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ON A COVERING PROBLEM FOR PARTIALLY
SPECIFIED SWITCHING FUNCTIONS

M. Schkolnick

Carnegie-Mellon University

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M. Schkolnick
Carnegie-Mellon University
Pittsburgh, Pa. 15213

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Abstract

We consider the problem of finding the minimum number $K(n,c)$ of total switching functions of n variables necessary to cover the set of all switching functions which are specified in at most c positions. We find an exact solution for $K(n,2)$ and an upper bound for $K(n,c)$ which is better than a previously known upper bound by an exponential factor.

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1. Introduction

The problem considered here can be stated as follows:

P1: Given the set F of all c -specified boolean functions of n variables, i.e., all functions which are specified in at most c positions, to find the cardinality $K(n,c)$ of a set G of total functions such that

P1-1: For all f in F , there is a g in G such that g covers f , i.e., if $f(x)$ is specified then $g(x) = f(x)$.

P1-2: $K(n,c) = |G|$, is minimal.

This problem relates the number of additional exterior connections (besides input and output) that are required in a circuit which is to be c -universal. (A circuit is c -universal if it is capable of simulating the behavior of any partial function which is specified in c or less points of its domain.)

This problem was studied in [1] in connection with adaptive networks, where an upper bound for $K(n,c)$ was shown to be

$$K(n,c) \leq \sum_k \binom{m}{p+k\delta}$$

where $m = 2^n$, $p = \lfloor c/2 \rfloor \bmod \delta$, $\delta = m+1-c$

This upper bound agrees with the exact solutions for $c=1$ (i.e., $K(n,1)=2$) and $c=2^n-1$ (i.e., $K(n,2^n-1)=2^{2^n-1}$). For $c=2$ we have $\delta=2^n-1$ and, for any $n > 1$, $p=1$ so

$$K(n,2) \leq \sum_k \binom{2^n}{1+k(2^n-1)} = \binom{2^n}{1} + \binom{2^n}{2^n} = 2^n + 1$$

and in general, for small c , this bound is of the order of $2^{nc/2}$.

In this note we show that for $c=2$, $K(n,2) = O(n)$ and present an upper bound which, for fixed c is a power of n .

2. An Exact Solution for $K(n,2)$

Consider the following problem:

P2: Given n and c , find the dimension $s(n,c)$ of a vector space over $GF(2)$ such

that there is a set P of at least 2^n vectors in it satisfying:

P2-1: $(\forall p_1, p_2, \dots, p_c) \in P, (\forall b_1, b_2, \dots, b_c) \in \{0,1\}, p_1^{b_1} p_2^{b_2} \dots p_c^{b_c} \neq 0$

P2-2: $s(n,c)$ is minimal

Notation: We will use the following convention

1) $(\forall a, b, \dots, z) \in M$ means for all elements $a, b, \dots, z$ in M

2) $p^b =$ if $b=1$ then p else $\sim p$

The first result we present shows that essentially, P1 and P2 are equivalent problems.

Lemma 1: For all $c > 1$, $K(n,c) = s(n,c)$.

Proof: We show that any solution to P1 satisfying P1-1 is a solution to P2 satisfying P2-1 and conversely. This implies that the minimality conditions are also satisfied.

Let $G = \{g_1, g_2, \dots, g_{K(n,c)}\}$ be a solution to P1 satisfying P1-1. Consider the set $P = \{p(x) = (g_1(x), g_2(x), \dots, g_{K(n,c)}(x)) \mid x \in \{0,1\}^n\}$. Let $x, y \in \{0,1\}^n$ with $x \neq y$. Then $p(x) \neq p(y) \Rightarrow (\forall g) \in G, g(x) \neq g(y)$. But since $c > 1$, this implies that there is a c -specified function f with $0 = f(x) \neq f(y) = 1$ which is not covered by any $g \in G$ which is a contradiction. Thus $p(x) \neq p(y)$, which shows that $|P| = 2^n$.

Assume now that there are c different elements $p_1, p_2, \dots, p_c$ in P such that, for some $b_1, b_2, \dots, b_c \in \{0,1\}$, $p_1^{b_1} p_2^{b_2} \dots p_c^{b_c} = 0$. Let $p_j = p(x_j) = (g_1(x_j), g_2(x_j), \dots, g_{K(n,c)}(x_j))$ for some n -tuple $x_j \in \{0,1\}^n$. Let f be a c -specified function such that $f(x_j) = b_j$ for $j = 1, 2, \dots, c$. Since $p_1^{b_1} p_2^{b_2} \dots p_c^{b_c} = 0$, for each $k = 1, 2, \dots, K(n,c)$, there is a j , $1 \leq j \leq c$ such that $g_k(x_j) = 1 - b_j$. Thus, for this value of j we have $g_k(x_j) \neq f(x_j)$ so g_k does not cover f . Since

this holds for all k , we have that G does not satisfy P1-1, a contradiction. Thus, P2-1 is satisfied.

Conversely, let P be a set of 2^n s -dimensional vectors $P = \{p_0, p_1, p_2, \dots, p_{2^n-1}\}$ satisfying P2-1. Consider the set $G = \{g_1, g_2, \dots, g_s\}$ of boolean functions of n variables defined as follows:

For each $1 \leq j \leq s$, $(\forall i) \in \{0, 1, \dots, 2^n-1\}$, $g_j((i_2)_1, (i_2)_2, \dots, (i_2)_n) = (p_i)_j$ where i_2 denotes the binary representation of i with n bits, $(i_2)_r$ denotes the r -th bit and for an s -dimensional vector p , $(p)_r$ denotes the r -th component.

Let f be a c -specified function of n variables. Without loss of generality, assume that f is specified at $((i_2)_1, (i_2)_2, \dots, (i_2)_n)$ for $i = 0, 1, \dots, c-1$. We claim there is at least one g which covers f . Define, for $i = 0, 1, \dots, c-1$, $b_i = f((i_2)_1, (i_2)_2, \dots, (i_2)_n)$. Since P satisfies P2-1, $p_0^{b_0} \wedge p_1^{b_1} \wedge \dots \wedge p_{c-1}^{b_{c-1}} \neq 0$. Thus, there is a $j \in \{1, 2, \dots, s\}$ such that, for all $i \in \{0, 1, \dots, c-1\}$, $(p_i^{b_i})_j = 1$. (Note that $p_i^{b_i}$ is either p_i or its complement, and this means the j -th component of this vector is 1.) This means that $(p_i)_j = b_i$. By the definition of b_i and the definition of G we have

$$g_j((i_2)_1, (i_2)_2, \dots, (i_2)_n) = f((i_2)_1, (i_2)_2, \dots, (i_2)_n)$$

for all $i \in \{0, 1, \dots, c-1\}$. Thus, $g_j \in G$ covers f . This completes the proof of Lemma 1. $\blacksquare$

Now we focus our attention to Problem 2. In what follows, we assume s is restricted to be even and we will show that $K(n, 2)$ can be determined exactly (to within 1). We first prove an auxiliary result. Since P2 can be interpreted as: Find the smallest s such that there are at least 2^n points in the s -cube satisfying P2-1, we will now show that the search for points in the s -cube satisfying P2-1 can be reduced to the set of all points in the middle plane (i.e., having weight $s/2$).

Lemma 2: Let $c = 2$, s be an even positive number, and P be a set of s -dimensional vectors satisfying P2-1. Then, there is a set Q of s -dimensional vectors, each of which has weight $s/2$ and such that $|Q| = |P|$, satisfying P2-1.

Proof: We can assume, without loss of generality, that all vectors in P have weight $\geq s/2$. (It is clear that changing a vector by its complement in any set satisfying P2-1 also produces a set satisfying P2-1.) If all vectors have weight $s/2$ we have proved the lemma. Assume then that P contains t vectors $p_1, p_2, \dots, p_t$ with maximal weight $u > s/2$. We will construct a set P' such that all vectors in it will have weights w such that $s/2 \leq w < u$. Since $u - s/2$ is finite this will prove the lemma.

Choose any set of t vectors $q_1, q_2, \dots, q_t$ with the property that $q_i < p_i$ for $i = 1, 2, \dots, t$ and such that the weight of each q_i is $u-1$.

Claim The set $P' = P \cup \{q_1, q_2, \dots, q_t\} - \{p_1, p_2, \dots, p_t\}$ is the required set.

To show the claim, we first note that there are always t vectors q_i as above. This follows directly from the relationship which exists between points in the s -cube.

Next we show that for any p_j , $j = 1, 2, \dots, t$ and for any p^b , $p \in P - \{p_1, p_2, \dots, p_t\}$, $w(p_j p^b) \geq 2$, where $w(p)$ denotes the weight of a boolean vector p . This follows because $w(p_j p^b) = w(p_j) + w(p^b) - w(p_j + p^b) \geq u + (s-u+1) - (s-1) = 2$. We then have that $w(g_j p^b) = w(p_j(\sim a_j) p^b) = w(p_j p^b) + w(\sim a_j) - w(p_j p^b + \sim a_j) \geq 2 + (s-1) - s = 1$ and so $g_j p^b \neq \underline{0}$. (Here a_j is an atom such that $a_j < p_j$ and $q_j = p_j(\sim a_j)$.) Similarly, $w(\sim q_j p^b) = w((\sim p_j + a_j) p^b) = w(\sim p_j p^b + a_j p^b) \geq w(\sim p_j p^b) \geq 1$.

This means that any vector q and any vector in $P - \{p_1, \dots, p_t\}$ satisfies P2-1. Clearly, any two vectors in $P - \{p_1, \dots, p_t\}$ satisfy P2-1, so it remains to be shown that any two vectors in $\{q_1, q_2, \dots, q_t\}$ satisfy P2-1.

We have $w(\sim q_i \sim q_j) = w((\sim p_i + a_i)(\sim p_j + a_j)) \geq w(\sim p_i \sim p_j) \geq 1$.

Also $w(\sim q_i q_j) \geq 1$ since $q_i \neq q_j$ and $w(q_i) = w(q_j) > s/2$. Finally,

$$w(q_i q_j) = w(p_i(\sim a_i) p_j(\sim a_j)) = w(p_i p_j) + w(\sim a_i \sim a_j) - w(p_i p_j + \sim a_i \sim a_j).$$

Since $w(p_i) = w(p_j) = u > s/2$,

$$w(p_i p_j) = w(p_i) + w(p_j) - w(p_i + p_j) \geq (s/2 + 1) + (s/2 + 1) - (s - 1) = 3$$

So $w(q_i q_j) \geq 3 + (s - 2) - s = 1$. This completes the proof of the lemma. ■

Lemma 2 makes the conditions in P2-1 to reduce to

$$(\forall p_1, p_2) \in P, p_1 p_2 \neq \emptyset \text{ and } \sim p_1 \sim p_2 \neq \emptyset$$

(The other two conditions which imply $p_1 < p_2$ or $p_2 < p_1$ are satisfied trivially if $w(p_1) = w(p_2)$). But these conditions are equivalent to saying that p_1 or p_2 are each the complement of the other. Since the maximum number of points with weight $s/2$, satisfying this condition is

$$1/2 \binom{s}{s/2} \quad \text{we have shown:}$$

Theorem 1: The solution to problem P2, for $c=2$, is given by $\underline{s}$ satisfying

$$\underline{s} = \min_s [1/2 \binom{s}{s/2} \geq 2^n].$$
■

$$\text{Since } 1/2 \binom{s}{s/2} \simeq \frac{2^s}{(2\pi s)^{0.5}}, \quad \underline{s} = O(n)$$

Thus we get $K(n, 2) = O(n)$ as was to be shown.

3. A Polynomial Bound on $K(n, c)$

In this section we will show that for each c , $K(n, c)$ grows not more than with a polynomial of n , namely $K(n, c) \leq 2^c n^{c-1}$. This is a substantial improvement over the previously mentioned bound. To obtain this bound we will construct a set G of functions satisfying P1-1. The construction is a modification of one suggested to the author by R. Rivest who pointed out the existence of polynomial bounds for this problem.

Let U and V be sets of functions of $n-1$ variables. Let $U \times V$ be the set of functions of n variables defined as $U \times V = \{f | \exists u \in U, \exists v \in V, \forall (b_2, \dots, b_n) \in \{0,1\}^n, f(0, b_2, \dots, b_n) = u(b_2, \dots, b_n), f(1, b_2, \dots, b_n) = v(b_2, \dots, b_n)\}$. Note that $|U \times V| = |U||V|$. Let $U = \{u_1, u_2, \dots, u_p\}$ and $V = \{v_1, v_2, \dots, v_p\}$ be sets of functions of $n-1$ variables with $p = |U| = |V|$. Let $U + V$ be the set of p functions of n variables defined as $U + V = \{f_i | \forall (b_2, b_3, \dots, b_n) \in \{0,1\}^n, f_i(0, b_2, \dots, b_n) = u_i(b_2, \dots, b_n), f_i(1, b_2, \dots, b_n) = v_i(b_2, \dots, b_n)\}$.

Let $\mathcal{G}(n, c)$ be a set of functions satisfying P1-1 for some n and c . $\mathcal{G}(n, c)$ can be constructed as follows:

1) Find all $\mathcal{G}(n-1, d)$, for $d = 1, \dots, c$.

$$2) \mathcal{G}(n, c) = \{\mathcal{G}(n-1, c) + \mathcal{G}(n-1, c)\} \cup \bigcup_{k=1}^{c-1} \mathcal{G}(n-1, k) \times \mathcal{G}(n-1, c-k).$$

The following is an immediate consequence of this definition.

Lemma 3: The set $\mathcal{G}(n, c)$ constructed as above satisfies P1-1.

From the above construction we get the following recurrence for $K(n, c)$:

$$K(n, c) \leq K(n-1, c) + \sum_{1 \leq k \leq c-1} K(n-1, k) \cdot K(n-1, c-k)$$

Using this recurrence we now show

$$\text{Theorem 2: } K(n, c) \leq 2^c n^{c-1}.$$

Proof: For $c=1$ we know $K(n, 1) = 2$ so the theorem holds. Assume the result holds for all values of the second parameter less than c . Then, using the above recurrence,

$$K(n, c) \leq K(n-1, c) + \sum_{1 \leq k \leq c-1} 2^{k(n-1)^{k-1}} \cdot 2^{c-k(n-1)^{c-k-1}}$$

Since the term inside the summation does not depend on k we get a new recurrence:

$$K(n, c) \leq K(n-1, c) + 2^c (c-1) (n-1)^{c-2}, \text{ so}$$

$$K(n, c) \leq 2^c (c-1) \sum_{j=1}^{n-1} j^{c-2} \leq 2^c (c-1) (n-1)^{c-1} / (c-1) \leq 2^c n^{c-1}$$

which proves the theorem. I

Since the number of control lines to select any of the $K(n,c)$ functions is $\log K(n,c)$ we get as a corollary:

Corollary 1: The number of exterior connections (besides those used for input) to a c -universal circuit is no more than $(c-1) \log n + c$.

Conclusions

In this note we have reexamined the problem of the number of exterior connections needed to control a circuit which is to be c -universal. For $c = 2$ we have found an exact solution and shown an upper bound for this number in the general case. The small bound found (of the order of $c \log n$ for the number of exterior connections) makes the implementation of these circuits very practicable.

References

- [1] T. Lang and M. Schkolnick, "The Minimization of Control Variables in Adaptive Systems," in Theory of Machines and Computations, Z. Kohavi and A. Paz (eds.), Academic Press, 1971.