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STATISTICAL ENERGY ANALYSIS FOR DESIGNERS

PART I. BASIC THEORY

CAMBRIDGE COLLABORATIVE

PREPARED FOR

AIR FORCE FLIGHT DYNAMICS LABORATORY

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This report describes the theoretical development of Statistical Energy Analysis (SEA). The historical development of the subject and an extensive annotated bibliography are included to add perspective to the technical discussions. Those aspects of random vibration theory are reviewed first. Then, the coupled vibration of randomly excited resonators is discussed. Using these results as a basis, the SEA model of coupled systems is introduced, and the consequences of the model are derived. From these basic relationships, a scheme for the prediction of vibration levels in coupled, randomly excited, resonantly vibrating systems is developed.

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STATISTICAL ENERGY ANALYSIS FOR DESIGNERS

PART I. BASIC THEORY

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1-E

FOREWORD

The research described in this report was performed by Cambridge Collaborative, Cambridge, Massachusetts for the Aerospace Dynamics Branch, Vehicle Dynamics Division, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio under Contract F33615-72-C-1196, Project No. 1370, "Dynamic Problems in Flight Vehicles," and Task No. 137002 "Flight Vehicle Vibration Control." Mr. A. R. Basso and later Mr. J. D. Willenborg were the Project Engineers at the Air Force Flight Dynamics Laboratory.

Part I of this report covers the basic theory of statistical energy analysis methods while Part II covers the application of statistical energy analysis.

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This report describes the theoretical development of Statistical Energy Analysis (SEA). The historical development of the subject and an extensive annotated bibliography are included to add perspective to the technical discussions. Those aspects of random vibration theory are reviewed first. Then, the coupled vibration of randomly excited resonators is discussed. Using these results as a basis, the SEA model of coupled systems is introduced, and the consequences of the model are derived. From these basic relationships, a scheme for the prediction of vibration levels in coupled, randomly excited, resonantly vibrating systems is developed.

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GLOSSARY OF SYMBOLS

VARIABLES AND PARAMETERS

a	strength of impulse of force (2.1)
A	proportionality factor for power flow (3.1), wall area (3.3)
ΔA_k	area associated with lattice point in k-space (2.2)
b	modal susceptance (2.2)
B	mechanical susceptance (2.2), bending rigidity (2.3), power flow-proportionality factor (3.1)
c	wave speed (see subscript section)
CC	confidence coefficient (4.3)
d	mean free path (2.4), determinant of power flow equations (3.1)
D	system determinant for pure tone excitation (3.1)
$D(\Omega)$	directivity function for reverberant field (3.3)
E	total vibrational energy - kinetic plus potential
$\mathcal{E}$	energy density (3.3)
f	cyclic frequency
G	mechanical conductance (2.2), gyroscopic coupling parameter (3.1)
h	plate or beam thickness (2.2)
H	frequency response function (3.1)
I	wave intensity (2.4)
$J_o(x)$	bessel function (2.4)
k	wavenumber = 2π/wavelength

GLOSSARY OF SYMBOLS (CONTINUED)

K	mechanical stiffness
KE	kinetic energy (2.1)
ℓ	dimension of beam or rectangular plate (2.2)
$\ell(t)$	time varying load (2.1)
L	amplitude of sinusoidal load (2.1), projected length of junction (3.3)
m	mean value (4.2), surface density of wall (3.4)
M	mass
n	modal density
N	mode count in band $\Delta\omega$
p	pressure, usually of sound field
P	system perimeter (2.4), fourier transform of p (2.3)
PE	potential energy (2.1)
Q	quality factor (2.1)
r	damping of continuous system (2.2), distance from origin (4.2), ratio of limit of confidence interval to mean (4.3)
r_D	distance from source to reverberant field (2.4)
R	mechanical resistance, vector distance (2.4)
S	spectral density
t	time
T_R	reverberation time
U	volume velocity
v	mechanical velocity

GLOSSARY OF SYMBOLS (CONTINUED)

V	system volume (3.4), velocity amplitude (2.1)
w	beam width (4.1)
x	spatial coordinate
X	mechanical reactance
y	resonator displacement (2.1), distributed system displacement (2.2)
Y	mechanical admittance (2.2), modal displacement (2.2)
z	displacement of coupled mode (3.1), distance from neutral axis of beam (4.1)
Z	mechanical impedance
α	complex natural frequency (2.1)
γ	normalized gyroscopic coupling parameter (3.1), ratio of specific heats (4.1), incomplete gamma function (4.3)
Γ	gamma function (4.3), proportionality constant (3.2)
δ_{mn}	Kronecker delta; = 1 for m=n, 0 otherwise (2.2)
$\delta(x)$	delta function (2.2)
Δ	frequency bandwidth (2.1)
$\Delta\omega$	noise or averaging bandwidth (2.1)
ε	mechanical strain (4.1)
ϕ	probability density (4.3), wave orientation (4.1), vibration phase angle (2.1)
η	loss factor (2.1), coupling loss factor (3.2)
κ	radius of gyration (2.3), normalized stiffness coupling (3.1)

GLOSSARY OF SYMBOLS (CONTINUED)

λ	coupling parameter (3.1), square root of mass ratio (3.1)
μ	inverse of normalized variance (4.3), normalized mass coupling (3.1)
ω	radian frequency
Ω	wave orientation (3.3), solid angle (3.3), natural frequency of coupled mode (3.1)
Π	power
ψ	mode shape (2.2)
ρ	surface density (2.2), system density (2.2)
σ	standard deviation (4.2)
τ	transmissibility (3.3)
ξ	normalized frequency variable (2.2)

OPERATORS

Arg(...)	phase angle of
∂	partial differential
∇^2	Laplacian or nabla
max(...)	maximum value of
$\Lambda(\partial/\partial x)$	spatial differential operator representing elastic restoring forces
$\langle\ldots\rangle_{\nu}$	average with respect to parameter ν
i	rotator in complex plane by $\pi/2$

GLOSSARY OF SYMBOLS (CONTINUED)

SUPERSCRIPTS

M	moment
(b)	blocked
s	source

SUBSCRIPTS

ϕ	phase
b	beam, bending
p	plate
σ,α	modal order indices
1,2	system order indices
R	room, reverberant
g	group
o	ambient
coup	coupling
s	surface, source
D	direct

CHAPTER 1. THE DEVELOPMENT OF STATISTICAL ENERGY ANALYSIS

1.0 Introduction

This report is a presentation of the basic theory and procedure for application of a branch of study of dynamical systems called Statistical Energy Analysis, which we shall refer to as "SEA". The name SEA was coined in the early 1960's to emphasize certain aspects of this new field of study. Statistical emphasizes that the systems being studied are presumed to be drawn from statistical populations having known distributions of their dynamical parameters. Energy denotes the primary variable of interest. Other dynamical variables such as displacement, pressure, etc., are found from the energy of vibration. The term Analysis is used to emphasize that SEA is a framework of study, rather than a particular technique.

Statistical Vibration Analysis. Statistical approaches in dynamical analysis have a long history. In mechanics, we are most familiar with their application to the vibration that is random in time of a deterministic system. We shall use this analysis in parts of Chapters 2 and 3. It is useful to emphasize here that the important feature of SEA is the description of the vibrating system as a member of a statistical population or ensemble, not whether or not the temporal behavior is random.

Traditional analyses of mechanical system vibration of machines and structures have been directed at the lower few resonant modes because these modes tend to have the greatest displacement response in many instances, and the frequencies of excitation were fairly low. Of course, the vibration of walls and floors and their high frequency sound radiation have been of interest for a long time, but mechanical and structural engineers have been generally unaware of or unconcerned with this work. The advent of fairly large and lightweight aerospace structures, and their attendant high frequency broadband loads, has meant much more attention to higher order modal analysis for purposes of predicting structural fatigue, equipment failure and noise production.

A characteristic of higher order mode analysis, however, is basic uncertainty in modal parameters. The resonance frequencies and mode shapes of these modes show great sensitivity to small details of geometry and construction. In addition, the computer programs used to evaluate the mode shapes and frequencies are known to be rather inaccurate for the higher order modes, even for rather idealized systems. In light of these uncertainties, a statistical model of the modal parameters seems quite natural and appropriate.

If there is cause for statistical approaches from the nature of the dynamical problem, there is equal motivation from the viewpoint of application. Mechanical and structural designers are often faced with making environmental and response estimates at a stage in a project where structural detail is not known. These estimates are made for the qualification of equipments and the design of isolation, damping, or structural configurations to protect equipment and protect the integrity of the structure. Highly detailed analyses requiring specific knowledge of shape, construction, loading functions, etc., are not appropriate. Simpler statistical analytical estimates of response to environment that preserve parameter dependence (such as damping, average panel thickness, etc.) are appropriate to the designer's need at this stage.

Inspirations for the SEA Approach. There is experience in dealing with dynamical systems described by random parameters. Two notable examples that have served as "touchstones" in early developments of SEA are the theory of room acoustics, and statistical mechanics. Room acoustics deals with the excitation of systems of very many degrees of freedom (there may be over a million modes of oscillation of a good sized room in the audible frequency range) and the interactions between such systems (sound transmission through a wall is an example). The analyses are carried out using both modal and wave models. The very large number of degrees of freedom is an advantage from a statistical viewpoint -- it tends to diminish the fluctuations in prediction of response. We shall show why this is so in Chapter 4.

Statistical mechanics deals with the random motion of systems with either a few or very many degrees of freedom. However, it is random motion of a very special type, which we may call "maximally disordered." In this state of vibration, all modes, whether they resonate at frequencies near each other or are far apart, tend to have equal energy of vibration and to have incoherent motion. We must add the proviso, "ignoring quantum effects." The energy of the modes is, aside from a universal constant, the system temperature. The state of equal modal energy is spoken of as "equipartition of energy". In SEA, we sometimes make the equipartition assumption for modes that resonate in the same frequency band, but not for all modes.

Statistical mechanics, and its related science, heat transfer, also teach us that thermal (random vibration) energy flows from hotter to cooler systems, and that the rate of flow is proportional to temperature (modal energy) difference. In Chapter 3, we show that this result also applies to dynamical systems excited by broad band noise sources. Not only that, but since we also show in Chapters 3 and 4, that narrow band sources are equivalent to broad band sources when system averages are taken, the result can be generalized, with proper care, to pure tones.

The study of the statistical mechanics of electrical circuits shows that a resistor at known temperature is equivalent to a thermal reservoir (or temperature bath). The interaction of the dynamical system with this reservoir is represented as a white noise generator in circuit theory. We can turn the argument around and say that a damper (mechanical resistance element), in conjunction with a noise generator, represents a thermal reservoir, and we should not be surprised when certain "thermal" results develop from its analysis.

The Advantages and Limitations of Statistical Analysis. One advantage of statistical analysis of systems may be seen from the practical aspects of room acoustics. If one truly has 10^6 modes to deal with, even the m.s. pressure associated with each, changing with time as the flute gives way to the tympani, would be a hopeless mass of information to assimilate. What one does instead is to describe the field by a few coherent features of the modal pattern (direct field and a few early reflections) and the incoherent energy (reverberant field) totalled into a few frequency bands. Thus, instead of 10^6 measures of the sound field (which would be incomplete in any case without the coherence data), we are able to describe the sound field effectively by 10-20 measures.

The statistical analysis also allows for much simpler description of the system, whether one describes the field by modes or waves. In the former case, modal density, average modal damping, and certain averages of modal impedance to sound sources are required. In a wave description, such parameters as mean free path for waves, surface and volume absorption, and general geometric configuration are required. The number of input parameters is generally in balance with the number of measures (10-20) to be taken.

The most obvious disadvantage of statistical approaches is that they give statistical answers, which are always subject to some uncertainty. In very high order systems, this is not a great problem. Many of the systems we may wish to apply SEA to, however, may not have enough modes in certain frequency bands to allow predictions with an acceptable degree of certainty. To keep track of this, we may calculate the variance as well as the mean, and also calculate the confidence for prediction intervals. This is discussed in Chapter 4.

In addition to hard and fast computational problems, there are certain difficulties in the psychology of statistical methods. A designer is not dealing with a gas of complicated molecules in random collision -- he is concerned about predicting the structural response of a wing, for which he has engineering drawings, to a loading environment, for which he has flight data. Instead of following a deterministic calculation (probably computer based) it is suggested that he will get a "better" estimate if he represents the wing as a flat plate of a certain average thickness and total area! His incredulity may be imagined. But he must remember that his knowledge of the wing at the 50th mode of vibration may be just as well represented by the flat plate as it is by his drawings. Also, the answers he gets by SEA will be in a form that is usable to him, generally retaining parameter dependence that will allow him to interpret the effect of certain simple design changes on response level.

To close this introductory section, it may be revealing to quote from M.L. Mehta, a theoretical physicist concerned with applying statistical methods to large nuclei. "Statistical theory does not predict the detailed level sequence [resonance frequencies] of any one nucleus, but it does describe the...degree of irregularity...that is expected to occur in any nucleus....Here we shall renounce knowledge of the system itself...There is a reasonable expectation, though no rigorous mathematical proof, that a system under observation will be described correctly by an ensemble average.... If this particular [system] turns out to be far removed from the ensemble average, it will show that...[the system]...possesses specific properties of which we are not aware. This, then, will prompt us to try to discover the nature and origin of these properties." The problems we face seem to be universal.

1.1 A Brief Historical Survey

Beginnings. The earliest work in the development of SEA as an identifiable entity were two independent calculations in 1959 by R. H. Lyon and P. W. Smith, Jr. In England, on an NSF postdoctoral fellowship, Lyon calculated the power flow between two lightly coupled, linear resonators excited by independent white noise sources. He found that the power flow was proportional to the difference in uncoupled energies of the systems and that it always flowed from the system of higher to lower modal energy.

The other calculation was by Smith at BBN working under U.S. Air Force support. Smith calculated the response of a resonator excited by a diffuse, broad band sound field, and found that the response of the system reached a limit when the radiation damping of the resonator exceeded its internal damping, but that this limit did not depend on the precise value of the radiation damping.

This result of Smith's was somewhat surprising since many workers regarded an acoustic noise field simply as a source of broad band random excitation. When a resonator, excited by broad band noise, has its internal damping reduced to zero, the response diverges, i.e., goes to infinity. The limit involved in Smith's result was of course due to the reaction of the sound field itself on the resonator.

After Lyon joined BBN in the fall of 1960, it developed that Smith's limiting vibration amounted to an equality of energy between the resonator and the average modal energy of the sound field. The two calculations were consistent, and power would flow between resonators until equilibrium would develop. If the coupling were strong enough compared to internal damping, equipartition would result. But how specifically did the wave-field result of Smith relate to the two-mode interaction that Lyon had studied?

To answer this question, Lyon and Maidanik wrote the first paper that may be said to be an SEA publication, although the name SEA had not then been coined. This paper combined Lyon's work in England with extensions to make it able to deal with the kind of problem Smith had analyzed. Formulas for the interaction of a single mode of one system with many modes of another were developed, and experimental studies of a beam (few mode system) with a sound field (multi modal system) were reported. This work also showed the importance of the basic SEA parameters for response prediction: modal density, damping, and coupling loss factor.

Early Extensions and Improvements of the Theory. Almost immediately, the activity in SEA split along two lines. One line was the clarification of basic assumptions and improvement in the range of approximation to real system performance. The second line was the application of SEA to other systems. The earliest application was to sound-structure interaction, largely a result of Smith's work, but also because it seemed "obvious" that SEA would work best when a sound field, with all of its many degrees of freedom was involved. Very soon, however, applications were also made to structure-structure interactions.

One basic assumption in SEA had to deal with light coupling. How much of a restriction did this represent? Also, what were the uncoupled systems? Two independent studies, by Ungar and Scharton, showed that the light coupling assumption was unnecessary if the uncoupled systems were defined as the blocked system, meaning that the other system was held fixed while the system being considered was allowed to vibrate. Also, the energy flow relations were valid whether one was using the "blocked" energies of the system as the driving force or the actual energy of each subsystem with the coupling intact. Of course, the constant of proportionality would be different for these two calculations.

The question of quantifying the uncertainty in the prediction of energy flow was examined by Lyon, who developed a theory of response variance and prediction intervals for SEA calculations. The calculation of variance for structure-structure interactions in which relatively few modes participate in the energy sharing process was first included in a paper by Lyon and Eichler.

An important extension of the two system theory was made by Eichler, who developed predictions from energy distribution for three systems connected in tandem. The practical application of SEA often involves the flow of vibratory energy through several intervening "substructures" before it gets to a vibration sensitive area. It is important, therefore, to be able to predict the energy distribution for such cases.

Improvements in Range of Application. The earliest work on structure-structure vibration transmission was undertaken with Air Force sponsorship and was concerned with electronic package vibration. An early paper on this topic by Lyon and Eichler dealt with plate and beam interaction, very similar to an example discussed in Chapter 4, and two plates connected together. A subsequent paper on a three element, plate-beam-plate system by Lyon and Scharton made use of the earlier formulation of three element systems by Eichler and some plate edge-admittance calculations, also by Eichler.

The basic SEA theory was pretty much directly applicable to these new systems. The major problem was evaluation of modal densities and coupling loss factors for various interacting junctions between systems. For example, the radiation of sound by reinforced plates was evaluated by Maidanik, and a similar study of the radiation of sound by cylinders was undertaken by Manning and Maidanik. We have already mentioned the plate-edge admittance calculations of Eichler. These, along with earlier (pre-SEA) calculations of force and moment impedances of beams and plates have allowed a wide variety of structural coupling loss factors to be evaluated. Quite recently, a series of soil-foundation impedance evaluations by Kurzweil allows one to apply SEA to certain structure-ground vibration problems.

Modal densities of acoustical spaces have been studied for quite a long time. Also, the modal densities of some flat and curved panel structures pre-date SEA. However, the activity in SEA has motivated work in modal density evaluation. For example, the modal density of cylinders has been studied by Heckl, Manning, Chandiramani, Miller, and Szechenyi. The modal density of cones has been calculated by Chandiramani, and by Wilkinson for curved sandwich panels. Generally, modal density prediction has not been as difficult as the calculation of coupling loss factors.

Understanding and prediction of system damping has improved very little over the years since SEA began. In most part, the improvements that have taken place were not particularly related to SEA work, although the work by Heckl on plate boundary absorption and by Ungar and Maidanik on air pumping along riveted beams were generally related to SEA. Despite this work, our ability to predict damping in built-up structures is still based largely on empiricism.

Other Developments. We should also note certain developments that might be termed "sword into plowshares" activity. Most early applications of SEA have been aerospace related because that is where the problems have been (and still are) and DOD and NASA were paying the bills. There have been recent applications of SEA to certain architectural and building problems that are worthy of note. In England, Crocker and his associates have carried out studies of sound transmission through double septum walls, following up on some earlier work by White and Powell. More recently, Ver has applied SEA to the prediction of sound transmission through floors consisting of a main slab with another lighter slab floated on many small point springs. Rinsky has also studied sound transmission through double and single stud-reinforced walls using SEA.

Finally, the attempts to better understand the theoretical basis for SEA and the limiting effect of its assumptions on the range of applications has continued. The most notable effort along these lines has been by Bogdanoff and Zeman. Other work includes a recent thesis by Lotz and calculations comparing ideal deterministic with averaged systems by Scharton, Manning and Remington. So far, the attempts at further reducing the number of assumptions that one must make in SEA have not been very effective, but we must not be discouraged by this. Every step, even the small ones taken to this point, have been very productive in extending the reach and usefulness of SEA.

1.2 The General Procedures of SEA

In the following chapters, we will derive the basic equations of SEA and give motivation for the modeling and computational procedures. To provide an overview of this process, however, in this section we describe the way that SEA calculations are made, and the steps that are necessary to arrive at a prediction of response. Hopefully, this will provide a framework for a better understanding of the purpose of the later chapters and how the various elements of SEA fit together.

Model Development. In its simplest elements, SEA results in a procedure for calculating the flow and storage of vibrational energy in a complex system. The energy storage elements are groups of "similar modes". Energy input to each of the storage elements comes from a set of external (usually random) sources. Energy is dissipated by mechanical damping in the system, and transferred between the storage elements. The analysis is essentially that of linear R-C circuit theory with energy playing a role analogous to electric charge and modal energy taking the role of electrical potential. A typical SEA model is drawn in Fig. 1.1 and the analogous electrical circuit that might be used to represent it is shown in Fig. 1.2. We shall not make direct use of this electrical analogy in this report, but rather directly work with the simultaneous equations.

The fundamental element of the SEA model is a group of "similar" energy storage modes. These modes are usually modes of the same type (flexural, torsional, etc.) that exist in some section of the system which we may call a "subsystem" (an acoustic volume, a beam, a bulkhead, etc.). In selecting the modal group, we are concerned that it meets the criteria of similarity and significance. Similarity means that we expect the modes of this group to have nearly equal excitation by the sources, coupling to modes of other subsystems, and damping. If these criteria are met, they will also have nearly equal energy of vibration. Significance means that they play an important role in the transmission, dissipation, or storage of energy. Inclusion of an "insignificant" modal group will not cause errors in the calculations, but may needlessly complicate the analysis.

Input power from the environment, labelled Π_{in}, may result from a turbulent boundary layer, acoustical noise, or mechanical excitation. It is usually computed for some frequency band, possibly a one-third or full octave band. In order to evaluate Π_{in}, we need to know certain input impedances for the subsystem. The important requirement is that this input power not be sensitive to the state of coupling between subsystems. If it is, then the system providing the power (a connecting structure for example) has important internal dynamics and should be modeled as another subsystem.

The dissipation of power for each subsystem Π_{diss} represents energy truly lost to the mechanical vibration and will depend only on the amount of energy stored in that subsystem. It may be truly dissipated by friction or viscosity, or it may be merely radiated away into the air or surrounding structure. The important proviso is that this power cannot be returned to the system. If it can, then it is part of the power flow through coupling elements and will require the addition of another subsystem or coupling path to the diagram.

The transmitted power Π_{12}, represents the rate of energy exchange between subsystems 1 and 2. All energy quantities that we deal with here are time averaged values. There may be very large temporal variations in power flow between the subsystems, even larger than the average flow, but these are ignored. The transmitted power depends on the difference in modal energy of the two subsystems and the strength of the coupling between them.

Parameter Evaluation. Evaluation of the quantities that appear in Fig. I.1 and that are discussed in the preceding paragraphs require the evaluation of certain parameters, which we may call SEA parameters, that mostly predate SEA. We group them as "energy storage" and "energy transfer" parameters. Energy storage is determined by the number of available modes N_1, N_2...for each subsystem in the chosen frequency band $\Delta\omega$. The ratio of N to $\Delta\omega$ is called the modal density n of the subsystem and is frequently used in SEA calculations instead of the mode count N.

Energy transfer parameters include the input impedance to the system for the determination of input power, the loss factor, which relates subsystem energy to dissipated power, and the coupling loss factor relating transmitted power to subsystem modal energy. In the following paragraphs we provide a brief indication of how these parameters are usually evaluated.

Modal density may be measured by exciting the subsystem with a pure tone and gradually varying the frequency. As a resonance is encountered a maximum in response will occur. If a chart of response amplitude as a function of frequency is drawn, these peaks may be counted. This technique may miss some modes and is, therefore, not perfectly reliable. Experimental methods have been devised to reduce the number of modes missed, but the error cannot be completely eliminated.

The most commonly used way of evaluating modal density is simply to calculate it from theoretical formulas. Most systems have modal densities that may be calculated in terms of relatively simple gross parameters, such as overall dimensions, and the average speed of waves in the system. A few examples of the calculation of modal density are given in Chapter 2.

The dissipation of energy is measured by the loss factor of the system (defined in Chapter 2). Unless the dissipation mechanism is of a very particular type, it is more reliable to measure the loss factor than it is to calculate it. This is done by measuring the rate of energy decay in the system when the excitation has been removed, or by measuring the response bandwidth of individual resonance peaks. The relation of both of these measures to the loss factor is developed in Chapter 2.

The input power from the environment is sometimes measured, but more often calculated. It may be measured by isolating the subsystem of known damping (loss factor) and observing its response to the environment. By equating input to dissipated power, Π_{in} is known. The input power may be calculated by evaluating the load exerted on the subsystem by the environment and the impedance of the subsystem to this load. Such calculations may be quite involved, but a simple example is given in Chapter 2 for flat plates excited by a point force.

The coupling loss factor is the parameter governing transmitted power. It is defined in Chapter 3 and has been measured for some systems, although it is often calculated. It is also related to quantities that may have been calculated or measured for other reasons - the junction impedances of mechanical systems, the transmission loss of walls, the acoustic radiation resistance of a piston, and other similar measures. Since it in general depends on both subsystems, and the variety of systems of application for SEA is very large, the number of coupling loss factors that we may be concerned with increases as the square of the number of subsystems on the list. The better strategy would appear, therefore, to express the coupling loss factor where possible in terms of subsystem impedances, as we have done in Chapter 3, and then list the impedances.

Calculation of Response. After the model has been decided upon and the parameters evaluated, the final step is to calculate the response. The first part of this calculation is the evaluation of the vibrational energy in the various energy storage elements or mode groups. As noted above, this amounts to solving a set of linear algebraic equations, one equation to each subsystem. A simple example is worked out in Chapter 3. These average energies will depend on the various input power values, the modal densities, and the coupling and dissipative loss factors.

On the basis of the formalism adopted in Chapters 3 and 4, the most direct relation between vibrational energy and another dynamical variable is with the velocity of motion. By formulas developed in the analysis, the velocity may in turn be related to still other variables -- displacement, strain, stress, pressure, etc. These variables are still forms of a spatially averaged response. However, on the basis of system geometry and mode shapes, estimates can also be made of the spatial distribution of response.

The variability just referred to occurs even with the SEA assumption of equal energy for every mode in the subsystem. In addition, however, there is variability because we deal with a particular system in the laboratory, not an ensemble of systems. As noted in the introduction to this chapter, the ensemble average will not fit each member exactly, and we may expect some variation in the various parameters that we have been discussing. The analysis of these variations is an important part of the response prediction process and tells us how much reliance we may place on a response estimate based on the average behavior. The discussion of this topic is developed in Chapter 4.

1.3 Future Developments of SEA

The future development of SEA is, in large part, likely to be a continuation of certain features of its development to date, i.e., its application to a larger group of subsystems. This group will surely include new structures and seismic (ground vibration) systems, and possibly water waves and ship or offshore structures. Such applications will increase the glossary of SEA parameters and make SEA even more useful than it has been until now.

The wider use of SEA is also likely to involve a larger group of professionals within each subject area. This means that research workers and analysts, test engineers, and designers are all likely to use it as one tool among many that are available for dynamical analysis. We may expect to see SEA become more computerized, not only for the purpose of solving the simultaneous equations governing response energy, but also for evaluating coupling loss factors for very complex structures using finite element methods.

Finally, we may expect to see developments in the basic theory of the statistical analysis of systems that will greatly expand the conceptual framework of SEA. To be more specific, let us see how SEA has expanded upon the framework of statistical mechanics, and what may develop in this regard in the future.

Statistical mechanics represents the most disordered state of motion possible for a system. Every mode throughout the system has the same average energy. This includes modes in different subsystems and those that resonate at different frequencies. Thermodynamics is the macroscopic counterpart to this totally equilibrium system of analysis.

The possibility for different temperatures in different parts of a system is allowed in the kinetic theory version of statistical mechanics and in "non-equilibrium" thermodynamics. The modes within each subsystem all have the same energy, but the average modal energies of the subsystems may be different.

This latter situation is the one we deal with in SEA, except that we proceed even farther from equilibrium by allowing modes in different frequency bands to have differing energies. Basically, we can get away from this because the systems that we deal with are presumed linear, and modes that resonate in different frequency bands may be considered to be uncoupled from each other. Thus, SEA represents still another step away from the "maximally disordered" state described by statistical mechanics.

At this point, it is logical to ask whether there is another step to be made in this sequence toward a model that is less disordered than the current SEA model that would provide use with useful answers in situations in which SEA has limitations. One answer might be -- forget about this chain of logic, go all the way back to the deterministic system. That is a possible answer, but it may not be the most useful one.

The most glaring deficiency of SEA is its inability to deal with modal coherence effects. In simplest terms, modal coherence may lead to such phenomena as "direct waves," discussed in Chapter 2. It is not clear at present whether this represents the "next logical step" in the chain that we spoke of, but it bears examination.

A second deficiency of SEA is its assumption that all modes that resonate in $\Delta\omega$ are equally probable over that interval, and that their resonance frequencies are unaffected by the values of the resonance frequencies of the other modes. There have already been some developments on improving upon this hypothesis and we may expect more. We are on more solid ground in predicting developments here than we are in our concern for modal coherence effects.

1.4 Organization of the Report

This report is concerned with presenting the basic theoretical elements of SEA. We may cite these as:

(1) The theory of energy exchange between two resonators. This is a problem in random vibration theory.

(2) The extension of the two resonator interaction to the interaction of two systems having modal behavior. This extension requires that we introduce the basic idea of statistical populations of systems.

(3) The representation of system interactions by the SEA model with the attendant necessity to identify and evaluate SEA parameters.

(4) The calculation of average energy. This is exemplified by applying the formulas to some fairly simple cases.

(5) The calculation of variance in response and the use of variance to generate estimation intervals and their associated confidence coefficients.

Since the emphasis in the report is the explication of SEA, we have included generous discussion of the ideas of energy storage and transfer, statistical ensembles of systems, isolated vs. coupled systems, etc. Our intention is to provide the most clear cut statement on these matters that can be made at present. The reader will have to decide how well we have succeeded.

Since our attention is on SEA, however, we do not spend time on other subjects that we employ in the various examples. Thus, for example, we use the equations of bending motion of plates and beams, but we do not derive them. We also use certain input impedances to beams, plates and sound fields, but these are not derived. For the most part, such derivations are readily available to the student, and in any case, in the annotated bibliography, guidance is given to where this information may be found.

The reader will notice that no references are given in the text, which is an unusual procedure for a report. In adopting this procedure, we do not wish to deny credit to anyone, rather we have followed regular textbook practice of providing a bibliography, but not breaking the presentation with references. Partly this procedure is a matter of self-discipline; sometimes it is very tempting for an author to try to avoid a difficult matter by slipping in a reference.

Chapter 2 deals with some fundamental ideas in the energy of vibration of single and multi degree of freedom systems. The storage of kinetic and potential energy by modes in free and forced vibration, and the decay or rate of energy removal due to damping are discussed, partly as a review of this important subject and partly to establish some of the basic vocabulary of the work to follow. Also, since both modal and wave descriptions of vibration are employed, energy storage, dissipation, and flow in a wave description are reviewed also.

Chapter 3 is concerned with developing the theory of average power flow (average in both a temporal and ensemble sense) between single and multi degree of freedom systems. First, power flow between two simple resonators is calculated. The important ideas of a blocked system is introduced and power flow is calculated in terms of both blocked and coupled system energies. The idea of averaged modal interaction as a white noise source is also introduced.

The latter sections of Chapter 3 are the central to our discussion. The very important ideas of blocked systems, ensemble averages of system interactions, and the definition and use of the coupling loss factor are introduced here. The use of reciprocity for development of certain useful relations for evaluation of the coupling loss factor is also discussed. The chapter ends with some elementary applications of the SEA relations.

Chapter 4 completes this part of the report and is concerned with the problems of estimating response, based on the average energy distribution calculations. The first part of this problem is estimating response variables of more particular interest than energy -- displacement, stress, etc. The second is the development of estimation intervals and their associated confidence coefficients, particularly when statistical analysis of variance indicates that the standard deviation is an appreciable fraction of the mean.

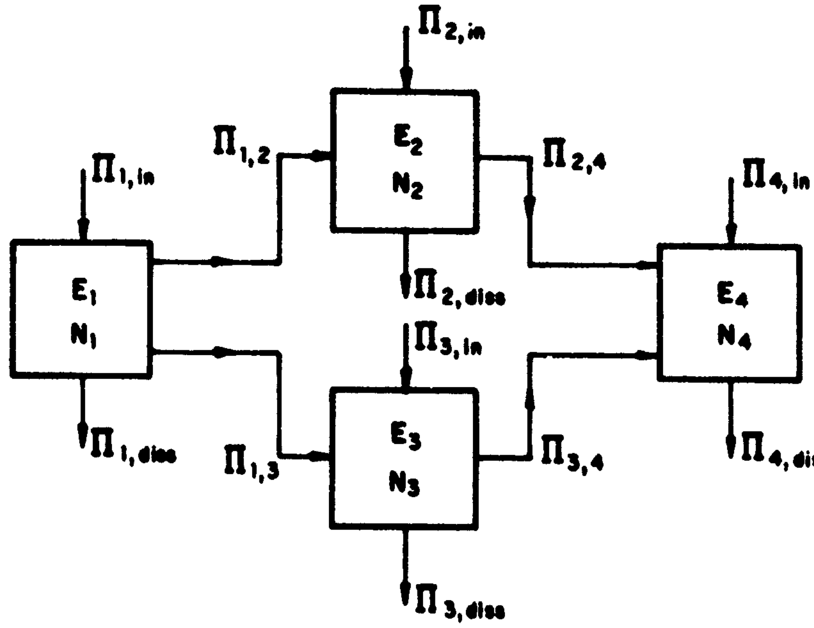

FIG 1.1

TYPICAL FORM OF SEA SYSTEM MODEL

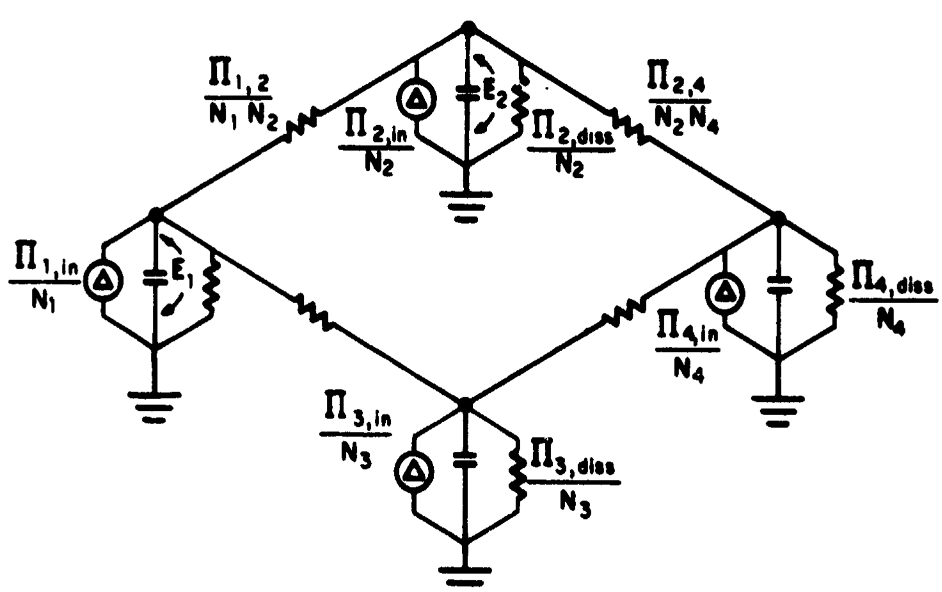

FIG 1.2

AN R-C ELECTRICAL CIRCUIT THAT MAY BE USED TO REPRESENT THE SYSTEM OF FIG 1.1

CHAPTER 2 ENERGY DESCRIPTION OF VIBRATING SYSTEMS

2.0 Introduction

This report is concerned with the exchange of energy in coupled vibrating systems. This chapter introduces the use of energy variables in describing the vibration of systems. Common ways of analyzing such systems employ modal oscillator or wave descriptions of the motion. A major goal of this chapter is to show how energy analysis applies to both of these models and how certain relations between them are revealed by the use of energy variables.

The chapter considers free vibration and both sinusoidal and random excitation of vibrating systems. This is a very large topic, but we are only interested in certain aspects of it. We specialize our interest to linear systems and to some interesting measures of response that are particularly related to vibrational energy. These measures are mean square response, spatial and temporal coherence and admittance functions.

The chapter begins with a discussion of the energetics of modal resonators in free vibration. The cases of sinusoidal and random excitation are studied next. The role of damping in causing decay of free vibration and bandwidth in forced vibration is of particular interest. Systems having many modes of vibration are studied next, and the notions of modal density and average admittance are introduced. At this point, the very important and pervasive idea of statistical modeling of real systems is introduced for the first time.

Vibrating systems modeled as collections of free and forced waves are introduced next. Certain concepts unique to such a description like energy velocity and wave impedance are introduced. Descriptors that are common to wave and modal descriptions such as mass and damping are also discussed. In paragraph (2.4) of this chapter, we point out some relationships between modal quantities, such as modal density and their wave descriptor counterparts, such as energy velocity.

The goal in this mode-wave interplay is to develop a way of thinking about vibrating systems that allows one shift back and forth between these two viewpoints, exploiting the one that is best suited to the problem at hand.

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2.1 Modal Resonators

To begin our discussion, we examine the energetics of the simple linear resonator shown in Fig. 2.1. We shall show later in this chapter that this system is a useful model of the dynamics of the modal amplitudes of multi-degree-of-freedom (dof) systems.

The dashpot or mechanical resistance R in Fig. 2.1 produces a force, -- $R\dot{y}$, opposite in direction to the velocity $\dot{y}$ of the mass M. The spring or stiffness element K produces a force -- Ky opposite in direction to displacement y from the equilibrium position of the mass. These forces, in combination with the force $\ell(t)$ applied by an external agent, results in an acceleration of the mass.

$$\ell(t) - R\dot{y} - Ky = M\ddot{y} \quad ,$$

or, more conventionally,

$$\ddot{y} + \omega_0\eta\dot{y} + \omega_0^2\, y = \ell(t)/M \qquad (2.1.1)$$

where $\omega_0 \equiv \sqrt{K/M}$ (natural radian frequency) and $\eta \equiv R/\omega_0 M$ (loss factor).

Free Vibration - No Damping. To study the case of free vibration, we set $\ell(t) = 0$ in Eq. (2.1.1). For the moment, we also neglect damping by setting $\eta = 0$. We then have

$$\ddot{y} + \omega_0^2 y = 0, \qquad (2.1.2)$$

which has the two solutions $\cos \omega_0 t$ and $\sin \omega_0 t$. The general solution, therefore, is

$$y + A \cos \omega_0 t + B \sin \omega_0 t = C \sin (\omega_0 t+\phi) \qquad (2.1.3)$$

We say that $\omega_0 = 2\pi f_0$ is the radian frequency of free, undamped oscillation of the resonator. The amplitudes A and B (or C and ϕ) are real numbers but otherwise arbitrary. They are determined if y and $\dot{y}$ are known at any time.

The kinetic energy of the mass at any time is

$$KE = \frac{1}{2} M \dot{y}^2 = \frac{1}{2} MC^2\omega_0^2 \cos^2(\omega_0 t + \phi) \tag{2.1.4}$$

and the potential energy in the stiffness element is

$$PE = \frac{1}{2} K y^2 = \frac{1}{2} KC^2 \sin^2(\omega_0 t + \phi) \ . \tag{2.1.5}$$

The sum of these is

$$E = KE + PE = \frac{1}{2} KC^2$$

which is time independent and dependent only on the peak amplitude of vibration. Since we have an isolated system vibrating without damping, it is evident that its vibrational energy should not vary in time.

The displacement and velocity repeat themselves in a period $1/f_0$. If we average the kinetic energy and the potential over such a period, we get

$$\langle KE \rangle_{\sim} = \langle PE \rangle = \frac{1}{4} KC^2 = \frac{1}{2} E \tag{2.1.6}$$

Thus, the time average kinetic and potential energies are both equal to 1/2 of the energy of vibration. We shall make considerable use of this relation in the work that follows.

Free Vibration with Damping. When damping is present, then $\eta \neq 0$ and we have

$$\ddot{y} + \omega_0 \eta \; \dot{y} + \omega_0^2 \, y = 0 \qquad (2.1.7)$$

and if we assume a form of solution $y \sim e^{\alpha t}$, we find that α must equal one of the two values

$$\alpha = - \frac{1}{2} \; \omega_0 \eta \pm i\omega_d \qquad (2.1.8)$$

where $\omega_d \equiv \omega_0 \sqrt{1-(\eta^2/4)}$. The solution for $y(t)$ in this case is then

$$y(t) = C \; e^{-\frac{1}{2}\omega_0 \eta t} \sin(\omega_d t + \phi). \qquad (2.1.9)$$

In this case, the oscillation occurs at radian frequency ω_d and the amplitude of the oscillations decreases exponentially in time due to the extraction of energy by damping. If the loss factor η is 0.5 or smaller, then ω_d is very nearly equal to ω_0 and the period of damped oscillations is essentially the same as that for undamped oscillations.

The potential energy of vibration in this case is

$$PE = \frac{1}{2} \, K \, y^2 = \frac{1}{2} \; KC^2 \, e^{-\omega_0 \eta t} \sin^2 (\omega_d t + \phi) \qquad (2.1.10)$$

The kinetic energy is a little more complicated. It is

$$KE = \frac{1}{2}\ M\ \dot{y}^2 = \frac{1}{2}\ M\omega_0^2\ e^{-\omega_0 \eta t} \left[-\frac{\omega_d}{\omega_0} \cos\ (\omega_d t + \phi) + \frac{1}{2}\ \eta\ \sin\ (\omega_d t + \phi) \right]^2 \qquad (2.1.11)$$

These expressions simplify considerably if we average over a cycle of oscillation, neglecting the slight change in amplitude in this period due to the exponential multiplier. The result is

$$\langle PE \rangle_{\sim} = \frac{1}{4}\ KC^2 e^{-\omega_0 \eta t} = \langle KE \rangle_{\sim} = \frac{1}{2}\ \langle E \rangle_{\sim} \qquad (2.1.12)$$

In free damped vibration, for which $\eta \leq 0.5$, we have the same relations between kinetic, potential, and total energy that we obtain for undamped vibration. Note that for both damped and undamped vibration

$$\langle y^2 \rangle_{\sim} = \langle \dot{y}^2 \rangle_{\sim} / \omega_0^2. \qquad (2.1.13)$$

The loss factor η is simply related to other standard measures of damping. For example, by its definition, $\eta = 1/Q$, where Q is the resonator quality factor, much used in electrical engineering. From Eq. (2.1.9), the amplitude decays as $e^{-\pi\eta t/T}$ and, therefore, $\pi\eta$ is the logarithmic decrement (nepers per cycle). Finally, the oscillations cease when $\eta \to 2$ so that η is twice the critical damping ratio.

From Eq. (2.1.12), we have

$$\langle E \rangle_{\sim} = E_0 e^{-\omega_0 \eta t} \qquad (2.1.14)$$

A measure of damping widely employed in acoustics is the reverberation time T_R, which is the time required for the vibrational energy to decrease by a factor of 10^{-6}. Thus,

$$e^{-\omega_0 \eta T_R} = 10^{-6}$$

which results in

$$T_R = \frac{2.2}{f_0 \eta} \quad . \tag{2.1.15}$$

Sinusoidal Forced Vibration. If the applied force $\ell(t)$ in Fig. 2.1 is sinusoidal at radian frequency ω, then it is convenient to use the exponential form

$$\ell(t) = |L|\cos(\omega t + \psi) = \mathrm{Re}\{|L|\exp[-i\omega t - i\psi]\} \tag{2.1.16}$$

where Re {...} means "real part of". The dynamical equations are linear and one can, therefore, consider response to the actual excitation as a combination of (complex) response to the complex excitation and its complex conjugate. This means that we may express the response variables in terms of complex exponentials also.

Accordingly, without loss of generality, we describe the complex force and velocity as

$$\ell(t) + L\, e^{\grave{}\, i\omega t}$$

$$\dot{y}(t) = V\, e^{-i\omega t} \tag{2.1.17}$$

where L and V are complex numbers of the form $|L|e^{-i\psi}$ and $|V|e^{-i\alpha}$, respectively. Differentiation and integration of these functions is particularly simple

$$\frac{d}{dt}\ell(t) = i\omega\, L\, e^{-i\omega t} = i\omega\ell$$

$$\int \ell(t)dt = (-i\omega)^{-1}L \quad e^{-i\omega t} = \ell/(-i\omega) \qquad (2.1.18)$$

Substituting for ℓ and $\dot{y}$ in Eq. (2.1.1), we obtain

$$L = V(-i\omega_0 M) \; \{(\omega/\omega_0-\omega_0/\omega) + i\eta\} \equiv VZ \qquad (2.1.19)$$

where Z is mechanical impedance of the resonator. This can also be expressed by an admittance $Y = 1/Z$,

$$V/L \equiv Y = \{\omega_0\eta\; M - i(\omega M - K/\omega)\;\}^{-1} \qquad (2.1.20)$$

Since the magnitudes of L and V are the peak force and velocity, we can graph Eq. (2.1.20) as shown in Fig. 2.2.

The average power supplied to the resonator by the source is $\Pi = \langle\ell\dot{y}\rangle_t$. When variables are described as complex variables, the time average of their product is easily expressed by their complex amplitudes, as follows:

$$\Pi = \langle\ell\dot{y}\rangle_t = \frac{1}{2}\; Re\;(LV^*) = \frac{1}{2}\; |L|^2\; Re(Y)^*$$

$$= \frac{1}{2}\; |V|^2\; Re\;(Z)\;. \qquad (2.1.21)$$

where ( )* denotes the complex conjugate. Since

$$\mathrm{Re}(Y) = \mathrm{RE}(1/Z) = \mathrm{RE}(Z)/|Z|^2 = \omega_0\eta\ M\ |Y|^2, \qquad (2.1.22)$$

the input power has the frequency dependence shown in Fig. 2.2. The maximum value is $\Pi = 1/2|L|^2/\omega_0\eta\ M = \langle \ell^2 \rangle_t/R$, the dissipation of a system with a resistance only.

At frequencies $\omega \neq \sqrt{K/M}$, the power diminishes, reaching 1/2 its value when

$$|Y|^2 = \frac{1}{2(\omega_0\eta M)^2} = \frac{1}{(\omega_0\eta M)^2\{1+(\omega^2-\omega_0^2)^2/\omega_0^2\ \omega^2\eta^2\}} \qquad (2.1.23)$$

which occurs when $\omega = \omega_0 \pm \omega_0\eta/2$ (assuming $\eta \leq 0.3$), as shown in Fig. 2.2. These frequencies are both the limits for half-power and the boundaries for simple forms of dynamical behavior. When $\omega < \omega_0(1-\eta/2)$, the admittance is adequately represented by neglecting R and M and keeping the stiffness term only

$$Y = -i\frac{\omega}{K} \quad [\omega < \omega_0(1-\eta/2)] \qquad (2.1.24)$$

we denote this region as "stiffness controlled".

For frequencies $\omega < \omega_0(1+\eta/2)$, the admittance is usefully approximated by the mass term only

$$Y \approx i/\omega M \quad [\omega > \omega_0(1+\eta/2)] \qquad (2.1.25)$$

The region is called "mass controlled". The intermediate region is called "damping controlled". These regions are noted in Fig. 2.2 also. These simplified limiting forms of behavior of the resonator are of great importance in our consideration of systems with many degrees of freedom.

It is possible to discuss sinusoidal excitation very extensively, and this is done in many textbooks. For our purposes here, however, we will only note some simple relations. First note that the relative phase of the velocity $\dot{y}$ with respect to the force f is the phase of the admittance function Y, which is the same as the phase of Z^*,

$$\operatorname{Arg}\{Z^*\} = \tan^{-1}|\{\omega - \omega_0^2/\omega\}/\omega_0\eta|$$

$$= \tan^{-1}|2(\omega - \omega_0)/\omega_0\eta| \ . \tag{2.1.26}$$

It is clear from this that the phase is changing rapidly as the system goes through resonance-the more quickly the smaller the damping. This is in contrast to the behavior of the amplitude which has a horizontal slope at $\omega=\omega_0$. For this reason, resonance can frequently be more accurately determined from examination of the phase rather than the amplitude of response.

The second point is concerned with mean square response at resonance:

$$\langle\dot{y}^2\rangle = \frac{1}{2} \; |V|^2 = \frac{1}{2} \; |L|^2/\omega_0^2\eta^2M^2 \qquad (\omega=\omega_0) \tag{2.1.27}$$

Since $y = (V/-i\omega)e^{-i\omega t}$ and $\ddot{y} = -i\omega \, V \, e^{-i\omega t}$, then when $\omega=\omega_0$,

$$\langle y^2\rangle = \frac{1}{2} \; |V|^2/\omega_0^2 = \langle\dot{y}^2\rangle/\omega_0^2$$

$$\langle y^2\rangle = \frac{1}{2} \; \omega_0^2|V|^2 = \omega_0^2\langle\dot{y}^2\rangle, \tag{2.1.28}$$

the same relations as found for free vibrations. These relations do not hold outside the damping controlled region, however, but they do hold to an acceptable degree of accuracy for $|\omega - \omega_0| < 1/2 \quad \omega_0\eta$.

Random Excitation. Most of the applications of SEA that concern us are situations in which the excitation by the loading environment is random. We should emphasize here, however, that the critical feature of SEA is that we assume a statistical model for the system being excited, not necessarily for the excitation. Thus, it is perfectly proper to apply SEA to systems excited by pure tones if a statistical model of the system and the description of its response using energy variables are appropriate. Nonetheless, the existence of random excitation generally means that much less averaging of system parameters is necessary and consequently less variability of response from the calculated mean in any particular situation may be expected.

There is no single satisfactory definition of a stationary random signal from all viewpoints, but one that has particular appeal from an experimental viewpoint can be readily developed. Imagine a filter having the frequency response shown in Fig. 2.3 and that the load function $\ell(t)$ is applied to the input of this filter. Now let the bandwidth Δf of the filter become very small. If the mean square output of the filter becomes proportional to Δf, then the force $\ell(t)$ is random. Note that a pure tone does not satisfy this requirement since the mean square output would be independent of bandwidth as long as the frequency of the tone were in the pass band of the filter. A similar statement can be made for any deterministic, periodic signal.

This statement not only supplies a definition of a stationary random signal that fits out intuition and is mathematically respectable, it also provides a direct indication of how the frequency decomposition of a random signal is effected. The mean square (m.s.) force corresponding to the band of frequencies is, therefore:

$$\langle \ell^2 \rangle_{\Delta f} = S_\ell \, \Delta f \qquad (2.1.29)$$

S_ℓ being the factor of proportionality. It is usually the case that this proportionality factor depends on the center frequency f, so it is written $S_\ell(f)$, and it is called the power spectral density (psd) of the random variable $\ell(t)$.

Let us suppose that the psd of $\ell(t)$ has been determined for all values of f and the result has been plotted as in Fig. 2.4. We also suppose that $\ell(t)$ is applied to two filters like the one in Fig. 2.3, except they are centered at frequencies f_1 and f_2. We now combine the output of the two filters to yield

$$\langle \ell^2 \rangle = S_\ell(f_1)\,\Delta f + S_\ell(f_2)\,\Delta f , \qquad (2.1.30)$$

since the mean squares of two time functions containing different frequency components are simply added together to form the mean square of the sum. Proceeding in this way, the mean square output of a filter that has unity gain from frequency f_1 to f_2 is simply

$$\langle \ell^2 \rangle = \int_{f_1}^{f_2} S_\ell(f)\,df , \qquad (2.1.31)$$

or, if the filter has a gain G(f) instead of unity, the dependence is

$$\langle \ell^2 \rangle = \int_{f_1}^{f_2} S_\ell(f)\,G(f)\,df. \qquad (2.1.32)$$

The total unfiltered m.s. value of the loading function is found by setting $f_1 = 0$ and $f_2 = \infty$ in Eq. (2.1.31).

Let us suppose that the loading force in Eq. (2.1.1) is a noise excitation having a psd S_ℓ that is constant. Such a noise signal is called "white noise". The m.s. force produced within a very narrow frequency band df is, therefore:

$$\langle \ell^2 \rangle_{df} = S_\ell\,df. \qquad (2.1.33)$$

The m.s. velocity response of the resonator to this force at frequency f is given by

$$\langle\dot{y}^2\rangle_{df} = S_\ell df\ |Y|^2 \tag{2.1.34}$$

According to Eq. (2.1.32), therefore, the psd of $\dot{y}$ is given (aside from a constant) by Fig. 2.2, and the total m.s. velɔcity is found from

$$\langle\dot{y}^2\rangle = \int_0^\infty S_\ell |Y|^2 df = S_\ell \int_0^\infty |Y|^2 df \tag{2.1.35}$$

for white noise.

It is clear from the form of Fig. 2.2 that most of the contribution to the integral in Eq. (2.1.35) will come from the region denoted "damping controlled". This observation leads to a useful concept of "equivalent bandwidth", Δ_e. This is the bandwidth of a system with a rectangular pass band (as in Fig. 2.3) that has a constant admittance determined by the damping along, $Y_{eq} = (\omega_0 \eta M)^{-1}$, that has the same response to the white noise excitation that the actual system does. Thus, by this definition,

$$\begin{aligned}\langle\dot{y}^2\rangle &= S_\ell \Delta_e (\omega_0 \eta M)^{-2} \\ &= \frac{1}{2\pi} S_\ell (\omega_0 \eta M)^{-2} \int_0^\infty \frac{d\omega}{1+(\omega^2-\omega_0^2)^2/\eta^2\omega^2\omega_0^2}\end{aligned} \tag{2.1.36}$$

This integrand is simplified by changing variables to $\xi=2(\omega-\omega_0)/\eta\omega_0$ and noting that the greatest contribution to the integrand is at $\omega=\omega_0$ or $\xi=0$, Thus,

$$\int_{0}^{\infty} \cdots \rightarrow \frac{\eta\omega_0}{2}\int_{-\infty}^{\infty} \frac{d\xi}{1+\xi^2} \quad \frac{\pi\eta\omega_0}{2}$$

and, therefore,

$$\Delta_e = \frac{\pi}{2} \eta f_0 , \tag{2.1.37}$$

We note that this bandwidth is greater than the "half power" bandwidth in Fig. 2.2 by the factor of $\pi/2$. The "replacement" of the resonator by this equivalent filter becomes a very useful approximation in many situations.

There is an alternative time domain definition of white noise that is useful. In this case, we consider the force to be made up of a series of impulses of strength $\pm a$, occurring randomly in time as shown in Fig. 2.5. When this excitation is applied to the system in Fig. 2.1, the mass is given a sequence of changes in velocity. Let each change in velocity be $\Delta v = a/M$. These occur randomly in time with random sign. The corresponding change in energy of vibration is

$$\Delta E = \frac{1}{2}M\{(y+\Delta v)^2-\dot{y}^2\}= M \dot{y} \Delta v + \frac{1}{2}M(\Delta v)^2 \tag{2.1.38}$$

If we average this over a sequence of impulses, the term $\langle y\Delta v\rangle$ will vanish since a positive Δv is just as likely as a negative one. The average energy increment is, therefore, $1/2\ M \langle\Delta v\rangle^2 = a^2/2M$.

If the average rate of impulses per second is ν, then the power fed into the resonator is $\nu a^2/2M$. But this must also equal the dissipated power $\langle y^2\rangle \ \omega_0\eta$ M. We have, therefore, an estimate for the m.s. velocity,

$$\langle\dot{y}^2\rangle = \nu a^2/2\omega_0\eta M^2 = S_\ell \Delta_e (\omega_0\eta M)^{-2}, \tag{2.1.39}$$

which leads to $S_\ell(f)=2\nu a^2$, which is a constant. More importantly, however, this relation gives us an additional physical interpretation of the spectral density function for white noise.

If we return to Eq. (2.1.36), the expression for m.s. displacement is simply

$$<y^2> = \frac{1}{2\pi} S_\ell \; (\omega_0 \eta M)^{-2} \int_0^\infty \frac{d\omega}{\omega^2 [1+(\omega^2-\omega_0^2)^2/\eta^2\omega^2\omega_0^2]}$$

(2.1.40)

and if the same assumptions are made regarding the level of damping are made as previously, we get

$$<y^2> \;\; = <\dot{y}^2> / \omega_0^2 \quad , \tag{2.1.41}$$

If we try to calculate the m.s. acceleration in the same way, we run into difficulty, since the integral

$$<\ddot{y}^2> \;\; = \frac{1}{2\pi} S_\ell (\omega_0 \eta M)^{-2} \int_0^{f_{max}} \frac{\omega^2 d\omega}{[1+(\omega^2-\omega_0^2)^2/\eta^2\omega^2\omega_0^2]}$$

(2.1.42)

will not converge as $f_{max} \to \infty$. For large ω, the integrand is simply $\omega_0^2\eta^2$. We can solve the problem by "subtracting out" this part, and obtain

$$<\ddot{y}^2> = \omega_0^2 <\dot{y}^2> + S_\ell \; f_{max}/M^2 \tag{2.1.43}$$

where the first term represents the damping controlled acceleration and second part is the mass controlled acceleration. In most instances of random excitation, the resonant part will dominate and we can simply use

$$\langle y^2 \rangle \approx \omega_0{}^2 \langle \dot{y}^2 \rangle \tag{2.1.44}$$

We have seen that free vibration and the resonant controlled response to sinusoidal and random excitation all produce the same relations between m.s. displacement, velocity and acceleration. This is very useful to us since it allows us to transform from one response variable to another without concern for the exact nature of the excitation as long as the general requirements of resonant dominated response for the validity of these relations are met.

2.2 Modal Analysis of Distributed Systems

The systems of interest to the mechanical designer are much more complicated than a linear resonator. In real systems, the stiffness, inertia and dissipation are all distributed over the space occupied by the structure. A displacement of the system that increases the potential energy is resisted by the elastic restoring forces. A rate of change in the displacement is resisted by the damping forces, and these forces, along with the loading excitation cause acceleration of the mass elements.

If we represent the generalized displacement of the system by v, then an equivalent statement of the above is

$$\rho\ddot{y} + r\dot{y} + \Lambda y = p, \tag{2.2.1}$$

where ρ is the mass density, r is a viscous resistance coefficient and Λ is a linear operator consisting of differentiations with respect to space. In the case of a flat plate, for example, ρ is the mass per unit area of the plate and $\Lambda = B\,\nabla^4$, where B is the bending rigidity. The use of simple viscous damping is a valuable simplification and does not affect the utility of our results as long as the damping is fairly small.

When this system is bounded, with well defined boundary conditions, the solution is frequently sought by expansion in eigenfunctions ψ_n, which are solutions to the equation

$$\frac{1}{\rho} \Lambda \psi_n = \omega_n^2 \psi_n \tag{2.2.2}$$

where the functions ψ_n satisfy the same boundary conditions that y does, and the quantities ω_n^2 are determined by the boundary conditions. The response and the excitation are then expanded in these functions

$$y = \sum_n Y_n(t)\ \psi_n(x)$$

$$p/\rho = \sum_n L_n(t)\ \psi_n(x) \tag{2.2.3}$$

If we multiply Eq. (2.2.2) by ψ_m and integrate over the region of the structure and subtract from this the same equation with the indices reversed, we get

$$\int \{\psi_m \Lambda \psi_n - \psi_n \Lambda \psi_m\}\ dx = (\omega_n^2 - \omega_m^2) \int \psi_m \rho(x) \psi_n\ dx. \tag{2.2.4}$$

When $n = m$, this is satisfied in a trivial fashion. When $n \neq m$, there are certain specific (but nevertheless very useful) conditions under which the differential operator has the property that

$$\int (\psi_m \Lambda \psi_n - \psi_n \Lambda \psi_m)\ dx$$

will vanish. This means that

$$\int \psi_m \rho\ \psi_n\ dx$$

must vanish, a condition that is referred to as orthogonality of the eigenfunctions ψ_m with a weighting function ρ.

A convenient normalization of the amplitude of the eigenfunctions is simply

$$\frac{1}{M}\int \psi_m \rho \, \psi_n \, dx \equiv \langle \psi_m \psi_n \rangle_\rho = \delta_{m,n} \tag{2.2.5}$$

which we may think of as a mass density weighted average of the product $\psi_m\psi_n$. For systems in which the mass density is uniform, this becomes a simple average over the spatial coordinates.

If we now place the expansions of Eq. (2.2.3) into Eq. (2.2.1), we obtain

$$\rho \sum_n \left(\ddot{Y}_n + \frac{r}{\rho}\dot{Y}_n + \omega_n^2 Y_n\right) \psi_n = \rho \sum_m L_m \psi_m \tag{2.2.6}$$

We can simplify this immediately if we also assume that $r(x)$ is proportional to $\rho(x)$; $r = \Delta\rho$. We can do this on two counts. To take a somewhat cynical view, since we have introduced the damping in a rather ad hoc fashion, we can feel free to configure it any way we like. The other basis for the assumption is that research studies of this problem have shown that the consequence of this assumption is to ignore a degree of inter-modal coupling that is less significant than other forms of coupling that we will be concerned about.

We now multiply Eq. (2.2.6) by $\psi_n(x)$ and integrate over the system domain. Using Eq. (2.2.5), we obtain

$$M\{\ddot{Y}_m + \Delta\dot{Y}_m + \omega_m{}^2 Y_m\} = L_m(t) \tag{2.2.7}$$

Thus, each modal response amplitude obeys the equation of a linear resonator of the sort discussed in paragraph 2.1. This result, in conjunction with the spatial orthogonality of the mode shapes according to Eq. (2.2.5), leads us to the concept of a complex dynamical system as a group of independent resonators of mass M, stiffness $\omega_m^2 M$ and mechanical resistance $M\Delta$.

Since there is a one-to-one correspondence between the modes and the ω_n's, we can use the latter to keep track of the modes. An example should be helpful here. For the two-dimensional, simply supported, isotropic and homogeneous rectangular flat plate of dimensions ℓ_1 by ℓ_2, drawn in Fig. 2.6, the mode shapes are

$$\psi_{n_1,n_2} = 2 \sin \frac{n_1 \pi x_1}{\ell_1} \sin \frac{n_2 \pi x_2}{\ell_2} \tag{2.2.8}$$

and ω_n^2 is

$$\omega^2_{n_1,n_2} = \left[\left(\frac{n_1\pi}{\ell_1}\right)^2 + \left(\frac{n_2\pi}{\ell_2}\right)^2\right]^2 \kappa^2 c_\ell^2$$

$$\equiv (k_1^2+k_2^2)^2 \kappa^2 c_\ell^2 \equiv k^4 \kappa^2 c_\ell^2 \tag{2.2.9}$$

where κ is the radius of gyration of the plate cross-section, $c_\ell = \sqrt{Y_p/\rho_m}$ is the longitudinal wavespeed in the plate material, where Y_p is the plate Young's modulus and ρ_m is the material density and n_1 and n_2 are integers.

Eq. (2.2.9) suggests a very convenient ordering of the modes, which is shown in Fig. 2.7. By inspection, we can see that each point in this "wave-number lattice" corresponds to a mode. Further, the distance from the origin to that point will determine the value the resonance frequency of the mode ω_n. The main value of this ordering is that it enables us for example to count the modes that will resonate in some frequency interval. It will not always be as convenient as it is in this case, but the ordering by some parameter is a necessary condition to allow us to do the counting at all.

When the ordering indices form a lattice as shown in Fig. 2.7, then each lattice point corresponds to an area $\Delta A_k = \pi^2/A_p$, where $A_p = \ell_1 \ell_2$ is the area of the plate. As we increase the wavenumber from k to $k + \Delta k$, we include a new area $1/2\ \pi k \Delta k$. On the average, this will include $1/2\ \pi k \Delta k/\Delta A_k$ new modes. Thus, the average number of modes per unit increment of wavenumber is

$$n(k) = \frac{\pi k \Delta k}{2\Delta A_k \Delta k} = \frac{\pi k}{2\Delta A_k} \qquad (2.2.10)$$

which we may call the modal density in wavenumber.

To find the modal density in frequency $n(\omega)$, which tells us how many modes on the average are encountered when we increase frequency by 1 unit, we use the relation $n(\omega)\Delta\omega = n(k)\ \Delta\ k$, or

$$n(\omega) = \frac{\pi k}{2\Delta A_k} \cdot \frac{\Delta k}{\Delta \omega} = \frac{\pi k}{2c_g \Delta A_k} \qquad (2.2.11)$$

where we have used the result from elementary physics that $\Delta\omega/\Delta k$ is the group velocity for waves in a system that has a phase velocity ω/k.

For a flat plate, the group velocity c_g is twice the phase velocity $c_b = \omega/k = \sqrt{\omega \kappa c_\ell}$, so that the modal density in cycles per second (hertz) is

$$n(f) = n(\omega)\frac{d\omega}{df} = \frac{2\pi^2 \omega A_p}{4c_b^2 \pi^2} = \frac{A_p}{2\kappa c_\ell} = \frac{\sqrt{3} A_p}{h c_\ell} \qquad (2.2.12)$$

where we have used $\kappa = h/2\sqrt{3}$, the radius of gyration for a homogeneous plate of thickness h. Note that the modal density in this case is independent of frequency. As an example, consider a plate that has an area of 10 ft.^{2} and a thickness of 1/8 in. (approximately 10^{-2} ft.). Then, since $c_\ell \simeq$ 17,000 ft./sec. (for steel or aluminum), we have

$$n(f) = \frac{(1.7)\cdot 10}{(.01)\ (17{,}000)} \simeq 0.1 \text{ mode/Hz} \qquad (2.2.13)$$

For this plate, one mode is encountered on average whenever the excitation frequency is increased by 10 Hz. A plate with a larger area or smaller thickness will have a modal density that is greater than this.

The kinetic energy of vibration is

$$\frac{1}{2}\int dx\ \rho\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\int dx_{m'}\ \Sigma_n \dot{Y}_m(t)\dot{Y}_n(t)\rho\psi_m(x)\psi_n(x)$$

$$= \frac{1}{2} M \sum_m \dot{Y}_m^2(t) \ , \qquad (2.2.14)$$

where we have used the orthogonality relation. Thus, the kinetic energies of the modes add separately to give the KE of the system. Since the kinetic and potential energies of each resonator are equal at resonance

$$\left(\langle \dot{y}_n^2 \rangle = \omega_n^2 \ \langle y_n^2 \rangle\right) \ ,$$

the total energies of the modes simply add to form the total system energy.

Response of System to Point Force Excitation. We now imagine that a point force of amplitude L_0 is applied at a location x_s on the structure. If we solve for the modal amplitudes from Eq. (2.2.3), we get

$$L_m = \frac{1}{M} \int p\psi_m \, dx = L_0 \, \psi_m(x_s)/M \,, \tag{2.2.15}$$

where M is the system mass. If the excitation and response are proportional to $e^{-i\omega t}$, Eq. (2.2.7) becomes

$$M(\omega_n^2 - i\omega \quad \omega_n \eta - \omega^2) \; Y_n = L_0 \psi_n(x_s) \quad , \tag{2.2.16}$$

so that the formal expression for the response is

$$y(x,t) = \frac{L_o e^{-i\omega t}}{M} \sum_n \frac{\psi_n(x_s)\psi_n(x)}{\omega_n^2 - \omega^2 - i \;\; \omega_n \omega \eta} . \tag{2.2.17}$$

We shall look at some ways of simplifying this complicated result.

The velocity at the excitation point x_s is $-i\omega y(x_s)$. The ratio of this velocity to the applied force is the input conductance of the system.

$$\frac{-i\omega Y(x_s,\omega)}{L_o} = \frac{-i\omega}{M} \sum_n \frac{\psi_n^2(x_s)}{\omega_n^2 - \omega^2 - i\omega_n \eta}$$

$$\equiv G - iB \quad . \tag{2.2.18}$$

where G, the real part of the sum is the conductance and B, the imaginary part, is the susceptance. By rationalizing the complex in Eq. (2.2.18), we get

$$G = \sum_n a_n(x_s)g_n(\omega) \quad , \tag{2.2.19}$$

where $a_n = \psi_n^2(x_s)/\omega M\eta$ and $g_n=(\xi^2+1)^{-1}$, where $\xi=(\omega_n-\omega)/\omega\eta$. The susceptance is

$$B = \sum_n a_n(x_s)b_n(\omega) \tag{2.2.20}$$

where $b_n=\xi(\xi^2+1)^{-1}$. In deriving these relations, we have assumed that the damping is small enough so that the individual modal admittance $a_n(g_n-ib_n)$ is quite sharp in frequency.

The total admittance Y is a rapidly fluctuating function of frequency. We can simplify this result however, if we consider averages of Y with respect to the variable ξ. Such an average is appropriate if we are exciting the system with a band of noise, since the response to noise having a uniform spectral density from ω_1 to ω_2 is the same as the average of the m.s. response to a pure tone as may be seen by referring to Eqs. (2.1.32) and (2.1.35).

Another possibility is to assume that the system itself is "random". That is, that the exact mode shapes and resonance frequencies are not known, either because of random irregularities in their construction, or because the detailed calculation procedures are not accurate enough to calculate them. In this case, we assume that the resonance frequencies ω_n are uniformly distributed over some frequency interval. Such an approach is one example of statistical modeling, a central theme in Statistical Energy Analysis. In the work that follows we take this latter approach, assuming that ω is fixed and that ω_n is the random variable.

If the interval of resonance frequency uncertainty is $\Delta\omega$, then if $\Delta\omega >> \frac{\pi}{2}\,\omega\eta$, the average over ω_n will give

$$\langle g_n \rangle_{\omega_n} \frac{\omega\eta}{2\Delta\omega} \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2+1} = \frac{\pi\omega\eta}{2\Delta\omega} \qquad (2.2.21)$$

which, interestingly enough, is simply the ratio of the modal bandwidth to the averaging bandwidth. In the interval $\Delta\omega$, the number of modes that can contribute to the average is $n(\omega)\Delta\omega$. Thus, from Eq. (2.2.19), the average conductance is

$$\langle G \rangle_{\omega_n, y_s} = \frac{n\Delta\omega}{\omega\eta\, M} \cdot \frac{\pi}{2\Delta\omega}\, \omega\eta \langle \psi^2 \rangle = \frac{\pi}{2} \frac{n(\omega)}{M} \qquad (2.2.22)$$

where the average on ψ^2 is over the mass distribution of the system, which is of course the same as a spatial average for a uniform mass density.

Eq. (2.2.22) is a useful and general result for multi-modal systems. In the case of a flat plate, $n(\omega)=A_p/4\pi\,\kappa c_\ell$, and $M=\rho_s A_p$, where ρ_s is the surface density of the plate. Then,

$$\langle G \rangle = (8\rho_s\, \kappa c_\ell)^{-1} \qquad (2.2.23)$$

which is also the admittance of an infinite plate. It very often happens that average impedance functions of finite systems are the same as those for the same system infinitely extended.

The average susceptance is of less interest, but we note that when the modal density is constant, then the average susceptance will vanish because the integral of $b_n(\xi)$ vanishes. When the modal density is not constant, the calculation is more complicated, and has been dealt with in the references.

Before leaving the discussion of impedance, let us consider the same problem from the point of view of noise excitation. If the force in Eq. (2.2.15) has a flat spectrum S_ℓ over the band $\Delta\omega$, then the power fed into any one mode can be found from the dissipation $\omega_0 \eta M \langle y^2 \rangle$ is, according to Eq. (2.1.39), just $S_\ell \psi_n^2(x_s)/4M$. If the bandwidth of the noise is $\Delta\omega$ (radians/sec) or $\Delta\omega/2\pi$ Hz then the number of modes randomly excited is $n\Delta\omega$ and the power input to the system, averaged over source location is

$$\langle \Pi \rangle \quad \frac{\pi}{2} \quad \frac{S_\ell \Delta\omega}{2\pi} \cdot \frac{n}{M} = \langle f^2 \rangle \quad \langle G \rangle \tag{2.2.24}$$

where again $\langle G \rangle = \pi n/2M$. Thus, we can view the conductance as a measure of the number of modes that are available to absorb energy from a noise source. On this basis, it is apparent that there should be a close tie between the average conductance and modal density. This relation can in fact be exploited as a way of measuring modal density when the modes are so closely packed or the damping is so large that counting resonance peaks is not feasible.

2.3 Dynamics of Infinite Systems

When the system is infinitely extended, then an alternative formulation is needed. The differential equation governing the motion is still given by Eq. (2.2.1). We now assume, however, that the mass and damping distributions are uniform (ρ and r = constant) and that the linear differential operator Λ is a simple polynomial in the spatial derivatives, $\Lambda(\partial/\partial x_i)$ with constant coefficients. With these assumptions, we may assume a "wave" solution to the equation for unforced motion in the form

$$y \sim e^{i(\vec{k}.\vec{x} - \omega t)} \quad . \tag{2.3.1}$$

With this substitution, the equation of motion becomes the "dispersion relation" between frequency and wave number:

$$-\rho\omega^2 - i\omega r + \Lambda(ik_i) = 0 \tag{2.3.2}$$

Let us consider some simple examples of Eq. (2.3.2). For the undamped string, ρ = lineal density, $r = 0$, and $\Lambda = -T(\partial/\partial x)^2$. Thus, Eq. (2.3.2) becomes

$$Tk^2 = \rho\omega^2, \text{ or } k = \pm\omega/c , \tag{2.3.3}$$

where $c=\sqrt{T/\rho}$ is the speed of free waves on the string. In the case of undamped bending motions of a thin beam, we would have

$$\Lambda(\partial/\partial x) = B\ (\partial/\partial x)^4 \tag{2.3.4}$$

where $B = \rho\ c_\ell^2\kappa^2$ is the bending rigidity of the beam. In this case the dispersion relation is

$$k^4 = \omega^2/\kappa^2 c_\ell^2 - (\omega/c_b)^4 \tag{2.3.5}$$

the same as Eq. (2.2.9). The parameter c_b is the phase velocity for bending waves on the beam. In the case of a two dimensional plate, the wave vector $\vec{k}$ in Eq. (2.3.1) has the two components (k_1, k_2). For either the beam or the plate, the phase velocity c_b is a function of frequency

$$c_b = \sqrt{\omega\ \kappa\ c_\ell} \tag{2.3.6}$$

and the system is said to be dispersive. When damping is included, the propagation constant k is complex and an attenuated wave results.

Energy variables are of great importance in infinite, free wave systems just as they are for finite, modal descriptions. In the infinite system, we are interested in energy density, the energy of vibration per unit length, area, or volume depending on the dimensionality of the system. It is shown in advanced textbooks that for the system that we are considering, the kinetic and potential energy densities of free waves are equal, so that the total energy density is just twice the kinetic energy density, which is $\rho(\partial y/\partial t)^2$.

The intensity I of a free wave is equal to the power flowing through a unit width (or area) of the system due to that wave as it propagates. If this power flows for 1 second, then the amount of structure that has filled with energy is numerically equal to $c_g = d\omega/dk$, the energy velocity. If $\mathcal{E}$ is the energy density, then the energy that passed the reference location is $\mathcal{E}c_g = I$. From the dispersion relation, Eq. (2.3.2) with r=0, the energy velocity is

$$c_g = \frac{d\omega}{dk} = \frac{i}{2\rho\omega} \quad \Lambda'(ik) \ . \tag{2.3.7}$$

Since $E = \rho(\partial y/\partial t)^2$,

$$I = -\frac{i\omega}{2} \ \langle y^2 \rangle_t \quad \Lambda'(ik) \ . \tag{2.3.8}$$

Let us consider some examples of the use of Eq. (2.3.8) for some familiar systems. In the case of the string, $\Lambda(ik) = -T(ik)^2$ and $\Lambda'(ik) = -2T(ik)$. Thus,

$$I_{string} = -\frac{i\omega}{2}\langle y^2 \rangle_t \ (-\rho c^2) \ 2ik = \rho c \ \langle (\partial y/\partial t)^2 \rangle_t \tag{2.3.9}$$

In the case of the beam, $\Lambda = \rho\ \kappa^2 c_\ell^2 (ik)^4$ and $\Lambda' = 4\rho\ \kappa^2 c_\ell^2 (ik)^3$. Thus

$$I_{beam} = \frac{i}{2\omega} < \left(\frac{\partial y}{\partial t}\right)^2 >_t 4\rho\ \kappa^2 c_\ell^2 (ik)^3 = 2\rho c_b < \left(\frac{\partial y}{\partial t}\right)^2 >_t \tag{2.3.10}$$

In both of these cases, the intensity is a mean square velocity of motion of the system times an impedance term of the form ρc, where ρ is the density and c is a wavespeed.

When damping is included, its most important effect is to cause the propagation constant to become complex. The new dispersion relation is found by the substitution,

$$k(\omega) \rightarrow k\left[\omega\left(1 + \frac{i\eta}{2}\right)\right] \quad , \tag{2.3.11}$$

Thus, for the string,

$$e^{ik_s x} = e^{i\frac{\omega}{c}[1+(i\eta/2)]x} = e^{i\frac{\omega x}{c}} e^{-\frac{\omega\eta}{2c} x} \quad \text{(string)} \tag{2.3.12a}$$

and for flexural motion of the beam ($k_b \equiv \omega/c_b$)

$$e^{ik_b x} = e^{i\sqrt{\frac{\omega(1+(i\eta/2))}{\kappa c_\ell}}\, x}$$

$$= e^{i\frac{\omega}{c_b} x} e^{-\frac{\omega\eta}{4c_b}} \quad \text{(beam)} \tag{2.3.12b}$$

From this, it is clear that the form of the dispersion relation will affect the rate at which the wave decays in space as damping is added. For a non-dispersive system, the attenuation is $2\pi\eta$ nepers or $27.3\,\eta$ dB per wavelength. It is clear that if we used the energy velocity, the attenuation factors in Eqs. (2.3.12a) and (2.3.12b) would be formally the same since $c_g = 2\,c_b$ for the plate.

Most of the structures of interest to structural designers consist of segments of beams and plates, so that we shall place most of our emphasis on such systems. In paragraph 2.2, we studied the impedance looking into a finite plate, and found that the average impedance was real and related to the modal density and the mass in a particularly simple fashion. We now want to carry out this same calculation for the infinite plate.

The two dimensional flat plate has the equation of motion Eq. (2.2.1) with

$$\Lambda = \rho\kappa^2\, c_\ell^2 \left[\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2\right]^2$$

The point force of strength $L_0 e^{-i\omega t}$ is assumed to act at x = 0. The problem is to calculate the motion y at x = 0 and form the ratio of $(\partial y/\partial t)$ to the force to find the admittance of the plate. We solve the problem by using two-dimensional Fourier transforms

$$p(\vec{x}) = \frac{1}{(2\pi)^2}\iint dk_1\, dk_2\, e^{i\vec{k}.\vec{x}} P(\vec{k})$$

$$P(\vec{k}) = \iint dx_1\, dx_2\, e^{-i\vec{k}.\vec{x}}\, p(\vec{x}) \tag{2.3.13}$$

with similar relations between y(x)and its transform Y(k). Since p(x) acts only at x = 0 with strength L_0, the second integral above is simple $P(k) = L_0$. Placing the transform into the equation of motion

$$[-\omega^2(1+i\eta)\rho + \Lambda(ik_1,ik_2)]\, Y(\vec{k}) = L_0 \tag{2.3.14}$$

which gives

$$y(0) = \frac{L_0}{(2\pi)^2}\iint dk_1\, dk_2 \quad \frac{1}{\Lambda(ik) - \omega^2\,(1+i\eta)\rho} \tag{2.3.15}$$

Since there is no azimuthal dependence in the (k_1,k_2) integration, we replace the area element by $dk_1\, dk_2 \rightarrow 2\pi k\, dk$ and have a simple integration over the magnitude k.

$$y(0) = \frac{L_0}{2\pi}\int_0^\infty k\, dk \, \frac{1}{\rho\kappa^2 c_\ell^2 [k^4 - k_b^4(1+i\eta)]}$$

$$= \frac{L_0}{4\pi\rho\kappa^2 c_\ell^2}\int_0^\infty d\xi \, \frac{1}{\xi^2 - \xi_b^2(1+i\eta)}$$

where $\xi = k^2$ and $\xi_b = k_b^2$. Thus,

$$y(0) = \frac{L_0}{8\pi\rho\kappa^2 c_\ell^2}$$

$$\times \int_{-\infty}^{\infty} \frac{d\xi}{[\xi - \xi_b(1+i\eta/2)]\;[\xi+\xi_b(1+i\eta/2)]} \tag{2.3.16}$$

or,

$$y(0) = \frac{L_0/(-i\omega)}{8\rho\kappa c_\ell}$$

allowing η to vanish, and the path of integration is taken as in Fig. 2.8. Thus, the ratio of velocity $-i\omega\, y(0)$ to the force F_0 is the infinite plate admittance

$$Y_\infty = (8\ \rho\kappa c_\ell)^{-1} \tag{2.3.17}$$

which is the impedance that we found for the finite plate when an average was taken over modal response and source location.

A mean square force $\langle \ell^2 \rangle$ applied at a point on an infinite plate will, therefore, inject an amount of power

$$\Pi_{in} = \langle \ell^2 \rangle\ Y_\infty \tag{2.3.18}$$

into the plate. This power will propagate at the energy velocity outward from the point of excitation as a circular wave. When a boundary is encountered, a reflection occurs and the energy propagates unimpeded until another boundary is encountered. The average distance traversed between reflections is called the "mean free path", d, and is given by $d = \pi A_p/P$, where P is the perimeter of the plate and A_p is the area of the plate. The attenuation rate (in space), according to Eq. (2.3.12b) is $\omega\eta/2c_g$ nepers per unit length, or $4.34\omega\eta$ dB per second (c_g is the energy velocity defined by Eq. (2.3.7)).

For systems in which the energy is contained in the propagation of free waves, the damping is frequently expressed in terms of the time (referred to as the reverberation time T_R) required for the energy to decay by 60 dB. Thus, $4.34\ \omega\eta\ T_R = 60$ or,

$$T_R = \frac{2.2}{f\,\eta}, \qquad (2.3.19)$$

a result that we found earlier for single dof and modal resonators. Measurement of decay rate or reverberation time is a commonly used procedure for determining loss factor. The decay rate procedure applies equally well whether our model of the system is one of a collection of modes or a group of waves rebounding within the system boundaries.

In Chapter 3, we will be concerned with the dynamics of interacting systems. As a preview, let us consider here the interaction between the single dof system and a finite plate shown in Fig. 2.9. We wish to calculate the mean square velocity of the resonator as a result of its attachment to a plate which is vibrating randomly.

The resonator consists of a mass M, stiffness K, and dashpot resistance R, configured as shown. A diffuse reverberant vibrational field (equal wave intensity in all directions) is assumed to exist on the plate, resulting in a transverse velocity v. At the point of resonator attachment x_s, the plate velocity is v_s. The velocity of the resonator mass is v_M. The reaction force ℓ resulting from compression of the spring K is due to a difference in velocities v_s and v_M:

$$\ell = K \int (v_s - v_M)\, dt = M \frac{dv_M}{dt} + R\, v_M \qquad (2.3.20)$$

The velocity v_s is equal to the velocity that would exist if there were no resonator v, less the "reaction" or "induced" velocity v_r that is induced by the force; $v_r = \ell\langle G\rangle$, where $\langle G\rangle$ is the plate admittance $(8\rho_s\, \kappa c_\ell)^{-1}$. In using $\langle G\rangle$, we assume that the modal density of the plate n_p is sufficiently high so that several plate modes resonate within the combined equivalent bandwidth of plate and resonator modes:

$$\frac{\pi}{2}\, \omega(\eta_p + \eta_0)\, n_p \gg 1.$$

We now differentiate Eq. (2.3.20) with respect to time to obtain

$$M \frac{d^2v_M}{dt^2} + R \frac{dv_M}{dt} + K\, v_M = K\, v - K\, \ell \, \langle G \rangle \qquad (2.3.21)$$

Substituting for ℓ from Eq. (2.3.20), we get

$$\frac{d^2v_M}{dt^2} + \omega_0(\eta_0+\eta_{coup}) \frac{dv_M}{dt} + \omega_0^2(1+\eta_0\eta_{coup})v_M=\omega_0^2 v \qquad (2.3.22)$$

where $\omega_0 \equiv \sqrt{K/M}$, $\eta_0 = R/\omega_0 M$ and η_{coup} is the combination of of parameters $\omega_0 M \langle G \rangle$. Eq. (2.3.22) is the equation for the response of a resonator to a random base excitation velocity v. The effects of the plate are expressed by modified damping and resonance frequency of the resonator. If v is assumed to have a flat spectrum $\langle v^2 \rangle/\Delta\omega$ over a bandwidth $\Delta\omega$, then from the discussion following Eq. (2.1.35), the response of the resonator is

$$\langle v_M^2 \rangle = \frac{\pi}{2} \frac{\omega_0}{\eta_0 + \eta_{coup}} \frac{\langle v^2 \rangle}{\Delta\omega} \qquad (2.3.23)$$

or,

$$M \langle v_M^2 \rangle = \frac{\eta_{coup}}{\eta_0 + \eta_{coup}} \frac{M_p \langle v^2 \rangle}{n_p \Delta\omega} \qquad (2.3.24)$$

The term $M_p \langle v^2 \rangle / n_p \Delta\omega$ is simply the vibrational energy of the plate divided by the number of modes that resonate in the band $\Delta\omega$. The ratio represents, therefore, the average energy per mode of vibration within that band. Since the ratio $\eta_{coup}/(\eta_{coup} + \eta_0)$ is always less than or equal to one, Eq. (2.3.24) says that the average energy of vibration of the resonator cannot exceed the average modal (resonator) energy of the plate. These resonator energies will in fact be equal when the loss factor due to the coupling between the resonator and the plate η_{coup} (also called the coupling loss factor) is large compared to the internal damping of the resonator. This tendency toward energy equality is an example of "equipartition of energy", a principle that is well known to students of statistical mechanics.

2.4 Modal-Wave Duality

In paragraphs 2.2 and 2.3 of this chapter, we have discussed modal and wave descriptions of finite systems. These two ways of describing the motion are in most cases fully equivalent to each other, but that does not mean that the choice of description is arbitrary. Certain aspects of structural vibration are much more readily interpreted in one description than in the other. The effect of damping at the boundaries of a plate is better described by wave reflection processes. The spatial variations in vibration amplitude are better described by the modal analysis. But we must emphasize, that it is always possible, at least in principle, to arrive at the same conclusions by either approach.

We have already demonstrated an important example of modal-wave duality, the average point conductance of finite plates. We have found that the conductance computed by averaging over modes of a finite plate is the same as that found by considering only waves radiated outward by driving an infinite plate at one point. The fact that these waves reflect from the boundaries and may return to the drive point is presumed to be unimportant in affecting the drive point impedance because they will have random phases, or be incoherent with the excitation, particularly if the drive point is assumed to be randomly located over the surface of the structure.

Another aspect of this duality is in the description of damping. We found in discussing resonators in paragraph (2.1) that the damping was closely associated with resonator (or modal) bandwidth and decay of vibrations in time. In discussing waves, in paragraph (2.2) we found that the damping was related to the spatial decay of energy. However, if this decay in space were related to the time required for the energy to propagate from one point to the other, the time decays in both descriptions were related to the loss factor in the same way. Thus, decay rate of energy forms a link between the two descriptions even though modal bandwidth is a very difficult concept to explain by a wave analysis and spatial decay of vibration is an equally difficult concept to explain using a modal description.

In the chapters that follow, particularly in examples dealing with applications of SEA, we find that certain quantities that enter the modal description for example, are equivalent to other quantities in the wave description. For example, modal energy is usually equivalent to a spatial energy density. For example, in a plate, the energy is $\rho_s\langle v^2\rangle$, whereas the average modal energy for the same plate is $(4\pi\ \kappa c_\ell)\ \rho_s\ \langle v^2\rangle$. In a sound field, the energy density is $\langle p^2\rangle/\rho c^2$, and the average modal energy for the same sound field is $(2\pi^2 c^3/\omega^2)\langle p^2\rangle\ \rho\ c^2$, where c is the speed of sound and p is the pressure fluctuation.

Other equivalences exist between coupling loss factors, appropriate for modal systems, and junction impedances, appropriate for wave descriptions. An example of this equivalence was used in the example of the combined plate-resonator systems of Fig. 2.9, where we found $\eta_{coup}\ \omega_0 M\ \langle G\rangle$. We shall encounter more examples of this in the chapters to come.

In the remainder of this section, we shall discuss a very important aspect of wave-mode duality. This is the problem of the "coherent" and "incoherent" wave fields in a plate. In paragraph 2.3 we began this discussion, but we wish to delve further into the matter at this point. Let us begin by considering the wave description first.

Let us suppose that a point force located at position x_s on a very large plate excites the plate in a bandwidth $\Delta\omega$. If the rms force is L, then the power injected into the plate at x_s is simply

$$\Pi_{in} = L^2 Y_\infty = L^2\langle G\rangle \qquad (2.4.1)$$

where $\langle G\rangle$ is given by Eq. (2.3.17). This energy flows radially outward from this point, so that at a distance r

from x_s, the mean square velocity is found from the intensity relation

$$\rho\, c_g \langle v_D^2 \rangle\, 2\pi r = \Pi_{in}. \tag{2.4.2}$$

This is the velocity that would exist on an infinite plate without damping. If damping is present, the energy is reduced by the factor exp $[-\eta\omega r/c_g]$, so that

$$\langle v_D^2 \rangle = \frac{\Pi_{in}}{2\pi\ \rho\ c_g}\ \frac{1}{r}\, e^{-\omega\eta r/c_g} \tag{2.4.3}$$

This part of the plate motion is called the "direct field" of the source and has a "geometric attenuation" of 3 dB per double distance and an attenuation due to damping that increases linearly with r. The direct field is the only contribution to plate vibration if the plate is infinite in extent or if η is so large that the vibration has nearly ceased by the time the direct wave reaches the boundary of the plate.

When the energy in the direct field reflects from the boundary, then if the boundary is not perfectly regular, it is frequently assumed that coherence with the direct field is lost, particularly as the number of reflections mounts. The motion v_R corresponding to this reflected energy is called the "reverberant" field and is determined by the power dissipated by it:

$$\Pi_R = \Pi_{in} \exp\ [-\omega\eta d/2c_g] = M \langle v_R^2 \rangle\ \omega\eta \tag{2.4.4}$$

where d is the mean free path. This leads to an expression for the reverberant velocity

$$\langle v_R^2 \rangle = \Pi_{in} \exp\ [-\omega\eta d/2c_g]/\omega\eta M. \tag{2.4.5}$$

Since the direct and reverberant velocity components are assumed to be incoherent, the total mean square velocity of vibration is obtained by simple addition,

$$\langle v^2 \rangle = \langle v_D^2 \rangle + \langle v_R^2 \rangle. \qquad (2.4.6)$$

Near the point of excitation, the direct field will dominate, but at large distances, the reverberant field will dominate. The "boundary" between the direct and reverberant fields is defined as the distance r_D for which these fields are equal

$$r_D = \frac{\omega\eta \; M}{2\pi \; \rho \; c_g} \qquad (2.4.7)$$

The total mean square velocity pattern is shown in Fig. 2.10.

We can also analyze this situation by a modal description. From paragraph 2.2, the response of a plate to a point force is

$$V = \frac{i \; \omega \; L_0}{M} \sum_m \frac{\psi_m(x) \;\; \psi_m(x_s)}{\omega^2(1+i\eta)-\omega_m^2} \qquad (2.4.8)$$

Let us assume that the plate is simply supported so that the mode shapes are

$$\psi_m(x) \;\; \psi_m(x_s)=\frac{1}{4} \prod_{i=1}^{2} \left(e^{ik_i x_i}-e^{-ik_i x_i}\right) \left(e^{ik_i x_i^s}-e^{-ik_i x_i^s}\right)$$

$$(2.4.9)$$

There are 16 terms in this product, each representing a plane wave having one of the force wave vectors shown in Fig. 2.11.

The phase factors are the "dot products" of these wave vectors with the four position vectors shown in Fig. 2.12. These position vectors are the difference vectors between the point source and the observation point and its three images in the coordinate axes.

If the drive point is excited at frequency ω, then because of the modal response bandwidth, we will assume that all modes within a band $\Delta\omega = \pi\omega\eta/2$ will be excited, and that modes outside this bandwidth are not excited. As we sum over the circle of excited modes in k-space (the quarter circle of Fig. 2.7 becomes a full circle because of the addition of the images of $\vec{k}$ in Fig. 2.11), some of the terms will fluctuate wildly in phase as we sum over the indices m_i, while others will combine because the phase variation between them is small. The smallest phase variation will occur for the smallest vector, $\vec{x} - \vec{x}_s$, as seen in Fig. 2.12. We shall assume that only these terms contribute to the "coherent" part of the summation in Eq. (2.4.8).

We, therefore, replace Eq. (2.4.9) by its coherent part

$$\psi_m(x)\ \psi_m(x_s) \approx \frac{1}{4}\, e^{i\vec{k}\cdot\vec{R}} \tag{2.4.10}$$

where $R \equiv \vec{x} - \vec{x}_s$, and $\vec{k}$ now varies over the complete circle in Fig. 2.11. If the phase variation in going from one lattice point (allowable wave) to another when summing over m is less than $\pi/2$, then the summation in Eq. (2.4.8) may be replaced by an integration over angle in k-space.

$$v(x) = -\frac{i\omega L_0}{4M} \sum_m \frac{\exp\left(\frac{i}{c}\,\omega_m \cdot R\cos\theta\right)}{\{\omega_m - \omega(1+i\eta/2)\}\ \{\omega_m + \omega(1+i\eta/2)\}} \tag{2.4.11}$$

where θ is the angle between $\vec{k}$ and $\vec{R}$.

Let us note that $M = \rho A_p = \rho\pi^2/\Delta A_k$. We can now write $\Delta A_k = k\ \Delta k\ \Delta\theta$, and the summation in Eq. (2.4.11) becomes

$$v(x) = -\frac{i\omega L_0}{4\pi^2\rho} \sum_m \frac{\exp\left(\frac{i\omega_m}{c} R\cos\theta\right) k\,\Delta k\,\Delta\theta}{\{\omega_m-\omega(1+i\eta/2)\}\{\omega_m+\omega(1+i\eta/2)\}}$$

$$\to -\frac{i\omega L_0}{4\pi^2\rho} \int_0^\infty \frac{k\,dk}{\{\omega_m-\omega(1+i\eta/2)\}\{\omega_m+\omega(1+i\eta/2)\}}$$

$$\times \int_0^{2\pi} \exp\left(i\frac{\omega_m}{c} R\cos\theta\right) d\theta. \qquad (2.4.12)$$

The second integral is simply $J_0(k\,R)$, while the first integral is approximately

$$v(x) = -\frac{i\,L_0}{4\pi^2\rho} \int_0^\infty \frac{k\,dk\,J_0(k\,R)}{\omega_m-\omega(1+\,i\eta/2)} = \frac{L_0}{2\rho c^2} J_0\left(\frac{\omega R}{c_b}\right) \qquad (2.4.13)$$

The condition that the phase between terms vary less than $\pi/2$ is most severly tested when $\vec{k}$ and $\vec{R}$ have the same direction ($\cos\theta=1$). The number or resonantly excited modes is $\pi/2\ \omega\eta n_s$. The average distance between these modes in k-space, therefore, is

$$2\pi k/\ \frac{\pi}{2}\ \omega\eta n_s = 4/n_s\eta c_b,$$

and our phase requirement is

$$4R/n_s\eta c_b < \pi/2; \quad R < \frac{\omega\ \eta M}{16\rho\ c_g}. \qquad (2.4.14)$$

which is consistent with, but more stringent than, Eq. (2.4.7).

The Bessel function $J_0(\omega R/c_b)$ in Eq. (2.4.13) has the asymptotic form

$$J_0 \to \sqrt{\frac{2c_b}{\pi\omega R}} \cos\left(\frac{\omega R}{c_b} - \frac{\pi}{4}\right) \tag{2.4.15}$$

so that the mean square velocity in the coherent waves varies as 1/R, which is the same as our result for the "direct field" in the wave model of the plate response.

The incoherent part of the velocity field is found from Eq. (2.4.8) by adding the mean square values of modal response incoherently. The mean square value of each term is

$$\frac{\omega^2 L_0^2}{2M^2} \cdot \frac{\pi}{2} \quad \omega\eta n_s \cdot \frac{1}{\omega^2\eta^2} = \frac{L_0^2}{2} \cdot \frac{1}{\omega\eta M} \cdot \frac{\pi}{2}\ \frac{n_s}{M}$$

$$= \frac{1}{2}\ L_0^2\ G/\omega\eta \quad M, \tag{2.4.16}$$

which is the same as the result in Eq. (2.4.5) if we assume that very little dissipation of the direct field occurs before the first reflection from a boundary.

We have shown here that the direct and reverberant fields that arise naturally in the wave analysis have their direct counterparts in the "coherent" and "incoherent" components of the summations of the modal description. Such correspondences are very useful in that they allow us to interpret some of the phenomena described by a wave picture in terms of their effects on a model analysis. For example, the transmission of vibrational energy through a connection between two plates will impose a degree of coherence between the modes of the receiving plate. But if the bulk of the vibrational energy may be judged to be in the reverberant field, we may treat the modes of the "source" and "receiving" structures to be incoherent with each other.

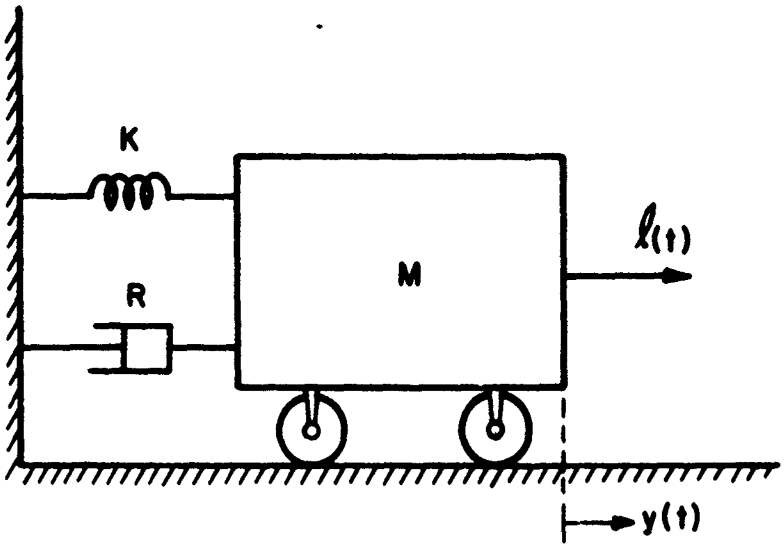


FIG. 2.1
LINEAR RESONATOR

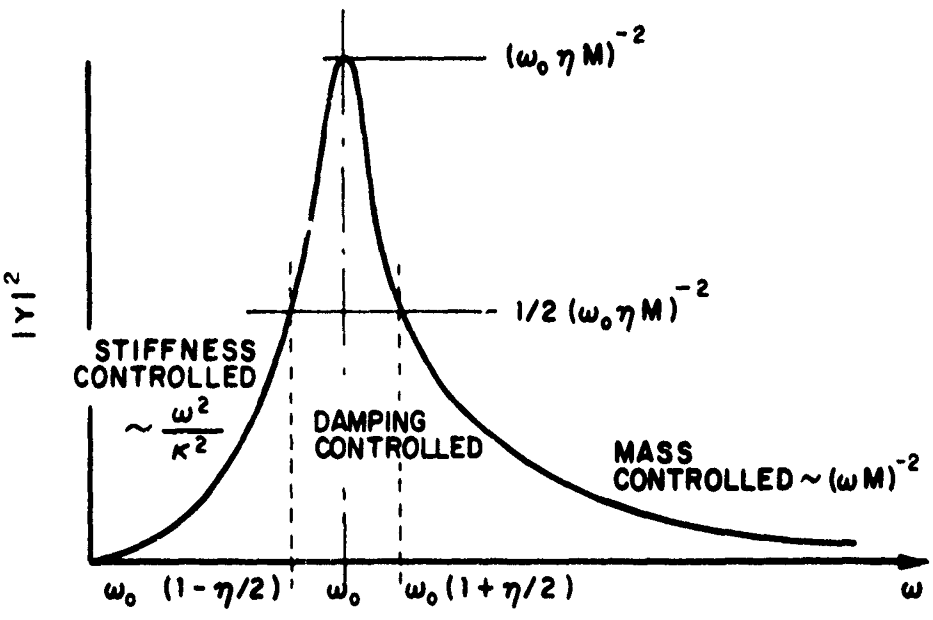


FIG. 2.2
ADMITTANCE OF LINEAR RESONATOR

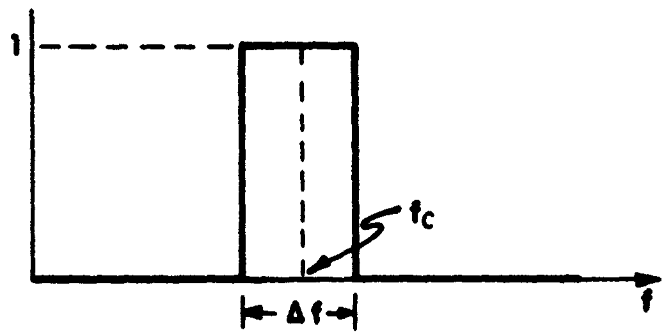


FIG 2.3

FREQUENCY RESPONSE OF IDEAL RECTANGULAR FILTER

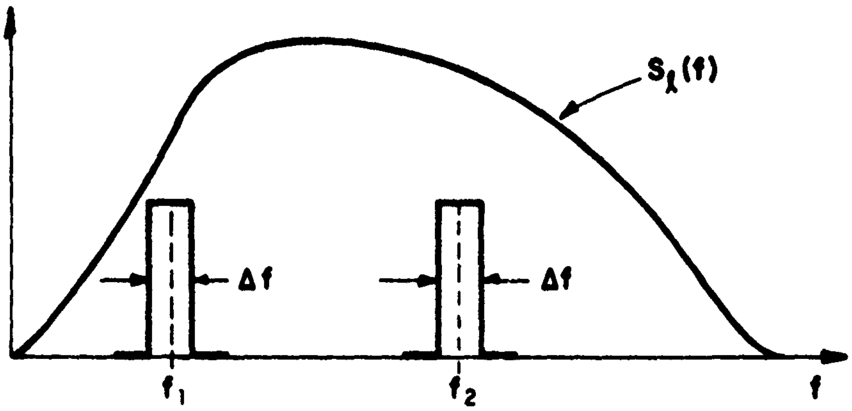


FIG 2.4

SAMPLING OF LOADING SPECTRUM BY TWO NARROW BAND FILTERS

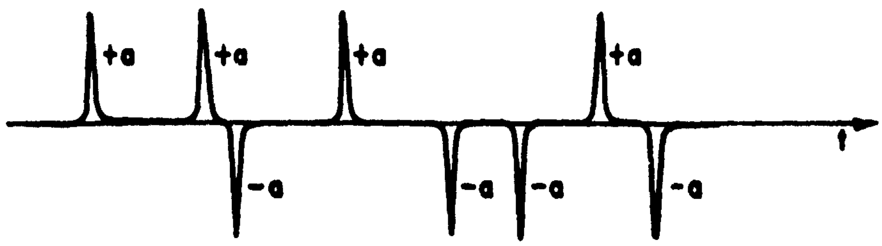


FIG 2.5

TEMPORAL REPRESENTATION OF WHITE NOISE

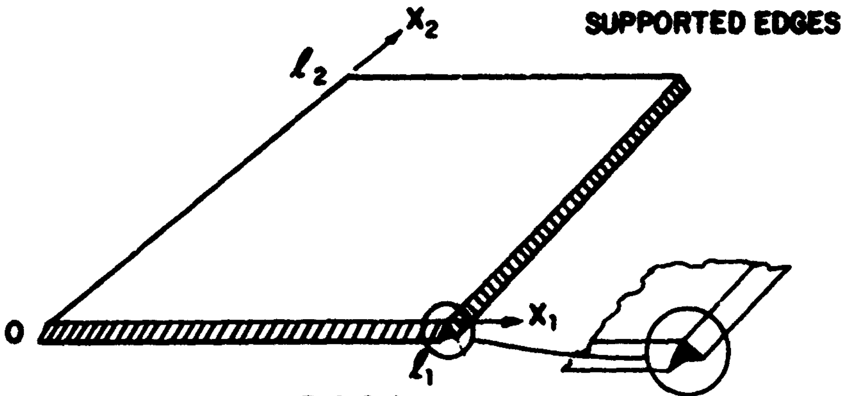


FIG 2.6
COORDINATES OF RECTANGULAR PLATE

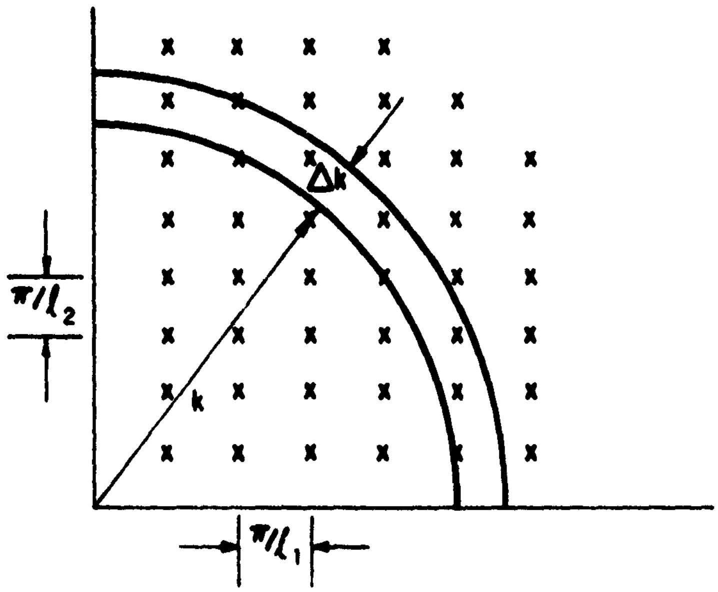


FIG 2.7
WAVE NUMBER LATTICE FOR RECTANGULAR SUPPORTED PLATE

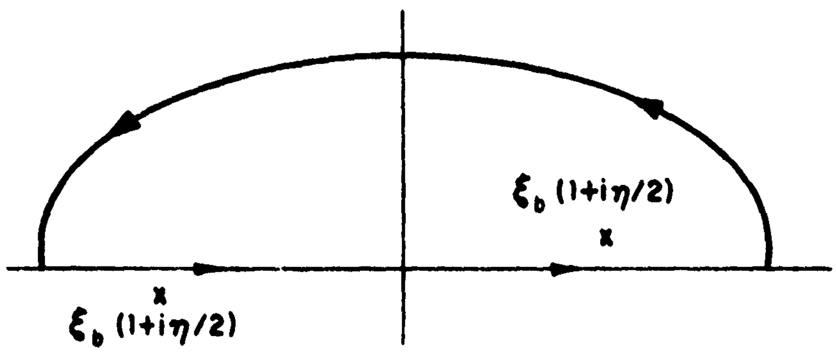


FIG 2.8
CONTOUR FOR EVALUATION OF INTEGRAL IN EQUATION 1.3.16

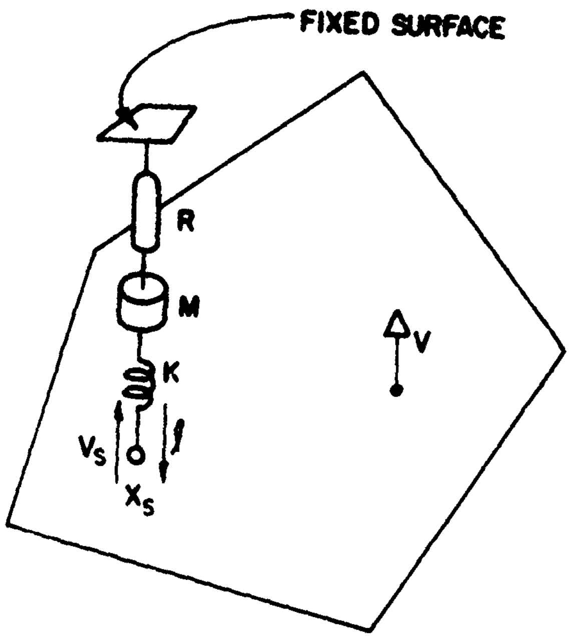


FIG 2.9
INTERACTION OF SINGLE RESONATOR AND FINITE PLATE

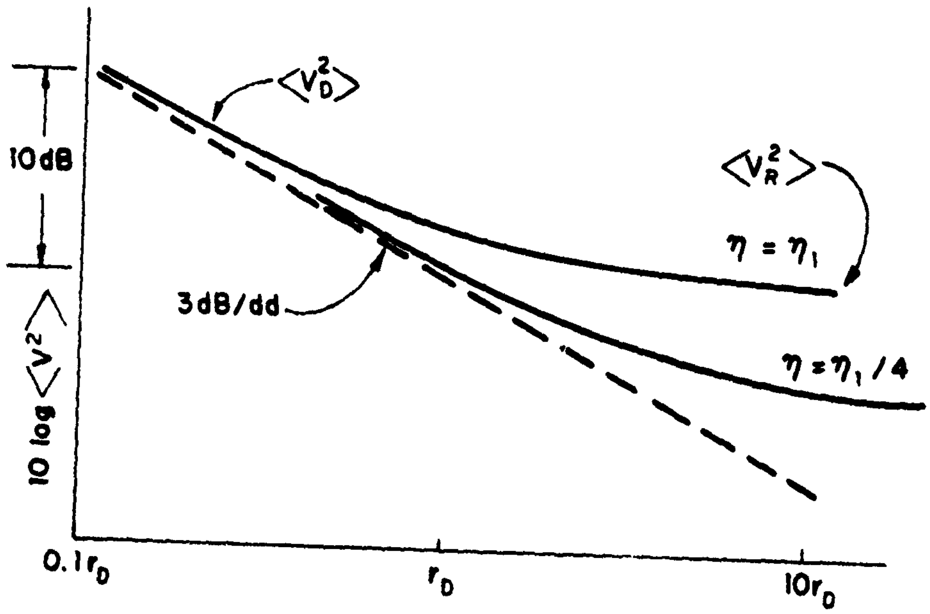


FIG 2.10
DIRECT AND REVERBERANT FIELDS ON ELASTIC PLATE

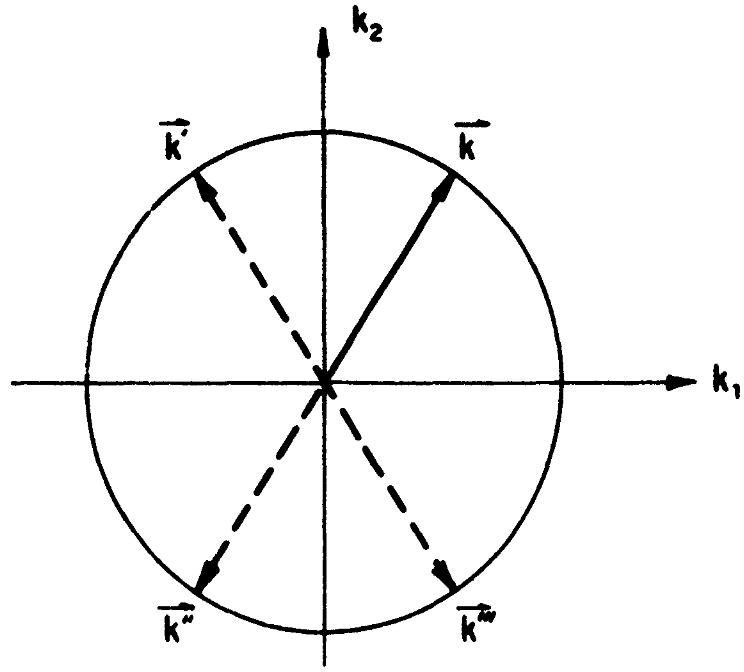


FIG 2.11

CONSTRUCTION OF 4 WAVE VECTORS
CORRESPONDING TO A SINGLE MODE

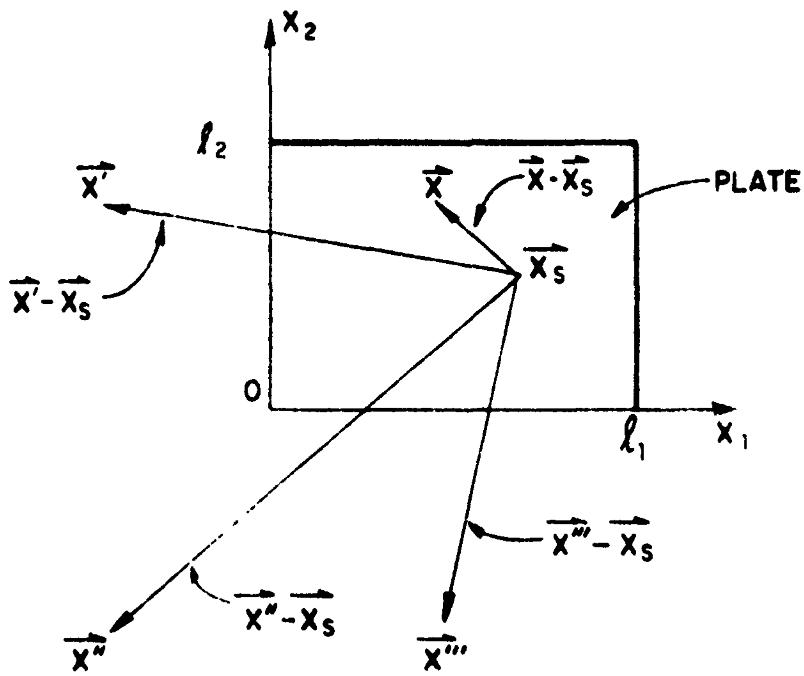


FIG 2.12

CONFIGURATION OF SOURCE POINT AND THE
OBSERVATION POINT AND ITS IMAGES

CHAPTER 3. ENERGY SHARING BY COUPLED SYSTEMS

3.0 Introduction

In SEA, we are principally concerned with systems that may be subdivided into subsystems that are directly excited, and other subsystems that are indirectly excited. For example, a vehicle excited by a turbulent boundary layer may be thought of as a single system excited by random noise, but we are more likely inclined to treat it as a collection of subsystems. One of these will be the exterior shell of the vehicle, which is directly in contact with the turbulence. A second system will include interior structures, such as bulkheads, shelves, etc. which are excited through their connections to the exterior shell. In this case, such a breakdown is immediate because of the nature of these structural elements and their configuration.

When the subsystems have been defined, we find that the effect of their connections is to provide coupling between them. This coupling is the mechanism by which vibrational energy is transferred from directly excited to indirectly excited systems. If a modal description has been applied to the subsystems, then intermodal coupling forces result from the connection. If a wave description has been used, the coupling is more naturally represented by impedances at the junction between the structure. In certain cases, one finds it convenient to use a mixed description, one of the subsystems being treated as a wave system and the other as a modal or resonator system. It is, therefore, quite important that we be able to shift our viewpoint between these descriptions as the need arises.

An important key to use of SEA is the definition of subsystems. This is a part of the modeling process that we shall deal with later on. Even for "obvious" choices of subsystems, the breakdown has an arbitrary element to it. In many cases, structural elements are the subsystems, but in others, modal (or wave) types may be the key. The shell of a vehicle has modes that represent mainly transverse or flexural vibration and modes of longitudinal or torsional in-plane motion. It may be necessary to treat transverse and in-plane modes as two separate subsystems, particularly if their excitation by the environment or their interaction with the rest of the structure is markedly different.

We begin the chapter by discussing the interaction of (sub) systems described in modal terms. Single resonator systems are dealt with first, and some important basic theories of SEA are derived. Then, multi-modal systems are studied. Various expressions for the energy flow are presented in this section.

In paragraph 3.1 of this chapter, we study interactions of systems using a wave description. Impedances and energy densities become the important interaction and dynamical quantities here. The definition of transmissibility coefficients is also made, a parameter that is widely used in acoustical analysis of interacting systems. Equivalences between the parameters of modal and wave descriptions are then drawn. We also show how reciprocity arguments may be used to develop response estimates which result in the same results as previously found. Reciprocity methods are very powerful in obtaining equivalences between response parameters in mechanical systems and one of the goals of this chapter is to show how they fit into the general SEA framework.

We complete the chapter with the analysis of some fairly simple systems that have practical importance. This serves to illustrate the methods developed in the chapter, and also to show how we begin to develop a catalog of SEA parameters as a result of the applications that have been made of SEA.

3.1 Energy Sharing Among Resonators

We began our discussion of vibration in Chapter 2 with a single dof system. Although quite artificial, we were able to develop basic ideas from the 1 dof system that were applicable to much more complex systems. This approach will be followed in the present chapter also. We shall study the energy interactions of the system shown in Fig. 3.1, consisting of two simple linear resonators coupled by conservative elements. We shall find that much more complex systems can be represented by this system in the following sections.

The equation of motion of mass M_1 is given by calculating all forces on it, including those arising from the motion of mass M_2.

In this instance, this is best done by using the Lagrangian. The kinetic energy of the system is:

$$KE = \frac{1}{2} M_1 \dot{y}_1^2 + \frac{1}{2} M_2 \dot{y}_2^2 + \frac{1}{8} M_c (\dot{y}_1 + \dot{y}_2)^2$$

The potential energy of the system is:

$$PE = \frac{1}{2} K_1 y_1^2 + \frac{1}{2} K_2 y_2^2 + \frac{1}{2} K_c (y_1 - y_2)^2 \qquad (3.1.1)$$

The equation of motion in the absence of velocity dependent forces are given by

$$\frac{d}{dt} \frac{\partial KE}{\partial \dot{y}_i} - \frac{\partial PE}{\partial y_i} = L_i \qquad (3.1.2)$$

Using Eq. (3.1.1) in Eq. (3.1.2) results in velocity independent equations of motion. In addition, we include the damping force $R\dot{y}$ and the gyroscopic coupling force $G\dot{y}$, to obtain

$$(M_1 + \frac{1}{4} M_c) \ddot{y}_1 + R_1\dot{y}_1 + (K_1 + K_c) y_1$$

$$= L_1 + K_c y_2 + G\dot{y}_2 - \frac{1}{4} M_c \ddot{y}_2 \qquad (3.1.3a)$$

$$(M_2 + \frac{1}{4} M_c) \ddot{y}_2 + R_2\dot{y}_2 + (K_2 + K_c) y_2$$

$$= L_2 + K_c y_1 - G\dot{y}_1 - \frac{1}{4} M_c \ddot{y}_1 \qquad (3.1.3b)$$

These equations clearly display the effect of motion in one system causing forces (and power input to or absorption from) the other.

In application to real systems, the question invariably arises: "What are the uncoupled structures?" From a mathematical viewpoint, we may obtain uncoupled equations by "clamping" system (2) [setting $y_2 = \dot{y}_2 = \ddot{y}_2 = 0$ in Eq. (3.1.3a)] and by clamping system (1) in Eq. (3.1.3b). An alternative is to reduce the coupling parameter to zero, i.e., setting $K_c = M_c = G = 0$ in both equations. Either is possible, but note that the effective mass and stiffness of each resonator will change in the process of decoupling if we use the latter approach. Since we do not want to change the resonator we deal with in the process of removing the coupling, we prefer the "clamping" definition of the uncoupled system.

The clamping or blocking concept is also appealing from an experimental viewpoint. A welded junction between a pipe and a plenum chamber for example is not readily identified in terms of the coupling parameters of Eq. (3.1.3). Nevertheless, we can imagine reducing the displacements of one element to zero by suitable clamping arrangement, even though it might be a very difficult thing to do from a practical viewpoint. If stress variables are used then the systems are decoupled by being cut apart. Such a thought experiment is of particular relevance in the event that one has a finite element computer model of each subsystem since the decoupling can be done by some simple instructions. In this monograph we will consistently define the decoupled forms of Eq. (3.1.3) or its counterparts to be found by setting the response variables of the other systems to be zero.

Eq. (3.1.3) displays more symmetry when rewritten as

$$\ddot{y}_1 + \Delta_1\dot{y}_1+\omega_1^2 y_1+ \frac{1}{\lambda} [\mu\ddot{y}_2-\gamma\dot{y}_2-\kappa y_2] = \ell_1 \qquad (3.1.4a)$$

$$\ddot{y}_2 + \Delta_2\dot{y}_2+\omega_2^2 y_2+ \lambda [\mu\ddot{y}_1+\gamma\dot{y}_1-\kappa y_1] = \ell_2 \qquad (3.1.4b)$$

where

$$\Delta_i = R_i/\omega_i(M_i+M_c/4), \omega_i^2 = (K_i+K_c)/(M_i+M_c/4),$$

$$\mu = (M_c/4)(M_1+M_c/4)^{-\frac{1}{2}}(M_2+M_c/4)^{-\frac{1}{2}}$$

$$\gamma = G/(M_1+M_c/4)^{\frac{1}{2}}, (M_2+M_c/4)^{-\frac{1}{2}},$$

$$\kappa = K_c/(M_1+M_c/4)^{\frac{1}{2}} (M_2+M_c/4)^{\frac{1}{2}}$$

and $$\lambda = (M_1+M_c/4)^{\frac{1}{2}}(M_2+M_c/4)^{\frac{1}{2}}.$$

Note that the equations could be made completely symmetric (that is, λ would vanish) by further defining variables $y_i' = y_i(M_i+M_c/4)^{\frac{1}{2}}$. Also note $\ell_i = L_i/(M_i+M_c/4)$.

In Chapter 2, we were interested in the energy of vibration of a single resonator excited by "white" or broad band noise. In this Chapter we are concerned with the response of two resonators when both ℓ_1 and ℓ_2 are independent white noise sources. Since Eqs. (3.1.4) are linear, such an analysis is fairly straightforward, but it is instructive to go through it in some detail. Since we want to apply these results to energy flow problems, we shall be particularly interested in calculating the power flow between the resonators.

Before we treat Eqs. (3.1.4) in all of their detail, however, let us consider a particularly simple case: that for $\Delta_1=\Delta_2 = \Delta$, $\omega_1=\omega_2=\omega_0$, $\lambda=1$, and $\mu=\gamma=0$. Thus, we have two identical resonators with stiffness coupling only. If we add and subtract Eqs. (3.1.4), we obtain,

$$\ddot{z}_1 + \Delta\dot{z}_1 + \Omega_1^2 z_1 = g_1 \tag{3.1.5a}$$

$$\ddot{z}_2 + \Delta\dot{z}_2 + \Omega_2^2 z_2 = g_2 \tag{3.1.5b}$$

when

$$z_1=y_1+y_2,\ z_2=y_1-y_2,\ g_1=\ell_1+\ell_2,\ g_2=\ell_1-\ell_2,$$

$$\Omega_1^2=\omega_0^2-\kappa,\ \Omega_2^2=\omega_0^2+\kappa.$$

From elementary dynamics, we know that if we start this system vibrating without damping or other excitation present, that the general free vibration response of either mass is of the form

$$y_1(t)=A \sin(\Omega_1 t+\phi_2) + B \sin(\Omega_2 t+\phi_2), \qquad (3.1.6)$$

If A and B are nearly equal in magnitude (this will occur if one mass is held fixed while the other is displaced and then released) then the motion of either mass is of the form

$$y_1(t) \approx C \sin\left(\frac{\kappa}{\omega_0} t+\psi_a\right) \sin(\omega_0 t+\psi_b), \qquad (3.1.7)$$

which represents an oscillation that is modulated by an envelope that varies at radian frequency κ/ω_0. Indeed, this beating phenomenon is one of the best known and most remarkable features of coupled oscillators.

If the resonators have damping Δ, however, then they will have a decaying motion represented by an additional "modulation" of Eq. (3.1.7) by the term $e^{-\Delta t/2}$ [see Eq. (2.1.9)]. If the damping is very light, so that $\kappa/\omega_0 >> \Delta$, then the beating phenomenon will still be apparent but if the damping is very strong compared to the coupling, so that $\Delta >> \kappa/\omega_0$, then the energy of vibration will be dissipated before the beating oscillations can take place. We shall find that quite parallel conclusions may be drawn when the resonators are excited by a white noise source.

Let us now return to Eqs. (3.1.5) and set $\ell_2=0$. Then by examining $\langle\dot{y}_2^2\rangle$, we can determine the power that is flowing into the indirectly excited resonator. When $\ell_2=0$, then $g_1 = g_2$ and the responses y_1, y_2 will be identical, except that their response curves will be shifted as shown in

Fig. 3.2. When there is no coupling, then $z_1 = z_2$ and $\dot{y}_2 = 1/2\ (\dot{z}_1 - \dot{z}_2) = 0$. When the coupling is such that $\kappa/\omega_0 << \Delta$, then the spectrum of $\dot{y}_2$ is very nearly the difference between the spectra $S_{\dot{z}_1}$ and $S_{\dot{z}_2}$, as shown in Fig. 3.2. The response y_2 is weak in this case, very broad band but with a slight double peak in its spectrum that would show a slight beating effect. The response $\dot{y}_2 = 1/2\ (\dot{z}_1 + \dot{z}_2) = \dot{z}_1$ has the spectrum $S_{\dot{y}_1} \approx S_{\dot{z}_1}$.

When the coupling is increased, the two spectra shift farther apart, and the difference spectrum is increased, as shown. This means that $\langle \dot{y}_2^2 \rangle$ is increasing with the increase in coupling. The spectrum of $\dot{y}_2$ is even more strongly double peaked, and fluctuations at the difference frequency κ/ω_0 (beats) will become more pronounced. Also, since z_1 and z_2 are no longer identical, $\langle \dot{y}_1^2 \rangle$ will be diminished.

Finally, when the coupling is strong enough so that the resonances have shifted beyond the damping bandwidth ($\kappa/\omega_0 > \Delta$), the response $\dot{y}_1$ and $\dot{y}_2$ are statistically independent since they do not share common frequency components. Accordingly, $\langle \dot{y}_1^2 \rangle = \langle \dot{y}_2^2 \rangle = \langle \dot{z}_1^2 \rangle / 2$. Thus, each resonator dissipates 1/2 of the power supplied by the white noise source ℓ_1. This power depends only on the mass of resonator "1" and not on the coupling [see discussion in connection with Eq. (2.1.39)]. Now the "beating" phenomenon will be quite strong implying an oscillation of energy in the system. This flow is in addition to the steady power flow from resonator "1" to resonator "2" required to supply the power dissipated by resonator $\Delta \langle \dot{y}_2^2 \rangle$.

Let us now return to the more complex system represented by Eq. (3.1.3) and (3.1.4). We assume that both L_1 and L_2 are independent white noise sources. The power supplied by the source L_1 is $\langle L_1\ \dot{y}_1 \rangle$, or

$$\langle L_1 \dot{y}_1 \rangle = (M_1 + \tfrac{1}{4} M_c)\ \langle \ddot{y}_1 \dot{y}_1 \rangle + R_1 \langle \dot{y}_1^2 \rangle + (K_1 + K_c) \langle y_1 \dot{y}_1 \rangle$$

$$- K_c\ \langle y_2 \dot{y}_1 \rangle - G\ \langle \dot{y}_2 \dot{y}_1 \rangle + \tfrac{1}{4} M_c \langle \ddot{y}_2 y_1 \rangle \qquad (3.1.8)$$

Since $\langle \frac{d}{dt}\ \dot{y}_1^2 \rangle = \langle \frac{d}{dt}\ y_1^2 \rangle = 0$ for stationary processes, the first and third terms on the right hand side of Eq. (3.1.8) vanish. The term $R\langle\dot{y}_1^2\rangle$ represents dissipation by the damper R_1 and the other terms represent power flow into the coupling elements.

In a similar fashion, the power supplied by source 2 is

$$\langle L_2\dot{y}_2\rangle = R\ \langle\dot{y}_2^2\rangle + K_c\ \langle y_1\dot{y}_2\rangle + G\ \langle\dot{y}_1\dot{y}_2\rangle + \tfrac{1}{4}M_c\ \langle\ddot{y}_1\dot{y}_2\rangle$$

(3.1.9)

Using the fact that

$$\langle\frac{d}{dt}\ \dot{y}_1\dot{y}_2\rangle = \langle\ddot{y}_1y_2\rangle + \langle\dot{y}_2y_1\rangle = 0$$

and

$$\langle\frac{d}{dt}\ \dot{y}_1\dot{y}_2\rangle = \langle\ddot{y}_1\dot{y}_2\rangle + \langle\ddot{y}_2\dot{y}_1\rangle = 0,$$

we can add Eqs. (3.1.8) and (3.1.9) to obtain

$$\langle L_1\dot{y}_1\rangle + \langle L_2\dot{y}_2\rangle = R_1\ \langle\dot{y}_1^2\rangle + R_2\ \langle\dot{y}_2^2\rangle \qquad (3.1.10)$$

which states that the total power supplied is dissipated in the two damping elements; the coupling elements in the form presented are non-dissipative.

We also note from Eq. (3.1.8) or (3.1.9) that the power flow from resonator 1 to resonator 2 is simply

$$\Pi_{12} = -\ K_c\ \langle y_2\dot{y}_1\rangle - G\ \langle\dot{y}_2\dot{y}_1\rangle - \tfrac{1}{4}M_c\ \langle\ddot{y}_2\dot{y}_1\rangle. \qquad (3.1.11)$$

Our next task is to evaluate these averages in terms of the spectral densities of the sources and the system parameters. This evaluation may be done in the time or frequency domain.

In the frequency domain calculation, we introduce the complex system frequency response $H_{pq}(\omega)$ defined as the response y_p to a force $L_q = e^{-i\omega t}$. Thus, all time derivatives in Eq. (3.1.3) are replaced by $\frac{d}{dt} \rightarrow -i\omega$, and we have

$$H_{11}(\omega) = (M_1+\tfrac{1}{4}M_c)^{-1}\ (-\omega^2-i\omega\Delta_2+\omega_2^2)/D \qquad (3.1.12)$$

$$H_{21}(\omega) = \lambda(M_1+\tfrac{1}{4}M_c)^{-1}(\omega^2\mu+i\omega\gamma + \kappa)/D \qquad (3.1.12b)$$

$$H_{12}(\omega) = (M_2+\tfrac{1}{4}M_c)^{-1}\lambda^{-1}(\omega^2\mu-i\omega\gamma + \kappa)/D \qquad (3.1.12c)$$

$$H_{22}(\omega) = (M_2+\tfrac{1}{4}M_c)^{-1}(-\omega^2-i\omega\Delta_1 + \omega_1^2)/D \qquad (3.1.12d)$$

where D is the system determinant given by

$$D = \omega^4(1-\mu^2) + i\omega^3(\Delta_1+\Delta_2)-\omega^2(\omega_1^2+\omega_2^2+\Delta_1\Delta_2+\gamma^2+2\mu\kappa)$$

$$- i\omega(\Delta_1\omega_2^2+\Delta_2\omega_1^2) + (\omega_1^2\omega_2^2-\kappa^2). \qquad (3.1.13)$$

Using these transfer functions, the various second moments of the dynamical variables can be generated by integration over frequency from $-\infty$ to $+\infty$:

$$\langle y_1^2\rangle = S_{L_1}\int |H_{11}|^2\, d\omega + S_{L_2}\int |H_{12}|^2\, d\omega \qquad (3.1.14a)$$

$$\langle y_2^2\rangle = S_{L_2}\int |H_{22}|^2\, d\omega + S_{L_1}\int |H_{21}|^2\, d\omega \qquad (3.1.14b)$$

$$\langle \dot{y}_1^2 \rangle = S_{L_1} \int \omega^2 |H_{11}|^2 \, d\omega + S_{L_2} \int \omega^2 |H_{12}|^2 \, d\omega \quad (3.1.14c)$$

$$\langle \dot{y}_2^2 \rangle = S_{L_2} \int \omega^2 |H_{22}|^2 \, d\omega + S_{L_1} \int \omega^2 |H_{21}|^2 \, d\omega \quad (3.1.14d)$$

$$\langle y_2 \dot{y}_1 \rangle = S_{L_1} \int -i\omega^3 H_{11} \, H_{21}^* \, d\omega + S_{L_2} \int -i\omega \, H_{12} \, H_{22}^* d\omega$$

$$(3.1.14e)$$

$$\langle \dot{y}_2 \dot{y}_1 \rangle = S_{L_1} \int \omega^2 \, H_{11} \, H_{21}^* \, d\omega + S_{L_2} \int \omega^2 \, H_{12}^* \, H_{22} \, d\omega$$

$$(3.1.14f)$$

$$\langle \ddot{y}_2 \dot{y}_1 \rangle = S_{L_1} \int -i\omega^3 \, H_{11}^* \, H_{21} \, d\omega + S_{L_2} \int -i\omega^3 \, H_{12}^* \, H_{22} \, d\omega$$

$$(3.1.14g)$$

The final three relations in Eqs. (3.1.14 e, f, g) are the terms we need to determine power flow, the first four, Eqs. (3.1.14a, b, c, d) are needed in evaluating the energy of vibration. In these equations we have used the "two-sided" spectrum $S_L(\omega)$ which has a range from $\omega=-\infty$ $\omega=+\infty$ rather than the more "physical" one-sided spectral density in cyclic frequency f which has a range from $f = 0$ to $f = \infty$. Since the integral over the range of each must produce the same mean square value, i.e.,

$$\langle y^2 \rangle = \int_{-\infty}^{\infty} S_y(\omega) \, d\omega = \int_0^{\infty} S_y(f) \, df, \qquad (3.1.15)$$

the relation between them must be

$$S_y(f) = 2\ S_y(\omega)\ \frac{d\omega}{df} = 4\pi\ S_y(\omega)\quad . \tag{3.1.16}$$

We will not bother to evaluate the integral of Eqs. (3.1.14) here, since they are of standard form. Even so, the form of the relations is quite complicated. They are found to be:

$$\langle y_2\dot{y}_1\rangle = -\left\{\frac{\pi S_{L_1}}{\Delta_1(M_1+\frac{1}{4}M_c)} - \frac{\pi S_{L_2}}{\Delta_2(M_2+\frac{1}{4}M_c)}\right\} \tag{3.1.17a}$$

$$\times\ \frac{\Delta_1\Delta_2[\mu(\Delta_1\omega_2^2+\Delta_2\omega_1^2) + \gamma(\omega_2^2-\omega_1^2) + \kappa(\Delta_1+\Delta_2)]}{(M_1+\frac{1}{4}M_c)^{\frac{1}{2}}(M_c+\frac{1}{4}M_c)^{\frac{1}{2}}d}$$

$$\langle\dot{y}_2\dot{y}_1\rangle = -\left\{\frac{\pi S_{L_1}}{\Delta_1(M_1+\frac{1}{4}M_c)} - \frac{\pi S_{L_2}}{\Delta_2(M_2+\frac{1}{4}M_c)}\right\}$$

$$\times\ \frac{\Delta_1\Delta_2\ [\gamma(\Delta_1\omega_2^2+\Delta_2\omega_1^2) - \kappa(\omega_2^2-\omega_1^2)]}{(M_1+\frac{1}{4}M_c)^{\frac{1}{2}}\ (M_2+\frac{1}{4}M_c)^{\frac{1}{2}}\ d}$$

$$- \left\{ \frac{\pi S_{L_1}}{M_1+\frac{1}{4}M_c} + \frac{\pi S_{L_2}}{M_2+\frac{1}{4}M_c} \right\}$$

$$\times \frac{\mu[(\Delta_1\omega_2^2+\Delta_2\omega_1^2)(\omega_1^2+\omega_2^2+\Delta_1\Delta_2+\gamma^2+2\mu\kappa)-(\Delta_1+\Delta_2)(\omega_1^2\omega_2^2-\kappa^2)]}{(M_1+\frac{1}{4}M_c)^{\frac{1}{2}}(M_2+\frac{1}{4}M_c)^{\frac{1}{2}}(1-\mu^2)\ d}$$

$$+\left\{ \frac{\omega_2^2\pi S_{L_1}}{M_1+\frac{1}{4}M_c} + \frac{\omega_2^2\pi S_{L_2}}{M_2+\frac{1}{4}M_c} \right\} \frac{\mu(\Delta_1\omega_2^2+\Delta_2\omega_1^2)}{(M_1+\frac{1}{4}M_c)^{\frac{1}{2}}(M_2+\frac{1}{4}M_c)^{\frac{1}{2}}d}$$

(3.1.17b)

$$\langle \ddot{y}_2\dot{y}_1 \rangle = \left\{ \frac{\pi S_{L_1}}{\Delta_1(M_1+\frac{1}{4}M_c)} - \frac{\pi S_{L_2}}{\Delta_2(M_2+\frac{1}{4}M_c)} \right\} \frac{\Delta_1\Delta_2}{(M_1+\frac{1}{4}M_c)^{\frac{1}{2}}(M_2+\frac{1}{4}M_c)^{\frac{1}{2}}d}$$

$$\times \left[\frac{\mu}{1-\mu^2}(\Delta_1\omega_2^2+\Delta_2\omega_1^2)(\omega_1^2+\omega_2^2+\Delta_1\Delta_2+\gamma^2+2\mu\kappa) \right.$$

$$\left. -(\Delta_1+\Delta_2)(\omega_1^2\omega_2^2-\kappa^2)+\kappa(\Delta_1\omega_2^2+\Delta_2\omega_1^2) \right]$$

$$- \left\{ \frac{\pi S_{L_1}}{M_1 + \frac{1}{4} M_c} + \frac{\pi S_{L_2}}{M_2 + \frac{1}{4} M_c} \right\}$$

$$\times \quad \frac{\gamma[(\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2)(\omega_1^2 + \omega_2^2 + \Delta_1 \Delta_2 + \gamma^2 + 2\mu\kappa) - (\Delta_1 + \Delta_2)(\omega_1^2 \omega_2^2 - \kappa^2)]}{(M_1 + \frac{1}{4} M_c)^{\frac{1}{2}} \quad (M_2 + \frac{1}{4} M_c)^{\frac{1}{2}} \quad (1 - \mu^2) \ d}$$

$$+ \left\{ \frac{\omega_2^2 \pi S_{L_1}}{M_1 + \frac{1}{4} M_c} + \frac{\omega_2^2 \pi S_{L_1}}{M_2 + \frac{1}{4} M_c} \right\} \quad \frac{\gamma[\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2]}{(M_1 + \frac{1}{4} M_c)^{\frac{1}{2}} \quad (M_2 + \frac{1}{4} M_c)^{\frac{1}{2}} \ d}$$

(3.1.17c)

The quantity d that appears in these equations is given by:

$$d = \Delta_1 \Delta_2 [(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2)] + \mu^2 \ (\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2)^2$$

$$+ \ (\gamma^2 + 2\mu\kappa)(\Delta_1 + \Delta_2)(\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2) + \kappa^2 (\Delta_1 + \Delta_2)^2 .$$

(3.1.18)

These expressions are now placed into Eq. (3.1.11) to evaluate the power flow. Incredibly, they simplify enormously (although they are still relatively complicated). The result of the substitution is

$$\Pi_{12} = A \left[\frac{\pi S_{L_1}}{\Delta_1 (M_1 + \frac{1}{4} M_c)} - \frac{\pi S_{L_2}}{\Delta_2 (M_2 + \frac{1}{4} M_c)} \right] \quad (3.1.19)$$

where

$$A = \frac{\Delta_1 \Delta_2}{(1-\mu^2) d} \Big\{ \mu^2 [\Delta_1 \omega_2^4 + \Delta_2 \omega_1^4 + \Delta_1 \Delta_2 (\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2]$$

$$+ (\gamma^2 + 2\mu\kappa) [\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2] + \kappa^2 (\Delta_1 + \Delta_2) \Big\} \quad (3.1.20)$$

and d is given by Eq. (3.1.18).

Comparing Eqs. (3.1.4) to Eq. (2.1.1), we see that $\Delta_1 = \omega_1 \eta_1$ and $\Delta_2 = \omega_2 \eta_2$, where the η's are the loss factors of each uncoupled resonator. Thus, when $x_2 = \dot{x}_2 = 0$, the term is

$$\frac{\pi S_{L_1}(\omega)}{\Delta_1 (M_1 + \frac{1}{4} M_c)} = \frac{S_{L_1}(f)}{4 \omega_1 \eta_1 (M_1 + \frac{1}{4} M_c)} = \langle \dot{y}_1^2 \rangle \quad (M_1 + \tfrac{1}{4} M_c) \quad (3.1.21)$$

according to Eqs. (2.1.39) and (2.1.40), the energy of resonator "1". Similarly

$$\frac{\pi S_{L_2}(\omega)}{\Delta_2 (M_2+\frac{1}{4}M_c)} = (M_2+\frac{1}{4}M_c) \quad \langle \dot{y}_2^2 \rangle \tag{3.1.22}$$

is the energy of resonator "2" when $y_1=\dot{y}_2=0$. Thus, Eq. (3.1.19) states that the power flow from resonator "1" to resonator "2" is

(a) directly proportional to the difference in decoupled energy of the resonators, where decoupling is defined by constraining the resonator, the energy of which is not being evaluated

(b) since the quantity A is positive definite, the average power flow is from the resonator of greater to lesser energy

(c) the quantity A is also symmetric in the system parameters so that an equal difference of resonator energies in either direction ($1\rightarrow2$ or $2\rightarrow1$) will result in an equal power flow, we may say that power flow is reciprocal.

The concepts of power flow between resonators being proportional to the difference in uncoupled energies is useful. When we are exciting a structure with noise so that the input power is known, then since S_F/M is proportional to input power, this interpretation of Eq. (3.1.19) is the appropriate one. However, we often do not know the input power, we only know the m.s. response values $\langle y^2 \rangle$ and $\langle y^2 \rangle$ by measurements taken on the system as it vibrates in its coupled state. We need to relate Eq. (3.1.19) to these values also. To do this, we require calculations of $\langle y_1^2 \rangle$, $\langle y_2^2 \rangle$, $\langle \dot{y}_1^2 \rangle$ and $\langle \dot{y}_2^2 \rangle$ from Eqs. (3.1.14). The results of the calculations are as follows:

$$\langle y_1^2 \rangle = \frac{\pi S_{L_1}}{(M_1+\frac{1}{4}M_c)^2(\omega_1^2\omega_2^2-\kappa^2)d} \Big\{ \Delta_2\omega_2^2[(\omega_1^2-\omega_2^2)^2$$

$$+ (\Delta_1+\Delta_2)(\Delta_1\omega_2^2+\Delta_2\omega_1^2)] + \mu^2[\omega_2^4(\Delta_1\omega_2^2+\Delta_2\omega_1^2)]$$

$$+ (\gamma^2+2\mu\kappa)[\omega_2^2(\Delta_1+\Delta_2)] - \kappa^2[(\Delta_1+\Delta_2)(\Delta_2^2-2\omega_2^2)$$

$$+ (\Delta_1\omega_2^2+\Delta_2\omega_1^2)]\Big\} + \frac{\pi S_{L_2}}{(M_1+\frac{1}{4}M_c)(M_2+\frac{1}{4}M_c)(\omega_1^2\omega_2^2-\kappa^2)\,d}$$

$$\times \Big\{ \mu^2[\omega_1^2\omega_2^2\,(\Delta_1\omega_2^2+\Delta_2\omega_1^2)] + (\gamma^2+2\mu\kappa)\,[\omega_1^2\omega_2^2(\Delta_1+\Delta_2)]$$

$$+ \kappa^2\,[\Delta_1\Delta_2(\Delta_1+\Delta_2) + (\Delta_1\omega_1^2+\Delta_2\omega_2^2)]\Big\} \qquad (3.1.23a)$$

$$\langle \dot{y}_1^2 \rangle = \frac{\pi S_{L_1}}{(M_1+\frac{1}{4}M_c)^2(1-\mu^2)d} \Big\{ \Delta_2[(\omega_1^2-\omega_2^2)^2+(\Delta_1+\Delta_2)(\Delta_1\omega_2^2+\Delta_2\omega_1^2)]$$

$$- \mu^2[\omega_2^4(\Delta_1+\Delta_2)+(\Delta_2^2-2\omega_2^2)(\Delta_1\omega_2^2+\Delta_2\omega_1^2)]$$

$$+ (\gamma^2 + 2\mu\kappa)\ [\Delta_1\omega_2^2+\Delta_2\omega_1^2] + \kappa^2[\Delta_1+\Delta_2]\Big\}$$

$$+ \frac{\pi S_{L_2}}{(M_1+\frac{1}{4}M_c)(M_2+\frac{1}{4}M_c)(1-\mu^2)d}\Big\{\mu^2\ [\Delta_1\omega_2^4+\Delta_2\omega_1^4$$

$$+ \Delta_1\Delta_2\ (\Delta_1\omega_2^2+\Delta_2\omega_1^2)] + (\gamma^2+ 2\mu\kappa)[\Delta_1\omega_2^2+\Delta_2\omega_1^2+\kappa^2(\Delta_1+\Delta_2)]\Big\}$$

(3.1.23b)

The expressions for $\langle y_2^2\rangle$ and $\langle\dot{y}_2^2\rangle$ are found by revising subscripts in these two equations.

We note from Eqs. (3.1.23) that in coupled vibration $\langle\dot{y}^2\rangle \neq \omega_1^2\langle y_1^2\rangle$ so that a detailed equality of potential and kinetic energy has been lost. Nevertheless, by direct substitution.

$$(M_1+\tfrac{1}{4}M_c)\ \langle\dot{y}_1^2\rangle - (M_2+\tfrac{1}{4}M_c)\ \langle\dot{y}_2^2\rangle = (K_1+K_c)\langle y_1^2\rangle-(K_2+K_c)$$

$$\langle y_2^2\rangle = \frac{\Delta_1\Delta_2}{d}\ [(\omega_1^2-\omega_2^2)^2+(\Delta_1+\Delta_2)(\Delta_1\omega_2^2+\Delta_2\omega_1^2)]$$

$$\left\{\frac{\pi S_{L_1}}{\Delta_1(M_1+\frac{1}{4}M_c)} - \frac{\pi S_{L_2}}{\Delta_2(M_2+\frac{1}{4}M_c)}\right\}$$

(3.1.24)

Thus, we arrive at the remarkable conclusion that the power flow is proportional to the difference in the actual kinetic energy, or potential energy, or total energy. This is a very useful result because it allows us to calculate power flow from input power to the system or from measured vibration. We are assured that we will get the correct answer whether we base our calculations on the uncoupled systems or on the actual vibrational energy of the coupled systems.

If the average energy of resonator "1" is E_1 and the average energy of resonator "2" is E_2, then the power flow between them is, according to Eqs. (3.1.19) and (3.1.24)

$$\Pi_{12} = B(E_1-E_2) \tag{3.1.25}$$

where

$$B=\frac{\mu^2[\Delta_1\omega_2^4+\Delta_2\omega_1^4+\Delta_1\Delta_2(\Delta_1\omega_2^2+\Delta_2\omega_1^2)]+(\gamma^2+2\mu\kappa)(\Delta_1\omega_2^2+\Delta_2\omega_1^2)+\kappa^2(\Delta_1+\Delta_2)}{(1-\mu^2)[(\omega_1^2-\omega_2^2)^2+(\Delta_1+\Delta_2)(\Delta_1\omega_2^2+\Delta_2\omega_1^2)]}$$

(3.1.26)

Let us suppose that resonator "2" has no direct excitation, so that $S_{L_2} = 0$. Then, the power dissipated by the resonator must equal the power transferred from "1" to "2"; $\Pi_{2,diss}=\Pi_{12}\rightarrow\Delta_2 E_2 = B(E_1-E_2)$ or,

$$\frac{E_2}{E_1} = \frac{B}{\Delta_2+B} \tag{3.1.27}$$

This relation shows that the largest value E_2 can have is E_1, which occurs when the coupling (determined by B) is strong compared to the damping Δ_2.

When the two resonators are identical and have stiffness coupling only, then from Eq. (3.1.26)

$$B = \frac{2\Delta\kappa^2}{4\Delta^2\omega_0^2} = \frac{\kappa^2}{2\Delta\omega_0^2} \tag{3.1.28}$$

and,

$$\frac{E_2}{E_1} = \frac{\kappa^2/2\Delta^2\omega_0^2}{1+\kappa^2/2\Delta^2\omega_0^2}$$

which shows that $E_2 \to E_1$ as $\omega_0\Delta/\kappa \to 0$ and $E_2 \to 0$ as $\kappa \to 0$, which is consistent with our earlier discussion of this system.

We can now make some additional points to those already made in the discussion following Eq. (3.1.22). They are:

(d) the power flow is also proportional to the actual vibrational energies of the system, the constant of proportionality being B, defined by Eq. (3.1.26)

(e) the parameter B is positive definite and symmetric in system parameters; the system is reciprocal and power flows from the more energetic resonator to the less energetic one

(f) if only one resonator is directly excited, the greatest possible value of energy for the second resonator is that of the first resonator.

These six points form the basis for our studies of energy flow in mechanical and acoustical systems, since the calculation of power flow between resonators according to the scheumes indicated in Eqs. (3.1.19) and (3.1.25) will be fundamental to our study of more complex systems. We will have to make some additional assumptions as we proceed, but we could not begin without these (conceptually) simple relationships.

Before we leave this subject, let us develop an additional result of modal interaction that will be useful to us in dealing with multi-dof systems. We derive the average power into resonator 2 when the only active source is L_1 and ω_1 is allowed to vary randomly over an interval $\Delta\omega$ that includes ω_2. To do this calculation, we see from Eq. (3.1.19) that for any particular value of ω_1, the power flow is $\Pi_{12} = A\ \pi S_{L_1}/\Delta_1(M_1+1/4M_c)$ and that an average over ω_1 requires that we evaluate $\langle A\rangle_{\omega_1}$. As long as the interval $\Delta\omega$ is not too large compared to the values of ω_1 and ω_2, we can replace both by ω, the center frequency of the averaging band, everywhere except in the $(\omega_1^2-\omega_2^2)^2$ term in d, since this is the term that is sensitive to the relative values of ω_1 and ω_2. We then form the average

$$\left\langle\frac{1}{d}\right\rangle_{\omega_1} = \frac{1}{\Delta_1\Delta_2}\left\langle\frac{1}{(\omega_1^2-\omega_2^2)^2+\omega^2\bar{\Delta}^2}\right\rangle_{\omega_1} = \frac{1}{\Delta_1\Delta_2}\ \frac{1}{\Delta\omega}\ \frac{\pi}{2}\ \frac{1}{\omega^2\bar{\Delta}} \tag{3.1.29}$$

where we have used the result leading to Eq. (2.1.37), and have defined

$$\bar{\Delta}=(\Delta_1+\Delta_2)\,[1+(\mu^2\omega^2+\gamma^2+2\mu\kappa+\kappa^2/\omega^2)/\Delta_1\Delta_2]^{\frac{1}{2}}. \tag{3.1.30}$$

We see that $\bar{\Delta}$ is very nearly the total damping of the two resonators when the coupling is very weak but is more complicated for stronger coupling. The average power flow then, is

$$\langle \Pi_{12} \rangle_\omega = \frac{\pi S'_{L_2}}{M_2 + \frac{1}{4} M_c} \tag{3.1.31}$$

and

$$S'_{L_2} = S_{L_1} \frac{M_2 + \frac{1}{4} M_c}{M_1 + \frac{1}{4} M_c} \frac{\pi \Delta_2}{2\Delta\omega} \frac{\lambda}{(1 + \lambda)^{\frac{1}{2}}} \tag{3.1.32}$$

with

$$\lambda \equiv [\mu^2\omega^2 + (\gamma^2 + 2\mu\kappa) + \kappa^2/\omega^2]/\Delta_1\Delta_2 \; . \tag{3.1.33}$$

and we have assumed $\mu^2 \ll 1$ and $\Delta/\omega \ll 1$.

The average interaction, therefore, is equivalent to a white noise source acting on resonator 2 of spectral density S'_{L_2} . This effective spectral density S'_{L_2} is proportional to S_{L_1}, to the ratio of the effective bandwidth $\pi\Delta_2/2$ of resonator 2, to the averaging bandwidth $\Delta\omega$, to the mass ratio of the systems, and to the strength of the system coupling as measured by $\lambda(1+\lambda)^{-\frac{1}{2}}$. The replacement of modal interactions averaged over system parameters with white noise sources is an important device in the development of SEA methods.

3.2 Energy Exchange in Multi-Degree-of-Freedom Systems

We now consider two subsystems that are connected together. We combine the analysis of distributed systems in paragraph 2.2 with that for two single resonators in paragraph 3.1 to develop a theory of multi-modal interactions. The actual system that we are interested in is the situation in Fig. 3.3a, in which the two subsystems are responding to their own excitation and the interaction "forces". Each of the sub-systems vibrates "on its own" when the other system is blocked or clamped as shown in Figs. 3.3b,c, and can be analyzed according to the methods of paragraph 2.2. Thus, the equations for the blocked subsystems are:

$$\ddot{y}_i^{(b)} + (r_1/\rho_1)\dot{y}_i^{(b)} + \Lambda_1 y_i^{(b)} = p_i/\rho_i \quad i=1,2 \qquad (3.2.1)$$

As in Paragraph 2.2, the operators have eigenfunctions and eigenvalues

$$\Lambda_i \; \psi_{i\alpha} = \omega_{i\alpha}^2 \; \psi_{i\alpha} \qquad (3.2.2)$$

with, of course

$$\langle \psi_{i\alpha} \psi_{i\beta} \rangle_{\rho_i} = \delta_{\alpha\beta} \quad . \qquad (3.2.3)$$

The boundary conditions satisfied for ψ_{ia} include the clamped condition on subsystem $j \neq 1$.

We now suppose that the spectral densities of the forces

$$L_{ia}(t) = \int dx_i p\psi_{ia}$$

is flat (white) over a finite range of frequency $\Delta\omega$, and that within this band there are $N_i = n_i \Delta\omega$ modes of each subsystem, where $n_i(\omega)$ is the modal density of the i^{th} subsystem. The modes for these subsystems may be illustrated as shown in Fig. 3.4a. Each mode group represents a model of the subsystem. In the applications of SEA, this model has some very particular properties, which we now list and discuss:

1. Each mode is assumed to have a natural frequency $\omega_{i\alpha}$ that is uniformly probable over the frequency interval $\Delta\omega$. This means that each subsystem is a member of a population of systems that are generally physically similar, but differ enough to have randomly distributed parameters. The assumption is based on the fact that nominally identical structures or acoustical spaces will have uncertainties in modal parameters, particularly at higher frequencies.

2. We assume that every mode in a subsystem is equally energetic, and that its amplitudes

$$Y_{i\alpha}(t) = \int \rho_i y_i \psi_{i\alpha} dx_i / M$$

are incoherent, that is,

$$\langle Y_{i\alpha} Y_{i\beta} \rangle_t = \delta_{\alpha\beta} \langle Y_{i\alpha}^2 \rangle.$$

This assumption requires that we select mode groups for which this should be approximately correct, at least, and is an important guide to proper SEA modeling. It also implies that the excitation functions L_i are drawn from random populations of functions that have certain similarities (such as equal frequency and wavenumber spectra) but are individually incoherent.

3. As a matter of convenience, we may also assume that the damping of each mode in a subsystem is

the same. This is not essential, but it greatly simplifies the formalism and tends to be nearly true for reasonably complex subsystems.

The conditions just described are the basis for the word "statistical" in SEA. The concept of systems derived by selection of individual modes and modal excitations from random populations is of greater importance than the use of random (or noise) excitation. Indeed, we can use SEA to good accuracy in situations in which the excitation is a pure tone if there are a sufficient number of modes in interaction to provide "good statistics". We shall see an example of this in Chapter 4.

We now "unblock" the system and consider the new equations of motion, which are

$$\ddot{y}_i+(r_i/\rho_i)\dot{y}_i+\Lambda_i y_i=[p_i+\mu_{ij}(x_i,x_j)\ddot{y}_j$$

$$+\ (-)^j\gamma_{ij}(x_i,x_j)\dot{y}_j+\kappa_{ij}y_j]/\rho_i$$

$$[i\neq j;\ i,j=1,2] \tag{3.2.4}$$

One now expands these 2 equations in the eigenfunctions $\psi_{i\alpha}(x_i)$ to obtain

$$M_1[\ddot{Y}_\alpha+\Delta_1\dot{Y}_\alpha+\omega_\alpha^2 Y_\alpha]=L_\alpha+\sum_\sigma[\mu_{\alpha\sigma}\ddot{Y}_\sigma+\gamma_{\alpha\sigma}\dot{Y}_\alpha+\kappa_{\alpha\sigma}Y_\sigma] \tag{3.2.5a}$$

$$M_2[\ddot{Y}_\alpha+\Delta_2\dot{Y}_\sigma+\omega_\sigma^2 Y_\sigma]=L_\sigma+\sum_\alpha[\mu_{\sigma\alpha}\ddot{Y}_\alpha\ \ \gamma_{\sigma\alpha}\dot{Y}_\alpha\ \ \kappa_{\sigma\alpha}Y_\alpha] \tag{3.2.5b}$$

where we now reserve the indices $\alpha,\beta,\ldots$ for subsystem 1 and $\sigma,\tau,\ldots$ for subsystem 2, and we have also set $\Delta_i = r_i/\rho_i$. The mass of subsystem i is M_i in these equations. The coupling parameters are

$$\mu_{\alpha\sigma} = \int_{\text{boundary}} \mu_{12}[x_1, x_2(x_1)]\psi_\alpha(x_1)\psi_\sigma[x_2(x_1)]\, dx_1, \text{etc.} \tag{3.2.6a}$$

$$\mu_{\sigma\alpha} = \int_{\text{boundary}} \mu_{21}[x_1(x_2), x_2]\psi_\alpha[x_1(x_2)]\psi_\sigma(x_2)dx_2, \text{ etc.} \tag{3.2.6b}$$

where the integrations are taken along the boundary between the subsystems and, therefore, over the same range of x_1, x_2 in both integrals. Conservative coupling requirements are met by $\mu_{\sigma\alpha} = \mu_{\alpha\sigma}, \gamma_{\alpha\sigma} = \gamma_{\sigma\alpha}, \kappa_{\alpha\sigma} = \kappa_{\sigma\alpha}$ or $\mu_{12} = \mu_{21}$, $\gamma_{12} = \gamma_{21}$, and $\kappa_{12} = \kappa_{21}$.

The coupled systems may now be represented as shown in Fig. 3.4b, with the interaction lines showing the energy flows that result from the coupling. Suppose we are interested in the energy flow between resonator (mode) α of subsystem 1 and mode σ of subsystem 2. We showed at the end of paragraph 3.1 that energy modal pair interaction, when averaged over the ensemble of systems will act like a white noise generator. Thus, modes α and σ have energies E_α and E_σ (yet to be determined) as a result of these noise generators. Further, the modal energies of subsystem 1 modes are all equal, so that $E_\alpha = E_1$ = const. and $E_\sigma = E_2$ = constant. (Note that E_1 and E_2 are modal energies). Under these circumstances we have for the inter-modal power flow

$$\Pi_{\alpha\sigma} = \langle B_{\alpha\sigma} \rangle_{\omega_\alpha, \omega_\sigma} (E_1 - E_2) \quad , \tag{3.2.7}$$

according to Eq. (3.1.25). The average value of $B_{\alpha\sigma}$ is found in a manner completely analogous to the result in Eq. (3.1.29) to be

$$\langle B_{\alpha\sigma}\rangle_{\omega_\alpha,\omega_\sigma} = \frac{\pi}{2}\,\frac{\Delta_1\Delta_2}{\Delta\omega}\,\langle\lambda\rangle_{\alpha,\sigma} \tag{3.2.8}$$

where λ is defined by Eq. (3.1.33) and the average is taken with respect to frequencies $\omega_\alpha, \omega_\sigma$.

The total power flow from all N_1 modes of subsystem 1 to mode σ of subsystem 2 is, therefore,

$$\Pi_{i,\sigma} = \langle B_{\alpha\sigma}\rangle\ N_1(E_1-E_2)\quad .$$

Finally, the total power from subsystem 1 to subsystem 2 is found by summing over the N_2 modes of subsystem 2;

$$\Pi_{12} = \langle B_{\alpha\sigma}\rangle\ N_1N_2\ (E_1-E_2)$$

$$= \frac{\pi}{2}\ \Delta\omega\ \frac{\mu^2\omega^2+(\gamma^2+2\mu\kappa)+\kappa^2/\omega^2}{\delta\omega_1\delta\omega_2}(E_1-E_2) \tag{3.2.9}$$

where $\delta\omega_i = \Delta\omega/N_i$ is the average frequency separation between modes.

Several interesting features may be seen in this result. The first is that the power flow is proportional to the bandwidth $\Delta\omega$, a result that agrees with our intuition. Secondly, the power flow is proportional to the difference in average actual modal energies of the two subsystems, which by now we also expect. We recall from Fig. 3.2 and Eq. (3.1.7) that κ/ω is the frequency shift produced by stiffness coupling alone. We may suspect (and more detailed analysis confirms) that the numerator of the fraction in Eq. (3.2.9) is the square of the frequency deviation produced by the coupling. The ratio of this to the square of the geometric mean of the average modal spacings of the two systems is a measure of the strength of the power flow.

Let us define the total energy of the subsystems by $E_{1,tot}$ and $E_{2,tot}$. Then, clearly since $E_1=E_{1,tot}/N_1$ and $E_2=E_{2,tot}/N_2$,

$$\Pi_{12}=\langle B_{\alpha\sigma}\rangle N_1N_2 \left[\frac{E_{1,tot}}{N_1} - \frac{E_{2,tot}}{N_2}\right] \equiv \omega\eta_{12} \left[E_{1,tot} - \frac{N_1}{N_2} E_{2,tot}\right] \tag{3.2.10}$$

where $\eta_{12}\equiv\langle B_{\alpha\sigma}\rangle N_2/\omega$. If we define $\eta_{21}=N_1\eta_{12}/N_2$, then

$$\Pi_{12} = \omega \ (\eta_{12}E_{1,tot}-\eta_{21}E_{2,tot}) \quad . \tag{3.2.11}$$

The quantities η_{12} and η_{21} are called the <u>coupling loss factors</u> for the systems 1, 2. By introducing them, we have lost the symmetry of the energy flow coefficients of the preceding equations, but the coupling loss factor has strong physical appeal. Notice that $\omega\eta_{12}\ E_{1,tot}$ represents the power lost by subsystem 1 due to coupling, just as the quantity $\omega\eta_1\ E_{1,tot}$ represents the power lost to subsystems 1 by damping, as measured by the damping loss factor $\eta=\Delta/\omega$. Of course, if system 1 is connected to another system with energy $E_{2,tot}$, it will receive an amount of power $E_{2,tot}\omega\eta_{21}$. The basic relationship

$$n_1\eta_{12} = n_2\eta_{21} \tag{3.2.12}$$

is very useful in practical situations of power flow calculations.

We may now use these results to calculate the still unknown system energies. Consider the system of Fig. (3.4b) to be more simply represented as shown in Fig. 3.5. Then the power flow equations for systems 1 and 2 respectively are

$$\Pi_{1,in}=\Pi_{1,diss}+\Pi_{12}=\omega[\eta_1 E_{1,tot}+\eta_{12}E_{1,tot}-\eta_{21}E_{2,tot}] \tag{3.2.13a}$$

$$\Pi_{2,in}=\Pi_{2,diss}+\Pi_{21}=\omega[\eta_2 E_{2,tot}+\eta_{21}E_{2,tot}-\eta_{12}E_{1,tot}] \tag{3.2.13b}$$

First, consider the case where only one of the systems is directly excited by an external source. Set $\Pi_{2,in}=0$, and accordingly, from Eq. (3.2.13b) we get

$$\frac{E_{2,tot}}{E_{1,tot}} = \frac{\eta_{12}}{\eta_2+\eta_{21}} = \frac{n_2}{n_1}\ \frac{\eta_{21}}{\eta_2+\eta_{21}} \tag{3.2.14}$$

Which is the multi-modal equivalent of Eq. (3.1.27). A solution of Eq. (3.2.13) in terms of both power sources is

$$E_1 = \{\Pi_{i,in}\ (\eta_2+\eta_{21})\ +\ \Pi_{2,in}\eta_{21}\}/\ \omega D \tag{3.2.15a}$$

$$\dot{E}_2 = \{\Pi_{2,in}(\eta_1+\eta_{12}) + \Pi_{1,in}\eta_{12}\}/\omega D \tag{3.2.15b}$$

$$D = (\eta_1+\eta_{12})(\eta_2+\eta_{21}) - \eta_{12}\eta_{21} \ . \tag{3.2.15c}$$

The coupling loss factor or its various equivalent expressions is a measure of inter-modal forces at the system junction, averaged over frequency and over the modes of the interacting systems. Sometimes this calculation may be carried out directly, but more often it is found useful to adopt another view of the interaction - that developed by the impingement of waves on the boundary. Consequently, we now proceed to a discussion of the coupling of wave bearing systems.

3.3 Reciprocity and Energy Exchange in Wave Bearing Systems.

In Paragraph 3.2, we described the vibrations of the systems that we are examining in terms of modes. An alternative formulation of the problem in terms of waves exists that sometimes offers both conceptual and practical computational advantages over a modal analysis. This approach was touched on in paragraph 2.3, wherein we found the dispersion relation between wavenumber and frequency in the form of Eq. (2.3.2). We examined the admittance function for a flat plate, both finite and infinite, to a point force transverse to the plane of the plate.

In paragraph 2.3, we also discussed the vibration of a simple resonator, shown in Fig. 2.9 as a result of its attachment to a plate undergoing random vibration. We found that the energy of the resonator was given by Eq. (2.3.24), which is identical in form to Eq. (3.2.14), where (the resonator is subsystem 2 and the plate is subsystem 1), one has $n_2\Delta\omega = 1$ (one resonator mode), and $E_{1,tot} = M_p\langle v_p^2\rangle$ and $\eta_{coup} = \eta_{21}$. Thus, the quantity η_{coup} that we calculated on the basis of an interaction of the resonator with many modes of the plate is just the coupling loss factor of Paragraph 3.2. But, it is also the same as would be calculated on the basis of the resonator interacting with an infinite plate [See Eq. (2.2.23)]. Consequently, it is often advantageous to develop the equations for coupling loss factors on the basis of interaction with an infinite system, since such calculations may be much simpler than they would be on a modal basis.

We should also make another point about our use of the term "blocked" as it applies to both interacting modal and wave systems. We define the blocked system to be that system that results when the other system has a vanishing response for all time. Thus, for example, when a sound field described by its pressure response interacts with a structure described by its displacement, the "blocked" sound field is that field that occurs when the structure is absolutely rigid. The "blocked" structural vibration is that vibration which occurs when the pressure vanishes, i.e., the in vacuo condition of the structure. We would ordinarily think of such a vibration as "free", but we must distinguish between the "blocked" or "uncoupled" interaction, and our ordinary ideas of free or constrained boundary conditions. The interpretation depends on the variables that we use to define "response" in the adjoining system.

We saw in Eq. (2.4.9) and in Fig. 2.11 that we could form four simple waves out of an eigenfunction of the rectangular plate. Although such a replacement is not universal, it is useful to think of it as a typical example of wave-mode duality. The wave number lattice for normal modes such as shown in Fig. 2.7 may be re-interpreted as a distribution of waves by reflecting the lattice points in the various coordinate axes (or planes) of the wave number space. Incoherence of modes means incoherence of waves in different directions. Equality of modal energy is translated into equality of intensity for wave groups contained by equal angles of spheres or circles in k-space as shown in Fig. 3.6. A cylinder has a more complicated locus for bending modes than that shown in Fig. 3.6 but in this case also, the equivalence between waves and modes is essentially as shown here.

The arguments for coherence and power are based upon the decomposition of modes into waves as developed in Paragraph 2.4. The relation between modal energy and power is obtained by simply noting that equal modal response amplitudes will result in equal amplitudes of each wave in the decomposition of Eq. (2.4.9). Secondly, if the modal responses are incoherent, the wave motions for adjacent points on the wave number circle in Fig. 3.6 will be incoherent. There is still the possibility that there will be appreciable coherence between waves in different directions at any point, since there is functional relationship between the phase of any one wave component of a mode and the 3 (for a two-dimensional mode) other components. However, as noted in the discussion preceding Eq. (2.4.10), this phase coherence is quite small unless the source and observation points are close together. If the source may be assumed to be randomly located over the surface (or volume) of the system, even this coherence will vanish.

We imagine, therefore, a diffuse wave field (in the SEA sense) possessing energy distributed uniformly over the frequency interval $\Delta\omega$. The energy density (Energy per unit "area") of this field is $\Delta\mathcal{E}$. The various parcels of energy are carried by the waves with an energy velocity c_g as defined by Eq. (2.3.7), and the power is distributed among the various directions according to an intensity

$$d(\Delta I) = \Delta\mathcal{E}\, c_g \; D(\Omega)\, d\Omega/\Omega_t \qquad (3.3.1)$$

where $d(\Delta I)$ is the intensity of wave energy in the interval of directions $d\Omega$, $D(\Omega)$ is a weighting function that essentially measures the distribution of area in the interval between k, $k + \Delta k$ as a function of Ω in Fig. 3.6(b), and Ω_t is the total range of the angle Ω. One must have $\langle D(\Omega)\rangle_\Omega = 1$, and it often occurs that $D(\Omega) \equiv 1$ (this is called "isotropy").

Reciprocity. There is a general principle that applies to systems composed of linear, passive and bilateral elements, called reciprocity, that is quite useful in our discussions of wave interactions. If we imagine for the moment that the system is cut up into a very large number of tiny masses, springs and dashpots (dampers) these descriptors mean the following:

(a) linear: the mechanical response of each element is directly proportional to the force (including its sense) causing the response

(b) passive: the only sources operative are those explicitly set aside as sources in the equations of motion; no element of the system can generate energy.

(c) bilateral: in transferring forces from one neighbor to the next, a reversal of roles between the neighbors as to force interactions will result in an exact reversal of the relative motions. Notice that gyrators are excluded from the system at this level, even though the interaction of system modes may end up as a gyroscopic interaction.

The statement of reciprocity may be made with references to Fig. 3.7. If at any two terminal pairs of a reciprocal system we generate a drop ℓ at terminal pair 1 and measure a flow U through a "wire" connecting the terminal pair 2, then if a drop p' is applied to terminal pair 2, a short circuit flow v' will be developed at terminal pair 1. The statement of reciprocity is

$$\frac{v'}{p'} = \frac{U}{\ell} \tag{3.3.2a}$$

The only restriction is that p', U and ℓ,v' be conjugate variables, that is, their time average product equals the power flow appropriate to that set of terminals. Clearly

$$\left|\frac{v'}{p'}\right| = \left|\frac{U}{\ell}\right| \tag{3.3.2b}$$

and if p' and ℓ are noise signals with identical spectral shapes, then

$$\frac{\langle v'^2 \rangle}{\langle p'^2 \rangle} = \frac{\langle U^2 \rangle}{\langle \ell^2 \rangle} \tag{3.3.2c}$$

Similar statements are easily derived for the systems shown in Figs. 3.8 and 3.9. They are

$$\frac{\langle p^2 \rangle}{\langle v^2 \rangle} = \frac{\langle \ell'^2 \rangle}{\langle U'^2 \rangle} \qquad (3.3.3)$$

for system B, and

$$\frac{\langle U^2 \rangle}{\langle v^2 \rangle} = \frac{\langle \ell'^2 \rangle}{\langle p'^2 \rangle} \qquad (3.3.4)$$

for system C. These various statements have their applications depending on whether it is more natural for us to apply forces or velocities at one location, and measure forces or velocities at the second location.

Systems Connected at a Point. As a simple example of reciprocity consider the force exerted on a point clamp at the edge of a plate, as shown in Fig. 3.10, due to vibration of the plate. Let $\ell(t)$ be a band limited noise load applied to an arbitrary point x_s on the surface of the plate, and let ℓ_B be the load that results on the clamp. The load ℓ will inject an amount of power $\langle \ell^2 \rangle G_p$ into the plate (G_p is the mechanical conductance for the plate) and this will result in a m.s. plate velocity $\langle v_p^2 \rangle$ given by

$$\langle v_p^2 \rangle = \langle \ell^2 \rangle G_p / \omega \eta_p M_p \qquad (3.3.5)$$

where η_p is the loss factor of the plate modes and M_p is the plate mass. Now we say that

$$\langle \ell_B^2 \rangle = \Gamma \langle v_p^2 \rangle \qquad (3.3.6)$$

where Γ is the parameter that we seek.

In the reciprocal situation, we drive the clamp with a prescribed noise velocity $v'_B(t)$ that has the same band limited spectrum that $\ell(t)$ has. This velocity source generates an amount of power $\langle v'^2_B \rangle R_1$, ($R_1$ is the input resistance at the edge of the plate) which results in a plate m.s. velocity $\langle v'^2_p \rangle$ given by

$$\langle v'^2_p \rangle = \langle v'^2_B \rangle R_1/\omega\eta_p M_p \tag{3.3.7}$$

If we now apply the reciprocity statement of Eq. (3.3.4), or Fig. 3.9, we obtain $\langle \ell^2_B \rangle / \langle \ell^2 \rangle = \langle v'^2_p \rangle / \langle v'^2_B \rangle$, or

$$\Gamma = R_1/G_{1,\infty}. \tag{3.3.8}$$

Let us now extend this example by connecting a second plate to the first at the edge point under discussion, requiring that they move together in a transverse way but (to keep the matter simple) no moment is transmitted. Again, we let $\ell_1(t)$ be the band limited noise, that generates a m.s. velocity on plate 1 given by $\langle v_1^2 \rangle$. We are interested in finding the velocity $\langle v_2^2 \rangle = \Gamma' \langle v_1^2 \rangle$ that results from this motion. Clearly

$$\langle v_2^2 \rangle = \Gamma' \langle v_1^2 \rangle = \Gamma' \langle \ell'^2 \rangle G_{1,\infty}/\omega\eta_1 M_1 \tag{3.3.9}$$

If we now apply the load $\ell_2'(t)$ (same spectrum as ℓ) at the position where v_2 was measured, we will inject an amount of power $\langle \ell_2'^2 \rangle G_{2,\infty}$ into plate 1 through the junction (call it Π'_{21}) and a certain amount will be dissipated. From our earlier definitions of loss factors and coupling loss factors, this fraction is

$$\Pi'_{21} = \langle \ell_2'^2 \rangle G_{2,\infty} \eta_{21}/(\eta_{21} + \eta_2). \tag{3.3.10}$$

This power, transmitted into plate 1 will produce a m.s. velocity

$$\langle v_1'^2 \rangle = \Pi_{21}'/\omega \eta_1 M_1 . \tag{3.3.11}$$

Applying the reciprocity condition,

$$\Gamma' = \frac{G_{2,\infty}}{G_{1,\infty}} \frac{\eta_{21}}{\eta_2+\eta_{21}} = \frac{\langle v_2^2 \rangle}{\langle v_1^2 \rangle} \tag{3.3.12}$$

If the conditions are such that $G_{2,\infty} = (\pi/2(n_2/M_2)$ and $G_{1,\infty} = (\pi/2)(n_1/M_1)$. Then Eq. (3.3.11) can be rewritten as

$$\frac{M_2 \langle v_2^2 \rangle}{n_2} = \frac{M_1 \langle v_1^2 \rangle}{n_1} \frac{\eta_{21}}{\eta_2+\eta_{21}} , \tag{3.3.13}$$

which, of course, is the same result as Eq. (3.2.14). In a sense, this offers an alternative derivation of the basic power flow relations of SEA and it avoids discussion of either modes or waves. Of course, to evaluate $G_{1,\infty}$ and $G_{2,\infty}$, one must have either a modal or wave model. Also, to evaluate η_{21}, one must either do a modal analysis, following the chain from Eqs. (3.2.4, 5, 6), through Eq. (3.2.9) and (3.2.10), or proceed by wave analysis.

We can relate the power flow from plate 1 to plate 2 to the blocked force given by Eqs. (3.3.6) and (3.3.8), by subdividing the situation shown in Fig. 3.12a with the two problems shown in Fig. 3.12b. The actual force ℓ that the function applies to plate 1 is equal to the blocked force ℓ_B, less the force induced by the velocity, $\ell_1 = vZ_1$. Since our convention is that upward forces and velocities are positive, the upward velocity v imposed at the boundary will result in upward force vZ, on the edge of plate 1. The motion v will also produce an upward force vZ_2 on plate 2 and a consequent downward reaction force on the edge of plate 1. Consequently,

$$\ell = - vZ_2 = \ell_B + vZ_1$$

or

$$\langle \ell_B^2 \rangle = \langle v^2 \rangle \; |Z_1+Z_2|^2 \tag{3.3.14}$$

We now relate $\langle \ell_B^2 \rangle$ to $\langle v_1^2 \rangle$ by Eq. (3.3.6). We can relate $\langle v^2 \rangle$ to $\langle v_2^2 \rangle$ by a relation similar to that of Eq. (3.3.7);

$$\langle v_2^2 \rangle = \langle v^2 \rangle \, R_2 / \, \omega M_2 \eta_2 \; . \tag{3.3.15}$$

We may then combine Eqs. (3.3.6, 8, 14, 15) to obtain

$$\frac{M_2 \langle v_2^2 \rangle}{n_2} = \frac{M_1 \langle v_1^2 \rangle}{n_1} \left\{ \frac{2}{\pi} \; \frac{R_1 R_2}{|Z_1+Z_2|^2} \right\} \frac{1}{\omega \eta_2} \tag{3.3.16}$$

In many ways, Eq. (3.3.16) is the multi-dof equivalent of Eq. (3.1.19) since $M\langle v_1^2 \rangle$ is determined in Eq. (3.3.16) by the power injected into system 1, and the power flow coefficient in the relation

$$a_{12} = \frac{2}{\pi \omega \eta_1} \; \frac{R_1 R_2}{|Z_1+Z_2|^2} \tag{3.3.17}$$

is the "coupling loss factor" for the uncoupled system energies. It satisfies the same reciprocity relation that η_{12} does - namely

$$n_1 a_{12} = n_2 a_{21} \quad . \tag{3.3.18}$$

The quantity a_{12} will take on various forms depending on the nature of the interacting systems, and its relation to η_{12} will depend on the system. We shall come back to this in paragraph 3.4.

The result of Eq. (3.3.17) for the system in Eq. 3.11 is typical. Since η_{12} must ultimately be determined by a_{12}, the coupling loss factor is usually representable in terms of certain junction impedances or averages over such impedances. If we had allowed moment constraints between the systems, then our impedances would have become matrices involving both force and moment terms, but the structure of the result would remain the same. The impedances that enter Eq. (3.3.17), $Z_1=R_1-iX_1$ and $Z_2=R_2-iX_2$ may be evaluated on either a modal or wave basis. As we saw in Chapter 2, there are circumstances in which a junction impedance can be quite simply calculated on a wave basis if resonance frequencies are averaged over a band of frequencies and if the drive point is also randomly located. Thus, one important use of wave analysis in SEA is in the calculation of junction impedances.

Systems Connected along a Line. The coupling loss factor is fairly easy to derive when the systems are connected at one or more points. Another relatively tractable situation is when the connection is along a line (for two dimensional systems) or along a surface (for three dimensional systems), if the dimensions of the line (or surface) are large compared to the length of a free wave in the system. In this case, a set of waves in the interval $d\Omega$, shown in Fig. 3.13 will be partially transmitted and partially reflected at the junction. In acoustics, the ratio of the transmitted to incident power is called the transmission coefficient $\tau(\Omega)$ and will in general depend on the direction Ω that the incident waves are travelling with respect to the bounding line or surface.

By our hypothesis, therefore, the total transmitted power will be

$$\Pi_{12} = \int_{\Omega_{inc}} \tau(\Omega)\ d(\Delta I)\ L_p(\Omega) \tag{3.3.19}$$

where $d(\Delta I)$ is given by Eq. (3.3.1), Ω_{inc} is the range of angles of propagation that impinge upon the boundary, and $L_p(\Omega)$ is the projection of the boundary length (or area) presented to the waves travelling in direction Ω. Since Π_{12} is the transmitted power due to energy $A_1 \Delta\mathcal{E}_1 = E_1$, where A_1 is the area (or volume) of system 1, we have

$$\eta_{12} = \frac{\Pi_{12}}{\omega A_1 \Delta\mathcal{E}_1} = \frac{c_g}{\omega A_1 \Omega_t} \int_{\Omega_i} \tau(\Omega)\ L_p(\Omega) D(\Omega) d\Omega. \tag{3.3.20}$$

For example, in a 2-dimensional isotropic system, let the transmitting boundary be a straight line of length L and Ω be the angle of wave incidence, as shown in Fig. 3.14. Then $D(\Omega) = 1$, $\Omega_i = \pi$, $\Omega_t = 2\pi$.

$$\eta_{12} = \frac{c_g L}{2\pi\omega A_1} \int_{\pi/2}^{\pi/2} \tau(\Omega)\ \cos\Omega\ d\Omega = \frac{c_g L}{2\omega A_1} \langle \tau(\Omega) \cos\Omega \rangle_{\Omega_i}$$

[2-dim'l] (3.3.21)

Of course, the transmissibility must be known, but this information is usually available from existing sources, or is calculable. In the 3-dimensional acoustic case, $c = c_g =$ speed of sound; $A_1 \to V_1$, the room volume and $L \to A_w$, the area of the adjoining partition. Also $\Omega_t = 4\pi$ (steradians) and $\Omega_i = 2\pi$ steradians. Then,

$$\eta_{12} = \frac{cA_w}{2\omega V_1} \langle \tau(\Omega) \cos\phi \rangle_{\Omega_i} \text{ [3-dim'l]} \tag{3.3.22}$$

where ϕ is the angle between the wave vector and the normal to the panel, and $-10 \log \langle\tau(\Omega)\cos\phi\rangle_\Omega$ is the ordinary transmission loss (TL) of building acoustics. In this situation, therefore, we would be advised to evaluate η_{12} from existing data sources on wall transmission loss.

3.4 Some Sample Applications of SEA.

Here, we present some applications of SEA that demonstrate its usefulness in relatively simple situations that are of practical interest. The emphasis here is in the use of relations developed in paragraph 3.3, rather than in development of the theory. Of course, every new application will generate formulas or data for the parameters that we are interested in, such as modal densities, coupling loss factor, and damping loss factor.

Resonator Excited by a Reverberant Sound Field. As the first example we consider a resonator formed by the small piston of mass M_0 in the wall of a room as shown in Fig. 3.15. The piston is supported by a spring of stiffness $M\omega_0^2$ and a dashpot having a mechanical resistance $\omega_0\eta_0M_0$. We shall call the room system "R" and the resonator system "0". Then, according to Eq. (3.2.14)

$$\frac{E_0}{E_R} = \frac{1}{n_R\Delta\omega} \frac{\eta_{OR}}{\eta_0+\eta_{OR}} \tag{3.4.1}$$

where η_{OR} is the coupling loss factor from the resonator to the room. We evaluate η_{OR} by using Eq. (3.3.17). The impedance Z_R is the radiation impedance of the piston "looking into" the room

$$Z_R = R_{rad} - i\omega\, M_{rad} \quad . \tag{3.4.2}$$

These quantities are well known for the piston radiator. The impedance "looking into" the resonator was given in Eq. (2.1.19),

$$Z_0 = \omega_0\eta_0 M_0 - i\omega\ M_0(1-\omega_0^2/\omega^2) \tag{3.4.3}$$

Thus, averaging a_{OR} (defined by Eq. (3.3.17)) over the band $\Delta\omega$ gives

$$\omega_0\langle a_{OR}\rangle_{\Delta\omega} = \frac{2\Delta\omega}{\pi}\left\{\frac{R_{rad}\omega_0\eta_0 M_0}{|\omega_0\eta_0 M_0+R_{rad}-i\omega(M_0+M_{rad})+i\omega_0^2 M_0/\omega|^2}\right\}_{\Delta\omega}$$

$$= \frac{\omega_0\eta_0 R_{rad}}{\omega_0\eta_0\ M_0+R_{rad}} \tag{3.4.4}$$

and substituting ω_0 $\langle a\rangle$ into Eq. (3.3.16) ,

$$M_0\ \langle v_0^2\rangle = \frac{\langle p_R^2\rangle\ V_R}{\rho c^2}\ \frac{1}{n_R\Delta\omega}\ \frac{R_{rad}}{\omega_0\eta_0 M_0+R_{rad}} \tag{3.4.5}$$

where we have used $\eta_2=\eta_0$. Comparing Eqs. (3.4.5) and (3.4.1), we see that

$$\eta_{OR} = R_{rad}/\omega M_0\quad . \tag{3.4.6}$$

The coupling loss factor for the piston resonator, therefore, is determined by the radiation resistance of the piston "looking into" the room. When $R_{rad}>>\omega_0\eta_0 M_0$, the energy of the resonator will equal the average model energy of the sound level.

Two Beams in Longitudinal Vibration. As a second example, consider the problem of two long beams connected by a fairly weak spring, as shown in Fig. 3.16. If beam 1 is directly excited to energy $M_1 \ \langle v_1^2 \rangle$ and beam 2 is only excited through the connection, then by our assumptions, the energy flow will be fairly weak into beam 2. We may use the result expressed by Eq. (3.3.16) directly, where we define system 1 to the beam 1 and the spring, and system 2 to be beam 2 alone. Then, the average mechanical impedance looking into beam 2 is

$$R_{2,\infty} \ (\rho_2 c_2 A_2)^2 \ \frac{\pi}{2} \ \frac{n_2}{M_2} = \frac{1}{2} \ \rho_2 c_2 A_2 = Z_2 = R_2 \qquad (3.4.7)$$

The impedance looking into system 1 is complex

$$Z_1 = \frac{R_{1,\infty}(iK/\omega)}{R_{1,\infty} + i \ K/\omega} \approx \frac{ik}{\omega} \ \frac{(K/\omega)^2}{R_{1,\infty}} \qquad (3.4.8)$$

Using $R_1 = R_e(Z_1)$ and $K/\omega << R_{1,\infty}, R_{2,\infty}$, we get

$$\frac{M_2 \langle v_2^2 \rangle}{n_2} = \frac{M_1 \langle v_1^2 \rangle}{n_1} \ \frac{2}{\pi \omega \eta_n} \ \frac{4K^2/\omega^2}{\rho_1 c_1 A_1 . \rho_2 c_2 A_2} \qquad (3.4.9)$$

In this case, $a_{12} \approx \eta_{12}$ (modal energy of system 2 << modal energy of system 1) and we can say

$$\eta_{12} = \frac{8}{\pi} \ \frac{K^2/\omega^2}{\rho_1 c_1 A_1 . \rho_2 c_2 A_2} \ . \qquad (3.4.10)$$

Note the similarity between this result and that of Eq. (3.1.28).

Two rooms coupled by a limp curtain. As a final example in this section, we consider an application of Eq. (3.3.22). We want to evaluate η_{12} for the system shown in Fig. 3.17. Two rooms, of volume V_1 and V_2 are separated by a "wall" that has a mass density m and area A_w. The average transmission coefficient for this situation is known to be approximately

$$<\tau(\Omega)\ \cos\phi>_{\Omega_i} = \frac{1}{3}\left(\frac{\rho c}{\omega m}\right)^2 . \tag{3.4.11}$$

Thus

$$\eta_{12} = \frac{\rho^2 c^3 A_w}{\omega^3 m^2 V_1} . \tag{3.4.12a}$$

The energy expression Eq. (3.2.14) in this case becomes simply

$$<p_2^2>\ V_2 = \ <p_1^2>\ V_1\ \frac{\eta_{12}}{\eta_2+\eta_{21}} . \tag{3.4.13}$$

Where $\eta_2 = 2.2/fT_R^{(2)}$, $T_R^{(2)}$ is the reverberation time of room 2, and

$$\eta_{21} = \frac{\rho^2 c^3 A_w}{6\omega^3 m^2 V_2} . \tag{3.4.12b}$$

Equation (3.4.13) predicts that $p_1^2 = p_2^2$ when losses in V_2 due to transmission of sound through the wall exceed the losses due to sound absorption in room 2.

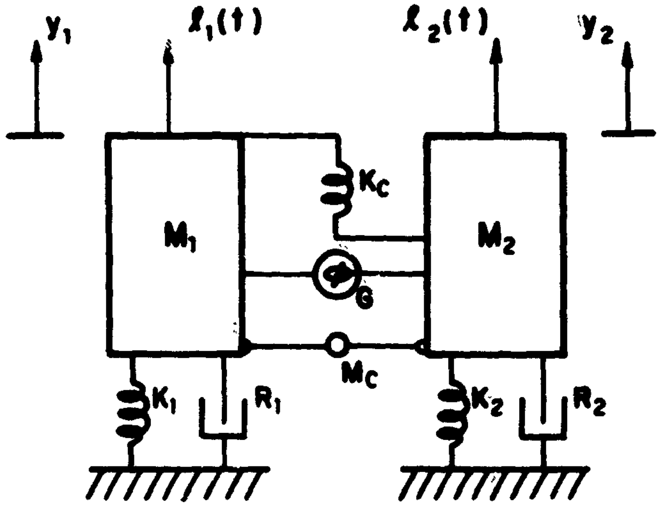


FIG 3.1

TWO LINEAR RESONATORS COUPLED BY SPRING, MASS, AND GYROSCOPIC ELEMENTS

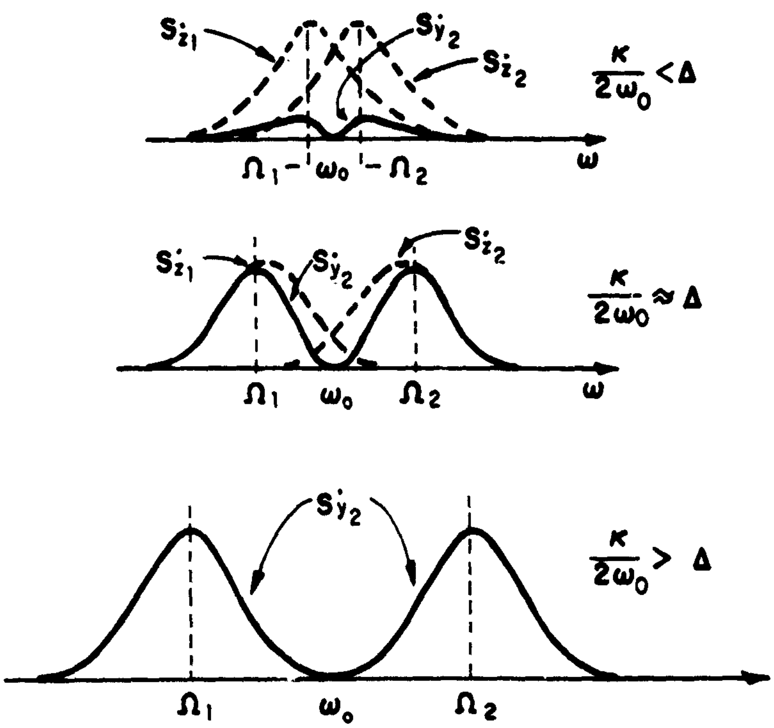


FIG 3.2

SPECTRAL DENSITY OF INDIRECTLY EXCITED RESONATOR VELOCITY AS A FUNCTION OF THE DEGREE OF COUPLING

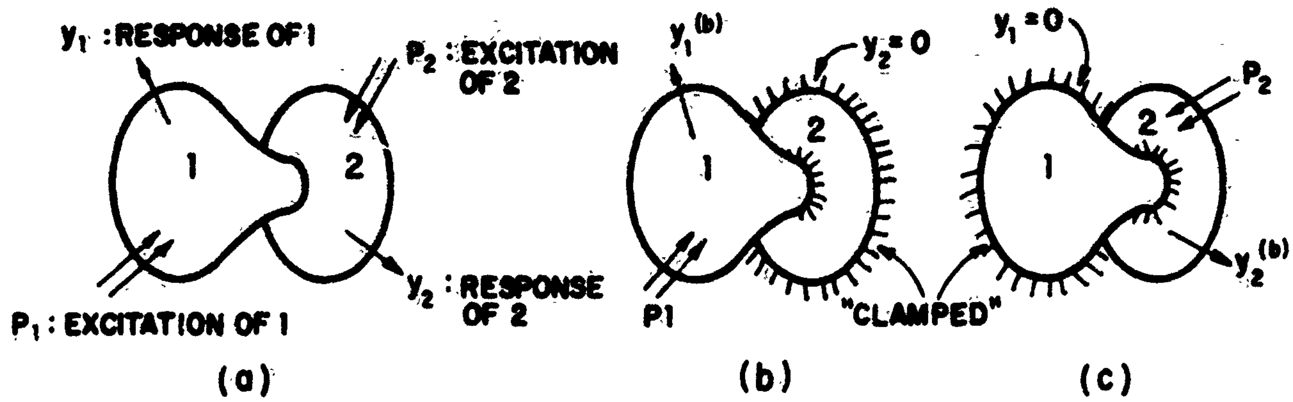


FIG 3.3

(a) SYSTEM OF INTEREST; TWO SUBSYSTEMS INDEPENDENTLY EXCITED AND RESPONDING BOTH TO EXCITATION AND COUPLING FORCES (b) SYSTEM 1 VIBRATING IN RESPONSE TO ITS OWN EXCITATION, SYSTEM 2 IS BLOCKED, (c) SYSTEM 2 IS VIBRATING IN RESPONSE TO ITS EXCITATION, SYSTEM 1 IS BLOCKED.

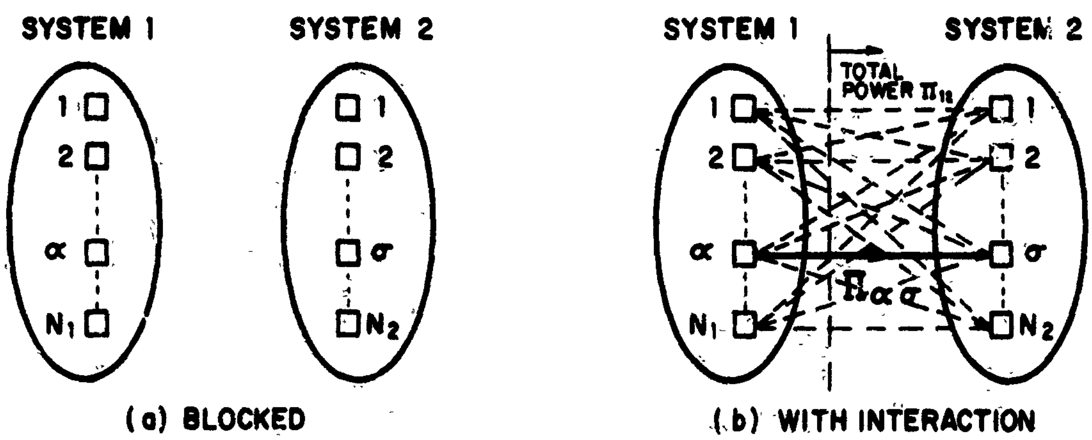


FIG 3.4

(a) SHOWS MODAL SETS WHEN THERE IS NO INTERACTION, AS ACHIEVED BY CONDITIONS SHOWN IN FIG 3.3 (b,c).
(b) SHOWS MODE PAIR INTERACTIONS THAT OCCUR WHEN ACTUAL CONDITIONS ARE OBTAINED AT THE JUNCTION

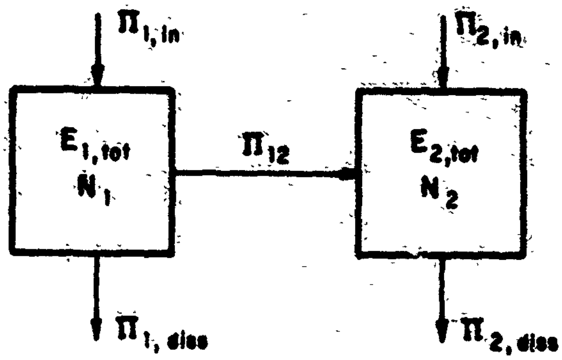


FIG 3.5

AN ENERGY TRANSFER AND STORAGE MODEL THAT CAN REPRESENT THE SYSTEM SHOWN IN FIG 3.4 (b) WHEN THE SEA MODEL IS USED THE DEFINING PARAMETERS ARE THE MASSES M_1, M_2; THE LOSS FACTORS η_1, η_2; THE MODAL DENSITIES $n_1 = N_1/\Delta\omega$ $n_2 = N_2/\Delta\omega$ AND THE COUPLING LOSS FACTOR η_{12}

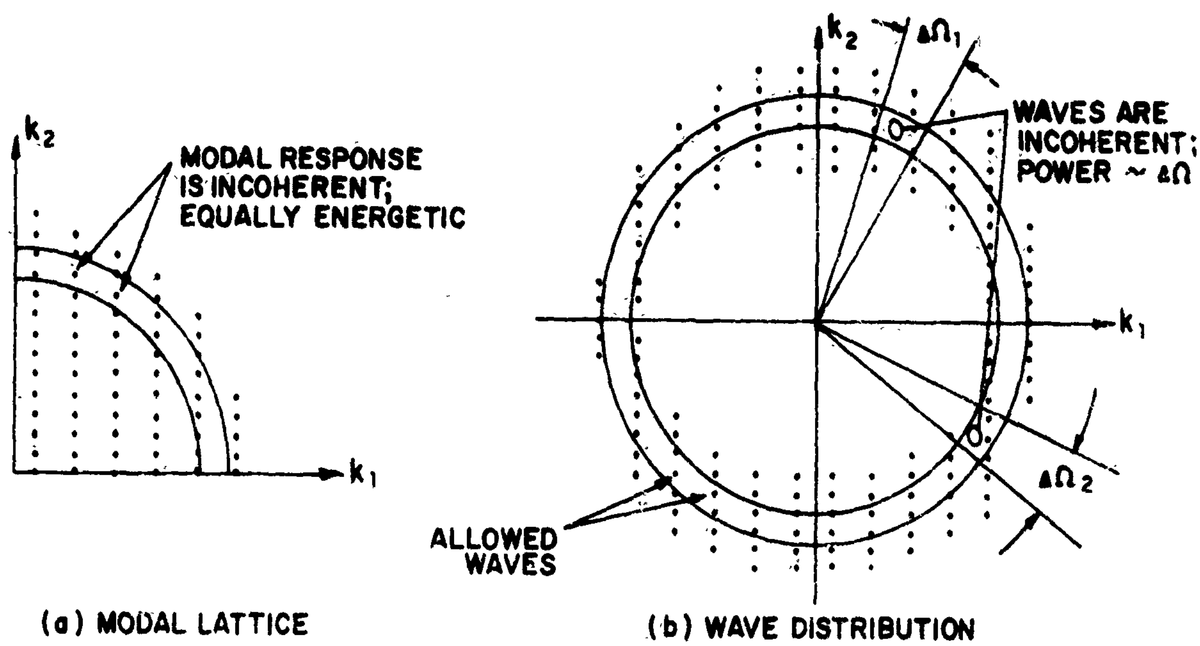


FIG 3.6

EQUIVALENCE OF MODAL AND WAVE COHERENCE ASSUMPTIONS IN SEA

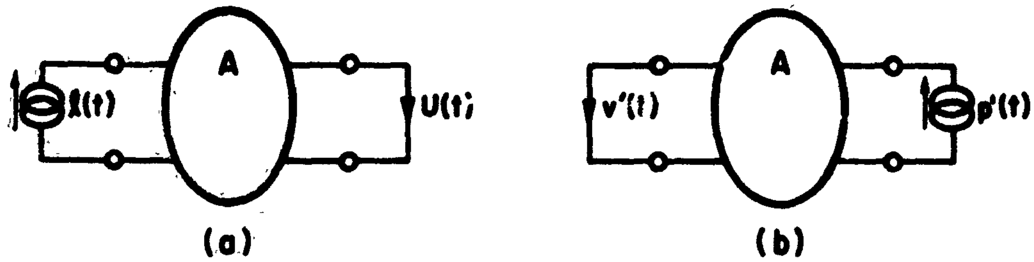


FIG 3.7

IF THE SYSTEM A IS RECIPROCAL, THEN IF ℓ AND p' ARE "PRESCRIBED DROP" SOURCES AT FREQUENCY ω, THEN ONE HAS $U/\ell = v'/p'$

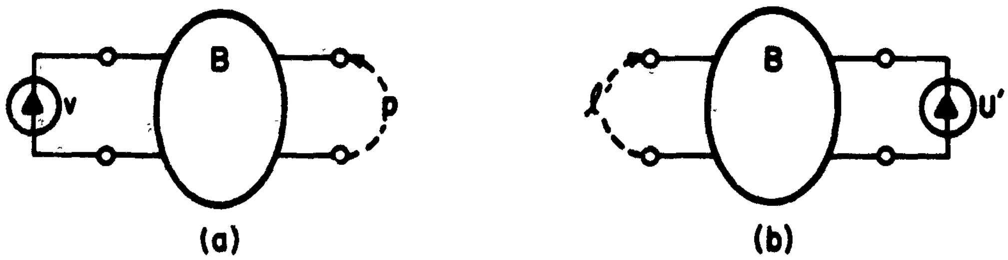


FIG 3.8

SYSTEM B IS THE SAME AS SYSTEM A WITH THE TWO TERMINAL PAIRS "OPEN" THE RECIPROCITY STATEMENT IN THIS CASE IS $p/v = \ell'/U'$

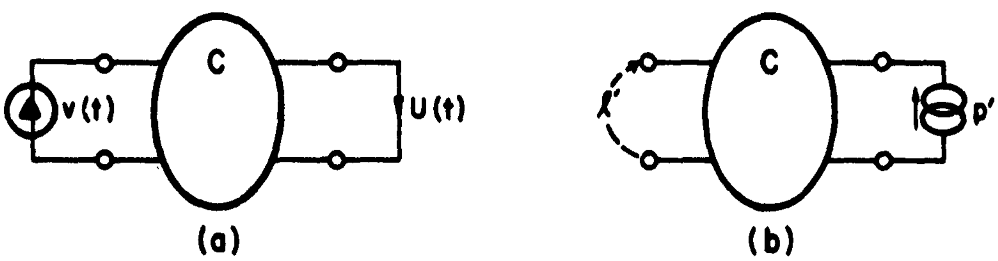


FIG 3.9

SYSTEM C IS THE SAME AS SYSTEM A WITH TERMINAL PAIR 1 OPEN – CIRCUITED AND TERMINAL PAIR 2 SHORT-CIRCUITED. THE RECIPROCITY STATEMENT IS NOW $U/v = \ell'/p'$.

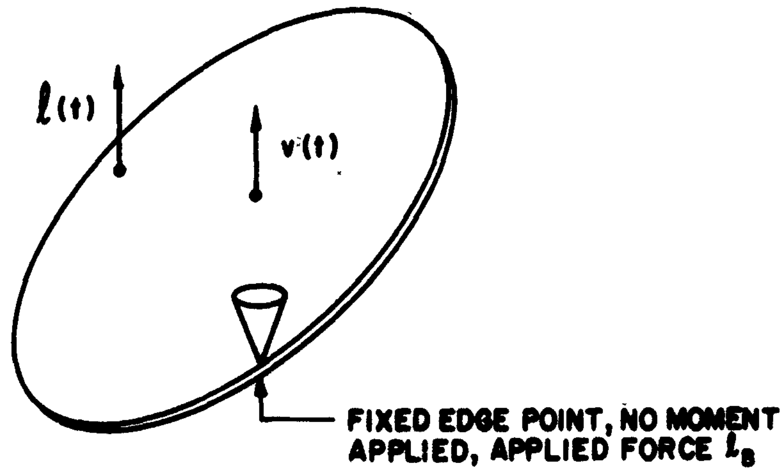


FIG. 3.10

PLATE DRIVEN BY POINT LOAD NOISE SOURCE WITH SECOND POINT ON PLATE EDGE FIXED WITHOUT MOMENT RESTRAINT

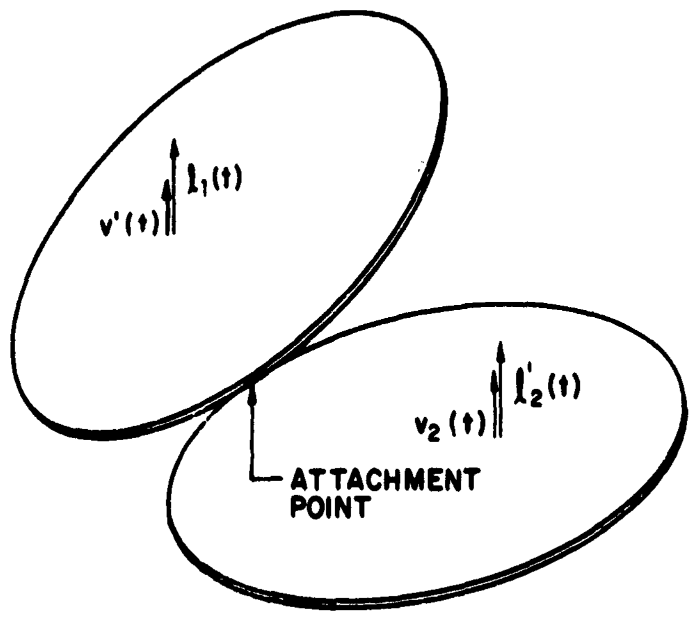


FIG. 3.11

THE PLATE IN FIG. 3.10 IS NOW JOINED TO ANOTHER PLATE AT ONE POINT ALONG THE EDGE WITH NO MOMENT COUPLING OR RESTRAINT

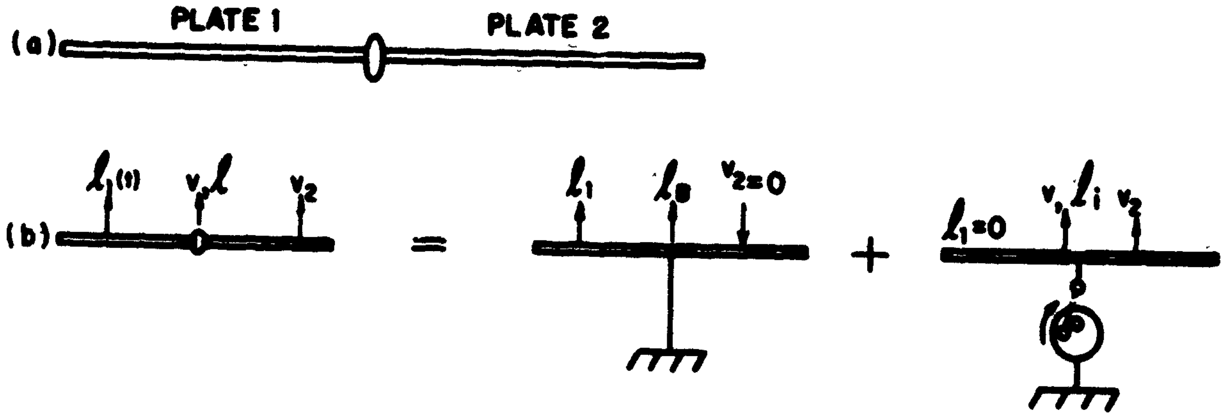


FIG 3.12

SKETCH OF 2-PLATE SYSTEM OF FIG 3.11 SHOWING HOW THE ACTUAL INTERACTION IS DEVELOPED AS A COMBINATION OF 2 IDEALIZED PROBLEMS

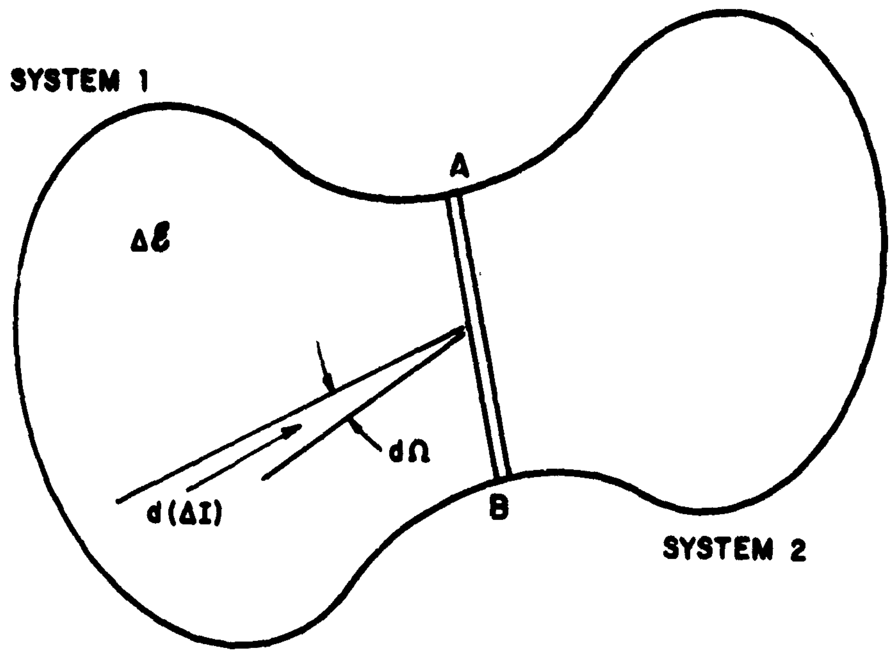


FIG 3.13

ENERGY FLOW TOWARD SYSTEM JUNCTION IN THE ANGULAR INTERVAL dΩ

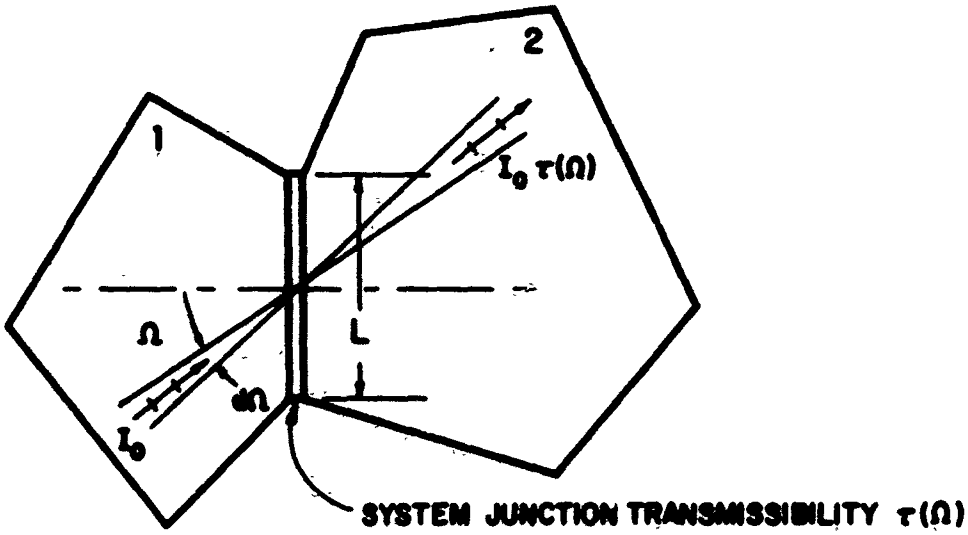


FIG 3.14

SKETCH SHOWING TRANSMISSION OF POWER BETWEEN TWO SYSTEMS THROUGH A LINEAR JUNCTION

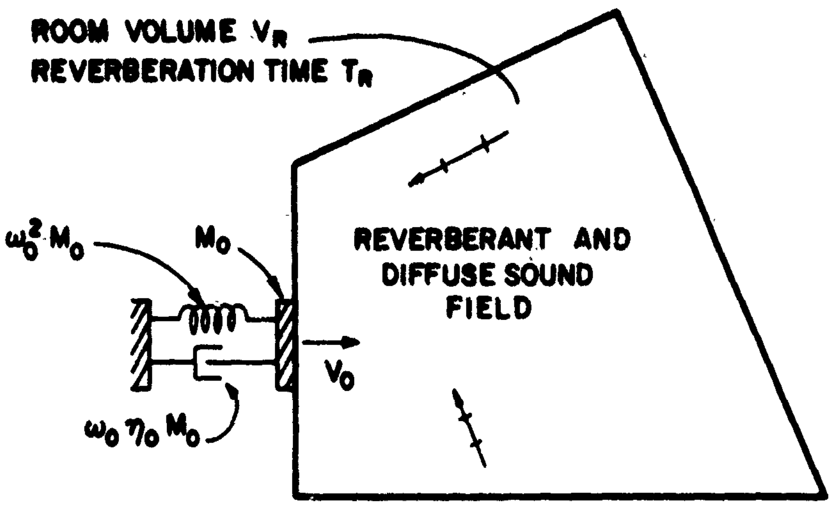


FIG 3.15

A PISTON RESONATOR IN THE WALL OF A REVERBERATION CHAMBER INTERACTING WITH THE CONTAINED SOUND FIELD

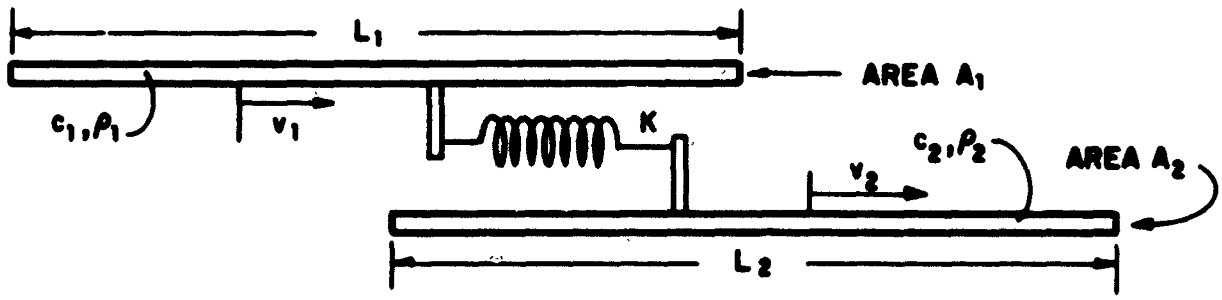


FIG 3.16

TWO BEAMS IN LONGITUDINAL VIBRATION
COUPLED BY A SPRING OF STIFFNESS K

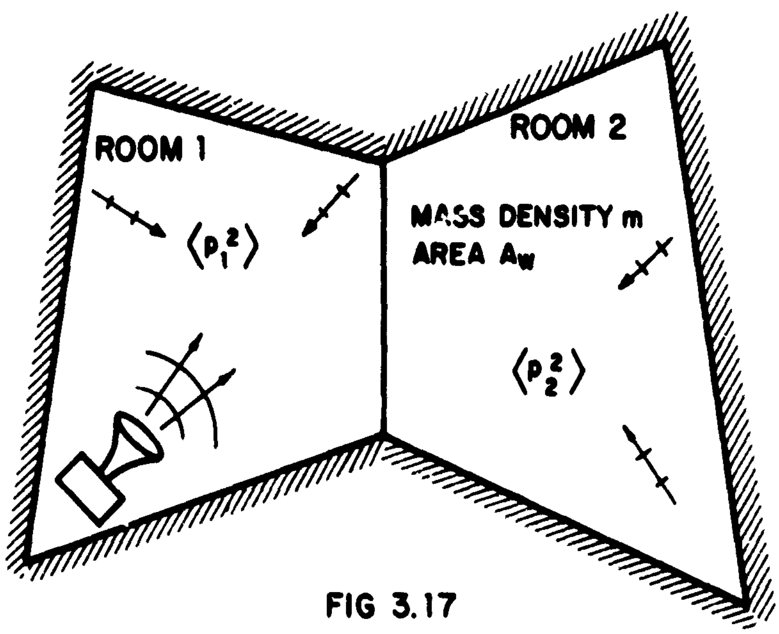


FIG 3.17

TWO ROOMS COUPLED BY LIMP WALL OF SURFACE DENSITY m

CHAPTER 4. THE ESTIMATION OF RESPONSE IN STATISTICAL ENERGY ANALYSIS

4.0 Introduction

In the preceding chapters, we have shown how the energy of vibration may be estimated in an average sensed based on system parameters that are fundamentally descriptive of energy storage and transfer capabilities of the systems. The use of energy quantities has the great advantage that sound, vibration and other resonant systems all are described by the same variables. Consequently, our viewpoint has great generality which is useful when we must deal with systems of such complexity.

This generality, however, has its shortcomings. We must remember that our motivation for these studies is usually not directed to the energy of oscillation, as such. Systems do not make too much noise, or fatigue, or have component malfunction because they contain too much energy, but because they move, are strained or are stressed too much or too often. Consequently, we must interpret the energy of oscillation in terms of dynamical variables of engineering interest such as velocity, strain, pressure, etc. We have already done this to a degree in earlier chapters, particularly in paragraphs 3.4. In paragraph 4.1 we shall expand on this important aspect of response estimation.

The calculations and formulas of energy prediction in Chapter 3 are directed toward predicting average energies, the average taken over the set of systems forming the ensemble of "similar" systems constructed according to the SEA model described in paragraph 3.2. Since we have a population of systems, the energy actually realized by any one system (the one sitting in the laboratory, for example) will not be precisely equal to the average energy as calculated by the SEA formulas. We can get an idea of how much our system may deviate from the average system by calculating a standard deviation (s.d.) of the system energy. This has, in fact, been done for some cases, and we can give some general ideas for estimating the variance (square of the s.d.) for practical situations. The calculation of response variance is discussed in paragraph 4.2.

The calculation of standard deviation is reassuring to one using the mean energy as an estimate only if the s.d. is a small fraction of the mean. Then we know that most realizations of the system (again including the one sitting in the laboratory) will have a response that is close to the mean. But what if the standard deviation is equal to the mean, or greater? In this event, the probability that any one realization is close to the mean will be small and the mean value loses its worth as an estimate.

In such a situation, it is common to calculate the probability that a realization will occur within some stated interval of response amplitudes. The probability associated with this "confidence interval" is called the "confidence coefficient". In order to carry out these calculations, however, we need a probability distribution for the energy. This distribution is introduced and the estimation procedures are derived from it in paragraph 4.3.

The distributions used for generating confidence intervals tend to be fairly accurate of the "middle range" of response levels, but are generally fairly inaccurate in predicting response having a very small probability. If we have a special interest in either the greatest or smallest response levels that may occur, we usually must take a different approach. Paragraph 4.4 describes the estimation of the probability of very large values (having small probability) in a particular situation. It may be argued that the problem treated in paragraph 4.4 is not properly an SEA problem, because it deals with the coherent properties of modal response. Nevertheless, the entire approach of describing modal responses by their m.s. values and adopting a statistical view of the population of systems being considered is very much within the SEA framework.

4.1 Mean Value Estimates of Dynamical Response

We can make useful estimates of the mean square amplitude of response based on the energy calculations in Chapter 3. We first consider systems like those discussed in paragraphs 2.3 and 3.4, in which only one mode of one system is interacting with a group of modes of another. This one mode may actually be a single dof system like that shown in Fig. 2.1 or it may be a mode that we single out for particular examination within a multi-modal system. We may do this by restricting the excitation bandwidth so that this is the only system mode that is resonant.

Single Mode Response

Suppose that the multimodal system 1 is excited by noise sources of bandwidth $\Delta\omega$. System 1 has total energy $E_{1,tot}$ and N_1 modes that resonate in this bandwidth. System 2 has only one mode with mode shape $\psi_2(x_2)$. According to paragraph 3.2, the energy of this mode will be given by

$$E_2 = E_1 \frac{\eta_{21}}{\eta_2+\eta_{21}} \tag{4.1.1}$$

where η_{21} may be determined from averages over interaction parameters as in Eq. (3.2.8) or from boundary impedances as in paragraph 3.3. Let us assume that this has been done, and we now want to find the actual response at location x_2.

From Eq. (2.2.3) we have

$$y_2(x,t) = Y_2(t)\ \psi_2(x), \tag{4.1.2}$$

with the general result that

$$E_2 = \int_X dx\ \rho\langle\dot{y}^2\rangle_t = M_2\langle\dot{Y}_2^2(t)\rangle_t = M_2\omega_2^2\ \langle Y_2^2(t)\rangle_t \tag{4.1.3}$$

where M_2 is the mass of system 2 and ω_2 is its resonance frequency (which might just as well be taken as ω, since ω_2 is within $\Delta\omega$ by hypothesis). Thus

$$
\langle y_2^2(x,t)\rangle_t = \frac{E_2}{\omega^2 M_2}\,\psi_2^2\ (x) = \left[\frac{E_{1,tot}}{N_1}\ \frac{\eta_{21}}{\eta_2+\eta_{21}}\right]\frac{1}{\omega^2 M_2}\,\psi_2^2(x)
$$

(4.1.4)

Thus, our statistical model of the system does not restrict us in any way from determining the spatial distribution of response. Of course, the normalization of the ψ_2 function still ensures that

$$
\langle y_2^2\rangle_{\rho,t} = E_2/\omega^2 M_2 .
$$

In Fig. 4.1 we show the form of the distribution of y_2^2 for a typical mode shape of a beam. The distribution of $\langle y_2''\rangle^2$ which would be proportional to the surface strain in the beam, is also sketched.

<u>Multi-Modal Response</u>. When there are several resonant modes (N_2) of (indirectly excited) system 2 in the band $\Delta\omega$, then Eq. (4.1.1) has the form

$$
\frac{E_{2,tot}}{N_2} \equiv E_2 = \frac{E_{1,tot}}{N_1}\ \frac{\eta_{21}}{\eta_2+\eta_{21}} \tag{4.1.5}
$$

and the m.s. response of the system in time and space (or mass) is

$$
\langle y_2^2\rangle_{\rho,t} = E_{2,tot}/\omega^2 M_2, \tag{4.1.6}
$$

which we may regard as our estimate of the system response. Note, however, that if such an estimate were made in the situation of Fig. 4.1, it would be rather poor, since the actual value of $<y^2>_t$ only equals the estimate $<y^2>_{\rho,t}$ at a finite number of points and oscillates around the estimate without converging on it.

In the case of two or three dimensional systems with many modes of vibration, the function $<y_2^2>_t$ tends to converge on the estimate of Eq. (4.1.6) in many important instances. To demonstrate this, we consider a rectangular supported plate which has modes

$$\psi(x) = 2 \sin k_1 x_1 \sin k_2 x_2 \tag{4.1.7}$$

where

$$k_1 = \sigma_1 \pi/\ell_1, k_2 = \sigma_2 \pi/\ell_2; \sigma_1 \sigma_2 = 1,2...$$

The modes that resonate within the interval $\Delta k = \Delta\omega/c_g$ are shown in the hatched region of Fig. 4.2.

As a result of our model of the system interaction, spelled out in paragraph 3.2, we can make the following observations about the response of system 2.

(1) The response may be written in the form

$$y_2(x,t) = \sum_{\sigma} Y_\sigma(t) \psi_\sigma(x) \tag{4.1.8}$$

(2) The modal response amplitudes $Y_\sigma(t)$ are incoherent;

$$<Y_\sigma(t) Y_\tau(t)>_t = <Y_\sigma^2(t)>_t \delta_{\sigma,\tau} \tag{4.1.9}$$

(3) **The modes of system 2 are equally energetic**

$$\langle Y_\sigma^2(t)\rangle_t = \langle Y_2^2(t)\rangle_t = E_2/\omega^2 M_2 \tag{4.1.10}$$

Thus, we can say that

$$\langle Y_2^2(x,t)\rangle_t = \sum_\sigma \sum_\tau \langle Y_\sigma(t) Y_\tau(t)\rangle_t \psi_\sigma(x)\ \psi_\tau(x)$$

$$= \langle y_2^2(t)\rangle_t \sum_\sigma \psi_\sigma^2(x), \tag{4.1.11}$$

where the allowed values of σ are determined by the modes of system 2 that resonate in $\Delta\omega$.

The result of Eq. (4.1.11) is quite general. For the specific two-dimensional system at hand, we can write, therefore,

$$\langle y_2^2\rangle_t/\langle y_2^2\rangle_{t,\rho} = 4N_2^{-1} \sum_\sigma \sin^2 k_1x_1\ \sin^2 k_2x_2$$

$$= N_2^{-1} \sum_\sigma (1-\cos 2k_1x_1)(1-\cos 2k_2x_2)$$

$$\tag{4.1.12}$$

We evaluate the sum by replacing it with an integral over ϕ in the hatched region of Fig. 4.2. We can see that if dN_2 is the number of modes in the rectangle bounded by Δk and $d\phi$, then $dN_2/N_2 = 2d\phi/\pi$, since the modes are uniformly distributed in ϕ. We may then write

$$\langle y_2^2\rangle_t/\langle y_2^2\rangle_{t,\rho} = \frac{2}{\pi}\int_0^{\pi/2}\Big[1-\cos(2kx_1\sin\phi)\cos(2kx_2\cos\phi)$$

$$+ \cos(2kx_1\sin\phi)\cos(2kx_2\cos\phi)\Big]\, d\phi$$

$$= 1-J_0(2kx_1)-J_0(2kx_2) + J_0(2kr) \tag{4.1.13}$$

where $r = \sqrt{x_1^2+x_2^2}$. We have evaluated the integral for the region near x_1, $x_2 = 0$. Since the sum in Eq. (4.1.12) is unaffected by the substitution $x_1 \rightarrow \ell_1 - x_1$, $x_2 \rightarrow \ell_2 - x_2$, the total pattern is as shown in Fig. 4.3.

Thus, we can estimate a spatially varying temporal mean square response of a system even with rather extensive assumptions regarding the system interactions. Patterns of the kind shown in Fig. 4.3 have been developed for a variety of systems, and are available from the literature. They are usually of interest if one wishes to avoid locating an item or making measurements at a point where the response is excessively high or otherwise unrepresentative.

Wave estimates. The use of wave models for SEA response estimates extends the power of SEA greatly. We can develop response patterns like the one shown in Fig. 4.3 equally well using a diffuse wave field model like that discussed in paragraphs 2.4 and 3.3. The more important application of wave notions employs the use of average impedance functions at system boundaries for the evaluation of coupling loss factors as discussed in paragraph 3.3.

As an example, consider the system shown in Fig. 4.4a, a beam cantilevered to a flexible plate. Along the line of connection of these systems there will be very little displacement (we shall assume there is none) because of the very high in-plane impedances of these systems. Nevertheless, power will be transmitted by torques (moments) and rotational motion of the junction. In order to evaluate the coupling loss factor, we must find the power transferred between the systems.

The "decoupled" systems are produced by eliminating the rotation of the contact line as shown in Fig. 4.4b. The modes of the beam for this condition are readily found - the beam has clamped - free boundary conditions. The plate, however, has a very complicated boundary condition - the boundary conditions around its outer edges and clamped along a finite line in the interior! It would require a complicated computer routine to find its modes.

Nevertheless, we can invoke two principles from our work thus far that allow us to solve this problem. The first is that of reciprocity of the coupling loss factor

$$N_p \eta_{pb} = N_b \eta_{bp} \tag{4.1.14}$$

which allows us to use η_{bp}, the beam to plate coupling loss factor (which is easier to calculate) to evaluate η_{pb}. Secondly, the moment impedance "looking into" the plate may be evaluated by considering the plate to be infinite. This has the effect of eliminating reflected energy in the plate from returning to the junction. In a manner completely analogous to the results in paragraphs 2.2 and 2.3 for the admittance for a transverse force, the moment impedance, averaged over the location of the junction of the plate and over resonance frequencies of the interacting systems in the band $\Delta\omega$, is given by the infinite system impedance, which is known.

In paragraph 3.3, we obtained a general result that, in terms of actual boundary impedances, the power flow between systems is given by

$$\Pi_{bp} = \omega a_{bp} E^{(b)}_{b,tot} \tag{4.1.15}$$

where

$$\omega a_{bp} = \frac{2}{\pi n_b} \frac{R_b R_p}{|Z_b + Z_p|^2} \tag{4.1.16}$$

On the other hand, we have the general result that (see Eq. 3.2.10)

$$\Pi_{bp} = \omega\eta_{bp}(E_{b,tot} - \frac{N_b}{N_p} E_{p,tot}) \tag{4.1.17}$$

It must be emphasized that Eqs. (4.1.15) and (4.1.17) are fully equivalent and apply to all conditions of system size, damping, etc. In general η_{bp} and a_{bp} are different functions. However, if the receiving system (the plate in this instance) is made very large, while its loss factor η_p is held fixed, the term $N_b\ E_{p,tot}/N_p$ will vanish in Eq. (4.1.17). If we use the infinite plate impedances in Eq. (4.1.16),

$$a_{bp} = \frac{2}{\eta\omega n_b} \frac{R_b R_p(\infty)}{|Z_b+Z_p(\infty)|^2} \tag{4.1.18}$$

The relation between η_{bp} and a_{bp} is then readily found. Since the input power to the system is independent of the junction,

$$\Pi_{i,in} = E^{(b)}_{1,tot}\ \Delta_1 = E_{1,tot}\Delta_1 + \omega a_{12}\ E^{(b)}_{1,tot} \tag{4.1.19}$$

we have

$$\omega\eta_{bp}\ E_{b,tot} = \omega\eta_{bp} \left[1 - \frac{\omega a_{bp}}{\Delta_b}\right] E^{(b)}_{b,tot}$$

$$= \omega a_{bp} E^{(b)}_{b,tot} \tag{4.1.20}$$

and, consequently,

$$\eta_{bp} = a_{bp}\ (1 - \omega a_{bp}/\Delta_b)^{-1} \tag{4.1.21}$$

When we deal with a single beam mode, then $\Delta_b = R_b/M_b$ and the average of a_{bp} over the band is

$$\langle a_{bp}\rangle_{\Delta\omega} = \frac{2}{\pi\omega}\ \frac{R_b R_p(\infty)}{(R_b + R_p)^2} \cdot \frac{\pi}{2}\ \frac{R_b + R_p}{M_b} . \qquad (4.1.22)$$

Substituting into Eq. (4.1.21) then gives

$$\omega\eta_{bp} = R_p(\infty)/M_b \quad . \qquad (4.1.23)$$

On the other hand, when the modes of the beam are dense so that $n_b\Delta_b >> 1$, the beam "looks infinite", and one has

$$\frac{\omega a_{bp}}{\Delta_b} = \frac{2}{\pi}\ \frac{R_b(\infty) R_p(\infty)}{|Z_b(\infty)+Z_p(\infty)|^2}\ \frac{1}{n_b\Delta_b} << 1 \qquad (4.1.24)$$

and referring to Eq. (4.1.21), we get

$$\eta_{bp} + a_{bp} = \frac{2}{\pi n_b}\ \frac{R_b(\infty) R_p(\infty)}{|Z_b(\infty)+Z_p(\infty)|^2} . \qquad (4.1.25)$$

Where we have used $Z_p(\infty) = R_p(\infty) - i\,X_p(\infty)$ to emphasize that these are input impedance functions for the infinitely extended system. Extending the receiving system affects the value of a_{bp}, but does not affect the value of η_{bp}, since η_{bp} depends on a mode-mode coupling coefficient $\langle B_{\alpha\sigma}\rangle$ in the form [see Eq. (3.2.10)]

$$\omega\eta_{bp} = \langle B_{bp}\rangle\, n_p \quad \Delta\omega. \tag{4.1.26}$$

As the plate gets larger, the coupling $\langle B_{bp}\rangle$ diminishes inversely with plate area since the quantities $\mu^2, \gamma^2, \kappa^2$ are inversely proportional to the mass of the receiving system according to Eq. (3.1.4) (or plate area) and n_p is proportional to plate area. Thus, while Eq. (4.1.16) applies to any situation, Eq. (4.1.21) is only correct when infinite system parameters are used for the receiving system.

In the system of Fig. 4.4, the infinite system junction impedances are

$$Z_b^M = \rho_b c_b^2 S_b \kappa_b^2 c_{fb}^{-1}(1+i) \tag{4.1.27}$$

and

$$Z_p^M = 16\rho_s \kappa_p^2 c_p^2/\omega(1-i\Gamma) \tag{4.1.28}$$

where ρ_b, c_b, κ_b are the material density, longitudinal wavespeed, and radius of gyration for the beam with the same parameters for the plate with a "p" subscript. S_b is the cross-sectional area of the beam, $c_{fb} = \sqrt{\omega\kappa_b c_b}$ is the flexural phase speed on the beam and ρ_s the mass per unit area of the plate. The quantity Γ is a moment susceptance parameter that depends on the shape of the junction. If we assume that $Z_p^M \gg Z_b^M$, then $R_p|Z_p|^{-2} \to G_p$, the plate moment conductance and the result is independent of Γ. With this assumption and assuming the beam and plate are cut from the same material, we get

$$\eta_{bp} \to w/4\ell, \tag{4.1.29}$$

a surprisingly simple result. We have also used the modal density of the beam

$$n_b(\omega) = \ell/2\pi c_{fb} \quad . \tag{4.1.30}$$

The coupling loss factor from plate to beam is, according to Eq. (3.2.12)

$$\eta_{pb} = n_b \eta_{bp}/n_p \quad , \tag{4.1.31}$$

where $n_p = A_p/4\pi\kappa_p c_p$.

Using Eq. (4.1.29), we obtain an estimate for the average response of the beam for bands of noise,

$$\langle v_b^2 \rangle = \langle v_p^2 \rangle \frac{M_p}{M_b} \; \frac{n_b}{n_p} \; \frac{\eta_{bp}}{\eta_b + \eta_{bp}} \tag{4.1.32}$$

In Fig. 4.5 we show a comparison of this simple theoretical result for the two limiting experimental conditions, first for $\eta_{bp} \gg \eta_b$ and then for $\eta_{bp} \ll \eta_b$. Modal energy equipartition between the beam and plate would be expected to obtain in the first instance, but not in the second.

Strain Response. As an illustration of how energy response may be expressed in strain or stress of the system, we have already shown that the pressure within a sound field is given by

$$\frac{\langle p^2 \rangle V_R}{\rho_o c^2} = \langle v^2 \rangle \;\; M_R \tag{4.1.33}$$

where $\langle v^2 \rangle$ is the m.s. velocity of the fluid (air) particles. Using $\rho_0 c^2 = \gamma P_0$, this expression may also be written as

$$\frac{\langle p^2 \rangle}{(\gamma P_0)^2} = \langle (\frac{\delta\rho}{\rho_0})^2 \rangle = \frac{\langle v^2 \rangle}{c^2} \qquad (4.1.34)$$

where $\delta\rho/\rho_0$ is the volumetric strain (dilatation) of the fluid since γP_0 is the volumetric stiffness (bulk modulus) of an ideal gas. The equation states that the m.s. strain equals the m.s. mach number (ratio of particle velocity to sound speed) of the particles.

Quite a similar result also obtains for plates. In a plate of thickness h, the strain distribution across the plate thickness is given by (see Fig. 4.6)

$$\varepsilon(z) = \frac{2z}{h} \varepsilon_m \qquad (4.1.35)$$

where ε_m is the strain at the free surface of the plate, where it is a maximum. The strain energy density of a layer of plate material dz thickness is $1/2\, dz \varepsilon^2(z)\, Y_0$, where Y_0 is Young's modulus, so that the total strain energy density is

$$\text{PE density} = Y_o \int_{-h/2}^{h/2} dz\, \frac{1}{2} z^2\, \frac{4}{h^2}\, \varepsilon_m^2 = \frac{2\kappa^2}{h}\, Y_0 \langle \varepsilon_m^2 \rangle \qquad (4.1.36)$$

If this is equated to the kinetic energy density $\frac{1}{2}\rho_p h \langle v^2 \rangle$, we get

$$\langle\varepsilon_m^2\rangle \frac{h^2}{4\kappa^2} \frac{\langle v^2\rangle}{c_\ell^2} = 3 \frac{\langle v^2\rangle}{c_\ell^2} \text{ (homogeneous)} \qquad (4.1.37a)$$

where we have used $\kappa^2 = h^2/12$ for a homogeneous plate. For a sandwich plate with all its stiffness at the surface, $\kappa = h/2$ and one has

$$\langle\varepsilon_m^2\rangle = \langle v^2\rangle / c_\ell^2 \text{ (sandwich).} \qquad (4.1.37b)$$

The results of Eq. (4.1.37) are entirely analogous to that of Eq. (4.1.34) and allow us to develop mean square strain estimates from energy (or velocity) estimates.

4.2 Calculation of Variance in Temporal Mean Square Response

In paragraphs 3.3, 3.4 and 4.1 we have shown how average energy estimates of system response may be developed and how m.s. response estimates are derived from the energy. We have seen, however, in Figs. 4.1, 4.3 and 4.5 that the average energy may not be a good estimate of m.s. response at a particular location on the system or in a frequency band in which the number of modes is too low. In this section, we shall develop estimates for the deviation of realized response from the mean in terms of the standard deviation (square root of variance) from the mean square response. Note that the variance that we are discussing is from one "similar" system to another, or from one location to another, not temporal variance. If we are dealing with noise signals, we assume that the averaging time is sufficiently long to remove variations due to this cause.

The variance in response between the mean and any member of the population is produced by four major effects.

(1) The modal energies of the directly excited system may not be equal, because of the spatial dependence of the actual sources of excitation and the internal coupling of the blocked system may not be great enough to ensure equipartition.

(2) The actual number of resonant interacting modes in each realization of the coupled system will vary.

(3) The strength of the coupling parameters will fluctuate from one realization to another because of slight changes in modal shape at the junction of the systems.

(4) The response at the selected observation position will vary because the observation position is randomly located and because the mode shapes change from one realization to another.

The way in which the variance of response depends on these four factors is not obvious, and indeed, has not been worked out for many cases of interest. There is a considerable area of interesting research work that needs to be done in analyzing variance of interacting systems. We should also remark here that we are required to use modal analyses to calculate variance. The wave-mode duality is useful for mean value estimates, but a wave analysis of variance by its nature disregards spatial coherence effects that are essential to the calculation of variance.

Modal Power Flow and Response. To calculate variance in the modal energies, we return to the system of Fig. 3.4. The power flow from all of the resonators of system 1 to resonator σ is (system 2 has no external excitation)

$$\Pi_{1\sigma} = \sum_{\alpha} A_{\alpha\sigma} E_{\alpha}^{(b)} = \sum_{\alpha} B_{\alpha\sigma} (E_{\alpha}-E_{\sigma}) \qquad (4.2.1)$$

where $A_{\alpha\sigma}$ is given by Eq. (3.1.20) and $B_{\alpha\sigma}$ is given by Eq. (3.1.26) with appropriate subscripts entered. If the damping of the "σ" is Δ_{σ}, then the dissipation of energy in this mode is $E_{\sigma}\Delta_{\sigma}$ and we have

$$E_{\sigma} = \sum_{\alpha} E_{\alpha}^{(b)} (A_{\alpha\sigma}/\Delta_{\sigma}) = (\sum_{\alpha} E_{\alpha} B_{\alpha\sigma}) (\Delta_{\sigma}+\sum_{\alpha} B_{\alpha\sigma})^{-1}. \qquad (4.2.2)$$

If we are interested in the velocity response, then we saw from Eq. (4.1.10) that

$$\langle v_\sigma^2 \rangle_t = (E_\sigma/M_2)\psi_\sigma^2(x_2) \qquad (4.2.3)$$

where x_2 is the observation position. Thus, the total m.s. response of system 2 observed at position x_2 is just

$$\langle v_2^2(x_2,t)\rangle_t = \frac{1}{M}\,\Sigma_{\sigma_2}\,\psi_\sigma^2\,(x_2)\,\sum_\alpha E_{(\alpha)}^{(b)}A_{\alpha\sigma}/\Delta\sigma \qquad (4.2.3a)$$

$$= \frac{1}{M_2}\sum_\sigma \psi_\sigma^2(x_2)\left(\sum_\alpha E_\alpha B_{\alpha\sigma}\right)\Big/\left(\Delta_\sigma + \sum_\alpha B_{\alpha\sigma}\right) \qquad (4.2.3b)$$

In writing Eq. (4.2.3) we have now relaxed the assumption of Chapter 3 that the modes of system 1 are equally energetic, but we continue to assume that modal responses are temporally incoherent.

The four factors listed above that contribute to variance are specifically shown in Eq. (4.2.3). The variance in modal energies (1) is in $E_\alpha^{(b)}$ or E_α. The coupling variance strength (3) is represented by the quantities $A_{\alpha\sigma}$ or $B_{\alpha\sigma}$, and the observation variance (4) is incorporated in ψ_σ^2. The variation in number of interacting modes (2) is represented by the summation in α. Our problem is to find the variance in this sum on a knowledge (or reasonable assumptions) about the statistics of the terms entering the equation.

To simplify matters a bit, let us first concentrate on the single mode response $\langle v_\sigma^2\rangle_t$ so that only a single sum over α is involved. The appearance of $B_{\alpha\sigma}$ in the denominator of Eq. (4.2.3b) is awkward from an analytical point of view, so that we would prefer to use Eq. (4.2.3a). However, since the coupling factors also enter the denominator of $A_{\alpha\sigma}$ (but not of $B_{\alpha\sigma}$), we cannot avoid this problem. Another reason for preferring Eq. (4.2.3a) is that we might be able to estimate the statistics of $E_\alpha^{(b)}$ better than E_α, since the former does not contain contamination by coupling effects. Nevertheless, we are essentially forced to use Eq. (4.2.3b) since it alone allows us to use an important result in the statistics of summed random processes that is discussed in the following paragraphs.

There are two situations in which Eq. (4.2.3b) can simplify sufficiently to allow one to make calculations. The first is where $\sum_\alpha B_{\alpha\sigma} >> \Delta_\sigma$. In this event, we have

$$\langle v_\sigma^2 \rangle_t = (\psi_\sigma^2(x_2)/M_2) \sum_\alpha E_\alpha B_{\alpha\sigma} / \sum_\alpha B_{\alpha\sigma} \; . \qquad (4.2.4)$$

In this case, the energy of the σ mode is a weighted average of the energy of the modes of system 1 with weighting coefficients $b_{\alpha\sigma} = B_{\alpha\sigma}/\sum_\alpha B_{\alpha\sigma}$. Notice that this condition, which is essentially an equipartition condition, does not require that $B_{\alpha\sigma} >> \Delta_\sigma$, and is thus a much weaker requirement on the coupling than when one has two single dof systems interacting.

The second situation for which one can calculate variance is when $\Delta_\sigma >> \sum_\alpha B_{\alpha\sigma}$. In this case

$$\langle v_\sigma^2 \rangle_t = (\psi_\sigma^2 / M_2 \Delta_\sigma) \sum_\alpha E_\alpha B_{\alpha\sigma} \qquad (4.2.5)$$

We note that $B_{\alpha\sigma}$ is given by Eq. (3.1.26) with appropriate subscripts. We will write it here in the form

$$B_{\alpha\sigma} = \frac{\lambda_{\alpha\sigma}}{\xi_{\alpha\sigma}^2 + 1} \; \frac{\Delta_\alpha \Delta_\sigma}{\Delta_\alpha + \Delta_\sigma} \qquad (4.2.6)$$

where $\lambda_{\alpha\sigma}$ is given by Eq. (3.1.33) and $\xi_{\alpha\sigma} = 2\,(\omega_\sigma - \omega_\alpha)/(\Delta_\alpha + \Delta_\sigma)$, in a manner analogous to the development in paragraph 2.2. If we now assume $\Delta_\alpha = \Delta_1 =$ const (all modes of system 1 have equal damping), then according to Eq. (2.2.21),

$$\langle B_{\alpha\sigma}\rangle\omega_{\alpha} = \langle\lambda_{\alpha\sigma}\rangle\omega_{\alpha} \ \frac{\Delta_1\Delta_\sigma}{\Delta_1+\Delta_\sigma} \ \frac{\pi(\Delta_1+\Delta_\sigma)}{2\Delta\omega} \tag{4.2.7}$$

which reproduces Eq. (3.2.8)

In the present instance, we are interested in the variance of the sum in Eq. (4.2.5). The assumption $\Delta_\sigma=\Delta_1$ has the effect of causing each of the functions $(\xi^2_{\alpha\sigma}+1)^{-1}$ to have the same shape as a function of frequency. We can plot Eq. (4.2.5) then as shown in Fig. 4.7. It appears as a sum of pulses, of strength

$$C_{\alpha\sigma} = \frac{(\psi^2_\sigma/M_2)\Delta_1\Delta_\sigma}{\Delta_1+\Delta_\sigma} \ E_\alpha\lambda_{\alpha\sigma}. \tag{4.2.8}$$

If we assume that the resonance frequencies are uniformly probable over the interval $\Delta\omega$ and are independent of each other, then the standard deviation of Eq. (4.2.5) may be shown to be

$$\sigma^2_{v^2_\sigma} = \frac{\pi}{2}\, n_1(\omega)\ (\Delta_1+\Delta_\sigma) \left[\frac{(\psi^2_\sigma/M_2)\Delta_1\Delta_\sigma}{\Delta_1+\Delta_\sigma}\right]^2 \langle E^2_\alpha\rangle_\alpha \ \langle\lambda^2_{\alpha\sigma}\rangle_\alpha. \tag{4.2.9}$$

This is the "important result" referred to above. Consequently, using Eq. (4.2.7) in the average of Eq. (4.2.5), we get

$$\frac{\sigma^2_{v^2_\sigma}}{m^2_{v^2_\sigma}} = \left\{n_1 \frac{\pi}{2} (\Delta_1+\Delta_2)\right\}^{-1} \frac{\langle E^2_\alpha\rangle_\alpha}{\langle E_\alpha\rangle^2_\alpha} \ \frac{\langle\lambda^2_{\alpha\sigma}\rangle_\alpha}{\langle\lambda_{\alpha\sigma}\rangle^2_\alpha}, \tag{4.2.10}$$

showing the explicit dependence of variance on uncertainty in the modal energy of the directly excited system and in the coupling parameters. This relation also shows the effect of the number of overlapping modes in reducing the ratio of variance to the square of the mean.

Finally, if we refer to Eq. (3.2.10) we see that the quantity $N_1 \langle B_{\alpha\sigma} \rangle_\alpha \equiv \omega\eta_{21}$. The standard deviation in the sum $\sum_\alpha B_{\alpha o}$ will tell us something about the uncertainty in the coupling loss factor η_{21}. If for the present discussion only, we define $\omega\eta_{21} = \sum_\alpha B_{\alpha o}$ then $m_\eta = \langle \eta_{21} \rangle_\alpha$ is the coupling loss factor we have been dealing with. Its variance is then given by

$$\sigma_\eta^2 = \frac{1}{\omega^2} \langle \lambda_{\alpha\sigma}^2 \rangle \left(\frac{\Delta_1 \Delta_\sigma}{\Delta_1 + \Delta_\sigma} \right)^2 \frac{\pi}{2} n_1 \quad (\Delta_1 + \Delta_\sigma)$$

and

$$\frac{\sigma_\eta^2}{m_\eta^2} = \frac{1}{\frac{\pi}{2} n_1 (\Delta_1 + \Delta_2)} \quad \frac{\langle \lambda_{\alpha\beta}^2 \rangle}{\langle \lambda_{\alpha\beta}^2 \rangle^2} \tag{4.2.11}$$

Returning to Eq. (4.2.5), if the observation position x_2 is located randomly in space, then we have additional factor of $\langle \psi_\sigma^4 \rangle / \langle \psi_\sigma^2 \rangle^2$ in Eq. (4.2.10) since ψ_σ^2 must be regarded as a random amplitude factor. If we assume that system 1 is excited by a randomly located point force at x_1, then according to Eq. (2.2.19), the energy of the α^{th} mode will vary as $\psi_\alpha^2(x_1)$. If we assume that the systems are joined at a point x_1' in system 1 coordinates and x_2' in system 2 coordinates, then according to Eqs. (3.2.6) and (3.2.9), we may expect $\lambda_{\alpha\sigma}$ to vary as $\psi_\alpha^2 (x_1') \psi_\sigma^2(x_2')$. Thus, the greatest variation that we can expect in the response of a single mode of system 2 is

$$\frac{\sigma^2_{v_\sigma^2}}{m^2_{v_\sigma^2}} = \left\{n_1 \frac{\pi}{2} (\Delta_1+\Delta_2)\right\}^{-1} \left[\frac{\langle\psi_1^4\rangle}{\langle\psi_1^2\rangle^2}\right]^2 \left[\frac{\langle\psi_2^4\rangle}{\langle\psi_2^2\rangle^2}\right]^2 \qquad (4.2.12)$$

where ψ_1 and ψ_2 are typical mode shapes of system 1 and system 2 respectively. We may expect that in any particular system, the variance will be less than that implied by Eq. (4.2.12).

Now suppose that we have a group of modes of system 2 represented by a modal density $n_2(\omega)$. We have a combination $n_2\Delta\omega$ independent response functions, each of which has a ratio of variance to square of mean given by Eq. (4.2.12). The total ratio of variance to square of mean for multimodal response of system 2 is, therefore,

$$\frac{\sigma^2_{v_2}}{m^2_{v^2}} = \left\{n_1 n_2 \frac{\pi}{2} (\Delta_1+\Delta_2)\Delta\omega\right\}^{-1} \left[\frac{\langle\psi_1^4\rangle}{\langle\psi_1^2\rangle^2}\right]^2 \left[\frac{\langle\psi_2^4\rangle}{\langle\psi_2^2\rangle^2}\right]^2 \qquad (4.2.13)$$

Notice that this result is completely symmetric in the properties of system 1 and 2. We should expect, therefore, that for systems excited and observed at point locations, the variance in response will be independent of which system is used as a "source" and which is used as a "receiver".

It will be apparent to the reader that a substantial number of assumptions have been made in order to get the result expressed by Eq. (4.2.13). Most of these assumptions, with notable exception of the assumptions of uniform damping, $\Delta_\alpha=\Delta_1$ and $\Delta_\alpha=\Delta_2$, would have the effect of reducing response variance. We, therefore, may reasonably regard Eq. (4.2.13) as a conservative estimate of the variance in the work that follows in that it estimates more variance than may, in fact, occur.

4.3 Calculation of Confidence Coefficients

We see from Eq. (4.2.13) that increasing the number of modal interactions by increasing the combined modal bandwidth $\pi/2\ (\Delta_1+\Delta_2)$, and increasing the number of modes observed by increasing the excitation bandwidth $\Delta\omega$ will both cause a reduction in variance of response. In this section, we show how the variance estimates may be used to predict the fraction of measurements that will fall within a certain interval of values, generally based on the predicted mean.

The Probability Density. If the total probability density $\phi(\theta)$ for the observed response $\langle v^2\rangle_t=\theta$ were known, then the probability that θ would fall within the "confidence interval" $\theta_1 < \theta < \theta_2$ is simply

$$CC = \int_{\theta_1}^{\theta_2} \phi(\theta)\ d\theta \qquad (4.3.1)$$

where the (fiducial) probability CC is called the confidence coefficient. Since we do not know what the function $\phi(\theta)$ is, this may seem rather useless. We do know, however, that θ has positive values only, and we can estimate its mean m_θ and its standard deviation σ_ϕ from paragraph 4.2.

A probability density that meets our requirements and has tabulated integrals is the "gamma density"

$$\phi(\theta) = \theta^{\mu-1}\exp(-\theta/\lambda)/\lambda^{\mu}\Gamma(\mu) \qquad (4.3.2)$$

where $\mu = m_\theta^2/\sigma_\theta^2$ and $\lambda=\sigma_\theta^2/m_\theta$ and $\Gamma(\mu)$ is the gamma function. If we change variables to $y = \theta m_\theta/\sigma_\theta^2$, then Eq. (4.3.1) becomes

$$CC = \frac{1}{\Gamma(\mu)} \int_{\theta_1 m_\theta/\sigma_\theta^2}^{\theta_2 m_\theta/\sigma_\phi^2} dy \; y^{\mu-1} e^{-y} = \Gamma^{-1}(\mu) \left[\gamma\left(\mu, \frac{\theta_2 m_\theta}{\sigma_\theta^2}\right) - \gamma\left(\mu, \frac{\theta_1 m_\theta}{\sigma_\theta^2}\right) \right] \tag{4.3.3}$$

where $\gamma(\mu,B)$ is the incomplete gamma function defined by

$$\gamma(\mu,B) \equiv \int_0^B y^{\mu-1} e^{-y} \, dy \tag{4.3.4}$$

and is tabulated in standard mathematical handbooks.

Confidence Intervals. First, let us consider a simple "exceedance" type of confidence interval. That is, search for a value of $\theta_{max} \equiv r m_\theta$ such that the probability (CC) that any realized value of $\theta < \theta_{max}$ is known. The equation to be solved then is

$$CC = \Gamma^{-1}(\mu) \; \gamma(\mu, r\mu) \tag{4.3.5}$$

and the result is a line of constant CC as a function of r and μ. The solution to Eq. (4.3.5) as found numerically is graphed in Fig. 4.8.

From Fig. 4.8 we see that if the ratio of the standard deviation to the mean (as computed from Eq. (4.2.13) for example) is equal to unity, then we may expect a measured response, selected at random, to have a value less than 2.5dB more than the mean in 80% of the cases. The measurement should be less than 5 dB more than the mean in 95% of the cases and less than 7 dB more than the mean in 99% of the cases. In many situations, this is precisely the way that we would want the estimate stated, particularly if we were concerned about the response exceeding some prescribed value that might lead to component failure, or excessive speech interference, or acoustical detectability.

Also note, that since the calculation of $\sigma_\theta^2/m_\theta^2$ is to be carried out in frequency bands $\Delta\omega$, or for individual modes that resonate at different frequencies, the form of the confidence interval is that of a spectrum. One would usually plot an "average estimate", based upon the methods of section 4.1 as in Fig. 4.5, then plot a second spectrum determined by 10 log r for each frequency band. One would then look upon the second spectrum as an "upper bound" for the data, with a degree of confidence determined by the selected confidence coefficient.

As a second example of a confidence interval, consider one that brackets the calculated mean. Such interval might be stated as "in the 500 Hz octave band, I estimate that in 95% of the cases, the measured acceleration level is within ±5 dB of the average estimate of -10 dB re 1g." The "cases" referred to here, of course, are reported measurements with the loading and observation points varied consistent with the calculations of mean and variance, and for a population of systems. Obviously, one rarely has this population to deal with; there is only one system in the laboratory, or at most 2 or 3. The system that one has might be very good or very poor from a viewpoint of variance, but we must hope that the estimate for the population will be useful in dealing with it. This situation is directly analogous to gambling. One only has a particular realization of all possible hands of cards before him, but the pattern of betting, calculation of odds and strategy is made on the basis of all possible hands and draws.

The evaluation of this "bracketing" interval is made by setting $\theta_2=rm_\theta$ and $\theta_1=m_\theta/r$ in Eq. (4.3.3). The equation to be solved in this case is

$$CC = \Gamma^{-1}(\mu)\ \{\gamma\ (\mu,\mu\ r)-\gamma(\mu,\mu/r)\}. \tag{4.3.6}$$

The solution to Eq. (4.3.6) is shown in Fig. 4.9. For example, the value of normalized variance that would produce the hypothetical estimation interval of the preceding paragraph is $\sigma^2/m^2 = 0.2$. An interval of this type is very useful in determining the value of the mean alone as an estimate of response. Since one can usually only make measurements of structural vibration and acoustical noise to a repeatability of 1 or 2 dB, then if an estimation interval of $\pm$ 1 dB has a high confidence coefficient (80%, say) then the mean value is a very good estimate. We see from Fig. 4.9 that this requires that $\sigma^2/m^2 \leq 0.1$.

Example. As an illustration of the discussion in paragraph 4.2 and 4.3, we consider the system shown in Fig. 4.4 excited at a point and observed at a point. We derived an average energy estimated for this system in paragraph 4.1, which we may use to obtain m_θ. We now apply Eq. (4.2.13) to obtain an estimate of variance and then use Figs. 4.8 and 4.9 to determine confidence intervals.

First, the beam (System 2) is a one-dimensional system with modes of the form sin kx, and the plate (System 1) is two dimensional with modes of the form sin k_1x_1 sin k_2x_2. If we regard the locations x, x_1, x_2 as uniformly located over the structural surfaces, we find that

$$\frac{\langle\psi_2^4\rangle}{\langle\psi_2^2\rangle^2} = \frac{\langle(\sin kx)^4\rangle_x}{\langle(\sin kx)^2\rangle_x^2} = \frac{3}{2} \text{ (one-dimensional)} \tag{4.3.7a}$$

$$\frac{\langle\psi_1^4\rangle}{\langle\psi_1^2\rangle^2} = \frac{\langle(\sin kx_1)^4\rangle\langle(\sin kx_2)^4\rangle}{\langle(\sin kx_1)^2\rangle\langle(\sin kx_2)^2\rangle^2} = \frac{9}{4} \text{ (two-dimensional)} \tag{4.3.7b}$$

Obviously, the corresponding quantity for three-dimensional systems is $(3/2)^3 = 27/8$.

Eq. (4.3.7) gives us two of the factors in Eq. (4.2.13). To obtain the third, we must say more about the system. However, note that the spatial variance alone contributes a factor $(3/2)^2$ $(9/4)^2 = 11$, so that we must have of the order of 10 modes interacting or sampled to bring σ^2/m^2 down to the order of 1 or 100 modes to get down to $\sigma^2/m^2 = 0.1$. In general, the higher the dimensionality of the system, the greater is the contribution to variance from the spatial sampling factors. To counteract this, however, the higher dimensional systems normally have greater modal density, and this reduces variance.

Suppose that the beam in Fig. 4.4 is made of aluminum and is 1.3 m long, .05 m wide, and .003 m thick. The average frequency separation between modes for this beam is according to (4.1.23)

$$\delta f_b = c_{fb}/\ell = 130 \sqrt{f/1000} \ . \tag{4.3.8}$$

Thus, at 1000 Hz, the average separation is 130 Hz and at 250 Hz it is 65 Hz. We assume that the plate is also of aluminum, is .003 m thick, and has an area of 1.7 m^2. The average frequency separation for plate modes is then

$$\delta f_p = hc_\ell/\sqrt{3}A = 5H_z \ . \tag{4.3.9}$$

Also, a reasonable value for loss factor for both systems is $\eta_1=\eta_2=.01$. Thus,

$$\frac{\pi}{2}\ (\Delta_1+\Delta_2) = \frac{\pi}{2}\ f\ (\eta_b+\eta_p) = 10^{-2}\ \pi f,\ \mathrm{Hz} \tag{4.3.10}$$

Let us assume that our data is taken in octave bands, so that $\Delta f = f/\sqrt{2}$. Thus, combining Eqs. (4.3, 8, 9, 10), we get

$$\left\{n_1 n_2\ \frac{\pi}{2}\ (\Delta_1+\Delta_2)\Delta\omega\right\}^{-1} = \frac{130\sqrt{f/1000}\ 5}{10^{-2}\pi f.\ \ f/\sqrt{2}} = 1000 f^{-3/2} \tag{4.3.11}$$

Thus, at f=100 Hz, this factor is 1 and at 10,000 Hz it is 10^{-3}. The product of this modal count term and the spatial factors is presented in Fig. 4.9 for the example. We have also sketched the variance expected when there is no uncertainty in modal amplitude on the beam, reducing the variance by a factor of $(2/3)^2$, from Eq. (4.3.7a). This is appropriate if for example we make response measurements at the free end of the beam and connect the plate to the base of the beam.

Now we can use this estimate of variance to establish a mean bracketing confidence interval in each frequency band. If this is done for a confidence coefficient of 0.8, we obtain the result shown in Fig. 4.11. We have used the estimate for the mean from Fig. 4.5 in the case where $n_{bp} < \eta_b$, which is not quite the case for one example since $n_{bp} = w/4\ell \simeq .05/5 \simeq .01 \simeq \eta_b$. The confidence interval in no way depends on the estimate of the mean, however.

Altogether then, our analysis of variance, in combination with a mean estimate and the calculation of confidence intervals based on normalized variance, provides us with a way of making estimates of response for real systems that includes variations in a realistic manner. The weakest link of this chain at present is the estimation of variance. There is good reason to believe, for example, that the location of resonance frequencies among the population of "similar" systems is not well described by a distribution that assumes that modes occur with equal probability along the frequency axis. Such a distribution predicts a high probability for very closely separated modes, whereas the normal coupling among modes will tend to "split" them, as demonstrated by Fig. 3.2. Thus, interacting groups of modes should have a lower probability of very small differences in resonance frequencies, and this should result in smoother distributions of response and lower variance.

4.4 Coherence Effects - Pure Tone and Narrow Band Response.

In this final section, we consider the effects on our analysis of the bandwidth of excitation $\Delta\omega$. When $\Delta\omega$ is very broad, then we have shown that certain frequency integrations such as those of paragraphs 2.1 and 3.1 become quite simple, and that the expected variance in system response calculated in paragraph 4.2 is reduced as the noise bandwidth increases. When $\Delta\omega$ gets smaller than the average spacing between modes, we may expect that variance will increase because of the smaller number of modes included in the averages. When $\Delta\omega$ becomes of the order of a modal bandwidth, other complications (or possibly simplifications) may develop.

Narrow band excitation of resonator. Let us suppose that the excitation $\ell(t)$ of the single dof system of Fig. 2.1 is the pure tone signal $L\,e^{-i\omega t}$. The response, according to Eq. (2.1.20) is

$$v(t) = LY\,e^{-i\omega t} = Le^{-i\omega t}(-i\omega_o M)^{-}\left[i\eta + \left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\right]^{-1} \tag{4.4.1}$$

In Chapter 2, we averaged $|Y|^2$ over an interval $\Delta\omega$ (large compared to $\omega_o\eta$) to determine the noise response of the resonator. Let us now suppose that ω is fixed and we are selecting resonators out of a population having resonance frequencies ω_o uniformly distributed over the interval $\Delta\omega$ (this is one feature of the SEA model introduced in paragraph 3.2). Then the average m.s. response will be

$$\langle v^2\rangle_{\omega_o} = \frac{1}{2}|L|^2 \langle|Y|^2\rangle_{\omega_b} = \frac{1}{2}|L|^2 \frac{\Delta_e}{\Delta\omega} \frac{1}{(\omega\eta\ M)^2} \tag{4.4.2}$$

where $\Delta_e = (\pi/2)\omega\eta$, the effective bandwidth. This is quite a slowly varying function of ω, so that if we now say that $\ell(t)$ is a narrow band signal of bandwidth $\delta\omega$ so that $1/2\ L^2 \to 2S_\ell(\omega)\delta\omega$, we have (remember that spectral densities in ω are defined from $-\infty < \omega < +\infty$).

$$\langle v^2\rangle_\omega = \frac{2S_\ell(\omega)\quad(\delta\omega/\Delta\omega)\quad\Delta_e}{(\omega\eta\ M)^2} \tag{4.4.3}$$

which should be compared to Eq. (2.1.39). We may say that the effect of restricting the bandwidth to $\Delta\omega$, is to reduce the effective spectral density of the excitation in the ratio $\delta\omega/\Delta\omega$. When $\delta\omega \to \Delta\omega$, we obtain the original response of the resonator to the noise of bandwidth $\Delta\omega$. The response of a resonator to a signal of any bandwidth is of the same form as a noise response, if we average over the resonance frequency of the resonator.

Interacting Modes. The result of Eq. (4.4.3) allows us to interpret the result of Eq. (3.1.32) in an interesting way. The interaction of mode "2" with a single mode "1" was found to be equivalent to excitation of mode by a white noise source. Let us re-examine this result in the light of Eq. (4.4.3). To do this, imagine for simplicity that the resonators in Fig. 3.1 have gyroscopic coupling only. Then, noise excitation with spectral density S_{ℓ_1} of resonator 1 with $y_2 \equiv 0$, results in a m.s. velocity of this resonator given by

$$\langle v_1^2 \rangle = \frac{\pi S_{\ell_1}(\omega)}{\Delta_1 M_1^2} \qquad \frac{S_{\ell_1}(f)}{4\Delta_1 M_1^2} \tag{4.4.4}$$

This m.s. velocity has an equivalent bandwidth $\Delta_{e,1} = \pi\Delta_1/2$, since its spectral form is that of the admittance of resonator 1.

From Eq. (3.1.3b) a m.s. velocity $\langle v_1^2 \rangle$ will produce a m.s. force on resonator 2 given by $\langle v_1^2 \rangle G^2 = \langle v_1^2 \rangle \gamma^2 M_1 M_2$. This m.s. force is "spread" over a bandwidth $\Delta_{e,1}$, so that its spectral density is

$$2\, S_{\ell_2}(\omega) = \langle v_1^2 \rangle \gamma^2 M_1 M_2/\Delta_{1e} = \frac{\pi S_{\ell_1}(\omega)}{\Delta_1 M_1^2} \cdot \frac{\gamma^2 M_1 M_2}{\Delta_{e,1}} \tag{4.4.5}$$

Thus

$$S_{\ell_2} = S_{\ell_1} \frac{M_2}{M_1} \frac{\gamma^2}{\Delta_1 \Delta_2} \left(\frac{\pi}{2} \Delta_2 \right) \frac{1}{\Delta_{e,1}} \tag{4.4.6}$$

is the spectral density. Since, according to Eq. (4.4.3), such a signal is equivalent to a broad band noise over a band $\Delta\omega$, with spectrum $S_{\ell_2}\,\Delta_{e1}/\Delta\omega$, the equivalent broad band excitation produced by the interaction is

$$S_{\ell_2}\,\frac{\Delta_{e\,1}}{\Delta\omega} = S'_{\ell_2} = S_{\ell_1}\,\frac{M_2}{M_1}\,\frac{\pi\Delta_2/2}{\Delta\omega}\,\lambda \qquad (4.4.7)$$

where $\lambda=\gamma^2/\Delta_1\Delta_2$ for gyroscopic coupling only, which is consistent with Eq. (3.1.32) for weak coupling.

We have shown two important features of the interaction with this calculation. First, we can properly account for the interaction when the coupling is weak ($\lambda<<1$) as excitation of the indirectly excited system by noise, through a filter of bandwidth Δ_{1e} and transfer magnitude determined by the coupling strength, if the resonance frequency of the receiving system is allowed to take random values over the interval $\Delta\omega$. Secondly, if the effective bandwidth of the excitation of system 2 is Δ_{e1}, then if more than one mode may be excited simultaneously, we may expect important coherence effects to arise from such narrow band excitation. We now discuss just what those effects may be.

Coherence Effects in Multi-Modal Response to Band-Limited Noise. We suppose that the system described by Eq. (2.2.1) is excited by noise source of bandwidth $\Delta\omega$. This source may be a vibration exciter or a damped resonator driven by noise as discussed in the preceding paragraphs. The displacement response is of the form given in Eq. (2.2.3). The density average of m.s. displacement is, of course

$$\langle y^2\rangle_{\rho,t} = \sum_m \langle y_m^2(t)\rangle_t \qquad (4.4.8)$$

We now assume that all modes in the interval $\delta\omega$ have the same damping, with effective bandwidth Δ_e. If we excite this system at a location where all modes have nearly the same amplitude (on a free plate, this would correspond to a corner), then we can assume that the m.s. response of each mode is identical.

Accordingly,

$$\langle y^2 \rangle_{\rho,t} = N \langle y^2 \rangle_t \tag{4.4.9}$$

where N is the number of resonantly excited modes and $\langle y^2 \rangle_t$ is the m.s. response of each.

Eq. (4.4.9) gives the average response of the system in the spirit of paragraph 4.1, assuming completely incoherent response. Let us suppose, however, that the response of every mode were perfectly coherent, as it would be under sinusoidal excitation ($\Delta\omega << \Delta_e$). In this event, the m.s. response is at any point is

$$\langle y^2 \rangle_t = \sum_{m,n} \langle Y_m(t) \; Y_n(t) \rangle_t \; \psi_m(x) \psi_n(x) . \tag{4.4.10}$$

If there is a location where every mode has an antinode ($\psi = \psi_{max}$) and every modal vibration is in phase, then the m.s. response at that location is

$$\max \langle y^2 \rangle_t = N^2 \; \langle y^2 \rangle_t \quad \psi^2_{max} \tag{4.4.11}$$

The ratio of the rms response at this location to the incoherent rms response is

$$\frac{y_{max}}{y_{rms}} = \psi_{max} \sqrt{N} \tag{4.4.12}$$

Thus, if N is large, this coherent "hot spot" of response may be substantially greater than the incoherent estimate.

Let us now assume that the noise bandwidth $\Delta\omega$ is large compared to Δ_e. The modal density of the system is $n(\omega)$. Thus, the size of each group of coherent excited modes in $n\Delta_e$, which by presumption must be large for coherent effects to be of any importance. The coherent response maximum, therefore, according to Eq. (4.4.11), is $(n\Delta_e)^2 <Y^2>_t \psi_{max}^2$, and a m.s. response of $n\Delta_e <Y^2>_t$, according to Eq. (4.4.9). Each possible coherent peak, therefore, exists in an incoherent "background" given by $n(\Delta\omega-\Delta_e)$, so that the total response at a hot spot is

$$\max <y^2>_t = (n\Delta_e)^2 \psi_{max}^2 <Y^2>_t + n(\Delta\omega-\Delta_e) <y^2>_t \tag{4.4.13}$$

The ratio of maximum to average response then is

$$\frac{\max <y^2>_t}{<y^2>_{x,t}} = n\frac{\Delta_e^2}{\Delta\omega}\psi_{max}^2 + \left(1-\frac{\Delta_e}{\Delta\omega}\right) \quad ; \ \Delta\omega > \Delta_e \tag{4.4.14a}$$

$$= n\Delta_\ell \psi_{max}^2 \quad ; \ \Delta\omega < \Delta_e \tag{4.4.14b}$$

As an example, consider the plate of the example in section 4.3. In that example $n\Delta_e = (\pi/2)\, f\eta_p/\delta f_p = 10^{-2}\pi f/10$. At 1000 Hz, the number of coherent modes would be ~ 3. At 10,000 Hz, we would have 31 modes in each coherent group. The statistical response concentrations for this plate under 1/3 OB and pure tone response ($\psi_{max}=2$ for 2-dimensional mode) at 1000 Hz are

$$(1/3\ \mathrm{OB})\frac{\max <y^2>_t}{<y^2>_{x,t}} = \pi \cdot \frac{\pi}{250}\cdot 4 + \left(1-\frac{10^{-2}\pi\cdot 10^3}{250}\right) \simeq 1$$

$$(\text{pure tone}) \quad \frac{\max\langle y^2\rangle_t}{\langle y^2\rangle_{x,t}} = \pi \cdot 4 = 13$$

At 1000 Hz, there is no discernible response concentration due to coherence for 1/3 OB excitation, but a concentration factor of about 3.5 will occur for pure tone excitation.

At 10,000 Hz, we have

$$(1/3 \text{ OB}) \quad \frac{\max\langle y^2\rangle_t}{\langle y^2\rangle_{x,t}} = \frac{\pi^2 \cdot 10^2 \cdot 4}{2500} + \left(1 - \frac{\pi \cdot 10^{-2} \cdot 10^3}{2 \cdot 2500}\right)$$

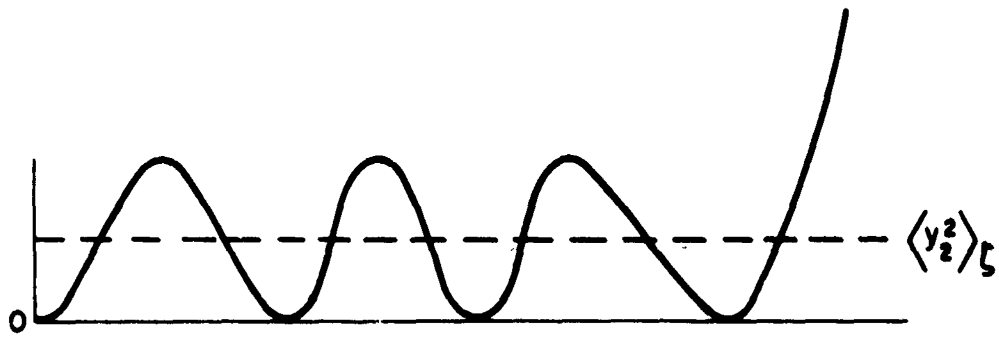
$$\simeq 1.6 + 1 = 2.6$$

$$(\text{pure tone}) \quad \frac{\max\langle y^2\rangle_t}{\langle y^2\rangle_{x,t}} = \pi \cdot 10 \cdot 4 = 126$$

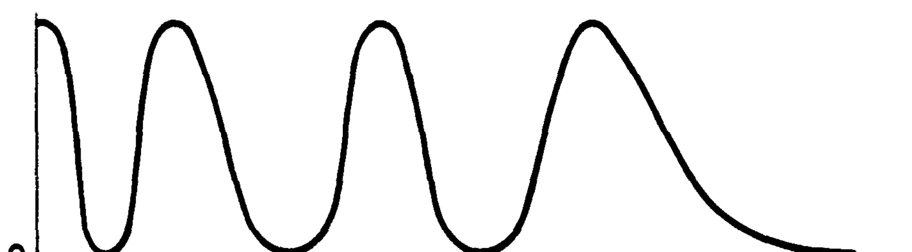
The concentration for the band of noise is still negligible, but for pure tone excitation, the response concentration is greater than 11.

It is clear that statistical response concentrations are sizeable. We have not discussed the probability that such a coherent peak, as represented by Eq. (4.4.12), will occur. Although the point is not completely resolved, it appears that such concentrations have a good likelihood of occurring. As a practical matter, such coherence effects will be important when large systems with high modal density and moderate to high damping are excited by pure tones. When these "statistical" concentrations occur, they may be considerably larger than the "stress concentrations" that occur near boundaries as exemplified by Fig. 4.3.

(a) MODE SHAPE $\psi_2(x_2)$ FOR CLAMPED-FREE BEAM



(b) M.S. DISPLACEMENT OF CLAMPED-FREE BEAM; $\psi_2^2(x_2)$



(c) M.S. SURFACE STRAIN OF CLAMPED-FREE BEAM; $\psi_2''^2(x_2)$

FIG 4.1

SPATIAL DISTRIBUTION OF RESPONSE FOR CLAMPED-FREE BEAM

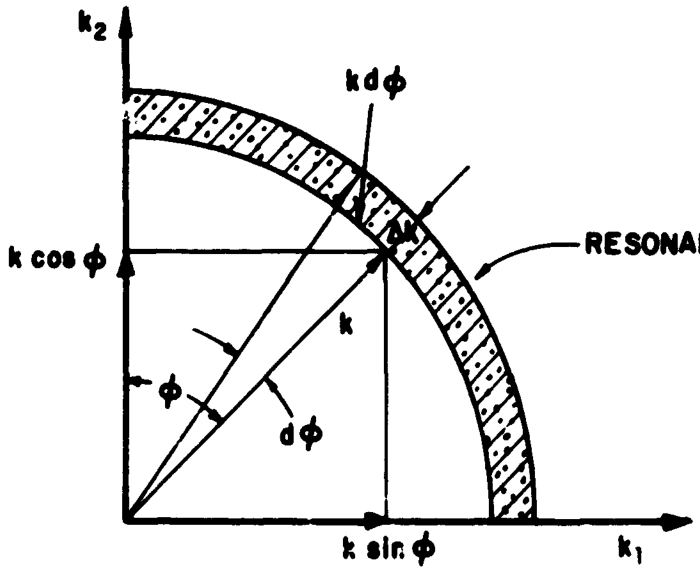


FIG. 4.2

DISTRIBUTION OF 2-DIMENSIONAL MODES EXCITED BY BAND OF NOISE $\Delta\omega$

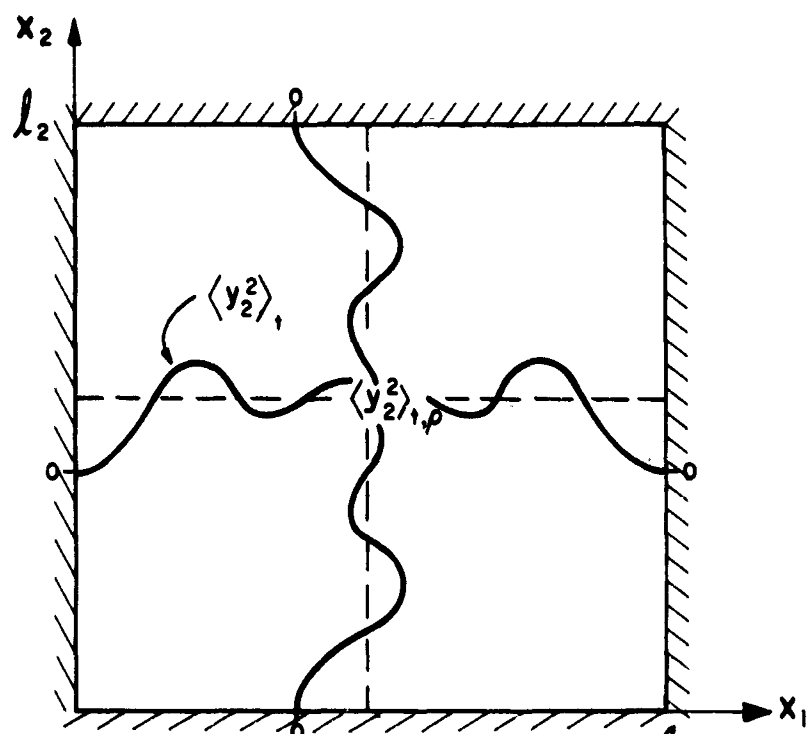


FIG. 4.3

MULTI-MODAL TEMPORAL MEAN SQUARE RESPONSE OF RECTANGULAR PLATE

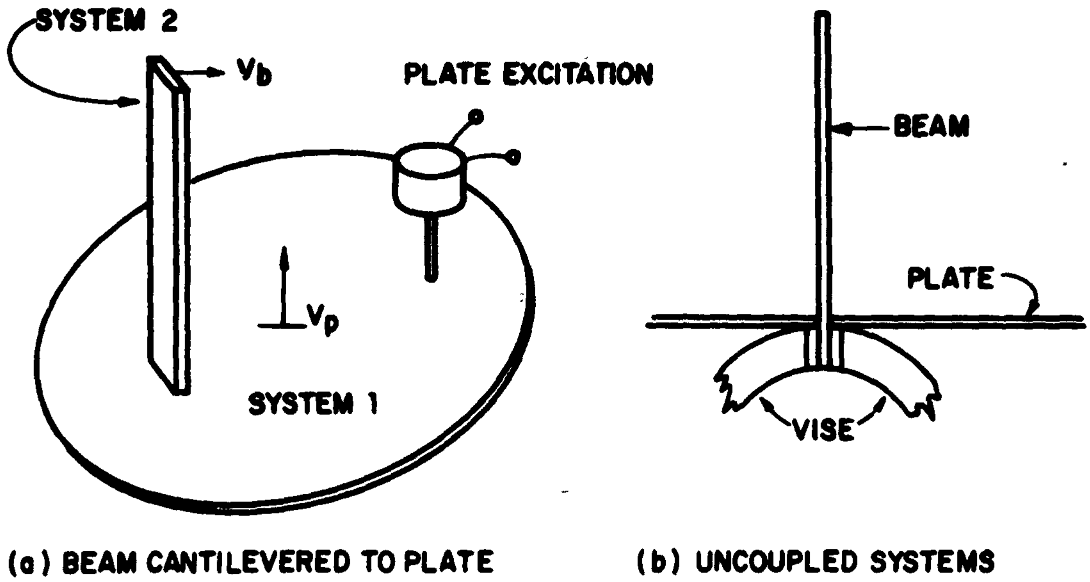


FIG 4.4

BEAM–PLATE SYSTEM, BEAM INDIRECTLY EXCITED

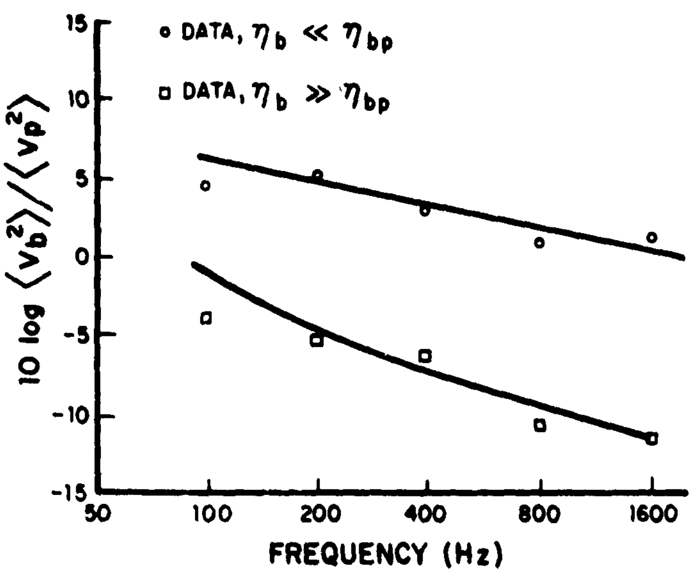


FIG 4.5

COMPARISON OF MEASURED RESPONSE RATIOS WITH THOSE PREDICTED BY EQ. (4.1.25)

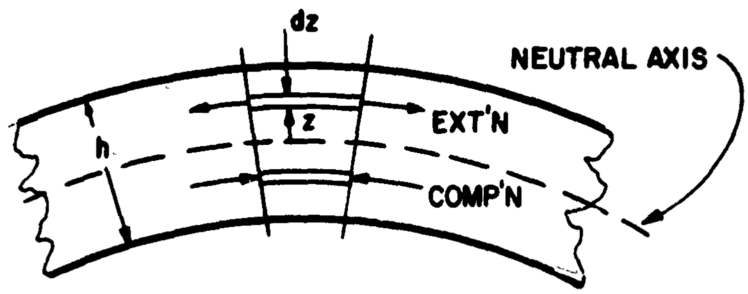


FIG 4.6

BENDING STRAIN IN A BEAM OF THICKNESS h

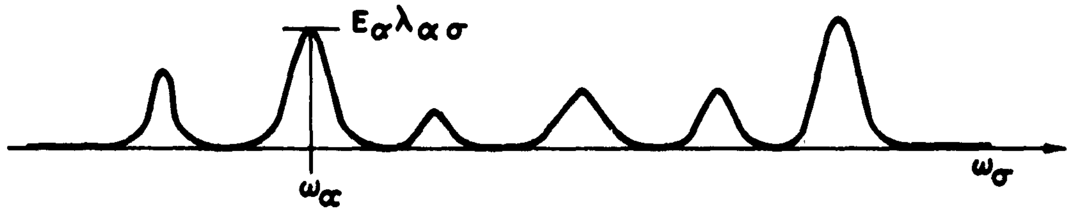


FIG 4.7

THE SUM IN EQ.(4.2.5) AS A FUNCTION OF ω_σ

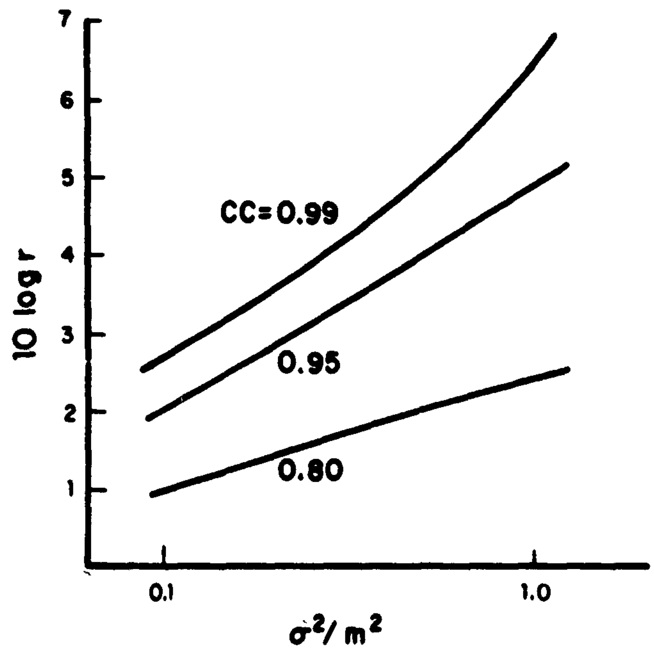


FIG 4.8

UPPER BOUND OF SIMPLE EXCEEDANCE ESTIMATION INTERVALS AS A FUNCTION OF NORMALIZED VARIANCE

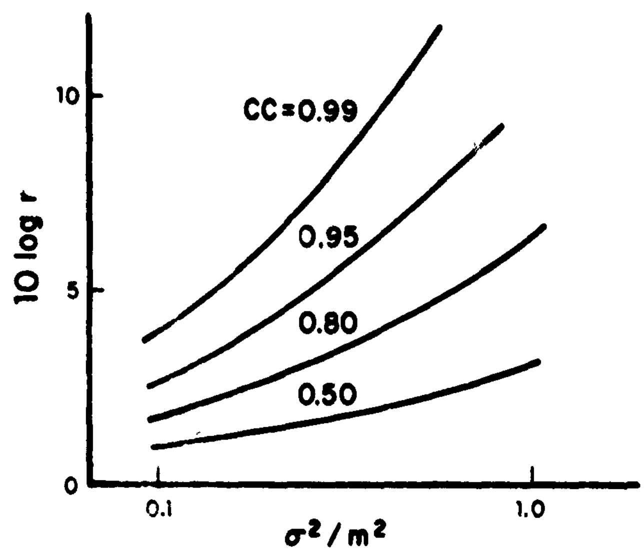


FIG 4.9

HALF-WIDTH OF MEAN-BRACKETING ESTIMATION INTERVALS AS A FUNCTION OF NORMALIZED VARIANCE

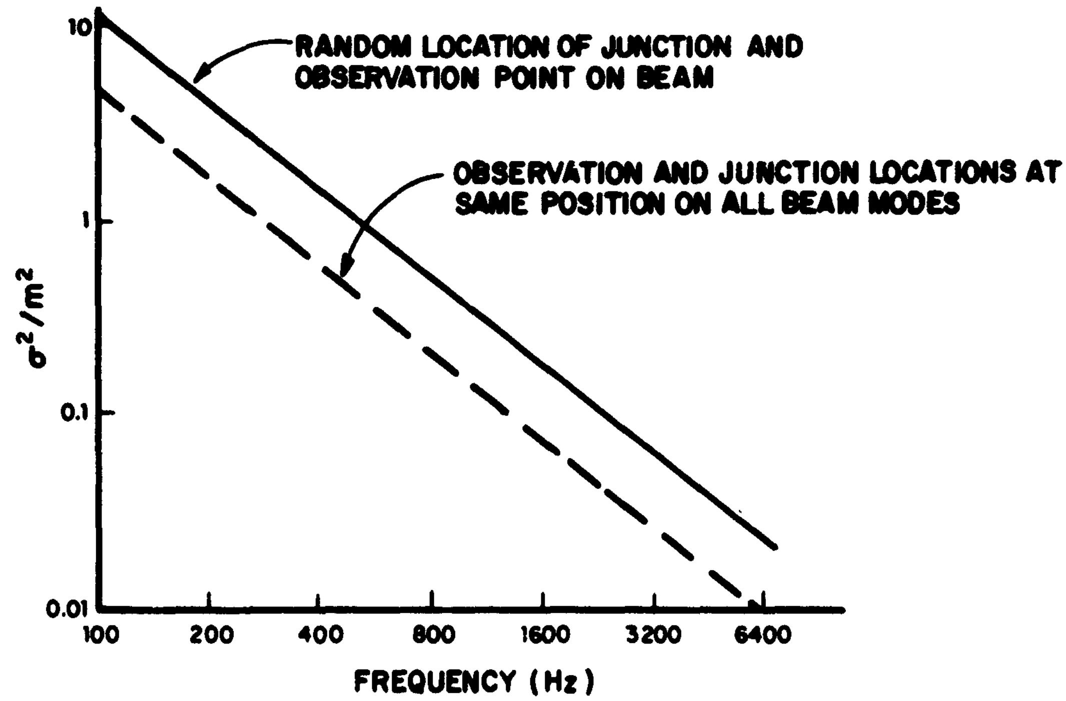


FIG 4.10

NORMALIZED VARIANCE AS A FUNCTION OF FREQUENCY FOR BEAM-PLATE EXAMPLE IN SECTION 4.3

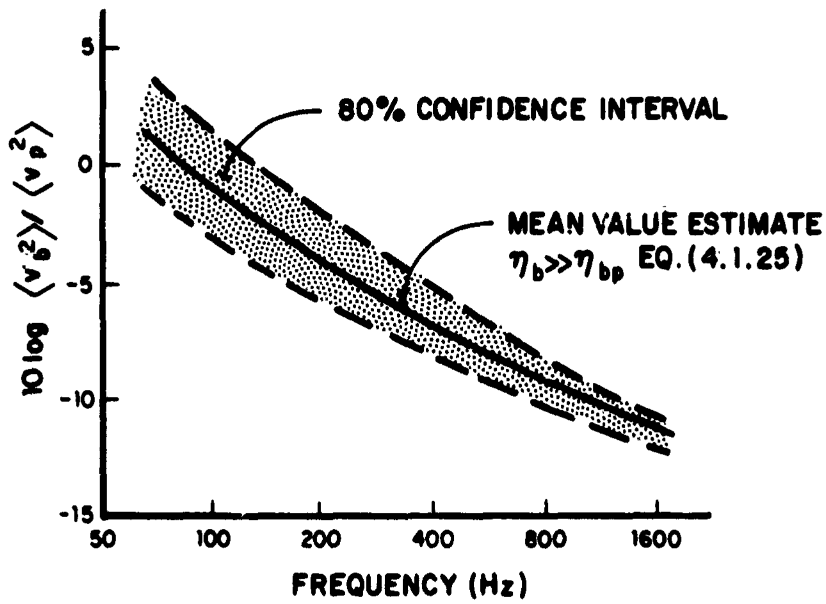


FIG 4.11

MEAN BRACKETING 80% CONFIDENCE INTERVAL BASE ON FIG 4.9 FOR SYSTEM OF FIG 4.4, MEAN VALUE ESTIMATE OF FIG 4.5 AND VARIANCE OF FIG 4.10 (DASHED LINE)

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(bj) E. E. Ungar, "Energy Dissipation at Structural Joints, Mechanisms and Magnitudes," U.S. Air Force FDL-TDR-64-98 (1964).

The application of SEA to sound transmission through double walls is reported in the following articles:

(bk) M. J. Croker, et al., "Sound and Vibration Transmission Through Panels and Tie Beams Using Statistical Energy Analysis", Trans. ASME: J. Engineering Ind. (Aug 1971) pp 775-782.

(bl) P. H. White and A. Powell, "Transmission of Random Sound and Vibration Through A Rectangular Double Wall", J. Acoust. Soc. Am., Vol. 40, No. 4, pp 821-832 (1965).

(bm) A. Rinsky, "The Effects of Studs and Cavity Absorption on the Sound Transmission Loss of Plasterboard Walls", Sc.D. Thesis, MIT Dept. of Mech. Eng., February 1972.

An attempt to elucidate implications of the SEA model in terms of classical vibration analysis is presented in

(bn) J. L. Zeman and J. L. Bogdanoff, "A Comment on Complex Structural Response to Random Vibrations", AIAA Journal, Vol. 7, No. 7, pp 1225-1231 (July 1969).

and also in

(bo) W. Gersch, "Mean Square Responses in Structural Systems", J. Acoust. Soc. Am., Vol 48, No. 1 (Pt. 2) pp 403-413 (1970).

(bp) W. Gersch, "Average Power and Power Exchange in Oscillators", J. Acoust. Soc. Am., Vol. 46, No. 5 (Pt. 2) pp 1180-1185 (1969).

The work of Lotz referred to is

(bq) R. Lotz, "Random Vibration of Complex Structures", Ph.D. Thesis, MIT Dept. of Mechanical Engineering, June 1971.

A comparison of the effect of power flow of various assumptions regarding modal frequency statistics (or occurrence in frequency) is given in (ap). A report in which calculations for a deterministic system may be compared with the SEA calculations is

(br) J. E. Manning and P. J. Remington, "Statistical Energy Methods", BBN Report, No. 2064, Submitted March 26, 1971 to NASA.

1.2 The General Procedures of SEA

Discussions of "similar" mode groups may be found in (aa, bc) and in

(bs) J. E. Manning, "A Theoretical and Experimental Model-Study of the Sound-Induced Vibration Transmitted to a Shroud-Enclosed Spacecraft". BBN Report 1891, submitted May 1, 1970 to NASA.

The equivalent RC circuit for energy flow is discussed in (am, at). The power injecting properties of various random loading environments are discussed in (bd) and also in

(bt) R. H. Lyon "Boundary Layer Noise Response Simulation with a Sound Field", Chapter 10 of Acoustical Fatigue in Aerospace Structures, Ed. by W. J. Trapp and D. M. Forney (Syracuse University Press, Syracuse, N. Y., 1965).

(bu) R. H. Lyon, Random Noise and Vibration in Space Vehicles Shock and Vibration Information Center, U. S. Dept. of Defense, 1967).

(bv) I. Dyer, "Response of Plates to a Decaying and Convecting Random Pressure Field". J. Acoust. Soc. Am., Vol. 31, No. 7 pp 922-928 (1965).

The relative magnitudes of time averaged and oscillating power flow between coupled resonators is discussed in

(bw) D. E. Newland, "Calculation of Power Flow Between Coupled Oscillators", J. Sound Vib., Vol. 3, No. 3, pp 262-276 (1966).

A brief summary of the problems of measuring various SEA parameters is given in

(bx) R. H. Lyon, "Analysis of Sound-Structural Interaction by Theory and Experiment", Contribution to Proceedings of Purdue Noise Control Conference, July 14-16, 1971, Purdue University.

Modal density measurements of cylinders are presented in (bb, bs). Measurements for flat plates are given in (ar, at), and for mass loaded plates in

(by) R. W. Sevy and D. A. Earls, "The Prediction of Internal Vibration Levels of Flight Vehicle Equipments Using Statistical Energy Methods," U.S. Air Force Technical Report AFFDL-TR-69-54, January 1970.

Calculations of modal densities are given in (ba through bh) for a variety of structural elements or subsystems. Input impedances to beams and plates are found in (ao, ay, av). Input impedances to sound fields may be found in (am, aw, ax, bd, bg), and in

(bz) R. H. Lyon, "Statistical Analysis of Power Injection and Response in Structures and Rooms" J. Acoust. Soc. Am., Vol. 45, No. 3, pp 545-565 (1969).

The derivation of transmission loss and its relation to coupling loss factor may be found in

(ca) I. L. Ver and C. I. Holmes, "Interaction of Sound Waves with Solid Structures", Chapter 11 of Noise and Vibration Control, Ed. by L. L. Beranek (McGraw-Hill Book Co., Inc. New York 1971).

(cb) C. I. Holmes, "Sound Transmission Through Structures: A Review" M.Sc. Thesis, John Carroll University, Cleveland, Ohio 1969.

A useful "dictionary" of transmission loss for various practical wall structures may be found in

(cc) R. D. Berendt, G. E. Winzer and C. B. Burrough, "A Guide to Airborne, Impact and Structure Borne Noise Control in Multifamily Dwellings." U. S. Dept. of Housing and Urban Development, Washington, D. C., September 1967.

The relation of vibrational velocity to stress in a variety of structural elements is presented in

(cd) F. V. Hunt, "Stress and Strain Limits on the Attainable Velocity in Mechanical Vibration", J. Acoust. Soc. Am., Vol. 32, No. 9, pp 1123-1128 (1960).

(ce) E. E. Ungar, "Maximum Stresses in Beams and Plates Vibrating at Resonance", J. Eng. Ind. Vol. 84, No. 1, pp 149-155 (February 1962).

Analysis of response variance is developed in (aq, ar, as, bz). A basic discussion of confidence intervals and coefficients may be found in

(cf) A. M. Mood, Introduction to the Theory of Statistics (McGraw-Hill Book Co., Inc., New York, 1950). Chapter 11.

1.3 Future Developments of SEA

The range of professional workers making use of SEA is discussed in (aa). That reference prompted additional discussion.

(cg) J. L. Zeman and J. L. Bogdanoff, Letter to the Editor. The Shock and Vibration Digest, Vol. 3, No. 1, p 2 (1971).

(ch) C. T. Morrow, "Can Statistical Energy Analysis be Applied to Design?" The Shock and Vibration Digest, Vol. 3, No. 5, Pt. 1 (1971).

and a reply

(ci) R. H. Lyon, Letter to the Editor. The Shock and Vibration Digest, Vol. 3, No. 10, pp 3-4 (1971).

A computer solution of simultaneous equations in SEA has been carried out and is reported in

(cj) D. M. Wong, "Transmission of Noise and Vibration Between Coupled Cylinders," Lockheed Sunnyvale Report, 1969.

The development of a range of models more complex than the SEA models presented here that have greater "information capacity" has been discussed in

(ck) R. H. Lyon, "Application of a Disorder Measure to Acoustical and Structural Models" J. Eng. Ind., Trans. ASME, Vol. 93, Ser. B, No. 3, pp 814-818 (August 1971).

The allowance for modal interference effects the modal resonance frequency spacing has been discussed in (al, bz).

1.4 Organization of the Report

Various surveys of SEA have been published for various audiences. They include (aa, ao, at, bq, br, bx) and also

(cl) R. H. Lyon and G. Maidanik, "Statistical Methods in Vibration Analysis", AIAA Journal, Vol. 2, No. 6, pp 1015-1024 (1964).

(cm) P. A. Franken and R. H. Lyon, "Estimation of Sound-Induced Vibrations by Energy Methods, with Applications to the Titan Missile", Shock and Vibration Bulletin, No. 31, Part III, pp 12-16 (1963).

(cn) R. H. Lyon, "Basic Notions in Statistical Energy Analysis," Contribution to Synthesis of Vibrating Systems, Ed. by V. H. Neubert and J. P. Raney, (ASME, New York, 1971).

(co) P. W. Smith, Jr. and R. H. Lyon, "Sound and Structural Vibration", NASA Contractor Report CR-160, March 1965.

Chapter 2. Energy Description of Vibrating Systems

2.0 Introduction

Additional discussion of energy variables in vibration may be found in (co). The relation of impedance functions to energy variables is discussed in

(cp) E. A. Guillemin, Introductory Circuit Theory, (Chapman and Hall, Ltd, New York 1953) Chapter 7.

Discussion of the relation between modal and wave descriptions may be found also in (ag, am, co).

2.1 Modal Resonators

Solutions for the single dof vibrator may be found in (ab, ad, co, cp). Relations among various measures of damping may be found in (ay). Reverberation time as a measure of the damping of sound waves is discussed in (ah, co). The idea of viewing variuos frequency regions of response is "mass-controlled", etc. is discussed in (ca, co). Random excitation of simple resonators is extensively discussed in (ab, ac). The notion of a white noise generator as a constant power source is in (am). The removal of the divergent mass law response of a mode to white noise may be found in

(cq) R. H. Lyon, "Shock Spectra for Statistically Modeled Structures", Shock and Vibration Bulletin, No. 40, Part 4, pp 17-23, December 1969.

2.2 Modal Analysis of Distributed Systems

The theory of the vibration of distributed systems by using modal expansions may be found in (an) and also in

(cr) P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill Book Col, Inc., New York, 1953).

When damping is not proportional to the mass density an irreducible modal coupling results which is extensively discussed in (bn). Modal lattices are used extensively in acoustics (ag, ah) and in x-ray diffraction theory and solid state physics:

(cs) L. Brillouin, Wave Propagation in Periodic Structures (McGraw-Hill Book Co., Inc., New York, 1946).

An elementary discussion of energy and group velocity may be found in (ai). The expansion of the structure admittance function as a sum of modal admittances is given in (bz). The point input resistance of an infinite plate is derived in (ay, co). The use of conductance measurement as a means of evaluating modal density was first proposed in (bx).

2.3 Dynamics of Infinite Systems

The development of dispersion relations and their interpretation in terms of phase and group velocities may be found in

(ct) L. M. Brekhovskikh, Waves in Layered Media (Academic Press, New York 1960). Derivations for the "stiffness operator" for strings, beams, and sound fields may be found in

(cu) P. M. Morse, Vibration and Sound, Second Edition, (McGraw-Hill Book Co., Inc. 1948). The theory of Fourier integrals and the use of contour integrals to evaluate the inverse transform may be found in (cr). The derivation of mean free path for a two dimensional system is to be found in (co). The discussion of resonator interaction with a finite plate is adapted from (bo).

2.4 Modal-Wave Duality

The decomposition of a wave field into "direct" and "reverberant" components is discussed in (ag, ah, ay, bz, co). The formation of the direct field as a result of modal coherence may be found in (ag). Various integral expressions for and relations between bessel functions may be found in (cr).

Chapter 3. Energy Sharing by Coupled Systems

A discussion of the environmental excitation of structures may be found in (bd, bu) and in

(cv) R. H. Lyon, "An Energy Method for Prediction of Noise and Vibration Transmission", Shock and Vibration Bulletin, No. 33, Part II pp 13-25 (February 1964).

The definition of uncoupled systems has been an important element in several studies (bc, bk, br, bs). An interesting application of reciprocity to develop engineering formulas for noise transmission is

(cw) M. A. Heckl and E. J. Rathe, "Relationship Between the Transmission Loss and the Impact-Noise Isolation of Floor Structures", J. Acoust. Soc. Am., Vol 35, No. 11, pp 1825-1830 (1963).

3.1 Energy Sharing Among Resonators

The discussion in this section is taken from (ap), which in turn is based on

(cx) T. D. Scharton, "Random Vibration of Coupled Oscillators and Coupled Structures," Sc.D. Thesis, MIT Dept. of Mechanical Engineering, 1965.

Additional discussion of the effects of blocking vibrating systems may be found in (bq, bn) and in

(cy) D. E. Newland, "Power Flow Between A Class of Coupled Oscillators," J. Acoust. Soc. Am., Vol. 43, No. 3, pp 553-559 (1968).

The example of two resonators coupled by stiffness only is taken from (cn). The behavior of coupled resonators in free vibration may be found in (ad). The analysis of power flow in this section and in (ao) is carried out in the "frequency domain". For a time domain analysis, see (am). Integrations over the frequency response functions in Eq. (3.1.14) may be found in (ab). The result of Eq. (3.1.19) was found in (am) for weak coupling, although the result (for weak coupling) was also applied to relations like Eq. (3.1.25). The essential correctness of Eq. (3.1.25) must be found from (ao, ap, cx). The equivalence of average modal interaction and white noise excitation was first put forward in (am).

3.2 Energy Exchange in Multi-Degree-of-Freedom Systems

Additional discussion of the assumptions involved in forming the "uncoupled" modes for the blocked condition may be found in (bn). A discussion of the treatment of damping as a random parameter is to be found in (cg). Quite a thorough analysis of sound-structural coupling, carried out on a completely modal basis is to be found in

(cz) F. Fahy, "Vibration of Containing Structures by Sound in the Contained Fluid", J. Sound Vib., Vol. 10, No. 3 pp 490-512 (1969).

(da) F. Fahy, "Response of a Cylinder to Random Sound in the Contained Fluid", J. Sound Vib., Vol. 13, No. 2, pp 171-194 (1970).

Discussions of the generation of multi-modal interactions from the two-mode interaction are to be found in (am, ao, at). The first published use of a diagram like that of Fig. 3.5 appears to be (cv).

3.3 Reciprocity and Energy Exchange in the Wave Description

For a discussion of diffusion, reverberation and energy density in sound fields, see (ah, cu). The general conditions for reciprocity in continuous systems are given in (cr, ct) and for "lumped" systems in (cp). The use of reciprocity to develop response estimates of the SEA form may be found in (cv). The use of impedance relations to determine coupling loss factors may be found in (am, as, au, bc). A study of two plates connected at a point is presented in (bq). The relation between coupling loss factor and transmission coefficients (or transmission loss) is developed in (bk, cn) and in

(db) R. H. Lyon, Aerospace Noise Vibration. Notes for a Lecture series offered by Bolt Beranek and Newman, 1966.

Transmission loss data are to be found in (cc).

3.4 Some Sample Applications of SEA

This discussion of the excitation of a resonator by a sound field may be compared to that in (an). The analysis of coupled beams may be compared with similar discussions in (ap, br) and

(dc) H. G. Davies, "Exact Solutions for the Response of Some Coupled Multimodal Systems", J. Acoust. Soc. Am., Vol. 51, No. 1 (Pt. 2), pp 387-392 (1971).

The relation between "mass law" transmission loss and the coupling loss factor is discussed in (bk, db).

Chapter 4. The Estimation of Response in Statistical Energy Analysis.

4.0 Introduction

Additional discussion of the requirements for response estimation may be found in (aa, ao, bu, cg, ch, ci) and in

(dd) J. L. Bogdanoff, "Meansquare Approximate Systems and Their Application in Estimating Response in Complex Disordered Linear Systems", J. Acoust. Soc. Am., Vol. 38, No. 2, pp 244-252 (1965).

General source books for statistical estimation procedures are (bf) and

(de) H. Cramer, Mathematical Methods of Statistics (Princeton University Press, Princeton, 1946).

4.1 Mean Value Estimates of Dynamical Response

The mean square pressure in reverberant sound fields has been studied extensively in

(df) R. V. Waterhouse, "Statistical Properties of Reverberant Sound Fields", J. Acoust. Soc. Am., Vol. 43, No. 6, pp 1436-1444 (1968).

Spatial coherence effects in multi-modal response in the neighborhood of the drive point are discussed in (bz) and in (dg) L. Wittig, "Random Vibration of Point-Driven Strings and Plates", Ph.D. Thesis, MIT Dept. of Mech. Eng., May 1971.

The problem of adequate sampling of a reverberant field for reducing uncertainty in the energy content of the field has been discussed by

(dh) R. V. Waterhouse and D. Lubman, "Discrete Versus Continuous Averaging in a Reverberant Sound Field", J. Acoust. Soc. Am. Vol. 48, No. 1 (Pt.1) pp 1-5 (1970).

The beam plate system studied in this section is similar to that in (as). There has been much confusion regarding the use of infinite system impedances to evaluate coupling loss factors in finite systems, as was done in (am, as, an, aw). Hopefully the discussion in this section will clarify the procedure. The expression of system strain in terms of a particle "Mach number" is found in (cd, ce).

4.2 Calculation of Variance in the Temporal Mean Square Response.

The analysis of variance is an extensive and important topic in statistical theory (cf, de). The procedures used in this section were first developed in (ag, ar, as) and are based on the theory of variance of Poisson pulse processes as presented in

(di) S. O. Rice, "Mathematic Analysis of Random Noise". Contribution to Noise and Stochastic Processes, Ed. by N. Wax (Dover Publications, Inc. New York, 1954).

The effects of temporal variation in multi dof system variance are discussed by

(dj) H. Andres, "A Law Describing the Random Spatial Variation of Sound Fields in Rooms and Its Application to Sound Power Measurements." Acustica, Vol. 16, p 278-294 (1965). Translation by I. L. Ver, BBN-TIR-70, (1968).

Eq. (4.2.9) is found from a relation in (di). Additional discussion of the role of mode shapes and sampling strategies for locations of excitation and observation points in affecting response variance may be found in (as, bz). The use of multiple observation points or line averages to reduce variance are discussed in (dh). Modifications to the variance calculation for non-Poisson intervals between resonance frequencies are discussed in (bz). Note that the effective number of modal interactions given in Eq. (4.2.13) is not the total number of possible interactions N_1N_2 as suggested in (cz, da).

4.3 Calculation of Confidence Coefficients

The approach and terminology in this section are taken from (cf). The gamma and related distributions are discussed in (cf, de) and in

(dk) E. Parzen, Stochastic Processes (Holden-Day, Inc., San Francisco, 1962).

The two particular confidence intervals derived in this section were first presented in (as, cv). The incomplete gamma function is discussed in

(dl) A. Erdelyi, et al., Higher Transcendental Functions, Vol. 2 (McGraw-Hill Book Co., Inc. 1953). Chapter IX.

Confidence coefficients for the determination of environmental levels based on loads and response data are used in

(dm) P. T. Mahaffey and K. W. Smith, "A Method for Predicting Environmental Vibration Levels in Jet-Powered Vehicles", S. & V. Bulletin, No. 28, Pt. 4, pp 1-14 (1960).

4.4 Coherence Effects - Pure Tone and Narrow Band Response

The discussion in this section is based on

(dn) R. H. Lyon, "Spatial Response Concentrations in Extended Structures". Trans. ASME, J. Eng. Ind., November 1967.

Additional discussion of extreme value statistics for multi-dimensional sinusoids may be found in

(do) R. H. Lyon, "Statistics of Combined Sine Waves" J. Acoust. Soc. Am., Vol. 48, No. 1 (Pt. 2) pp 145-149 (1970).