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ADJUSTMENT OF MEAN EARTH ELLIPSOID PARAMETERS

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13. ABSTRACT

The report describes the details of an adjustment technique used to find parameters of a mean earth ellipsoid. The parameters considered include: the equatorial radius, geocentric gravitational constant, the flattening, equatorial gravity, and the geoid potential. Various adjustment models are described. The linearized observation equations are formed and the specific adjustment procedure is given. The procedure used incorporates apriori estimates of quantities determined from satellite and gravimetric data. Results, as originally reported by Rapp (1974), are given.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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Foreword

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1. Introduction

Recent estimates of the dynamical parameters of the mean earth ellipsoid assuming a rotational equipotential ellipsoid as the figure of reference have been given by Rapp (1974). The parameters which may be estimated independently, with realistic values for the standard deviations of these estimates, were described and their adjusted values from a weighted least squares adjustment were presented. The purpose of this paper is to describe the details of the adjustment procedure.

The parameters considered were the rotational velocity of the earth, ω; gravitational constant times the mass of the solid earth, excluding the atmosphere, kM_E; equatorial gravity, γ_e; the dynamical form factor, J_2; scale factor for lengths, R_0; and the equatorial radius, a. ω was considered to be well determined, and treated as a known constant; while the a-priori estimates of the other five parameters were treated as their 'observed' values, with estimated standard deviations, σ, as follows (Rapp, ibid, Table 3):

$\omega = 7.292,115,146,7 \times 10^{-5}$ radians/sec.

$kM_E = 3.986,004,6 \times 10^{14}$ m^3/sec^2; $\sigma_{kM_E} = 0.000,002.5 \times 10^{14}$ m^3/sec^2

$\gamma_e = 978,030.9$ mgals; $\sigma_{\gamma_e} = 1.0$ mgals
$= 9.780,309$ m/sec^2 $= 0.000,010$ m/sec^2

$J_2 = 1082.635 \times 10^{-6}$; $\sigma_{J_2} = 0.011 \times 10^{-6}$

$R_0 = 6,363,676.0$ m; $\sigma_{R_0} = 4.0$ m

$a = 6,378,128.0$ m; $\sigma_a = 6.0$ m (1.1)

As we need only four parameters, including ω, to completely define the mean earth ellipsoid, we need two equations to relate the six parameters in equation (1.1). The two equations finally chosen for performing the adjustment of the six parameters to get their best minimum variance estimates, were (Rapp, ibid, equations (8) and (9)):

$$F_1(R_0, a, J_2, \gamma_e) = 0 \tag{1.2a}$$

$$F_2(kM_E, a, J_2, \gamma_e) = 0 \tag{1.2b}$$

The potential on the geoid, W_0, due to the mass of the solid earth, is related to kM_E and R_0 as:

$$W_0 = kM_E/R_0, \tag{1.3}$$

and its a-priori value could be determined from equation (5) of Rapp (1974). We could have therefore used W_0 as one of the six parameters instead of R_0. The adjustment could then be carried out based on the equations

$$F_3(W_0,\ a,\ J_2,\ \gamma_e) = 0$$

$$F_4(kM_E,\ a,\ J_2,\ \gamma_e) = 0 \tag{1.4}$$

The adjustment of these six parameters could have also been performed directly using equation (2-61) on page 67, and equation (2-73) on page 69 of Heiskanen and Moritz (1967). These equations may be functionally expressed as:

$$F_5(W_0,\ a,\ J_2,\ kM_E) = 0$$

$$F_6(\gamma_e,\ a,\ J_2,\ kM_E) = 0 \tag{1.5}$$

The adjustment was, in fact, done initially using equations (1.5), and later using equations (1.4). However, equations (1.2) were chosen for the final computations, as it was considered that while the a-priori estimates of a, J_2, γ_e could essentially be arrived at independently, the a-priori estimates of R_0 and kM_E could perhaps be obtained with less correlation between them, as compared to the correlation between a-priori estimates of W_0 and kM_E.

The details of the adjustment of the mean earth ellipsoid parameters will be described in this paper using equations (1.2) as the mathematical model relating the six parameters.

2. Mathematical Model

Equations (1.2) may be obtained by first rewriting equations (2-61) and (2-73) of Heiskanen and Moritz (1967) as in Rapp (ibid, equations (6) and (7)):

$$\frac{1}{R_0} = \frac{W_0}{kM_E} = \frac{\tan^{-1} e'}{ae} + \frac{1}{3}\,\frac{\omega^2 a^2}{kM_E}\ , \tag{2.1}$$

and

$$kM_E = \gamma_e\, ab\left(1 - m - \frac{m}{6}\,\frac{e' q_0'}{q_0}\right)^{-1} \tag{2.2}$$

where b is the semi-minor axis of the earth ellipsoid related to the semi-major axis 'a', and the flattening 'f' through:

$$b = a(1-f); \tag{2.3}$$

and e, e′ are the first and second eccentricities of the ellipsoid given by:

$$e^2 = (a^2 - b^2)/a^2 = 2f - f^2$$

$$e'^2 = (a^2 - b^2)/b^2 = 2f - f^2/(1-f)^2 \qquad (2.4)$$

m is given by (Heiskanen and Moritz, 1967, equation (2-70)):

$$m = \omega^2 a^2 b/kM_E = \omega^2 a^3 (1-f)/kM_E \qquad (2.5)$$

q_o and q_o' are given by (ibid, equations (2-58), (2-67) and (2-71)):

$$q_o = \frac{1}{2}\left[\left(1 + \frac{3}{e'^2}\right)\tan^{-1} e' - \frac{3}{e'}\right] \qquad (2.6)$$

$$q_o' = 3\left(1 + \frac{1}{e'^2}\right)\left(1 - \frac{1}{e'}\tan^{-1} e'\right) - 1 \qquad (2.7)$$

We now substitute for kM_E from equation (2.2) in equation (2.1), which becomes:

$$\frac{\tan^{-1} e'}{ae} + \frac{1}{3}\,\frac{\omega^2}{\gamma_e (1-f)}\left(1 - m - \frac{m}{6}\,\frac{e' q_o'}{q_o}\right) - \frac{1}{R_o} = 0 \qquad (2.8)$$

and we also rewrite equation (2.2) as:

$$\gamma_e a^2 (1-f)\left(1 - m - \frac{m}{6}\,\frac{e' q_o'}{q_o}\right)^{-1} - kM_E = 0 \qquad (2.9)$$

Equations (2.8) and (2.9) are in the form of equations (1.2), except that f should be expressed in terms of J_2. We should also express e, e′, q_o and q_o' explicitly in terms of f, which could then be related to J_2. Finally, we should also express m in equation (2.5) in terms of the parameters of the adjustment. We will do this by using expansions in power series, where we will generally retain terms up to the third power of flattening and terms of similar order, and neglect higher order terms. Here we note that m and $\frac{\omega^2 a}{\gamma_e}$ are of the same order as flattening.

We first note from equation (2.4) that:

$$e'^2 = (2f - f^2)(1 + 2f + 3f^2 + \ldots)$$

$$= 2f + 3f^2 + 4f^3 + \ldots,$$

$$e'^4 = 4f^2 + 12f^3 + \ldots,$$

$$e'^6 = 8f^3 + \ldots, \qquad (2.10)$$

and that

$$e'/e = (1-f)^{-1} = 1 + f + f^2 + f^3 + \dots$$

Next, by expanding $\tan^{-1} e'$ in a power series in terms of e'

$$\tan^{-1} e' = e' - \frac{e'^3}{3} + \frac{e'^5}{5} - \frac{e'^7}{7} + \dots$$

in equations (2.6) and (2.7) and by simple manipulations as in Heiskanen and Moritz equations (2-101), (2-102) and (2-103), we get:

$$q_0 = 2\left(\frac{1}{3\cdot 5} e'^3 - \frac{2}{5\cdot 7} e'^5 + \frac{3}{7\cdot 9} e'^7 - \frac{4}{9\cdot 11} e'^9 + \dots\right)$$

$$= \frac{2}{15} e'^3 \left(1 - \frac{6}{7} e'^2 + \frac{5}{7} e'^4 - \frac{20}{33} e'^6 + \dots\right), \qquad (2.11)$$

$$q_0' = 6\left(\frac{1}{3\cdot 5} e'^2 - \frac{1}{5\cdot 7} e'^4 + \frac{1}{7\cdot 9} e'^6 - \frac{1}{9\cdot 11} e'^8 + \dots\right)$$

$$= \frac{2}{5} e'^2 \left(1 - \frac{3}{7} e'^2 + \frac{5}{21} e'^4 - \frac{5}{33} e'^6 + \dots\right) \qquad (2.12)$$

and now expanding $1/q_0$ in a power series in terms of e', we get from equations (2.11) and (2.12):

$$\frac{e' q_0'}{q_0} = 3\left(1 - \frac{3}{7} e'^2 + \frac{5}{21} e'^4 - \frac{5}{33} e'^6 + \dots\right)\left(1 + \frac{6}{7} e'^2 + \frac{1}{49} e'^4 + \frac{128}{11319} e'^6 + \dots\right)$$

$$= 3\left(1 + \frac{3}{7} e'^2 - \frac{16}{147} e'^4 - \frac{233}{1617} e'^6 + \dots\right), \qquad (2.13)$$

which on expressing e' in terms of f from equations (2.10) becomes:

$$\frac{e' q_0'}{q_0} = 3\left[1 + \frac{3}{7}(2f + 3f^2 + 4f^3 + \dots) - \frac{16}{147}(4f^2 + 12f^3 + \dots) - \frac{233}{1617}(8f^3 + \dots) + \dots\right]$$

$$= 3\left(1 + \frac{6}{7} f + \frac{125}{147} f^2 - \frac{1204}{1617} f^3 + \dots\right) \qquad (2.14)$$

Also,

$$\frac{\tan^{-1} e'}{e} = \frac{e'}{e}\left(1 - \frac{e'^2}{3} + \frac{e'^4}{5} - \frac{e'^6}{7} + \dots\right),$$

which on using equations (2.10) becomes:

$$\frac{\tan^{-1} e'}{e} = (1+f+f^2+f^3+\ldots)(1-\frac{2}{3}f-\frac{1}{5}f^2-\frac{8}{105}f^3+\ldots)$$

$$= 1+\frac{1}{3}f+\frac{2}{15}f^2+\frac{2}{35}f^3+\ldots \qquad (2.15)$$

Now, substituting for kM_E from equation (2.9) in equation (2.5), the expression for m becomes:

$$m = \frac{\omega^2 a}{\gamma_e}\left[1 - m\left(1+\frac{1}{6}\frac{e'q_0'}{q_0}\right)\right]$$

$$\Rightarrow \quad m = \frac{\omega^2 a}{\gamma_e}\left[1+\frac{\omega^2 a}{\gamma_e}\left(1+\frac{1}{6}\frac{e'q_0'}{q_0}\right)\right]^{-1}$$

$$= \frac{\omega^2 a}{\gamma_e}\left[1+\frac{\omega^2 a}{\gamma_e}\left(1+\frac{1}{2}+\frac{3}{7}f+\ldots\right)\right]^{-1}, \text{ using equation (2.14),}$$

$$= \frac{\omega^2 a}{\gamma_e}\left[1-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}\left(1+\frac{2}{7}f-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}+\ldots\right)+\ldots\right], \qquad (2.16)$$

which is an expression accurate up to third order terms.

Further,

$$m^2 = \frac{\omega^4 a^2}{\gamma_e^2}\left(1-3\frac{\omega^2 a}{\gamma_e}+\ldots\right)$$

$$m^3 = \frac{\omega^6 a^3}{\gamma_e^3}+\ldots \qquad (2.17)$$

The expression raised to the power of -1 in equation (2.9) may now be expanded by using expression for $e'q_0'/q_0$ from equation (2.14), and for m and its powers from equations (2.16) and (2.17) as:

$$\left(1-m-\frac{m}{6}\frac{e'q_0'}{q_0}\right)^{-1}$$

$$=1+m\left(1+\frac{1}{6}\frac{e'q_0'}{q_0}\right)+m^2\left(1+\frac{1}{6}\frac{e'q_0'}{q_0}\right)^2+m^3\left(1+\frac{1}{6}\frac{e'q_0'}{q_0}\right)^3+\ldots$$

$$=1+\frac{\omega^2 a}{\gamma_e}\left(1-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}-\frac{3}{7}\frac{\omega^2 a}{\gamma_e}f+\frac{9}{4}\frac{\omega^4 a^2}{\gamma_e^2}+\ldots\right)\left(\frac{3}{2}+\frac{3}{7}f+\frac{125}{294}f^2+\ldots\right)$$

$$+\frac{\omega^4 a^2}{\gamma_e^2}\left(1-3\frac{\omega^2 a}{\gamma_e}+\ldots\right)\cdot\frac{9}{4}\left(1+\frac{4}{7}f+\ldots\right)+\frac{\omega^6 a^3}{\gamma_e^3}\cdot\frac{27}{8}+\ldots$$

$$=1+\frac{3}{2}\frac{\omega^2 a}{\gamma_e}\left(1+\frac{2}{7}f+\frac{125}{441}f^2+\ldots\right), \qquad (2.18)$$

as the remaining terms up to third order cancel out.

Finally, we may express the second term in equation (2.8) by using expressions for $e'q_0'/q_0$ and m from equations (2.14) and (2.16) as in equation (2.18) above:

$$\frac{1}{3}\frac{\omega^2}{\gamma_e(1-f)}\left(1-m-\frac{m}{6}\frac{e'q_0'}{q_0}\right)$$

$$=\frac{1}{3}\frac{\omega^2}{\gamma_e}(1+f+f^2+\ldots)\left[1-\frac{\omega^2 a}{\gamma_e}\left(\frac{3}{2}+\frac{3}{7}f-\frac{9}{4}\frac{\omega^2 a}{\gamma_e}+\ldots\right)\right]$$

$$=\frac{1}{3}\frac{\omega^2}{\gamma_e}\left[1+f+f^2-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}\left(1+\frac{9}{7}f-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}+\ldots\right)+\ldots\right], \tag{2.19}$$

where we have retained terms in the square brackets up to second order only, as these are being multiplied by ω^2/γ_e, which is already smaller than a third order term.

Inserting equations (2.15), (2.18) and (2.19) in equations (2.8) and (2.9), we get:

$$\frac{\tan^{-1}e'}{ae}+\frac{1}{3}\frac{\omega^2}{\gamma_e}\left[1+f+f^2-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}\left(1+\frac{9}{7}f-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}+\ldots\right)+\ldots\right]-\frac{1}{R_0}=0 \tag{2.20}$$

or

$$\frac{1}{a}\left(1+\frac{1}{3}f+\frac{2}{15}f^2+\frac{2}{35}f^3+\ldots\right)+\frac{1}{3}\frac{\omega^2}{\gamma_e}\left[1+f+f^2-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}\left(1+\frac{9}{7}f-\frac{3}{2}\frac{\omega^2 a}{\gamma_e}+\ldots\right)+\ldots\right]-1/R_0=0$$

and

$$\gamma_e a^2(1-f)\left[1+\frac{3}{2}\frac{\omega^2 a}{\gamma_e}\left(1+\frac{2}{7}f+\frac{125}{441}f^2+\ldots\right)\right]-kM_E=0 \tag{2.21}$$

$$\Rightarrow\quad \gamma_e a^2(1-f)+\frac{3}{2}\omega^2 a^3\left(1-\frac{5}{7}f-\frac{1}{441}f^2+\ldots\right)-kM_E=0 \tag{2.22}$$

Equations (2.21) and (2.22) are now explicitly in the form of equations (1.2):

$$F_1(R_0,\ a,\ J_2,\ \gamma_e)=0$$

$$F_2(kM_E,\ a,\ J_2,\ \gamma_e)=0 \tag{1.2}$$

except that f has yet to be expressed in terms of J_2. We first write from equations (2-90) and (2-92′) of Heiskanen and Moritz (1967):

$$J_2=\frac{1}{3}e^2\left(1-\frac{2}{15}\frac{me'}{q_0}\right), \tag{2.23}$$

which on substituting for e^2 from equation (2.4), and for $1/q_0$ from equation (2.11) becomes:

$$J_2 = \frac{1}{3}(2f - f^2)\left[1 - \frac{m}{e'^2}\left(1 + \frac{6}{7}e'^2 + \frac{1}{49}e'^4 + \ldots\right)\right]$$

and using equations (2.4) and (2.10) to express e and e′ in terms of f, we get:

$$J_2 = \frac{2}{3}f\left(1 - \frac{f}{2}\right) - \frac{m}{3}(1 - 2f + f^2)\left(1 + \frac{12}{7}f + \frac{130}{49}f^2 + \ldots\right)$$

$$= \frac{2}{3}f\left(1 - \frac{f}{2}\right) - \frac{m}{3}\left(1 - \frac{2}{7}f + \frac{11}{49}f^2 + \ldots\right), \qquad (2.24)$$

which is an expression accurate up to third order terms.

Equation (2.24) may now be solved for f as:

$$f\left(1 - \frac{f}{2}\right) = \frac{3}{2}J_2 + \frac{m}{2}\left(1 - \frac{2}{7}f + \frac{11}{49}f^2 + \ldots\right),$$

which on substituting for m from equation (2.5) becomes:

$$f = \left[\frac{3}{2}J_2 + \frac{\omega^2 a^3 (1-f)}{2kM_E}\left(1 - \frac{2}{7}f + \frac{11}{49}f^2 + \ldots\right)\right] \Big/ \left(1 - \frac{f}{2}\right) \qquad (2.25)$$

Equation (2.25) enables computation of f to a desired degree of accuracy from the 'observed' values of the earth parameters in equation (1.1) by iteration with a starting value of f from:

$$f \simeq \frac{3}{2}J_2 + \frac{\omega^2 a^3}{2kM_E}$$

About 8 iterations were adequate to ensure that two successive values of f did not differ more that 1×10^{-15}.

It is now possible to rigorously compute the misclosures in the mathematical model given in the form of closed expressions by equations (2.8) and (2.9) by substituting for the 'observed' values of the earth parameters from equation (1.1).

However, for the later evaluation of partials we would be using the alternative form of the mathematical model given by equations (2.21) and (2.22). But as these equations involve f, we need a direct expression for f in terms of a, J_2 and γ_e, which we may obtain from equation (2.25).

We first rewrite equation (2.25) by using the alternate expression for m from equation (2.16):

$$f = \frac{1}{1-f/2}\left[\frac{3}{2}J_2 + \frac{\omega^2 a}{2\gamma_e}\{1 - \frac{3}{2}\frac{\omega^2 a}{\gamma_e}(1+\frac{2}{7}f - \frac{3}{2}\frac{\omega^2 a}{\gamma_e} + \ldots)\}(1 - \frac{2}{7}f + \frac{11}{49}f^2 + \ldots)\right]$$

We then collect terms and simplify by retaining terms up to third order only as follows:

$$f = (1 + \frac{f}{2} + \frac{f^2}{4} + \ldots)\left[\frac{3}{2}J_2 + \frac{\omega^2 a}{2\gamma_e}\{1 - \frac{2}{7}f - \frac{3}{2}\frac{\omega^2 a}{\gamma_e} + \frac{11}{49}f^2 + \frac{9}{4}\frac{\omega^4 a^2}{\gamma_e^2} + \ldots\}\right]$$

$$= \frac{3}{2}J_2(1 + \frac{f}{2} + \frac{f^2}{4} + \ldots) + \frac{\omega^2 a}{2\gamma_e}[1 + \frac{3}{14}f - \frac{3}{2}\frac{\omega^2 a}{\gamma_e} + \frac{65}{196}f^2 - \frac{3}{4}\frac{\omega^2 a}{\gamma_e}f + \frac{9}{4}\frac{\omega^4 a^2}{\gamma_e^2} + \ldots] \tag{2.26}$$

Equation (2.26) now expressed f in terms of a, J_2 and γ_e in a form suitable for obtaining partials.

We may now summarize Sec. 2 by noting that the mathematical model in equations (1.2):

$$F_1(R_0, a, J_2, \gamma_e) = 0$$

$$F_2(kM_E, a, J_2, \gamma_e) = 0 \tag{1.2}$$

is given by closed expressions in equations (2.8) and (2.9):

$$\frac{\tan^{-1}e'}{ae} + \frac{1}{3}\frac{\omega^2}{\gamma_e(1-f)}(1 - m - \frac{m}{6}\frac{e'q_0'}{q_0}) - \frac{1}{R_0} = 0 \tag{2.8}$$

$$\gamma_e a^2(1-f)(1 - m - \frac{m}{6}\frac{e'q_0'}{q_0})^{-1} - kM_E = 0 \tag{2.9}$$

where e, e′ are computed by closed expressions involving f in equations (2.4); q_0 and q_0' are computed by closed expressions involving e and e′ in equations (2.6) and (2.7); m is computed from closed expressions involving f and the values of ω, a, kM_E in equation (2.6); while f is obtained by iteration in equation (2.25) to the desired degree of accuracy from the values of J_2, ω, a, kM_E.

The mathematical model in equations (2.8) and (2.9) may be put in the alternate form for obtaining partials in equations (2.21) and (2.22), which may be functionally expressed as:

$$F_1(R_0, a, f(a, J_2, \gamma_e), \gamma_e) = 0 \tag{2.27a}$$

$$F_2(kM_E, a, f(a, J_2, \gamma_e), \gamma_e) = 0 \tag{2.27b}$$

and the expression for f in terms of a, J_2 and γ_e suitable for obtaining partials is given by equation (2.26).

3. Partial Derivatives

We will consider here the partial derivatives of the functions F_1 and F_2 in equations (2.27) with respect to a, J_2 and γ_e. The partial derivatives with respect to R_0 and kM_E will be discussed later.

The partial derivatives $\partial(F_1, F_2)/\partial(a, J_2, \gamma_e)$ may be functionally expressed from equation (2.27) as:

$$\frac{\partial F_1}{\partial a} = \frac{\partial F_1}{\partial a} + \frac{\partial F_1}{\partial f} \cdot \frac{\partial f}{\partial a}$$

$$\frac{\partial F_1}{\partial J_2} = \frac{\partial F_1}{\partial f} \cdot \frac{\partial f}{\partial J_2}$$

$$\frac{\partial F_1}{\partial \gamma_e} = \frac{\partial F_1}{\partial \gamma_e} + \frac{\partial F_1}{\partial f} \cdot \frac{\partial f}{\partial \gamma_e}, \tag{3.1}$$

and similarly:

$$\frac{\partial F_2}{\partial a} = \frac{\partial F_2}{\partial a} + \frac{\partial F_2}{\partial f} \cdot \frac{\partial f}{\partial a}$$

$$\frac{\partial F_2}{\partial J_2} = \frac{\partial F_2}{\partial f} \cdot \frac{\partial f}{\partial J_2}$$

$$\frac{\partial F_2}{\partial \gamma_e} = \frac{\partial F_2}{\partial f} \cdot \frac{\partial f}{\partial \gamma_e} \tag{3.2}$$

We first consider the partials of f with respect to a, J_2 and γ_e; and we use one step earlier in equation (2.26) for differentiation, i.e.

$$f\left(1 - \frac{f}{2}\right) = \frac{3}{2} J_2 + \frac{\omega^2 a}{2\gamma_e}\left(1 - \frac{2}{7} f - \frac{3}{2}\frac{\omega^2 a}{\gamma_e} + \frac{11}{49} f^2 + \frac{9}{4}\frac{\omega^4 a^2}{\gamma_e^2} + \ldots\right) \tag{3.3}$$

We now differentiate equation (3.3) with respect to a:

$$\frac{\partial f}{\partial a}(1 - f) = \frac{\omega^2}{2\gamma_e}\left(1 - \frac{2}{7} f - 3\frac{\omega^2 a}{\gamma_e} + \frac{11}{49} f^2 + \frac{27}{4}\frac{\omega^4 a^2}{\gamma_e^2} + \ldots\right)$$

$$\frac{\omega^2 a}{2\gamma_e}\left(-\frac{2}{7}\frac{\partial f}{\partial a} + \frac{22}{49} f \frac{\partial f}{\partial a} + \ldots\right)$$

$$\Rightarrow \quad \frac{\partial f}{\partial a} = \frac{\omega^2}{2\gamma_e}\left(1 - \frac{2}{7} f - \frac{3\omega^2 a}{\gamma_e} + \frac{11}{49} f^2 + \frac{27}{4}\frac{\omega^4 a^2}{\gamma_e^2} + \ldots\right)\left[1 - f + \frac{\omega^2 a}{7\gamma_e}\left(1 - \frac{11}{7} f + \ldots\right) + \ldots\right]^{-1} \tag{3.4}$$

Noting that ω^2/γ_e is already smaller than third power of flattening, we may simplify equation (3.4) as:

$$\frac{\partial f}{\partial a} = \frac{\omega^2}{2\gamma_e}\left(1 - \frac{2}{7}f - \frac{3\omega^2 a}{\gamma_e} + \ldots\right)\left(1 - f + \frac{\omega^2 a}{7\gamma_e} + \ldots\right)^{-1}$$

$$= \frac{\omega^2}{2\gamma_e}\left(1 + \frac{5}{7}f - \frac{22}{7}\frac{\omega^2 a}{\gamma_e} + \ldots\right) \qquad (3.5)$$

We now differentiate equation (3.3) with respect to J_2:

$$\frac{\partial f}{\partial J_2}(1 - f) = \frac{3}{2} + \frac{\omega^2 a}{2\gamma_e}\left(-\frac{2}{7} + \frac{22}{49}f + \ldots\right)\frac{\partial f}{\partial J_2}$$

$$\Rightarrow \frac{\partial f}{\partial J_2} = \frac{3}{2}\left[1 - f + \frac{\omega^2 a}{7\gamma_e}\left(1 - \frac{11}{7}f + \ldots\right) + \ldots\right]^{-1} \qquad (3.6)$$

Differentiating equation (3.3) again with respect to γ_e:

$$\frac{\partial f}{\partial \gamma_e}(1 - f) = -\frac{\omega^2 a}{2\gamma_e^2}\left(1 - \frac{2}{7}f - \frac{3\omega^2 a}{\gamma_e} + \frac{11}{49}f^2 + \frac{27}{4}\frac{\omega^4 a^2}{\gamma_e^2} + \ldots\right)$$

$$+ \frac{\omega^2 a}{2\gamma_e}\left(-\frac{2}{7} + \frac{22}{49}f + \ldots\right)\frac{\partial f}{\partial \gamma_e}$$

$$\Rightarrow \frac{\partial f}{\partial \gamma_e} = -\frac{\omega^2 a}{2\gamma_e^2}\left(1 - \frac{2}{7}f - \frac{3\omega^2 a}{\gamma_e} + \frac{11}{49}f^2 + \frac{27}{4}\frac{\omega^4 a^2}{\gamma_e^2} + \ldots\right)\left[1 - f + \frac{\omega^2 a}{7\gamma_e}\left(1 - \frac{11}{7}f + \ldots\right) + \ldots\right]^{-1} \qquad (3.7)$$

By comparing equation (3.4) and (3.7), we easily see:

$$\frac{\partial f}{\partial \gamma_e} = -\frac{a}{\gamma_e}\frac{\partial f}{\partial a} \qquad (3.8)$$

We now obtain the partials $\partial F_1/\partial f$ and $\partial F_2/\partial f$ from equations (2.21) and (2.22) as:

$$\frac{\partial F_1}{\partial f} = \frac{1}{3a}\left(1 + \frac{4}{5}f + \frac{18}{35}f^2 + \ldots\right) + \frac{1}{3}\frac{\omega^2}{\gamma_e}\left(1 + 2f - \frac{27}{14}\frac{\omega^2 a}{\gamma_e} + \ldots\right) \qquad (3.9)$$

$$\frac{\partial F_2}{\partial f} = -\gamma_e a^2 + \frac{3}{2}\omega^2 a^3\left(-\frac{5}{7} - \frac{2}{441}f + \ldots\right)$$

$$= -\gamma_e a^2\left[1 + \frac{15}{14}\frac{\omega^2 a}{\gamma_e}\left(1 + \frac{14}{2205}f + \ldots\right)\right] \qquad (3.10)$$

We now assemble the partials $\partial(F_1, F_2)/\partial(a, J_2, \gamma_e)$ using equations (3.1) and (3.2), where we get the direct partials of F_1 and F_2 with respect to a and γ_e from equations (2.21) and (2.22).

$$\frac{\partial F_1}{\partial a} = -\frac{1}{a^2}\frac{\tan^{-1}e'}{e} - \frac{\omega^4}{2\gamma_e^2}(1 + \frac{9}{7}f - \frac{3\omega^2 a}{\gamma_e} + \ldots) + \frac{\partial F_1}{\partial f}\cdot\frac{\partial f}{\partial a} \tag{3.11}$$

where the expressions for $\partial F_1/\partial f$ and $\partial f/\partial a$ are given in equations (3.9) and (3.5).

$$\frac{\partial F_1}{\partial J_2} = \frac{\partial F_1}{\partial f}\cdot\frac{\partial f}{\partial J_2} \tag{3.12}$$

where expressions for $\partial F_1/\partial f$ and $\partial f/\partial J_2$ are given in equations (3.9) and (3.6).

$$\frac{\partial F_1}{\partial \gamma_e} = -\frac{1}{3}\frac{\omega^2}{\gamma_e^2}(1 + f - \frac{3\omega^2 a}{\gamma_e} + \ldots) + \frac{\partial F_1}{\partial f}\cdot\frac{\partial f}{\partial \gamma_e} \tag{3.13}$$

where expressions for $\partial F_1/\partial f$ and $\partial f/\partial \gamma_e$ are given in equations (3.9) and (3.8).

Similarly, using equations (2.22) and (3.2), we get:

$$\frac{\partial F_2}{\partial a} = 2\gamma_e a(1-f) + \frac{9}{2}\omega^2 a^2(1 - \frac{5}{7}f - \frac{1}{441}f^2 + \ldots) + \frac{\partial F_2}{\partial f}\cdot\frac{\partial f}{\partial a} \tag{3.14}$$

where expressions for $\partial F_2/\partial f$ and $\partial f/\partial a$ are given in equations (3.10) and (3.5).

$$\frac{\partial F_2}{\partial J_2} = \frac{\partial F_2}{\partial f}\cdot\frac{\partial f}{\partial J_2} \tag{3.15}$$

where expressions for $\partial F_2/\partial f$ and $\partial f/\partial J_2$ are given in equations (3.10) and (3.6). And finally

$$\frac{\partial F_2}{\partial \gamma_e} = a^2(1-f) + \frac{\partial F_2}{\partial f}\cdot\frac{\partial f}{\partial \gamma_e} \tag{3.16}$$

where expressions for $\partial F_2/\partial f$ and $\partial f/\partial \gamma_e$ are given in equations (3.10) and (3.8).

4. The Adjustment Procedure

The mathematical model expressed by equations (1.2)

$$F_1(R_o, a, J_2, \gamma_e) = 0$$

$$F_2(kM_E, a, J_2, \gamma_e) = 0 \tag{1.2}$$

may be represented for generalized adjustment following Uotila (1967, Sec. 11) as:

$$F(\overset{a}{L}_F, \overset{a}{L}_X) = 0 \tag{4.1}$$

where

$$\overset{a}{L}_F{}' = (\overset{a}{R}_o, k\overset{a}{M}_E) \tag{4.2}$$

and

$$L_X' = (\overset{a}{a}, \overset{a}{J}_2, \overset{a}{\gamma}_e) \tag{4.3}$$

the superscript a denoting the adjusted value of the parameters.

The subdivision of the parameters in two groups represented by vectors L_F and L_X may appear arbitrary, but it is convenient for assigning their weight matrices. The variance-covariance matrix, Σ_X, for the parameters L_X is:

$$\Sigma_X = \begin{bmatrix} \sigma_a^2 & & 0 \\ & \sigma_{J2}^2 & \\ 0 & & \sigma_{\gamma_e}^2 \end{bmatrix} \tag{4.4}$$

where the a-priori variances σ_a^2, σ_{J2}^2, $\sigma_{\gamma_e}^2$ are taken from equation (1.1), and their covariances are considered to be zero, as their 'observed' value estimates in equation (1.1) are considered to have been arrived at independently.

Assuming variance of unit weight, σ_o^2, as unity, the weight matrix, P_X, of the parameters L_X, is given by:

$$P_X = \sigma_o^2 \Sigma_X^{-1} = \begin{bmatrix} 1/\sigma_a^2 & & 0 \\ & 1/\sigma_{J2}^2 & \\ 0 & & 1/\sigma_{\gamma_e}^2 \end{bmatrix} \tag{4.5}$$

The variance-covariance matrix, Σ_F, for the parameters L_F is:

$$\Sigma_F = \begin{bmatrix} \sigma_{R_o}^2 & \rho_{R_o, kM_E} \cdot \sigma_{R_o} \cdot \sigma_{kM_E} \\ \rho_{R_o, kM_E} \cdot \sigma_{R_o} \cdot \sigma_{kM_E} & \sigma_{kM_E}^2 \end{bmatrix} \tag{4.6}$$

where the a-priori variances $\sigma_{R_o}^2$, $\sigma_{kM_E}^2$ are taken from equation (1.1), and the covariance between R_o and kM_E is given by:

$$\rho_{R_o, kM_E} \cdot \sigma_{R_o} \cdot \sigma_{kM_E},$$

where ρ_{R_o, kM_E} is the correlation coefficient between the 'observed' values, i.e. a-priori estimates of R_o and kM_E in equation (1.1). It is now possible to assign various values to this correlation coefficient, say from 0.1 to 0.9 and watch its effect on the adjustment. This will be discussed later.

Irrespective of the value of $\rho_{R_0, kM_\varepsilon}$ in equation (4.6), the weight matrix, P_F, of the parameters L_F, is given by:

$$P_F = \sigma_0^2 \Sigma_F^{-1} = \Sigma_F^{-1} \tag{4.7}$$

as the variance of unit weight, σ_0^2, was unity.

The linearized form of the mathematical model in equation (4.1), using the first derivatives in the Taylor's series expansion, is given by:

$$B_F V_F + B_{Fx} + W_F = 0 \tag{4.8}$$

where

$$B_F = \begin{bmatrix} \partial F_1/\partial R_0 & 0 \\ 0 & \partial F_2/\partial kM_\varepsilon \end{bmatrix} \tag{4.9}$$

$$B_{Fx} = \begin{bmatrix} \partial F_1/\partial a & \partial F_1/\partial J_2 & \partial F_1/\partial \gamma_e \\ \partial F_2/\partial a & \partial F_2/\partial J_2 & \partial F_2/\partial \gamma_e \end{bmatrix} \tag{4.10}$$

and

$$W_F = F(L_F^b, L_x^b) \tag{4.11}$$

where the superscript b in the vectors L_F, L_x represent their 'observed' value estimates in equation (1.1). The partial derivatives matrices B_F and B_{Fx} are also evaluated using the 'observed' value of the parameters. The elements of B_{Fx} are computed as in equations (3.11) to (3.16). The elements of B_{Fx} are computed as in equations (3.11) to (3.16). The elements of B_F will be discussed later. V_F and V_x are the correction vectors to the 'observed' values, giving their adjusted values, i.e.

$$L_F^a = L_F^b + V_F \tag{4.12}$$

$$L_x^a = L_x^b + V_x \tag{4.13}$$

The adjusted values of the parameters are obtained for the minimum variance of their corrections, and for the fulfillment of the linearized form of the mathematical model in equation (4.8), i.e. for a minimum value of the Loss function φ given by:

$$\varphi = V_F' P_F V_F + V_x' P_x V_x - 2K_F' (B_F V_F + B_{Fx} V_x + W_F) \tag{4.14}$$

where the elements of vector K_F are Lagrange multipliers.

The correction vector, V_X, under a minimum value of σ_0, is given (Uotila, 1967, equation (230)) by:

$$V_X = -(B'_{FX} M_F^{-1} B_{FX} + P_X)^{-1} B'_{FX} M_F^{-1} W_F \,, \tag{4.15}$$

where

$$M_F = B_F P_F^{-1} B'_F \tag{4.16}$$

It is easy to see that if we have a special case of the mathematical model, which is linear in the parameters L_F, then the partial derivatives matrix B_F is given by:

$$B_F = -I$$

resulting in

$$M_F = P_F^{-1} \,, \quad M_F^{-1} = P_F \,,$$

and the correction vector V_X is then given by:

$$V_X = -(B'_{FX} P_F B_{FX} + P_X)^{-1} B'_{FX} P_F W_F \tag{4.17}$$

We recall that our mathematical model given by equations (2.8) and (2.9):

$$\frac{\tan^{-1} e'}{ae} + \frac{1}{3} \frac{\omega^2}{\gamma_e (1-f)} \left(1 - m - \frac{m}{6} \frac{e' q'_o}{q_o}\right) - \frac{1}{R_o} = 0 \tag{2.8}$$

and

$$\gamma_e a^2 (1-f) \left(1 - m - \frac{m}{6} \frac{e' q'_o}{q_o}\right)^{-1} - kM_E = 0 \tag{2.9}$$

is not linear in the parameters R_o and kM_E, but it is linear if we take the parameters of adjustment as $1/R_o$ and kM_E.

We may thus use the simplified equation (4.17), instead of equation (4.15) if we treat $1/R_o$ as one of the parameters of adjustment instead of R_o. We may then use the following equations instead of the equations (4.2), (4.6), (4.9):

$$L_F^{a\prime} = (1/R_o^a , \quad kM_E^a) \tag{4.2'}$$

$$\Sigma_F = \begin{bmatrix} \sigma^2_{1/R_o} & \rho_{1/R_o, kM_E} \cdot \sigma_{1/R_o} \cdot \sigma_{kM_E} \\ \rho_{1/R_o, kM_E} \cdot \sigma_{1/R_o} \cdot \sigma_{kM_E} & \sigma^2_{kM_E} \end{bmatrix} \tag{4.6'}$$

$$B_f = \begin{bmatrix} \partial F_1/\partial(1/R_o) & 0 \\ 0 & \partial F_2/\partial(kM_\varepsilon) \end{bmatrix} = -I \qquad (4.9')$$

and the equation (4.12) is explicitly given by:

$$\frac{1}{R_o^a} = \frac{1}{R_o^b} + v_{1/R_o} \qquad (4.12a')$$

$$kM_\varepsilon^a = kM_\varepsilon^b + v_{kM_\varepsilon} \qquad (4.12b')$$

The adjustment was actually done using equation (4.17), treating $1/R_o$ as the parameter for adjustment as in equations (4.2′), (4.6′), (4.9′) and (4.12′a). The 'observed' value for $1/R_o$ is simply obtained as:

$$(1/R_o)^b = 1/R_o^b$$

while the a-priori variance of $1/R_o$, σ^2_{1/R_o}, was obtained as:

$$\sigma^2_{1/R_o} = \left[\frac{\partial(1/R_o)}{\partial R_o}\right]^2 \sigma^2_{R_o} = \frac{1}{R_o^4}\sigma^2_{R_o} \qquad (4.18)$$

and, finally when the adjustment is completed with $1/R_o$ as a parameter, we get the adjusted value of R_o from:

$$R_o^a = 1/(1/R_o)^a \qquad (4.19)$$

and the standard deviation, $\sigma_{R_o^a}$, of the adjusted value of R_o from the standard deviation, $\sigma_{(1/R_o)^a}$, of the adjusted value of $1/R_o$ from:

$$\sigma^2_{R_o^a} = \left[\frac{\partial R_o}{\partial(1/R_o)}\right]^2 \sigma^2_{(1/R_o)^a} = R_o^4\,\sigma^2_{(1/R_o)^a}$$

$$\Rightarrow \quad \sigma_{R_o^a} = R_o^2\,\sigma_{(1/R_o)^a} \qquad (4.20)$$

Now, reverting to the correlation coefficient $\rho_{1/R_o,kM_\varepsilon}$ between $1/R_o$ and kM_ε, different weight matrices were initially used for P_f (equations (4.7) and (4.6′)) using the correlation coefficient from 0.1 to 0.9 in steps of 0.1, but the effect on the adjusted values was very little, the range of the adjusted values because of the various differing correlation coefficients being generally less than 1/15 of the standard deviation of the adjusted parameters. The final computations were therefore made using the value of this correlation coefficient as zero.

To complete the description of the adjustment procedure, the correction vector V_F is given by:

$$V_F = -P_F^{-1} B_F' K_F,$$

where

$$K_F = -M_F^{-1} (B_{Fx} V_x + W_F),$$

and, as in our mathematical model, with $1/R_0$ as the parameter for adjustment,

$$B_F = -I, \quad M_F^{-1} = P_F,$$

the correction vector V_F is now given by:

$$V_F = -(B_{Fx} V_x + W_F) \tag{4.21}$$

The variance-covariance matrix, Σ_x^a, of the adjusted parameters, L_x^a, is given from equation (4.17) by:

$$\Sigma_x^a = \sigma_0^2 (B_{Fx}' P_F B_{Fx} + P_x)^{-1} = (B_{Fx}' P_F B_{Fx} + P_x)^{-1} \tag{4.22}$$

since we retain the a-posteriori variance of unit weight as unity, as the degrees of freedom being 2 are not sufficient to obtain a revised estimate of the a-posteriori variance of unit weight. The standard deviations of the adjusted parameters L_x^a, i.e. σ_a^a, $\sigma_{J_2}^a$, $\sigma_{\gamma_e}^a$ is obtained as the square root of the variances, i.e. the diagonal elements in Σ_x^a.

The variance-covariance matrix, Σ_F^a, of the adjusted parameters L_F^a is obtained from:

$$\Sigma_F^a = \left[\frac{\partial L_F}{\partial L_x}\right] \Sigma_x^a \left[\frac{\partial L_F}{\partial L_x}\right]' = B_{Fx} \Sigma_x^a B_{Fx}' \tag{4.23}$$

as

$$\left[\frac{\partial L_F}{\partial L_x}\right] = B_{Fx}$$

since the mathematical model is linear in the parameters L_F, when $1/R_0$ is used as a parameter for adjustment. The standard deviations of the adjusted parameters L_F^a is obtained as before by taking the square root of the diagonal elements of Σ_F^a. The standard deviation of the adjusted values of R_0, i.e. $\sigma_{R_0}^a$ is obtained from equation (4.20).

While performing the adjustment of the parameters R_0, kM, a, J_2, γ_e of the mean earth ellipsoid, it is also of interest to obtain the adjusted value of the reverse flattening, $1/f$, and its standard deviation. The 'observed' or a-priori value of f was obtained in equation (2.25) by iteration from the a-priori estimates of J_2, a, kM_E in equation (1.1). The adjusted, or a-posteriori, value of f is similarly obtained by iteration from equation (2.25):

$$f = \left[\frac{3}{2} J_2 + \frac{\omega^3 a^3 (1-f)}{2kM_E} \left(1 - \frac{2}{7} f + \frac{11}{49} f^2 + \ldots\right)\right] / \left(1 - \frac{f}{2}\right) \tag{2.25}$$

where we use the adjusted value of the parameters J_2, a, kM_E ; and the adjusted value of reverse flattening is simply:

$$(1/f)^a = 1/f^a$$

The standard deviation, $\sigma_{(1/f)^a}$ of the adjusted value of the reverse flattening, $1/f$, is obtained rigorously from:

$$\sigma^2_{(1/f)^a} = \left\{\frac{\partial(1/f)}{\partial f}\right\}^2 \left[\frac{\partial f}{\partial(a, J_2, \gamma_e)}\right] \Sigma^a_x \left[\frac{\partial f}{\partial(a, J_2, \gamma_e)}\right]'$$

but as we notice from equations (3.5) to (3.7), $\partial f/\partial a$ and $\partial f/\partial \gamma_e$ are much smaller, by two and one orders of magnitude respectively, than $\partial f/\partial J_2$. It is, therefore adequate to compute $\sigma_{(1/f)^a}$ from:

$$\sigma_{(1/f)^a} = \left|\frac{1}{f^2} \frac{\partial f}{\partial J_2}\right| \sigma^a_{J_2} \tag{4.24}$$

As a practical detail of the adjustment, it may be mentioned that for the chosen units of ω, kM_E , J_2, R_o, a in equation (1.1), it is necessary to have the units of γ_e and σ_{γ_e} in m/sec^2 throughout the adjustment, instead of mgals. It is easy to see this from a consideration of the units in the mathematical model in equations (2.8) and (2.9). The initial value of γ_e and σ_{γ_e} in mgals is thus multiplied by 1×10^{-5}, and after the adjustment is completed, the adjusted values of γ_e and σ_{γ_e} are multiplied by 1×10^{5} to quote the results in mgals.

Finally to refine the adjusted values, say $\hat{L}_F$, $\hat{L}_x$, obtained by the procedure already described, i.e. by the first cycle of adjustment, we may carry out a second cycle of adjustment. The value of B_{Fx} in equation (4.10), say $\tilde{B}_{Fx}$, is then computed using $\hat{L}_F$, $\hat{L}_x$ in place of L^b_F, L^b_x which were used in the first cycle of adjustment. The value of W_F is, however updated for the second cycle of adjustment, say $\tilde{W}_F$, as:

$$\begin{aligned} \tilde{W}_F &= F(\hat{L}_F, \hat{L}_x) + B_F(L^b_F - \hat{L}_F) \\ &= F(\hat{L}_F, \hat{L}_x) - (L^b_F - \hat{L}_F) \end{aligned} \tag{4.25}$$

since $B_F = -I$ in our mathematical model with $1/R_o$ as a parameter for adjustment.

The second cycle of adjustment is then carried out to obtain the correction vectors, say $\tilde{V}_x$ and $\tilde{V}_F$ from equations (4.17) and (4.21), where we now use $\tilde{B}_{Fx}$ and $\tilde{W}_F$ in place of B_{Fx} and W_F. The adjusted values of the parameters, L_F^a. L_x^a, after the second cycle of adjustment are then obtained as:

$$L_x^a = \hat{L}_x + \tilde{V}_x ; \quad L_F^a = \hat{L}_F + \tilde{V}_F$$

The adjustment in two cycles was actually carried out in the initial stages, when the a-priori estimates of the parameters in equation (1.1) were yet being finalized. The adjusted values in the second cycle of adjustment were however not found to differ in any of the significant digits from the adjusted values in the first cycle of adjustment. Accordingly, in the final computations, the adjustment of the parameters was done only in one cycle, and not iterated.

We may now summarize Sec. 4 by noting that the parameters for adjustment may be divided into two groups denoted by vectors L_F and L_x:

$$L_F' = (1/R_e ,\ kM_E), \qquad (4.2')$$

and

$$L_x' = (a,\ J_2 ,\ \gamma_a), \qquad (4.3)$$

and their weight matrices P_F and P_x are obtained from equations (4.6′), (4.7) and (4.5). The separation of parameters enables correlation between $1/R_o$ and kM_E to be taken into account, though in the final computations it was considered to be adequate to treat this correlation as zero. Further, the use of $1/R_o$ as a parameter for adjustment in place of R_o makes the mathematical model linear with respect to parameters L_F resulting in simpler equations for the computation of the correction vectors, V_F and V_x, given by:

$$V_x = -(B_{Fx}' P_F B_{Fx} + P_x)^{-1} B_{Fx}' P_F W_F \qquad (4.17)$$

and

$$V_F = -(B_{Fx} V_x + W_F), \qquad (4.21)$$

where B_{Fx} and W_F are evaluated as in equations (4.10) and (4.11). The adjusted values of the parameters, L_F^a, L_x^a, is given by:

$$L_F^a = L_F^b + V_F \qquad (4.12')$$

$$L_x^a = L_x^b + V_x \qquad (4.13)$$

where: L_F^b, L_x^b are the a-priori estimates of the parameters given in equation (1.1).

The standard deviations of the adjusted parameters are obtained as the square roots of the diagonal elements of the variance-covariance matrices, Σ_F^a, Σ_x^a, given by:

$$\Sigma_x^a = (B'_{fx} P_f B_{fx} + P_x)^{-1} \qquad (4.22)$$

and

$$\Sigma_f^a = B_{fx} \Sigma_x^a B'_{fx} \qquad (4.23)$$

The standard deviation of the adjusted value of R_o is obtained as in equation (4.20), and the standard deviation of the adjusted value of the reverse flattening, 1/f, is obtained as in equation (4.24).

5. Adjusted Value of the Earth Parameters

The a-priori estimates of the earth parameters R_o, kM_E, a, J_2, γ_e have been given in equation (1.1). The considerations in arriving at these estimates, as well as their estimated standard deviations, have been described by Rapp (1974). As only four parameters, including ω, are needed to completely define an equipotential rotational ellipsoid, the a-priori estimates of the earth parameters in equation (1.1) may be adjusted to satisfy the mathematical model given by equations (2.8) and (2.9). The weight matrices for the parameters in this adjustment were formed based on the a-priori estimates of their standard deviations in equation (1.1).

The 'best' minimum variances estimates of the earth parameters, and their estimated standard deviations, due to the above adjustment, are then given (as also in Rapp, ibid, Table 3) by:

$kM_E = 3.986{,}003{,}4 \times 10^{14}\ m^3/sec^2$; $\qquad \sigma_{kM_E} = 0.000{,}002{,}3 \times 10^{14}\ m^3/sec^2$

$\gamma_e = 978{,}031.69$ mgals; $\qquad \sigma_{\gamma_e} = 0.77$ mgals

$J_2 = 1082.635 \times 10^{-6}$; $\qquad \sigma_{J_2} = 0.011 \times 10^{-6}$

$1/f = 298.256{,}36$; $\qquad \sigma_{1/f} = 0.001{,}47$

$R_o = 6{,}363{,}674.98$ m; $\qquad \sigma_{R_o} = 2.51$ m

$a = 6{,}378{,}139.00$ m; $\qquad \sigma_a = 2.51$ m

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