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F-CONVEX FUNCNIONS

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13. ABSTRACT

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F -CONVEX FUNCTIONS

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F-CONVEX FUNCTIONS

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ABSTRACT

Let F be a family of functions: $R^n \to R$. A function: $R^n \to R$ is called F-convex if it is supported, at each point, by some member of F. For particular choices of F one obtains the convex functions: $R^n \to R$ and the generalized convex functions in the sense of Beckenbach. F-convex functions are characterized and studied, retaining some essential results of classical convexity.

$\mathcal{F}$-CONVEX FUNCTIONS

Aharon Ben-Tal and Adi Ben-Israel

§1. INTRODUCTION

Let $\mathcal{F}$ be a family of functions: $R^n \to R$, depending on $(n+1)$ parameters $\{x^*, \xi^*\} \in R^n \times R$. A function $f: R^n \to R$ is called $\mathcal{F}$-convex if its graph is supported at each point by some member of $\mathcal{F}$, see Definition 2.1. For particular choices of $\mathcal{F}$, the $\mathcal{F}$-convex functions reduce to the ordinary proper convex functions (Example 2.2) and the sub $\mathcal{F}$-functions of Beckenbach (Example 2.3 and Proposition 2.4).

In this paper we study the basic properties of $\mathcal{F}$-convex functions.

Sections 2 and 3 contain definitions and examples.

Section 4 gives first order conditions (so called because they involve only first derivatives and the "gradients" $\{x_f^*, \xi_f^*\}$ defined in 3.2) for $\mathcal{F}$-convexity. For families $\mathcal{F} \in \underset{\sim}{A}$, see Definition 3.2, $\mathcal{F}$-convexity is characterized in Theorem 4.2 by an analog of the gradient inequality. The remaining results in Section 4 are conditions for $\mathcal{F}$-convexity or strict $\mathcal{F}$-convexity, in terms of the mapping: $x \to \{x_f^*(x), \xi_f^*(x)\}$ or the $\mathcal{F}$-gradient mapping: $x \to x_f^*(x)$, see Definition 3.3.

Second order conditions for F-convexity and strict F-convexity are given in Section 5. These conditions involve the matrix

$$H(x) = f_{xx}(x) - F_{xx}(x_f^*(x),\ \xi_f^*(x);\ x) \tag{5.1}$$

which in the classical case reduces to the Hessian matrix (Example 5.2). The main results here are Theorems 5.1 and 5.5. An analog of the differential inequality of Peixoto [14], characterizing sub-F functions, is obtained as a special case (Example 5.3).

Section 6 deals with the monotonicity properties of the F-gradient x_f^* of an F-convex function, where F belongs to certain classes defined in 6.1. The derivative of x_f^* is computed in Lemma 6.2, and the result is used, for the separable families (6.10), to establish that x_f^* is a P_0-function [P-function] if f is F-convex [strictly F-convex], see Theorem 6.4.

In a sequel paper we study the corresponding generalizations of conjugacy and duality in the sense of Fenchel [16]. These results involve a conjugate family F^*, and are hidden in the classical case by the fact that there $F = F^*$.

§2. F-CONVEX FUNCTIONS: DEFINITIONS AND EXAMPLES

2.1 Definitions

Let F be a family of functions: $R^n \to R$ with common domain X

$$X \triangleq \cap\{\text{dom } F: F \in F\} \tag{2.1}$$

and range

$$\Xi \triangleq \cup\{\text{range } F: F \in F\} \quad . \tag{2.2}$$

Let f be a function: $R^n \to R$ with domain

$$\text{dom } f \subset X \tag{2.3}$$

and let S be an open subset of dom f. Then f is called F-convex in S if for every $x \in S$, there exists an $F \in F$ such that

$$f(x) = F(x) \quad \text{and} \quad f(z) \geq F(z) \quad \text{for all} \quad x \neq z \in S , \tag{2.4}$$ [1]

in which case F is called a support of f: S at x. The function f is called strictly F-convex in S if strict inequality holds in (2.4) for all $x \neq z \in S$.

If there is no need to specify S, for example if $S = \text{dom } f$, the above names are abbreviated by omitting S, e.g., F-convex, support of f at x, etc.

---

[1] The name F-convex function was used recently ([15], p.241) to denote the sub-F functions, see Example 2.2

2.2 Example

Let F be the family of affine functions: $R^n \to R$, i.e.,

(2.5) $$F = \{F(.) = \langle x^*, \cdot \rangle - \xi^* : x^* \in R^n, \xi^* \in R\} .$$

Then a function $f: R^n \to R$ is F-convex if and only if it is a proper convex function, i.e., a convex function whose epigraph is a non empty set containing no vertical lines, ([16], §4).

2.3 Example

Let F be a family of continuous functions: $R \to R$ with domain $X = (a,b)$ and such that

(B) For any two distinct points in X, say,

$$a < x_1 < x_2 < b$$

and any two real numbers $\{y_1, y_2\}$, there is a unique $F \in F$ satisfying

$$F(x_i) = y_i , \qquad (i = 1,2) .$$

We call such an F a Beckenbach family in (a,b). E.F. Beckenbach [1] called a function $f: (a,b) \to R$ a sub-F function if for any two points

$$a < x_1 < x_2 < b$$

the member of F, F_{12}, defined by

$$F_{12}(x_i) = f(x_i), \qquad (i=1,2), \tag{2.6}$$

satisfies

$$f(x) \leq F_{12}(x), \qquad x_1 < x < x_2 .$$

M.M. Peixoto ([13],[14] Theorem 1) showed that if f is a sub-F function and $a < x_0 < b$ then there exist two functions

$$F_i \in F, \qquad (i=1,2),$$

such that

$$F_i(x_0) = f(x_0), \qquad (i=1,2),$$

$$F_2(x) \leq F_1(x) \leq f(x), \qquad (a < x < x_0),$$

and

$$F_1(x) \leq F_2(x) \leq f(x), \qquad x_0 < x < b .$$

(Furthermore, if the derivatives $f'(x_0)$, $F_1'(x_0)$ and $F_2'(x_0)$ exist, they are equal). Thus both F_1 and F_2 support f at x_0.

Therefore every sub-F function is F-convex. We will now prove the converse for Beckenbach families F.

2.4 Proposition

Let F be a Beckenbach family in (a,b). Then a function $f: (a,b) \to R$ is F-convex in (a,b) if and only if f is a sub-F function.

Proof.

The proof of the "if" part was cited above.

To prove "only if" suppose f is not a sub-F function, i.e., there are three points

$$a < x_1 < x_0 < x_2 < b$$

such that the function $F_{12} \in F$, defined by (2.6), satisfies

$$F_{12}(x_0) < f(x_0) . \tag{2.7}$$

Suppose that $F_0 \in F$ is a support of f at x_0, i.e.,

$$f(x_0) = F_0(x_0) \quad \text{and} \quad f(x) \geq F_0(x), \qquad a < x < b . \tag{2.8}$$

From (2.6), (2.7) and (2.8) it follows that F_{12} and F_0 intersect twice over the interval (a,b), contradicting (B). Therefore f is not F-convex. □

2.5 Example

Let $G(x,y,z)$ be a continuous function: $(a,b)\times R\times R \to R$, such that

(P1) For each $\{x_0, y_0, y_0'\} \in (a,b) \times R \times R$, the differential equation

(2.9) $$y'' = G(x,y,y'), \qquad (a < x < b) ,$$

has a unique solution $y = y(x)$ satisfying

(2.10) $$y(x_0) = y_0 , \qquad y'(x_0) = y_0' .$$

(P2) The solution of (2.9) is continuous with respect to the initial values y_0, y_0'.

(P3) For any two points $\{x_i, y_i\} \in (a,b) \times R$ $(i = 1,2)$ with $x_1 \neq x_2$, there is a unique solution of (2.9) satisfying

(2.11) $$y(x_i) = y_i , \qquad i = 1,2 .$$

Let F be the Beckenbach family of solutions of (2.9). M.M. Peixoto ([14] Theorem 2) showed that a function $f \in C^2(a,b)$ is a sub-F function if and only if

(2.12) $$f'' \geq G(x,f,f') , \qquad a < x < b .$$

2.6 Example

While sub-F functions are continuous ([1],[15] p. 242), an F-convex function need not be continuous in its domain, even if each $F \in F$ is continuous:

Let F be the family of functions: $R \to R$

$$F(x) = F(x^*,\xi^*;. x) = \xi^* \sin(e^{x^*|x|} - 1)$$

depending on the two parameters

$$x^* \geq 0\,, \qquad 0 \leq \xi^* < 1\,.$$

Then the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is $\mathcal{F}$-convex. Indeed, for every $x \neq 0$, the function $F = F(x^*, \xi^*; \cdot)$ defined by

$$x^* = \frac{\log(1+\frac{\pi}{2})}{|x|}\,, \qquad \xi^* = 1$$

supports f at x. Also every $F \in \mathcal{F}$ supports f at 0.

We show now that an $\mathcal{F}$-convex function inherits from $\mathcal{F}$ its lower semi continuity.

2.7 Proposition

Let $\mathcal{F}$ be a family of l.s.c. (= lower semi continuous) functions and let f be $\mathcal{F}$-convex in $\operatorname{dom} f$. Then f is l.s.c. in $\operatorname{dom} f$.

Proof.

Suppose f is not l.s.c.. Then there exists an $x \in \operatorname{dom} f$ such that

$$f(x) > \liminf_{y \to x} f(y)\,. \tag{2.13}$$

Let $F \in \mathcal{F}$ support f at x. Then

$$F(x) = f(x) > \liminf_{y \to x} f(y) \geq \liminf_{y \to x} F(y),$$

by (2.4) and (2.13),

contradicting the lower semi continuity of F. □

2.8 Notes

For further generalizations of convexity see the surveys in ([2], Chapter 4), [4], and ([15], Chapter VIII).

For functions of several variables, the analogs of the sub $\mathcal{F}$-functions are the subfunctions in particular the subharmonic functions; see [2] p. 146, [3] and [8], where applications to second order differential inequalities are surveyed.

§3. REQUIREMENTS ON $\mathcal{F}$

3.1 General

With Examples 2.2, 2.3 and 2.5 as our motivation, we consider from now on only families $\mathcal{F}$ of functions $F: R^n \to R$ depending continuously on $n+1$ parameters

$$\{x^*, \xi^*\} \in X^* \times \Xi^*$$

where the sets of parameters X^* and Ξ^* are given subsets of R^n and R respectively. The general member of $\mathcal{F}$ is thus denoted by

$$F(.) = F(x^*, \xi^*; \cdot), \quad (x^* \in X^*, \ \xi^* \in \Xi^*) , \tag{3.1}$$

with function values

$$F(x) = F(x^*, \xi^*; x), \qquad x \in X . \tag{3.2}$$

We assume that the mapping: $\{x^*, \xi^*\} \to F(x^*, \xi^*; \cdot)$ is one to one on $X^* \times \Xi^*$, i.e., $F(x^*, \xi^*; \cdot)$ is uniquely determined by $\{x^*, \xi^*\}$.

3.2 The class $\underset{\sim}{A}$

Let $D^k(X)$ denote the functions: $R^n \to R$ which are k times differentiable in X . If $\mathcal{F} \subset D(X)$ we define the set

$$Z \triangleq \cup \{\text{range} \begin{bmatrix} F \\ F_x \end{bmatrix} : F \in \mathcal{F}\} \subset R \times R^n \tag{3.3}$$

where F_x is the gradient of F with respect to x .

A family F of differentiable functions is said to be in class $\underset{\sim}{A}$, denoted by $F \in \underset{\sim}{A}$, if for every $x \in X$ and $\begin{bmatrix} \xi \\ y \end{bmatrix} \in Z$, the system

$$(3.4) \qquad \xi = F(x^*, \xi^*; x)$$

$$(3.5) \qquad y = F_x(x^*, \xi^*; x)$$

has a unique solution $\{x^*, \xi^*\} \in X^* \times \Xi^*$.

If $F \subset D(X)$ and if f and S are a function: $R^n \nrightarrow R$ and an open subset of $\operatorname{dom} f$ respectively, we denote by

$$(3.6) \qquad f \overset{S}{\approx} F$$

the facts

$$(D1) \qquad S \subset \operatorname{dom} f \subset X$$

$$(D2) \qquad f \in D(S)$$

$$(D3) \qquad \operatorname{range} \left\{ \begin{bmatrix} f(x) \\ f_x(x) \end{bmatrix} : x \in S \right\} \subset Z .$$

We abbreviate $f \overset{\operatorname{dom} f}{\approx} F$ by $f \approx F$.

If $F \in \underset{\sim}{A}$, $f \approx F$ and $x \in \operatorname{dom} f$ we denote by

$$(3.7) \qquad (x_f^*(x),\ \xi_f^*(x))$$

the unique solution of

$$(3.8) \qquad f(x) = F(x^*, \xi^*; x)$$

$$(3.9) \qquad f_x(x) = F_x(x^*, \xi^*; x) .$$

A family F of differentiable functions is said to be in <u>class</u> $\underset{\sim}{A}$, denoted by $F \in \underset{\sim}{A}$, if for every $x \in X$ and $\begin{bmatrix}\xi\\ y\end{bmatrix} \in Z$, the system

(3.4) $$\xi = F(x^*,\xi^*;\ x)$$

(3.5) $$y = F_x(x^*,\xi^*;\ x)$$

has a unique solution $\{x^*,\xi^*\} \in X^* \times \Xi^*$.

If $F \subset D(X)$ and if f and S are a function $R^n \to R$ and an open subset of $\mathrm{dom}\, f$ respectively, we denote by

(3.6) $$f \overset{S}{\approx} F$$

the facts

(D1) $$S \subset \mathrm{dom}\, f \subset X$$

(D2) $$f \in D(S)$$

(D3) $$\mathrm{range}\ \{\begin{bmatrix}f(x)\\ f_x(x)\end{bmatrix} : x \in S\} \subset Z .$$

We abbreviate $f \overset{\mathrm{dom}\, f}{\approx} F$ by $f \approx F$.

If $F \in \underset{\sim}{A}$, $f \approx F$ and $x \in \mathrm{dom}\, f$ we denote by

(3.7) $$(x_f^*(x),\ \xi_f^*(x))$$

the unique solution of

(3.8) $$f(x) = F(x^*,\xi^*;\ x)$$

(3.9) $$f_x(x) = F_x(x^*,\xi^*;\ x) .$$

3.3 The class $\underset{\sim}{C}$

A family F is said to be in class $\underset{\sim}{C}$, denoted by $F \in \underset{\sim}{C}$, if for every $\{x^*, x\} \in X^* \times X$ the function $F(x^*, \cdot; x)$ is a strictly decreasing function of $\xi^*, \xi^* \in \Xi^*$. In this case, we denote by $F^I(x, \cdot; x^*)$ the inverse function of $F(x^*, \cdot; x)$. It satisfies the identity

$$\xi = F(x^*, F^I(x, \xi; x^*); x), \; \xi \in \Xi \quad . \tag{3.10}$$

If $F \in \underset{\sim}{A} \cap \underset{\sim}{C}$, $f \approx F$ and $x \in \operatorname{dom} f$, then (3.8) gives

$$\xi^* = F^I(x, f(x); x^*) \tag{3.11}$$

which, substituted in (3.9), gives

$$f_x(x) = F_x(x^*, F^I(x, f(x); x^*); x) \quad . \tag{3.12}$$

The unique solution of (3.12) is then called the F-gradient of f at x, and is denoted by $x_f^*(x)$.

3.4 Example

Let F be the family (2.5) of affine functions: $R^n \to R$. Then

(a) $F \subset D(R^n)$, $F_x(x^*, \xi^*; x) = x^*$ for every $F \in F$ and $x \in R^n$, and (3.3) gives $Z = R \times R^n$.

(b) $F \in \underset{\sim}{A}$. For every $x \in R^n$ and $\begin{bmatrix}\xi\\ y\end{bmatrix} \in R \times R^n$, the unique solution of (3.4)-(3.5) is

$$x^* = y , \quad \xi^* = \langle y,x\rangle - \xi .$$

(c) $F \in \underset{\sim}{C}$.

(d) $f \approx F$ means that $f \in D(\text{dom } f)$.

(e) If $f \approx F$ then for every $x \in \text{dom } f$

(3.13) $$x_f^*(x) = f_x(x), \quad \xi_f^*(x) = \langle f_x(x), x\rangle - f(x) .$$

Thus the F-gradient of f, x_f^*, coincides here with its ordinary gradient f_x.

3.5 Example

Let ϕ be a given function: $X^* \times X \to R$ and let the family F consist of the functions $F(x^*,\xi^*;\cdot)$, $\{x^*,\xi^*\} \in X^* \times \Xi^*$, with values

(3.14) $$F(x^*,\xi^*;\, x) = \phi(x^*,x) - \xi^*, \quad x \in X.$$

Then:

(a) $F \in \underset{\sim}{A}$ if and only if the following two conditions hold:

(a1) $\phi(x^*,\cdot) \in D(X)$ for every $x^* \in X^*$.

(a2) For every $x \in X$, $y \in \bigcup_{x^*} \text{range } \phi_x(x^*,\cdot)$, the system

$$y = \phi_x(x^*,x)$$

has a unique solution x^*.

(b) $F \in \underset{\sim}{C}$.

(c) Let $F \in \underset{\sim}{A}$, $f \approx F$ and $x \in \operatorname{dom} f$. Then the F-gradient of f at x , $x_f^*(x)$, is the unique solution x^* of

$$f_x(x) = \phi_x(x^*,x) \; . \tag{3.15}$$

Also

$$\xi_f^*(x) = \phi(x_f^*(x),x) - f(x) \; . \tag{3.16}$$

A concrete example is the following family F defined by

$$F(x^*,\xi^*;x) \triangleq \sum_{i=1}^{n} x_i^* \, \phi^i(x_i) + \frac{1}{\sum_{i=1}^{n} x_i^* \phi^i(x_i)} - \xi^*$$

$$X^* = R_+^n \, , \qquad \Xi^* = R, \qquad X = \cap \operatorname{dom} \phi^i$$

where for every $i = 1,2,\dots,n$, $\phi^i \colon R \to R_+$ is differentiable and $\phi_{x_i}^i > 0$. The condition $f \approx F$ for this case is

$$f_{x_i} > 0 \qquad i = 1,2,\dots,n$$

or

$$f_{x_i} < 0 \qquad i = 1,2,\dots,n \, .$$

The F-gradient is

$$x_f^*(x) = \left(\frac{t^2(x)}{t^2(x) - 1}\right) \begin{bmatrix} f_{x_1}(x)/\phi_{x_1}^1(x_1) \\ \vdots \\ f_{x_n}(x)/\phi_{x_n}^n(x_n) \end{bmatrix} ,$$

where

$$t(x) \triangleq \frac{1}{2} \left\{ \sum_{i=1}^{n} \frac{f_{x_i}}{\phi_{x_i}^i} \phi^i + \left[\left(\sum_{i=1}^{n} \frac{f_{x_i}}{\phi_{x_i}^i} \phi^i \right) + 4 \right]^{1/2} \right\} .$$

§4. FIRST ORDER CONDITIONS FOR $\mathcal{F}$-CONVEXITY

In this section we give first order conditions (so-called because they involve only first derivatives and the "gradients" $\{x_f^*, \xi_f^*\}$ of f, see (3.7)) for $\mathcal{F}$-convexity, for families $\mathcal{F}$ in class $\underset{\sim}{A}$. These conditions use the extremal property of the supports implied by the inequality (2.4). First we require

4.1 Lemma

Let $\mathcal{F} \in \underset{\sim}{A}$, $f: R^n \to R$, and let $f \overset{S}{\approx} \mathcal{F}$. If $f: S$ is supported (by some $F \in \mathcal{F}$) at a point $x \in S$, then

$$(4.1) \qquad F(x_f^*(x), \xi_f^*(x); \cdot)$$

is the unique support of f at x.

Proof.

Let $F(x_0^*, \xi_0^*; \cdot) \in \mathcal{F}$ support $f: S$ at x, i.e.,

$$(4.2) \qquad h(z) \triangleq f(z) - F(x_0^*, \xi_0^*; z) \geq 0, \quad \forall z \in S,$$

and

$$(4.3) \qquad h(x) = f(x) - F(x_0^*, \xi_0^*; x) = 0.$$

Therefore $h(z)$ is minimized, in S, by $z = x$. Since S is open, this implies that x is a critical point of h, i.e.,

$$(4.4) \qquad h_z(x) = f_x(x) - F_x(x_0^*, \xi_0^*; x) = 0.$$

Since $F \in \underset{\sim}{A}$, a comparison of (4.3)-(4.4) and (3.8)-(3.9) shows that

$$\{x_0^*, \xi_0^*\} = \{x_f^*(x),\ \xi_f^*(x)\}$$

proving that (4.1) is the unique support at x. □

4.2 <u>Theorem</u>

Let $F \in \underset{\sim}{A}$, $f: R^n \to R$, and $f \overset{S}{\approx} F$. Then f is F-convex in S if and only if for every $x \in S$

$$f(z) \geq F(x_f^*(x), \xi_f^*(x); z), \quad \forall\ x \neq z \in S. \tag{4.5}$$

Furthermore, f is strictly F-convex in S if and only if for every $x \in S$

$$f(z) > F(x_f^*(x), \xi_f^*(x); z), \quad \forall\ x \neq z \in S. \tag{4.6}$$

<u>Proof</u>.

<u>If</u>. From (4.5) and (3.8) it follows, for any $x \in S$, that the function (4.1) supports $f: S$ at x. It is the unique support if (4.6) holds.

<u>Only if</u>. Let f be F-convex in S. Then, by Lemma 4.1, for any $x \in S$, the function (4.1) is the unique support of $f: S$ at x. The inequality (4.5) then follows from (2.4). Similarly (4.6) follows from the strict F-convexity of f. □

4.3 Example

Let F be the family (2.5) of affine functions: $R^n \to R$,

$$F = \{F(x^*,\xi^*;\cdot) = \langle x^*,\cdot\rangle - \xi^*: x^* \in R^n,\ \xi^* \in R\} .$$

Then, using (3.13), the inequality (4.5) reduces to

$$f(z) \geq \langle f_x(x),\ z - x\rangle + f(x) , \quad \forall\ x \neq z \in S ,$$

the classical gradient inequality.

4.4 Corollary

(a) Let $F \in \underset{\sim}{A}$, and let $f: R^n \to R$, $f \overset{S}{\gtrless} F$, be F-convex in S . Then f is strictly F-convex in S if and only if the mapping

$$(4.7) \qquad x \to \{x_f^*(x), \xi_f^*(x)\}$$

is one to one on S .

(b) Let, in addition, $F \in \underset{\sim}{C}$. Then f is strictly F-convex in S if and only if the mapping

$$(4.8) \qquad x_f^*: x \to x_f^*(x)$$

is one to one on S .

Proof.

From Lemma 4.1 it follows, for every $x \in S$, that the function (4.1) is the unique support of $f: S$ at x . By definition, f is strictly F-convex in S if, and only if, every support of $f: S$ supports f at exactly one point of S. This is equivalent to the mapping (4.7) being one to one on S.

To prove the last part, note that the additional hypothesis $F \in \underset{\sim}{C}$ implies

$$(4.9) \qquad [x \overset{1:1}{\to} x_f^*(x) \text{ on } S] \Longleftrightarrow [x \overset{1:1}{\to} \{x_f^*(x), \xi_f^*(x)\} \text{ on } S].$$

Indeed, the implication $\Longrightarrow$ is always true. Conversely, suppose that x_f^* is not one to one on S, i.e., there exist $x_1, x_2 \in S$, $x_1 \neq x_2$, such that

$$(4.10) \qquad x_f^*(x_1) = x_f^*(x_2) \overset{\Delta}{=} x_0^* .$$

Let $\xi_i^* \overset{\Delta}{=} \xi_f^*(x_i) = F^I(x_i, f(x_i); x_0^*)$ and let

$$F^i(\cdot) \overset{\Delta}{=} F(x_0^*, \xi_i^*; \cdot), \quad (i = 1,2) .$$

Then

$$F^i(x_i) = f(x_i), \quad F_x^i(x_i) = f_x(x_i), \qquad i = 1,2 .$$

Hence by Theorem 4.2, F^i supports f at x_i. If $\xi_1^* = \xi_2^*$, then this and (4.10) contradicts the fact that (x_f^*, ξ_f^*) is 1:1, established earlier. Thus suppose that $\xi_1^* > \xi_2^*$. This implies, since $F \in \underset{\sim}{C}$, that $F^1(z) < F^2(z) \quad \forall z \in S$. In particular

$$F^2(x_1) > F^1(x_1) = f(x_1)$$

contradicting the fact that F^2 is a support. $\square$

4.5 Theorem

Let $F \in \underset{\sim}{A} \cap \underset{\sim}{C}$, $f: R^n \to R$, and $f \overset{S}{\approx} F$. Then f is strictly F-convex in S if the following two conditions hold.

(a) The mapping x_f^* is one to one on S .

(b) For every $x \in S$ and for every sequence $\{z_k\} \subset S$ which either converges to a point $y \in \text{bdry}\, S$ or $\|z_k\| \to \infty$ there exists an $\hat{x} \in S$ such that

$$\limsup_{k\to\infty} \{F^I(z_k, f(z_k), x_f^*(x)) - F^I(\hat{x}, f(\hat{x}), x_f^*(x))\} \le 0 \tag{4.11}$$

where F^I is defined in §3.3.

Proof.

For any $x \in S$ consider the function

$$T(z) \triangleq F^I(z, f(z); x_f^*(x)) . \tag{4.12}$$

We show first that $z = x$ is a critical point of T .
Differentiating the identity

$$F(x^*, F^I(y, f(y); x^*); y) - f(y) = 0 \tag{4.13}$$

with respect to y we get

$$F_x(\cdot,\cdot;\cdot) + F_{\xi^*}(\cdot,\cdot;\cdot)\,[F_x^I(y,f(y);x^*) + F_\xi^I(y,f(y);x^*)f_x(y)] - f_x(y) = 0 \tag{4.14}$$

where

$$(\cdot,\cdot;\cdot) = (x^*, F^I(y, f(y);x^*); y) .$$

Now $F_{\xi^*} \neq 0$, since $F \in \underset{\sim}{C}$. Therefore, for $y = x$ and $x^* = x_f^*(x)$, it follows from (4.14) and (3.12) that

$$(4.15) \qquad F_x^I(x,f(x);\ x_f^*(x)) + F_\xi^I(x,f(x);\ x_f^*(x))\ f_x(x) = 0$$

which, by (4.12), is the same as $T_z(x) = 0$, proving that $z = x$ is critical.

Moreover, $z = x$ is the unique critical point of T in S. For suppose that $x \neq x' \in S$ is another critical point of T, i.e.

$$T_z(x') = F_x^I(x',f(x');\ x_f^*(x)) + F_\xi^I(x',f(x');x_f^*(x))f_x(x') = 0$$

implying that for $y = x'$ and $x^* = x_f^*(x)$, (4.14) reduces to

$$F_x(x_f^*(x),F^I(x',f(x');\ x_f^*(x));\ x') - f_x(x') = 0$$

which, together with (3.12), implies that

$$x_f^*(x') = x_f^*(x) ,$$

contradicting (a).

We show next that

$$(4.16) \qquad \sup\{T(z):\ z \in S\} = T(x) .$$

Indeed if this supremum occurs at some $z = y \in \text{bdry}\, S$ or if a supremizing sequence $\{z_k\}$ is such that $|z_k| \to \infty$ then the supremum is also attained at $\hat{x} \in S$, by (4.11). Therefore $z = \hat{x}$ is a critical point of T, proving that $\hat{x} = x$, since the latter is the unique critical point in S, and therefore (4.16) becomes

$$F^I(x,f(x);\ x_f^*(x)) > F^I(z,f(z);\ x_f^*(x)) \ , \quad \forall\ x \neq z \in S \ ,$$

which is the same as

$$f(z) > F(x_f^*(x),\ F^I(x,f(x);\ x_f^*(x));\ z), \quad \forall\ x \neq z \in S \ ,$$

proving that f is strictly F-convex in S, by theorem 4.2. □

4.6 Example

Consider the family

$$F = \{\phi(x^*,\cdot) - \xi^* : x^* \in X^*,\ \xi^* \in \Xi^*\} \ ,$$

of Example 3.5 and let $F \in A$, $f: R^n \to R$, and $f \underset{\approx}{S} F$. Then condition (b) of Theorem 4.5 follows from

(b1) For every $x^* \in \text{range}\{x_f^*(x): x \in S\}$ and every sequence $\{z_k\}$ as in Theorem 4.5(b),

$$\liminf_{k\to\infty}\{f(z_k) - \phi(x^*,z_k)\} = +\infty \ . \tag{4.17}$$ □

In particular, if

$$S = \text{dom}\, f = X = R^n$$

and

$$\limsup_{\|x\|\to\infty} \frac{\phi(x^*,x)}{\|x\|} < \infty \ , \quad \forall\ x^* \in \text{range}\ x_f^* \ . \tag{4.18}$$

then condition (b) of Theorem 4.5 is satisfied if

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty .$$

Note that (4.18) is trivially satisfied by the family F of affine functions. Hence, a differentiable function $f\colon R^n \to R$ is strictly convex if the following two conditions hold.

(a) The mapping

$$x \to f_x(x)$$

is one to one on R^n.

(b)
$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty . \qquad \square$$

As a concrete example of condition (b1) let F be the family of functions: $R^2 \to R$ given by

(4.19) $$F(x^*,\xi^*;\, x) = x_1^* e^{-x_1} + x_2^* x_2 e^{-x_1} - \xi^*$$

with $X = X^* = R^2$, $\Xi^* = R$.

Consider the function $f\colon R^2 \to R$

(4.20) $$f(x) = \frac{1}{2} e^{2x_1} + \frac{1}{2} x_2^2 e^{-x_1}$$

with $\operatorname{dom} f = R^2$. Then f is F-convex in R^2 since:

(a) The F-gradient

$$x_f^*(x) = \begin{bmatrix} -e^{3x_1} - \frac{1}{2} x_2^2 \\ x_2 \end{bmatrix}$$

is one-to-one, and

(4.21) $$\text{range } x_f^* = \{(x_1^*, x_2^*) \in R^2 : x_1^* + \frac{1}{2} x_2^{*2} < 0\}.$$

(b) $$f(z) - \phi(x^*, z) = (\frac{1}{2} e^{2z_1} + \frac{1}{2} z_2^2 e^{-z_1}) - (x_1^* e^{-z_1} + x_2^* z_2 e^{-z_1}) =$$

$$= \frac{1}{2} e^{2z_1} - (x_1^* + \frac{1}{2} x_2^{*2}) e^{-z_1} + \frac{1}{2} (z_2 - x_2^*)^2 e^{-z_1}$$

by (4.21) the coefficients of all exponents are positive and hence

$$\lim_{|z| \to \infty} [f(z) - \phi(x^*, z)] = \infty \qquad \forall\, x^* \in \text{range } x_f^* .$$

§5. SECOND ORDER CONDITIONS FOR $\mathcal{F}$-CONVEXITY

In this section we collect second order conditions (involving second derivatives) for $\mathcal{F}$-convexity.

5.1 Theorem

Let $F \in \underset{\sim}{A} \cap D^2(X)$, $f: R^n \to R$, $f \overset{S}{\approx} \mathcal{F}$ and $f \in D^2(S)$. Then:

(a) f is $\mathcal{F}$-convex in S only if, for every $x \in S$, the matrix

$$H(x) \triangleq f_{xx}(x) - F_{xx}(x_f^*(x),\ \xi_f^*(x);\ x) \tag{5.1}$$

is positive semi definite.[1]

(b) Let S be convex and let f and each $F \in \mathcal{F}$ be twice continuously differentiable in S. Then f is $\mathcal{F}$-convex in S if

$$\langle y, \int_0^1 (f_{xx}(x+sy) - F_{xx}(x_f^*(x), \xi_f^*(x);\ x+sy))y\,ds\rangle \geq 0 \ , \tag{5.2}$$

for every $x \in S$ and $y \in S - x$.

If strict inequality holds in (5.2), F is strictly $\mathcal{F}$-convex in S.

Proof.

(a) Let f be $\mathcal{F}$-convex in S. Then, for any $x \in S$, the function

$$h(z) \triangleq f(z) - F(x_f^*(x),\ \xi_f^*(x);\ z) \tag{5.3}$$

---

[1] A matrix $H \in R^{n\times n}$ is called here positive semi definite if

$$\langle Hz, z\rangle \geq 0, \quad \forall\, z \in R^n .$$

We do not mean by this that H is symmetric.

satisfies

(5.4) $h(x) = 0, \quad h_z(x) = 0, \quad$ by (3.8)-(3.9),

and

$$h(z) \geq 0, \quad \forall\, z \in S, \quad \text{by Theorem 4.2.}$$

Therefore $z = x$ minimizes h in S. Since S is an open set, it follows that

$$h_{zz}(x) = H(x)$$

is positive semi-definite.

(b) The function h of (5.3) satisfies

$$\begin{aligned} h(z) &= h(z) - h(x) - \langle h_z(x),\, z - x\rangle, \quad \text{by (5.4)},\\ &= \langle (h_z(x + t(z-x)) - h_z(x)),\, z-x\rangle, \quad \text{for some } 0 < t < 1,\\ &\quad \text{by a mean value theorem ([12], Theorem 3.2.2)},\\ &= \langle z - x,\, (\int_0^1 h_{zz}(x + st(z-x))ds)\, t(z-x)\rangle,\\ &\quad \text{by a mean value theorem ([12], Theorem 3.2.7)},\\ &= \frac{1}{t}\langle y,\, \int_0^1 (f_{xx}(x+sy) - F_{xx}(x_f^*(x), \xi_f^*(x); x+sy))y\,ds\rangle,\\ &\quad \text{where } y = t(z-x). \end{aligned}$$

Thus, (5.2) implies that

(5.5) $h(z) \geq 0, \quad \forall\, z \in S,$

proving that f is F-convex in S, by Theorem 4.2.

Similarly, strict inequality in (5.2) implies strict inequality in (5.5), hence strict $\mathcal{F}$-convexity. □

5.2 Example

Let $\mathcal{F}$ be the family (2.5) of affine functions: $R^n \to R$. Then the matrix $H(x)$ of (5.1) reduces to the Hessian of f

$$H(x) = f_{xx}(x)$$

and Theorem 5.1 gives the classical conditions for convexity in terms of the Hessian.

5.3 Example

Let $\mathcal{F}$ be the Beckenbach family of solutions of the second order differential equation

$$(2.9) \qquad y'' = G(x,y,y'), \qquad (a < x < b) ,$$

discussed in Example 2.5. Then (5.1) becomes

$$H = f'' - G(x,f,f') .$$

Now, suppose that $\mathcal{F} \subset C^2(X)$, $f \in C^2(S)$, then $H(x) > 0$ implies $H(x+sy) > 0$ for $0 < s < 1$ and y sufficiently close to x. Thus (5.2) is a strict inequality in some neighborhood of x, and we conclude that f is, locally, strictly $\mathcal{F}$-convex. By proposition 2.4 this implies that f is locally strictly sub-$\mathcal{F}$, which by ([1]

Theorem 7) implies that f is sub-F globally in (a,b). This result is the analog of [14], Theorem 3. To get the analogous result of ([14] Theorem 1), we need the implication $H(x) \geq 0 \Longrightarrow H(x+sy) \geq 0$, for $0 < s < 1$ and y sufficiently close to x, for which Peixoto's additional requirement, (P2) of Example 2.5, is needed (see Peixoto's proof of [14] Lemma 1).

5.4 Definition

A mapping $T\colon R^n \to R^n$ is called one to one on R^n if

(a) $x,y \in R^n$, $x \neq y \Rightarrow T(x) \neq T(y)$.

(b) The inverse images $T^{-1}(B)$ of bounded sets $B \subset R^n$ are bounded.

5.5 Theorem

Let $F \in \underset{\sim}{A} \cap \underset{\sim}{C} \cap C^2(R^n)$, $f\colon R^n \to R$, $f \in C^2(R^n)$ and $f \approx F$. Then f is strictly F-convex in R^n if the following two conditions hold

(a) The mapping x_f^* is one to one on R^n.

(b) For every $x \in R^n$, the matrix

$$H(x) = f_{xx}(x) - F_{xx}(x_f^*(x),\ F^I(x,f(x);\ x_f^*(x));\ x) \tag{5.6}$$

is positive definite. Conversely, if f is strictly F-convex in R^n then (a) holds and the matrix $H(x)$ is positive semi-definite for every $x \in R^n$.

Proof.

First we note, by (3.11), that (5.6) and (5.1) are the same.

For any $x \in R^n$ consider now the function

$$(4.12) \qquad T(z) = F^I(z, f(z); x_f^*(x)) .$$

As in the proof of Theorem 4.5 it follows from (a) that $z = x$ is the unique critical point of T in R^n.

Differentiating the identity (4.13) twice with respect to y we get, by using (4.15) and (3.12),

$$(5.7) \qquad T_{zz}(x) = \frac{1}{F_{\xi*}} H(x)$$

(where $F_{\xi*}$ is evaluated at $\{x_f^*(x), F^I(x,f(x); x_f^*(x)); x\}$). From (5.7), (b) and $F \in \underset{\sim}{C}$ it follows that $T_{zz}(x)$ is negative definite. Therefore $z = x$ is an isolated local maximizer of T, and its unique critical point in R^n.

Thus, by Leighton's Theorem [9], see also [17], $z = x$ is the global maximizer of T, i.e.,

$$F^I(x,f(x); x_f^*(x)) > F^I(z,f(z); x_f^*(x)), \quad \forall\ x \neq z \in R^n ,$$

which is the same as

$$f(z) > F(x_f^*(x), F^I(x,f(x); x_f^*(x)); z), \quad \forall\ x \neq z \in R^n ,$$

proving that f is strictly F-convex in R^n by Theorem 4.2. □

If f is strictly F-convex in R^n then (a) and (b) follow from Corollary 4.4 and Theorem 5.1(a) respectively.

5.6 <u>Example</u>

Let F and f be given by (4.19) and (4.20) respectively. Then the matrix (5.1) is positive definite

$$H(x) = \begin{bmatrix} 3e^{2x_1} & 0 \\ 0 & e^{-x_1} \end{bmatrix}$$

and f is strictly F-convex in R^2, by Theorem 5.5.

§6. MONOTONICITY OF F-GRADIENTS

In this section we prove monotonicity results for the F-gradien x_f^* of an F-convex function. We recall that a mapping $g: R^n \to R^n$ is a P-function [P_0-function] if for every $x,y \in \operatorname{dom} g$, $x \neq y$, there is an index $k = k(x,y) \in \{1,2,\ldots,n\}$ such that $(x_k - y_k)(g_k(x) - g_k(y)) > 0$ $[(x_k - y_k)(g_k(x) - g_k(y)) \geq 0$ and $x_k \neq y_k]$, see [10]. In particular, a mapping $g: R^n \to R^n$ is monoton [strictly monotone] if for every $x,y \in \operatorname{dom} g$, $x \neq y$, we have $\langle x-y, g(x) - g(y)\rangle \geq 0$ $[\langle x-y, g(x) - g(y)\rangle > 0]$. We also require the following

6.1 Definitions

A family F is said to be in class $\underset{\sim}{A}_1$, denoted by $F \in \underset{\sim}{A}_1$, if $F \in \underset{\sim}{A}$ and for every $\{x^*,\xi^*;\ x\} \in X^* \times \Xi^* \times X$ the derivatives in (6.1) are continuous and the matrix

$$(6.1) \qquad J(x^*,\xi^*;\ x) = \begin{bmatrix} F_{\xi^*}(x^*,\xi^*;\ x) & F^T_{x^*}(x^*,\xi^*;\ x) \\ F_{\xi^* x}(x^*,\xi^*;\ x) & F_{x^* x}(x^*,\xi^*;\ x) \end{bmatrix} \quad {}^{1)}$$

is nonsingular, say

---

1) This matrix is the Jacobian matrix of the function

$$\begin{bmatrix} F(\cdot,\cdot;\ x) \\ F_x(\cdot,\cdot;\ x) \end{bmatrix},$$

see (3.4)-(3.5).

$$\text{(6.2)} \qquad \det J(x^*,\xi^*;\ x) < 0 \ .$$

A family F is said to be in <u>class</u> $\underset{\sim}{A}_2$, denoted by $F \in \underset{\sim}{A}_2$, if $F \in \underset{\sim}{A}_1$ and for every $x \in X$ the matrix

$$\text{(6.3)} \qquad J_0(x) \triangleq \frac{1}{F_{\xi^*}} [F_{\xi^*} F_{x^*x} - F_{\xi^*x} F^T_{x^*}] \ ,$$

where all derivatives are evaluated at $\{x^*_f(x), \xi^*_f(x);\ x\}$, is positive definite.

6.2 <u>Lemma</u>

Let $F \in \underset{\sim}{A}_1 \cap \underset{\sim}{C}$, $f: R^n \to R$, $f \overset{S}{\approx} F$ and let f and each $F \in F$ be twice continuously differentiable in S. Then, for every $x \in S$,

$$\text{(6.4)} \qquad D_x x^*_f(x) = J_0(x)^{-1} H(x)$$

where $D_x x^*_f(x)$ denotes the derivative of x^*_f at x and J_0 and H are given by (6.3) and (5.1), respectively.

<u>Proof</u>.

For any $x \in S$ consider the system

$$\text{(3.8)} \qquad F(x^*,\xi^*;\ x) - f(x) = 0$$

$$\text{(3.9)} \qquad F_x(x^*,\xi^*;\ x) - f_x(x) = 0$$

which, since $F \in \underset{\sim}{A}$, has a unique solution $\{x^*_f(x), \xi^*_f(x)\}$. The implicit function theorem, applicable since $F \in \underset{\sim}{A}_1$, then gives

$$\begin{bmatrix} D_x\xi_f^*(x) \\ D_x x_f^*(x) \end{bmatrix} = \begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix}^{-1} \begin{bmatrix} f_x(x) - F_x(x_f^*(x), \xi_f^*(x);\ x) \\ f_{xx}(x) - F_{xx}(x_f^*(x), \xi_f^*(x);\ x) \end{bmatrix} \tag{6.5}$$

where the derivatives

$$\begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix}$$

are evaluated at $\{x_f^*(x), \xi_f^*(x);\ x\}$.

Using (3.9) and (5.1), we rewrite (6.5) as

$$F_{\xi^*} D_x \xi_f^*(x) + F_{x^*}^T D_x x_f^*(x) = 0 \tag{6.6}$$

$$F_{\xi^* x} D_x \xi_f^*(x) + F_{x^* x} D_x x_f^*(x) = H(x)\ . \tag{6.7}$$

Now $F_{\xi^*} \neq 0$ since $F \in \underset{\sim}{C}$. Eliminating $D_x\xi_f^*(x)$ from (6.6) and substituting in (6.7) gives

$$H(x) = \frac{1}{F_{\xi^*}} [F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T] D_x x_f^*(x)\ .$$

The proof is completed by showing that the matrix

$$[F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T]$$

is nonsingular, which follows since

$$\det[F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T] = F_{\xi^*}^{n-1} \det \begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix}, \tag{6.8}$$

by Sylvester's identity ([7], Section II.3),

$\neq 0$, since $F \in \underset{\sim}{C} \cap \underset{\sim}{A}_1$. □

6.3 Example

Let F be the family (2.5) of affine functions: $R^n \to R$.

Then

$$x^*_f(x) = f_x(x), \qquad \text{by Example 3.4,}$$

$$J_0(x) = I \quad \text{by (6.3) since } F_{x^*x} = I, \ F_{\xi^*x} = 0$$

and (6.4) reduces to the obvious

$$D_x f_x(x) = f_{xx}(x) \ . \tag{6.9}$$

If f is a convex [strictly convex] differentiable function, then its gradient f_x is monotone [strictly monotone] in $\operatorname{dom} f$. This is an immediate consequence of the gradient inequality (Example 4.3), and Theorem 4.2. Alternatively and less directly, the monotonicity of f_x can be shown to follow from (6.9) and the fact that f_{xx} is positive semi definite, see, e.g. [12], Theorem 5.4.3. Two other cases in which the factorization (6.4) is used to establish a monotonicity property of the F-gradient x^*_f, will now be given.

6.4 Theorem

Let $F \in \underset{\sim}{A}_2 \cap C^2(X)$ where $X = I_1 \times I_2 \times \ldots \times I_n$ is the product of open intervals $I_i \subset R$, $(i = 1,\ldots,n)$. Let each $F \in F$ be of the form

$$F(x^*,\xi^*;x) = \sum_{i=1}^{n} F^i(x^*_i, x_i) - \xi^* \tag{6.10}$$

where $F^i(x_i^*,\cdot): I_i \to R$ $(i = 1,2,\ldots,n)$. Let $f: R^n \to R$ be F-convex [strictly F-convex] with $\operatorname{dom} f \supset X$ and $f \in C^2(X)$. Then x_f^* is a P_0-function [P-function] in X.

Proof.

From (6.10), (6.3) and $F \in \underset{\sim}{A}_2$ it follows that

$$J_0(x) = F_{x^*x}$$

a diagonal, positive definite matrix. From (6.4) and Theorem 5.1(a) it therefore follows, for an F-convex function f, that $D_x x_f^*(x)$ is a P_0-matrix, (see [5],[6]), proving that x_f^* is a P_0-function, by [10], Corollary 5.3.

If f is strictly F-convex, then, by Corollary 4.4(b) (applicable since $F \in \underset{\sim}{C}$), it follows for any $x,y \in X$, $x \neq y$, that there is a $k = k(x,y) \in \{1,2,\ldots,n\}$ such that

$$x_k \neq y_k \quad \text{and} \quad x_f^*(x)_k \neq x_f^*(y)_k\,,$$

proving that x_f^* is a P-function. □

A special case of Theorem 6.4 is the following, one dimensional result:

6.5 Corollary

Let $F \in \underset{\sim}{A}_1 \cap \underset{\sim}{C}$ be a family of functions: $R \to R$, let $f: R \to R$, S an open subset of $\operatorname{dom} f$, and let f and each $F \in F$ be twice continuously differentiable in S. If f is F-convex in S then x_f^* is a nondecreasing function in S.

Proof.

Using (6.3), (6.8) and (6.1) we write

(6.11) $$J_0(x) = \frac{1}{F_{\xi *}} \det J(x_f^*(x),\ \xi_f^*(x);\ x)$$

$$> 0\ ,\ \text{by (6.2) and}\ F \in \underset{\sim}{C}.$$

Therefore

$$\frac{d}{dx}\, x_f^*(x) \geq 0\ ,\ \text{by (6.4) and Theorem 5.1(a).} \qquad \square$$

6.6 Corollary

Let F, f and S be as in Corollary 6.5, where S is an interval (a,b). If

$$f''(x) > F_{xx}(x_f^*(x),\ \xi_f^*(x),\ x),\quad x \in S\ ,$$

then f is strictly F-convex.

Proof.

From (6.4) and (6.11) we infer that x_f^* is 1:1 on (a,b). As in the proof of Theorem 5.6 this implies that $z = x$ is a local minimizer of $h(z) \triangleq f(z) - F(x_f^*(x), \xi_f^*(x);\ z)$ and that no other critical point exists in (a,b). Hence $z = x$ is the unique global minimizer of $h(z)$, which was previously shown to be equivalent to the strict F-convexity of f. $\square$

6.7 Corollary

Let F be as in Theorem 6.4, with $X = R^n$. A function $f: R^n \to R$ with $\operatorname{dom} f = R^n$, $f \in C^2(R^n)$, $f \approx F$ is strictly F-convex, if the matrix $H(x)$ is positive definite.

Proof.

Follows from (6.4) and Theorem 5.5. $\square$

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