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LECTURES ON TURBULENCE AND MIXING PROCESSES IN STRATIFIED FLUIDS

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13. ABSTRACT

This is a set of notes based on a set of lectures for a summer course "Vertical Exchange Processes in the Sea". Topic 1 gives a background discussion of basic fluid mechanical problems and simple turbulence theory. Topic 2 discusses mixing in a stably stratified fluid. Topic 3 is concerned with the Richardson number, the flux Richardson number and eddy coefficients of viscosity and buoyancy together with a discussion of mass and salt transfers at the mouth of an estuary and a determination of the depth of the halocline in an estuary. Topic 4 contains a discussion of the surface layer of the atmosphere. Topic 5 discusses density currents and wake collapse. Topic 6 involves problems in which the earth's rotation is important.

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Lectures on

TURBULENCE AND MIXING PROCESSES IN STRATIFIED FLUIDS

By

Robert R. Long

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Lectures on
Turbulence and Mixing Processes in Stratified Fluids

PREFACE

The notes in this report are based on a set of eleven lectures prepared for a summer course "Vertical Exchange Processes in the Sea", August 26-31, 1974, at Marstrand, Sweden, sponsored by the Nordiska Sommarskolen för forskarutbildning in collaboration with the Oceanographic Institute of the University of Gothenburg. I would like to take this opportunity to thank Gösta Walin, organizer of these seminars, for offering me the opportunity to participate in this program.

The other lecturer in this course was Prof. Claes Rooth of the University of Miami. Since my understanding of the myriad of practical oceanographic problems involving turbulence and vertical mixing processes is much inferior to that of Prof. Rooth, I have concentrated in these notes on basic fluid mechanical concepts with only passing references to such problems as pollution, water quality, ecological impacts, etc. which inspire interest in the subject of these lectures.

These notes presuppose an understanding of basic fluid mechanics, but they do not require any special awareness of turbulence research. Topic 1, in fact, presents the basic concepts required, including the equations of a stratified (Boussinesq) fluid, the energy equation for one-dimensional turbulence in a stratified fluid, instability of stratified, shearing flow, and a comparison of molecular and turbulent diffusion. The Appendix in Topic 1 has a rudimentary discussion of turbulence in homogeneous fluids.

Topic 2 discusses and compares experimental investigations of mixing processes in a stably stratified fluid and includes a unifying discussion of observations in experiments with and without shear. These experiments now appear to be reasonably good models of such atmospheric and oceanic phenomena as the erosion of inversions and thermoclines.

Topic 3 has a discussion of the Richardson number, Ri, the flux Richardson number Rf and eddy coefficients of viscosity and buoyancy diffusion. One basic result of recent experiments and observations is that the flux Richardson number tends to be a constant under moderately stable or very stable conditions (typical of atmosphere and oceans). This yields a valuable relationship for purposes of parameterization of fluxes of buoyancy and momentum. The last part of this topic applies the experiments of Topic 2 to the practical problem of mass and salt transfers into and out of estuaries and the way in which these transfers combine with vertical mixing processes to determine the depth of the halocline. These concepts are applied to the Baltic Sea.

One of the most useful concepts for understanding the surface layer of the atmosphere is the Monin-Obukhov theory. Little is known about its application to oceanic problems but it seems likely to be relevant to the upper mixed layer and to the layer near the bottom if a bottom current exists. This theory is discussed in Topic 4 for stable and unstable conditions. In the latter case a similarity theory has been advanced which has strong adherents, especially in the Soviet Union. An alternative theory is offered in these notes which seems to be in closer accord with observations in the laboratory and atmosphere.

Topic 5 is concerned with density currents and "wake collapse" and is relevant to currents of fresh water issuing from an estuary, the intrusion of a salt-water wedge into an estuary and the collapse of a region of turbulence in breaking internal waves.

Topic 6 discusses problems in which the earth's rotation is important, inc.uding mixing near the coast due to internal Kelvin waves and the problem of upwelling. Finally we include a list of symbols for easy reference and a list of participants in the course.

Robert R. Long

Lectures On

TURBULENCE AND MIXING PROCESSES IN STRATIFIED FLUIDS

Topic 1. Basic Ideas

1.1 Introduction

There is great interest on the part of geophysicists and engineers in fluid systems in which there are density variations. Such variations are typical of the atmosphere, seas, lakes and reservoirs and even when the variations are exceedingly small, they are almost always exceedingly important. In some cases, for example in the air above the warmed surface of the earth, there may be a layer in which the average density increases with height, i.e., the fluid is unstably stratified (Fig. 1.1). Then, because the density distribution is

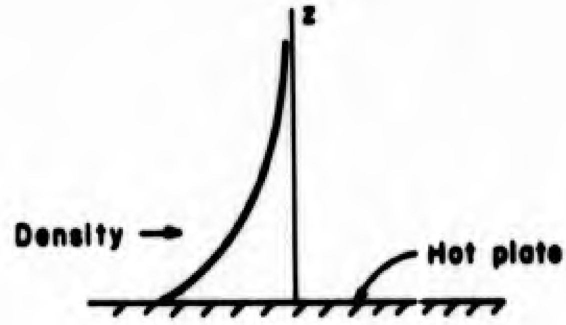

Fig. 1.1 Convection above a warm surface.

basically unstable, there is a strong tendency for light parcels of fluid near the ground to rise and heavier parcels to descend in a type of motion called thermal convection. The heating is sufficiently strong to maintain the average density increase with height close to the ground, but at some distance above the ground the mean density increase becomes very weak, or may even be replaced by a density decrease with height (Gille, 1967, p. 380). In the layer of density increase with height, the motion may have a regular cellular pattern in the form of Bénard cells (Bénard, 1901). More usually, especially in geophysical situations, the motion is turbulent or irregular, characterized by plumes or thermals. Turbulent convection is discussed at some length in Section 4.3.

Layers with mean density increase with height are rather rare in the atmosphere and in oceans and lakes. As we have mentioned, they occur in the atmosphere near the surface when the surface is giving off heat to the air (frequently in the daytime). Unstable layers occur in water near the water-air interface when the water is losing heat (e.g. by nocturnal cooling or evaporational cooling or when there is overriding cold air). Such layers may also occur near the ocean bottom in regions of formation of bottom water (Turner, 1973, p. 313). Usually, however, the mean density decreases (stably stratified) or is uniform with height. Except for a portion of Topic 4, we confine ourselves to the stable case and, especially, to the occurrence of turbulence and mixing in stably stratified fluids.

If a turbulent fluid has a mean vertical density gradient, turbulence will <u>tend</u> to reduce this density variation. As a simple example of this process, we can imagine a vessel filled with water with a stable temperature stratification - warm near the top becoming colder with depth. If we now induce turbulence by stirring mechanically, the density gradient will ultimately be wiped out (Section 1.6). If, on the other hand, the fluid is initially unstably stratified - cold above and warm below - convection will set in automatically and the density gradient will again be eliminated. In the first case the warmth near the top is transported downward; in the second case, the warmth near the bottom is transported upward. Thus there is always a flux of heat from warm regions to colder regions.

1.2 Governing Equations

Let us now write down the governing equations for the fluid systems considered in these notes. For a liquid[1] such as water in which the density variation is linearly related to temperature, for example, a reasonably accurate set of equations is

$$\frac{d\underset{\sim}{v}}{dt} = -\frac{1}{\rho}\nabla p - g\underset{\sim}{k} + \nu\nabla^2\underset{\sim}{v} \qquad (1.1)$$

$$\nabla \cdot \underset{\sim}{v} = 0 \qquad (1.2)$$

$$\frac{d\rho}{dt} = k_h \nabla^2 \rho \qquad (1.3)$$

where $\underset{\sim}{v} = (u, v, w)$ is the vector velocity at the point (x, y, z), ρ is density, p is pressure, $\underset{\sim}{k}$ is a unit vector in the vertical, g is gravity and ν and k_h are constant coefficients of viscosity and heat conduction. We have neglected the rotation of the earth although this will be important for larger scale phenomena (Monin and Yaglom, 1971, p. 406). Rotation is discussed under Topic 6.

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[1] We do not give a separate discussion for the compressible atmosphere. In the lowest few thousand feet the same equations are valid if we use for the density ρ in a liquid the potential density in the gas (Yih, 1965, p. 16). The latter is the density of a parcel of air when its pressure is changed adiabatically to a reference pressure, usually taken to be 10^6 dynes/cm^2.

In geophysical problems, ρ varies rather little and it is convenient to define a quantity called buoyancy b:

$$b = \frac{\rho - \rho_0}{\rho_0} g \tag{1.4}$$

where ρ_0 is some representative density, often taken to be $\rho_0 = 1\ \mathrm{gm/cm^3}$ in the case of water. Under the assumption that

$$\frac{\rho - \rho_0}{\rho_0} \ll 1 \tag{1.5}$$

we may neglect the difference between ρ_0 and ρ everywhere in the equations except in the gravity-force term and obtain

$$\frac{d\underset{\sim}{v}}{dt} = -\nabla p_* - b\underset{\sim}{k} + \nu \nabla^2 \underset{\sim}{v} \tag{1.6}$$

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{1.7}$$

$$\frac{db}{dt} = k_h \nabla^2 b \tag{1.8}$$

where $p_* = p/\rho_0 + gz$. The approximation we have used, namely (1.5), is called the Boussinesq approximation[1] (Boussinesq, 1903). Physically, in

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[1] There are some delicate points in a careful derivation of the Boussinesq approximation, especially as applied to compressible fluids. These are considered carefully by Spiegel and Veronis (1960). In some special problems certain phenomena of interest may be lost by making the approximation (Long, 1965), but this does not affect the subject of these notes.

Our treatment of the governing equations is very brief and the reader is referred to more careful discussions of the equations by Phillips (1966, p. 8-19), Monin and Yaglom (1971, pp. 421-425), Turner (1973, pp. 3-13).

making this approximation we have taken into account the tendency for a parcel lighter or heavier than its environment to rise or fall under the influence of gravity (since we do not assume $\frac{g(\rho-\rho_0)}{\rho_0}$ is small compared to dw/dt for example), but we have neglected all other effects of density variation, for example the small excess of momentum $\rho\underset{\sim}{v} - \rho_0\underset{\sim}{v}$ of a parcel because its density is ρ and not ρ_0. In natural circumstances this approximation is excellent in water where, for example, temperature variations cause density differences of order of one-tenth of one percent. It is also quite good in the lower atmosphere (Monin and Yaglom, 1971, p. 421).

In the sea, density variations are caused by both temperature and salinity. We assume a linear relationship, $\rho = \rho_0(1-\alpha T+\beta S)$ and we use the following form of the Boussinesq equations:

$$\frac{d\underset{\sim}{v}}{dt} = -\nabla p_* + (g\alpha T - g\beta S)\underset{\sim}{k} + \nu\nabla^2\underset{\sim}{v} \tag{1.8a}$$

$$\frac{dT}{dt} = k_h\nabla^2 T \tag{1.8b}$$

$$\frac{dS}{dt} = k_s\nabla^2 S \tag{1.8c}$$

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{1.8d}$$

where T is temperature and S is salinity and where the representative density ρ_0 is taken to be the density when $T = S = 0$; the quantities α and β are positive known constants, k_s is the molecular coefficient of salt diffusivity and is much smaller than $k_h \left(\frac{k_h}{k_s} \simeq 100\right)$. Notice that even when density variations are due to both heat and salt, if we can neglect molecular processes, we may use the simpler set in Eqs. (1.6)-(1.8) in terms of buoyancy $b = g(\beta S - \alpha T)$ after setting the diffusion terms $\nu \nabla^2 \underset{\sim}{v}$ and $k_h \nabla^2 b$ in Eqs. (1.6) and (1.8) equal to zero.

1.3 Energy Equation and Mean Momentum and Buoyancy Equations.

Let us now consider a simple situation[1] in which turbulence exists but mean quantities vary only with height z and not with horizontal distances x and y. Such turbulence is called horizontally homogeneous or one-dimensional, and this approximation is useful in many practical problems because horizontal variations are often negligible in natural circumstances. We also assume a mean velocity $\bar{u}$ along the x-axis with $\bar{v}$ and $\bar{w}$ both zero. We define disturbance quantities by

$$u = \bar{u} + u'$$

$$v = v'$$

$$w = w'$$

$$p_* = \bar{p} + p'$$

$$b = \bar{b} + b'$$

[1] For a fuller discussion, see Lumley and Panofsky (1964, pp. 67-75).

where an overbar denotes an average[1].

In one-dimensional turbulence the averaged equation of motion along the flow and the averaged buoyancy diffusion equation reduce to

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \tau}{\partial z}$$

$$\frac{\partial \bar{b}}{\partial t} = \frac{\partial q}{\partial z} \tag{1.9}$$

In the <u>momentum flux</u> τ and the <u>buoyancy flux</u> q the molecular contributions are almost always negligible and we may consider $\tau = -\overline{u'w'}$ and $q = -\overline{w'b'}$. Notice that in steady-state turbulent flows the fluxes are independent of height.

Let us now form the kinetic energy equation by multiplying Eq. (1.6) by $\underline{v}$ and averaging. We obtain[2]

$$\frac{\partial}{\partial t}(\overline{c'^2/2}) = -\frac{\partial}{\partial z}\left[\overline{w'(c'^2/2+p')} - \nu\frac{\partial}{\partial z}(\overline{c'^2/2})\right] + \tau\bar{u}_z + q - \epsilon \tag{1.10}$$

where c' is the turbulent speed, where we have used $\bar{u}_t = \tau_z$, and where

$$\epsilon = \nu\left[\overline{(\nabla u')^2 + (\nabla v')^2 + (\nabla w')^2}\right] \tag{1.11}$$

is the (positive) <u>dissipation function.</u>

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[1] The most basic choice from a theoretical viewpoint is an ensemble average, but here we may also use the more practical concept of an average over, say, a large horizontal area or in steady cases over a long time. For a careful discussion, see Monin and Yaglom (1971, pp. 205-218).

[2] The energy equation (1.10) is identical when Eq. (1.8a) is the governing equation.

1.4 Discussion of Energy Equation

The dissipation is proportional to the (small) viscosity coefficient ν, but this does not mean that ϵ is small compared to the other terms in Eq. (1.10) because such quantities as $(\overline{\nabla u'})^2$ are very large in turbulent flows. In homogeneous fluids, for example, the eddy velocity usually decreases as we consider smaller and smaller eddies but the size of the eddies decreases faster than u' in the sense that $\partial u'/\partial x$ increases. Thus $(\partial u'/\partial x)^2$ is very large and, in fact, tends to be inversely proportional to ν so that ϵ is independent of the viscosity. We may often safely assume[1] (see Appendix)

$$\epsilon \sim \sigma_u^3/\ell \qquad (1.12)$$

where σ_u is the rms turbulent horizontal velocity which is of the order of the velocity of the energy-containing eddies, and ℓ is the integral length scale which is a measure of the length scale of the energy-containing eddies.

Although molecular friction must be taken into account in forming the energy equation, there is ample reason to believe that molecular viscosity and molecular heat and salt diffusion are negligible for most other uses of the Boussinesq equations provided the Reynolds number, $\sigma_u \ell/\nu$, and Péclet numbers, $Pe = \sigma_u \ell/k_s$, $\sigma_u \ell/k_h$ are sufficiently large, perhaps over 200 (Topic 2). One consequence of this is that Eqs. (1.6)-(1.8) can then be used instead of Eqs. (1.8a)-(1.8d) and we do not have to consider the roles of temperature and salt separately.

The buoyancy flux q in Eq. (1.10) has a sign that is controlled by the correlation between vertical velocity and perturbation buoyancy in rising

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[1] The symbol $\sim$ reads: "is of the order of". Here we mean $\epsilon = \alpha_1 \sigma_u^3/\ell$ where α_1 is close to one and independent of σ_u and ℓ. Usually we have certain non-dimensional numbers tending to infinity or zero and then $P \sim R$ means that the ratio $P/R \to C$ where C is neither zero or infinite. Notice that C need not be close to one, although it frequently is or is assumed to be.

and falling parcels of air. If the mean density increases with height, as in turbulent thermal convection, rising parcels ($w' > 0$) are associated with lower density ($\rho' < 0$) and sinking parcels with higher density so that $q = -\overline{w'\rho'}$ is positive. Then Eq. (1.10) shows that this effect is a source of kinetic energy helping to drive the motion. The correlation coefficient is of order one. (Deardorff & Willis, 1967, p. 691, measured values around 0.5 and 0.6).

When the mean stratification is stable, q will have the opposite sign as rising parcels tend to be cool and falling parcels warm. One must be careful in the stable case, however, because of the possibility of wave motions contributing to w' and ρ'. Thus, if the fluid is at rest and stably stratified, it is capable of internal gravity-wave motion in which the basically level density surfaces move up and down in waves. Obviously, if the waves do not break, there will be no rupture of these surfaces and therefore, neglecting molecular conduction, no flux of heat or buoyancy despite sizable values of w' and ρ'. The correlation coefficient will be zero. If the waves break, there will be intermittent turbulence superimposed on the wave motion and q will be negative although the correlation coefficient may be much less than one. Negative q means that the kinetic energy tends to decrease. This is because it requires work to lift heavy parcels up and bring light parcels down. There is a tendency in doing this to increase potential energy at the expense of kinetic energy so that some of the other terms in Eq. (1.10) must be energy-producing if turbulent energy is to increase or be maintained.

It is useful to define available potential energy per unit mass (Long, 1970, p. 357) by considering it to be the kinetic energy per unit mass attained by a parcel of buoyancy $b = b' + \bar{b}(z)$ as it falls from the height z to the height ζ at which its buoyancy b is equal to the mean buoyancy $\bar{b}(\zeta)$ at that level. We have

$$b' = \bar{b}(\zeta) - \bar{b}(z) \tag{1.13}$$

$$= -\bar{b}_z \xi \tag{1.14}$$

approximately, where $\xi = z - \zeta$ and we have assumed that ξ is small compared with the length scale of the vertical variation of mean buoyancy[1]. Then neglecting disturbance pressure, we may write

$$\frac{dw}{dt} = \frac{d^2\xi}{dt^2} = -b' = \bar{b}_z \xi \tag{1.15}$$

because ζ is a Lagrangian quantity, so that $d\zeta/dt = 0$. Integrating, we get

$$\frac{w^2}{2} - \bar{b}_z \frac{\xi^2}{2} = \text{const} \tag{1.16}$$

Thus available potential energy may be defined as

$$V' = -\bar{b}_z \frac{\xi^2}{2} \text{ or } V' = \frac{b'\xi}{2} \tag{1.17}$$

We may also identify q with potential energy changes. The potential energy of a particle of volume V_0 and density ρ is $\rho g V_0 z$. Let us now define

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[1] This may not always, or even usually, be the case but our development here is only suggestive of the definition of Eq. (1.17).

<u>incremental potential energy</u>[1] as $\rho g V_0 z - \rho_0 g V_0 z$ so that this potential energy is zero when the particle has the characteristic density ρ_0. If we let V represent the incremental potential energy per unit mass, then, to within the Boussinesq approximation, $V = bz$. Since b is nearly conservative, putting $b = b' + \bar{b}$ and assuming no mean vertical velocity, we have

$$\overline{\frac{dV}{dt}} = \overline{b'w'} = -q \qquad (1.18)$$

is the average rate of increase of incremental potential energy per unit mass. We may identify this with available potential energy by differentiating (1.17) and again assuming $db/dt = 0$. We get

$$\frac{dV'}{dt} = \frac{1}{2}\frac{d\zeta}{dt} b' - \frac{1}{2}\zeta\frac{d\bar{b}}{dt} = \frac{1}{2} w'b' - \frac{1}{2}\zeta \bar{b}_z w' = w'b' \qquad (1.19)$$

so that

$$\overline{\frac{dV'}{dt}} = \overline{w'b'} = -q \qquad (1.20)$$

Comparing (1.18) and (1.20), we see that the average rate of increase of incremental potential energy and the average rate of increase of available potential energy are the same.

In application, we can conceive of an energy-containing eddy in the form of a whirl with horizontal axis of rotation, with velocity σ_u and diameter ℓ.

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[1] Some authors identify available potential energy with our incremental potential energy.

It will lift parcels from their level of origin a distance $\xi \sim \ell$ so that $V' \sim \sigma_b \ell$, where σ_b is the rms buoyancy fluctuation. If the turbulence is not decaying, the kinetic energy must be of this order or larger so

$$\frac{\sigma_b \ell}{\sigma_u^2} \leq O(1) \tag{1.21}$$

In a fully turbulent layer, ℓ is of order of the depth D of the layer[1]. Thus we see that if D increases and σ_u is maintained, <u>the layer must become more and more homogeneous.</u> (See Sections 2.5 and 2.6).

Let us now discuss the first and second terms on the rhs of Eq. (1.10). The term proportional to ν is negligible in the first term, which is called the <u>energy flux divergence.</u> In most literature on turbulence in stratified fluids (Proudman, 1953, p. 101, Phillips, 1966, p. 201) the role of the energy flux divergence is treated rather casually. Typically, the arguments neglect the energy flux divergence and in stable conditions the only energy source term is then $\tau \bar{u}_z$. It is then stated that the sink term q must not exceed the source term if the turbulence is maintained, so that

$$\frac{q}{\tau \bar{u}_z} = Rf < 1 \tag{1.22}$$

since ϵ is positive and non-zero. Rf is called the <u>flux Richardson number.</u>

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[1] The large eddies tend to be as large as the dimensions of the region of turbulent motion when there are no density gradients (see the Appendix and Tennekes and Lumley, 1972, p. 19). Here the potential energy may be as large as the kinetic energy but this should not change the order of magnitude of the eddy size. Experiments, for example those of Moore and Long (1971), have been run in which buoyancy forces are of the same order as the acceleration, i.e. $\sigma_b \sim \sigma_u^2 / \ell$. It was observed that the eddies filled the whole channel in this case.

In addition, it is possible to define an eddy viscosity K_a and eddy conductivity K_b by

$$\tau = K_a \bar{u}_z \ , \quad q = K_b \bar{b}_z \qquad (1.23)$$

so that the condition (1.22) becomes

$$Ri < \frac{K_a}{K_b} \ , \quad Ri = \frac{|\bar{b}_z|}{\bar{u}_z^{\,2}} \qquad (1.24)$$

Ri is called the gradient Richardson number and plays a considerable role in stratified flow theory. We question Eq. (1.22) below but even if the arguments were rigorous, the inequality (1.24) would be of limited usefulness because K_a/K_b varies strongly with Ri.

The above argument is completely incorrect in an important class of laboratory experiments (see Topic 2) in which a stratified fluid in a closed vessel is agitated by a vibrator. There is no mean flow, so $\bar{u}_z = \tau = 0$ and Ri and Rf are both infinite, yet turbulence and buoyancy fluxes exist even in the presence of very large density gradients (Turner, 1973, Chapter 9). Obviously in this experiment the flux-divergence term in the energy equation is the only source of kinetic energy. If shear exists, as we will see, the flux-divergence term tends to be of the same order as the shear term in Eq. (1.10) and this is the apparent reason for the usefulness of arguments that Rf has an upper limit for the existence of turbulence (see Topic 2). A commonly cited value is $Rf_c = 0.15$ (Turner, 1973). We will see in our discussion of Topic 3 that Rf_c is probably closer to 0.05 in natural circumstances. Indeed it appears that Rf_c is not a universal constant.

1.5 Instability of Stratified Shearing Flow

As indicated by the energy equation, a source of disturbance energy is in the shear of a flow pattern. This is most simply seen in the case of two fluids with a buoyancy jump and velocity jump across a surface of discontinuity. One finds that small waves on the interface will grow exponentially if the wave length is small enough, i.e. that such a surface is always unstable (Lamb, 1932, p. 373).

With general velocity and density distributions, the problem becomes more difficult but a famous theorem of Miles (1961) is helpful. It states that all velocity and density distributions are stable if $Ri > \frac{1}{4}$ everywhere in the flow. Accompanying this is a rule of thumb which says that the situation is usually unstable when $Ri < \frac{1}{4}$ somewhere (Hazel, 1972).

Hazel analyzed stability and instability when the velocity difference occurs over a thickness greater $(r > 1)$ or less $(r < 1)$ than the layer thickness for the density variation. If $r > 1$, the Richardson number falls to zero outside of the region of density variation even when Ri is large in the region of density variation. The theorem of Miles indicates that instability may occur and Hazel has shown that it does occur. We will see that this situation is found in experiments (Topic 2). It may also be common in oceans and other bodies of water.

The instability resulting when $Ri < \frac{1}{4}$ is called Kelvin-Helmholtz instability. A number of experiments have been run to investigate it (Scotti and Corcos, 1969, Thorpe, 1973) and they have shown that this instability can lead to turbulence. Its occurrence in clear air in the atmosphere as detected by ultra-sensitive radar (Browning and Watkins, 1970) and in the thermocline (Woods and Wiley, 1972) is well established. In the ocean, at least, the instability can be set off by the increase of shear locally due to the passage of long waves along a density interface. The wave roll-up is called billow turbulence and resembles the drawing in Fig. 1.2. The whole layer, then

Fig. 1.2. Billow Turbulence.

breaks down into a general region of turbulence with the layer of strong density variation considerably increased in depth. The Richardson number for the new density interface is just above $\frac{1}{4}$. This behavior has not been explained (Thorpe, 1973).

1.6 Molecular and Turbulent Diffusion

Our concern in these notes is with turbulent transport of various properties, but, by way of comparison, it is instructive to see how molecular mixing processes proceed. If we consider a container of water at rest, stratified by a variable distribution of salt, the salt will diffuse according to the equation

$$\frac{\partial S}{\partial t} = k_s \nabla^2 S \tag{1.25}$$

Without solving exactly, we can see that for a container of dimensions d, the time for diffusion T_s is yielded by

$$\frac{\Delta S}{T_s} \sim k_s \frac{\Delta S}{d^2} \tag{1.26}$$

or $T_s \sim d^2/k_s$. For salt, $k_s \sim 10^{-5} cm^2/sec$, so that for d = 30 cm, the time for the water to become homogeneous is of order 2-3 years! We can see from this example that molecular processes are very slow. Of course, salt has a particularly low coefficient of diffusion but even for heat the diffusion time in this experiment is of order of a week. They may be compared to turbulent diffusion. In the problem we are considering, the time scale in a turbulent situation will be a function of σ_u and ℓ or $T_t \sim \ell/\sigma_u$. If we stir to produce eddies of speed 10 cm/sec and if the eddies fill the box, T_t is of the order of seconds! It is not surprising, therefore, that turbulent diffusion of momentum, heat and additives dominates in almost all problems. We must sometimes be careful in laboratory experiments in which σ_u and ℓ may be quite small and in which strong stability locally may cause intermittent rather than fully developed turbulence (Crapper and Linden, 1974).

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APPENDIX[1]

SOME BASIC CONCEPTS IN TURBULENCE

Turbulence is a disordered, random type of motion that is usually found in the atmosphere and larger bodies of water and in larger laboratory equipment. It tends to occur when the Reynolds number $U_s L_s/\nu$ is large, where U_s is representative of the mean fluid velocity and L_s is representative of the size of the system. Turbulence is very effective in transport of momentum, heat and various additives and is always dissipative, i.e., is characterized by the flow of energy from kinetic energy to heat.

The theory of turbulence is poorly developed even in the simplest cases. The primary difficulty is that a mathematical approach always leads to more unknowns than there are equations to determine them. This is called the closure problem. The usual effort to overcome this is to make (very often ad hoc) assumptions to close the problem. Such assumptions frequently involve the definition and use of coefficients of eddy viscosity and eddy diffusivity (Eq. 1.23) because of the superficial resemblance between the way molecular motions transport properties like momentum and heat and the way turbulence transfers these quantities. In the analogy, the mixing length (usually proportional to the size of the eddies) replaces mean free path and turbulent velocity replaces the speed of the molecules so that one normally assumes for momentum, for example, $K_s \sim \ell\sigma_s$. One big difference, as we

[1] This Appendix draws heavily on Tennekes and Lumley (1972).

see, is that K_s is a property of the flow whereas ν is a property of the fluid. In addition, K_s is usually much larger than ν so that molecular friction is negligible. As an example, $K_s/\nu \sim 10^7$ in the lower atmosphere.

One of the most powerful tools for turbulence research is dimensional analysis and we can often find useful results with this technique. The approach always leads to undetermined constants which we estimate approximately from observation. One example of this is also of importance to the subject of these notes. An idealized version of the problem is the flow of a homogeneous fluid over a flat plate at $z = 0$. Most realistic from our viewpoint is when the plane is rough with a fairly uniform distribution of fairly uniform roughness elements of height h_0. If the flow is fast enough, it will be turbulent and we should be

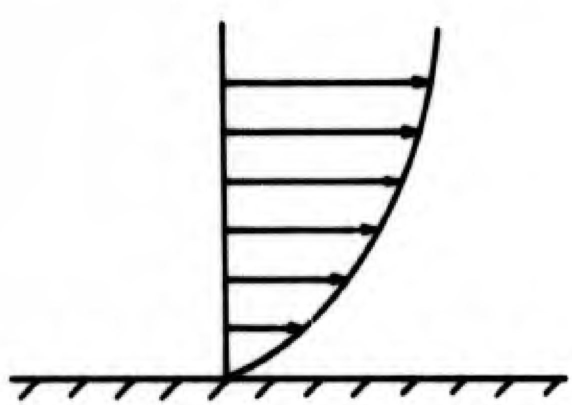

Fig. A.1. Flow over a flat plate.

able to neglect viscosity ν. Since the flow is steady, Eq. (1.9) yields $\partial\tau/\partial z = 0$ so $\tau = u_*^2$ is a constant which we take to be a characteristic quantity. u_* is called the friction velocity. If the fluid is infinite or, more practically, if the height H is large compared to a height of interest z, we may neglect H. If, tentatively, we also neglect h_o, all mean quantities will be a function of u_* and z. For example, the rms velocity $\sigma_u = \alpha_2 u_*$, where α_2 is a universal constant, so that σ_u does not vary with height. Also, the shear $\bar{u}_z = f(z, u_*)$ or

$$\frac{\bar{u}_z z}{u_*} = \text{const} = \frac{1}{\varkappa} \qquad \text{(A.1)}$$

where $\varkappa$ is called Von Kármáns constant and has a value of about 0.40. Integrating Eq. (A.1), we obtain

$$\bar{u} = \frac{u_*}{\varkappa} \ln z + \text{constant} \qquad \text{(A.2)}$$

Since we have neglected viscosity and the presence of the roughness elements, we cannot apply this to $z = 0$, obviously, or even too close to the roughness elements. Instead we choose the constant in (A.2) so as to yield

$$\bar{u} = \frac{u_*}{\varkappa} \ln \frac{z}{z_o} \qquad \text{(A.3)}$$

where z_o, called the roughness length, is the height at which the velocity would go to zero if the logarithmic law held down to that level (it does not). Experiment supports (A.3) (for example in pipe flows near the walls) and shows that z_o is proportional to the height of the roughnesses[1]. Notice that $\bar{u}$ involves

[1] Values of z_o were first found by Nikuradse (see Schlichting, 1955, chapter 20) in sand-roughened pipes. He found $z_o \simeq h_o/30$. In the lower atmosphere z_o varies from a few millimeters over snow cover to 4 cm over high grass, to a meter or two over cities. It sometimes varies with wind velocity, for example as the surface is deformed by the wind. An important example is a water surface where the roughness elements (waves) are generated by the wind (Plate, 1971, pp. 24-34).

the roughness but σ_u, $\bar{u}_z$, etc. do not. The reason is that the roughnesses do nothing more than account for the zero of the velocity profile, and in general far from the roughness elements, the fluid "feels" only the momentum stress and the distance from the plane. For example, the integral length scale $\ell = \alpha_3 z$, where α_3 is another universal constant. Thus the eddy size increases to the center, becoming proportional to the pipe diameter, i.e. $\ell \sim D$. This is a general tendency, i.e. the largest eddies become as large as possible and furthermore these are the eddies that contain the bulk of the energy.

In flow in a pipe, for example, it is possible to find a relationship between the friction velocity u_* and the mean speed at the center of the pipe U_0. This ratio turns out to involve the size of the roughnesses (or viscosity ν for a smooth pipe) so that it would not have been correct in this case to analyse the flow near the wall in terms of parameters U_0 and z. For this reason we normally choose τ and q when performing a dimensional analysis. The drag coefficient $c_d = \frac{\tau}{U_0^2/2}$ is often of interest (Section 2.3). It is a function of ν or the height of the roughness, but it varies very slowly and it is often possible to choose an average value and consider the stress as simply proportional to the square of the mean velocity.

It is characteristic of turbulence that we can develop the velocity field into Fourier components each of which can be regarded as an "eddy". Then we always find eddies of scales down to a certain minimum. The energy spectrum E(k) (k is wave number $k = \frac{2\pi}{\lambda}$, where λ is a wave length) typically may look as in Fig. A.2. E(k) have the property that the area shown is the kinetic energy in an interval of wave numbers k to k + dk. Then, integrating the roughness but σ_u, $\bar{u}_z$, etc. do not. The reason is that the roughnesses do nothing more than account for the zero of the velocity profile, and in general far from the roughness elements, the fluid "feels" only the momentum stress and the distance from the plane. For example, the integral length scale $\ell = \alpha_3 z$, where α_3 is another universal constant. Thus the eddy size increases to the center, becoming proportional to the pipe diameter, i.e. $\ell \sim D$. This is a general tendency, i.e. the largest eddies become as large as possible and furthermore these are the eddies that contain the bulk of the energy.

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$$\frac{u'^2+v'^2+w'^2}{2} = \int_0^\infty E(k)dk \tag{A.4}$$

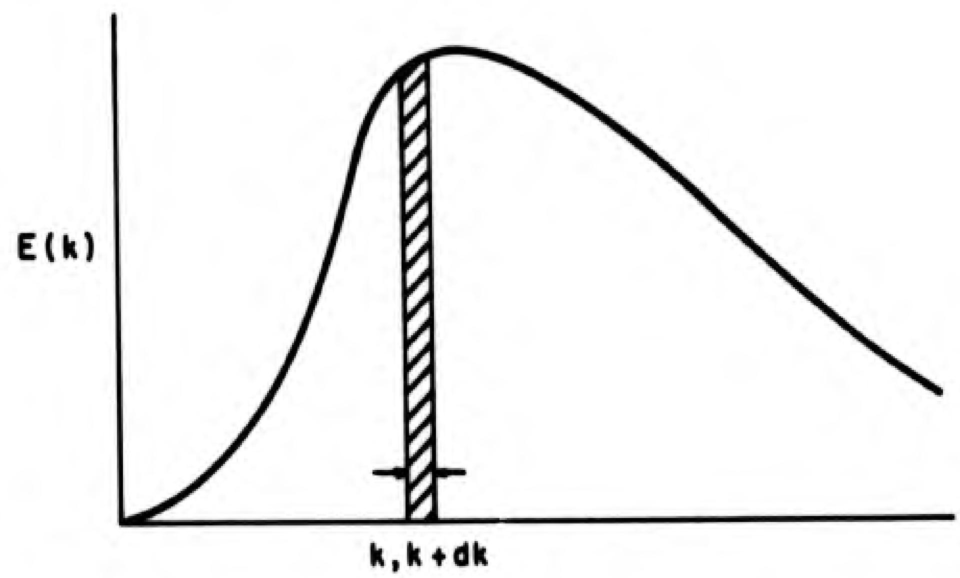

Fig. A.2. Energy spectrum as a function of wave number.

The peak of the curve corresponds to the integral length scale (not precisely defined here) or to the length ℓ of the energy-containing eddies.

We may discuss the smallest length scales. For these, viscosity, which is negligible for scales of length ℓ, becomes important in smoothing out velocity fluctuations. The small-scale fluctuations derive their energy from the larger eddies in a process of energy cascade arising from the non-linear terms in the equations of motions. The viscosity prevents infinitesimally small eddies by ending the energy cascade and turning the kinetic energy into heat. We might expect that at large Reynolds numbers the small viscosity would be too small to affect even the smallest scales. However, the non-linear terms counteract this by generating motion at scales small enough to be affected.

Since small length scales of motion have small time scales, one may assume that their properties are independent of the properties of large-scale motion and are functions only of the dissipation rate ϵ and of ν. Dimensional analysis then supplies the velocity and length scales called Kolmogorov microscales of length, time and velocity,

$$\ell_v = (\nu^3/\epsilon)^{\frac{1}{4}}, \quad t_v = (\nu/\epsilon)^{\frac{1}{2}}, \quad u_v = (\nu\epsilon)^{\frac{1}{4}} \qquad \text{(A.5)}$$

Notice that the Reynolds number $\ell_v u_v/\nu = 1$, which precisely implies the importance of viscosity for these scales of motion.

The dissipation rate can be related to the characteristics of the energy-containing eddies σ_u and ℓ. These suffer dissipation not by friction but by the non-linear process of passing energy down to smaller eddies. The dissipation rate ϵ should be a function, therefore, of the properties of the energy-containing eddies, say σ_u and ℓ, so $\epsilon \sim \sigma_u^3/\ell$ (Batchelor, 1953). Thus, dissipation proceeds at the rate dictated by the inviscid inertial behavior of the large eddies. This estimate is supported by experiment. One of the consequences is that the decay time of the eddy is one turnover time and so the turbulent process is as "dissipative" as possible.

Topic 2. Experiments Involving Erosion of Density Interfaces

2.1 Introduction.

Oceanographers, meteorologists and engineers have a considerable interest in problems related to the erosion and motion of interfaces between fluids of different densities and the related problems of momentum, heat and salt fluxes across these surfaces. Such interfaces occur frequently in the atmosphere and in oceans, lakes and reservoirs. An example of the practical importance of these studies is artificial destratification of reservoirs to improve water quality in the hypolimnion (stagnant region below the thermocline). One method involves the pumping of fluid from the hypolimnion through a tube and discharging the water into the epilimnion in a jet downward from the free surface. The heavy discharged water moves down to the interface and the mixing in the upper layer and the turbulence generated by the jet erodes the interface, causing it to weaken, move downward and eventually disappear (Brush, 1970).

Geophysical implications of mixing across density interfaces are numerous. In the oceans, for example, suddenly increased stress forces exerted by the wind at the water-air surface will increase the turbulence and cause the upper mixed layer to increase in depth at a rate dependent on the stress, the instantaneous depth of the layer, the density jump across the interface and, perhaps, other effects (Kato and Phillips, 1969, Kraus and Turner, 1967). In the atmosphere, inversions are common and the motion of these surfaces and heat, momentum and moisture fluxes across them are of great importance to our understanding of atmospheric turbulence and its parameterization in numerical models.

2.2 Experiments without Shear.

A number of laboratory experiments have been designed to help give an understanding of the problem. The first was by Rouse and Dodu (1955) and consisted of a vessel with two layers of liquid of different densities, as in Fig. 2.1.

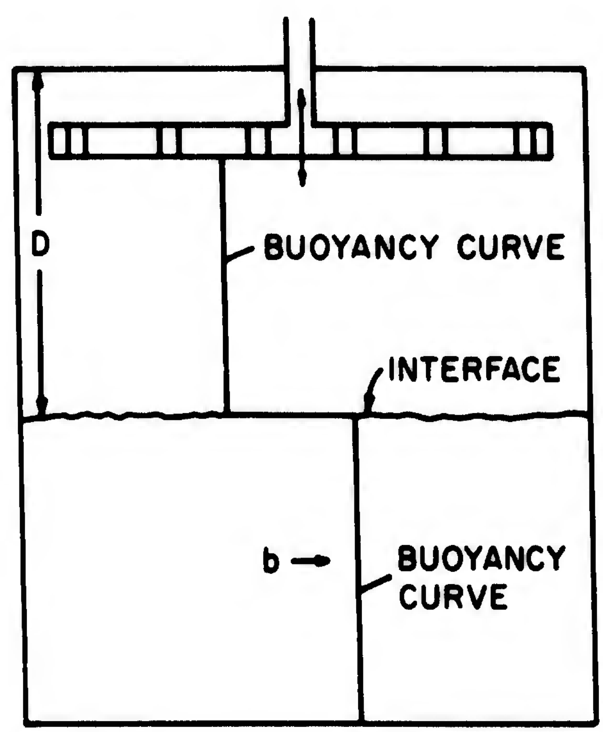

Fig. 2.1. Experiment of Rouse and Dodu.

A grid of metal bars was oscillated vertically in the upper layer with a small stroke a and observations were made of the entrainment velocity, u_e, or the downward velocity of propagation of the interface. In this and other figures the interface is shown as a discontinuity of density. Actually it is a thin layer of thickness 1-2 cm. The thickness is independent of the Richardson number[1] (Moore and Long, 1971, Wolanski, 1972, Crapper and Linden, 1974).

A detailed understanding of the turbulence in this experiment has not yet been achieved but Linden (1973) has performed allied experiments and has suggested that the large eddies in the upper mixed layer deflect the interface downward, storing potential energy. When this is released by upward motion, a portion of the heavier fluid is ejected into the homogeneous layer and then carried away by the turbulent eddies, leaving the interface sharp again.

The experiment of Rouse and Dodu is typical of those without shear and, as we have seen, the only source of energy is the energy flux divergence term of Eq. (1.10). We will see that similar experiments have been constructed with shear (Section 2.3) because this yields an energy supply which must be of great importance in oceanic and atmospheric motions. When interfaces exist in the presence of shear, the details of the turbulence are similar to those described by Linden and do not resemble Kelvin-Helmholtz wave-breaking (Moore and Long, 1971).

Cromwell (1960) constructed a similar experiment to simulate the pycnocline, but the first reliable data were obtained by Turner (1968). Turner ran two different experiments. One was similar to that described above; the other had stirring in both layers. In the first experiment the lower fluid was agitated and fluid was withdrawn from the stirred layer at a rate adjusted to

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[1] Very recent experiments in the Moore and Long apparatus support the conjecture by the author (Long, 1973) that the interface thickness is proportional to the depth of the mixed layer.

keep the interface at the same distance from the grid. The entrainment velocity is then defined by $Au_* = Q$ where Q is the volume withdrawn per unit time and A is the cross-sectional area of the tank. In the second experiment both layers are turbulent and, with the same stirring action by the two grids, the interface stays at mid-point.

Theoretical considerations of these experiments involve the concept of buoyancy, defined in Eq. (1.4). In all experiments, the density difference is very small so that the Boussinesq approximation is a very good one. As we see in Eqs. (1.6)-(1.8), this means that b is the only quantity involving density or gravity entering the analysis.

In the one-grid experiment with the upper level mixed, if the interface is allowed to move downward at speed u_* (no fluid added or subtracted), the buoyancy flux q satisfies the second equation in (1.9),

$$\frac{\partial q}{\partial z} = \frac{\partial \bar{b}}{\partial t} \tag{2.1}$$

where $\bar{b}$ is the mean buoyancy in the upper layer and z may be taken downward from the top of the upper layer. In Eq. (2.1) $\bar{b}$ changes little with height and we may write $\bar{b}(z,t)$ as

$$\bar{b}(z,t) = b_o(t) + \overline{\Delta b}(z,t)$$

where $\overline{\Delta b}(z,t)$ expresses the (small) variation of $\bar{b}$ over the depth of the "homogeneous" layer and b_o is the buoyancy at the top of the "homogeneous" layer. Then

$$\frac{\partial q}{\partial z} = \frac{\partial b_0}{\partial t} + \frac{\partial}{\partial t}[\overline{\Delta b}(z,t)] = -\frac{d}{dt}(\Delta b) + 0\left(\frac{\partial}{\partial t}\overline{\Delta b}\right)$$

where Δb is the buoyancy jump across the interface. Since $q = 0$ at $z = 0$, the flux at the interface is

$$q = -D\frac{d}{dt}(\Delta b) + 0\left(D\frac{\partial}{\partial t}\overline{\Delta b}\right)$$

Thus q is given by the first term with a relative error of order $\overline{\Delta b}/\Delta b$ where $\overline{\Delta b}$ is now the increment of $\bar{b}$ over the depth. This is observed to be very small whenever density interfaces occur. We give its order of magnitude in Eq. (2.33) for experiments with and without shear. Thus the flux at the interface is

$$q = -D\frac{d}{dt}(\Delta b)$$

or

$$q = -\frac{d}{dt}(D\Delta b) + u_* \Delta b \qquad (2.2)$$

Let us apply considerations of mass continuity to an exper'ment somewhat more general than this, in which there is initially a basic density variation $\rho(z)$. The upper portions are agitated and a homogeneous layer forms of density ρ_1 and depth D. In time dt the interface depth increases to D + dD. Conservation of mass yields

$$(\rho_1 + d\rho_1)(D + dD) = \rho(D)dD + \rho_1 D$$

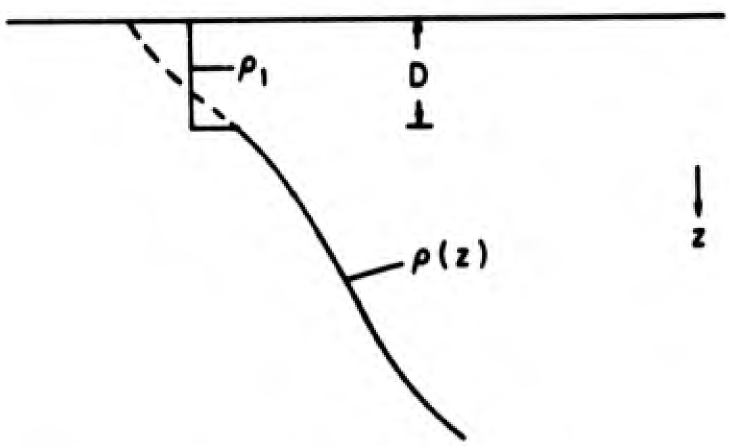

Fig. 2. 2. Erosion of a continuously stratified fluid.

This may be written

$$d(D\,\Delta b) = D\frac{db}{dD}\,dD \qquad (2.3)$$

where Δb is the buoyancy jump across the interface and b(z) is the original buoyancy distribution in the fluid at rest. In the present case, the lower fluid has a uniform density, so $db/dD = 0$ and $D\Delta b$ is constant. Therefore, from Eq. (2. 2), $q = u_* \Delta b$. This result may be used to define an entrainment velocity when both layers are agitated. In this case, if ρ_0 is the average of the densities of the two mixed layers, the buoyancy flux at the interface is $q = -(D/2)d(\Delta b)/dt$ and we may define the entrainment velocity to be

$$u_* = -\frac{D}{2\Delta b}\frac{d}{dt}(\Delta b) \qquad (2.4)$$

Quantities on the rhs are all easily measured.

There have been a number of recent experiments similar to those of Rouse and Dodu and of Turner, for example by Brush (1970). Equipment identical to that of Turner was constructed by Wolanski (1972), and the one- and two-grid experiments were run with stratification caused by heat, salt, sugar, suspensions of sediments and minute silica spheres. Additional experiments have been run in Turner's apparatus by Linden (1973), Crapper (1973) and Crapper and Linden (1974) using heat and salt.

An important result of the experiments by Turner (1968) may be expressed as

$$\frac{u_e}{\omega} = C\left[\frac{\omega^2}{(\Delta b)}\right]^n \qquad (2.5)$$

where ω is the frequency of the oscillating grid and C is independent of ω and Δb. A number of lengths are kept constant in the experiment so that the dimensional quantity C may vary with these. Turner found that for larger values of $\Delta b/\omega^2$, the exponent $n = 1$ when stratification is related to temperature differences and $n = 3/2$ when related to differences in salt content. Later investigations (Wolanski, 1972) have confirmed these results and, very recently, Crapper and Linden (1974) have shown rather convincingly that the difference in the values of n is due to the influence of the relatively large molecular conductivity k_h in the heating experiments (the coefficient k_s is much smaller for salt). It appears that whenever the $n = 1$ law describes the entrainment, the thin layer of strong density variation has an inner layer or core in which molecular diffusion is important. In support of this interpretation, Claes Rooth (Turner, 1973) in unpublished work has found a 3/2 dependence in heating experiments when larger turbulent velocities are generated, so that it appears well established that the 3/2 dependence is appropriate for larger Péclet numbers, $Pe = \sigma_u \ell / k_h$ or $\sigma_u \ell / k_s$, where σ_u and ℓ are the velocity and length units of the turbulence discussed in Topic 1. Crapper and Linden suggest a threshold value of $Pe \simeq 200$ when $\sigma_u = \sigma_u'$ and $\ell = \ell'$ are characteristic of the turbulence near the interface. The dependence on Reynolds number, Re, has not been established because of the rather small ranges of Re in the experiments, but both Wolanski (1972), who varied Re by a factor of three, and Crapper and Linden (1974) report very weak dependence, if any[1]. We may, therefore, write for large Pe and Re, and strong stability,

$$\frac{u_e}{\omega} = C_1 \frac{\omega^3}{(\Delta b)^{3/2}} \qquad (2.6)$$

where C_1 is a function of a, D, and a_1, a_2..., where a is the total stroke of the oscillating grid, and where a_1, a_2... are lengths characteristic of the grid. It is convenient to introduce a dimensionless quantity K_n by the definition

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[1] The independence of molecular quantities is common in turbulence as we learned in the Appendix to Topic 1. In Turner's original paper (1968), he expressed the belief that the n = 1 law was the fundamental one and that in some manner the very low diffusivity of salt caused the n = 3/2 law. He has changed his mind on the basis of the evidence we present here (Turner, 1973).

$$C_1 = \frac{a^4}{D^3} K_n\left(\frac{a}{D}, \frac{a}{a_1}, \frac{a}{a_2} \cdots\right) \qquad (2.7)$$

and, therefore,

$$\frac{u_e}{u_*} = K_n Ri^{*-\frac{3}{2}}, \quad Ri^* = \frac{D\Delta b}{u_*^2} \qquad (2.8)$$

where $u_* = \omega a$. It is likely (see Section 2.6) that K_n is independent of a/D when this ratio is small.

2.3 Experiments with Shear.

Several experiments have been constructed to introduce shearing currents in turbulent density-stratified systems in an effort to simulate atmospheric and oceanic phenomena. The first of these of direct relevance to our discussion was that of Kato and Phillips (1969). The apparatus was a large circular annular channel filled with salt water with an initial linear density gradient. A constant stress $\tau = u_*^2$ was applied by rotating a flat screen at the surface (Fig. 2.3).

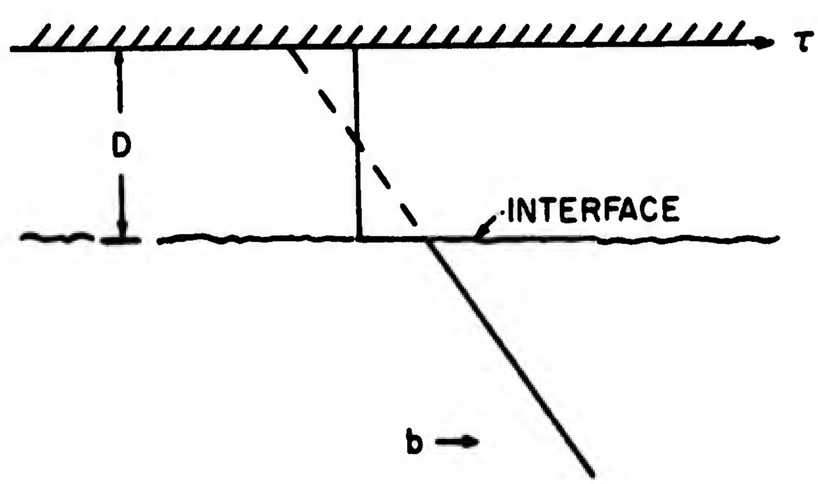

Fig. 2.3. Experiment of Kato and Phillips.

They found, for larger values of Ri^*,

$$\frac{u_b}{u_*} = K_1 Ri^{*-1} \qquad (2.9)$$

where Ri^* is of the same form as in Eq. (2.8) and Δb is the buoyancy jump from the upper mixed layer to the quiescent region below.

It is important for later purposes to present an analysis of the drag coefficient $2u_*^2/U^2$ where U is the speed of the screen in the experiment of Kato and Phillips. They found that U/u_* increased with time (or depth) with u_* held fixed. At first glance one might expect this to be an influence of the stable density distribution in the fluid system, but on closer consideration it seems more reasonable to neglect Δb entirely and consider the flow and turbulence in the upper layer as turbulent flow due to the motion of a rough plate at $z = 0$ with the interface at $z = D$ serving only to reduce the mean velocity to zero at that level. This is supported by a description by Kato and Phillips: "The movement [of a line of hydrogen bubbles] indicated that the mean velocity varied most rapidly near the screen and near the entrainment interface, being almost constant in the central region, where the velocity was typically about half that of the screen". The mean motion seems very close to that in turbulent plane Couette flow (Robertson, 1959), and a theory for the ratio U/u_* may be obtained by use of the technique of Izakson (1937) and Millikan (1938). We assume for the mean velocity near the screen (in a coordinate system moving with the screen)

$$\frac{\bar{U}}{u_*} = f\left(\frac{z}{z_0}\right) \qquad (2.10)$$

where z_o is the roughness length and $\bar{u}(z)$ is the mean velocity. In the interior we adopt the velocity-defect law (Monin and Yaglom, 1971)

$$\frac{U/2-\bar{u}}{u_*} = m_1\left(\frac{z}{D}\right)$$

The defect law should hold near $z = z_o$ so that we may match the two expressions for $\bar{u}$ in this region. Writing $U/2u_* = m_2\left(\frac{z_o}{D}\right)$, we get

$$m_2\left(\frac{z_o}{D}\right) - f\left(\frac{z}{z_o}\right) = m_1\left(\frac{z}{D}\right)$$

Putting $z_o/D = \eta$, $z/z_o = \theta$, we have

$$m_2(\eta) - f(\theta) = m_1(\eta\theta)$$

Differentiating with respect to η and θ, we get

$$\eta m_2' = m_1'\theta\eta$$

$$-\theta f' = m_1'\eta\theta$$

so that

$$\theta f'(\theta) = -\eta m_2'(\eta) = \text{constant}$$

This yields

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln\frac{z}{z_o}$$

where the constant κ may now be identified as Von Kármán's constant. We also get

$$\frac{U}{u_*} = -\frac{2}{\kappa} \ln \frac{z_0}{D} + A_1 \qquad (2.11)$$

where A_1 is another constant. Kato and Phillips give data on U(t) for some of their experiments and, although the paper contains an empirical equation for D(t), it does not agree with the data over the whole time period of the experiment. However, one case permits a comparison and this is shown in Fig. 2.4 in which the theoretical curve is

$$\frac{U}{u_*} = \frac{2}{\kappa} \ln D(t) + 5.78 \qquad (2.12)$$

The single constant was chosen to give U = 27.25 cm/sec at t = 200 sec. The solid curve in the figure is a plot of Eq. (2.12) in which D(t) is given by curve II, Fig. 5 of Kato and Phillips (1969). The data points are from Fig. 3 of Kato and Phillips. All experiments have the same buoyancy gradient. The agreement is remarkable and leaves little doubt[1] that U/u_* is independent of the Richardson number which varied by a factor of 100 over the course of the experiments. In subsequent discussions we suppress any dependence of quantities on z_0/D.

The argument may be reasonably extended to flow of air over the cool surface of the earth. An inversion will be present at some height and the momentum stress will be a function only of the depth, the roughness length and the wind speed at the height of the inversion. This has obvious usefulness in problems of parameterizing the momentum flux near the ground in numerical atmospheric models.

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[1]It also leaves little doubt that a recent suggestion to the contrary by the author is wrong (Long, 1973).

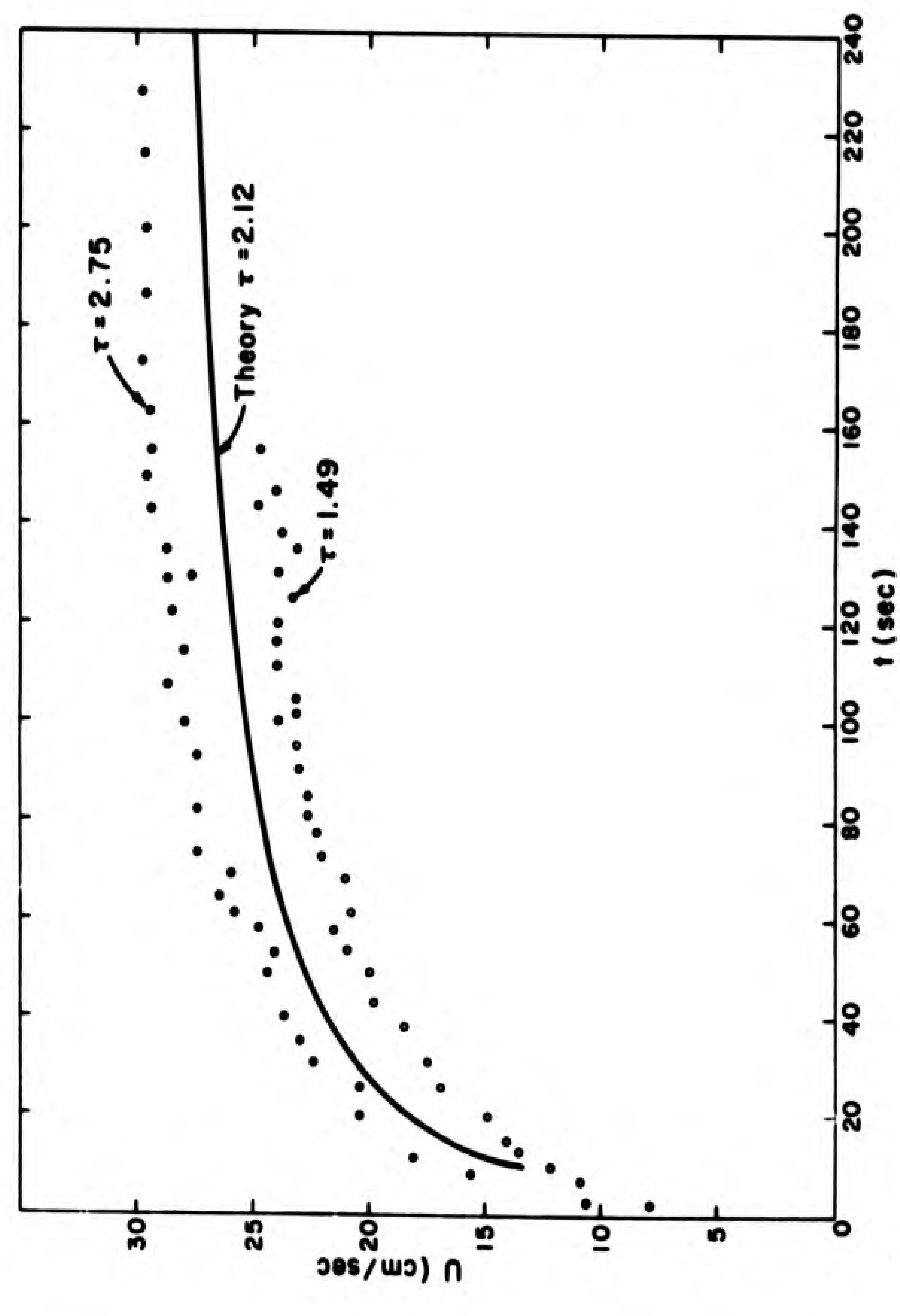

Fig. 2.4. Velocity of screen in Kato and Phillips experiment as a function of time.

An experiment by Moore and Long (1971) was constructed to permit a steady state. In a large channel shaped like a race track, fluid was injected from nearly horizontal jets at bottom (salt water) and top (fresh water) in opposite directions to obtain a shearing current (Fig. 2.5). Mean zero vertical

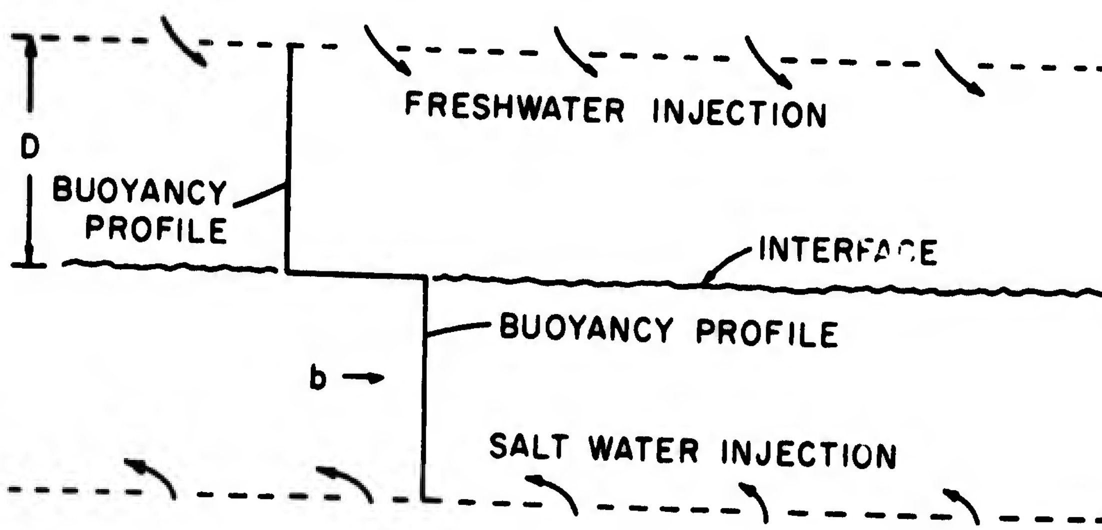

Fig. 2.5. Moore and Long experiment.

velocities were achieved by withdrawing equal volumes of fluid through numerous holes at bottom and top. At larger values of the density difference, two homogeneous layers existed at top and bottom with an interface in the middle. The salt water in the jets comes from a reservoir and the withdrawn fluid at the bottom is pumped back into the reservoir which is kept at a constant level. The jets at the top are of tap water and the slightly salty withdrawn fluid at the top is pumped to waste. Since the fluid returned from the bottom to the reservoir is somewhat less salty than that in the lower jets, salt must be added continually to keep the reservoir at a fixed density. The added salt is transported vertically by the turbulence. The salt flux is known, of course, and this can be used directly to compute the buoyancy flux. The experiment yielded

$$q = K_3 \frac{(\Delta u)^3}{D} \tag{2.13}$$

where Δu is the velocity difference between mean velocities measured near the top and bottom. If we define the entrainment velocity by $u_e \Delta b = q$, Eq. (2.13) yields the same result as in Kato and Phillips (Eq. 2.9) if, as seems very likely from the discussion of the Kato and Phillips experiment, $\Delta u/u_*$ is independent of the Richardson number, where u_*^2 is the constant momentum flux in the tank. Moore and Long also ran unsteady experiments similar to those of Kato and Phillips with a fluid with initial linear buoyancy gradient and subject to the system of jets and withdrawals at the bottom only. They obtained the result $D \propto t^{\frac{1}{2}}$. Since Eq. (2.9) may be written

$$\frac{dD}{dt} = K_1 \frac{u_*^3}{D\Delta b}$$

and since Eq. (2.3) indicates that $\Delta b \propto D$ in erosion of a linear density gradient, the behavior $D \propto t^{\frac{1}{3}}$ is consistent with Eq. (2.9).

The velocity observations by Moore and Long apparently relate to those found by Kato and Phillips. Moore and Long report a velocity distribution with strong shears at top and bottom and across the density interface region. This, of course, is exactly the behavior to be expected from plane Couette flow in each layer.

Finally, in a recent experiment by Wu (1973), the source of energy and shear was a current of air blowing over a vessel containing a two-fluid system. The apparatus and the shear produced is shown in Fig. 2.6. Wu also obtained Eq. (2.9) although his coefficient of proportionality was much smaller. He conjectured that this was because of the very different shear produced in a closed container.

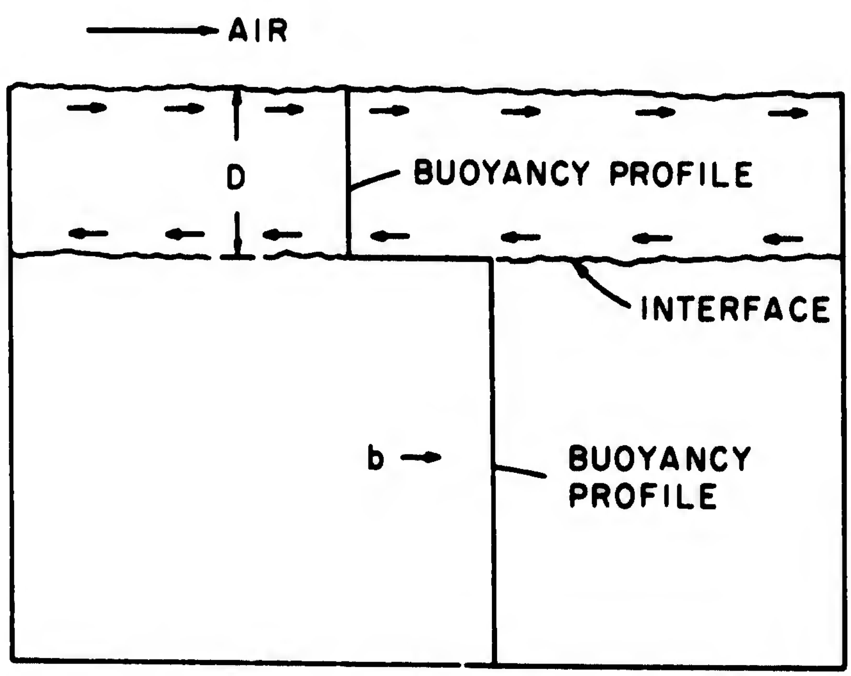

Fig. 2.6. Wu's experiment.

2.4 Comparison of Experiments with and without Shear.

The different dependence on Ri^* for the two experiments has been the source of perplexity (Turner, 1973, Linden, 1973) because the mixing processes appear to be very similar. Indeed, Linden has stated that the Kato and Phillips data are also consistent with a -3/2 behavior, although support for the statement seems lacking. In the rest of this topic, we attempt to contribute to a unified understanding of the two results.

Turner (1973) has made the valuable suggestion that the erosion of the interface should depend on the properties of the turbulence near the interface, in particular on the rms velocity σ_u' and the integral length scale ℓ' near the interface. Thus, he proposed the form

$$\frac{u_*}{\sigma_u'} = f(Ri), \quad Ri = \frac{\ell' \Delta b}{\sigma_u'^2} \tag{2.14}$$

where possible dependence on other quantities is suppressed and it is assumed that Pe and Re are large. In an attempt to determine the dependence on Ri from his density-interface experiments, in which σ_u' and ℓ' were not measured, Turner used unpublished experimental data by Thompson (see Turner, 1973, Crapper and Linden, 1974, for a description). Thompson used Turner's apparatus with a homogeneous fluid and one grid. He measured σ_u and ℓ at many levels, where σ_u is the rms velocity and ℓ is the integral lengthscale at depth z. As reported by Crapper and Linden, Thompson found that ℓ increased linearly with distance from the grid but was independent of ω. He also found that σ_u decreased with z but at a given z was proportional to ω. Although Thompson's experiment had no density variation, Turner (1973), Thorpe (1973) and Crapper and Linden (1974) have assumed that the results are directly applicable to the mixing experiments. Thus, at z = D they use

$$\frac{\sigma_u'}{\omega a} = C_2\left(\frac{a}{D}, \frac{a}{a_1}, \frac{a}{a_2} \dots\right) \tag{2.15}$$

$$\frac{\ell'}{D} = C_3\left(\frac{a}{a_1}, \frac{a}{a_2} \dots\right) \tag{2.16}$$

so that Eq. (2.8) may be written

$$\frac{u_*}{\sigma_u'} = K_4 \, Ri^{-\frac{3}{2}} \qquad K_4 = K_4\left(\frac{a}{D}, \frac{a}{a_1}, \frac{a}{a_2} \cdots\right) \tag{2.17}$$

Notice that the proportionality of σ_u' and ω follows from dimensional analysis but only when the fluid is homogeneous because the presence of an interface introduces a new quantity involving time, namely Δb.

We may also obtain a dependence on Ri for the shearing experiments. With shear, we have from Eq. (1.9),

$$\frac{\partial \tau}{\partial z} = \frac{\partial \bar{u}}{\partial t} \tag{2.18}$$

where $\bar{u}$ is the mean horizontal velocity at depth z. In the steady-state experiments of Moore and Long (1971), $\partial\tau/\partial z = 0$ and therefore τ is constant with height. Since $\tau = -\overline{u'w'}$, and since the correlation coefficient is very likely to be of order one in the homogeneous layers, it follows that $u_* = \tau^{\frac{1}{2}}$ is proportional to σ_u'. The interface introduces the length D and it seems reasonable that the eddies fill the whole depth as they would if the interface were a rigid surface. Thus we use $\ell' \sim D$ and obtain

$$\frac{u_*}{\sigma_u} = K_5 Ri^{-1} \tag{2.19}$$

for the Moore and Long experiment. In the experiment of Kato and Phillips, we may use Eq. (2.18) to obtain the increment of τ over the depth D. It is

$$\frac{\Delta\tau}{\tau} \sim \frac{U}{T_1 u_*^2} D \tag{2.20}$$

where T_t is the time period for a change of depth of order D so that $T_t \sim D/u_e$.
Therefore,

$$\frac{\Delta\tau}{\tau} \sim \frac{U}{u_*}\frac{u_e}{u_*} \sim Ri^{*-1} \qquad (2.21)$$

This reveals that the stress varies very little over the depth, $u_* \sim \sigma_u'$, and Eq. (2.19) again holds.

Thus, two different entrainment velocities, Eqs. (2.17) and (2.19) are indicated in the two cases even when the characteristics of the eroding eddies are the same and this is more perplexing than the difference in the exponent of Ri^*. The subsequent discussion of this topic questions the applicability of Thompson's experiment, in particular Eq. (2.15), to an experiment with a density interface and suggests that Eq. (2.17) is incorrect[1]. The difficulty is indicated by a simple analysis based on the assumption that the Ri^{-1} law or Eq. (2.19) is correct with or without shear. Since Eq. (2.8) is indicated by direct measurements and since $\ell' \sim D$ also seems reasonable in experiments without shear, we obtain

$$\frac{\sigma_u'}{\omega a} = K_5\, Ri^{*-\frac{1}{5}} \qquad (2.22)$$

instead of Eq. (2.15). Thus, if the assumptions are correct, there is a very weak dependence on the Richardson number which could not, of course, be revealed by Thompson's experiments because his experiments had no interface. Such a weak dependence is, nevertheless, capable of accounting for the difference in the power laws.

---

[1] Linden (1973) has attempted to derive the $Ri^{-\frac{3}{2}}$ law by order-of-magnitude arguments. If we accept the conclusions of these notes, Linden's argument must also be incorrect.

2.5 Energy Arguments.

The dependence of σ_u'/u_* on Ri^* in experiments without shear, as indicated in Eq. (2.22), may be obtained by a plausible argument. When there is shear, we have seen that experiment indicates

$$q \sim \frac{\sigma_u'^3}{D} \sim \frac{u_*^3}{D} \qquad (2.23)$$

Let us now evalutate q in the homogeneous layer near the interface. We get $q \sim \sigma_u'\sigma_b'$ where σ_b' is the rms buoyancy fluctuation near the interface, and we make the plausible assumption that the correlation is of order one. Thus

$$\frac{\sigma_u'^2}{\sigma_b' D} \sim 1 \qquad (2.24)$$

so that kinetic energy and available potential energy, $\sigma_b' D$ (see the discussion in Section 1.4) are of the same order in the "homogeneous" layer[1]. Although the layer has very little density variation, this result may be obtained by considering first an experiment with very strong turbulence imposed externally. Then $\sigma_u'^2/\sigma_b'\ell'$ will be very large. As we decrease the turbulence in successive experiments, this ratio will decrease. If turbulence continues to exist, the ratio has a limiting value, however, because $\sigma_u'^2/\sigma_b'\ell' < 1$ would imply that $T' < V'$ and the turbulence would die out as we discussed in Section 1.4. Thus $\sigma_u'^2/\sigma_b'\ell'$ should approach a constant as stability increases. The argument is equally valid with or without shear. Thus, using Eq. (2.24) when shear is

---

[1] We mean that the two energies are proportional. The constant of proportionality is actually very small (see the discussion at the end of Section 3.2).

absent to eliminate σ_b' in the relationship $q \sim u_* \Delta b \sim \sigma_u' \sigma_b'$, the -3/2 law leads to

$$\frac{u_e}{u_*} \sim \frac{\sigma_u'^3}{D u_* \Delta b} \sim \frac{u_*^3}{(D \Delta b)^{\frac{3}{2}}} \tag{2.25}$$

or

$$\frac{\sigma_u'}{u_*} \sim Ri^{*-\frac{1}{6}} \tag{2.26}$$

as in Eq. (2.22). The decrease of rms velocity with increase of Richardson number when density variations are present is probably caused by the weak density gradient in the layers that we have called "homogeneous". In a layer as a whole, the slight density variation still has dynamic importance as indicated by the proportionality of kinetic energy and available potential energy and by the fact that q, varying linearly in the layer, has a relatively large value near the interface. Such arguments have been advanced earlier by the author (Long, 1972, 1973).

The energy argument may be amplified. Rouse and Dodu (1955) and others (Kato and Phillips, 1969, Turner, 1973, Wu, 1973) have suggested that the Ri^{*-1} law implies that the change of potential energy is proportional to the energy supply by the external source. Thus, as we have seen in Section 1.4, the average rate of increase of potential energy per unit mass is q, so that the rate of increase of potential energy for the system is proportional to qD. In the Kato and Phillips experiment, for example, the rate of working of the external force is τU. If these are proportional

$$q \sim \frac{\tau U}{D} \sim \frac{u_*^3}{D}$$

as in Eq. (2.23). The same conclusion cannot be reached for cases without shear and on this basis it may be argued that the $Ri^{*-\frac{3}{2}}$ law does not conform to any simple energy argument. We may show, however, that the last conclusion is not correctly drawn. The energy equation, Eq. (1.10), for experiments with or without shear is

$$\frac{\partial}{\partial t}(\overline{c'^2/2}) = -\frac{\partial}{\partial z}[\overline{w'(c'^2/2+p')}] + \tau\bar{u}_z + q - \epsilon \qquad (2.27)$$

where $\bar{u}_z = 0$ in the experiment without shear. In the shearing experiments, the velocity difference is proportional to $\tau^{\frac{1}{2}}$, and the two energy-source terms, as well as the dissipation ϵ, are of order $\sigma_u'^3/D$ or $\sigma_u'^3/\ell'$ near the interface. If $q \sim u_* \Delta b$ is also of this order, we obtain $u_*/\sigma_u' \sim Ri^{-1}$ as in Eq. (2.19). When shear is absent, the single source term is the first term on the rhs of Eq. (2.27) and is also of order $\sigma_u'^3/\ell'$. The Ri^{-1} law again implies equality of all sink and source terms. The correct interpretation of experimental results thus seems to be that the turbulence has a character that causes potential energy to increase at a rate proportional to the rate at which kinetic energy is supplied to the region of the interface and not necessarily proportional to the rate of generation of kinetic energy at the external source.

An additional piece of information may be added in relation to experiments without shear. If we assume that the small buoyancy difference $\overline{\Delta b}$ across the "homogeneous" layer is of order of the rms buoyancy fluctuation (implying an eddy length scale of the order of the depth of the layer), Eq. (2.24) and Eq. (2.26) lead to

$$\frac{\overline{\Delta b}}{\Delta b} \sim Ri^{*-\frac{4}{3}} \tag{2.28}$$

This quantity was measured by Wolanski (1972) for the salt experiments (Fig. 2.7). There is good agreement with Eq. (2.28), especially at higher values of Ri^*. The ratio $\overline{\Delta b}/\Delta b$ is proportional to Ri^{*-1} in experiments with shear.

Finally, we remark that in both of the basic experiments considered in these notes, $q \sim u_* \Delta b \sim \sigma_u' \sigma_b'$ so that

$$\frac{u_*}{\sigma_u'} \sim \frac{\sigma_b'}{\Delta b} \sim \frac{\ell' \sigma_b' / \sigma_u'^2}{\ell' \Delta b / \sigma_u'^2} \sim Ri^{-1} \frac{V'}{T'} \tag{2.29}$$

where V' is available potential energy and T' is turbulent kinetic energy. The existence of turbulence implies that $V' \leq T'$, so that the experimental result $u_* / \sigma_u' \sim Ri^{-1}$, in both experiments shows that u_* / σ_u' <u>is a maximum consistent with the maintenance of the turbulence</u>.

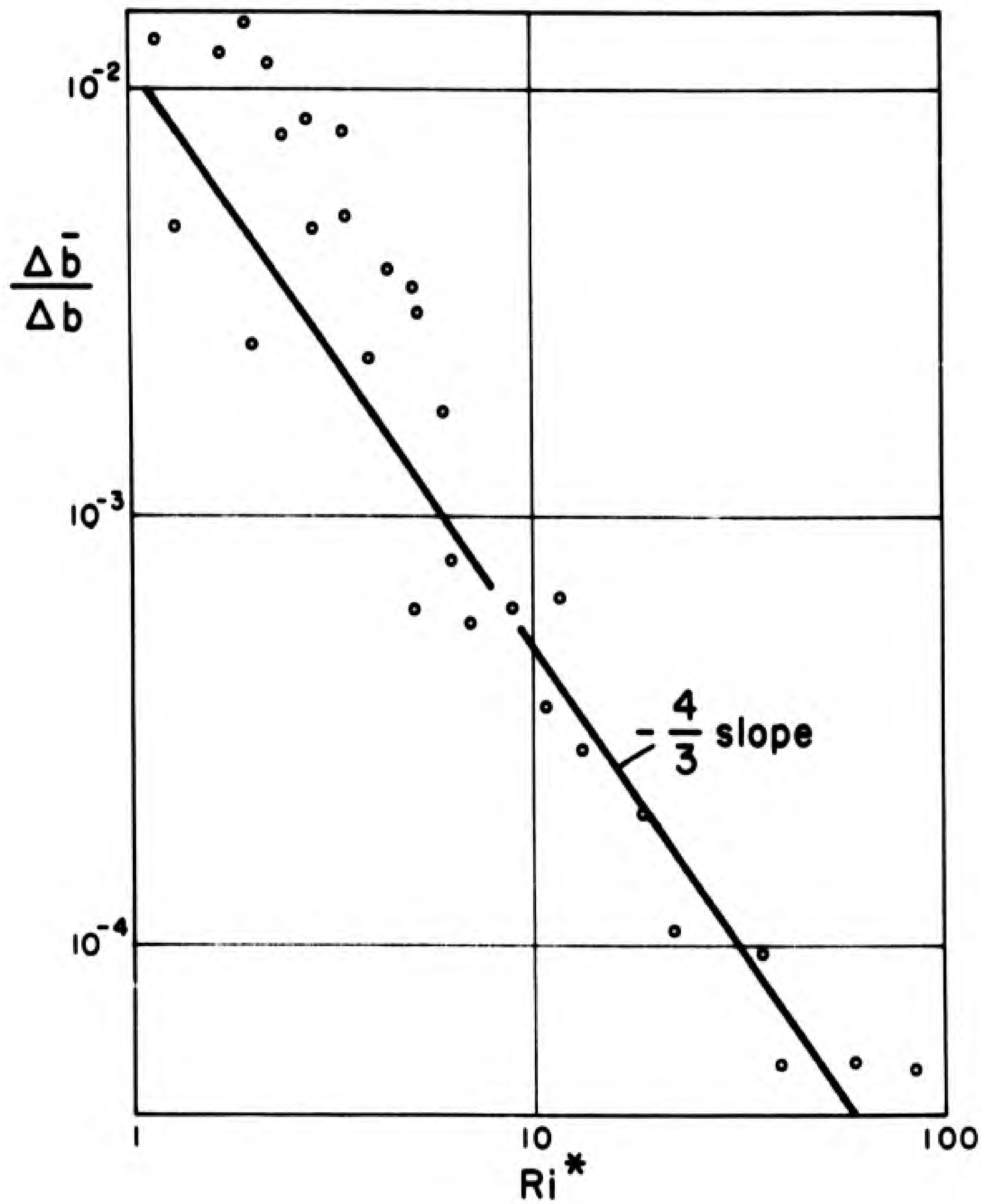

Fig. 2.7

Non-dimensional ratio of buoyancy differences across a "homogeneous" layer. The data are from Wolanski (1972).

2.6 Discussion of an Idealized Experiment.

It is instructive to discuss an idealized experiment, which has limiting behaviors close to the two basic experiments. This is a two-fluid system (Fig. 2.8)

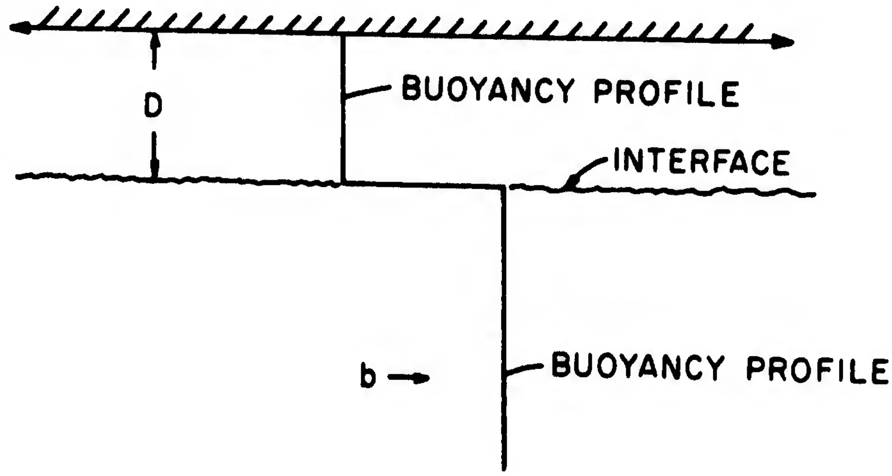

Fig. 2.8. Idealized experiment.

with a plate or screen oscillating back and forth in its plane. The densities are nearly uniform in each layer and the upper layer presumably deepens with time. The oscillation is produced by applying to the plate a stress $\tau = u_*^2$ which is constant in magnitude over each half cycle. The amplitude of the oscillation is a and may be large or small. If the Péclet and Reynolds numbers are large, it seems reasonable to assume

$$\frac{u_e}{u_*} = f(Ri^*, \frac{a}{D}) \qquad (2.30)$$

where $D\Delta b$ in Ri^* is constant from considerations of mass continuity. If we use the classical case of flow over a flat plate as a guide (Monin and Yaglom, 1971; see also the discussion in the Appendix to Topic 1), it seems likely that f is independent of viscosity, in the case of a smooth plate, or of the nature of the roughness in the case of a rough plate.

If we let $a/D \to \infty$, u_e/u_* will become independent of a/D and we obtain an experiment similar to that of Kato and Phillips (1969). Their results and the similar experiments by Moore and Long (1971) and by Wu (1973) indicate

$$\frac{u_e}{u_*} = K_7 Ri^{*-1} \qquad (2.31)$$

On the other hand, if we let $a/D \to 0$, we should again find independence of a/D and, since this is similar to Turner's experiment, we may write

$$\frac{u_e}{u_*} = K_8 Ri^{*-\frac{3}{2}} \qquad (2.32)$$

We are again led to two different laws for the entrainment velocity, but this simple experiment indicates that this should not be considered paradoxical since the two limits $a/D \to 0$ and $a/D \to \infty$ are very different. Notice that in a given experiment with fixed a of moderate size, the erosion is at first rapid as in Eq. (2.31), slowing down gradually and tending to the lower rate in Eq. (2.32) as D increases.

It is interesting to compare the time rates of change of D in the various experiments when a/D may be neglected in Eq. (2.30), i.e., when a is either very small (similar to Turner's experiment) or very large (as in Kato and Phillips). If there are two homogeneous layers, we have seen that $D\Delta b$ is constant in time so that in both experiments u_e is constant and $D \propto t$. In fact this is the behavior reported by Kraus and Turner (1967) in Turner's equipment and by Wu (1973) in an experiment with shear. On the other hand, if the undisturbed layer has a linear density gradient, Eq. (2.3) shows that $\Delta b \propto D$ so that the -3/2 law yields $D \propto t^{\frac{1}{4}}$ for experiments without shear and $D \propto t^{\frac{1}{3}}$ with shear (as observed by Kato and Phillips and Moore and Long). Linden has reported a lower speed u_e in Turner's equipment than in the Kato and Phillips experiment in the case of linear density gradient (private communication).

A variation of the idealized experiment of Fig. 2.8 is useful to apply the ideas of Section 2.5 to gain an appreciation of the inevitability of the appearance of homogeneous layers. In Fig. 2.9 there is fluid above and below the oscillating plate which now is porous to permit a buoyancy flux through it. The initial distribution is uniform buoyancy, $-\Delta b/2$, above and $+\Delta b/2$ below the plate. At the beginning of the experiment, eddies form near the plate of small dimensions $h = u_* t$. The Richardson number $Ri^* = h\Delta b/u_*^2$ is small so that buoyancy is unimportant dynamically and the erosion proceeds rapidly with $u_e \sim u_*$. Ri^* is increasing with time, however, because h is increasing, and when $Ri^* \sim Ri$ is of order one, $V' \sim \sigma_b' h \sim T' \sim u_*^2$. Since $\sigma_b' \sim \overline{\Delta b}$, we have $\overline{\Delta b}/\Delta b \sim Ri^{*-1} \sim 1$ so that the density difference

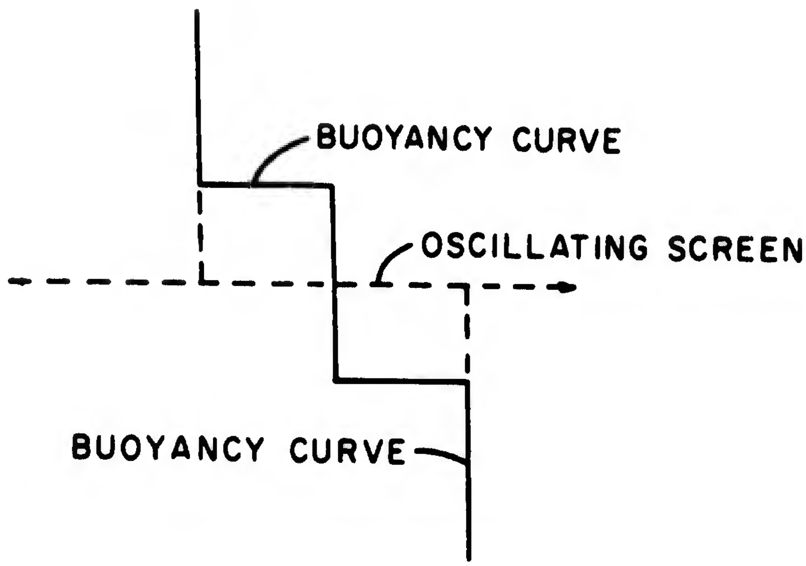

Fig. 2.9. Erosion in a two-fluid system.

across the mixing layer is not yet small. As time goes on, $Ri^* = D\Delta b/u_*^2$ eventually becomes large. However, V' is still of order T' so that $\sigma_b' D/\sigma_u'^2 \sim 1$. With an increase in depth, σ_b' or $\overline{\Delta b}$ must decrease and the homogeneous layer forms. The behavior, as we have seen, is

$$\frac{\overline{\Delta b}}{\Delta b} \sim Ri^{*-1} \quad \text{or} \quad \sim Ri^{*-\frac{4}{3}} \qquad (2.33)$$

depending on the relative amplitude of the oscillation.

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Topic 3. Application of Experimental Results and other Observations

3.1 Introduction.

The experimental observations and the analysis of the observations are of the greatest importance to our understanding of the basic nature of turbulence in stratified fluids. Equally important, however, is the application to practical problems, for example to the construction of numerical models of mixing processes in estuaries and seas, the development and maintenance of thermoclines in lakes and reservoirs, etc. It is not at all obvious that laboratory experiments have relevance to natural systems because it is not possible to model all non-dimensional numbers and it is not known whether it is possible to model those non-dimensional numbers that have important effects on the phenomena of interest. At first glance it is discouraging that the Reynolds numbers of the natural systems are always orders of magnitude greater than those in the laboratory. Nevertheless, as we discussed in Section 2.2, we can frequently create laboratory models in which the Reynolds numbers and Péclet numbers based on the properties of the eddies are large enough so that viscosity and conduction are not important in model or prototype. It then appears from our discussion below that we can infer relationships between the laboratory models and natural systems.

3.2 Parameterization and the Behavior of Eddy Viscosity and Eddy Diffusivity.

We are always faced with the closure problem when we deal with turbulent systems such as the atmosphere and oceans, and we may hope to use laboratory results or observations of the natural systems to help formulate assumptions that will close the problem. This is essentially an engineering problem in that we depart from the search for basic knowledge and seek useful rather than fundamental results. In one of the simplest examples, we may direct attention to a developing one-dimensional problem of turbulence in stratified shearing flow for which the two mean equations are those in Eq. (1.9):

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \tau}{\partial z} \tag{3.1}$$

$$\frac{\partial \bar{b}}{\partial t} = \frac{\partial q}{\partial z} \tag{3.2}$$

Obviously, we have an indeterminate problem with four unknowns and two equations. We may define coefficients of eddy viscosity and diffusion by the equations

$$\tau = K_m \frac{\partial \bar{u}}{\partial z} , \quad q = K_b \frac{\partial \bar{b}}{\partial z} \tag{3.3}$$

but this is not immediately helpful unless we can say something about K_m and K_b. Our hope is to "parameterize" K_m and K_b, i.e. to express them in terms of $\bar{u}$, $\bar{b}$, z and t or in some other more complicated way, to obtain a mathematically determined problem. Observations and experiment are helpful in this respect although no problem of this type has been determined without making rather arbitrary assumptions.

Let us first consider the eddy viscosity K_s. There is one case in which K_s can be determined exactly except for the appearance of a universal constant which, however, has a rather well determined value. This is the homogeneous steady flow over a plane considered in the Appendix of Topic 1. From Eq. (A.1) and the definition in Eq. (3.3), we obtain $K_s = \varkappa u_* z$ so that K_s increases linearly with distance from the plate. In general, we may set $K_s = \alpha_4 \sigma_u \ell$ where σ_u is rms horizontal velocity and ℓ is the integral length scale. In simple cases α_4 may be a universal non-dimensional constant. Of course, $K_s \sim \sigma_u \ell$ may also be obtained from use of the analogy between kinetic theory and eddy exchange of momentum in which σ_u is analogous to the velocity of the molecules and ℓ to the mean free path.

A somewhat different viewpoint is obtained by using a combination of Eqs. (3.1) and (3.3),

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial}{\partial z}\left(K_s \frac{\partial \bar{u}}{\partial z}\right)$$

and calculating the order of magnitude of K_s from this equation. We get

$$K_s \sim \bar{L}^2/\bar{T}$$

where $\bar{L}$ is the length scale of the mean motion and $\bar{T}$ is the time scale of the mean motion. In homogeneous fluids it is quite generally true that these scales are the same as those for the turbulence so that $\bar{L} \sim \ell$ and $\bar{T} \sim \ell/\sigma_u$ which again yields $K_s \sim \sigma_u \ell$. We have seen in Topic 1 that it is also useful in homogeneous fluids to assume that $\ell \sim \bar{L}$ is also of the order of the size of the container, i.e. the eddies tend to be as large as possible. Obviously with stability, vertical motions are inhibited and scales will tend to be smaller. If, however, nearly homogeneous layers appear, separated by density interfaces, it seems likely, as we assumed in Topic 2, that the eddies in such a layer will tend to be of the order of the depth of the layer.

The variation of K_s when stability is important may be discussed with reference to the Moore and Long experiment in Topic 2. The flow in each layer is similar to plane Couette flow, as we see in Fig. 3.1, and τ is constant. Then $K_s = \tau/\bar{u}_z$. We obtain small K_s near the top and bottom

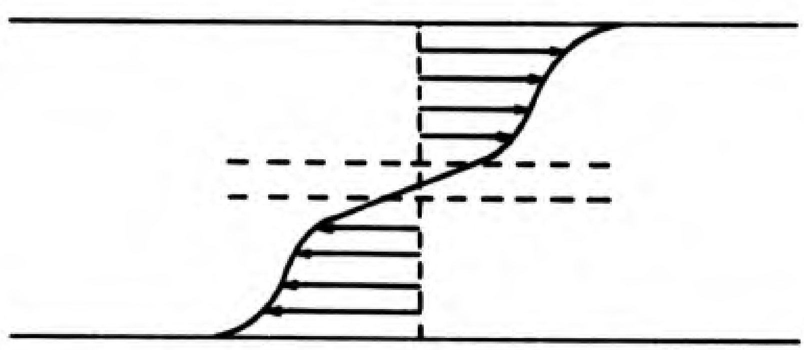

Fig. 3.1. Mean velocity in Moore and Long experiment.

and on both sides of the interface but quite large K_m near the middle of each "homogeneous"layer. Thus, K_m is reduced near rigid surfaces and near and in density interfaces. The thickness of the shear layer near the density interface is, perhaps, $h_i/2$, where h_i is the thickness of the interface layer, so that $K_m \cong \kappa u_* h_i/2 \cong u_* h_i/5$ near the interface. Also $h_i \cong D/6$ so that

$$K_m \cong 0.033\, u_* D$$

compared to $K_m \cong \frac{1}{2} \kappa u_* D \cong .20\, u_* D$ in the middle of the homogeneous layer. It is important to point out that K_m appears to be independent of the overall Richardson number when the Richardson number is large and this contrasts strongly with the behavior of the eddy coefficient of diffusion K_b as we will see. In regions of small, continuous Richardson number, Ellison and Turner (1960) found a small (30%) decrease of K_m from $Ri = 0$ to $Ri = 0.6$.

The variation of K_b may be inferred from the experiment of Moore & Long (1971). In the "homogeneous" layer, we would expect K_b to be similar to K_m because of the weak stability. Estimates in the atmosphere (Rider, 1954, Swinbank, 1960) indicate $0.8 < K_b/K_m < 1.3$ in neutral conditions (forced convection) in which temperature acts like a passive additive. In more accurate laboratory experiments, the ratio σ_n has been found to be higher, about 1.35 (Page, Schlinger, Breaux and Sage, 1952) but the Reynolds numbers were not nearly as large as in the atmosphere.

As Ri increases, Ellison and Turner (1960) found a decrease of K_b/K_m to 0.20 or so[1] at Ri = 0.6 and Proudman (1953) used data of Jacobsen (1913) in the Kattegat to obtain $K_b/K_m \sim .03\text{-}.05$ at Ri of 4-10. The situation in the atmosphere is somewhat different because observations in the surface layer seem to indicate that turbulence ceases when Ri rises above 0.2 or so. Lettau (1973) obtains an empirical relationship in which K_b/K_m is proportional to $Ri^{-\frac{2}{3}}$ up to values of Ri of 0.1 or so.

The earliest theory for K_b/K_m was given by Ellison (1957). As discussed by Ellison and Turner (1960), the theory may be put in the form

$$\frac{K_b}{K_m} = \frac{\alpha_n\left(1-\frac{Rf}{Rf_c}\right)}{(1-Rf)^2} \tag{3.4}$$

where Rf is the flux Richardson number $Rf = q/\tau\bar{u}_z = Ri(K_b/K_m)$, and Rf_c is a constant. All experiments indicate that Rf is bounded as Ri gets large; indeed, if $\tau\bar{u}_z$ is regarded as the important energy-supply term in the energy equation, Rf must always be less than 1. Therefore, for large Ri (assuming turbulence exists), Eq. (3.4) shows that $Rf \to Rf_c$ so that a critical value of Rf exists above which turbulence presumably dies out. The data suggest $Rf_c \cong 0.05\text{-}0.15$ as we discuss below.

At larger values of Ri, perhaps Ri > 0.1, the quantity $1-Rf/Rf_c$ is small so that we may write, approximately,

$$1-\frac{Rf}{Rf_c} = \epsilon, \quad \frac{K_b}{K_m} \cong \frac{\alpha_n \epsilon}{(1-Rf_c)^2} = \frac{\alpha_n}{(1-Rf_c)^2}\left[1-\frac{RiK_b}{Rf_c K_m}\right]$$

---

[1] In Topic 3, Ri denotes the gradient Richardson number, $\bar{b}_z/\bar{u}_z^2$

or

$$\frac{K_b}{K_m} = \frac{\alpha_n/(1-Rf_c)^2}{\left[1+\frac{\alpha_n}{(1-Rf_c)^2}\frac{Ri}{Rf_c}\right]} \tag{3.5}$$

The coefficient of Ri in Eq. (3.5) is 15-30 depending on the choice of Rf_c so that for $Ri \gg 0.1$ we would expect the simpler equation

$$\frac{K_b}{K_m} = \text{const}/Ri \tag{3.6}$$

to be reasonably accurate. This was indeed found to be true by Kullenberg (1971).

Indications from data of Kato and Phillips (1969), Moore and Long (1971) and Wu (1973) that for moderate or large Ri^*, $q/\tau\bar{u}_z$ is constant support the behavior $Rf \to Rf_c$. The data of Kato and Phillips yield for the middle of the homogeneous layer,

$$Rf_c = q/\tau\bar{u}_z \cong \frac{u_e \Delta b D}{2u_*^2 U} \cong 1.2\frac{u_*}{U} \cong .06 \tag{3.7}$$

where we have estimated $U/u_* \cong 20$. This estimate of Rf_c is rather low compared to some other estimates (Turner, 1973), but the wind tunnel data of Arya and Plate (1969) indicate that Rf increases with Ri, tending, however, to level off at $Rf \cong 0.06$ for $Ri > 0.1$. This agreement with the data of Kato and Phillips may merely mean that U/u_* happens to be similar in the experiments of Kato and Phillips and Arya and Plate.

We should not expect too much from an equation such as Eq. (3.5).

For example, the often-used value of $\alpha_n = 1.35$ is the value determined from heat transfer data in forced convection (turbulent motion without sensible buoyancy effects) in a pipe and is the value at the pipe center. However, it may be noted that α_n varies from 1.35 to approximately 1 near the wall of the pipe. Thus there is no reason why α_n should be regarded as a constant in application to flow near the ground in the atmosphere or near the ocean surface in water. It is also possible that Rf_c varies, perhaps as a function of height or depth.

The behavior of K_b/K_m is quite different in unstable conditions. The only measurements available are in the atmosphere where Businger (1969) has advanced the formula

$$\frac{K_b}{K_m} = \alpha_n \left(1-\beta_1 \frac{z}{L}\right)^{\frac{1}{4}} \qquad (3.8)$$

near the ground where $\alpha_n = 1.35$ and $\beta_1 = 15$. L is the Monin-Obukhov length, $L = \tau^{\frac{3}{2}}/q\kappa$ and is discussed at length in Topic 4. Very near the ground z/L is proportional to Ri (Eq. 4.9).

A number of investigators, beginning with Rossby and Montgomery (1935) have chosen, for stable conditions, the relationship

$$K_b = K_{bo}(1+\sigma Ri)^{-1} \qquad (3.9)$$

where σ is a constant and K_{bo} is the value of K_b in neutral conditions. This can be derived from Eq. (3.5) by ignoring the (weaker) variation of K_m with Ri. For example, Sundaram and Rehm (1973) have applied Eq. (3.9) to a one-dimensional model of a lake. Their numerical time-dependent integrations yield a thermocline. Eq. (3.9) has great limitations, however, in that Ri is small in deep water but K_b is certainly small there since it is protected by the thermocline from the disturbances of the mixed layer. This was overcome by Sundaram and Rehm by arbitrarily choosing a low value in deep levels. Sundaram and Rehm form Ri by using the (given) stress at the free surface instead of the local shear so that the approach contains no more dynamics than appears in Eq. (3.9).

The formula (3.6) has some support from laboratory experiments. In Moore and Long (1971) the value of K_b in the homogeneous layers is certainly of order of the rms velocity σ_u times the eddy length ℓ. We have shown that $\sigma_u \sim \Delta u$, $\ell \sim D$ so $K_{bo} \sim D\Delta u$. In the stable region K_b is of order $qh_1/\Delta b$ where h_1 is the thickness of the layer. But h_1 is proportional to D (the ratio is .16 approximately) and $q \sim (\Delta u)^3/D$ so that

$$\frac{K_b}{K_{bo}} = C_4 \, Ri^{*-1} \tag{3.10}$$

where $Ri^* = D\Delta b/(\Delta u)^2$. Since Ri^* and Ri are proportional, Eq. (3.10) is in agreement with Eq. (3.6).

If we adopt Eq. (3.9), we may compute σ from the data. Ellison and Turner (1960), for example, find a weak dependence of K_m on Richardson number so that we may safely take $K_{bo} = K_m$ in the range $0 \leq Ri \leq 0.6$. Their data for K_b/K_m yield $K_b/K_m = 0.2$ at $Ri = 0.6$ so that $\sigma \cong 7$.

Kullenberg (1971) has proposed a relationship for K_b by assuming a constant flux Richardson number $Rf = Rf_c$. Then

$$K_b = Rf_c \tau \frac{d\bar{u}}{dz} \Big/ \frac{d\bar{b}}{dz} \qquad (3.11)$$

If the flow is steady, the stress in the water τ equals the surface stress $\tau_0 = \frac{\rho_a}{\rho_w}\tau_a$ where ρ_a and ρ_w are the densities of water and τ_a is the stress exerted by the air. Kullenberg also assumed a constant drag coefficient in the expression $\tau_a = c_d \frac{U_a^2}{2}$. His result is

$$K_b = Rf_c \frac{c_d}{2} \frac{\rho_a}{\rho_w} U_a^2 \frac{d\bar{u}}{dz} \Big/ \frac{d\bar{b}}{dz} \qquad (3.12)$$

The data on the dispersion of die patterns (assuming the same coefficient of diffusion for heat and dye) fit well. Indeed, the theory assumes mainly that q is proportional to $\tau\bar{u}_z$ and this is verified well by experiment. He takes $Rf_c = .05$, which is close to the estimate made from the data of Kato and Phillips but rather low compared to other estimates, and $c_d = .001$. A limitation as far as parameterization is concerned is the assumption that the stress everywhere in the water is equal to the surface stress. According to Eq. (1.9) this would not permit time integration, for example. Also, of course, the assumption that $Rf = Rf_c$ is not permissible for $Ri < 0.1$.

We have mentioned in Section 1.4 the difficulty with the concept of a critical flux Richardson number above which turbulence is supposed to die out. In fact, as we have seen, turbulence exists at infinite values of Rf in the experiment of Turner (1968). We may overcome this difficulty by taking a somewhat different viewpoint. The flux Richardson number is the ratio of the potential energy term q and an energy-source term, $\tau \bar{u}_z$ in Eq. (1.10). There is, in addition, another possible source of energy in the energy flux divergence term and, indeed, this is the only source of energy when shear is absent as in Turner's experiment. The flux divergence term is of order σ_u^3/ℓ in a fully turbulent layer[1] and in all cases we may simply accept this as the order of the energy-source terms because evidence indicates that $\tau \bar{u}_z$ is also of this order in a fully turbulent layer when there is a shear. Instead of $Rf = q/\tau\bar{u}_z$, we may form the ratio $Rf^* = q/(\sigma_u^3/\ell)$. Since $q \sim \sigma_u \sigma_b$, where σ_b is rms buoyancy fluctuation,

$$Rf^* \sim \frac{\ell \sigma_b}{\sigma_u^2} \qquad (3.13)$$

and, therefore, Rf^*, as defined, is of the order of the ratio of available potential energy to turbulent kinetic energy. We have already argued (Eq. 1.21) that this has an upper limit whether shear is present or not so that Rf_c^* exists and, from physical considerations, should have the same value in experiments with or without shear and in natural circumstances in atmosphere and oceans. We may estimate Rf_c^* as

$$Rf_c^* \cong .35\frac{V'}{T'}$$

---

[1] We are not discussing turbulence in density interfaces in which the turbulence is presumably intermittent. In such a region, unknown intermittency factors (see Eq. 5.20) must be used in estimating quantities such as q and τ in terms of rms quantities σ_u, σ_b, and length ℓ.

where V' is available potential energy, $\sigma_b \ell/2$, and T' is kinetic energy. We have taken the correlation coefficient between w' and b' as 0.3 and $\sigma_w \cong 0.6\, \sigma_u$, $\sigma_v \cong 0.75\, \sigma_u$ (Arya and Plate, 1969). In the case of shear we estimate Rf_c from the experiment of Kato and Phillips:

$$Rf_c = \frac{\bar{q}}{\tau \bar{u}_z} \sim 63 \frac{V'}{T'} \frac{u_*}{U} \cong 1.2 \frac{u_*}{U}$$

where we have taken $D \cong 14\ell$ in accordance with measurements in a pipe (Schlichting, 1955), and $u_*^2 = 0.3\, \sigma_u \sigma_w$. Thus $V'/T' \cong 0.019$ and $Rf_c^* = 1/150$. If this critical value of Rf^* is universal, it would be a more useful concept than Rf_c which, according to Eq. (3.7) varies with the drag coefficient.

3.3 Mass and Salt Transfers and Halocline Depths in an Estuary.

There is an interesting application of the ideas discussed above and in Topics 1 and 2. This relates to circulations and the depth of the upper mixed layer in an estuary. The essential features of an estuary are that of a confined body of brackish water with a number of fluxes into and out of the basin. One is a flux of fresh water to the basin from precipitation (less evaporation) and river runoff. In addition, at the entrance or transition there is a double flow with salt water flowing in from the open ocean and brackish water flowing out. We have in mind a body of water such as the Baltic Sea which is connected to the salt water supply in the Kattegat by three narrow passages which we idealize to be a single sound with a sill as shown Fig. 3.2. One of the practical problems that inspire interest in the Baltic circulation is the dramatic decline in dissolved oxygen in the deep waters over the present century. It is possible that this is due to increased pollution, but it is also possible that the decrease is due to reduced vertical mixing because of a long-term increase of stability.

The theoretical model is similar to that of Stommel and Farmer (1953) and is based on Figures 3.2 and 3.3, which represent a deep basin

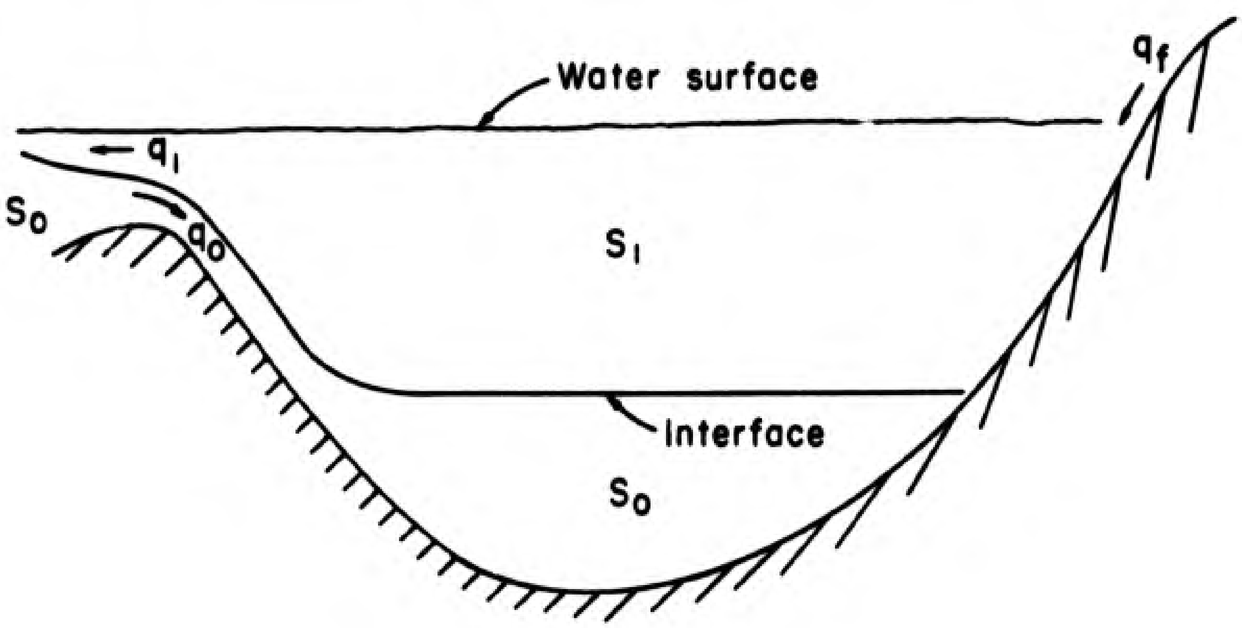

Fig. 3.2. Schematic picture of Baltic Sea model.

with a steady influx q_f of fresh water. The upper fluid in the basin is assumed to be well mixed at all times by the action of a steady wind exerting stress on the surface. The result is a kinematic stress in the water of magnitude u_*^2, where u_* may also be considered as typical of the turbulent velocities in the mixed layer. There is an influx q_0 of salt water of salinity S_0 over the sill in the sound and the upper mixed fluid of salinity S_1 and flux q_1 flows out over the saltier water. The driving effect, of course, is the fresh water influx and it is assumed that q_f is known.

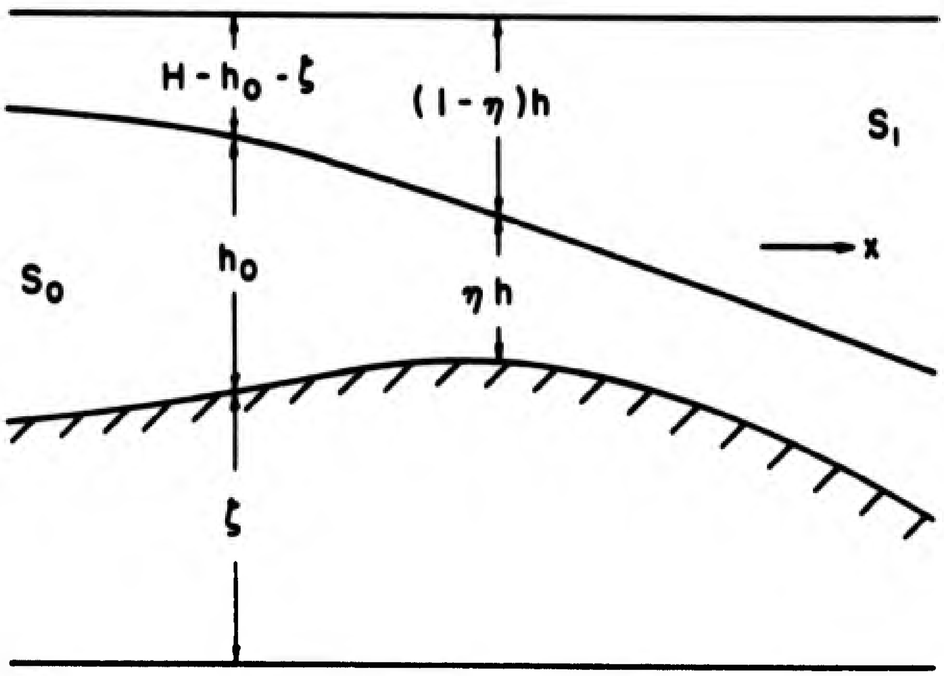

Fig. 3.3. Flow in the vicinity of the sill.

Using the Boussinesq approximation, we may immediately write down the two conservation equations for volume and salt

$$q_0 + q_f = q_1$$

(3.14)

$$q_0 S_0 \quad q_1 S_1$$

where the salinity S is the mass of salt per unit volume and where the q's are volume fluxes. We assume a linear relation between density and salinity, $\rho = \rho_{00}(1+\beta S)$, where β is a constant and ρ_{00} is the density of fresh water.

Let us now analyse the flow over the sill. With reference to Fig. 3.3, the two Bernoulli equations are

$$p_1 + \rho_1 \frac{u_1^2}{2} + \rho_1 gz = \text{const} \qquad (3.15)$$

$$p_0 + \rho_0 \frac{u_0^2}{2} + \rho_0 gz = \text{const} \qquad (3.16)$$

where we assume steady state, irrotational flow in both fluids, and quasi-horizontal motion. As discussed by Long (1954), the flow of a two-fluid system over a barrier is similar to that of a single fluid in which the free surface usually dips down with a non-zero slope at the crest, and the system changes from subcritical to supercritical flow from one side to the other. The interface is symmetric upstream and downstream (and the slope is zero at the crest) only when the flows are very weak or very strong. Accordingly, we will assume two-dimensional motion in the sound and a shape of the interface as in Fig. 3.3 in which $dh_0/dx \neq 0$ at the crest. We now assume hydrostatics and obtain

$$p_1 = \rho_1 g(H-z), \quad p_0 = \rho_1 g(H-h_0-\zeta) + \rho_0 g(h_0+\zeta-z) \qquad (3.17)$$

where H is the height of the free surface above an arbitrary level, ζ is the height of the bottom topography, h_o is the thickness of the lower layer and $h_1 = H - h_o - \zeta$ is the thickness of the upper layer. Substituting Eqs. (3.17) into Eqs. (3.15) and (3.16) and differentiating with respect to x, we obtain

$$\frac{dH}{dx} - \Delta\rho F_1^2\left(\frac{dH}{dx} - \frac{dh_o}{dx} - \frac{d\zeta}{dx}\right) = 0$$

(3.18)

$$\frac{dH}{dx} - \Delta\rho F_o^2\frac{dh_o}{dx} + \Delta\rho\left(\frac{dh_o}{dx} + \frac{d\zeta}{dx}\right) = 0$$

where

$$F_o^2 = \frac{q_o^2}{\ell^2 \Delta b h_o^3}, \quad F_1^2 = \frac{q_1^2}{\ell^2 \Delta b (H - h_o - \zeta)^3}$$

are the internal Froude numbers of the two layers. Eqs. (3.18) show that dH/dx is small, so that subtraction of the two equations (3.18) yields

$$F_o^2 \frac{dh_o}{dx} - \frac{dh_o}{dx} + \frac{d\zeta}{dx} + F_1^2\left(\frac{dh_o}{dx} + \frac{d\zeta}{dx}\right) = 0$$

At the sill where $d\zeta/dx = 0$, $dh_o/dx \neq 0$, we obtain

$$F_o^2 + F_1^2 = 1 \qquad (3.19)$$

or

$$\frac{q_o^2}{\eta^3} + \frac{q_1^2}{(1-\eta)^3} = h^3 \ell^2 \Delta b \qquad (3.20)$$

at the sill where h is the depth of the sill. Eq. (3.20) is the relation used by Stommel and Farmer (see Appendix). An additional equation is necessary. Stommel and Farmer used an extremum argument [equivalent to the assumption that S_1 is a maximum subject to the constraints in Eqs. (3.13), (3.14) and

(3.20) to obtain another relationship to close the problem. The condition of maximum S_1 was called "overmixed". The closure assumption in the present paper is based on observations by Ellison and Turner (1959) of the motion of density currents in which the internal Froude number based on the velocity, thickness and density difference of the current tends to be a constant. Analogously, we assume

$$\frac{q_0^2}{\ell^2 \Delta b h^3 \eta^3} = F_0^2 \qquad (3.21)$$

where F_0^2 is a constant. From Eq. (3.20) we see that both Froude numbers are then constants. If we now impose the condition that $\eta \to 1/2$ as $(S_0 - S_1) \to 0$ as argued by Stommel and Farmer and as observed in their experiments, we find that $F_0^2 = 1/2$ and we have the symmetric condition that both the Froude numbers are constant and equal.

Our conditions now allow us to solve for the unknowns η, q_0, q_1 and $(S_0 - S_1)$ in terms of q_f. However, we are also interested in the depth D of the mixed layer in the basin. If we regard the lower fluid as quiescent, this is determined by the balance between the downward entrainment velocity u_e of the halocline (Turner, 1973) and the upward velocity associated with the influx q_0. Thus, q_0 is proportional to u_e. As we saw in Topic 2, experiment indicates that u_e is proportional to the Richardson number, $D\Delta b/u_*^2 = Ri^*$, raised to the -3/2 power when there is pure stirring action without shear (Turner, 1968, Wolanski, 1972) and to the -1 power when there is a mean velocity (Kato and Phillips, 1969, Moore and Long, 1971). In an estuary, for example in the Baltic, the shear will effectively communicate the turbulence to the interface and we would expect the more rapid entrainment rate as found by Wu (1973) in his experiments with the erosion of an interface in a confined basin. We thus assume

$$q_o = \frac{AK_1 u_*^3}{(D\Delta b)} \qquad (3.22)$$

where A is the horizontal area of the halocline and K_1 is a constant.

If we non-dimensionalize by the following definitions:

$$Q_f^2 = \frac{q_f^2}{g\beta S_o h^3 \ell^2}, \quad Q_1^2 = \frac{q_1^2}{g\beta S_o h^3 \ell^2}, \quad Q_o^2 = \frac{q_o^2}{g\beta S_o h^3 \ell^2},$$

$$r = \frac{D}{h}\frac{(\beta S_o g)^{\frac{3}{2}} h^{\frac{5}{2}} \ell}{u_*^3 K_1 A}, \quad \frac{S_o - S_1}{S_o} = \Delta S \qquad (3.23)$$

we obtain

$$\eta = \frac{(1-\Delta S)^{\frac{2}{3}}}{1+(1-\Delta S)^{\frac{2}{3}}}, \quad Q_1 = \frac{Q_f}{\Delta S}, \quad Q_o = Q_f\left(\frac{1}{\Delta S} - 1\right)$$

$$r = \frac{1}{(1-\Delta S) Q_f}, \quad Q_f^2 = \frac{1}{2}\frac{(\Delta S)^3}{[1+(1-\Delta S)^{\frac{2}{3}}]^3} \qquad (3.24)$$

These are five equations in the five unknowns η, Q_1, Q_o, r, ΔS, and the problem is determined.

3.3. Discussion.

Results are shown in Figures 3.4 and 3.5. In Fig. 3.4, we see the plot of r vs. the discharge Q_f. There is a minimum depth of the halocline corresponding to $Q_{f_m} = .235$. For greater influx, the depth increases, becoming infinite at $Q_{f_c} = .707$ corresponding to a situation in which the basin is filled with fresh water. As Q_f decreases from Q_{f_m}, the interface also descends, again becoming infinite as $Q_f \to 0$ and $\Delta S \to 0$. This corresponds to a basin filled with water of salinity S_0. The lower curve in Fig. 3.4 reveals the variation of η. For large influx of fresh water, η is small and the outflow occupies almost the whole depth over the sill. As the influx of fresh water weakens, the depth of the lower layer increases until it equals the upper fluid depth at small values of Q_f.

Additional results are shown in Fig. 3.5. Interesting features are the maximum of Q_0 as the fresh water supply increases, and the monotonic decrease of S_1 with increasing fresh water supply. Both of these behaviors agree with predictions of Stommel and Farmer (1953) and Kullenberg (1955). As pointed out by Welander (1974), the increase of stability with increased fresh water supply is opposite to the conjecture of Fonselius (1969) that a decrease of fresh water is responsible for the observed increase of stability in the Baltic in this century.

It is of some interest to apply the theory for the depth of the halocline to the Baltic. We assume, arbitrarily, that the fluxes q_0 and q_1 all occur in the narrow passage between Helsingborg and Helsingör which has an average depth of 18 m and a width of 4 km. The area of the Baltic is $A \cong 3.1\ 10^{15}\ cm^2$,

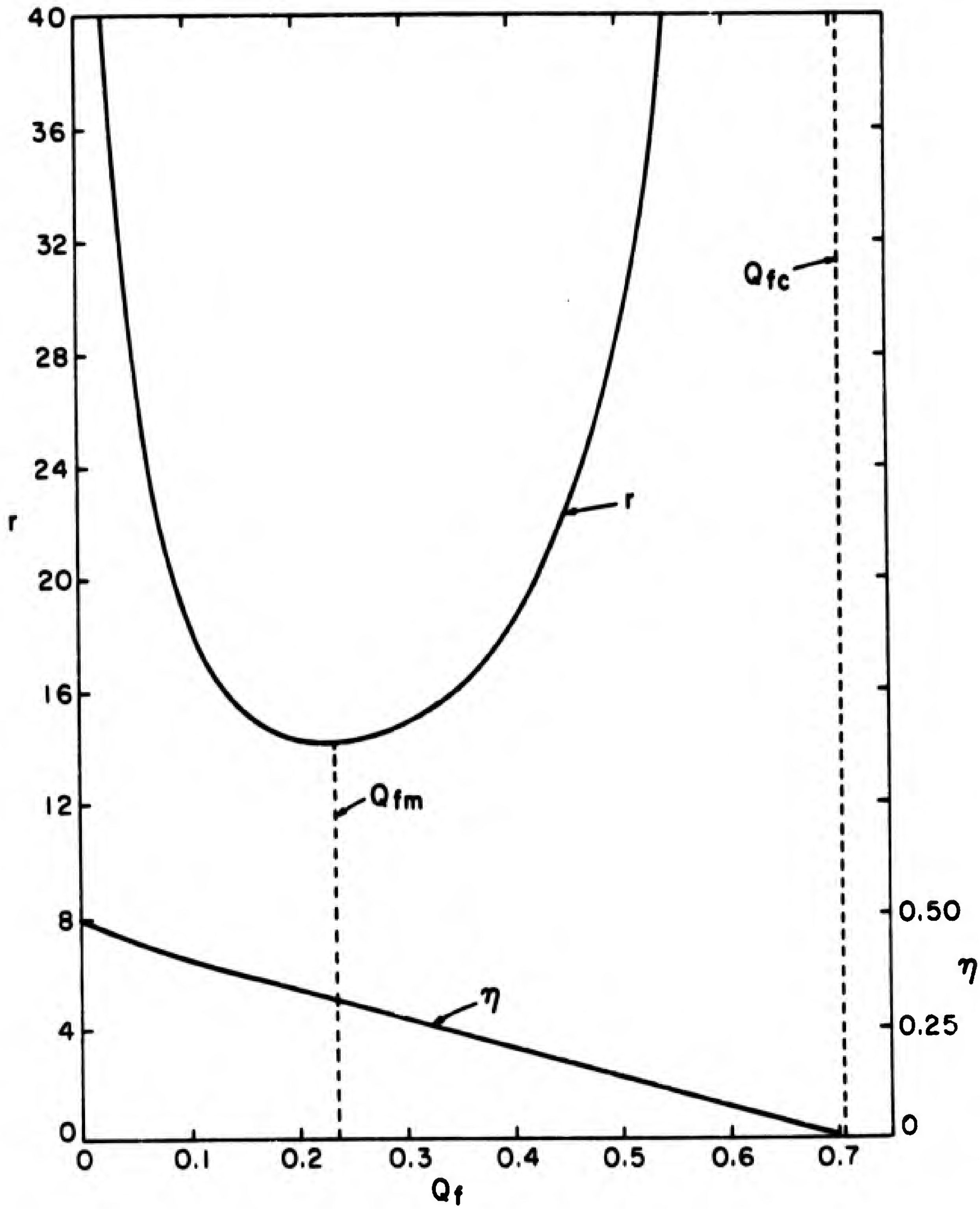

Fig. 3.4. The upper curve is the non-dimensional depth of the halocline as a function of the non-dimensional fresh-water influx. The lower curve is the ratio of the depths of the two layers.

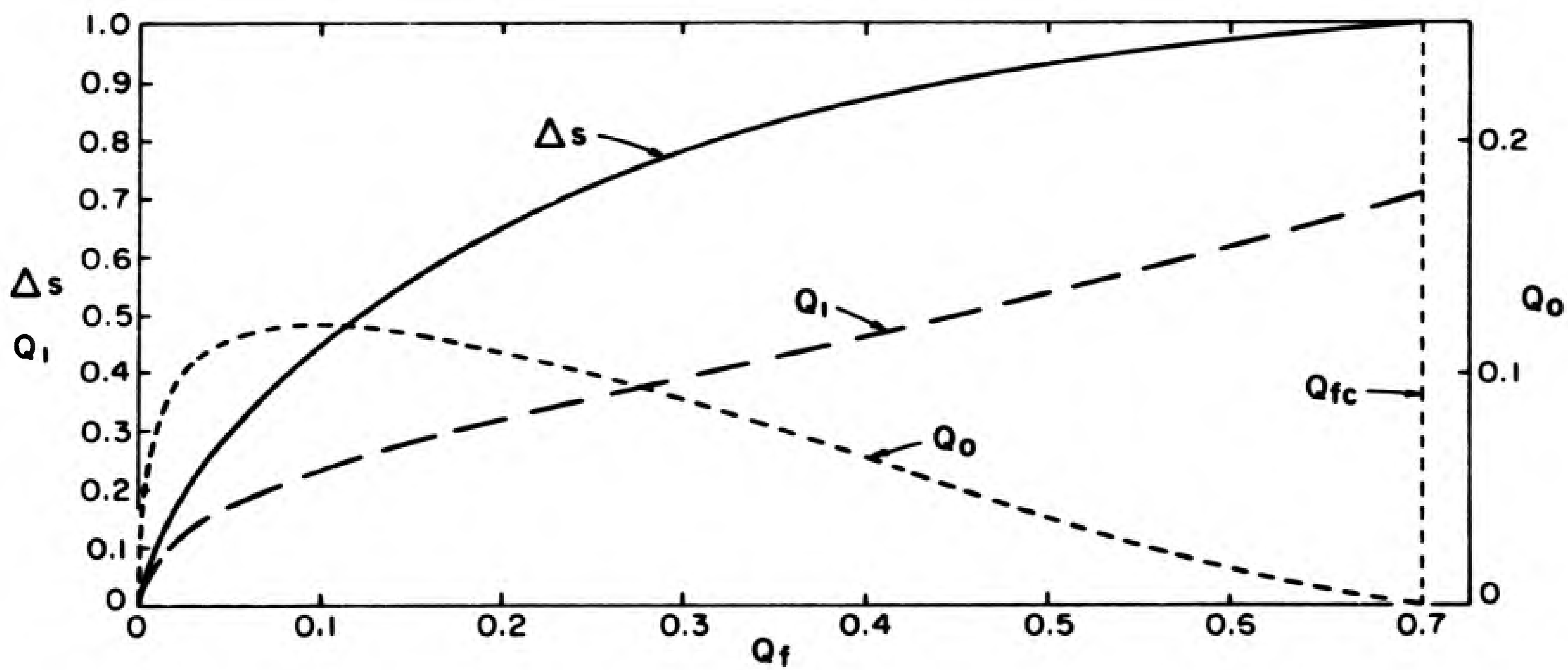

Fig. 3.5. Non-dimensional fluxes and salinity differences as functions of the non-dimensional fresh-water influx.

and Wu's data yield $K_s = .234$. Finally with $u_* \cong 1$ cm/sec, and $\beta g S_0 = 12$ cm/sec^2, the calculation leads to a minimum halocline depth of about 80 m which is as close to the observed 60 m as one could hope for considering the many uncertainties in the calculation.

A comparison of the theory for r or D may also be made with a recent experimental result of Welander (1974) who constructed an experiment shown schematically in Fig. 3.6. It consists of a model

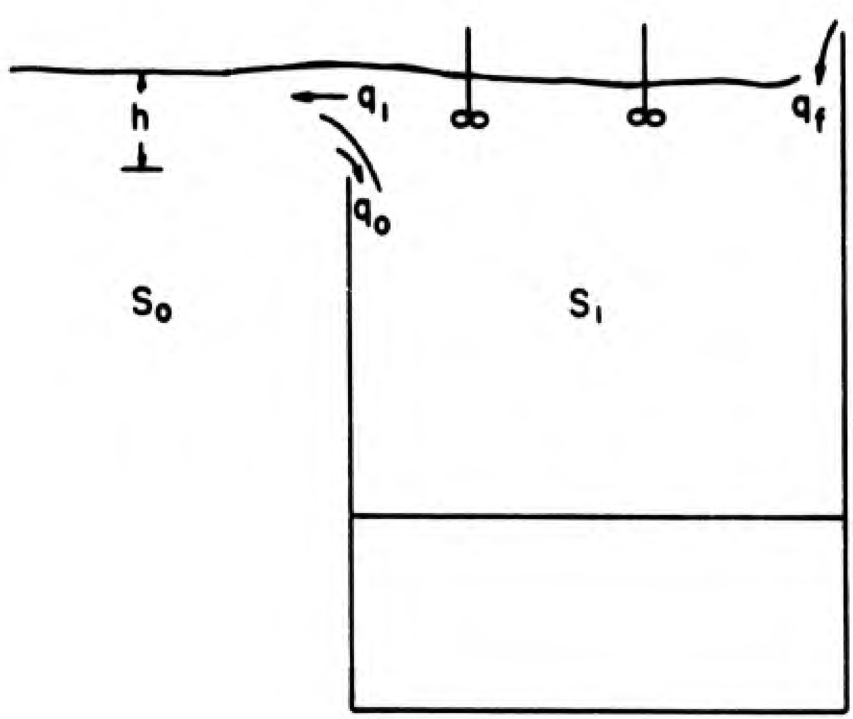

Fig. 3.6. Schematic picture of Welander's experiment.

basin with a fresh water supply and a double flow over the sill. The upper layer is stirred to make the layer homogeneous. The situation is similar to the model of this paper except that there is no shear, and a relation

$$q_o = \frac{A K_n u_*^4}{(D\Delta b)^{\frac{3}{2}}} \qquad (3.25)$$

must be used instead of Eq. (3.22) where $K_n \cong 10^{-3}$ (Wolanski, 1972). The definition

$$r_n = \frac{D}{h} \frac{(\beta S_o g)^{\frac{4}{3}} h^2 \ell^{\frac{2}{3}}}{u_*^{\frac{8}{3}} K_n^{\frac{2}{3}} A^{\frac{2}{3}}} \qquad (3.26)$$

replaces that in Eq. (3.23) and

$$r_n = \frac{1}{[\Delta S(1-\Delta S)^2 Q_f^2]^{\frac{1}{3}}} \qquad (3.27)$$

replaces that in Eq. (3.24). The result is used in Fig. 3.7 to make a comparison with the experimental data of Welander. In the comparison, constants are chosen to make the theory and the data agree near the minimum point. The theory reveals the same general behavior as the experiment especially with regard to a minimum of the depth. There is, to be sure, a large discrepancy between theory and experiment for small values of q_f. However, as the depth increases, the turbulent intensities greatly decrease at the interface and the downward entrainment velocity becomes very slow. We suggest that the experiments were not at a steady state and that a longer

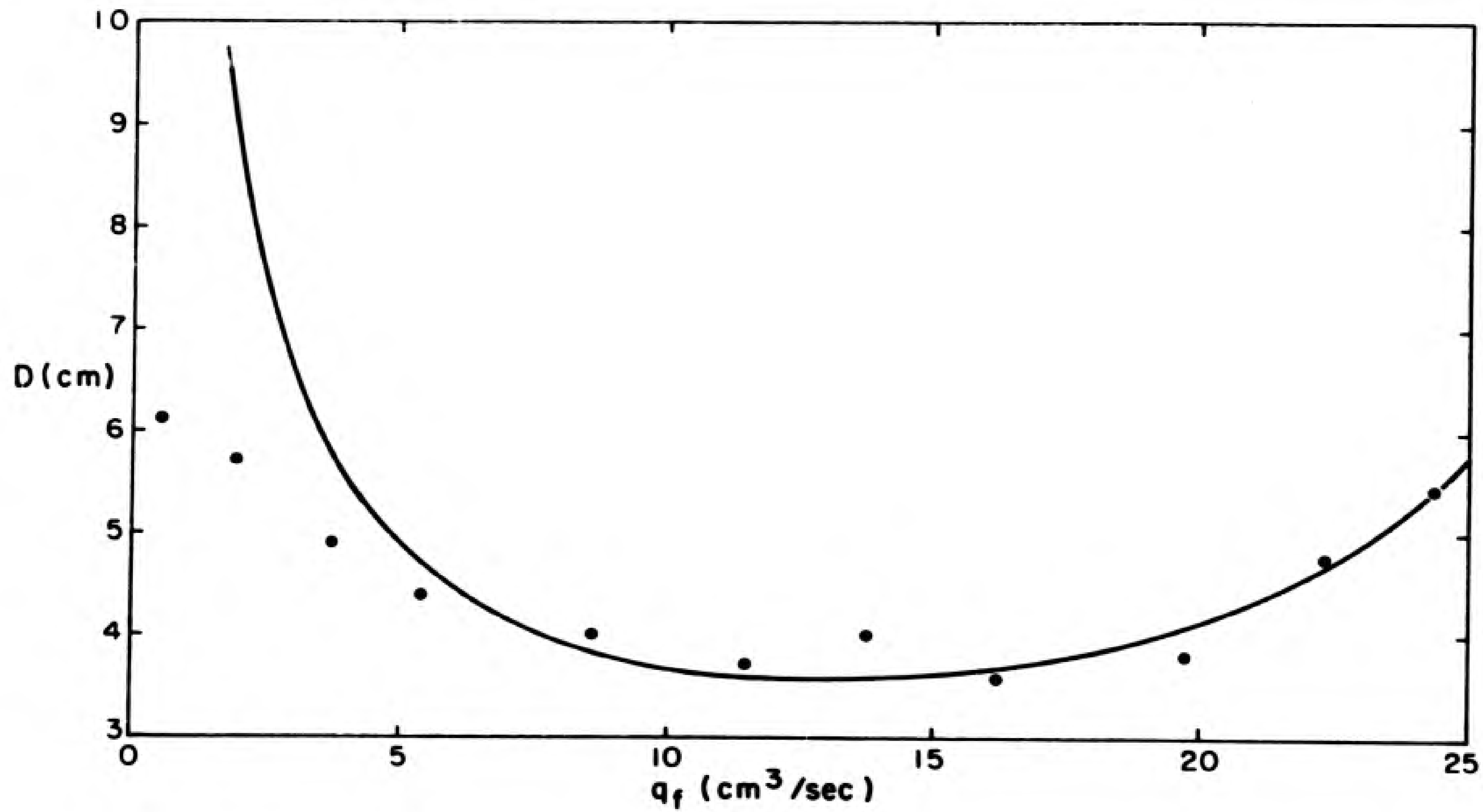

Fig. 3.7. Comparison of theory of the present paper and data of Welander for depth of halocline vs. fresh-water influx.

wait before the measurement of the depth would improve the agreement.

Finally, a comparison may be made with the theory of Stommel and Farmer (see Appendix). The theory led to the straight line in Fig. 3.8 and the agreement with experimental observations was quite good. The present theoretical curve approaches that of Stommel and Farmer for large S_1/S_0 and gives better agreement with the observations over the entire range of the experiments.

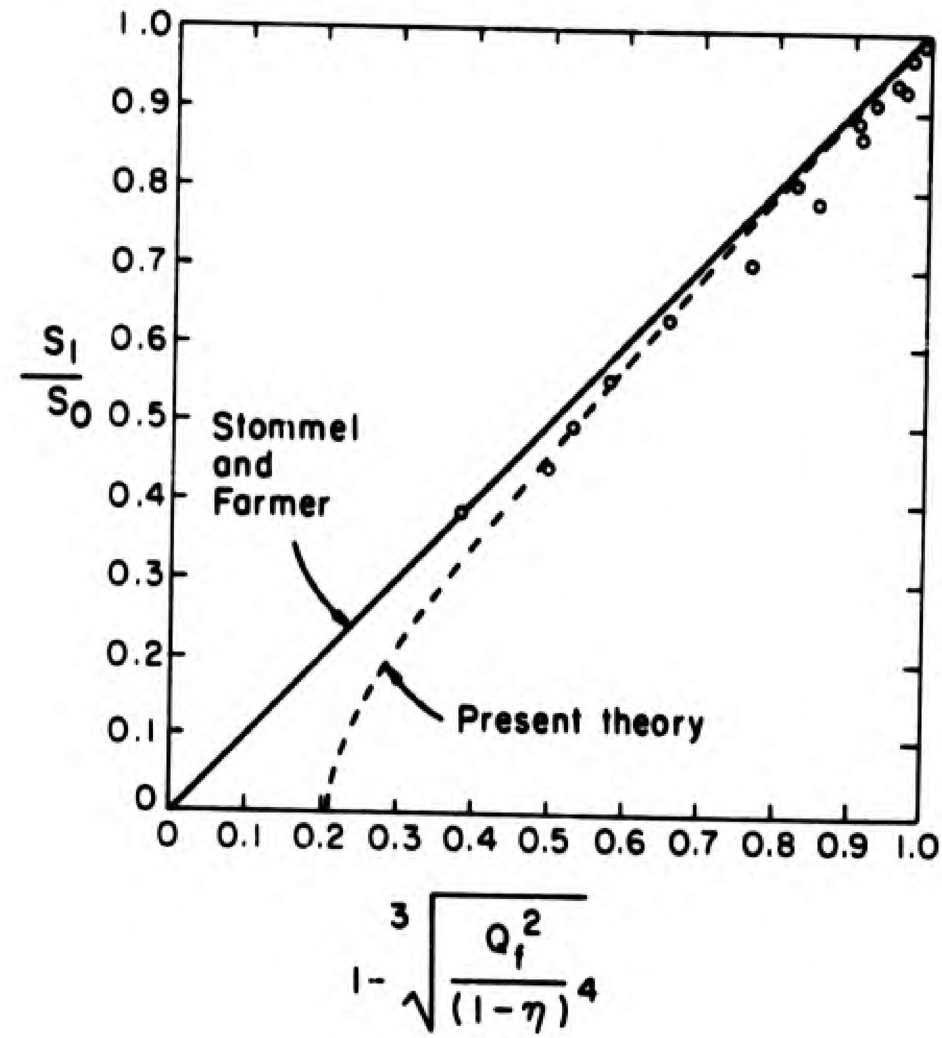

Fig. 3.8. Comparison of the theory of Stommel and Farmer and present theory with observational data of Stommel and Farmer.

APPENDIX

Our discussion of the critical condition adopted by Stommel and Farmer is not adequate for natural circumstances, for example at a control section for the Baltic which we have chosen to be the Öresund between Helsingör and Helsingborg (neglecting flow through the other Danish sounds). If we allow both ℓ and ζ to vary in Eq. (3.18), we obtain

$$\frac{dh_0}{dx}\left[1-F_0^2-F_1^2\right]+\frac{d\ell}{dx}\left[-F_0^2\frac{h_0}{\ell}+F_1^2\frac{h_1}{\ell}\right]+\frac{d\zeta}{dx}\left[1-F_1^2\right]=0 \qquad \text{(A)}$$

where we put $(H-h_0-\zeta) = h_1$. In our discussion above, we ignored variations in ℓ and assumed $d\zeta/dx = 0$ at the control section. In Stommel and Farmer's experiment on the other hand, the bottom is flat and the width ℓ is a minimum, i.e. $d\ell/dx = 0$ at the control section so that the critical condition may have been satisifed. At the Baltic control section $d\ell/dx = 0$ also, but the bottom slope is appreciable and we probably do not get the critical condition. It is possible that the critical condition does not apply in natural conditions but that the two streams adjust to yield constant Froude numbers in each layer. We may then proceed as above using the definitions in Eq. (3.23) and we obtain

$$\eta = \frac{R^{\frac{1}{3}}(1-\Delta S)^{\frac{2}{3}}}{1+R^{\frac{1}{3}}(1-\Delta S)^{\frac{2}{3}}}, \quad Q_1 = \frac{Q_f}{\Delta S}, \quad Q_0 = Q_f\left(\frac{1}{\Delta S}-1\right)$$

(B)

$$r = \frac{1}{(1-\Delta S)Q_f}, \quad Q_f^2 = \frac{F^2 R(\Delta S)^3}{\left[1+R^{\frac{1}{3}}(1-\Delta S)^{\frac{2}{3}}\right]^3}$$

instead of Eq. (3.24) where $R = F_1^2/F_o^2$. If $F_o^2 = F_1^2 = 1/2$, we obtain the results in Eq. (3.24). In general, however, we may allow ourselves the freedom to choose F_o^2 and F_1^2 empirically. It seems clear from the experiments of Stommel and Farmer, however, that $\eta \rightarrow 1/2$ as $\Delta S \rightarrow 0$ so that $R = 1$.

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Topic 4. Turbulence in the Surface Layer.

4.1 Introduction.

We have seen in the Appendix to Topic 1 that flow of a homogeneous fluid over a smooth or rough surface is determined by the friction velocity u_*, the height z and the depth of the fluid H. Viscosity does not enter when the surface is smooth and the roughness length z_0 does not enter when the surface is rough except to determine the boundary conditions at the underlying surface and thus to determine the absolute value of the mean velocity $\bar{u}(z)$. In considering the shear $\bar{u}_z$, for example, we have $\bar{u}_z = f(z, u_*, H)$ or

$$\frac{z\bar{u}_z}{u_*} = \varphi\left(\frac{z}{H}\right) \tag{4.1}$$

If $z \ll H$, the depth H is also unimportant, φ is constant and integration yields the logarithmic law.

In the lower atmosphere or in the upper portions of the sea, we may extend these ideas to include the presence of a buoyancy flux q as well as a momentum flux u_*^2 and to argue that quantities such as $\bar{b}_z$, $\bar{u}_z$, ℓ and rms fluctuations σ_u, σ_b are functions of z, u_*, q close to the ground or to the air-sea interface. There are some doubts about this theory because, as we will discuss below, experimental and theoretical investigations of convection in a stationary container indicate that the molecular coefficient of heat conduction plays a considerable role even when the thermal boundary layers are very thin. The boundary layer is the source of the thermals that arise and form the thermal elements and the thermal boundary layer has characteristics controlled by the molecular coefficients. Long (1974a, 1974b) has discussed this problem at length and concludes that for wind velocities over 1 m/sec in the atmosphere, molecular coefficients may be neglected; air speeds are commonly greater than this. In the oceans, the situation is somewhat different because molecular coefficients appear to be important for water velocities less than 5-10 cm/sec and such speeds are commonly observed.

The theory involving z, u_* and q is called the Monin-Obukhov theory (Monin and Yaglom, 1971) and is applied in both stable ($q < 0$) and unstable ($q > 0$) conditions. It yields, for example,

$$\bar{u}_z = \frac{u_*}{\kappa z} f\left(\frac{z}{L}\right) \qquad (4.2)$$

$$\bar{b}_z = \frac{q}{u_* z} \varphi\left(\frac{z}{L}\right) \qquad (4.3)$$

where $L = u_*^3/\kappa q$ is the Monin-Obukhov length introduced in Topic 3. (Von Kármán's constant is included to conform to usage.)

Very little is known about the functions f and φ, although there is strong evidence in the atmosphere (Monin and Yaglom, 1971, Chapter 7) and in the laboratory (Arya and Plate, 1969) that these and other "similarity laws" are very good approximations. Observations are lacking in the sea, but we may be reasonably confident that the similarity theory is valid near the surface in the water and in a layer near the bottom.

The Monin-Obukhov length may be examined in the mixing layer. There $\tau \sim \sigma_u^2$, $q \sim \sigma_u \sigma_b$ so that

$$L \sim \sigma_u^2/\sigma_b$$

As we have already discussed, $\sigma_u^2/\sigma_b \ell \sim 1$ in the layer so that $L \sim \ell \sim D$. We may find the coefficient of proportionality between L and D from the experiments of Kato and Phillips (1969). If we use the average flux of buoyancy in the mixed layer, $\bar{q}$, we have

$$Rf_c \cong \frac{\bar{q}D}{\tau U}$$

It follows from Eq. (3.7) that

$$\frac{L}{D} = \frac{\tau^{\frac{3}{2}}}{\bar{q}\kappa D} = \frac{\sqrt{c_d/2}}{\kappa Rf_c} \cong 2 \qquad (4.4)$$

where c_d is the drag coefficient, $c_d = 2u_*^2/U^2$. Alternatively, we may compute L from the definition using the estimate for Rf_c^* . We get

$$\frac{L}{\ell} = \frac{\tau^{\frac{3}{2}}}{q\ell\kappa} \cong \frac{(.18)^{\frac{3}{2}}\sigma_u^3}{\kappa\ q\ell} = \frac{.191}{Rf_c^*}$$

or $L/\ell \cong 28.6$. Using $D \cong 14\ell$, we again obtain $L/D \cong 2$. Kitaigorodskii (1960) found $L/D \cong 1.2$ for the mixed layer in the ocean and recently Sundaram (1973) has computed $L/D \cong 4$ for a lake.

In Eq. (4.2), we may fix z and u_* and allow q to tend to zero. Then density variations must become unimportant and $f\left(\frac{z}{L}\right) \to f(0) \to 1$. This suggests that for small z/L, we may expand f in a Taylor series. This yields

$$\bar{u}_z = \frac{u_*}{\kappa z}\left[1 + \beta_s \frac{z}{L}\right] \tag{4.5}$$

or

$$\bar{u} = \frac{u_*}{\kappa}\left\{\ln \frac{z}{z_0} + \beta_s \frac{(z-z_0)}{L}\right\} \tag{4.6}$$

where we truncate the series after the first two terms. This is called the log-linear law. The constant β_s may have different values when the situation is stable or unstable because the function may not be analytic at $z/L = 0$. Indeed when the fluid is stable, turbulence should be impeded and the velocity curve should be "steeper". When unstable, turbulence is increased and the velocity should increase more slowly with height than in the neutral case. Then $\beta_s > 0$ or $\beta_s < 0$ according as the fluid is stable or unstable. Observations (Arya and Plate, 1969) indicate $\beta_s \cong 10$ in the stable case and $\beta_s \cong -0.6$ in the unstable case.

The log-linear laws hold fairly well in the atmosphere to $z/L = 0.1$ in the stable case but only to $z/L = 0.03$ in the unstable case. The usefulness is unknown in the oceans but the theory should apply near the surface and again near the bottom if a bottom current exists.

The conclusion that the velocity should increase more rapidly with distance from a surface than in the neutral case is a reasonable explanation for observations by both Kato and Phillips (1969) and Moore and Long (1971) that the velocity profiles were very flat in the middle of the layer. This would require the velocity to increase more rapidly in the region near the plate or the bottom. In both experiments the interface behaves as a rigid surface so that (4.6) should also describe the mean velocity field in the "homogeneous" layer near the interface if z is distance from the interface.

4.2 The Stable Case.

Any mean quantity may be expressed by the similarity theory but only a few measurements have been made. We may, for example, consider $\bar{b}_z$. At small values of z

$$\bar{b}_z \sim \frac{b'}{z}$$

and $q \sim u'b' \sim u_* b'$ so that

$$\bar{b}_z \sim \frac{q}{u_* z}$$

Thus, near z = 0 we have

$$\bar{b}_z = \frac{q}{\kappa_1 u_* z} f\left(\frac{z}{L}\right), \; f(0) = 1$$

and the next approximation yields

$$\bar{b}_z = \frac{q}{\kappa_1 u_* z}\left[1 + \beta_b \frac{z}{L}\right] \tag{4.7}$$

where β_b and κ_1 are constants. Mean buoyancy has been measured in the stable case by Arya and Plate (1969) confirming Eq. (4.7) with $\beta_b \cong 17$. The finding that $\beta_b > \beta_m$ is expected. Thus

$$\frac{K_b}{K_m} = \frac{q\bar{u}_z}{\bar{b}_z u_*^2} \cong \alpha_n \left[\frac{1+\beta_m(z/L)}{1+\beta_b(z/L)}\right] \tag{4.8}$$

and, as we have seen, K_b/K_m decreases for $z/L > 0$, i.e. for stable conditions.

The Monin-Obukhov theory has been discussed and extended by many authors (see a long list of references for Chapter 7, Monin and Yaglom, 1971). For example, the Richardson number Ri is

$$Ri = \frac{\bar{b}_z}{\bar{u}_z^2} = \frac{z}{L}\varphi_1\left(\frac{z}{L}\right) \tag{4.9}$$

where $\varphi_1(0) = \text{const}$ so that with height above, the surface, Ri increases and the situation becomes more stable. We have seen that indications are strong that Rf approaches a constant value Rf_c as stability increases so that $q/\tau\bar{u}_z \to$ constant as z/L becomes large corresponding to a stable layer below the upper mixed level in the sea or in an inversion in the atmosphere. Thus

$$\bar{u}_z \to \text{const}\,\frac{q}{\tau} \tag{4.10}$$

in the stable layer so that we would expect a linear velocity profile over the stable layer as observed by Moore and Long (1971). The buoyancy variation with large z/L is more difficult to establish and we merely quote the result of Long (1970) that

$$N^2 = \bar{b}_z \to \text{const}\,\frac{q^4}{\tau^5}\,z^2 \tag{4.11}$$

Monin (1969) has found verification of the prediction in Eq. (4.11). His analysis of 40 hydrological stations in the central North Pacific shows that the law N(z) = const z is fulfilled quite satisfactorily in layers of the deep ocean below a depth of $1\frac{1}{2}$ - 2 km.

4.3 The Unstable Case.

Let us now discuss the unstable case in which the potential energy term q in Eq. (1.10) is an energy-source term. In the upper layers of the ocean, this corresponds to conditions of cooling at the surface which must be the general case in autumn and winter. The cold water sinks and the resultant mixing eventually destroys the thermocline.

Despite the importance in the oceans, we have almost no data there to form the background of this discussion so that we will emphasize meteorological observations in this section. Many investigators have explored the problem of the surface layer from an observational and theoretical viewpoint. A valuable reference is Monin and Yaglom (1971) which emphasizes contributions by scientists in the U.S.S.R. Many experiments have been constructed to study convection between two stationary horizontal surfaces, heated below and cooled above (Malkus, 1954a,b; Thomas and Townsend, 1957; Silveston, 1958; Croft, 1958; Townsend, 1959; Globe and Dropkin, 1959; Somerscales and Dropkin, 1966; Deardorff and Willis, 1967; Somerscales and Gazda, 1969), and one recent experiment also involves shear (Townsend, 1972).

If one divides the scientists interested in turbulent convection into two groups, one finds that the experimentalists all attach major importance to the molecular coefficients of viscosity and conduction, ν and k_h, whereas the atmospheric scientists believe that these quantities are unimportant in the surface layer. There are two reasons for this. One is that laboratory experiments are on much smaller scales than those characteristic of the atmosphere, so that the relevant Reynolds numbers or Rayleigh numbers (see below) in the laboratory are much smaller than those in the atmosphere. The second reason is that all experiments, except the recent one by Townsend, are without shear whereas the atmosphere is usually in appreciable mean motion. We illustrate the importance of the latter remark by considering the case of zero shear in which fluid is contained between horizontal smooth plates at $z = 0$ and $z = H$. The lower plate is heated and the upper plate is cooled. The temperature difference corresponds to an increment of buoyancy $2\Delta b$. If the reference density ρ_0 is the mean value of the density at $z = H/2$, we have $\bar{b} = 0$ at $z = H/2$.

Dimensional analysis leads to forms of the mean quantities in terms of several nondimensional parameters. The mean buoyancy gradient, for example, is

$$q = (\Delta b)^{\frac{4}{3}} k_h^{\frac{1}{3}} f_1(Pr, \xi_0) \qquad (4.12)$$

$$\bar{b}_z = (\Delta b/z) f_3(Pr, \xi_0, \xi) \qquad (4.13)$$

where z is the vertical coordinate and

$$Pr = \nu/k_h, \; \xi_0 = H(\Delta b)^{\frac{1}{3}}/k_h^{\frac{2}{3}}, \; \xi = z(\Delta b)^{\frac{1}{3}}/k_h^{\frac{2}{3}} \qquad (4.14)$$

In Eq. (4.14), Pr is the Prandtl number, ξ_0 is proportional to the cube root of the Rayleigh number,

$$Ra = H^3 \Delta b/\nu k_h \qquad (4.15)$$

and ξ may be regarded as the ratio of the height above the lower surface to the thickness $\delta_T \sim k_h^{\frac{2}{3}}/(\Delta b)^{\frac{1}{3}}$ of the thermal boundary layer. We confine attention, of course, to situations and regions in which ξ_0 and ξ are large.

Many measurements have been made of the buoyancy flux in Eq. (4.12). All experiments indicate a very weak dependence on ξ_0 when this number is large (Turner, 1973) and it seems likely that f_1 may be considered independent of ξ_0 at large Rayleigh numbers. It is then inescapable that q is directly influenced by the molecular coefficients even at the very high Rayleigh numbers characteristic of the atmosphere. Since the heat transport is by the eddies, they too are directly influenced by the molecular coefficients. This result is in contrast with the case of flow of a homogeneous fluid over a surface. As we have discussed in the Appendix of Topic 1, the molecular coefficient of viscosity is then <u>unimportant.</u> It seems likely, therefore, that ν and k_h may be neglected for moderate and large shears when convection also exists and that we may apply the Monin-Obukhov theory which we now write in the form,

$$\bar{b}_z = (q/\tau^{\frac{1}{2}} z)\varphi_3(\zeta), \quad \bar{u}_z = (\tau^{\frac{1}{2}}/z)\varphi_4(\zeta) \qquad (4.16)$$

where $\zeta = z/L$ and we have assumed that H is infinite.

We emphasize, however, that we should properly include two other numbers in the functions of Eqs. (4.16), namely Pr and an appropriately defined Reynolds number. For simplicity, we may set Pr equal to a constant, say Pr = 1, i.e. $k_h = \nu$, and ignore k_h. The Reynolds number is obviously based on the friction velocity and the length L, so that we may write

$$\bar{b}_z = (q/\tau^{\frac{3}{2}} z)\varphi_2(\zeta, R_*), \quad R_* = \tau^2/q\nu \tag{4.17}$$

Although (4.17) is more accurate than the expression for $\bar{b}_z$ in (4.16), we acknowledge that our experience with homogeneous fluids suggests that R_* is negligible when R_* is large. If R_* is small, or when the shear is small, we must be cautious. In the first place a small Reynolds number suggests quite naturally that molecular quantities may be important, and in the second place small R_* is associated with the case of zero or weak shear and our discussion of convection in a box indicates that ν and k_h cannot be neglected in this case.

We illustrate the importance of R_* by repeating the arguments of Priestley (1954) used to determine the behavior of $\bar{b}_z$ or of $\varphi_2(\zeta)$ when z is large. The contention is made that large ζ in Eqs. (4.16) means either large z or small τ so that the behavior of $\varphi_2(\zeta)$ as $\zeta \to \infty$ may be obtained by requiring that τ disappear from the analysis. Then, for large z, we have

$$\bar{b}_z = \text{const}\; q^{\frac{2}{3}} z^{-\frac{4}{3}} \tag{4.18}$$

Notice, however, that if we use the more accurate Eq. (4.17) instead of (4.16), we see that the argument uses the dangerous assumption that the Reynolds number R_* continues to be unimportant even when it is very small.

There is controversy concerning the prediction in (4.18) (and predictions of the forms of other mean quantities using the same argument). Soviet scientists believe that the predictions are "well satisfied" (Monin and Yaglom, 1971), while acknowledging some deviations at larger values of ζ. Western scientists have less confidence. There is a general belief that $\bar{b}_z \propto z^{-\frac{3}{2}}$, for example, as indicated by excellent atmospheric data of Dyer (1965) and Businger et al. (1971). These data seem to be good enough to distinguish clearly between the observed $z^{-\frac{3}{2}}$ behavior and the similarity theory of Eq. (4.18). Townsend (1959) proposed $\bar{b}_z \propto z^{-n}$ where $1.3 < n < 2.5$. Data of Croft (1958) yield $z^{-\frac{3}{2}}$.

The similarity arguments can be used to predict other quantities, for example, $\sigma_b \propto z^{-\frac{1}{3}}$ compared with measurements in which exponents range from -0.48 to -0.70 (Rossby, 1969; Somerscales and Gazda, 1969). Finally Townsend (1959) gives an interesting result for the "dissipation function" for buoyancy fluctuations δ. He finds $\delta \propto z^{-\frac{3}{2}}$ compared with $z^{-\frac{4}{3}}$ from similarity theory.

We may conclude that evidence in atmospheric and laboratory investigations does not inspire confidence in the similarity theories when heating effects are strong or the shear is weak. We have seen that all theories reduce to the argument that mean quantities should depend only on q and z when the shear is zero and that the argument was inspired by the success of the theory that only τ and z are important in shearing flow above a surface. In the latter case, however, τ is porportional to $(\Delta u)^2$ and is either weakly dependent on ν (smooth wall) or independent of ν (rough wall).

In the case of heating, however, q and, therefore, either σ_w or σ_b directly involve the molecular coefficients.

A theory, differing from the similarity theory, has been proposed by the author (Long, 1974a, b) and involves the behavior of a number of mean quantities including mean buoyancy gradient, mean velocity gradient, rms velocities and buoyancies, length and time scales, and eddy viscosity and conductivity. We are concentrating here on the properties of the mean buoyancy gradient, and we will therefore confine attention to a few aspects of the theory relevant to finding $\bar{b}_z$ in a region well above the thermal boundary layer. For simplicity, we will assume $\nu = k_h$ and $H = \infty$. For the time being, we may also consider the shear to be zero.

The similarity theory for $\Delta u = 0$ may be based on simple dimensional analysis once it has been decided that $\bar{b}_z$, for example, should depend only on q and z. On the other hand, as shown by Kraichnan (1962), it may also be based on estimates of orders of magnitude of certain quantities as $z \to \infty$ and this approach reveals the essential difference between the present theory and the similarity theory.

The eddy conductivity K_b may be defined by the equation $q = K_b \bar{b}_z$ and since q is a constant, the variation of $\bar{b}_z$ with height is determined by the variation of K_b. To find this variation, we notice that $q \sim \overline{w'b'}$ implies that $q \sim \sigma_w \sigma_b C_0$ where C_0 is the correlation coefficient. We know, however, that warm parcels rise and cold parcels descend so that C_0 should be of order one. This assumption is strongly supported by observations (Deardorff and Willis, 1967) in which C_0 is 0.5 - 0.6 over a wide range of Rayleigh numbers.

Both theories then should yield $q \sim \sigma_b \sigma_w$. The quantity σ_b may be estimated by assuming $\sigma_b \sim \bar{b}_z \ell$ where ℓ is a length over which the total buoyancy is conserved. It is reasonable that $\ell \sim z$ as is the case in turbulent shear flow above a surface. Indeed, $\ell \sim z$ implies $\ell \sim H$ for the container as a whole; this implies that the energy-containing eddies fill the entire container and this is observed in experiments. The three estimates $q \sim K_b \bar{b}_z$, $q \sim \sigma_b \sigma_w$, $\sigma_b \sim \bar{b}_z z$ lead to

$$K_b \sim z\sigma_w \tag{4.19}$$

and the problem of finding K_b and, therefore, $\bar{b}_z$ is reduced to finding σ_w. This is the point of departure for the present theory and the similarity theory. The latter, as shown by Kraichnan (1962) assumes $\sigma_w^2 \sim \sigma_b \ell \sim \sigma_b z$ which is a seemingly reasonable assumption that the kinetic and potential energies are of the same order, or that the vertical acceleration of a parcel is of the order of the buoyancy force. This estimate and the earlier estimates, lead to $\bar{b}_z \sim q^{\frac{2}{3}} z^{-\frac{4}{3}}$.

In the present theory the last order-of-magnitude estimate is not made. Instead experimental observations are used (Malkus, 1954b; Deardorff and Willis, 1967) that the kinetic energy averaged over the entire container is of order $H\Delta b$. This result holds with considerable accuracy over a range of Rayleigh numbers from 10^5 to 10^9. This has a physical interpretation that the vertical velocity of a parcel is of an order obtained by imagining that it conserves its density and rises freely in the unstable environment from an origin in the thermal boundary layer. It suggests, therefore, that $\sigma_w^2 \sim z\Delta b$ at any given level z. If we adopt this estimate, we obtain

$$\bar{b}_z = B(\Delta b)^{\frac{5}{6}} \nu^{\frac{1}{3}}/z^{\frac{3}{2}} \tag{4.20}$$

where B is a constant for zero shear but in general may be considered a function of $V = \Delta u/(\nu\Delta b)^{\frac{1}{3}}$. This contrasts with the $z^{-\frac{4}{3}}$ dependence of the similarity theory. In essence, it is easy to see that the basic difference in the two theories can be reduced to a difference in the estimate of the time scale T_e of the eddy motion. The present theory assumes that T_e depends on z and Δb only and the similarity theory assumes that T_e depends on q and z only. Actually, if one acknowledges that Δb is a more fundamental parameter than q, the present assumption is preferable if one also feels that vanishingly small molecular coefficients should not directly affect the time scale of the eddies. In any case, experiment and observation will be decisive. We have already stated that experimental observations of $\bar{b}_z$ are inconclusive, although it is fair to point out that Croft's observations (Croft, 1958) support Eq. (4.20). The atmospheric observations, of course, directly support Eq. (4.20).

The present theory is easily extended to obtain other mean quantities, for example $\sigma_b \propto z^{-\frac{1}{2}}$, and we have already seen that this is closer to observed behavior than the similarity theory. The present theory also yields $\delta \propto z^{-\frac{3}{2}}$ where δ is the "dissipation function" for buoyancy fluctuations and this agrees exactly with Townsend's observations.

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Topic 5. Density Currents and Wake Collapse

5.1 Introduction.

Frequently a fluid of a certain density moves under or over a fluid of lesser or greater density, respectively. Examples are currents of brackish water issuing from an estuary and flowing over salty water in the sea (Ekman, 1904), the intrusion of a salt-water wedge into an estuary, the spreading of oil over a water surface (Cochran and Scott, 1970), turbidity currents in seas, lakes and reservoirs as fluid heavy with sediment (suspended solids) moves from the streams and rivers pouring into the larger body of water (Kao, 1974), and cold air mass outbreaks in the atmosphere (Petterssen, 1969). In these cases it may perhaps be possible to consider each mass to be a fluid of uniform density. In another case of interest, the intruding fluid has a density corresponding to some intermediate level in an ambient fluid stratified with a variable density. An example is the spread of sewage effluent released from submerged outlets (outfalls) in the ocean. More basically, it occurs when a homogeneous mass of fluid collapses in a stratified environment (Wu, 1965, 1969) as in Fig. 5.1.

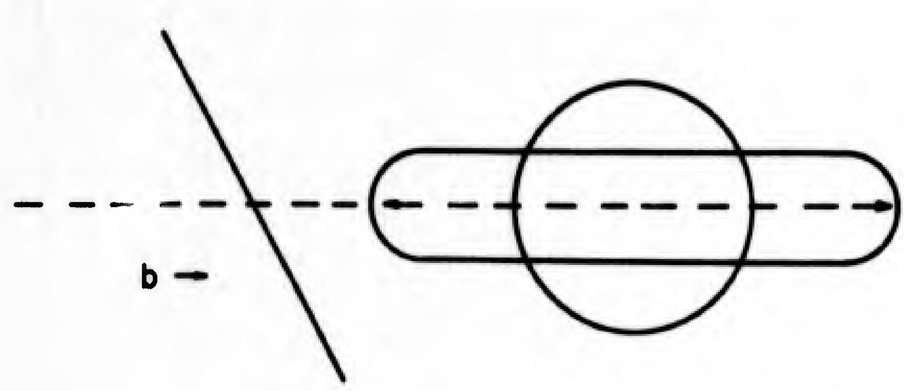

Fig. 5.1. Spread of a mass of homogeneous fluid in a stratified environment.

This may occur because the presence of a body, e.g. submarine moving in the thermocline, creates a homogeneous wake which then collapses (Schooley and Stewart, 1963) or when a patch of turbulence in the thermocline (perhaps a breaking wave) homogenizes the local fluid and then collapses (Woods,1968). The wake collapse is of special importance because the intermittent breaking of internal waves and the subsequent collapse of the patch is of undoubted importance in the transfer of heat and salt and possibly momentum in the seas and oceans.

5.2 Theories of Benjamin and Kao.

Wu has performed experiments on the collapse of a two-dimensional mixed region and has identified three stages of collapse: A short initial stage $Nt < 3$ of rapid collapse, where N is the Brunt-Väisälä frequency, $N^2 = -d\bar{b}/dz$ in the ambient fluid; a long principal stage $3 < Nt < 25$ of slower collapse and a final stage $Nt > 25$ in which friction and diffusion become important and the collapse is very slow.

The principal stage has been discussed very effectively by Benjamin (1968) and more recently by Kao (1974). The picture of this stage appears schematically in Fig. 5.2. The coordinate system moves with the density current so that there is an approaching current of strength U. If the flow is treated as inviscid, it is, in effect, steady flow past a body and thus there can be no drag. Since there are gravitational forces accelerating the mass, no steady state is possible. This implies that there is dissipation just as in

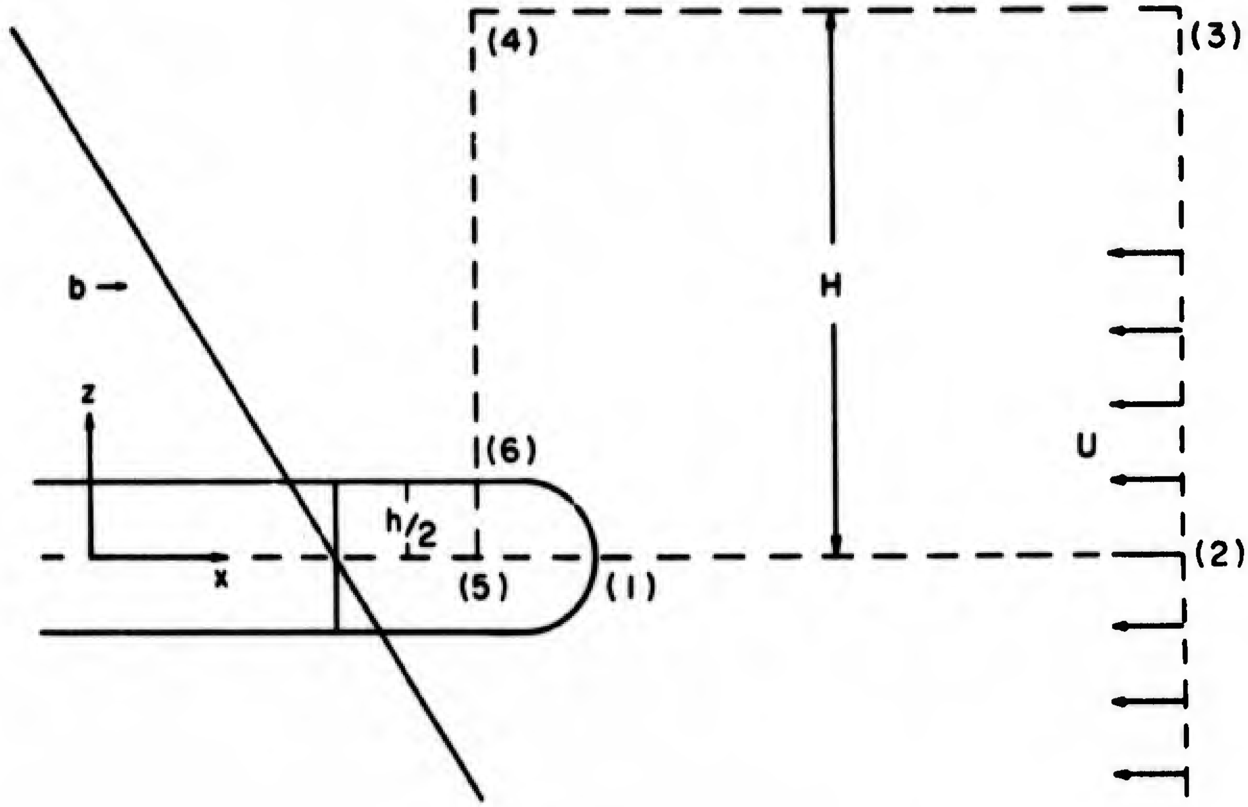

Fig. 5.2. Idealized model of wake collapse.

the case of an hydraulic jump in water flow. This means that the Bernoulli equation cannot be applied between points (1) and (6) for example on the "body". This dissipation is, of course, due to turbulence but as in the hydraulic jump it is possible to by-pass the difficulty and solve for the velocity of the current U without detailed knowledge. Following Benjamin (1968) and Kao (1974), we apply Bernoulli's equation on the horizontal streamline between the stagnation point (1) and point (2) far upstream in Fig. 5.2:

$$p_2 + \frac{\rho_0 U^2}{2} = p_1 \tag{5.1}$$

Since the pressure is hydrostatic far upstream, we get

$$p_3 = p_2 - \int_0^H \bar{\rho} g dz \qquad (5.2)$$

where H >> h. Obviously $p_3 = p_4$. Finally applying hydrostatics to the region far behind the nose, we obtain

$$p_5 = p_4 + \int_{h/2}^{H} \bar{\rho} g dz + \int_0^{h/2} \rho_0 g dz \qquad (5.3)$$

But the flow is zero inside the mass, so that $p_5 = p_1$ and

$$\rho_0 \frac{U^2}{2} = \int_0^{h/2} (\rho_0 - \bar{\rho}) g dz \qquad (5.4)$$

We may reasonably assume a linear density gradient near the mass so if

$$\bar{\rho} = \rho_0 + \left[\frac{d\bar{\rho}}{dz}\right]_0 z \qquad (5.5)$$

We get

$$U = \frac{Nh}{2}, \quad N = \left(-\frac{d\bar{b}}{dz}\right)_0^{\frac{1}{2}} \qquad (5.6)$$

The above analysis pertains to a strictly steady current. Wake collapse may be treated in the principal stage as quasi-steady so that the velocity is Nh/2 at any time when the height is h. The wake is centered at the origin $x = 0$. Considering half of it only, its length is $x(t)$ and height $h(t)$ with a constant volume so that $\alpha xh = V_o$ is constant where α denotes the shape factor. We assume an elongated wake with $\alpha = 1$. Since $dx/dt = U$, we have from Eq. (5.6)

$$\frac{dx}{dt} = \frac{NV_o}{2x} \quad \text{or} \quad x = V_o^{\frac{1}{2}}(Nt+k_1)^{\frac{1}{2}} \tag{5.7}$$

when k_1 is a constant. Although we are concerned only with the principal stage, we take $t = 0$ at the beginning of the experiment (Wu's experiment) when the mass was a circular cylinder. Thus $V_o = \frac{\pi x_o^2}{2}$ since we are only considering half of the volume. Eq. (5.7) becomes

$$\frac{x}{x_o} = \sqrt{\frac{\pi}{2}}\,(Nt + k_2)^{\frac{1}{2}} \tag{5.8}$$

where k_2 is a constant to be determined from the data or from a matching with the initial stage. It is approximately -0.9. The result is shown in Fig. 5.3 and we see a remarkable agreement with the theory.

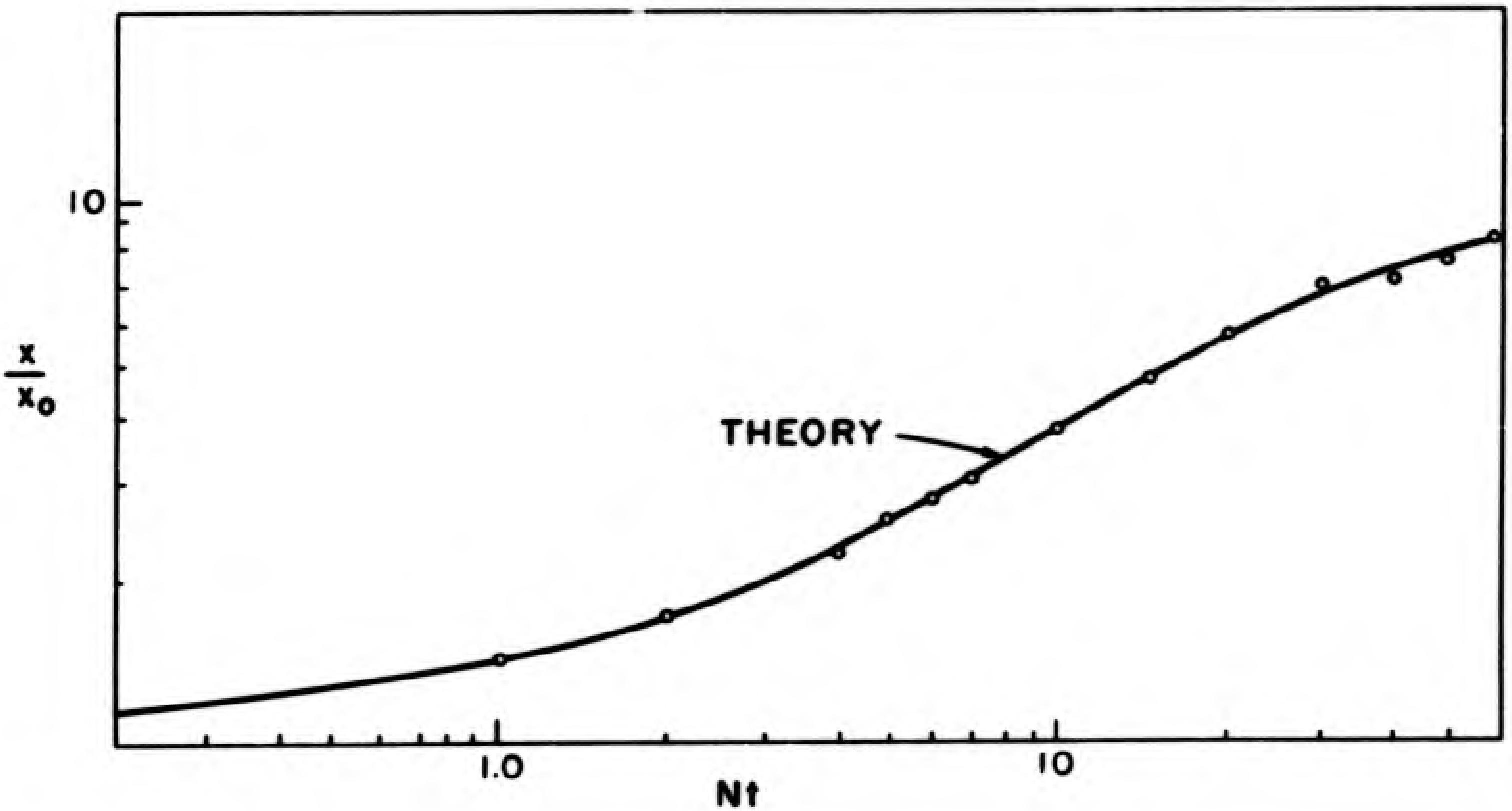

Fig. 5.3. Comparison of Kao's theory and Wu's observations.

5.3 Gravity Currents on a Sloping Bottom.

Of similar interest is the flow of a heavy fluid at speed U along a sloping floor (Fig. 5.4). This was investigated by Ellison and Turner (1959).

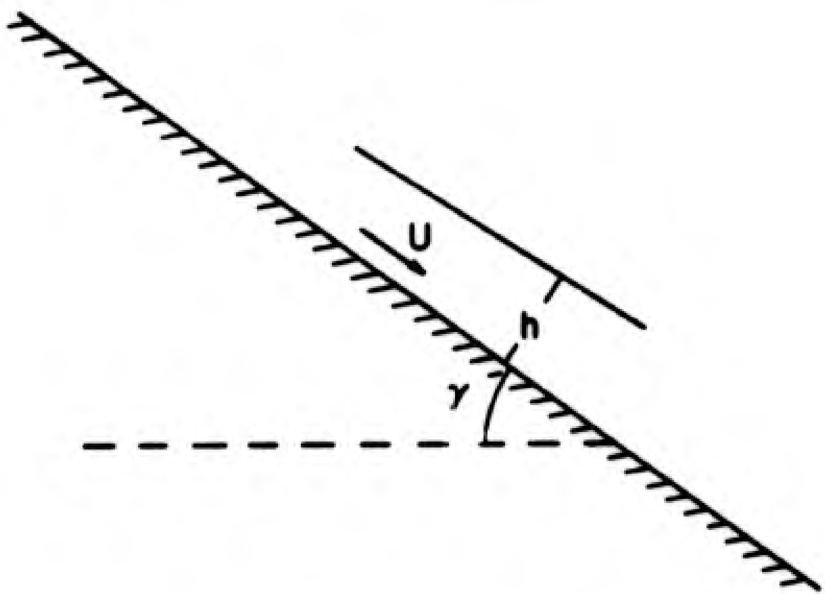

Fig. 5.4. Flow of density current.

As the heavier fluid moves, the lighter fluid is entrained into it causing the depth h to increase and the buoyancy difference Δb to decrease. Dimensional analysis suggests that $U = K_9 (h \Delta b)^{1/2}$, where K_9 is a function of the slope and this is verified by measurements where $K_9 = 2 - 3$ increasing with increasing slope from $\gamma = 0$ to $\gamma = 90°$. Of course, the entrainment falls off rapidly with increase of Ri.

5.4 Buoyancy Flux due to Wake Collapse.

Long (1970) has attempted to apply the concept of wake collapse to the exchange of buoyancy (heat or salt) in a density-stratified medium. We suppose that a turbulent region (or patch) of linear dimensions ℓ_p, volume V_0, is produced by the breaking of internal gravity waves. We assume the region is initially undisturbed with density given by $\bar{\rho}(z)$. Waves break and thoroughly mix the patch. It will then tend to flatten out at a level at which the environmental density equals the density of the patch. If the new level is different from that of the original center of mass of the material, heat will be transported vertically by this process. If z is taken at the original center of mass (Fig. 5.5), the average value of density for the mixed patch

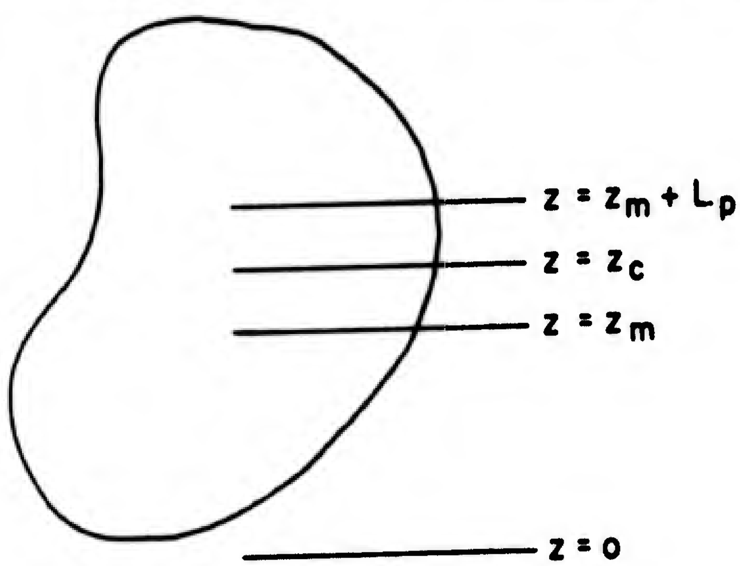

Fig. 5.5. Collapsing turbulent patch.

is

$$\bar{\rho}_p = \frac{1}{V_0}\iiint \bar{\rho}(z)dv = \frac{1}{V_0}\iiint\left[\bar{\rho}(z_c) + \bar{\rho}_z(z_c)(z-z_c) + \bar{\rho}_{zz}(z_c)\frac{(z-z_c)^2}{2} + \dots\right]dv \quad (5.9)$$

or

$$\bar{\rho}_p = \bar{\rho}(z_c) + \bar{\rho}_{zz}(z_c)A_3\ell_p^2 + \ldots \tag{5.10}$$

where A_3 is of order one. Expanding about $z = z_s$, we get

$$\bar{\rho}_p = \bar{\rho}(z_s) + \bar{\rho}_z(z_s)L_1 + \bar{\rho}_{zz}(z_s)\frac{L_1^2}{2} + \ldots + \bar{\rho}_{zz}(z_s)A_3\ell_p^2 + \ldots \tag{5.11}$$

where L_1 is the height of the centroid above the center of mass. The patch flattens out at height L_p above the center of mass given by $\bar{\rho}_p = \bar{\rho}(z_s + L_p) = \bar{\rho}(z_s) + \bar{\rho}_z(z_s)L_p + \ldots$, or

$$L_p \cong L_1 + \frac{\bar{\rho}_{zz}(z_s)A_3\ell_p^2}{\bar{\rho}_z(z_s)} \tag{5.12}$$

The distance L_1 is easily computed. Taking $z = 0$ at the centroid

$$L_1 = \frac{1}{M}\iiint \bar{\rho}(z)z\,dv \tag{5.13}$$

where M is the mass of the patch, or

$$L_1 = \frac{\iiint(\bar{\rho}(0) + \bar{\rho}_z(0)z + \frac{1}{2}z^2\bar{\rho}_{zz}(0) + \ldots)z\,dv}{\iiint(\bar{\rho}(0) + \bar{\rho}_z(0)z + \frac{1}{2}\bar{\rho}_{zz}(0)z^2 + \ldots)dv} \tag{5.14}$$

or

$$L_1 = \frac{B_1\bar{\rho}_z(0)\ell_p^5 + B_2\bar{\rho}_{zz}(0)\ell_p^6 + \ldots}{B_3\bar{\rho}(0)\ell_p^3 + B_4\bar{\rho}_{zz}(0)\ell_p^5 + \ldots} \tag{5.15}$$

where B_1, B_2 ... are of order one. Thus, since we take ℓ_t much less than the vertical scale of the motion

$$L_1 \sim \frac{\bar{\rho}_{zz}(0)\,\ell_t^2}{\bar{\rho}(0)} \tag{5.16}$$

The ratio of L_1 to $A_4\bar{\rho}_{zz}\ell_t^2/\bar{\rho}_z$ is of order

$$\frac{\bar{\rho}_z^2(0)}{\bar{\rho}(0)\bar{\rho}_{zz}(0)} \sim \frac{(\Delta\bar{\rho})^2}{\ell_t^2\rho_0}\,\frac{\ell_t^2}{\Delta\bar{\rho}} \sim \frac{\Delta\bar{\rho}}{\rho_0} \ll 1 \tag{5.17}$$

where $\Delta\bar{\rho}$ is density difference across the patch. Thus

$$L_t \sim \frac{\bar{b}_{zz}\,\ell_t^2}{\bar{b}_z} \tag{5.18}$$

and is proportional to the curvature of the buoyancy profile. Notice that the patch rises when the curvature is negative and falls when the curvature is positive. In either case, heat is transported downward. The flux of buoyancy is

$$q \sim K_t\bar{b}_z \tag{5.19}$$

where K_t is the coefficient of eddy diffusion and is given by the product of the mixing length L_t, the vertical velocity of the patch $w_t \sim |\bar{b}_z|^{\frac{1}{2}} L_t$ and the ratio γ of the volume of the turbulent patch to the whole volume, i.e. the intermittency factor (Hinze, 1959). Thus

$$q \sim \frac{\gamma \bar{b}_{zz}^{2} \ell_0^{4}}{|\bar{b}_z|^{3/2}} \tag{5.20}$$

This formula plays a considerable role in a theory of turbulence in stratified fluids by the author (Long, 1970). For example, we may use the theory to indicate that continuous density gradients tend to be unstable.

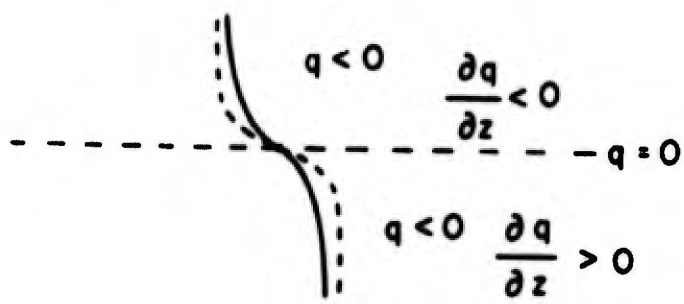

Fig. 5.6. Buoyancy profile.

Thus, in Fig. 5.6, suppose a profile (solid curve) develops an inflection point. Then the buoyancy flux distribution is as shown so that the mean equation

$$\frac{\partial \bar{b}}{\partial t} = \frac{\partial q}{\partial z}$$

leads to the development of a profile of the form of the dotted line. Thus, a discontinuity tends to form and the process continues until a sharp interface develops and some other process, perhaps Kelvin-Helmholtz instability, arises to transfer the heat.

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Topic 6. Problems Involving the Earth's Rotation

6.1 Introduction.

So far we have neglected the effect of the rotation of the earth on vertical mixing processes. However, for larger time scales, this must be taken into account. Thus, if the Coriolis force is important,

$$\frac{du}{dt} \leq 0(fu) \quad \text{or} \quad \bar{T}f \geq 0(1) \tag{6.1}$$

where f is the Coriolis parameter, $f = 2\Omega \sin\varphi$, in which Ω is the angular velocity of the earth and φ is latitude. $\bar{T}$ is the time period of the system, for example the duration of a wind of roughly the same speed and direction. In mid-latitudes, $f \cong 10^{-4} \sec^{-1}$.

The Coriolis force is proportional to the speed of a particle and is directed 90° to the right of the velocity. For example, as a particle of water begins to move under the influence of a wind we may, for extreme simplicity, neglect the pressure gradient force so that the equations of motion are

$$\frac{du}{dt} - fv = 0 \qquad \frac{dv}{dt} + fu = 0 \tag{6.2}$$

where u and v are velocities along the x and y axes. We now make a common approximation, namely that the Coriolis parameter $f = 2\Omega \sin\varphi$ is a constant. We choose a central latitude, φ_0 and expand f in a Taylor series. We get

$$f = 2\Omega \sin\varphi_0 + 2\Omega \cos\varphi_0 (\varphi - \varphi_0) + \ldots$$

The ratio of the neglected terms to $2\Omega \sin\varphi_0$ is, in mid-latitudes, of order

$$\varphi - \varphi_0 = \frac{\bar{L}}{a}$$

where $\bar{L}$ is the horizontal length scale of the motion and a is the radius of the earth. For large-scale oceanic and atmospheric phenomena $\bar{L}/a$ is not small but for small-scale motions and in seas and lakes the constant f assumption is very good. Using it, a solution of Eq. (6.2) is

$$v = v_0 \cos ft \qquad u = v_0 \sin ft \qquad (6.3)$$

where v_0 is the initial speed. This represents motion in a circle[1] with a period $2\pi f^{-1}$. Thus, as the water begins to move when the wind blows, the fluid will be deflected to the right looking in the direction of the motion.

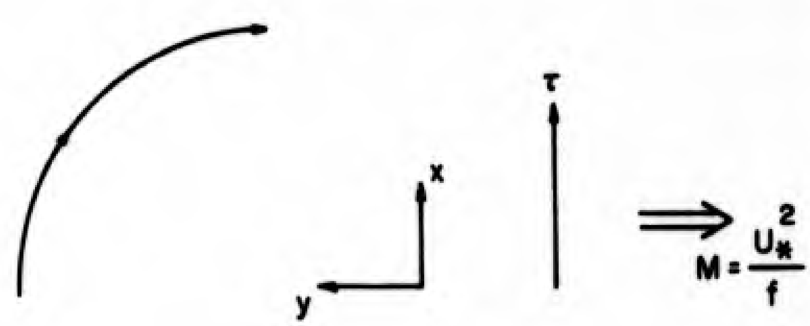

Fig. 6.1. Effect of rotation

[1] Clearly, if the radius of the circle $\frac{v_0}{f} \ll a$, the constant f assumption will be a good one.

In the absence of a pressure gradient, it will be moving to the right of the wind in $\frac{2\pi f^{-1}}{4}$ sec or approximately 4 hours. The pressure effect will slow down the turning so we would expect a somewhat longer time. Actually the water set in motion does move with an average velocity perpendicular and to the right of the wind after about 6 hours (Csanady, 1974). The theory of this phenomenon is due to Ekman (1905) and is easily derived. We consider a wind blowing uniformly over an infinite ocean inducing velocities $\bar{u}(z,t)$, $\bar{v}(z,t)$. The situation is homogeneous horizontally so no quantity (including mean pressure) varies with x, y. Therefore, the equations of mean motion are[1]

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -\frac{\partial}{\partial z}(\overline{u'w'}) \qquad (6.4)$$

$$\frac{\partial \bar{v}}{\partial t} + f\bar{u} = -\frac{\partial}{\partial z}(\overline{v'w'}) \qquad (6.5)$$

If we take the x-axis along the direction of the surface stress, we have $-\overline{u'w'} = u_*^2$, $-\overline{v'w'} = 0$ at $z = 0$ where we now take u_* independent of time. The net transports in the affected layer are

$$M_x = \int_0^\infty \bar{u}(z,t)dz, \quad M_y = \int_0^\infty \bar{v}(z,t)dz \qquad (6.6)$$

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[1] Notice that the fluid may have density variations both horizontally and vertically but this does not affect the accuracy of (6.4) and (6.5) to within the Boussinesq approximation. Thus the analysis may be applied to a stratified ocean.

Therefore, integrating Eqs, (6.4) and (6.5), we find

$$\dot{M}_x - fM_y = u_*^2 \qquad (6.7)$$

$$\dot{M}_y + fM_x = 0 \qquad (6.8)$$

or

$$\ddot{M}_y + f^2 M_y = -fu_*^2 \qquad (6.9)$$

If we assume an impulsive start for the wind at t = 0, we get

$$M_x = \frac{u_*^2}{f} \sin ft, \quad M_y = -\frac{u_*^2}{f}(1-\cos ft) \qquad (6.10)$$

We see that the transports oscillate in time about the mean values

$$\bar{M}_x = 0, \quad \bar{M}_y = -\frac{u_*^2}{f} \qquad (6.11)$$

with the inertia period $2\pi f^{-1}$. After a while, friction will damp out the oscillation and the steady transports will be given by Eq. (6.11). There is no transport in the direction of the wind. The transport is to the right as indicated in Fig. 6.1. This is called the Ekman drift.

The onset of the wind creates a wave-motion as we have just seen, but the situation is normally strongly affected by density stratification which yields another tendency for wave motion, namely up and down motions of the fluid under gravity in internal waves of period of order $\sqrt{H_0/\Delta b}$, where Δb is the buoyancy difference from top to bottom and H_0 is the depth of the fluid. In general then a disturbance of the free surface corresponding to an imposed atmospheric pressure increment or wind stress will set up wave motions which are a combination of these two effects: rotation and stability. These are called internal Kelvin waves and have been discussed by Csanady (1967a, b, 1968a, b, 1971) in connection with circulations in the Great Lakes in northern USA. Csanady confined his analysis to simple models, i.e. a two-layer system with constant total depth in a circular basin, etc. More recently Walin (1972a, b) investigated the same problem with the Baltic sea in mind and, although his results are less specific, he obtained useful information by scale analysis and approximate mathematical methods applied to realistic types of topography and meteorological forcing effects.

6.2 Preliminary Discussion.

In this section we will present sufficient background to permit a coherent discussion of the basic nature of Walin's investigation.

Hydrostatic Assumption

The hydrostatic assumption neglects vertical accelerations compared with gravity and the vertical pressure gradient. The significance of this is seen most easily by considering the case of a homogeneous fluid with a free surface of height $z = H(x,y,t) \sim H_o$. Integrating the vertical equation of motion (z-axis is directed upward), we get

$$p = -\rho g(z-H) + 0\left[\int_z^H \rho \frac{dw}{dt}\, dz\right] \tag{6.12}$$

If we insert this in the x-equation of motion, we get

$$\frac{du}{dt} = -g\frac{\partial H}{\partial x} + 0\left[\frac{1}{\bar{L}} H_0 \frac{dw}{dt}\right] \tag{6.13}$$

where $\bar{L}$ is the length scale of the horizontal motion. The continuity equation yields

$$\frac{dw}{dt} \sim \frac{\bar{U}\bar{W}}{\bar{L}} \sim \frac{\bar{U}^2 H_0}{\bar{L}^2} \tag{6.14}$$

where $\bar{U}$ is the order of the horizontal velocity and $\bar{W}$ is the order of the vertical velocity. If we now take the ratio of the error in (6.13) to the acceleration $du/dt \sim \bar{U}^2/\bar{L}$, we get a relative error of order $H_0^2/\bar{L}^2$. In applications, for example to the Baltic, $H_0^2/\bar{L}^2 \sim 10^{-6}$ if we take H_0 to be the depth ($\sim 10^4$ cm) and $\bar{L}$ to be the lateral dimensions ($\sim 10^7$ cm). Of course, we have assumed the largest possible scales of motion, but we believe that these are the important scales to direct attention to. If the fluid is stratified and rotating, the situation is similar, indeed the error in assuming hydrostatics may be even smaller. There are, of course, some phenomena for which the hydrostatic assumption may not be valid. Examples are water waves whose length is short compared to the depth and lee waves behind mountains in the atmosphere.

Barotropic and Baroclinic Motions

A fluid is called barotropic if the density is a function of the pressure. In water this is nearly equivalent to a fluid with uniform density. Thus, suppose we have a homogeneous rotating fluid moving slowly in the sense that the non-linear acceleration terms are small compared to the Coriolis accelerations. This implies a small Rossby number U_*/fL_*, where U_* is a velocity characteristic of the system and L_* is a characteristic length scale. The horizontal equations of motion are then

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{6.15}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{6.16}$$

where ρ is constant. If, in addition, the motion is steady (or more usefully if $\bar{T}f^{-1} >> 1$), we obtain geostrophic motion,

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{6.17}$$

$$fv = \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{6.18}$$

Then, assuming hydrostatics,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{6.19}$$

we obtain

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \tag{6.20}$$

so that the horizontal velocities are constant with height. Although barotropy is only one of the assumptions made, this type of motion is called barotropic motion in meteorological and oceanographic literature. If we also assume f is constant, Eqs. (6.17) and (6.18) reveal that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6.21)$$

so that the equation of continuity yields

$$\frac{\partial w}{\partial z} = 0 \qquad (6.22)$$

Thus w is constant with height. Since it is zero at the surface z = 0, it is zero everywhere. In particular, it is zero at the bottom z = H(x, y). Then the kinematic boundary condition at the bottom yields

$$w_b = u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = 0 \qquad (6.23)$$

so that the fluid moves parallel to the bottom contours.

When the fluid density varies (especially vertically) so that the pressure surfaces and density surfaces do not coincide in the presence of disturbances, the motion is then called baroclinic. Among other effects, we expect important variations of horizontal velocities with depth and internal gravity-wave motions. Baroclinic effects are typical of the atmosphere and oceans and lakes, but theory sometimes leads to barotropic motions, at least to the first order (Section 6.3).

6.3 Walin's Investigation

Walin (1972a,b) has made a very general investigation of a problem, first treated by Csanady, namely the response of a large lake or a sea similar to the Baltic, to surface disturbances caused by pressure and wind acting on the free surface. We will not repeat his analysis in detail but we will discuss the main assumptions, procedures and results.

The region under consideration is divided into three parts. One is the large interior region and the other the "thin" viscous (eddy friction) boundary layers near the free surface and near the bottom. In the interior he assumes (1) constant f, (2) hydrostatics, (3) Boussinesq approximation, (4) equal horizontal velocity scales $\bar{L}$ in both horizontal directions, (5) a vertical scale of the order of the depth of the fluid, H_o, (6) zero friction, (7) $\sqrt{H_o \Delta b} << f\bar{L}$, where Δb is the buoyancy difference from top to bottom, (8) weak meteorological disturbances.

Assumptions (1)-(4) are excellent approximations and need no further discussion. Assumption (5) may be dangerous if the fluid is continuously stratified, because the stability will tend to inhibit vertical motion and lead to a vertical length scale of smaller order than H_o. In fact, observations in the oceanic thermocline (Stommel and Federov, 1967, Woods, 1968) indicate a small-scale vertical structure in regions (thermocline) in which the density (averaged over several meters vertically) varies continuously. Experiments indicate, however, (Moore and Long, 1971) that continuous density gradients are only turbulent when overall Richardson numbers are of order one and, in this case, the turbulence extends over the whole depth of the fluid and the vertical scale is of order H_o. In the Baltic, the fluid is nearly homogeneous in the winter in the upper fluid and the motion extends from the surface to the permanent halocline at about half the average depth of the basin. In the summer there is an upper homogeneous layer above a seasonal thermocline of depth 20 - 25 m and the vertical scale is of this order. This, however, is a major portion of H_0 so that H_0 is probably an appropriate measure of the vertical scale.

In the interior of the fluid, assumptions (1)-(8) lead to a fundamental conclusion that the fluid in the interior (away from lateral boundaries) moves barotropically, in particular that the horizontal motion is independent of depth and is parallel to the depth contours. Assumption (6) is, of course, crucial for this result and we may examine it in detail. According to Svansson (1966) the kinematic stress in the air is $\tau_a = \alpha_3 \frac{\rho_w}{\rho_a} U_a^2$ or $\tau_w = \alpha_3 U_a^2$ where "w" denotes water and "a" denotes air and $\alpha_3 \cong 1.75\ 10^{-6}$. For a velocity $U_a = 7$ m/sec, typical of strong disturbances in summer, we get $\tau_w \cong .86$ cm^2/sec^2. In flow in a pipe the eddy viscosity $K_a \cong \frac{\kappa}{2} \tau^{\frac{1}{2}} H$ in the center, where H is the diameter and κ is Von Kármán's constant, so that we may estimate $K_a \cong 464$ cm^2/sec in the bulk of the upper layer (taking H or $H_0 \cong 25$m). The viscous term is then of order $K_a \bar{u}_{zz} \sim 7.4\ 10^{-4}$ cm/sec^2, if we take $\bar{u} \cong 10$ cm/sec. The Coriolis force, on the other hand, is then of order $f\bar{u} \sim 10^{-3}$ so that the frictional terms are in fact of the same order as the retained terms in the major portion of the fluid depth. The result indicates that the entire mixed layer is, in fact, the same as the Ekman layer in which friction dominates, and that Walin's assumption that the Ekman layers are thin compared to H_0 is not justified. In the simplest interpretation, the fluid depth is divided into two Ekman layers with strong changes of water velocity due to friction throughout the depth. The equivalence of the Ekman layer and the lower mixed layer in the atmosphere has been recognized for some time by meteorologists (Arya, 1974). We also note that recent theory indicates that the depth of the Ekman layer is also given as $D \sim 0.25\, u_*/f$ (Csanady, 1967a,b; Long, 1974; Arya, 1974). In the Baltic using $\tau_w \sim 1$ we get $D \sim 25$ m as observed. The two approaches are consistent.

With respect to assumptions (7) and (8), the former states in effect that internal wave speeds are much less than velocities in an inertia circle of the size of the basin. A calculation reveals the ratio of order 0.05 so the assumption is valid. The remaining assumption (8) leads to the linearization of the equations. It is shown a posteriori that the linearization is least valid for shorter time-scale phenomena in the interior and for longer time-scale phenomena near the coast.

Walin's analysis leads to a prognostic equation for the barotropic response when the additional assumption $T \gg f^{-1}$ is made, although this will result in a solution that has too little energy in the shorter time scales. As we approach the coast, we would expect that the length scale of the disturbances should decrease so that $\bar{L}$ is no longer an appropriate scaling. Assumption (7) then becomes questionable and, in fact, it is just in a narrow region near the coast of width L_c such that $\sqrt{H_0 \Delta b} \sim fL_c$ where we would expect assumption (7) to break down. It is in this region that the internal wave (baroclinic) phenomenon will be important. Walin attacks this problem by a boundary layer approach in which baroclinic disturbances are assumed to exist only in this narrow zone. The width L_c may be estimated by taking $\Delta b \sim 2$, $H_0 \sim 60$ m so that $L_c \sim 10^6$ cm or 10 km.

The baroclinic waves, also called long internal Kelvin waves, are thus confined to a region near the coast. They move at a speed $\sim (H_B \Delta b)^{\frac{1}{2}}$ where H_B is the depth in this region. The free waves have the interesting property that they travel in a counter-clockwise sense, i.e. southward along the Swedish coast of the Baltic. If friction is taken into account, low frequency waves are most affected.

Walin considers the effect of friction and forcing in detail. He assumes a forcing function sinusoidal in space and time with t_0 characteristic of the time scale and ℓ_0 characteristic of the space scale. The forcing function is such that ℓ_0 may be taken to be representative of the wave length of the coastal indentations (distance between successive bays). He finds that the amplitude is limited by friction but is a maximum when $\ell_0/t_0 \sim$ speed of a free Kelvin wave. This resonance condition may frequently be satisfied so that large amplitude waves are to be expected.

In his second paper, Walin discusses the practical application of his analysis, in particular the result that strong disturbances in the velocity field (and, hence, in the temperature field) should be found near the coast and that such disturbances should die off exponentially within 5-10 km from the coast. This has very important implications for the subject of these notes because the large-amplitude waves at the coast should lead to strong vertical mixing there. In addition, one would expect a coastal jet stream which would tend to carry away pollutants from their near-shore sources.

Walin gathered two months of data on temperature variations near the Swedish coast. He obtained temperature-depth soundings with thermistors for temperature and a pressure gauge for depth. The soundings were taken at various distances from shore; in addition, continuous temperature measurements were made of the mean temperature from 0 -10 m depth in a region 0 -600 m from the shoreline. The measurements show that near the coast the temperature decreases more or less continuously with depth but at distances greater than 10 km from the coast the upper 25 m are homogeneous and a well-developed thermocline exists. This is what one would expect if the region near the coast is continually disturbed by large-amplitude internal waves. This tends to confirm the presence of the baroclinic disturbances of the theory.

The soundings also reveal that the temperature variations from day-to-day are an order of magnitude (10 times) greater in the coastal region than in the interior. (Some soundings were made out to 24 km from shore.) These large fluctuations are associated with strong wind disturbances.

Finally, Walin notices that upwelling tends to be associated with off-shore winds and thus with an Ekman transport parallel to the coast rather than with a transport away from the coast as one would expect. He believes, however, that the coastal jet brings water from a region off the coast to the north of the section in which observations were taken. In that region the orientation of the coast is such that the transport is in fact perpendicular to it. The time for the water to move this distance is not larger than the time scale of the meteorological disturbances.

6.4 Upwelling.

We may discuss upwelling in a little more detail. The water in the oceans and seas and lakes is quite cold in the deep portions. This water normally does not affect surface layers except in regions of upwelling. This occurs, of course, when the Ekman transport is away from the coast. This displaced water must be replaced by water from below. It occurs when the wind blows parallel to the coast with the coast to the left of the wind. An example, cited by Rooth (1974) is the coast of Oregon where the prevailing winds are northerly. The water is, therefore, very cold in the region even in summer and this has profound effects on the weather. Also of great importance is the supply of nutrients from below in regions of upwelling. This leads to riotous biological activity along the upwelling zone.

Upwelling also occurs in limited bodies of water such as the Baltic (as we have seen) and the Great Lakes. According to Csanady (1974), the temperature of the left shore can drop from 20°C to 5°C very shortly after the beginning of a strong wind. The wind is generally southwesterly over the Great Lakes so the "warm" shores are the eastern shores of Lakes Michigan and Huron and the south shore of Lake Ontario. However, the wind directions are variable, of course, and alternate upwelling and down-wellings in the course of a summer, together with the coastal jet causes the whole coastal zone to be flushed out with water coming from the depths of the lakes. This, of course, is most important with respect to the pollution problem.

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LIST OF SYMBOLS

A	area
$A_1, A_2, \ldots$	constants
a	amplitude of grid or plate vibration; earth's radius $\sim 6.4\ 10^8$ cm; as a subscript denotes "air"
$a_1, a_2, \ldots$	lengths characteristic of vibrating grid
α	"coefficient of expansion" in expression $\rho = \rho_0(1-\alpha T+\beta S)$; shape factor
$\alpha_1, \alpha_2, \ldots$	constants of proportionality
α_n	ratio K_b/K_m in neutral conditions
B, B_1	constants
b	buoyancy: $b = \frac{\rho-\rho_0}{\rho_0} g$
b_0	buoyancy at top of homogeneous layer
b_1	rms buoyancy fluctuation near interface
b'	fluctuation buoyancy
$\bar{b}$	mean value of b
β	"coefficient of expansion" in expression $\rho = \rho_0(1-\alpha T+\beta S)$
β_1	constant
β_b	constant in the log-linear buoyancy law
β_m	constant in the log-linear mean velocity law
$C, C_1, \ldots$	constants
C_0	correlation coefficient
c_d	drag coefficient $= \tau/(U_0^2/2)$
c'	turbulent speed
cm	centimeter
γ	intermittency factor; angle
D	depth of homogeneous layer; pipe diameter
d	a length
dv	element of volume
Δ	increment symbol
Δb	buoyancy jump across interface
$\Delta\bar{b}$	buoyancy increment across "homogeneous" layer
$\Delta\bar{b}(z,t)$	difference between buoyancy at top of "homogeneous" layer and buoyancy at depth z.

ΔS	non-dimensional salinity difference
Δu	velocity difference
$\Delta\tau$	increment in τ over depth D
δ	increment symbol; dissipation function for buoyancy fluctuations
δ_T	thickness of thermal boundary layer
$E(k)$	spectrum function
ϵ	dissipation function : $\epsilon = \nu[\overline{(\nabla u')^2 + (\nabla v')^2 + (\nabla w')^2}]$
F_0, F_1	internal Froude numbers
f	Coriolis parameter $f = 2\Omega\sin\varphi$
ζ	level where mean density equals the particle density; z/L; height of the sill in a sound
g	acceleration of gravity (981 cm/sec^2)
H	total fluid depth; height of free surface
H_b	depth near coast
H_0	a length of the order of the fluid depth
h	u_*t small depth of developing mixing layer; thickness of density current or wake; depth of sill in a sound
h_1	depth of upper layer in two-fluid system
h_t	thickness of density interface
h_0	average height of roughness elements; depth of lower layer in two-fluid system
η	$= z_0/D$; ratio of depths of a two-fluid system
θ	$= z/z_0$
K	constant
$K_1, K_2, \ldots$	constants
K_b	eddy diffusivity of buoyancy $= q/\bar{b}_z$
K_{b0}	value of K_b in neutral conditions
K_m	eddy viscosity, $= \tau/\bar{u}_z$
K_n	constant in entrainment law of Turner (no shear)
K_s	constant in entrainment law of Kato and Phillips (shear)
k	wave number; $k = 2\pi/\lambda$
$\underset{\sim}{k}$	vertical unit vector
$k_1, k_2, \ldots$	constants
k_h	molecular coefficient of heat conduction

k_s	molecular coefficient of salt diffusion
κ	Von Kármán's constant ($\cong .40$)
κ_1	constant
L	Monin-Obukhov length = $\tau^{\frac{3}{2}}/\kappa q$
L_1	height of centroid above center of mass
L_c	width of narrow region near coast
L_p	level above center of mass at which turbulent patch flattens out
L_s	characteristic length
$\bar{L}$	horizontal length scale of mean motion
ℓ	integral length scale ~ length scale of energy-containing eddies; width of channel
ℓ_1	value of ℓ near interface
ℓ_0	length scale
ℓ_p	dimensions of turbulent patch
ℓ_v	length scale of dissipating eddies, i.e. Kolmogorov microscale of length: $\nu^{\frac{3}{4}}/\epsilon^{\frac{1}{4}}$
ℓ'	value of ℓ near interface in mixed layer
ℓn	natural logarithm
λ	wave length
M	mass of patch of turbulence
$\underset{\sim}{M}$	transport vector (Ekman transport)
m_1, m_2	functions
N	Brunt-Väisälä frequency: $N^2 = -\frac{d\bar{b}}{dz}$
n	exponent
ν	molecular coefficient of viscosity
ξ	$z-\zeta$: distance of a particle above the level ζ where the mean density equals the density of the parcel; $z(\Delta b)^{\frac{1}{3}}/k_h^{\frac{2}{3}}$
ξ_0	$H(\Delta b)^{\frac{1}{3}}/k_h^{\frac{2}{3}}$
O	order of magnitude symbol
O(1)	the ratio of two quantities P and R is O(1) if they are proportional
P	physical quantity
Pe	Péclet number based on velocity and length scale of energy-containing eddies. $Pe = \sigma_u \ell/k_h$, or $\sigma_u \ell/k_s$

Pr	Prandtl number ν/k_h
p	pressure
p'	fluctuation pressure
$\bar{p}$	mean value of p
p_*	$= p/\rho_0 + gz$
Q, Q_1, Q_0, Q_f	non-dimensional volume fluxes
q	buoyancy flux; if turbulent $q = -\overline{w'b'}$
q_1, q_0, q_f	volume fluxes in an estuary
R	physical quantity, ratio of Froude numbers: F_1^2/F_0^2
Re	Reynolds number based on velocity and length scale of energy-containing eddies. $Re = \sigma_u \ell/\nu$
Ra	Rayleigh number, $\frac{H^3 \Delta b}{\nu k_h}$
R_s	$\tau_*^2/q\nu$
R_τ	$\tau_*^{2} H/\nu$
Rf	flux Richardson number $= q/\tau \bar{u}_z$
Rf^*	$q \ell/\sigma_u^3$
Rf_c	value of Rf above which turbulence ceases
Ri	gradient Richardson number $\lvert\bar{b}_z\rvert/\bar{u}_z^2$; or $\ell' \Delta b/\sigma_u'^2$
Ri^*	overall Richardson number, $D\Delta b/u_*^2$ or $D\Delta b/(\Delta u)^2$
r	ratio of thickness of a shear layer to thickness of density "interface"; non-dimensional quantity defined in Eq. (3.23)
r_n	non-dimensional quantity defined in Eq. (3.26)
rhs	right hand side (of an equation)
rms	root-mean-square
ρ	density
ρ_a	density of air
ρ_0	characteristic density; density in the lower layer of a two-fluid system
ρ_{00}	density of fresh water
ρ_T	density of turbulent patch
ρ_w	density of water
ρ_1	density in homogeneous layer
ρ'	fluctuation density

S	salinity: mass of salt per unit volume. In a two-layer system, S_0 and S_1 are the salinities of the lower and upper layer, respectively.
sec	second
σ	constant; $z(\Delta b)^{\frac{1}{3}}/\nu^{\frac{2}{3}}$
σ_b	rms buoyancy fluctuation
σ_b'	rms buoyancy fluctuation near interface
$\sigma_u, \sigma_v, \sigma_w$	rms velocity fluctuations
σ_u'	value of σ_u near interface in mixed layer
T	temperature
$\overline{T}$	time scale of mean motion
T_e	eddy time scale
T_i	time scale for erosion of interface
T_κ	molecular diffusion time
T_t	turbulent diffusion time
T'	turbulent kinetic energy per unit mass
t	time
t_0	time scale
t_ν	time scale of dissipating eddies, i.e. Kolmogorov time scale: $t_\nu = (\nu/\epsilon)^{\frac{1}{2}}$
τ	momentum flux; if turbulent, $\tau = -\overline{u'w'}$
τ_a	stress exerted on a water surface by the wind
τ_0	surface stress
U	speed of scree; velocity of density current
U_a	wind speed
U_0	velocity at center of pipe
U_s, $\overline{U}$	characteristic horizontal velocity
u	velocity component along x-axis
$\bar{u}$	mean value of u
u_e	entrainment velocity
u_0	horizontal velocity in the lower layer of a two-fluid system
u_ν	velocity scale of dissipating eddies, i.e. Kolmogorov micro-scale of velocity: $u_\nu = (\nu\epsilon)^{\frac{1}{4}}$

u_1	horizontal velocity in the upper layer of a two-fluid system
u'	fluctuating horizontal velocity component
u_*	friction velocity ($= \tau^{\frac{1}{2}}$); ωa
ϕ	latitude
V	incremental potential energy per unit mass; $\Delta u/(\nu \Delta b)^{\frac{1}{3}}$
V_0	volume
V'	available potential energy per unit mass
v	velocity component along the y-axis
v_0	initial velocity
$\bar{v}$	mean value of v
$\bar{v}$	mean value of v
$\underset{\sim}{v}$	vector velocity
v'	fluctuating velocity component along y-axis
$\bar{W}$	characteristic vertical velocity
w	subscript denoting "water"; vertical velocity of fluid
w_b	vertical velocity at the (sloping) bottom
w_p	vertical velocity of patch
w'	fluctuating vertical velocity component
$\bar{w}$	mean value of w
x, y	horizontal coordinates
x_0	initial radius of collapsing wake
z_c	height of centroid of turbulent patch
z_0	roughness length
z_E	height of center of mass of turbulent patch
z_+	$z\tau^{\frac{1}{2}}/\nu$
Ω	angular speed of earth's rotation
ω	frequency of vibration of grid
∇	"del" operator
∇^2	Laplacean: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
overbar	indicates an average
prime	indicates a fluctuating quantity
$\sim$	order of magnitude symbol; proportionality symbol: when below another symbol it signifies a vector
$\cdot$	scalar (dot) product

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