke, central limit, or moment theorems, the expression for the normalized first-order correlation function $g^{(1)}(\Delta\phi)$ in [5, eq. 102], used in Michelson phase stellar interferometer measurements, and the expression for the normalized second-order correlation function $g^{(2)}(\Delta\phi)$ in [5, eq. 145], used in Hanbury Brown–Twiss intensity stellar interferometer measurements, are derived and shown to satisfy the simple relationship $g^{(2)}(\Delta\phi) = 1 + \left[g^{(1)}(\Delta\phi)\right]^2$. For Lambertian radiation (no intensity taper), the classic angular separation distance of $\Delta\phi_0 = 1.22\lambda_0/D$ is found for the first null of both the temporally averaged and spatially averaged first-order correlation functions, where λ_0 is the mean wavelength of the assumed narrow bandwidth of the measured radiation and $D = 2a$ is the diameter of the star. Although the stars are assumed to be Lambertian radiators throughout the analysis, if it is necessary to account for limb effects, this can be done by inserting the observed intensity taper of the star into the intensity parameter, as explained in [5].

Among the advantages of working with real-valued fields directly in the time domain are that explicit expressions are found for the quasi-monochromatic wave-packet fields radiated by the spherical star and that new criteria, much less restrictive than the Rayleigh distance for coherent sources, are revealed for the minimum distance at which the large-argument approximation for the Hankel function can be used to determine the radiated fields from the incoherent stellar sources. The terminology, concepts, and methodology used in the direct real-valued time-domain solution for the fields and correlation functions of spherical stars are first introduced by similarly solving the much simpler problem of a linear array of randomly excited scalar-field (acoustic) point sources.

5 A Derivation of Causality from Passivity for the Impedance Representation of Transmitting Antennas

Since this work with causality and passivity of antennas is documented in detail in the IEEE Transactions on Antennas and Propagation [6], we will limit the final report of this work to a review of the major results.

A powerful theorem for linear, time-invariant networks states that passivity of the network implies causality. This important theorem was first proven for input-impedance or input-admittance representations of networks by Youla, Castriota, and Carlin [7] using rigorous linear operator theory with Lebesgue measure and later by Zemanian [8, sec. 10.3] using Schwartz's approach to rigorous distribution theory. Both methods of proof are rather involved and removed from the physics, and require a substantial amount of preliminary mathematical development and analysis that may be prohibitive for the uninitiated, who nevertheless may be familiar with the definition of delta functions as a parametric limit of well-defined Riemann integrable functions.

Therefore, it is the main purpose of this work to provide within the input-impedance representation of transmitting antennas a self-contained, sufficiently general derivation, using only the basic mathematical tools familiar to most of the antenna engineering community, to prove that passivity of a linear, time-invariant, single-port transmitting antenna implies that the antenna is also causal. We assume realistic time-domain voltages and currents that are effectively time-limited and whose corresponding frequency-domain voltages and currents are effectively bandlimited. In addition, all voltages and currents are assumed to be bounded and Riemann integrable in both the time and frequency domains. Electrical stability of the antennas is defined and the necessary and sufficient condition on the time-domain input impedance for stability is determined. Linearity is defined by assuming the frequency-domain voltage is equal to the frequency-domain current multiplied by a frequency-domain input impedance. Within these assumptions of time-limited, bandlimited, bounded, Riemann integrable functions, all the important steps of the derivation are rigorously justified by referencing the relevant classical textbook theorems of differentiation and Riemann integration.

Remarkably, it is shown that a series resistance-reactance representation of the input impedance of an antiresonant passive antenna has a noncausal and nonpassive time-domain resistance and reactance, even though the total time-domain input impedance satisfies the passivity, causality, and stability conditions.

6 The Solution of an Electron moving in a Counterpropagating Laser Beam

The deceleration of relativistic electrons by intense counterpropagating optical laser beams have produced X-rays and, more recently, γ-rays in the laboratory. Although the detailed theoretical determination of X-ray and γ-ray production by the interaction of high-energy electrons with intense optical lasers may require the inclusion of quantum effects, a hybrid approach that incorporates quantum corrections into a classical solution can predict reasonable results. Because the classical Lorentz-Abraham-Dirac (LAD) equation of motion does not have a closed-form solution to the problem of an electron in a plane wave, nor is it amenable to a numerical solution, especially when many charges are involved, the more readily solvable Landau-Lifshitz (LL) approximation to the LAD equation of motion has become the classical equation of choice for this problem within much of the physics community.

Di Piazza [9, 10] and later Hadad et al. [11] have derived a closed-form solution to the LL approximate equation of motion for the problem of a plane-wave pulse scattered by a moving electron. These authors use a four-vector formulation of the LL equation of motion to determine the solution for the velocity components of the electron in terms of the retarded time parameter $\xi = \omega(t + z/c)$, where ω is the angular frequency of the plane wave, t is the time, c is the speed of light, and z is the time-dependent longitudinal coordinate of the electron (opposite the direction of propagation of a counterpropagating plane wave). To obtain numerical results for the velocity components as functions of time t, expressions for the proper time τ in terms of ξ are found in [11] and then the time t is found in terms of $\tau(\xi)$ by numerically integrating the relativistic factor $\gamma(\tau)$ with respect to the proper time τ.

In our work, the problem of the moving electron illuminated by linearly and circularly polarized counterpropagating plane-wave laser-beam pulses is solved directly with the three-vector electromagnetic field formulation of the LL approximate equation of motion. This approach has the advantage of maintaining transparency of the three-vector electromagnetic fields and velocities as well as the envelope function of the plane-wave pulse throughout the solution. In addition, the three-vector formulation reveals an explicit closed-form expression for the time t in terms of the retarded time parameter ξ without having to deal with the proper time τ. It also facilitates the derivation of a simple useful formula for the error in the radiated power introduced by the LL approximation.

The explicit closed-form expressions for the velocity components and relativistic factor as well as the radiated, kinetic, Schott, and total-supplied power are evaluated numerically for linearly and circularly polarized, uniform and sinusoidal-envelope plane-wave pulses with a laser-strength parameter $a_0 = 100$ and an initial electron relativistic factor $\gamma_0 = 1000$ in order to compare with the numerical results given in [11] for the velocity components and relativistic factor. (These large laser intensities and electron energies can produce measurable γ-ray radiation.) For this high-speed, high-intensity problem, it is found that the LL approximate solution predicts a radiated power that agrees very closely with the exact LAD expression for the radiated power evaluated with the approximate LL velocity and acceleration components — a result also predicted by the formula derived for the error in the LL radiated power. Despite the erroneous irreversibility of the Schott power introduced by the LL approximate solution, this irreversible Schott power is found nonetheless to be nearly equal to the reversible Schott power in the LAD equation of motion. These close agreements in radiated and Schott powers in the LL and LAD solutions strongly confirm the high accuracy of the LL approximation to the LAD solution for these particular high-speed-electron and high-intensity-laser results.

Interestingly, the radius of the computed scattering cross section of the high-speed electron in the counterpropagating high-intensity laser is much closer to the Compton wavelength than the classical electron radius, which is on the order of the radius of the Klein-Nishina scattering cross section for Compton scattering of photons from the electron. Also, as one might expect, the observed rapid changes in the power radiated and relativistic factor near the beginning of the uniform (rectangular-envelope) laser-beam pulse are greatly reduced by the sinusoidal-envelope pulse that begins continuously from a value of zero fields.

For relativistic electrons in high-intensity optical laser fields, quantum effects may appreciably alter the classical results. Therefore, a concise semi-classical determination is provided for the conditions on laser intensity and electron energies for deciding the importance of the three quantum effects that can significantly change the motion and radiation of the electron, namely quantum-vacuum electron-positron pair production, Compton scattering of the incident photons, and electron quantum recoil from photon emission ("inverse Compton scattering"). In addition, the a_0-γ-ω region of validity for the LL solution to be an accurate approximation to the LAD solution is determined by substituting the LL approximate solution into the exact expressions for the radiation power and momentum in the LAD equation of motion. Taken together, these conditions conclusively show that the LL approximation is an accurate solution to the LAD equation of motion except in the region of high enough values of the product $a_0\gamma$ that quantum recoil effects of the electron can dominate the solution. (Nevertheless, in the region where the LAD equation is accurate but the LL equation is not, the LAD equation, if solvable, could prove useful as an initial classical solution for incorporating quantum effects.) Two conditions are also found for this LL solution to be approximately equal to the Lorentz force (LF) solution (no radiation reaction). One of these two conditions reveals that the LF solution is never adequate and radiation reaction is always required if the electron has been subject to the plane-wave Lorentz force for a long enough time.

Though it has been more than a century since Lorentz and Abraham derived the classical equation of motion for an extended charged sphere, there continues to be considerable discussion and uncertainty in the literature concerning some of the more subtle aspects of the equation of motion and its derivation, specifically, the $4/3$ factor in the inertial mass term of the self force, the discrepancy between the kinetic power obtained from the self-force integral and from the self-power integral (as well as the relationship of this discrepancy to Poincaré stresses), the renormalization of the infinite mass of the classical extended charged sphere to a finite value as its radius is allowed to approach zero (to obtain the LAD equation of motion), and the noncausality (pre-acceleration and pre-deceleration) that arises in an otherwise well-behaved solution to the LAD equation of motion.

Consequently, the work begins with a critical review of the derivation of the Lorentz-Abraham (LA) equation of motion, explaining the root cause and remedy for the noncausality, the $4/3$ factor and Poincaré stresses, and the one remaining inconsistency introduced by renormalizing the mass of the charge as its radius approaches zero. The LL approximate solution is then derived simply and straightforwardly from a convenient three-vector form of the LAD equation of motion. The derivation manifestly separates the LL approximate radiation momentum-energy from the LL approximate Schott acceleration momentum-energy, the latter of which is no longer perfectly reversible in the LL approximate solution.

This research is presently being prepared for publication as an extended article in the Physical Review journal on Accelerators and Beams (PRAB).

7 Additional Published Research Accomplishment (Not Required under the Grant)

7.1 Maxwell's Definition and Interpretation of Electric Polarization Density

Maxwell's Treatise (and papers) have no concept of what today is call the electric polarization $\mathbf{P}$, which was introduced in the 1890's, and thus there is no concept of electric-polarization volume and surface charge densities, $-\nabla\cdot\mathbf{P}$ and $\hat{\mathbf{n}}\cdot\mathbf{P}$, in Maxwell's work, but only what we refer to today as electric charge density ρ_e satisfying the continuity equation, $\nabla\cdot\mathbf{J} = -\partial\rho_e/\partial t$. In Maxwell's Treatise, this electric charge density [12, Art. 31], which he also calls "electrification," or "free electricity," or just "electricity," is a fluid substance (continuum) [12, Art. 36]; and electric conduction current $\mathbf{J}$ is the "transference of electrification" [12, Art. 231].

Throughout vol. I of his Treatise, Maxwell defines and refers to electric polarization as the displacement $\mathbf{D}$. (In vol. II, Maxwell does not discuss electric polarization per se – just displacement.) He begins his discussion of electric polarization in Art. 59 of his Treatise where he says, "It is better, however, in considering the theory of dielectrics from the most general point of view, to distinguish between the electromotive intensity at any point and the electric polarization of the medium at that point, since these directed quantities, though related to one another, are not, in some solid substances, in the same direction. The most general expression for the electric energy of the medium per unit of volume is half the product of the electromotive intensity [$\mathbf{E}$] and the electric polarization [$\mathbf{D}$] multiplied by the cosine of the angle between their directions [$\mathbf{E}\cdot\mathbf{D}/2$ – see Art. 111].

In all fluid dielectrics the electromotive intensity and the electric polarization are in the same direction and in a constant ratio [$\mathbf{D} = \epsilon\mathbf{E}$ – see Arts. 68 and 111].” Maxwell reiterates this definition of electric polarization even more forcefully in Art. 111.

In Art. 60, Maxwell declares again that electric polarization is synonymous with electrical displacement: “The electric polarization of an elementary portion of a dielectric is a forced state into which the medium is thrown by the action of electromotive force, and which disappears when that force is removed. We may conceive it to consist in what we may call an electrical displacement, produced by the electromotive intensity. When the electromotive force acts on a conducting medium it produces a current through it, but if the medium is a nonconductor or dielectric, the current cannot flow through the medium, but the electricity is displaced within the medium in the direction of the electromotive intensity, the extent of this displacement depending on the magnitude of the electromotive intensity, so that if the electromotive intensity increases or diminishes, the electric displacement increases and diminishes in the same ratio. *The amount of the displacement is measured by the quantity of electricity which crosses unit of area, while the displacement increases from zero to its actual amount. This, therefore, is the measure of the electric polarization.*”

As a consequence of Maxwell’s definition of the measure of electric polarization as the vector $\mathbf{D}$, the electric polarization is zero wherever $\mathbf{D}$ is zero (not where only the present-day electric polarization vector $\mathbf{P}$ is zero). This is further confirmed by the words of Maxwell in Art. 62, “That the energy of electrification resides in the dielectric medium, whether that medium be solid, liquid, or gaseous, dense or rare, *or even what is called a vacuum*, provided it be still capable of transmitting electrical action. *That the energy in any part of the medium is stored up in the form of a state of constraint called electric polarization, the amount of which* [$\mathbf{D}$] *depends on the resultant electromotive intensity* [$\mathbf{E}$] *at the place* [$\mathbf{D} = \epsilon\mathbf{E}$].” In other words, according to Maxwell’s definition of electric polarization, there exists electric polarization even in the vacuum/ether if $\mathbf{D}$ is not equal to zero. Therefore, whenever Maxwell says that there is zero electric polarization in a solid dielectric or vacuum/ether, he means that $\mathbf{D}$ is zero in that region. Much of what Maxwell says about electric polarization in his Treatise becomes understandable only within the realization that Maxwell uses the term “electric polarization” synonymously with “electric displacement” or just “displacement.” Maxwell’s expression $(\mathbf{J} + \partial\mathbf{D}/\partial t) \times \mathbf{B}$ in Art. 619 for the “mechanical force” exerted by the magnetic induction $\mathbf{B}$ on a current carrying dielectric is consistent with his assumption that $\mathbf{D}$ is the measure of electric polarization even in the vacuum/ether where $\mathbf{D} = \epsilon_0\mathbf{E}$. (Today we would properly express this mechanical force as $(\mathbf{J} + \partial\mathbf{P}/\partial t) \times \mathbf{B}$ with $\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}$ such that the electric polarization force is zero in a vacuum where $\epsilon = \epsilon_0$ [13, Sec. 2.1.10].)

With Maxwell’s definition of electric polarization as the displacement $\mathbf{D}$, along with his divergence equation $\nabla \cdot \mathbf{D} = \rho_e$ [12, Arts. 83a or 612] and his normal-displacement boundary condition $D_{n2} + D_{n1} = \sigma_e$ (with $\hat{\mathbf{n}}_{1,2}$ defined into the media on either side of the surface) [12, Arts. 83a or 613], where σ_e is the surface charge density, his discussions of the surface charge associated with electric polarization in dielectrics become understandable. For example, consider Maxwell’s statement in Art. 62, “That the surface of any elementary portion into which we may conceive the volume of the dielectric divided must be conceived to be charged so that the surface-density at any point of the surface is equal in magnitude to the displacement through that point of the surface reckoned inwards [into the dielectric]. If the displacement is in the positive direction, the surface of the element will be charged negatively on the positive side of the element, and positively on the negative side. These superficial charges will in general destroy one another when consecutive elements are considered, except where the dielectric has an internal charge, or at the surface of the dielectric.” In Art. 111, Maxwell repeats, “Conceive any portion of the dielectric, large or small, to be separated (in imagination) from the rest by a closed surface, then we must suppose that on every elementary portion of this surface there is a charge measured by the total displacement of electricity through that element of surface reckoned inwards [into the dielectric].” Maxwell is saying that if a portion of the dielectric is imagined isolated from any other polarization $\mathbf{D}$, or if the dielectric is divided into volume elements separated by infinitesimally thin shells in which there is no displacement ($\mathbf{D} = 0$), that is, no Maxwell electric polarization, then there is negative surface charge ($-D_n$) on the positive side of the volume element and positive surface charge ($+D_n$) on the negative side of the volume element, where $\hat{\mathbf{n}}$ denotes the positive direction.

This interpretation is further confirmed at the beginning of Art. 325 where Maxwell summarizes, “We have seen that when electromotive force acts on a dielectric medium it produces in it a state which we have called electric polarization, and which we have described as consisting of electric displacement within the medium in a direction which, in isotropic media, coincides with that, of the electromotive force, combined with a superficial charge on every element of volume into which we may suppose the dielectric divided, which is negative on the side towards which the force acts, and positive on the side from which it acts.” In a modern-day version of Maxwell’s separated dielectric volume elements, we would imagine infinitesimally thin free-space (vacuum) separation shells in which $\mathbf{P} = 0$ but not $\mathbf{D} = 0$ because, unlike Maxwell, we would not consider the vacuum/ether as a dielectric containing electric polarization. In the case of volume elements separated by free space, $\mathbf{D}$ is continuous across the dielectric-vacuum interface and $\sigma_e = 0$ on the dielectric surfaces.

Although a source of confusion for past commentaries and histories of Maxwell’s Treatise, I would assert that none of these explanations of Maxwell concerning surface charge are incorrect given that his volume elements are separated not by infinitesimally thin free-space shells having zero $\mathbf{P}$ and nonzero $\mathbf{D}$, as one would imagine today, but by infinitesimally thin shells having zero $\mathbf{D}$. As discussed above, Maxwell did this because he assumed that $\mathbf{D}$ was the measure of electric polarization even in free space (the ether).

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Major Publications Resulting from the Grant

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[2] A.D. Yaghjian, “A Simplified Derivation of Causality from Passivity for the Impedance Representation of Transmitting Antennas,” accepted for publication in *IEEE Trans. Antennas Propagat.*, 6 pages, 2021.

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