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CHECKING PROOFS IN THE
METAMATHEMATICS OF FIRST ORDER
LOGIC

Mario Aiello, et al
Stanford University

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by

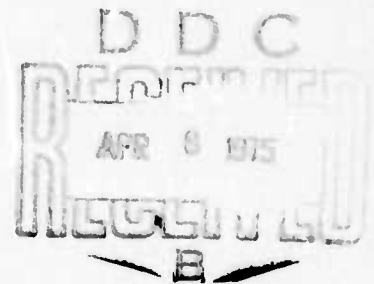
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and
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Abstract:

This is a report on some of the first experiments of any size carried out using the new first order proof checker FOL. We present two different first order axiomatizations of the metamathematics of the logic which FOL itself checks and show several proofs using each one. The difference between the axiomatizations is that one defines the metamathematics in a many sorted logic, the other does not.

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SECTION 1 INTRODUCTION

This paper represents a first attempt at the axiomatization of the metamathematics of a first order theory and at using the new proof checker FOL (First Order Logic). The logic which FOL checks is described in detail in the user manual for this program, Weyhrauch and Thomas 1974. It is based on a system of natural deduction described in Prawitz 1965, 1970.

Our motivation in axiomatizing the metamathematics of FOL was the desire to work on an example which could be used as a case study for projected features of FOL and, at the same time, had independent interest with respect to representing the proofs of significant mathematical results to a computer.

The eventual ability to clearly express the theorems of mathematics to a computer will require the facility to state and prove theorems of metamathematics. There are several clear examples:

a. *Axiom schemas.* How exactly do we express that

$$P(\theta) \wedge \forall n.(P(n) \supset P(n+1)) \supset \forall n.P(n)$$

is an axiom schema? We need to say: "If for any first order sentence P with one free variable y we denote by $P(n)$ the formula obtained from P by substituting n for y assuming n is free for y in P , then the sentence

$$P(\theta) \wedge \forall n.(P(n) \supset P(n+1)) \supset \forall n.P(n)$$

is an axiom of arithmetic".

b. *Theorem schemas.* The following kind of "theorem" is sometimes seen in set theory books

$$\forall x_1 \dots x_n \in S. \exists T. \forall u. ((x_1, \dots, x_n) \in T \supset \exists y. ((x_1, \dots, x_n, y) \in S)).$$

It asserts the existence of some particular projection of $n+1$ -tuples. In its usual formulation this is not a theorem of set theory at all, but a metatheorem which states that, for each n , the above sentence is a theorem. We do not know of any implementation of first order logic capable of expressing the above notion in a straightforward way.

c. *Subsidiary deduction rules.* Below we show how to prove that if there is a proof of $\forall x y.WFF$ then there is also a proof of $\forall y x.WFF$, where WFF is any well formed formula. We chose this task because it seemed simple enough to do, and is a theorem which may actually be used. The use of metatheorems as rules of inference by means of a reflection principle will be discussed in a future memo by Richard Weyhrauch. Eventually we hope to check some more substantial metamathematical theorems.

d. *Interesting mathematical theorems.* We present two examples. The first is any theorem about finite groups. The notion of finite group cannot be defined in the usual first order language of group theory. Thus many "theorems" are actually metatheorems, unless you axiomatize groups in set theory. The second theorem is the "duality principle" in projective geometry.

SECTION 2 THE AXIOM SYSTEM

In this section we present two axiomatizations of the metamathematics of first order logic. The main difference between them is that one is done in a many sorted first order logic and the other not. These axiomatizations represent an attempt at experimenting with proofs about properties of formulas and deductions. No effort has been spent on guaranteeing that the axioms are independent. It would not only have been uninteresting but also contrary to our basic philosophy. We wish to find axioms which naturally reflect the relevant notions. At the moment this axiomatization is far from being in its final form. Neither the extent of the notions involved nor the best way of expressing them is considered settled.

Section 2.1 The sorts

The sorts we have defined correspond to the basic notions of the metamathematics i.e. terms, formulas, individual variables, logical symbols, function symbols etc. and to the notions of the domains (strings and sequences of strings) in which the axiomatization has been defined. FOL (see Weyhrauch and Thomas 1974) allows the declaration of variables to be of a certain sort. In the formulas appearing in this paper the following declarations are assumed:

$g \ g1 \ g2 \ g3 \ g4 \ g5 \ g6$	range over the most general sort
$sq \ sq1 \ sq2 \ sq3 \ sq4 \ sq5 \ sq6 \in \text{SEQ}$	(SEQs are sequences of strings)
$pf \ pf1 \ pf2 \ pf3 \ pf4 \ pf5 \ pf6 \in \text{PROOFTREE}$	(PROOFTREEs are sequences representing derivations in FOL)
$s \ s1 \ s2 \ s3 \ s4 \ s5 \ s6 \in \text{STRING}$	(STRINGs are strings)
$t \ t1 \ t2 \ t3 \ t4 \ t5 \ t6 \in \text{TERM}$	(TERMs are strings representing terms)
$x \ x1 \ x2 \ x3 \ x4 \ x5 \ x6 \in \text{INDVAR}$	(INDVARs are strings representing individual variables)
$el \ el1 \ el2 \ el3 \ el4 \ el5 \ el6 \in \text{ELF}$	(ELFs are strings representing elementary formulas)
$f \ f1 \ f2 \ f3 \ f4 \ f5 \ f6 \in \text{FORM}$	(FORMs are well formed formulas)
$th \ th1 \ th2 \ th3 \ th4 \ th5 \ th6 \in \text{BEW}$	(BEWs are theorems of a first order theory)
$A \ A1 \ A2 \ A3 \ A4 \ A5 \ A6 \in \text{AXIOM}$	(AXIOMs are axioms of a particular theory)
$c0 \ c1 \ c2 \ c3 \ c4 \ c5 \ c6 \in \text{INDCONST}$	(INDCONSTs are individual constants)
$a \ a1 \ a2 \ a3 \ a4 \ a5 \ a6 \in \text{ATOM}$	(ATOMs are the individual constituents of a string)
$n \ n1 \ n2 \ n3 \ n4 \ n5 \ k \in \text{INTEGER}$	(INTEGERs are integers)
$nc \ nc1 \ nc2 \ nc3 \ nc4 \ nc5 \ nc6 \in \text{NUMERAL}$	(NUMERALs are numerals)

$sy\ sy1\ sy2\ sy3\ sy4\ sy5\ sy6 \in SYM$ (SYMs are logical symbols)
 $np\ np1\ np2\ np3\ np4\ np5\ np6 \in N_PLCSYM$ (N_PLCSYMs are symbols which have an *arity*)
 $fn\ fn1\ fn2\ fn3\ fn4\ fn5\ fn6 \in OPCODE$ (OPCODEs are function symbols)
 $P\ P1\ P2\ P3\ P4\ P5\ P6 \in PREDCONST$; PREDCONSTs are predicate symbols)

the partial order between these sorts is defined by the following FOL declarations:

```

MG SEQ      ≥ { STRING , PROOFTREE };
MG PROOFTREE ≥ { FORM };
MG STRING   ≥ { TERM , FORM , ATOM , VARSTRING };
MG TERM     ≥ { INDVAR };
MG FORM     ≥ { ELF , SENTCONST , PREDPARB , AXIOM , BEW };
MG BEW      ≥ { AXIOM };
MG ATOM     ≥ { INDCONST , SENTCONST , SYM , INTEGER , N_PLCSYM ,
              INDPAR , INDVAR , AUXSIGN , PREDCONSTO , PREDPARB };
MG INDCONST ≥ { NUMERAL };
MG SYM      ≥ { QUANT , SENTCONN };
MG N_PLCSYM ≥ { PREDCONST , OPCODE , PREDPAR };
  
```

Sorts are always predicates with one argument. The declaration

$MG\ SORT1 \geq \{ SORT2 , \dots , SORTn \}$

should be read as $SORT1$ is more general than $SORT2, \dots, SORTn$ and corresponds to the implicit axioms

$\forall g. SORT1(g) \supset SORT2(g),$

$\forall g. SORTn(g) \supset SORT1(g).$

The first declaration, for instance, says that strings and derivations are particular sequences of formulas. Strings are in fact sequences of length 1 and derivations are those sequences satisfying the predicate PROOFTREE.

Section 2.2 The domain of representation of the metamathematics

The basic notions of the metamathematics of first order logic have been axiomatized in terms of strings and sequences of strings. The primitive functions on them are concatenation (c for strings, cc for sequences) and selectors (car , cdr for strings and $scar$, $scdr$ for sequences). c and cc are infix operators.

2.2.1 Formulas and terms

Formulas and terms are represented by the string of symbols appearing in them. Terms are defined recursively as strings which either represent an individual variable or can be decomposed into $n+1$ substrings representing a function symbol of arity n , followed by n terms. The two predicates defining terms are:

TERMSEQ(θ,LAMBDA)

$\forall s.(\text{TERM}(s) \equiv \text{INDVAR}(s) \vee \exists n \text{ fn} . (\text{fn} = \text{car}(s) \wedge n = \text{arity}(\text{fn}) \wedge \text{TERMSEQ}(n, \text{cdr}(s))))$

$\forall n \ s.(\text{TERMSEQ}(n,s) \equiv ((\text{car}(s) = \text{LPARSYM}) \wedge ((\text{len}(s) \text{ gl } s) = \text{RPARSYM}) \wedge \exists n1 . (\text{TERM}(\text{substring}(s,2,n1)) \wedge \text{TERMSEQ}(n-1, \text{substring}(s,n1+1, \text{len}(s)-1))))))$

where the function $\text{substring}(s,m,n)$ (see appendix 1.2) returns the substring of s starting from its m -th element and ending with the n -th. $\text{len}(s)$ computes the length of s and $(n \text{ gl } s)$ selects the n -th element of s .

Well formed formulas (wffs) are represented as strings which either are elementary formulas (defined by the predicate ELF) or can be partitioned into substrings for formulas and logical connectives. Formulas are defined by:

$\forall s.(\text{ELF}(s) \equiv (s = \text{FALSESYM} \vee \text{PREDPARO}(s) \vee \exists n \ P. (P = \text{car}(s) \wedge n = \text{arity}(P) \wedge \text{TERMSEQ}(n, \text{cdr}(s))))),$

$\forall s.(\text{FORM}(s) \equiv (\text{ELF}(s) \vee \exists x \ f. (s = (x \text{ gen } f) \vee s = (x \text{ ex } f)) \vee \exists f1 \ f2. (s = (f1 \text{ dis } f2) \vee s = (f1 \text{ con } f2) \vee s = (f1 \text{ impl } f2)) \vee \exists f. s = \text{neg}(f))) ; ;$

gen is the infix operator that maps its arguments x and f into the string $(\text{FORALLSYM } c \ x) \ c \ f$ representing the well formed formula $\forall x.f$. The operator ex is used for the existential quantifier. dis , con and impl are the infix operators for the disjunction, conjunction and implication of two formulas. Finally, neg is the operator which maps a formula into its negation.

We could possibly represent wffs as structured objects (lists, trees, etc.) which contain all the information about the structure of the formula and do not require any parsing. This approach amounts to axiomatizing metamathematics in terms of the abstract syntax of first order logic, instead of strings of symbols. Both of these possibilities should be explored. We have chosen the first alternative because:

- 1) It is the most traditional, i.e. metamathematics, as it appears in logic books, is usually stated in terms of strings.
- 2) Axioms in terms of abstract syntax are simply theorems of the theory expressed in terms of strings. Thus the two representations look substantially the same with respect to "high level" theorems.
- 3) Ill-formed formulas can be mentioned. This is of course impossible in an axiomatization in terms of the abstract syntax.

The properties of wffs relevant to our theory have been defined by the predicates FR, FRN, GEB and SBT. FR(x,f) is true iff the variable x has at least one free occurrence in the wff f, while FRN(x,n,f) and GEB(x,n,f) are respectively true when the variable x occurs free or bound at the place n in the formula f. These predicates are defined in appendix 1.6. In addition, some generalized selector functions are defined, which evaluate the first or the k-th free occurrence of a variable in a wff, or the number of its free occurrences. The predicate SBT is then defined. It axiomatizes the notion of substitution of a term for any free occurrence of a variable in a wff.

$$\forall x \ t \ f1 \ f2. (SBT(x,t,f1,f2) = \begin{aligned} &\forall n1 \ n2 \ ((n2 = (\text{number of occurrences of } x \text{ in } f1) * (\text{length of } f1) - n1) \supset \\ &((\neg \text{INDVAR}(n1 \ \text{gl } f1) \supset (n1 \ \text{gl } f1) = (n2 \ \text{gl } f2)) \wedge \\ &(\text{INDVAR}(n1 \ \text{gl } f1) \supset ((\text{FRN}(x,n1,f1) \supset \text{SUBT}(f1,f2,n2)) \wedge \\ &(\neg \text{FRN}(x,n1,f1) \supset \text{INVART}(n1,f1,n2,f2)))))) \end{aligned})))$$

$$\forall t \ f2 \ n2. (\text{SUBT}(f1,f2,n2) = \forall x2 \ k \ ((k \ \text{gl } f1) = x2 \supset \text{FRN}(x2,n2 - (\text{length of } f1) - k, f2))))$$

$$\forall n \ f1 \ n1 \ f2 \ ((\text{INVART}(n,f1,n1,f2) = ((\text{GEB}(n \ \text{gl } f2,n1,f2) = \text{GEB}(n \ \text{gl } f1,n,f1)) \wedge (\text{FRN}(n \ \text{gl } f2,n1,f2) = \text{FRN}(n \ \text{gl } f1,n,f1)) \wedge (n1 \ \text{gl } f2) = (n \ \text{gl } f1))))$$

In the previous definition, n1 is any position in the string f1 and n2 is the corresponding position in f2. The auxiliary predicate SUBT states that the variables appearing in the term t substituted for a free occurrence of the variable x are still free. INVART defines which properties of f1 are still true for f2. If the term t is a variable, then SBT reduces to SBV.

$$\forall x1 \ x2 \ f1 \ f2. (\text{SBV}(x1,x1,f1,f2) = \begin{aligned} &\forall n \ ((\neg \text{INDVAR}(n \ \text{gl } f1) \supset (n \ \text{gl } f1) = (n \ \text{gl } f2)) \wedge \\ &(\text{INDVAR}(n \ \text{gl } f1) \supset ((\text{FRN}(x1,n,f1) \supset \text{FRN}(x2,n,f2)) \wedge \\ &(\neg \text{FRN}(x1,n,f1) \supset \text{INVARV}(n,f1,f2)))))) \end{aligned})))$$

$$\forall n \ f1 \ f2. (\text{INVARV}(n,f1,f2) = ((\text{GEB}(n \ \text{gl } f2,n,f2) = \text{GEB}(n \ \text{gl } f1,n,f1)) \wedge (\text{FRN}(n \ \text{gl } f2,n,f2) = \text{FRN}(n \ \text{gl } f1,n,f1)) \wedge (n \ \text{gl } f2) = (n \ \text{gl } f1))))$$

The proof of the equivalence of SBT and SBV when t is a variable is very simple. It is based on the fact that n2 coincides with n1 when the term t has length 1 (see appendix 4). The function sbt (sbv) evaluates to the string representing the result of substituting a term (variable) for every free occurrence of a variable in a given wff. sbt and sbv are defined from the predicates SBT and SBV as follows:

$$\forall x \ t \ f1 \ f2. (\text{SBT}(x,t,f1,f2) = \text{sbt}(x,t,f1)=f2)$$

$$\forall x1 \ x2 \ f1 \ f2. (\text{SBV}(x1,x2,f1,f2) = \text{sbv}(x1,x2,f1)=f2)$$

The problem of finding the best way of defining functions in FOL is crucial. In the axiom system given in this paper a uniform way has not been followed. In defining the substitution we are interested in properties of the functions sbt and sbv and in drawing conclusions from the fact that a substitution has been made. It is thus useful to have a predicate which defines the relation between formulas before and after a substitution instead of inferring it from the definitions of the functions (stated for example as a system of equations, as in Kleene 1952). One of the motivations of the present experiment was to explore different ways of defining functions. We do not yet have enough examples of proofs to make a clear statement about this matter.

2.2.2 Rules of inference, deductions and the notion of provability

The rules of inference are defined by the predicates in appendix 1.7. The rules with one premise, are expressed by means of a binary predicate whose arguments are two sequences of wffs (sq , pf) which satisfy PROOFTREE. The predicate is true iff pf is the scdr of sq and the first element of sq is a wff obtained by applying that particular deduction rule to the first wff of pf . The rules with more antecedents are defined in a similar way.

Derivations are recursively defined as sequences of wffs which either are a single wff or are obtained from one or more derivations by applying one of the deduction rules. The recursion is implicitly stated by saying that there exist objects of sort PROOFTREE which satisfy one of the predicates defining the rules of inference. These sequences represent the linearization of a deduction-tree and are defined as follows:

$$\begin{aligned} \forall sq. (\text{PROOFTREE}(sq) \equiv & \\ & (\text{FORM}(sq) \vee \\ & \exists pf. (\text{ORI}(sq, pf) \vee \text{ANDE}(sq, pf) \vee \text{FALSEE}(sq, pf) \vee \text{NOTI}(sq, pf) \vee \text{NOTE}(sq, pf) \vee \text{IMPLI}(sq, pf)) \vee \\ & \exists pf \ x \ t. (\text{GENI}(sq, pf, x, t) \vee \text{GENE}(sq, pf, x, t) \vee \text{EXI}(sq, pf, x, t)) \vee \\ & \exists pf1 \ pf2. (\text{ANDI}(sq, pf1, pf2) \vee \text{FALSEI}(sq, pf1, pf2) \vee \text{IMPLE}(sq, pf1, pf2)) \vee \\ & \exists pf1 \ pf2 \ x1 \ x2. \text{EXE}(sq, pf1, pf2, x1, x2) \vee \\ & \exists pf1 \ pf2 \ pf3. \text{ORE}(sq, pf1, pf2, pf3))) ;; \end{aligned}$$

A sequence of wffs is a prooftree if either it consists of a single wff or one of the following alternatives holds: there exists another prooftree and a one premise deduction rule has been applied; there exist two prooftrees and one of the two premises rules has been applied; finally, there are three prooftrees and the predicate defining the \vee -elimination rule is true. Note that the root of a prooftree is not necessarily a theorem in a given theory. A predicate **DEPEND** has been defined which is true if a given wff is a dependence for the root of a prooftree. The axioms about **DEPEND** allows to decide all the dependencies of a prooftree.

Since some of the deduction rules (the implication introduction, for instance) eliminate dependencies, not all the leaves of a prooftree pf are dependencies for a wff f such that $f = \text{scdr}(pf)$. The predicate **DEPEND** is true only for those leaves of the prooftree which the formula f actually depends on. Its definition is shown in appendix 1.8. The axioms **DEPEND** state which dependencies do not change by applying the deduction rules and are transferred from one prooftree to the other. The axioms **NDEPEND** state which rules discharge dependencies in a given prooftree.

Using this notion of dependence, the provability of a formula in a theory is defined as follows:

$$\forall f. (\text{BEW}(f) \equiv \exists sq. (\text{PROOFTREE}(sq) \wedge f = \text{scdr}(sq) \wedge \forall fl. (\text{DEPEND}(sq, fl) \supset \text{AXIOM}(fl)))));;$$

A wff f is a theorem in a given theory if there exists a prooftree whose first element is f and whose only dependencies are axioms in that theory. We have limited our attention to theories in which axioms have no free variables. This property is defined by the axiom:

$$\forall x \ t. (\text{AXIOM}(t) \supset \neg \text{FR}(x, t));;$$

Section 2.3 The main proof in the many sorted logic

The main theorem we have proved in this axiomatization of the metamathematics states that if $\forall x y.wff$ is provable in some theory, then $\forall y x.wff$ is also provable. We have chosen this theorem because, even if very simple, it involves basic notions of provability, substitution and universal quantification. Its proof is found in appendices 5.1-2. The theorem depends on the first three lines of the proof. The first step is a lemma stating that $\forall x wff.sbl(x,x,f)=wff$, i.e. substituting a variable x for any free occurrence of x in wff doesn't change that wff . Steps two and three give simple facts about sequences. The theorem is then proved by instantiating two other lemmas: 1) if $\forall x.wff$ is a theorem, then wff is also a theorem, 2) if wff is provable, then x cannot be free in the dependencies of the proof of wff and so $\forall x.wff$ is provable. This is of course true only for theories with no free variables in their axioms.

The only property of the inference rules used in this proof involves universal quantification. The restriction on the applicability of the \forall -introduction rule is that the variable to be universally quantified in a wff must not appear free in any of its dependencies. This restriction is reflected in our axiomatization by the predicate **APGENI**. In this proof **APGENI** is satisfied because if wff is provable, its dependencies are axioms with no free variables.

The following is an informal proof of the above theorems. If $\forall x.wff$ is provable, then there is a prooftree pf whose first string is $\forall x.wff$. The sequence $(\forall x.wff) cc pf$ is still a prooftree. It is obtained by applying the \forall -elimination rule. The application of this rule doesn't add any dependency to the prooftree. As its only dependencies are axioms, it follows from the definition of **BEW** that wff is a theorem. On the other hand, if wff is a theorem there exists a prooftree pf whose first element is wff . By applying the \forall -introduction rule to pf we obtain the prooftree $(\forall x.wff) cc pf$. This rule is applicable since theorems have no free variables in their dependencies. It follows that $\forall x.wff$ is a theorem. If $\forall x y.wff$ is provable then $\forall x.wff$ and wff are provable using the first lemma. Finally, we can quantify first over x and then over y , obtaining $\forall y x.wff$ as a theorem.

Section 2.4 Another axiomatization

A different axiomatization has been given in an earlier version of FOL where there was no facility for creating sorts. We present it here as we want to do some comparisons between proofs, and discuss some of the features of FOL. Some differences between the two axiomatizations are due to the new features available in FOL. They will be discussed in the next section. Here we only discuss the difference between the definition of formulas and terms. The list of all the axioms can be found in appendices 2.1-8.

In this axiomatization, formulas and terms are still represented as the string of the symbols appearing in them. They are defined as strings that can be decomposed into a sequence of substrings recording the construction of that formula or term from elementary formulas and individual variables, according to the usual formation rules (see appendix 2.5 for the list of axioms). These sequences are defined by the predicate **TERMSEQ** for terms and **FRR** for wff s. A sequence satisfies the predicate **TERMSEQ** if it represents the history of the construction of its first element (the term to be defined), starting from symbols, functions and individual variables. Similarly, a string is a wff if there exists a sequence which satisfies the predicate **FRR** and represents the history of the construction of that wff from elementary formulas and the logical connectives.

SECTION 3 THE PROOFS

In this section we look at the proofs appearing in the appendices, in order to explore the features of FOL that need improving and their use in carrying out formal proofs.

Section 3.1 A look at sorts

As already noted, the primary difference between the two axiomatizations we presented is the introduction of a many sorted logic. In the earlier version of FOL there was no facility for creating sorts, but it soon became evident that relativization of wffs to predicates was desirable. The notion of partially ordered sorts was a natural outgrowth. The axioms in the sorted logic are simpler and more readable and, most important, proofs are considerably shorter. First of all, in the axiomatization done in the earlier version of FOL the partial order of sorts wasn't explicit and was to be derived as a theorem. In the proofs shown in the appendices these theorems appear as dependencies. At the moment FOL has no facility for using already proved statements as lemmas in making new proofs. In FOL there is also the possibility of declaring for each function symbol the sorts of its arguments and of its value. These sorts were defined in the original version by additional axioms. For example, together with the definition of the functions *sbt* and *sbv*, the second axiomatization has two extra axioms.

$$\forall x \ t \ f1. ((\text{INDVAR}(x) \wedge \text{TERM}(t) \wedge \text{FORM}(f1)) \supset \text{FORM}(\text{sbt}(x,t,f1))) ; ;$$

$$\forall x1 \ x2 \ f1. ((\text{INDVAR}(x1) \wedge \text{INDVAR}(x2) \wedge \text{FORM}(f1)) \supset \text{FORM}(\text{sbv}(x1,x2,f1))) ; ;$$

Proofs are shorter in a many sorted logic. As an example, we can examine the two proofs in appendices 5.1-2 and 5.5-6. The second proof is longer because the explicit assumption that *x* is an individual variable and *f* is a wff must be made, and the symbol \supset must be introduced at the end of the proof, to discharge this assumption. Note that in this proof the statements labeled TH2 and TH3 appear as dependencies and the proof would have been even longer if we had proved them there. Another difference between the two proofs is that, in the second one, we had to use the axiom previously mentioned stating that the result of substituting a term *t* for every free occurrence of a variable in a wff is still a wff. The different axiomatization of wffs and terms only influences the length of the proofs in appendix 3.1,-2,-3. All the other proofs are shorter only due to the presence of sorts in FOL. Furthermore, note that proofs in the second axiomatization have more dependencies since all the theorems about the partial order of sorts have been assumed.

Section 3.2 The unify and tautology commands

FOL proofs are greatly simplified by the existence of the commands TAUT and TAUTEQ. They decide if a given formula is a tautological consequence of a specified set of wffs. The difference between TAUT and TAUTEQ is that the latter uses properties of the equality and the former doesn't. These commands make proofs shorter since they allow to decide every propositional sentence in one step. As a consequence, the rules of inference most frequently used manipulate quantifiers. The form of almost all the proofs we presented is the same. First of all, the right instantiations of the relevant axioms and theorems are done. Then the propositional consequences are asserted by using TAUT and TAUTEQ. The tautology commands cannot of course manipulate the quantifiers appearing in

statements. Hence, the statements produced by them have quantifiers as main symbols or it is necessary to introduce a quantifier to proceed in the proof. After the right introductions or eliminations have been done to them, the tautology commands are used again. This process is iterated until the completion of the proof.

The command UNIFY decides if a given wff can be obtained by instantiation of quantified variables or introduction of them for free occurrences of variables or terms in a second wff. The code for this command has been written by Ashok Chandra and is still in an experimental stage. In the proofs presented here, this command has been essentially used for the simultaneous introduction of the existential quantifier. As an example, consider the following assumption:

1 $\forall x.(P(x) \supset (Q(f) \wedge \forall l.R(l)))$ (1) ASSUME

the command

unify $\exists x.(P(x) \supset \exists f.(Q(f) \wedge R(g(l))))$, 1;

deduces in a single step

2 $\exists x.(P(x) \supset \exists f.(Q(f) \wedge R(g(l))))$ (6) UNIFY 1

A good example of a combined use of these features is found in appendix 3.3:

19 FRR((x1 gen f) cc SQ) (SEQUENCE ((x1 gen f) cc SQ) ^ (((x1 gen f) cc U) ^ SLAMBDA ^ (ELF (scar((x1 gen f) cc SQ)) v (FRR(scdr((x1 gen f) cc SQ))) ^ $\exists s1 s2.(STRING(s1) \wedge (STRING(s2) \wedge (scar((s1 gen f) cc SQ) = NEG(s1) \wedge find(1, s1, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 dis s2) \wedge find(2, s1 c s2, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 con s2) \wedge find(2, s1 c s2, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 impl s2) \wedge find(2, s1 c s2, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 gen s2) \wedge (INDVAR(s1) \wedge find(1, s2, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 ex s2) \wedge (INDVAR(s1) \wedge find(1, s2, scdr((x1 gen f) cc SQ))))))))) --- VE WFF1 (x1 gen f) cc SQ$

20 STRING(x1) ^ (STRING(f) ^ (scar((x1 gen f) cc SQ) = NEG(x1) ^ find(1, x1, scdr((x1 gen f) cc SQ))) v (scar((x1 gen f) cc SQ) = (x1 dis f) ^ find(2, x1 c f, scdr((x1 gen f) cc SQ))) v (scar((x1 gen f) cc SQ) = (x1 con f) ^ find(2, x1 c f, scdr((x1 gen f) cc SQ))) v (scar((x1 gen f) cc SQ) = (x1 impl f) ^ find(2, x1 c f, scdr((x1 gen f) cc SQ))) v (scar((x1 gen f) cc SQ) = (x1 gen f) ^ (INDVAR(x1) ^ find(1, f, scdr((x1 gen f) cc SQ))) v (scar((x1 gen f) cc SQ) = (x1 ex f) ^ (INDVAR(x1) ^ find(1, f, scdr((x1 gen f) cc SQ)))))) (1 2 3 4 5 6 7 8 11) --- TAUTEQ 1:19

21 $\exists s1 s2.(STRING(s1) \wedge (STRING(s2) \wedge (scar((s1 gen f) cc SQ) = NEG(s1) \wedge find(1, s1, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 dis s2) \wedge find(2, s1 c s2, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 con s2) \wedge find(2, s1 c s2, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 impl s2) \wedge find(2, s1 c s2, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 gen s2) \wedge (INDVAR(s1) \wedge find(1, s2, scdr((x1 gen f) cc SQ))) \vee (scar((x1 gen f) cc SQ) = (s1 ex s2) \wedge (INDVAR(s1) \wedge find(1, s2, scdr((x1 gen f) cc SQ))))))$ (1 2 3 4 5 6 7 8 11) --- UNIFY 20

Line 19 is the instantiation of an axiom. Line 20 is generated by the command,

TAUTEQ 19: #2#2#2#2#2#1#1[s1←f : s2←x1] 1:19;

note how the use of the FOL subpart designators allows us to mention the desired subpart of 19, without having to retype it. In addition we can do the appropriate substitutions. Line 21 is just a use of UNIFY:

```
UNIFY 19:22222 20;
```

Because we can mention the conclusion, without writing it down explicitly, the amount of typing necessary is severely reduced. Without UNIFY, line 21 would have required two \exists -introductions and the commands would have been:

```
 $\exists$  20 x1 ← s1 OCC 1,2,3,4,7,8,11,12,15,16,19,20,23,24;
```

```
 $\exists$  20 f ← s2 OCC 1,5,9,13,17,18,21,22;
```

We do not enter into a detailed discussion of the command UNIFY. It is our intension to do it elsewhere. It should be thought of as the routine which handles quantifiers in "simple" inferences. As seen above, the saving to a user can be large.

SECTION 4 CONCLUSION

The desire to represent mathematics in a computer in a feasible way certainly requires the facility to discuss metamathematical notions. The axiomatization presented here only treats the syntactic part of the problem. Any mention of the models involved needs the addition of set theory to the axiomatization. However, it is clear from the simple theorems we proved that any practical system needs more extensive features even to do a satisfactory job of writing down the theorems we might want.

An important point for future work is how (in a practical way) to use these theorems. Consider for instance:

$$\forall x_1 x_2 f. (\text{BEW}(x_1 \text{ gen } (x_2 \text{ gen } f)) \supset \text{BEW}(x_2 \text{ gen } (x_1 \text{ gen } f)))$$

What we mean by reflection principle is a rule of FOL which says:

$$\frac{\text{//BEW}(f)}{\text{//}f} \quad \begin{array}{l} \text{//in meta FOL} \\ \text{// in FOL} \end{array}$$

That is, if in the axiomatization of the metamathematics of FOL, we can prove the existence of an FOL proof of f , then we can assert f in FOL. Suppose we have a proof in FOL of $\forall x y. wff$. Then instantiating the above theorem gives us

$$\text{BEW}(x \text{ gen } (y \text{ gen } wff)) \supset \text{BEW}(y \text{ gen } (x \text{ gen } wff))$$

Since we started with a proof of $\forall x y. wff$ in FOL and BEW represents the proof predicate for FOL, we can conclude $\text{BEW}(x \text{ gen } (y \text{ gen } wff))$. Using modus ponens we get $\text{BEW}(y \text{ gen } (x \text{ gen } wff))$, and using the above rule we can conclude $\forall y x. wff$ in FOL.

The exact form of such a rule requires more examples of proofs and is one of the main reasons for doing the example in the memo. It is not just a proof checking exercise, but a case study for fundamental questions of representing mathematical information in a computer. Using metamathematics also prepares the way for more comprehensive systems which can formally discuss how they reason. That is exactly what the metamathematics is good for.

APPENDIX 1

THE AXIOMS IN THE MANY SORTED LOGIC

1.1 Natural numbers

AXIOM NUMB:

- $\forall n1\ n2\ n3. (n1=n2 \supset (n1=n3 \supset n2=n3)),$
- $\forall n1\ n2. (n1=n2 \supset succ(n1)=succ(n2)),$
- $\forall n1. \emptyset \neq succ(n1),$
- $\forall n1\ n2. (succ(n1)=succ(n2) \supset n1=n2),$
- $\forall n1. n1 \cdot \emptyset = n1,$
- $\forall n1\ n2. n1 \cdot succ(n2) = succ(n1 \cdot n2),$
- $\forall n1. n1 * \emptyset = \emptyset,$
- $\forall n1\ n2. n1 * succ(n2) = (n1 * n2) \cdot n1 ;;$

AXIOM INDCT:

$$(F(\emptyset) \wedge \forall n. (F(n) \supset F(n+1))) \supset \forall n. F(n) ;;$$

AXIOM DEFN:

- $\forall n. (succ(n)-1) = n,$
- $\forall n1\ n2. succ(n1) - n2 = n1 - (n2 - 1),$
- $\forall n1\ n2\ n3. (n1 < n2 \equiv \exists n3. (n3 \neq \emptyset \wedge n1 \cdot n3 = n2)),$
- $\forall n1\ n2. (n1 \leq n2 \equiv (n1 < n2) \vee (n1 = n2)),$
- $\forall n1\ n2. (n2 > n1 \equiv n1 < n2),$
- $\forall n1\ n2. (n2 \geq n1 \equiv n1 \leq n2) ;;$

1.2 The set of symbols

AXIOM SYM:

$$\forall a. (SYM(a) \equiv a = LPARSYM \vee a = RPARSYM \vee a = ORSYM \vee a = ANDSYM \vee a = IMPSYM \vee a = FALSESYM \vee a = FORALLSYM \vee a = EXISTSYM) ;;$$

1.3 Strings

AXIOM STRING:

- $\forall s. s = car(s) \ c \ cdr(s),$
- $\forall s1\ s2. (s1 = LAMBDA \supset car(s1 \ c \ s2) = car(s2)),$
- $\forall s1\ s2. (s1 \neq LAMBDA \supset car(s1 \ c \ s2) = car(s1)),$
- $\forall s1\ s2. (s1 = LAMBDA \supset cdr(s1 \ c \ s2) = cdr(s2)),$
- $\forall s1\ s2. (s1 \neq LAMBDA \supset cdr(s1 \ c \ s2) = cdr(s1)),$
- $\forall s. (s \ c \ LAMBDA = LAMBDA \ c \ s),$
- $\forall s. s \ c \ LAMBDA = s,$
- $\forall s1\ s2\ s3. (s1 \ c \ (s2 \ c \ s3) = (s1 \ c \ s2) \ c \ s3),$
- $\forall a. (len(a) = 1 \vee a = LAMBDA),$
- $\forall s. len(s) \geq 0,$
- $\forall s1\ s2. len(s1 \ c \ s2) = len(s1) + len(s2),$
- $\forall s. (len(s) = 1 \supset ATOM(s)),$
- $\forall s. \emptyset \text{ gl } s = LAMBDA,$

$\forall s. \quad 1 \text{ gl } s = \text{car}(s) ,$
 $\forall s \ n. \quad ((n > 1) \Rightarrow ((n \text{ gl } s) = ((n-1) \text{ gl } \text{cdr}(s)))) ;;$

AXIOM SUBSTRDEF:

$\forall n1 \ n2 \ s1 \ s2 \quad (\text{SUBSTP}(s1, s2, n1, n2) = (\text{len}(s2) = n2 - n1 + 1 \wedge (\forall n (n \geq n1 \wedge n \leq n2 \Rightarrow n \text{ gl } s1 = (n - n1 + 1) \text{ gl } s2)))) ,$
 $\forall n1 \ n2 \ s1 \ s2 \quad (\text{SUBSTP}(s1, s2, n1, n2) = \text{substring}(s1, n1, n2) = s2) ,$
 $\forall s1 \ s2. \quad (\text{SUBS}(s1, s2) = \exists n1 \ n2. \text{SUBSTP}(s1, s2, n1, n2)) ;;$

The value of $\text{substring}(s1, n1, n2)$ is the substring of $s1$ whose first element is the $n1$ th element of $s1$ and whose last element is the $n2$ th element.

AXIOM DISEQ:

$\forall g1 \ g2. \quad (\neg(g1 = g2) = g1 \neq g2) ;;$

AXIOM EQS:

$\forall s1 \ s2. \quad (\forall n (n \text{ gl } s1 = n \text{ gl } s2) = s1 = s2) ;;$

AXIOM COMP:

$\forall f. \quad \alpha(f) = (\text{LPARSYM } c \ f) \in \text{RPARSYM} ,$
 $\forall f1 \ f2. \quad f1 \text{ dis } f2 = (\alpha(f1) \in \text{ORSYM}) \in \alpha(f2) ,$
 $\forall f1 \ f2. \quad f1 \text{ impl } f2 = (\alpha(f1) \in \text{IMPSYM}) \in \alpha(f2) ,$
 $\forall f. \quad \text{neg}(f) = (f \text{ impl } \text{FALSESYM}) ,$
 $\forall f1 \ f2. \quad f1 \text{ con } f2 = (\alpha(f1) \in \text{ANDSYM}) \in \alpha(f2) ,$
 $\forall x \ f2. \quad x \text{ gen } f2 = (\text{FORALLSYM } c \ x) \in f2 ,$
 $\forall x \ f2. \quad x \text{ ex } f2 = (\text{EXISTSYM } c \ x) \in f2 ;;$

1.4 Formulas

AXIOM TERM:

$\forall n \ s. \quad (\text{TERMSEQ}(n, s) = (\exists n1 \ (\text{TERM}(\text{substring}(s, 1, n1)) \wedge \text{TERMSEQ}(n-1, \text{substring}(s, n1+1, \text{len}(s)))))) ,$

AXIOM WFF:

$\forall s. \quad (\text{TERM}(s) = \text{INDVAR}(s) \vee \exists n \ \text{fn}. (\text{fn} = \text{car}(s) \wedge n = \text{arity}(\text{fn}) \wedge \text{TERMSEQ}(n, \text{cdr}(s)))) ;;$
 $\forall s. \quad (\text{ELF}(s) = (\neg \text{FALSESYM} \vee \text{PREDPARO}(s) \vee \exists n \ P (P = \text{car}(s) \wedge n = \text{arity}(P) \wedge \text{TERMSEQ}(n, \text{cdr}(s)))) ,$
 $\forall s. \quad (\text{FORM}(s) = (\text{ELF}(s) \vee \exists x \ f ((s = x \text{ gen } f) \vee (s = x \text{ ex } f)) \vee \exists f1 \ f2 ((s = f1 \text{ dis } f2) \vee (s = f1 \text{ con } f2) \vee (s = f1 \text{ impl } f2)) \vee \exists f. s = \text{neg}(f))) ;;$

1.5 Sequences

AXIOM SEQ:

$\forall sq \quad sq = \text{car}(sq) \text{ cc } \text{scdr}(sq) ,$
 $\forall sq1 \ sq2. \quad (sq1 = \text{SLAMBDA} \Rightarrow \text{scar}(sq1 \text{ cc } sq2) = \text{scar}(sq2)) ,$
 $\forall sq1 \ sq2. \quad (sq1 \neq \text{SLAMBDA} \Rightarrow \text{scar}(sq1 \text{ cc } sq2) = \text{scar}(sq1)) ,$
 $\forall sq1 \ sq2. \quad (sq1 = \text{SLAMBDA} \Rightarrow \text{s cdr}(sq1 \text{ cc } sq2) = \text{s cdr}(sq2)) ,$

$\forall sq1\ sq2. (sq1 \neq SLAMBDA \supset scdr(sq1\ cc\ sq2) = scdr(sq1)\ cc\ sq2),$
 $\forall sq. sq\ cc\ SLAMBDA = SLAMBDA\ cc\ sq,$
 $\forall sq. sq\ cc\ SLAMBDA = sq,$
 $\forall sq1\ sq2\ sq3. (sq1\ cc\ (sq2\ cc\ sq3) = (sq1\ cc\ sq2)\ cc\ sq3),$
 $\forall s. (slen(s) = 1 \wedge s = SLAMBDA),$
 $\forall sq. slen(sq) \geq 0,$
 $\forall sq1\ sq2. slen(sq1\ cc\ sq2) = slen(sq1) + slen(sq2),$
 $\forall sq. 0\ sgl\ sq = SLAMBDA,$
 $\forall sq. 1\ sgl\ sq = scar(sq),$
 $\forall n\ sq. ((n > 1) \supset ((n\ sgl\ sq) = ((n-1)\ sgl\ scdr(sq)))) ;;$

AXIOM SUBSEQDEF:

$\forall n1\ n2\ sq1\ sq2. (SUBSEP(sq1, sq2, n1, n2) \equiv (slen(sq2) = n2 - n1 + 1 \wedge$
 $(\forall n. (n \geq n1 \wedge n \leq n2 \supset n\ sgl\ sq2 = (n - n1 + 1)\ sgl\ sq1))))),$
 $\forall n1\ n2\ sq1\ sq2. (SUBSEP(sq1, sq2, n1, n2) \equiv subseq(sq1, n1, n2) = sq2),$
 $\forall sq1\ sq2. (SUBSSE(sq1, sq2) \equiv \exists n1\ n2. (SUBSEP(sq1, sq2, n1, n2)))) ;;$

AXIOM EQSQ:

$\forall sq1\ sq2. (\forall n. (n\ sgl\ sq1 = n\ sgl\ sq2) \supset sq1 = sq2) ;;$

1.6 Free and bound variables and the substitution

AXIOM BOUNDV:

$\forall x\ n\ f. (GEB(x, n, f) \equiv \exists s1\ s2\ f1. (slen(s1) + 1 < n \wedge n < (slen(f) - slen(s2)) \wedge$
 $(x = n\ gl\ f) \wedge ((f = (s1\ c\ ((x\ gen\ f1)\ c\ s2))) \vee (f = (s1\ c\ ((x\ ex\ f1)\ c\ s2)))))) ;;$

AXIOM FREEV:

$\forall x\ n\ f. (FRN(x, n, f) \equiv (x = (n\ gl\ f) \wedge \neg GEB(x, n, f))),$
 $\forall x\ f. (FR(x, f) \equiv \exists n. FRN(x, n, f));;$

AXIOM FIRSTFRDF:

$\forall x\ n\ f. (FIRSTFREE(x, n, f) \equiv (FRN(x, n, f) \wedge \forall n1. (x = n1\ gl\ f \supset (n1 \geq n \vee GEB(x, n1, f))))),$
 $\forall x\ n\ f. (FIRSTFREE(x, n, f) \equiv firstfreeocc(x, f) = n) ;;$

AXIOM KFREEOCCDF:

$\forall x\ k\ n\ f. (KTHFREEOCC(x, k, n, f) \equiv ((k = 0 \wedge n = 1) \vee$
 $(n = slen(f) \wedge \forall n2. (n2 > kthfreeocc(x, k-1, f) \supset \neg FRN(x, n2, f))) \vee$
 $(FRN(x, n, f) \wedge \forall n1. ((n1 < k \wedge n1 > 0) \supset \exists n2. (n2 < n \wedge KTHFREEOCC(x, n1, n2, f)))))),$
 $\forall x\ k\ n\ f. (KTHFREEOCC(x, k, n, f) \equiv kthfreeocc(x, k, f) = n),$
 $\forall x\ k\ n\ f. (KTHFREEOCC(x, k, n, f) \supset numbfreeocc(x, n, f) = k),$
 $\forall x\ k\ n\ f. (numbfreeocc(x, n, f) = k \supset (KTHFREEOCC(x, k, n, f) \vee$
 $(n < kthfreeocc(x, k, f) \wedge n > kthfreeocc(x, k-1, f)))) ;;$

AXIOM SUBSTDF:

$\forall x\ t\ f1\ f2. (SBT(x, t, f1, f2) \equiv \forall n1\ n2. ((n2 = (numbfreeocc(x, n1, f1) * (slen(t) - 1)) + n1) \supset$
 $((\neg INDVAR(n1\ gl\ f1) \supset n1\ gl\ f1 = n2\ gl\ f2) \wedge$
 $(INDVAR(n1\ gl\ f1) \supset ((FRN(x, n1, f1) \supset SUBT(t, f2, n2)) \wedge$
 $(\neg FRN(x, n1, f1) \supset INVART(n1, f1, n2, f2)))))),$
 $\forall t\ f2\ n2. (SUBT(t, f2, n2) \equiv \forall x2\ k. (((k\ gl\ t) = x2) \supset FRN(x2, n2 - (slen(t) - k), f2))),$
 $\forall n\ f1\ n1\ f2. (INVART(n, f1, n1, f2) \equiv ((GEB(n1\ gl\ f2, n1, f2) \equiv GEB(n\ gl\ f1, n, f1)) \wedge$
 $(FRN(n1\ gl\ f2, n1, f2) \equiv FRN(n\ gl\ f1, n, f1)) \wedge n1\ gl\ f2 = n\ gl\ f1))$
 $\forall x\ t\ f1\ f2. (SBT(x, t, f1, f2) \vee sbt(x, t, f1) = f2) ;;$

AXIOM SUBDEF:

$\forall x1\ x2\ f1\ f2. (SBV(x1,x2,f1,f2) \equiv \forall n. ((\neg INDVAR(n\ gl\ f1) \supset n\ gl\ f1 = n\ gl\ f2) \wedge$
 $(INDVAR(n\ gl\ f1) \supset ((FRN(x1,n,f1) \supset FRN(x2,n,f2)) \wedge$
 $(\neg FRN(x1,n,f1) \supset INVAVR(n,f1,f2))))),$
 $\forall n\ f1\ f2. (INVAVR(n,f1,f2) \equiv ((GEB(n\ gl\ f2,n,f2) \equiv GEB(n\ gl\ f1,n,f1)) \wedge$
 $(FRN(n\ gl\ f2,n,f2) \equiv FRN(n\ gl\ f1,n,f1)) \wedge n\ gl\ f2 = n\ gl\ f1)),$
 $\forall x\ x1\ f1\ f2. (SBV(x,x1,f1,f2) \equiv sbv(x,x1,f1)=f2));;$

1.7 Rules of inference

AXIOM ANDIRUL:

$\forall sq\ pf1\ pf2. (ANDI(sq,pf1,pf2) \equiv \exists f1\ f2. (scdr(sq)=(pf1\ cc\ pf2) \wedge scar(sq)=f1\ con\ f2 \wedge$
 $f1=scar(pf1) \wedge f2=scar(pf2))),$
 $\forall sq\ pf. (ANDE(sq,pf) \equiv \exists f1\ f2. (scdr(sq)=pf \wedge scar(sq)=f1 \wedge ((f1\ con\ f2)=scar(pf)) \vee$
 $(f2\ con\ f1)=scar(pf))));;$

AXIOM FALSERUL:

$\forall sq\ pf1\ pf2. (FALSEI(sq,pf1,pf2) \equiv \exists f1. ((scdr(sq)=(pf1\ cc\ pf2)) \wedge$
 $(scar(sq)=FALSESYM) \wedge (\neg f1=scar(pf1)) \wedge (f1=scar(pf2))))),$
 $\forall sq\ pf. (FALSEE(sq,pf) \equiv \exists f. ((scar(pf)=FALSESYM) \wedge f=scar(sq) \wedge scdr(sq)=pf));;$

AXIOM IMPLRUL:

$\forall sq\ pf1\ pf2. (IMPLE(sq,pf1,pf2) \equiv \exists f1\ f2. ((scdr(sq)=(pf1\ cc\ pf2)) \wedge$
 $(scar(pf1) \supset (f1\ impl\ f2)) \wedge (scar(sq)=f2 \wedge (scar(pf2) = f1))))),$
 $\forall sq\ pf\ f1. (IMPLID(sq,pf,f1) \equiv (scdr(sq)=pf \wedge \exists f2. ((scar(sq)=(f1\ impl\ f2)) \wedge$
 $(f2=scar(pf)) \wedge \exists n. (f1 = (n\ sgl\ pf))))),$
 $\forall sq\ pf. (IMPLI(sq,pf) \equiv \exists f. IMPLID(sq,pf,f1));;$

AXIOM NEGRUL:

$\forall sq\ pf\ f. (NOTID(sq,pf,f) \equiv (scdr(sq)=pf \wedge scar(sq)=f \wedge (scar(pf)=FALSESYM) \wedge$
 $\exists n. (n\ sgl\ pf)=f)),$
 $\forall sq\ pf. (NOTI(sq,pf) \equiv \exists f. NOTID(sq,pf,f)),$
 $\forall sq\ pf\ f. (NOTED(sq,pf,f) \equiv (scdr(sq)=pf \wedge (scar(pf) = FALSESYM) \wedge$
 $\exists n. ((n\ sgl\ pf)=f) \wedge (f \equiv \neg scar(sq))))),$
 $\forall sq\ pf. (NOTE(sq,pf) \equiv \exists f. NOTED(sq,pf,f));;$

AXIOM ORRUL:

$\forall sq\ pf. (ORI(sq,pf) \equiv (scdr(sq)=pf \wedge \exists f1\ f2. ((scar(sq)=(f1\ dis\ f2)) \wedge$
 $f1=scar(pf)) \vee (f2=scar(pf)))),$
 $\forall sq\ pf1\ pf2\ pf3\ f1\ f2. (ORED(sq,pf1,pf2,pf3,f1,f2) \equiv (scdr(sq)=(pf1\ cc\ pf2\ cc\ pf3)) \wedge$
 $(scar(pf1)=(f1\ dis\ f2) \wedge \exists f3. (scar(pf2)=f3) \wedge scar(sq)=f3 \wedge$
 $(scar(pf3)=f3)) \wedge \exists n1. (n1\ sgl\ pf2)=f1) \wedge \exists n1. (n1\ sgl\ pf3)=f2))),$
 $\forall sq\ pf1\ pf2\ pf3. (ORE(sq,pf1,pf2,pf3) \equiv \exists f1\ f2. ORED(sq,pf1,pf2,pf3,f1,f2));;$

AXIOM EXRUL:

$\forall sq\ pf\ x\ f. (EXI(sq,pf,x,f) \equiv \exists f1. ((scdr(sq)=pf1) \wedge (scar(sq)=(x\ ex\ f1)) \wedge$
 $scar(pf)=sbt(x,f1))),$
 $\forall sq\ pf1\ pf2\ x1\ x2\ f1. (EXED(sq,pf1,pf2,x1,x2,f1) \equiv ((scdr(sq)=(pf1\ cc\ pf2)) \wedge$
 $(scar(pf1)=(x1\ ex\ f1)) \wedge (scar(sq)=scar(pf2)) \wedge$
 $\exists n. ((n\ sgl\ pf2)=sbt(x1,x2,f1) \wedge EXAPPL(x2,pf2,f1))))),$
 $\forall sq\ pf1\ pf2\ x1\ x2. (EXE(sq,pf1,pf2,x1,x2) \equiv \exists f1. EXED(sq,pf1,pf2,x1,x2,f1)),$
 $\forall x\ pf\ f. (EXAPPL(x,pf,f) \equiv (\neg FR(x,scar(pf)) \wedge \neg FR(x,f) \wedge \forall f1. (DEPEND(pf,f1) \supset$
 $\neg FR(x,f1)))));;$

AXIOM GENRUL:

$\forall sq \ sq1 \ x \ t. \quad (GENE(sq, sq1, x, t) = (scdr(sq) = sq1 \wedge PROOFTREE(sq1) \wedge$
 $\quad \exists f. (scar(sq1) = x \wedge gen \ f \wedge scar(sq) = sbt(x, t, f))))),$
 $\forall sq \ sq1 \ x1 \ x2. \quad (GENI(sq, sq1, x1, x2) = (scdr(sq) = sq1 \wedge PROOFTREE(sq1) \wedge$
 $\quad \exists f. (scar(sq) = x1 \wedge gen \ f \wedge scar(sq1) = sbt(x1, x2, f) \wedge APGENI(x2, sq1))))),$
 $\forall x \ sq. \quad (APGENI(x, sq) = (\forall f. (DEPEND(sq, f) \supset \neg FR(x, f))) \wedge PROOFTREE(sq)),$
 $\forall pf. \exists x. \quad APGENI(x, pf));;$

1.8 Deduction

AXIOM PROOF:

$\forall sq. \quad (PROOFTREE(sq) = (FORM(sq) \vee$
 $\quad \exists pf. (ORI(sq, pf) \vee ANDE(sq, pf) \vee FALSEE(sq, pf) \vee NOTI(sq, pf) \vee NOTE(sq, pf) \vee$
 $\quad \quad \quad IMPLI(sq, pf)) \vee$
 $\quad \exists pf \ x \ t. (GENI(sq, pf, x, t) \vee GENE(sq, pf, x, t) \vee EXI(sq, pf, x, t)) \vee$
 $\quad \exists pf1 \ pf2. (ANDI(sq, pf1, pf2) \vee FALSEI(sq, pf1, pf2) \vee IMPLE(sq, pf1, pf2)) \vee$
 $\quad \exists pf1 \ pf2 \ x1 \ x2. EXE(sq, pf1, pf2, x1, x2) \vee$
 $\quad \exists pf1 \ pf2 \ pf2 \ pf3. ORE(sq, pf1, pf2, pf3));;$

AXIOM DEPNDG:

$\forall sq \ f. \quad (DEPEND(sq, f) \supset SUBSSE(f, sq)) ,$
 $\forall sq \ f. \quad (f = sq \supset DEPEND(sq, f));;$

AXIOM DEPEND:

$\forall pf \ pf1 \ f. \quad ((pf1 = scdr(pf) \supset (DEPEND(pf, f) = DEPEND(pf1, f))) =$
 $\quad (ORI(pf, pf1) \vee ANDE(pf, pf1) \vee FALSEE(pf, pf1) \vee$
 $\quad \exists f1. ((NOTID(pf, pf1, f1) \vee NOTED(pf, pf1, f1) \vee IMPLID(pf, pf1, f1)) \wedge f1 \neq f) \vee$
 $\quad \exists x \ t. (GENI(pf, pf1, x, t) \vee GENE(pf, pf1, x, t) \vee EXI(pf, pf1, x, t)))) ,$
 $\forall pf \ pf1 \ pf2 \ f. \quad (((pf1 \ cc \ pf2 = scdr(pf)) \vee (pf2 \ cc \ pf1 = scdr(pf))) \supset$
 $\quad (DEPEND(pf, f) = ((DEPEND(pf1, f) \vee DEPEND(pf2, f)))) =$
 $\quad (ANDI(pf, pf1, pf2) \vee FALSEI(pf, pf1, pf2) \vee IMPLE(pf, pf1, pf2) \vee$
 $\quad \exists x1 \ x2 \ f1. (EXED(pf, pf1, pf2, x1, x2, f1) \wedge f \neq f1)) ,$
 $\forall pf \ pf1 \ pf2 \ pf3 \ f. \quad (((((pf1 \ cc \ (pf2 \ cc \ pf3)) = scdr(pf)) \vee$
 $\quad ((pf1 \ cc \ (pf3 \ cc \ pf2)) = scdr(pf)) \vee$
 $\quad ((pf2 \ cc \ (pf1 \ cc \ pf3)) = scdr(pf)) \vee$
 $\quad ((pf2 \ cc \ (pf3 \ cc \ pf1)) = scdr(pf)) \vee$
 $\quad ((pf3 \ cc \ (pf1 \ cc \ pf2)) = scdr(pf)) \vee$
 $\quad ((pf3 \ cc \ (pf2 \ cc \ pf1)) = scdr(pf))) \supset$
 $\quad (DEPEND(pf, f) = (DEPEND(pf1, f) \vee DEPEND(pf2, f) \vee$
 $\quad \quad \quad DEPEND(pf3, f)))) =$
 $\quad \exists f1 \ f2. (ORED(pf, pf1, pf2, pf3, f1, f2) \wedge f \neq f1 \wedge f \neq f2) ;;$

AXIOM NDEPND:

$\forall pf1 \ pf2 \ f. \quad ((NOTID(pf1, pf2, f) \vee NOTED(pf1, pf2, f) \vee IMPLID(pf1, pf2, f)) \supset$
 $\quad \neg DEPEND(pf1, f)) ,$
 $\forall pf1 \ pf2 \ pf3 \ x1 \ x2 \ f. \quad EXED(pf1, pf2, pf3, x1, x2, f) \supset \neg DEPEND(pf1, f) ,$
 $\forall pf1 \ pf2 \ pf3 \ pf4 \ f1 \ f2. \quad (ORED(pf1, pf2, pf3, pf4, f1, f2) \supset \neg DEPEND(pf1, f1) \wedge$
 $\quad \neg DEPEND(pf1, f2));;$
 $\forall f. \quad (BEW(f) = \exists sq. (PROOFTREE(sq) \wedge f = scar(sq) \wedge \forall f1. (DEPEND(sq, f1) \supset$
 $\quad \quad \quad AXIOM(f1)))));;$

AXIOM THEORY:

$\forall x \ f. \quad (AXIOM(f) \supset \neg FR(x, f));;$

AXIOM INVVAR:
Vs. $\exists x. \forall n.$

$n \text{ gls } \neq x ; ;$

APPENDIX 2

THE AXIOMS IN THE LOGIC

2.1 Natural numbers

AXIOM NUMB:

- $\forall n1\ n2\ n3. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2) \wedge \text{INTEGER}(n3)) \supset (n1=n2 \supset (n1=n3 \supset n2=n3))),$
- $\forall n1\ n2. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2)) \supset (n1=n2 \supset \text{succ}(n1)=\text{succ}(n2))),$
- $\forall n1. (\text{INTEGER}(n1) \supset 0 \neq \text{succ}(n1)),$
- $\forall n1\ n2. (\text{INTEGER}(n1) \wedge \text{INTEGER}(n2)) \supset (\text{succ}(n1) = \text{succ}(n2) \supset n1 = n2),$
- $\forall n1. (\text{INTEGER}(n1) \supset n1 + 0 = n1),$
- $\forall n1\ n2. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2)) \supset n1 + \text{succ}(n2) = \text{succ}(n1 + n2)),$
- $\forall n1. (\text{INTEGER}(n1) \supset n1 * 0 = 0),$
- $\forall n1\ n2. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2)) \supset n1 * \text{succ}(n2) = (n1 * n2) + n1);;$

AXIOM INDCT:

- $(F(0) \wedge \forall x. (\text{INTEGER}(x) \supset (F(x) \supset F(x+1)))) \supset \forall x. (\text{INTEGER}(x) \supset F(x));;$

AXIOM DEFM:

- $\forall n. (\text{INTEGER}(n) \supset (\text{succ}(n)-1)=n),$
- $\forall n1\ n2. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2)) \supset \text{succ}(n1)-n2=n1-(n2-1)),$
- $\forall n1\ n2\ n3. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2) \wedge \text{INTEGER}(n3)) \supset$
 $(n1 < n2 \supset \exists n3 (n3 \neq 0 \wedge n1 + n3 = n2))),$
- $\forall n1\ n2. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2)) \supset (n1 \leq n2 \equiv (n1 < n2) \vee (n1 = n2))),$
- $\forall n1\ n2. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2)) \supset (n2 > n1 \equiv n1 < n2)),$
- $\forall n1\ n2. ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2)) \supset (n2 \geq n1 \equiv n1 \leq n2)),$

2.2 The set of symbols

AXIOM SYM:

- $\forall a. (\text{SYM}(a) \equiv a = \text{LPARSYM} \vee a = \text{RPARSYM} \vee a = \text{ORSYM} \vee a = \text{ANDSYM} \vee a = \text{IMPSYM} \vee$
 $a = \text{FALSESYM} \vee a = \text{FORALLSYM} \vee a = \text{EXISTSYM});;$

2.3 Strings

AXIOM STRING:

- $\forall s. (\text{STRING}(s) \supset s = \text{car}(s) \text{ c } \text{cdr}(s)),$
- $\forall s1\ s2. ((\text{STRING}(s1) \wedge \text{STRING}(s2)) \supset (s1 = \text{LAMBDA} \supset \text{car}(s1 \text{ c } s2) = \text{car}(s2))),$
- $\forall s1\ s2. ((\text{STRING}(s1) \wedge \text{STRING}(s2)) \supset (s1 \neq \text{LAMBDA} \supset \text{car}(s1 \text{ c } s2) = \text{car}(s1))),$
- $\forall s1\ s2. ((\text{STRING}(s1) \wedge \text{STRING}(s2)) \supset (s1 = \text{LAMBDA} \supset \text{cdr}(s1 \text{ c } s2) = \text{cdr}(s2))),$
- $\forall s1\ s2. ((\text{STRING}(s1) \wedge \text{STRING}(s2)) \supset (s1 \neq \text{LAMBDA} \supset \text{cdr}(s1 \text{ c } s2) = \text{cdr}(s1))),$
- $\forall s. ((\text{STRING}(s) \supset (s \text{ c } \text{LAMBDA} = \text{LAMBDA} \text{ c } s)),$
- $\forall s. (\text{STRING}(s) \supset (s \text{ c } \text{LAMBDA} = s)),$
- $\forall s1\ s2\ s3. ((\text{STRING}(s1) \wedge \text{STRING}(s2) \wedge \text{STRING}(s3)) \supset (s1 \text{ c } (s2 \text{ c } s3) = (s1 \text{ c } s2) \text{ c } s3)),$
- $\forall s. (\text{STRING}(s) \supset (\text{len}(s) = 1 \vee s = \text{LAMBDA})),$
- $\forall s. (\text{STRING}(s) \supset \text{len}(s) \geq 0),$
- $\forall s1\ s2. ((\text{STRING}(s1) \wedge \text{STRING}(s2)) \supset \text{len}(s1 \text{ c } s2) = \text{len}(s1) + \text{len}(s2)),$
- $\forall s. (\text{STRING}(s) \supset (\text{len}(s) = 1 \supset \text{ATOM}(s))),$

$\forall s.$ (STRING(s) \supset \exists gl s=LAMBDA),
 $\forall s.$ (STRING(s) \supset \exists gl s=car(s)),
 $\forall s n.$ ((STRING(s) \wedge INTEGER(n)) \supset ((n>1) \supset ((n gl s)=((n-1) gl cdr(s))))),

AXIOM SUBSTRDEF:

$\forall n1 n2 s1 s2.$ ((INTEGER(n1) \wedge INTEGER(n2) \wedge STRING(s1) \wedge STRING(s2)) \supset
 (SUBSTP(s1,s2,n1,n2) = (len(s2)=n2-n1+1 \wedge ($\forall n.n \geq n1 \wedge n \leq n2 \supset$
 n gl s1=(n-n1+1) gl s2))))),
 $\forall n1 n2 s1 s2.$ ((INTEGER(n1) \wedge INTEGER(n2) \wedge STRING(s1) \wedge STRING(s2)) \supset
 (SUBSTP(s1,s2,n1,n2) = substring(s1,n1,n2)=s ,
 $\forall s1 s2.$ ((STRING(s1) \wedge STRING(s2)) \supset (SUBS(s1,s2) = \exists n1 n2.SUBSTP(s1,s2,n1,n2)))));

AXIOM DISEQ:

$\forall g1 g2.$ (\neg (g1=g2) = g1 \neq g2) ;;

AXIOM EQS:

$\forall s1 s2.$ ((STRING(s1) \wedge STRING(s2)) \supset ($\forall n.$ (INTEGER(n) \supset (n gl s1=n gl s2)) = s1=s2));;

AXIOM COMP:

$\forall f.$ (FORM(f) \supset (e(f)=(LPARSYM c f) c RPARSYM),
 $\forall f1 f2.$ ((FORM(f1) \wedge FORM(f2)) \supset (f1 dis f2)=(e(f1) c ORSYM) c e(f2)),
 $\forall f1 f2.$ ((FORM(f1) \wedge FORM(f2)) \supset (f1 impl f2)=(e(f1) c IMPSYM) c e(f2)),
 $\forall f.$ (FORM(f) \supset neg(f)=(f impl FALSESYM)),
 $\forall f1 f2.$ ((FORM(f1) \wedge FORM(f2)) \supset (f1 con f2)=(e(f1) c ANDSYM) c e(f2)),
 $\forall x f2.$ ((INDVAR(x) \wedge FORM(f2)) \supset (x gen f2)=(FORALLSYM c x) c f2),
 $\forall x f2.$ ((INDVAR(x) \wedge FORM(f2)) \supset (x ex f2)=(EXISTSYM c x) c f2) ;;

2.4 Sequences

AXIOM SEQ:

$\forall sq.$ (SEQUENCE(sq) \supset sq=scar(sq) cc scdr(sq)),
 $\forall sq1 sq2.$ ((SEQUENCE(sq1) \wedge SEQUENCE(sq2)) \supset (sq1=SLAMBDA \supset
 scar(sq1 cc sq2)=scar(sq2))),
 $\forall sq1 sq2.$ ((SEQUENCE(sq1) \wedge SEQUENCE(sq2)) \supset (sq1 \neq SLAMBDA \supset
 scar(sq1 cc sq2)=scar(sq1))),
 $\forall sq1 sq2.$ ((SEQUENCE(sq1) \wedge SEQUENCE(sq2)) \supset (sq1=SLAMBDA \supset
 scdr(sq1 cc sq2)=scdr(sq2))),
 $\forall sq1 sq2.$ ((SEQUENCE(sq1) \wedge SEQUENCE(sq2)) \supset (sq1 \neq SLAMBDA \supset
 scdr(sq1 cc sq2)=scdr(sq1) cc sq2)),
 $\forall sq.$ (SEQUENCE(sq) \supset sq cc SLAMBDA=SLAMBDA cc sq),
 $\forall sq.$ (SEQUENCE(sq) \supset sq cc SLAMBDA=sq),
 $\forall sq1 sq2 sq3.$ ((SEQUENCE(sq1) \wedge SEQUENCE(sq2) \wedge SEQUENCE(sq3)) \supset
 (sq1 cc (sq2 cc sq3)=(sq1 cc sq2) cc sq3))
 $\forall s.$ (STRING(s) \supset (slen(s)=1 \vee s=SLAMBDA)),
 $\forall sq.$ (SEQUENCE(sq) \supset slen(sq) \geq 0),
 $\forall sq1 sq2.$ ((SEQUENCE(sq1) \wedge SEQUENCE(sq2)) \supset slen(sq1 cc sq2)=slen(sq1)+slen(sq2)),
 $\forall sq.$ (SEQUENCE(sq) \supset \exists gl sq=SLAMBDA),
 $\forall sq.$ (SEQUENCE(sq) \supset \exists gl sq=scar(sq)),
 $\forall n sq.$ ((INTEGER(n) \wedge SEQUENCE(sq)) \supset ((n>1) \supset ((n gl sq)=((n-1) gl scdr(sq)))));

AXIOM SUBSEQDEF:

$\forall n1 n2 sq1 sq2.$ ((INTEGER(n1) \wedge INTEGER(n2) \wedge SEQUENCE(sq1) \wedge SEQUENCE(sq2)) \supset

$$\begin{aligned} & (\text{SUBSEP}(sq1, sq2, n1, n2) \equiv (\text{slen}(sq2) = n2 - n1 + 1 \wedge (\forall n. (n \geq n1 \wedge n \leq n2 \Rightarrow \\ & n \text{ sgl } sq2 = (n - n1 + 1) \text{ sgl } sq1))))), \\ \forall n1 \ n2 \ sq1 \ sq2. & ((\text{INTEGER}(n1) \wedge \text{INTEGER}(n2) \wedge \text{SEQUENCE}(sq1) \wedge \text{SEQUENCE}(sq2)) \Rightarrow \\ & (\text{SUBSEP}(sq1, sq2, n1, n2) \equiv \text{subseq}(sq1, n1, n2) = sq2)), \\ \forall sq1 \ sq2. & ((\text{SEQUENCE}(sq1) \wedge \text{SEQUENCE}(sq2)) \Rightarrow (\text{SUBSSE}(sq1, sq2) \equiv \\ & \exists n1 \ n2. (\text{SUBSEP}(sq1, sq2, n1, n2))))); \end{aligned}$$

AXIOM EQSQ:

$$\forall sq1 \ sq2. ((\text{SEQUENCE}(sq1) \wedge \text{SEQUENCE}(sq2)) \Rightarrow (\forall n. (n \text{ sgl } sq1 = n \text{ sgl } sq2) \Rightarrow sq1 = sq2));$$

2.5 Formulas

AXIOM FIND:

$$\begin{aligned} \forall sq. & (\text{FIND}(0, \text{LAMBDA}, sq) \equiv \text{SEQUENCE}(sq)), \\ \forall n \ s \ sq. & (\text{FIND}(n, s, sq) \equiv \text{INTEGER}(n) \wedge \text{STRING}(s) \wedge \text{SEQUENCE}(sq) \wedge \\ & \exists n \ s1 \ s2. (\text{INTEGER}(n) \wedge \text{STRING}(s1) \wedge \text{STRING}(s2) \wedge (\theta < s \wedge s < \text{slen}(sq)) \wedge \\ & (s1 = (n \text{ sgl } sq)) \wedge (s = (s1 \text{ c } s2)) \wedge \text{FIND}(n-1, s2, sq))); \end{aligned}$$

AXIOM FINDTOP:

$$\begin{aligned} \forall sq. & (\text{FINDTOP}(0, \text{SLAMBDA}, sq) \equiv \text{SEQUENCE}(sq)), \\ \forall n \ s \ sq. & (\text{FINDTOP}(n, s, sq) \equiv \text{INTEGER}(n) \wedge \text{STRING}(s) \wedge \text{SEQUENCE}(sq) \wedge \\ & \exists s1 \ s2. (\text{STRING}(s1) \wedge \text{STRING}(s2) \wedge (s1 \neq \text{LAMBDA}) \wedge (s = (s1 \text{ c } s2)) \wedge \\ & (s = \text{scar}(sq)) \wedge \text{FINDTOP}(n-1, s2, \text{scar}(sq)))); \end{aligned}$$

AXIOM TERM:

$$\begin{aligned} \forall sq. & (\text{TERMSEQ}(sq) \equiv \text{SEQUENCE}(sq) \wedge ((\text{slen}(sq) = 1 \wedge \text{INDVAR}(1 \text{ sgl } sq)) \vee \\ & (\text{slen}(sq) > 1 \wedge \text{TERMSEQ}(\text{scdr}(sq)) \wedge (\text{INDVAR}(\text{scar}(sq)) \vee \\ & \exists n \ s \ \text{INTEGER}(n) \wedge \text{STRING}(s) \wedge (s = \text{car}(\text{scar}(sq)) \wedge \text{OPCONST}(s) \wedge n = \text{arity}(s) \wedge \\ & \text{FIND}(n, \text{cdr}(\text{scar}(sq)), \text{scdr}(sq)))))), \end{aligned}$$

$$\forall t. (\text{TERM}(t) \equiv \text{STRING}(t) \wedge \exists sq. (\text{TERMSEQ}(sq) \wedge t = \text{car}(sq)));$$

AXIOM WFF:

$$\forall f. (\text{ELF}(f) \equiv \text{STRING}(f) \wedge (f = \text{FALSESYM} \vee \text{PREDPAR}(f) \vee \exists n \ sq. (\text{INTEGER}(n) \wedge \text{SEQUENCE}(sq) \wedge \text{PREDPAR}(\text{car}(f)) \wedge n = \text{arity}(\text{car}(f)) \wedge \text{TERMSEQ}(sq) \wedge \text{FINDTOP}(n, \text{cdr}(f), sq)));$$

$$\begin{aligned} \forall sq. & (\text{FRR}(sq) \equiv \text{SEQUENCE}(sq) \wedge (sq \neq \text{SLAMBDA}) \wedge (\text{ELF}(\text{scar}(sq)) \vee \\ & (\text{FRR}(\text{scdr}(sq)) \wedge \exists s1 \ s2. (\text{STRING}(s1) \wedge \text{STRING}(s2) \wedge \\ & (((\text{scar}(sq) = \text{neg}(s1) \wedge \text{FIND}(1, x1, \text{scdr}(sq))) \vee \\ & (\text{scar}(sq) = (s1 \text{ dis } s2) \wedge \text{FIND}(2, (s1 \text{ c } s2), \text{scdr}(sq))) \vee \\ & (\text{scar}(sq) = (s1 \text{ con } s2) \wedge \text{FIND}(2, (s1 \text{ c } s2), \text{scdr}(sq))) \vee \\ & (\text{scar}(sq) = (s1 \text{ impl } s2) \wedge \text{FIND}(2, (s1 \text{ c } s2), \text{scdr}(sq))) \vee \\ & (\text{scar}(sq) = (s1 \text{ gen } s2) \wedge \text{INDVAR}(s1) \wedge \text{FIND}(1, s2, \text{scdr}(sq))) \vee \\ & (\text{scar}(sq) = (s1 \text{ ex } s2) \wedge \text{INDVAR}(s1) \wedge \text{FIND}(1, s2, \text{scdr}(sq))))))), \\ \forall f. & (\text{FORM}(f) \equiv \text{STRING}(f) \wedge \exists sq. (\text{FRR}(sq) \wedge f = \text{scar}(sq))); \end{aligned}$$

2.6 Free and bound variables and the substitution

AXIOM BOUNDV:

$$\begin{aligned} \forall x \ n \ f. & (\text{GEB}(x, n, f) \equiv \text{INDVAR}(x) \wedge \text{INTEGER}(n) \wedge \text{FORM}(f) \wedge \exists s1 \ s2 \ f1. (\text{STRING}(s1) \wedge \\ & \text{FORM}(f1) \wedge \text{STRING}(s2) \wedge \text{len}(s1) + 1 < n \wedge n < (\text{len}(f) - \text{len}(s2)) \wedge \\ & (x = n \text{ gl } f) \wedge ((f = (s1 \text{ c } ((x \text{ gen } f1) \text{ c } s2)) \vee (f = (s1 \text{ c } ((x \text{ ex } f1) \text{ c } f3)))))); \end{aligned}$$

AXIOM FREEV:

$\forall x n f. (FRN(x,n,f) \equiv INDVAR(x) \wedge INTEGER(n) \wedge FORM(f) \wedge x=(n \text{ gl } f) \wedge \neg GEB(x,n,f)),$
 $\forall x f. (FR(x,f) \equiv \exists n.(INTEGER(n) \wedge FRN(x,n,f)));\;$

AXIOM FIRSTFRDF:

$\forall x n f. (FIRSTFREE(x,n,f) \equiv FRN(x,n,f) \wedge \forall n1.(INTEGER(n1) \wedge x=n1 \text{ gl } f \supset (n1 \geq n \vee GEB(x,n1,f))))),$
 $\forall x n f. (FIRSTFREE(x,n,a) \equiv \text{firstfree}(x,f)=n);\;$

AXIOM KFREEOCCDF:

$\forall x k n f. (KTHFREEOCC(x,k,n,f) \equiv (INDVAR(x) \wedge INTEGER(k) \wedge INTEGER(n) \wedge FORM(f) \wedge (k=B \wedge n=B) \vee (n=\text{len}(f) \wedge \forall n2.((INTEGER(n2) \wedge n2 > \text{kthfreeocc}(x,k-1,f)) \supset \neg FRN(x,n2,f))) \vee (FRN(x,n,f) \wedge \forall n1.((INTEGER(n1) \wedge (n1 < k \wedge n1 > B)) \supset \exists n2.(INTEGER(n2) \wedge n2 < n \wedge KTHFREEOCC(x,n1,n2,f))))))),$
 $\forall x k n f. (KTHFREEOCC(x,k,n,f) \equiv \text{kthfreeocc}(x,k,f)=n),$
 $\forall x k n f. (KTHFREEOCC(x,k,n,f) \supset \text{numbfreeocc}(x,n,f)=k),$
 $\forall x k n f. (\text{numbfreeocc}(x,n,f)=k \supset (KTHFREEOCC(x,k,n,f) \vee (n < \text{kthfreeocc}(x,k,f) \wedge n > \text{kthfreeocc}(x,k-1,f))));\;$

AXIOM SUBSTDF:

$\forall x t f1 f2. (SBT(x,t,f1,f2) \equiv ((INDVAR(x) \wedge TERM(t) \wedge FORM(f1) \wedge FORM(f2)) \supset \forall n1 n2.((INTEGER(n1) \wedge INTEGER(n2) \wedge n2=\text{numbfreeocc}(x,n1,f1) * (\text{len}(t)-1)) * n1 \supset ((\neg INDVAR(n1 \text{ gl } f1) \supset n1 \text{ gl } f1 = n2 \text{ gl } f2) \wedge (INDVAR(n1 \text{ gl } f1) \supset ((FRN(x,n1,f1) \supset SUBT(t,f2,n2)) \wedge (\neg FRN(x,n1,f1) \supset INVART(n1,f1,n2,f2))))))))),$
 $\forall t f2 n2. (SUBT(t,f2,n2) \equiv (TERM(t) \wedge FORM(f2) \wedge INTEGER(n2) \wedge \forall x2 k.((INDVAR(x2) \wedge INTEGER(k) \wedge ((k \text{ gl } t)=x2) \supset FRN(x2,n2-(\text{len}(t)-k),f2))))),$
 $\forall n1 f1 n2 f2. (INVART(n1,f1,n2,f2) \equiv (INTEGER(n1) \wedge FORM(f1) \wedge INTEGER(n2) \wedge FORM(f2) \wedge (GEB(n2 \text{ gl } f2,n2,f2) \equiv GEB(n1 \text{ gl } f1,n1,f1)) \wedge (FRN(n2 \text{ gl } f2,n2,f2) \equiv FRN(n1 \text{ gl } f1,n1,f1)) \wedge n2 \text{ gl } f2 = n1 \text{ gl } f1)),$
 $\forall x t f1 f2. ((INDVAR(x) \wedge TERM(t) \wedge FORM(f1) \wedge FORM(f2)) \supset (SBT(x,t,f1,f2) \equiv \text{sbt}(x,t,f1)=f2)),$
 $\forall x t f1. ((INDVAR(x) \wedge TERM(t) \wedge FORM(f1)) \supset FORM(\text{sbt}(x,t,f1)));\;$

AXIOM SUBDEF:

$\forall x1 x2 f1 f2. (SBV(x1,x2,f1,f2) \equiv ((INDVAR(x1) \wedge INDVAR(x2) \wedge FORM(f1) \wedge FORM(f2)) \supset \forall n.(INTEGER(n) \supset ((\neg INDVAR(n \text{ gl } f1) \supset n \text{ gl } f1 = n \text{ gl } f2) \wedge (INDVAR(n \text{ gl } f1) \supset ((FRN(x1,n,f1) \supset FRN(x2,n,f2)) \wedge (\neg FRN(x1,n,f1) \supset INVAV(n,f1,f2))))))))),$
 $\forall n f1 f2. (INVAV(n,f1,f2) \equiv (INTEGER(n) \wedge FORM(f1) \wedge FORM(f2) \wedge (GEB(n \text{ gl } f2,n,f2) \equiv GEB(n \text{ gl } f1,n,f1)) \wedge FRN(n \text{ gl } f2,n,f2) \equiv FRN(n \text{ gl } f1,n,f1)) \wedge n \text{ gl } f2 = n \text{ gl } f1)),$
 $\forall x1 x2 f1 f2. ((INDVAR(x1) \wedge INDVAR(x2) \wedge FORM(f1) \wedge FORM(f2)) \supset (SBV(x1,x2,f1,f2) \equiv \text{sbv}(x1,x2,f1)=f2)),$
 $\forall x1 x2 f1. ((INDVAR(x1) \wedge INDVAR(x2) \wedge FORM(f1)) \supset FORM(\text{sbv}(x1,x2,f1)));\;$

2.7 Rules of inference

AXIOM ANDIRUL:

$\forall sq \text{ pf1 pf2. } (ANDI(sq, pf1, pf2) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf1) \wedge PROOFTREE(pf2) \wedge \exists f1 f2. (scdr(sq) = (pf1 \text{ cc } pf2) \wedge scar(sq) = f1 \text{ con } f2 \wedge FORM(f1) \wedge FORM(f2) \wedge f1 = scar(pf1) \wedge f2 = scar(pf2))))),$
 $\forall sq \text{ pf. } (ANDE(sq, pf) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf) \wedge \exists f1. (scdr(sq) = pf \wedge FORM(f1) \wedge ((scar(sq) \text{ con } f1) = scar(pf)) \vee ((f1 \text{ con } (scar(sq)) = scar(pf))))));;$

AXIOM FALSERUL :

$\forall sq \text{ pf1 pf2. } (FALSEI(sq, pf1, pf2) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf1) \wedge PROOFTREE(pf2) \wedge \exists f1. ((scdr(sq) = (pf1 \text{ cc } pf2)) \wedge (scar(sq) = FALSESYM) \wedge FORM(f1) \wedge (neg(x) = scar(pf1)) \wedge (x1 = scar(pf2))))),$
 $\forall sq \text{ pf. } (FALSEE(sq, pf) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf) \wedge (scar(pf) = FALSESYM) \wedge scdr(sq) = pf));;$

AXIOM IMPLRUL :

$\forall sq \text{ pf1 pf2. } (IMPLE(sq, pf1, pf2) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf1) \wedge PROOFTREE(pf2) \wedge \forall f1. ((scdr(sq) = (pf1 \text{ cc } pf2)) \wedge FORM(f1) \wedge (scar(pf1) = (f1 \text{ impl } (scar(sq))) \wedge (scar(pf2) = f1))))),$
 $\forall sq \text{ pf f1. } (IMPLID(sq, pf, f1) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf) \wedge scdr(sq) = pf \wedge FORM(f1) \wedge \exists f2. ((scar(sq) = (f1 \text{ impl } x2)) \wedge FORM(f1) \wedge (f2 = scar(pf)) \wedge \exists n. (INTEGER(n) \wedge f1 = (n \text{ sgl } pf))))));;$
 $\forall sq \text{ pf. } (IMPLI(sq, pf) \equiv \exists f \text{ IMPLID}(sq, pf, f));;$

AXIOM NEGRUL:

$\forall sq \text{ pf f1. } (NOTID(sq, pf, f1) \equiv (scdr(sq) = pf \wedge SEQUENCE(sq) \wedge PROOFTREE(pf) \wedge FORM(f1) \wedge \exists n. ((scar(pf) = FALSESYM) \wedge scar(sq) = neg(f1) \wedge INTEGER(n) \wedge ((n \text{ sgl } pf) = f1))))),$
 $\forall sq \text{ pf. } (NOTI(sq, pf) \equiv \exists f. \text{NOTID}(sq, pf, f)),$
 $\forall sq \text{ pf f1. } (NOTED(sq, pf, f1) \equiv (scdr(sq) = pf \wedge SEQUENCE(sq) \wedge PROOFTREE(pf) \wedge FORM(f) \wedge \exists n. ((scar(pf) = FALSESYM) \wedge INTEGER(n) \wedge ((n \text{ sgl } pf) = neg(scar(sq)))))),$
 $\forall sq \text{ pf. } (NOTE(sq, pf) \equiv \exists f. \text{NOTED}(sq, pf, f));;$

AXIOM ORRUL:

$\forall sq \text{ pf. } (ORI(sq, pf) \equiv (scdr(sq) = pf \wedge SEQUENCE(sq) \wedge PROOFTREE(pf) \wedge \exists f1 f2. ((scar(sq) = (f1 \text{ dis } f2)) \wedge FORM(f1) \wedge FORM(f2) \wedge (f1 = scar(pf)) \vee (f2 = scar(pf))))),$
 $\forall sq \text{ pf1 pf2 pf3 f1 f2. } (ORED(sq, pf1, pf2, pf3, f1, f2) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf1) \wedge PROOFTREE(pf2) \wedge PROOFTREE(pf3) \wedge FORM(f1) \wedge FORM(f2) \wedge (scdr(sq) = (pf1 \text{ cc } (pf2 \text{ cc } pf3)) \wedge (scar(pf1) = (f1 \text{ dis } f2)) \wedge (scar(pf2) = scar(sq)) \wedge (scar(pf3) = scar(sq)) \wedge \exists n1. (n1 \text{ sgl } pf2) = f1 \wedge \exists n2. (n2 \text{ sgl } pf3) = f2))))),$
 $\forall sq \text{ pf1 pf2 pf3. } (ORE(sq, pf1, pf2, pf3) \equiv \exists f1 f2. \text{ORED}(sq, pf1, pf2, pf3, f1, f2));;$

AXIOM EXRUL :

$\forall sq \text{ pf x f. } (EXI(sq, pf, x, f) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf) \wedge INDVAR(x) \wedge TERM(f) \wedge \exists f1. ((scdr(sq) = pf1) \wedge (scar(sq) = (x \text{ ex } f1)) \wedge FORM(f1) \wedge scar(pf) = sbf(x, f1))))),$
 $\forall sq \text{ pf1 pf2 x1 x2 f1. } (EXED(sq, pf1, pf2, x1, x2, f1) \equiv (SEQUENCE(sq) \wedge PROOFTREE(pf1) \wedge INDVAR(x1) \wedge INDVAR(x2) \wedge (scdr(sq) = (pf1 \text{ cc } pf2)) \wedge FORM(f1) \wedge (scar(pf1) = (x1 \text{ ex } f1)) \wedge (scar(sq) = scar(pf2)) \wedge \exists n. ((n \text{ sgl } pf2) = sbt(x1, x2, f1) \wedge INTEGER(n) \wedge \text{EXAPPL}(x2, pf2, f1))))),$
 $\forall sq \text{ pf1 pf2 x1 x2. } (\text{EXE}(sq, pf1, pf2, x1, x2) \equiv \text{EXED}(sq, pf1, pf2, x1, x2)),$
 $\forall x \text{ pf f. } (\text{EXAPPL}(x, pf, f) \equiv (INDVAR(x) \wedge PROOFTREE(pf) \wedge FORM(f) \wedge \neg \text{FR}(x, scar(pf)) \wedge$

APPENDIX 3

THE PROOF OF "IF f IS A WFF ALSO $\exists x.f$ IS A WFF"

3.1 FOL commands and printout in the many sorted logic commands

```

VE WFF1, x gen f;
TAUTEQ (x gen f = x gen f) v (x gen f = x ex f);
UNIFY --:2#2#1, -;
TAUT ---:01, 1:-;

```

proof

```

1 FORM(x gen f)=(ELF(x gen f)v(3x1 f1.((x gen f)=(x1 gen f1)v(x gen f)=(x1 ex f1))v
  (3f1 f2.((x gen f)=(f1 dis f2)v((x gen f)=(f1 con f2)v(x gen f)=(f1 impl f2)))v
  3f1.(x gen f)=neg(f1)))
2 (x gen f)=(x gen f)v(x gen f)=(x ex f)
3 3x1 f1.((x gen f)=(x1 gen f1)v(x gen f)=(x1 ex f1))
4 FORM(x gen f)

```

3.2 FOL commands in the earlier axiomatization

```

DECLARE INDVAR A U;
label hpt1;
ASSUME FORM(f) ^ INDVAR (x1);
label teo1;
ASSUME V f s.(SEQUENCE(sq)^sq # SLAMBDA => (STRING(s) => (s cc sq) # SLAMBDA));
label teo2;
ASSUME V s sq.(STRING(s)^SEQUENCE(sq) = scar(s :c sq) = s);
label teo3;
ASSUME V s sq.(STRING(s)^SEQUENCE(sq) = scdr(s cc sq) = sq);
label teo4;
ASSUME V sq.(SEQUENCE(sq)^sq # SLAMBDA = find(1,scar(sq),sq));
label teo5;
ASSUME V f x.(FORM(f)^INDVAR(x) => STRING(x gen f));
label teo6;
ASSUME V s sq.(STRING(s)^SEQUENCE(sq) => SEQUENCE(s cc sq));
label teo7;
ASSUME V x.(INDVAR(x) => STRING(x));

V0 WFF2 f;
LABEL ass1;
taut 3sq.(FRR(sq)^f=scar(sq)) 1:-;
ASSUME FRR(SQ) ^ f = SCAR(SQ);
V0 WFF1 SQ;
V0 teo1 SQ ,x1 gen f;

```

```

Ve leo2 x1 gen f ,SQ;
Ve leo3 x1 gen f ,SQ;
Ve leo4      SQ;
Ve leo5 f ,x1;
Ve leo7 x1;
Ve Wff1 (x1 gen f) cc SQ;

TAUTEQ --:#2#2#2#2#2#1#1[s1←f : s2←x1] 1:-;
unify --:#2#2#2#2#2 -;
Ve leo6 x1 gen f ,SQ;
Ve WFF2 x1 gen f ;
tauteq --:#2#2#1[sq←(x1 gen f) cc SQ] 1:-;
unify --:#2#2 -;
taut FORM(x1 gen f) 1:-;
∃e ass1,-,SQ;
⇒i hpt1,-;
∀l -,x1,f;
    
```

3.3 Printout of the proof in the earlier axiomatization

- 1 FORM(f)∧INDVAR(x1) (1) --- ASSUME
- 2 ∀sq s ((SEQUENCE(sq)∧sq/SLAMBDA)⇒(STRING(s)⇒(s cc sq)/SLAMBDA)) (2) --- ASSUME
- 3 ∀s sq ((STRING(s)∧SEQUENCE(sq)) ⇒scar(s cc sq)=s) (3) --- ASSUME
- 4 ∀s sq ((STRING(s)∧SEQUENCE(sq)) ⇒scdr(s cc sq)=sq) (4) --- ASSUME
- 5 ∀sq ((SEQUENCE(sq)∧sq/SLAMBDA) ⇒find(1,scar(sq),sq)) (5) --- ASSUME
- 6 ∀f x.((FORM(f)∧INDVAR(x))⇒STRING(x gen f)) (6) --- ASSUME
- 7 ∀s sq.((STRING(s)∧SEQUENCE(sq))⇒SEQUENCE(s cc sq)) (7) --- ASSUME
- 8 ∀x.(INDVAR(x)⇒STRING(x)) (8) --- ASSUME
- 9 FORM(f)=(STRING(f)∧∃sq (FRR(s)∧f=scar(sq))) --- VE WFF2 f
- 10 ∃sq.(FRR(sq)∧f=scar(sq)) (1 2 3 4 5 6 7 8) --- TAUT 1:9
- 11 FRR(SQ)∧f=scar(SQ) (11) --- ASSUME
- 12 FRR(SQ) ⇒ (SEQUENCE(SQ)∧(SQ/SLAMBDA)∧(ELF(scar(SQ))∨(FRR(scdr(SQ))∧∃s1 s2.(STRING(s1)∧(STRING(s2)∧((scar(SQ)=NEG(s1)∧find(1,s1,scdr(SQ)))∨((scar(SQ)=s1 dis s2)∧find(2,s1 c x2,scdr(SQ)))∨((scar(SQ)=(s1 con s2)∧find(2,s1 c s2,scdr(SQ)))∨((scar(SQ)=(s1 impl s2)∧find(2,s1 c s2,scdr(SQ)))∨((scar(SQ)=(s1 gen s2)∧(INDVAR(s1)∧find(1,s2,scdr(SQ)))∨(scar(SQ)=(s1 ex s2)∧(INDVAR(s1)∧find(1,s2,scdr(SQ)))))))))))))) --- VE WFF1 SQ
- 13 (SEQUENCE(SQ)∧SQ/SLAMBDA)⇒(STRING(x1 gen f)⇒(x1 gen f) cc SQ)/SLAMBDA (2) --- VE 2 SQ,x1 gen f

- 14 (STRING(x1 gen f) ^ SEQUENCE(SQ)) = scdr((x1 gen f) cc SQ) = (x1 gen f)
 (3) --- VE 3 x1 gen f, SQ
- 15 (STRING(x1 gen f) ^ SEQUENCE(SQ)) = scdr((x1 gen f) cc SQ) = SQ (4) --- VE 4 x1 gen f, SQ
- 16 (SEQUENCE(SQ) ^ SQ) / SLAMBDA = find(1, scdr(SQ), SQ) (5) --- VE 5 SQ
- 17 (FORM(f) ^ INDVAR(x1)) = string(x1 gen f) (6) --- VE 6 f, x1
- 18 INDVAR(x1) = STRING(x1) (8) --- VE 8 x1
- 19 FRR((x1 gen f) cc SQ) = (SEQUENCE((x1 gen f) cc SQ) ^ ((x1 gen f) cc U)) /
 SLAMBDA ^ (ELF(scdr((x1 gen f) cc SQ)) v (FRR(scdr((x1 gen f) cc SQ))) ^
 ∃s1 s2. (STRING(s1) ^ (STRING(s2) ^ ((scdr((s1 gen f) cc SQ) = NEG(s1) ^
 find(1, s1, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (s1 dis s2) ^
 find(2, s1 c s2, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (s1 con s2) ^
 find(2, s1 c s2, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (s1 impl s2) ^
 find(2, s1 c s2, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (s1 gen s2) ^ (INDVAR(s1) ^
 find(1, s2, scdr((x1 gen f) cc SQ))) v (scdr((x1 gen f) cc SQ) = (s1 ex s2) ^ (INDVAR(s1) ^
 find(1, s2, scdr((x1 gen f) cc SQ))))))))) --- VE WFF1 (x1 gen f) cc SQ
- 20 STRING(x1) ^ (STRING(f) ^ ((scdr((x1 gen f) cc SQ) = NEG(x1) ^
 find(1, x1, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (x1 dis f) ^
 find(2, x1 c f, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (x1 con f) ^
 find(2, x1 c f, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (x1 impl f) ^
 find(2, x1 c f, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (x1 gen f) ^ (INDVAR(x1) ^
 find(1, f, scdr((x1 gen f) cc SQ))) v (scdr((x1 gen f) cc SQ) = (x1 ex f) ^ (INDVAR(x1) ^
 find(1, f, scdr((x1 gen f) cc SQ))))))))) (1 2 3 4 5 6 7 8 11) --- TAUTEQ 1:19
- 21 ∃s1 s2. (STRING(s1) ^ (STRING(s2) ^ ((scdr((s1 gen f) cc SQ) = NEG(s1) ^
 find(1, s1, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (s1 dis s2) ^
 find(2, s1 c s2, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (s1 con s2) ^
 find(2, s1 c s2, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (s1 impl s2) ^
 find(2, s1 c s2, scdr((x1 gen f) cc SQ))) v ((scdr((x1 gen f) cc SQ) = (s1 gen s2) ^ (INDVAR(s1) ^
 find(1, s2, scdr((x1 gen f) cc SQ))) v (scdr((x1 gen f) cc SQ) = (s1 ex s2) ^ (INDVAR(s1) ^
 find(1, s2, scdr((x1 gen f) cc SQ))))))))) (1 2 3 4 5 6 7 8 11) --- UNIFY 20
- 22 (STRING(x1 gen f) ^ SEQUENCE(SQ)) = SEQUENCE((x1 gen f) cc SQ) (7) --- VE 7 x1 GEN f, SQ
- 23 FORM(x1 gen f) = (STRING(x1 gen f) ^ ∃sq. (FRR(sq) ^ (x1 gen f) = scdr(sq))) --- VE WFF2 x1 gen f
- 24 FRR((x1 gen f) cc SQ) ^ (x1 gen f) = scdr((x1 gen f) cc SQ) (1 2 3 4 5 6 7 8 11) TAUTEQ 1:23
- 25 ∃sq. (FRR(sq) ^ (x1 gen f) = scdr(sq)) (1 2 3 4 5 6 7 8 11) --- UNIFY 24
- 26 FORM(x1 gen f) (1 2 3 4 5 6 7 8 11) --- TAUT 1:25
- 27 FORM(x1 gen f) (1 2 3 4 5 6 7 8) --- ∃E 10 26 U
- 28 (FORM(f) ^ INDVAR(x1)) = FORM(x1 gen f) (2 3 4 5 6 7 8) --- ∃I 1 27
- 29 ∀f x1. ((FORM(f) ^ INDVAR(x1)) = FORM(x1 gen f)) (2 3 4 5 6 7 8) --- VI 28 x1 ← x1 f ← f

APPENDIX 4

THE PROOF OF THE EQUIVALENCE BETWEEN SBV AND SBT FOR VARIABLES

4.1 FOL commands in the many sorted logic

LABEL ARITH1; ASSUME $\forall n x. (n * (\text{len}(x) - 1) = 0)$;
 LABEL ARITH2; ASSUME $\forall n. (0 = n)$;
 LABEL ARITH3; ASSUME $\forall x. (\text{len}(x) - 1) = 0$;
 LABEL ARITH4; ASSUME $\forall n. (n - 0) = n$;
 LABEL STRING1; ASSUME $\forall x. |g| x = x$;

Proof of the First Lemma: $\forall x f n. (\text{SUBT}(x, f, n) \Rightarrow \text{FRN}(x, n, f))$

LABEL HPTLEM; ASSUME $\text{SUBT}(x, f, n)$;
 $\forall e \text{ SUBSTDF1 } x, f, n$;
 TAUT $-\#2, -\#1, -\#$;
 $\forall e -x, l$;
 $\forall e \text{ STRING1 } x$; substr - in --;
 $\forall e \text{ ARITH3 } x$; substr - in --;
 $\forall e \text{ ARITH4 } n$; substr - in --;
 TAUTEQ $\text{FRN}(x, n, f), \text{HPTLEM} \#1$:-;
 $\Rightarrow \text{HPTLEM}$,-;
 LABEL LEMMA1; $\forall i -x, f, n$;

Proof of the Second Lemma: $\forall n f1 f2. (\text{INVART}(n, f1, n, f2) = \text{INVARV}(n, f1, f2))$

$\forall e \text{ SUBSTDF2 } n, f1, n, f2$;
 $\forall e \text{ SUBDEF1 } n, f1, f2$;
 TAUT $-\#1 = -\#1, -\#1, -\#$;
 LABEL LEMMA2; $\forall i -n, f1, f2$;

Proof of the Main Theorem: $\forall x1 x2 f1 f2. (\text{SBT}(x1, x2, f1, f2) \Rightarrow \text{SBV}(x1, x2, f1, f2))$

LABEL HPT; ASSUME $\text{SBT}(x1, x2, f1, f2)$;
 $\forall e \text{ SUBSTDF0 } x1, x2, f1, f2$;
 TAUT $-\#2, \text{HPT}, -\#$;
 $\forall e -n1, n1$;
 $\forall e \text{ ARITH1, numbfreeocc}(x1, n1, f1), x2$; substr - in --;
 $\forall e \text{ ARITH2, } n1$;
 $\forall e \text{ SUBDEFO } x1, x2, f1, f2$;
 $\forall e \text{ LEMMA1 } x2, f2, n1$;
 $\forall e \text{ LEMMA2, } n1, f1, f2$;
 TAUTEQ $-\#1 \#2 \#1 [n \leftarrow n1], \text{HPT} \#1$:-;
 $\forall i -n1 \leftarrow n$;
 TAUTEQ $-\#1, \text{HPT} \#1$:-;
 $\Rightarrow \text{HPT}$,-;
 $\forall i -x1, x2, f1, f2$;

4.2 Printout of the proof in the many sorted logic

- 1 $\forall n x.(n*(len(x)-1))=0$ (1)
- 2 $\forall n.(0*n)=n$ (2)
- 3 $\forall x.(len(x)-1)=0$ (3)
- 4 $\forall n.(n-0)=n$ (4)
- 5 $\forall x.(1 \text{ gl } x)=x$ (5)
- 6 SUBT(x,f,n) (6)
- 7 $SUBT(x,f,n)=\forall x2 k.((k \text{ gl } x)=x2 \supset FRN(x2,n-(len(x)-k),f))$
- 8 $\forall x2 k.((k \text{ gl } x)=x2 \supset FRN(x2,n-(len(x)-k),f))$ (6)
- 9 $(1 \text{ gl } x)=x \supset FRN(x,n-(len(x)-1),f)$ (6)
- 10 $(1 \text{ gl } x)=x$ (5)
- 11 $x=x \supset FRN(x,n-(len(x)-1),f)$ (5 6)
- 12 $(len(x)-1)=0$ (3)
- 13 $x=x \supset FRN(x,n-0,f)$ (3 5 6)
- 14 $(n-0)=n$ (4)
- 15 $x=x \supset FRN(x,n,f)$ (3 4 5 6)
- 16 $FRN(x,n,f)$ (3 4 5 6)
- 17 $SUBT(x,f,n) \supset FRN(x,n,f)$ (3 4 5)
- 18 $\forall x f n.(SUBT(x,f,n) \supset FRN(x,n,f))$ (3 4 5)
- 19 $INVART(n,f1,n,f2)=((GEB(n \text{ gl } f2,n,f2)=GEB(n \text{ gl } f1,n,f1)) \wedge ((FRN(n \text{ gl } f2,n,f2)=FRN(n \text{ gl } f1,n,f1)) \wedge (n \text{ gl } f2)=(n \text{ gl } f1)))$
- 20 $INVARV(n,f1,f2)=((GEB(n \text{ gl } f2,n,f2) \neq GEB(n \text{ gl } f1,n,f1)) \wedge ((FRN(n \text{ gl } f2,n,f2) \neq FRN(n \text{ gl } f1,n,f1)) \wedge (n \text{ gl } f2) \neq (n \text{ gl } f1)))$
- 21 $INVART(n,f1,n,f2) \neq INVARV(n,f1,f2)$
- 22 $\forall n f1 f2.(INVART(n,f1,n,f2) \neq INVARV(n,f1,f2))$
- 23 SBT(x1,x2,f1,f2) (23)
- 24 $SBT(x1,x2,f1,f2)=\forall n1 n2.(n2=((numbfreeocc(x1,n1,f1)*(len(x2)-1))+n1) \supset ((\neg INDVAR(n1 \text{ gl } f1) \supset (n1 \text{ gl } f1) \neq (n2 \text{ gl } f2)) \wedge (INDVAR(n1 \text{ gl } f1) \supset ((FRN(x1,n1,f1) \supset SUBT(x2,f2,n2)) \wedge (\neg FRN(x1,n1,f1) \supset INVART(n1,f1,n2,f2))))))$

- 25 $\forall n1\ n2.(n2=((\text{numbfreeocc}(x1,n1,f1))*(\text{len}(x2)-1))+n1)\supset((\neg\text{INDVAR}(n1\ \text{gl}\ f1)\supset(n1\ \text{gl}\ f1)=(n2\ \text{gl}\ f2))\wedge(\text{INDVAR}(n1\ \text{gl}\ f1)\supset((\text{FRN}(x1,n1,f1)\supset\text{SUBT}(x2,f2,n2))\wedge(\neg\text{FRN}(x1,n1,f1)\supset\text{INVART}(n1,f1,n2,f2))))))$ (23)
- 26 $n1=((\text{numbfreeocc}(x1,n1,f1))*(\text{len}(x2)-1))+n1)\supset((\neg\text{INDVAR}(n1\ \text{gl}\ f1)\supset(n1\ \text{gl}\ f1)=(n1\ \text{gl}\ f2))\wedge(\text{INDVAR}(n1\ \text{gl}\ f1)\supset((\text{FRN}(x1,n1,f1)\supset\text{SUBT}(x2,f2,n1))\wedge(\neg\text{FRN}(x1,n1,f1)\supset\text{INVART}(n1,f1,n1,f2))))))$ (23)
- 27 $(\text{numbfreeocc}(x1,n1,f1))*(\text{len}(x2)-1)=0$ (1)
- 28 $n1=(0+n1)\supset((\neg\text{INDVAR}(n1\ \text{gl}\ f1)\supset(n1\ \text{gl}\ f1)=(n1\ \text{gl}\ f2))\wedge(\text{INDVAR}(n1\ \text{gl}\ f1)\supset((\text{FRN}(x1,n1,f1)\supset\text{SUBT}(x2,f2,n1))\wedge(\neg\text{FRN}(x1,n1,f1)\supset\text{INVART}(n1,f1,n1,f2))))))$ (1 23)
- 29 $(0+n1)=n1$ (2)
- 30 $\text{SBV}(x1,x2,f1,f2)=\forall n((\neg\text{INDVAR}(n\ \text{gl}\ f1)\supset(n\ \text{gl}\ f1)=(n\ \text{gl}\ f2))\wedge(\text{INDVAR}(n\ \text{gl}\ f1)\supset((\text{FRN}(x1,n,f1)\supset\text{FRN}(x2,n,f2))\wedge(\neg\text{FRN}(x1,n,f1)\supset\text{INVARV}(n,f1,f2))))))$
- 31 $\text{SUBT}(x2,f2,n1)\supset\text{FRN}(x2,n1,f2)$ (3 4 5)
- 32 $\text{INVART}(n1,f1,n1,f2)=\text{INVARV}(n1,f1,f2)$
- 33 $(\neg\text{INDVAR}(n1\ \text{gl}\ f1)\supset(n1\ \text{gl}\ f1)=(n1\ \text{gl}\ f2))\wedge(\text{INDVAR}(n1\ \text{gl}\ f1)\supset((\text{FRN}(x1,n1,f1)\supset\text{FRN}(x2,n1,f2))\wedge(\neg\text{FRN}(x1,n1,f1)\supset\text{INVARV}(n1,f1,f2))))$ (1 2 3 4 5 23)
- 34 $\forall n((\neg\text{INDVAR}(n\ \text{gl}\ f1)\supset(n\ \text{gl}\ f1)=(n\ \text{gl}\ f2))\wedge(\text{INDVAR}(n\ \text{gl}\ f1)\supset((\text{FRN}(x1,n,f1)\supset\text{FRN}(x2,n,f2))\wedge(\neg\text{FRN}(x1,n,f1)\supset\text{INVARV}(n,f1,f2))))))$ (1 2 3 4 5 23)
- 35 $\text{SBV}(x1,x2,f1,f2)$ (1 2 3 4 5 23)
- 36 $\text{SBT}(x1,x2,f1,f2)\supset\text{SBV}(x1,x2,f1,f2)$ (1 2 3 4 5)
- 37 $\forall x1\ x2\ f1\ f2.(\text{SBT}(x1,x2,f1,f2)\supset\text{SBV}(x1,x2,f1,f2))$ (1 2 3 4 5)

4.3 FOL commands in the earlier axiomatization

LABEL ARITH1; ASSUME $\forall n\ x.((\text{INTEGER}(n) \wedge \text{INDVAR}(x))\supset(n*(\text{len}(x)-1)=0))$;
 LABEL ARITH2; ASSUME $\forall n. (\text{INTEGER}(n) \supset (0*n=n))$;
 LABEL ARITH3; ASSUME $\forall x. (\text{INDVAR}(x) \supset ((\text{len}(x)-1)=0))$;
 LABEL ARITH4; ASSUME $\forall n. (\text{INTEGER}(n) \supset (n-0)=n)$;
 LABEL STRING1; ASSUME $\forall x. (\text{INDVAR}(x) \supset 1\ \text{gl}\ x=x)$;

Proof of the First Lemma:

$\forall x\ n\ f.((\text{INDVAR}(x) \wedge \text{INTEGER}(n) \wedge \text{FORM}(f) \wedge \text{SUBT}(x,f,n)) \supset \text{FRN}(x,n,f))$

LABEL HPTLEM; ASSUME $\text{INDVAR}(x)\wedge\text{FORM}(f)\wedge\text{INTEGER}(n)\wedge\text{SUBT}(x,f,n)$;

LABEL FACT ; ASSUME $\text{INTEGER}(1)$;

$\forall\ \text{SUBSTDF1},x,f,n$;

TAUT $-\text{:}2*2*2*2,-,-,-,-$;

$\forall\ -,x,1$;

$\forall\ \text{STRING1},x$; TAUT $-\text{:}2,\text{HPTLEM}:-;\text{substr} - \text{in} -$;

$\forall\ \text{ARITH3},x$; TAUT $-\text{:}2,\text{HPTLEM}:-;\text{substr} - \text{in} -$;

$\forall \bullet$ ARITH4,n; TAUT \rightarrow :#2,HPTLEM:-;substr - in ---;
 TAUTEQ FRN(x,n,f),HPTLEM:-;
 \supset I HPTLEM,-;
 LABEL LEMMA1; \forall I -,x,f,n;

Proof of the Second lemma : $\forall k f1 f2.(INVART(k,f1,k,f2) \bullet INVARY(k,f1,f2))$

$\forall \bullet$ SUBSTDF2,k f1,k,f2;
 $\forall \bullet$ SUBDEF1 ,k,f1,f2;
 TAUT \rightarrow :#1 = -:#1,-,-;
 LABEL LEMMA2; \forall i -,k,f1,f2;

Proof of the Main Theorem

$\forall x1 x2 f1 f2.((INDVAR(x1) \wedge INDVAR(x2) \wedge FORM(f1) \wedge FORM(f2) \wedge SBT(x1,x2,f1,f2)) \rightarrow$
 $SBV(x1,x2,f1,f2))$

LABEL HPT; ASSUME $INDVAR(x1) \wedge INDVAR(x2) \wedge FORM(f1) \wedge FORM(f2) \wedge SBT(x1,x2,f1,f2)$;

LABEL THTERM; ASSUME $\forall x2.(INDVAR(x2) \supset TERM(x2))$;

$\forall E$ THTERM,x2;

LABEL THNFRO; ASSUME $\forall x1 n1 f1.INTEGER(numbfreeocc(x1,n1,f1))$;

$\forall \bullet$ SUBSTDF0,x1,x2,f1,f2;
 TAUT \rightarrow :#2#2#2#2,HPT:-;
 $\forall E$ -,n1,n1;

LABEL AUX;ASSUME $INTEGER(n1)$;

$\forall E$ THNFRO,x1,n1,f1;

$\forall \bullet$ ARITH1,numbfreeocc(x1,n1,f1),x2; TAUT \rightarrow :#2,HPT:-;substr - in -----;
 $\forall \bullet$ ARITH2,n1;TAUT \rightarrow :#2,HPT:-;SUBSTR-IN ---;

TAUTEQ \rightarrow :#2,HPT:-;
 $\forall \bullet$ SUBDEF0 x1,x2,f1,f2;

$\forall \bullet$ LEMMA1,x2,f2,n1;

$\forall \bullet$ LEMMA2,n1,f1,f2;

TAUTEQ \rightarrow ---:#2#2#1#2[n \leftarrow n1], HPT :-;

\supset I AUX,-;

\forall i -,n1;

TAUTEQ -----:#1, HPT:-;

\supset I HPT,-;

\forall I -,x1,x2,f1,f2;

4.4 Printout of the proof in the earlier axiomatization

1 $\forall n x.((INTEGER(n) \wedge INDVAR(x)) \supset (n * (len(x) - 1)) = 0)$ (1)

2 $\forall n.(INTEGER(n) \supset (0 \cdot n) = n)$ (2)

3 $\forall x.(INDVAR(x) \supset (len(x) - 1) = 0)$ (3)

- 4 $\forall n.(\text{INTEGER}(n) \Rightarrow (n-0)=n)$ (4)
- 5 $\forall x.(\text{INDVAR}(x) \Rightarrow (1 \text{ gl } x)=x)$ (5)
- 6 $\text{INDVAR}(x) \wedge (\text{FORM}(f) \wedge (\text{INTEGER}(n) \wedge \text{SUBT}(x,f,n)))$ (6)
- 7 $\text{INTEGER}(1)$ (7)
- 8 $\text{SUBT}(x,f,n) \equiv (\text{TERM}(x) \wedge (\text{FORM}(f) \wedge (\text{INTEGER}(n) \wedge \forall x_2 k.((\text{INDVAR}(x_2) \wedge (\text{INTEGER}(k) \wedge (k \text{ gl } x) = x_2)) \Rightarrow \text{FRN}(x_2, n - (\text{len}(x) - k), f))))))$
- 9 $\forall x_2 k.((\text{INDVAR}(x_2) \wedge (\text{INTEGER}(k) \wedge (k \text{ gl } x) = x_2)) \Rightarrow \text{FRN}(x_2, n - (\text{len}(x) - k), f))$ (6)
- 10 $(\text{INDVAR}(x) \wedge (\text{INTEGER}(1) \wedge (1 \text{ gl } x) = x)) \Rightarrow \text{FRN}(x, n - (\text{len}(x) - 1), f)$ (6)
- 11 $\text{INDVAR}(x) \Rightarrow (1 \text{ gl } x) = x$ (5)
- 12 $(1 \text{ gl } x) = x$ (5 6 7)
- 13 $(\text{INDVAR}(x) \wedge (\text{INTEGER}(1) \wedge x = x)) \Rightarrow \text{FRN}(x, n - (\text{len}(x) - 1), f)$ (5 6 7)
- 14 $\text{INDVAR}(x) \Rightarrow (\text{len}(x) - 1) = 0$ (3)
- 15 $(\text{len}(x) - 1) = 0$ (3 5 6 7)
- 16 $(\text{INDVAR}(x) \wedge (\text{INTEGER}(1) \wedge x = x)) \Rightarrow \text{FRN}(x, n - 0, f)$ (3 5 6 7)
- 17 $\text{INTEGER}(n) \Rightarrow (n-0)=n$ (4)
- 18 $(n-0)=n$ (3 4 5 6 7)
- 19 $(\text{INDVAR}(x) \wedge (\text{INTEGER}(1) \wedge x = x)) \Rightarrow \text{FRN}(x, n, f)$ (3 4 5 6 7)
- 20 $\text{FRN}(x, n, f)$ (3 4 5 6 7)
- 21 $(\text{INDVAR}(x) \wedge (\text{FORM}(f) \wedge (\text{INTEGER}(n) \wedge \text{SUBT}(x,f,n)))) \Rightarrow \text{FRN}(x, n, f)$ (3 4 5 7)
- 22 $\forall x f n.((\text{INDVAR}(x) \wedge (\text{FORM}(f) \wedge (\text{INTEGER}(n) \wedge \text{SUBT}(x,f,n)))) \Rightarrow \text{FRN}(x, n, f))$ (3 4 5 7)
- 23 $\text{INVART}(k, f_1, k, f_2) \equiv (\text{INTEGER}(k) \wedge (\text{FORM}(f_1) \wedge (\text{INTEGER}(k) \wedge (\text{FORM}(f_2) \wedge ((\text{GEB}(k \text{ gl } f_2, k, f_2) \equiv \text{GEB}(k \text{ gl } f_1, k, f_1)) \wedge ((\text{FRN}(k \text{ gl } f_2, k, f_2) \equiv \text{FRN}(k \text{ gl } f_1, k, f_1)) \wedge (k \text{ gl } f_2) = (k \text{ gl } f_1)))))))$
- 24 $\text{INVARV}(k, f_1, f_2) \equiv (\text{INTEGER}(k) \wedge (\text{FORM}(f_1) \wedge (\text{FORM}(f_2) \wedge ((\text{GEB}(k \text{ gl } f_2, k, f_2) \equiv \text{GEB}(k \text{ gl } f_1, k, f_1)) \wedge ((\text{FRN}(k \text{ gl } f_2, k, f_2) \equiv \text{FRN}(k \text{ gl } f_1, k, f_1)) \wedge (k \text{ gl } f_2) = (k \text{ gl } f_1))))))$
- 25 $\text{INVART}(k, f_1, k, f_2) \equiv \text{INVARV}(k, f_1, f_2)$
- 26 $\forall k f_1 f_2.(\text{INVART}(k, f_1, k, f_2) \equiv \text{INVARV}(k, f_1, f_2))$
- 27 $\text{INDVAR}(x_1) \wedge (\text{INDVAR}(x_2) \wedge (\text{FORM}(f_1) \wedge (\text{FORM}(f_2) \wedge \text{SBT}(x_1, x_2, f_1, f_2))))$ (27)
- 28 $\forall x_2.(\text{INDVAR}(x_2) \Rightarrow \text{TERM}(x_2))$ (28)

- 29 $\text{INDVAR}(x2) \supset \text{TERM}(x2)$ (28)
- 30 $\forall x1\ n1\ f1.\text{INTEGER}(\text{numbfreeocc}(x1,n1,f1))$ (30)
- 31 $\text{SBT}(x1,x2,f1,f2) = ((\text{INDVAR}(x1) \wedge (\text{TERM}(x2) \wedge (\text{FORM}(f1) \wedge \text{FORM}(f2)))) \supset \forall n1\ n2. ((\text{INTEGER}(n1) \wedge (\text{INTEGER}(n2) \wedge n2 = ((\text{numbfreeocc}(x1,n1,f1) * (\text{len}(x2) - 1)) * n1))) \supset ((\neg \text{INDVAR}(n1\ g1\ f1) \supset (n1\ g1\ f1) = (n2\ g1\ f2)) \wedge (\text{INDVAR}(n1\ g1\ f1) \supset ((\text{FRN}(x1,n1,f1) \supset \text{SUBT}(x2,f2,n2)) \wedge (\neg \text{FRN}(x1,n1,f1) \supset \text{INVART}(n1,f1,n2,f2)))))))$
- 32 $\forall n1\ n2. ((\text{INTEGER}(n1) \wedge (\text{INTEGER}(n2) \wedge n2 = ((\text{numbfreeocc}(x1,n1,f1) * (\text{len}(x2) - 1)) * n1))) \supset ((\neg \text{INDVAR}(n1\ g1\ f1) \supset (n1\ g1\ f1) = (n2\ g1\ f2)) \wedge (\text{INDVAR}(n1\ g1\ f1) \supset ((\text{FRN}(x1,n1,f1) \supset \text{SUBT}(x2,f2,n2)) \wedge (\neg \text{FRN}(x1,n1,f1) \supset \text{INVART}(n1,f1,n2,f2))))))$ (27 28 30)
- 33 $(\text{INTEGER}(n1) \wedge (\text{INTEGER}(n1) \wedge n1 = ((\text{numbfreeocc}(x1,n1,f1) * (\text{len}(x2) - 1)) * n1))) \supset ((\neg \text{INDVAR}(n1\ g1\ f1) \supset (n1\ g1\ f1) = (n1\ g1\ f2)) \wedge (\text{INDVAR}(n1\ g1\ f1) \supset ((\text{FRN}(x1,n1,f1) \supset \text{SUBT}(x2,f2,n1)) \wedge (\neg \text{FRN}(x1,n1,f1) \supset \text{INVART}(n1,f1,n1,f2))))))$ (27 28 30)
- 34 $\text{INTEGER}(n1)$ (34)
- 35 $\text{INTEGER}(\text{numbfreeocc}(x1,n1,f1))$ (30)
- 36 $(\text{INTEGER}(\text{numbfreeocc}(x1,n1,f1)) \wedge \text{INDVAR}(x2)) \supset (\text{numbfreeocc}(x1,n1,f1) * (\text{len}(x2) - 1)) = 0$ (1)
- 37 $(\text{numbfreeocc}(x1,n1,f1) * (\text{len}(x2) - 1)) = 0$ (1 27 28 30 34)
- 38 $(\text{INTEGER}(n1) \wedge (\text{INTEGER}(n1) \wedge n1 = (0 * n1))) \supset ((\neg \text{INDVAR}(n1\ g1\ f1) \supset (n1\ g1\ f1) = (n1\ g1\ f2)) \wedge (\text{INDVAR}(n1\ g1\ f1) \supset ((\text{FRN}(x1,n1,f1) \supset \text{SUBT}(x2,f2,n1)) \wedge (\neg \text{FRN}(x1,n1,f1) \supset \text{INVART}(n1,f1,n1,f2))))))$ (1 27 28 30 34)
- 39 $\text{INTEGER}(n1) \supset (0 * n1) = n1$ (2)
- 40 $(0 * n1) = n1$ (1 2 27 28 30 34)
- 41 $(\text{INTEGER}(n1) \wedge (\text{INTEGER}(n1) \wedge n1 = n1)) \supset ((\neg \text{INDVAR}(n1\ g1\ f1) \supset (n1\ g1\ f1) = (n1\ g1\ f2)) \wedge (\text{INDVAR}(n1\ g1\ f1) \supset ((\text{FRN}(x1,n1,f1) \supset \text{SUBT}(x2,f2,n1)) \wedge (\neg \text{FRN}(x1,n1,f1) \supset \text{INVART}(n1,f1,n1,f2))))))$ (1 2 27 28 30 34)
- 42 $(\neg \text{INDVAR}(n1\ g1\ f1) \supset (n1\ g1\ f1) = (n1\ g1\ f2)) \wedge (\text{INDVAR}(n1\ g1\ f1) \supset ((\text{FRN}(x1,n1,f1) \supset \text{SUBT}(x2,f2,n1)) \wedge (\neg \text{FRN}(x1,n1,f1) \supset \text{INVART}(n1,f1,n1,f2))))$ (1 2 27 28 30 34)
- 43 $\text{SBV}(x1,x2,f1,f2) = ((\text{INDVAR}(x1) \wedge (\text{INDVAR}(x2) \wedge (\text{FORM}(f1) \wedge \text{FORM}(f2)))) \supset \forall n. (\text{INTEGER}(n) \supset ((\neg \text{INDVAR}(n\ g1\ f1) \supset (n\ g1\ f1) = (n\ g1\ f2)) \wedge (\text{INDVAR}(n\ g1\ f1) \supset ((\text{FRN}(x1,n,f1) \supset \text{FRN}(x2,n,f2)) \wedge (\neg \text{FRN}(x1,n,f1) \supset \text{INVARV}(n,f1,f2)))))))$
- 44 $(\text{INDVAR}(x2) \wedge (\text{FORM}(f2) \wedge (\text{INTEGER}(n1) \wedge \text{SUBT}(x2,f2,n1)))) \supset \text{FRN}(x2,n1,f2)$ (3 4 5 7)
- 45 $\text{INVART}(n1,f1,n1,f2) = \text{INVARV}(n1,f1,f2)$
- 46 $(\neg \text{INDVAR}(n1\ g1\ f1) \supset (n1\ g1\ f1) = (n1\ g1\ f2)) \wedge (\text{INDVAR}(n1\ g1\ f1) \supset ((\text{FRN}(x1,n1,f1) \supset \text{FRN}(x2,n1,f2)) \wedge (\neg \text{FRN}(x1,n1,f1) \supset \text{INVARV}(n1,f1,f2))))$ (1 2 3 4 5 7 27 28 30 34)
- 47 $\text{INTEGER}(n1) \supset ((\neg \text{INDVAR}(n1\ g1\ f1) \supset (n1\ g1\ f1) = (n1\ g1\ f2)) \wedge (\text{INDVAR}(n1\ g1\ f1) \supset ((\text{FRN}(x1,n1,f1) \supset \text{FRN}(x2,n1,f2)) \wedge (\neg \text{FRN}(x1,n1,f1) \supset \text{INVARV}(n1,f1,f2))))))$ (1 2 3 4 5

7 27 28 30)

- 48 $\forall n1. (\text{INTEGER}(n1) \supset ((\neg \text{INDVAR}(n1 \text{ gl } f1) \supset (n1 \text{ gl } f1) = (n1 \text{ gl } f2)) \wedge (\text{INDVAR}(n1 \text{ gl } f1) \supset ((\text{FRN}(x1, n1, f1) \supset \text{FRN}(x2, n1, f2)) \wedge (\neg \text{FRN}(x1, n1, f1) \supset \text{INVARV}(n1, f1, f2))))))$ (1 2 3 4 5 7 27 28 30)
- 49 $\text{SBV}(x1, x2, f1, f2)$ (1 2 3 4 5 7 27 28 30 34)
- 50 $(\text{INDVAR}(x1) \wedge (\text{INDVAR}(x2) \wedge (\text{FORM}(f1) \wedge (\text{FORM}(f2) \wedge \text{SBT}(x1, x2, f1, f2)))))) \supset \text{SBV}(x1, x2, f1, f2)$ (1 2 3 4 5 7 28 30 34)
- 51 $\forall x1 \ x2 \ f1 \ f2. ((\text{INDVAR}(x1) \wedge (\text{INDVAR}(x2) \wedge (\text{FORM}(f1) \wedge (\text{FORM}(f2) \wedge \text{SBT}(x1, x2, f1, f2)))))) \supset \text{SBV}(x1, x2, f1, f2))$ (1 2 3 4 5 7 28 30)

APPENDIX 5

THE PROOF THAT UNIVERSAL QUANTIFIER CAN BE INTERCHANGED

5.1 FOL commands for the main lemma in the many sorted logic

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LABEL TH1; ASSUME  $\forall x_1 x_2 f_1 f_2. (SBT(x_1, x_2, f_1, f_2) \supset SBV(x_1, x_2, f_1, f_2));$ 
 $\forall e$  TH1,  $x, x, f_1, sbt(x, x, f_1);$ 
 $\forall e$  SUBSTDF3  $x, x, f_1, sbt(x, x, f_1);$ 
 $\forall e$  SUBDEFC  $x, x, f_1, sbt(x, x, f_1);$ 
tauteq  $:=#2, 1:=;$ 
 $\forall e$   $\neg, n;$ 
 $\forall e$  FREEVO,  $x, n, f_1;$ 
 $\forall e$  FREEVO,  $x, n, sbt(x, x, f_1);$ 
 $\forall e$  SUBDEF1  $n, f_1, sbt(x, x, f_1);$ 
tauteq  $(n \text{ gl } f_1) = (n \text{ gl } sbt(x, x, f_1)), 11, 17, 18;$ 
 $\forall i$   $\neg, n;$ 
 $\forall e$  EQS  $f_1, sbt(x, x, f_1);$ 
tauteq  $sbt(x, x, f_1) = f_1, \neg, \neg;$ 
 $\forall i$   $\neg, x, f_1 \leftarrow i;$ 

```

5.2 Printout of the proof in the many sorted logic

- 1 $\forall x_1 x_2 f_1 f_2. (SBT(x_1, x_2, f_1, f_2) \supset SBV(x_1, x_2, f_1, f_2)) \quad (1)$
- 2 $SBT(x, x, f_1, sbt(x, x, f_1)) = SBV(x, x, f_1, sbt(x, x, f_1)) \quad (1)$
- 3 $SBT(x, x, f_1, sbt(x, x, f_1)) = sbt(x, x, f_1) = sbt(x, x, f_1)$
- 4 $SBV(x, x, f_1, sbt(x, x, f_1)) = \forall n. ((\neg \text{INDVAR}(n \text{ gl } f_1) \supset (n \text{ gl } f_1) = (n \text{ gl } sbt(x, x, f_1))) \wedge (\text{INDVAR}(n \text{ gl } f_1) \supset ((\text{FRN}(x, n, f_1) \supset \text{FRN}(x, n, sbt(x, x, f_1))) \wedge (\neg \text{FRN}(x, n, f_1) \supset \text{INVARV}(n, f_1, sbt(x, x, f_1)))))$
- 5 $\forall n. ((\neg \text{INDVAR}(n \text{ gl } f_1) \supset (n \text{ gl } f_1) = (n \text{ gl } sbt(x, x, f_1))) \wedge (\text{INDVAR}(n \text{ gl } f_1) \supset ((\text{FRN}(x, n, f_1) \supset \text{FRN}(x, n, sbt(x, x, f_1))) \wedge (\neg \text{FRN}(x, n, f_1) \supset \text{INVARV}(n, f_1, sbt(x, x, f_1))))) \quad (1)$
- 6 $(\neg \text{INDVAR}(n \text{ gl } f_1) \supset (n \text{ gl } f_1) = (n \text{ gl } sbt(x, x, f_1))) \wedge (\text{INDVAR}(n \text{ gl } f_1) \supset ((\text{FRN}(x, n, f_1) \supset \text{FRN}(x, n, sbt(x, x, f_1))) \wedge (\neg \text{FRN}(x, n, f_1) \supset \text{INVARV}(n, f_1, sbt(x, x, f_1))))) \quad (1)$
- 7 $\text{FRN}(x, n, f_1) = (x = (n \text{ gl } f_1) \wedge \text{GEB}(x, n, f_1)) \quad \forall e$ FREEVO x, n, f_1
- 8 $\text{FRN}(x, n, sbt(x, x, f_1)) = (x = (n \text{ gl } sbt(x, x, f_1)) \wedge \text{GEB}(x, n, sbt(x, x, f_1)))$
- 9 $\text{INVARV}(n, f_1, sbt(x, x, f_1)) = ((\text{GEB}(n \text{ gl } sbt(x, x, f_1), n, sbt(x, x, f_1)) = \text{GEB}(n \text{ gl } f_1, n, f_1)) \wedge ((\text{FRN}(n \text{ gl } sbt(x, x, f_1), n, sbt(x, x, f_1)) = \text{FRN}(n \text{ gl } f_1, n, f_1)) \wedge (n \text{ gl } sbt(x, x, f_1) = (n \text{ gl } f_1)))$
- 10 $(n \text{ gl } f_1) = (n \text{ gl } sbt(x, x, f_1)) \quad (1)$

- 11 $\forall n.(n \text{ gl } f1) = (n \text{ gl } \text{sbl}(x,x,f1)) \text{ (1)}$
 12 $\forall n.(n \text{ gl } f1) = (n \text{ gl } \text{sbl}(x,x,f1)) = f1 = \text{sbl}(x,x,f1)$
 13 $\text{sbl}(x,x,f1) = f1 \text{ (1)}$
 14 $\forall x f.\text{sbl}(x,x,f) = f \text{ (1)}$

5.3 FOI commands for the theorem in the many sorted logic

LABEL FIRSTLEMMA;
 ASSUME $\forall x f.\text{sbl}(x,x,f) = f$;

LABEL THEON1;
 ASSUME $\forall f \text{ sq}.\text{scar}(f \text{ cc } \text{sq}) = f$;
 LABEL THEON2;
 ASSUME $\forall f \text{ sq}.\text{schr}(f \text{ cc } \text{sq}) = \text{sq}$;

Proof of the Lemma: $\text{BEW}(x \text{ gen } f) \supset \text{BEW}(f)$

LABEL HPT;
 ASSUME $\text{BEW}(x \text{ gen } f)$;

LABEL THTAUT;
 \forall FIRSTLEMMA x, f ;

\forall PROVABLE $x \text{ gen } f$;
 TAUT $-\text{:}\#2, -, \text{HPT}$;
 LABEL HPAUX;
 $\exists - , \text{sq}$;

\forall GENRULO $f \text{ cc } \text{sq}, \text{sq}, x, x$;
 LABEL THN1;
 \forall THEON1 f, sq ;
 \forall THEON2 f, sq ;
 TAUTEQ $---\text{:}\#2\#2\#2\#1\{f1 \leftarrow f\}, 1\text{:}-$;
 UNIFY $----\text{:}\#2\#2\#2, -$;
 TAUTEQ $-----\text{:}\#1, 1\text{:}-$;

\forall PROOF $f \text{ cc } \text{sq}$;
 LABEL GENI1;
 \forall GENI($f \text{ cc } \text{sq}, \text{sq}, x, x$) , $--$, EXI($f \text{ cc } \text{sq}, \text{sq}, x, x$) ;
 UNIFY $---\text{:}\#2\#2\#2\#1, -$;
 LABEL PROOFTR;
 TAUT $---\text{:}\#1, 1\text{:}-$;

\wedge HPAUX $\text{:}\#2\#2$;
 $\forall - , f1$;

```

V0 DEPENDO f cc sq, sq, f1;
UNIFY --:#2#2#2#2#2, GENE1 ;

TAUTEQ DEPEND(f cc sq, f1) => AXIOM (f1) ,1:-;
Vi -, f1←f1;
TAUTEQ THN1:#2 = THN1:#1 ,THN1;
^i PROOFTR, -, -- ;
LABEL USEFUL;
V0 PROVABLE f;
UNIFY --:#2 ,--;
TAUT --:#1,1:-;
LABEL CITHI;
=>I HPT,-;
Proof of the Lemma: BEW(f) => BEW(x gen f)

LABEL HPT1;
ASSUME BEW(f);

TAUT USEFUL:#2 , -,HPT1,USEFUL;
3e -,sq;

^e --:#2#2;
V0 -, f1;
V0 GENRUL2 x,sq;
V0 THEORY x,f1;
TAUTEQ --:#2#1#1[f←f1] ,HPT1:-;
Vi -, f1←f1;
TAUT ----:#1 ,HPT1:-;

V0 GENRUL1 ((x gen f) cc sq) , sq ,x,x ;
LABEL THN2;
V0 THEON1 x gen f , sq ;
V0 THEON2 x gen f , sq ;
TAUTEQ ----:#2#2#2#1[f ← f] , THTAUT,HPT1:-;
UNIFY ----:#2#2#2 , -;
TAUTEQ ----:#1 , THTAUT,HPT1:-;

V0 PROOF (x gen f) cc sq ;
LABEL GEN1;
Vi -- , GENE((x gen f) cc sq,sq,x,x) , EXI((x gen f) cc sq,sq,x,x) ;
UNIFY --:#2#2#2#1 , - ;

LABEL PROOFTR1;
TAUT ----:#1 , HPT1:-,THTAUT;

V0 DEPENDO (x gen f) cc sq, sq, f1;
3I GEN1 ,x←f OCC 3 6 9,x←x1 OCC 2 4 6;

TAUTEQ DEPEND((x gen f) cc sq, f1) => AXIOM (f1) ,THTAUT,HPT1:-;

```

```

Vi -,f1←f1;
TAUTEQ THN2:≡2 = THN2:≡1 ,THN2;
Λi PROOFTRI, -, -- ;
V• PROVABLE x gen f;
UNIFY -:≡2 ,--;
TAUT --:≡1 ,THTAUT,HPT1:-;
LABEL C2TH1;
≡I HPT1,-;
≡I C1TH1,C2TH1;
LABEL TH1;
Vl -,x,f;
V• TH1 x1,x2 gen f;
V• TH1 x2,f;
V• TH1 x1,f;
V• TH1 x2,x1 gen f;
TAUT ----:≡1 ≡ -:≡1 , TH1:-;
Vi -,x1,x2,f;

```

5.4 Printout of the proof of the theorem in the many sorted logic

- 1 $\forall x f.sbt(x,x,f)=f$ (1)
- 2 $\forall f sq.scar(f cc sq)=f$ (2)
- 3 $\forall f sq.scdr(f cc sq)=sq$ (3)
- 4 BEW(x gen f) (4)
- 5 $sbt(x,x,f)=f$ (1)
- 6 BEW(x gen f)= $\exists sq$ (PROOFTREE(sq) \wedge ((x gen f)= $scar(sq) \wedge \forall f1.(DEPEND(sq,f1) \supset AXIOM(f1))$)))
- 7 $\exists sq.(PROOFTREE(sq) \wedge ((x gen f)= $scar(sq) \wedge \forall f1.(DEPEND(sq,f1) \supset AXIOM(f1))$)))$ (4)
- 8 PROOFTREE(sq) \wedge ((x gen f)= $scar(sq) \wedge \forall f1.(DEPEND(sq,f1) \supset AXIOM(f1))$) (8)
- 9 GENE(f cc sq,sq,x,x) \neg ($scdr(f cc sq)=sq \wedge (PROOFTREE(sq) \wedge \exists f1.(scar(sq)=(x gen f1) \wedge scar(f cc sq)=sbt(x,x,f1))$)))
- 10 $scar(f cc sq)=f$ (2)
- 11 $scdr(f cc sq)=sq$ (3)
- 12 $scar(sq)=(x gen f) \wedge scar(f cc sq)=sbt(x,x,f)$ (1 2 3 4 8)
- 13 $\exists f1.(scar(sq)=(x gen f1) \wedge scar(f cc sq)=sbt(x,x,f1))$ (1 2 3 4 8)
- 14 GENE(f cc sq,sq,x,x) (1 2 3 4 8)
- 15 PROOFTREE(f cc sq)=(FORM(f cc sq) \vee ($\exists pf.(ORI(f cc sq,pf) \vee (ANDE(f cc sq,pf) \vee (FALSEE(f cc sq,pf) \vee (NOTI(f cc sq,pf) \vee (NOTE(f cc sq,pf) \vee IMPLI(f cc sq,pf))))))$))) \vee

$(\exists pf\ x\ t.(GENI(f\ cc\ sq, pf, x, t) \vee (GENE(f\ cc\ sq, pf, x, t) \vee EXI(f\ cc\ sq, pf, x, t))) \vee$
 $(\exists pf1\ pf2.(ANDI(f\ cc\ sq, pf1, pf2) \vee (FALSEI(f\ cc\ sq, pf1, pf2) \vee IMPLE(f\ cc\ sq, pf1, pf2))) \vee$
 $\exists pf1\ pf2\ x\ t.EXE(f\ cc\ sq, pf1, pf2, x, t) \vee \exists pf1\ pf2\ pf3.ORE(f\ cc\ sq, pf1, pf2, pf3))))))$

- 16 $GENI(f\ cc\ sq, sq, x, x) \vee (GENE(f\ cc\ sq, sq, x, x) \vee EXI(f\ cc\ sq, sq, x, x))$ (1 2 3 4 8)
- 17 $\exists pf\ x\ t.(GENI(f\ cc\ sq, pf, x, t) \vee (GENE(f\ cc\ sq, pf, x, t) \vee EXI(f\ cc\ sq, pf, x, t)))$ (1 2 3 4 8)
- 18 $PROOFTREE(f\ cc\ sq)$ (1 2 3 4 8)
- 19 $\forall f1.(DEPEND(sq, f1) \supset AXIOM(f1))$ (8)
- 20 $DEPEND(sq, f1) \supset AXIOM(f1)$ (8)
- 21 $PROOFTREE(f\ cc\ sq) \supset (PROOFTREE(sq) \supset ((sq = scdr(f\ cc\ sq) \supset (DEPEND(f\ cc\ sq, f1) \supset$
 $DEPEND(sq, f1))) \supset (ORI(f\ cc\ sq, sq) \vee (ANDE(f\ cc\ sq, sq) \vee (FALSEE(f\ cc\ sq, sq) \vee$
 $(\exists f1.((NOTID(f\ cc\ sq, sq, f) \vee (NOTED(f\ cc\ sq, sq, f) \vee IMPLID(f\ cc\ sq, sq, f))) \wedge f \neq f1) \vee$
 $\exists x\ t.(GENI(f\ cc\ sq, sq, x, t) \vee (GENE(f\ cc\ sq, sq, x, t) \vee EXI(f\ cc\ sq, sq, x, t))))))))))$
- 22 $\exists x\ t.(GENI(f\ cc\ sq, sq, x, t) \vee (GENE(f\ cc\ sq, sq, x, t) \vee EXI(f\ cc\ sq, sq, x, t)))$ (1 2 3 4 8)
- 23 $DEPEND(f\ cc\ sq, f1) \supset AXIOM(f1)$ (1 2 3 4 8)
- 24 $\forall f1.(DEPEND(f\ cc\ sq, f1) \supset AXIOM(f1))$ (1 2 3 4 8)
- 25 $f = scar(f\ cc\ sq)$ (2)
- 26 $PROOFTREE(f\ cc\ sq) \wedge (f = scar(f\ cc\ sq) \wedge \forall f1.(DEPEND(f\ cc\ sq, f1) \supset AXIOM(f1)))$ (1 2 3 4 8)
- 27 $BEW(f) \equiv \exists sq.(PROOFTREE(sq) \wedge (f = scar(sq) \wedge \forall f1.(DEPEND(sq, f1) \supset AXIOM(f1))))$
- 28 $\exists sq.(PROOFTREE(sq) \wedge (f = scar(sq) \wedge \forall f1.(DEPEND(sq, f1) \supset AXIOM(f1))))$ (1 2 3 4)
- 29 $BEW(f)$ (1 2 3 4)
- 30 $BEW(x\ gen\ f) \supset BEW(f)$ (1 2 3)
- 31 $BEW(f)$ (31)
- 32 $\exists sq.(PROOFTREE(sq) \wedge (f = scar(sq) \wedge \forall f1.(DEPEND(sq, f1) \supset AXIOM(f1))))$ (31)
- 33 $PROOFTREE(sq) \wedge (f = scar(sq) \wedge \forall f1.(DEPEND(sq, f1) \supset AXIOM(f1)))$ (33)
- 34 $\forall f1.(DEPEND(sq, f1) \supset AXIOM(f1))$ (33)
- 35 $DEPEND(sq, f1) \supset AXIOM(f1)$ (33)
- 36 $APGENI(x, sq) \equiv (\forall f.(DEPEND(sq, f) \supset \neg FR(x, f)) \wedge PROOFTREE(sq))$
- 37 $AXIOM(f1) \supset \neg FR(x, f1)$
- 38 $DEPEND(sq, f1) \supset \neg FR(x, f1)$ (31 33)

- 39 $\forall f1. (DEPEND(sq, f1) \supset \neg FR(x, f1))$ (31 33)
- 40 $APGENI(x, sq)$ (31 33)
- 41 $GENI((x \text{ gen } f) \text{ cc } sq, sq, x, x) = (scdr((x \text{ gen } f) \text{ cc } sq) = sq \wedge (PROOFTREE(sq) \wedge \exists f1. (scar((x \text{ gen } f) \text{ cc } sq) = (x \text{ gen } f1) \wedge (scar(sq) = sbl(x, x, f1) \wedge APGENI(x, sq))))))$
- 42 $scar((x \text{ gen } f) \text{ cc } sq) = (x \text{ gen } f)$ (2)
- 43 $scdr((x \text{ gen } f) \text{ cc } sq) = sq$ (3)
- 44 $scar((x \text{ gen } f) \text{ cc } sq) = (x \text{ gen } f) \wedge (scar(sq) = sbl(x, x, f) \wedge APGENI(x, sq))$ (1 2 3 31 33)
- 45 $\exists f1. (scar((x \text{ gen } f) \text{ cc } sq) = (x \text{ gen } f1) \wedge (scar(sq) = sbl(x, x, f1) \wedge APGENI(x, sq)))$ (1 2 3 31 33)
- 46 $GENI((x \text{ gen } f) \text{ cc } sq, sq, x, x)$ (1 2 3 31 33)
- 47 $PROOFTREE((x \text{ gen } f) \text{ cc } sq) = (FORM((x \text{ gen } f) \text{ cc } sq) \vee (\exists pf. (ORI((x \text{ gen } f) \text{ cc } sq, pf) \vee (ANDE((x \text{ gen } f) \text{ cc } sq, pf) \vee (FALSEE((x \text{ gen } f) \text{ cc } sq, pf) \vee (NOTI((x \text{ gen } f) \text{ cc } sq, pf) \vee (NOTE((x \text{ gen } f) \text{ cc } sq, pf) \vee IMPLI((x \text{ gen } f) \text{ cc } sq, pf)))))) \vee (\exists pf \ x1 \ f. (GENI((x \text{ gen } f) \text{ cc } sq, pf, x1, t) \vee (GENE((x \text{ gen } f) \text{ cc } sq, pf, x1, t) \vee EXI((x \text{ gen } f) \text{ cc } sq, pf, x1, t))) \vee (\exists pf1 \ pf2. (ANDI((x \text{ gen } f) \text{ cc } sq, pf1, pf2) \vee (FALSEI((x \text{ gen } f) \text{ cc } sq, pf1, pf2) \vee IMPLI((x \text{ gen } f) \text{ cc } sq, pf1, pf2))) \vee (\exists pf1 \ pf2 \ x1 \ f EXE((x \text{ gen } f) \text{ cc } sq, pf1, pf2, x1, t) \vee \exists pf1 \ pf2 \ pf3. ORE((x \text{ gen } f) \text{ cc } sq, pf1, pf2, pf3))))))$
- 48 $GENI((x \text{ gen } f) \text{ cc } sq, sq, x, x) \vee (GENE((x \text{ gen } f) \text{ cc } sq, sq, x, x) \vee EXI((x \text{ gen } f) \text{ cc } sq, sq, x, x))$ (1 2 3 31 33)
- 49 $\exists pf \ x1 \ f. (GENI((x \text{ gen } f) \text{ cc } sq, pf, x1, t) \vee (GENE((x \text{ gen } f) \text{ cc } sq, pf, x1, f) \vee EXI((x \text{ gen } f) \text{ cc } sq, pf, x1, t)))$ (1 2 3 31 33)
- 50 $PROOFTREE((x \text{ gen } f) \text{ cc } sq)$ (1 2 3 31 33)
- 51 $PROOFTREE((x \text{ gen } f) \text{ cc } sq) \supset (PROOFTREE(sq) \supset ((sq = scdr((x \text{ gen } f) \text{ cc } sq) \supset (DEPEND((x \text{ gen } f) \text{ cc } sq, f1) \text{ cc } sq, f1) \text{ cc } sq, f1))) \vee (ORI((x \text{ gen } f) \text{ cc } sq, sq) \vee (ANDE((x \text{ gen } f) \text{ cc } sq, sq) \vee (FALSEE((x \text{ gen } f) \text{ cc } sq, sq) \vee (\exists f1. ((NOTID((x \text{ gen } f) \text{ cc } sq, sq, f1) \vee (NOTED((x \text{ gen } f) \text{ cc } sq, sq, f1) \vee IMPLID((x \text{ gen } f) \text{ cc } sq, sq, f1))) \wedge f1))) \vee (\exists x1 \ f. (GENI((x \text{ gen } f) \text{ cc } sq, sq, x1, t) \vee (GENE((x \text{ gen } f) \text{ cc } sq, sq, x1, t) \vee EXI((x \text{ gen } f) \text{ cc } sq, sq, x1, t))))))$
- 52 $\exists x1 \ f. (GENI((x \text{ gen } f) \text{ cc } sq, sq, x1, t) \vee (GENE((x \text{ gen } f) \text{ cc } sq, sq, x1, f) \vee EXI((x \text{ gen } f) \text{ cc } sq, sq, x1, t)))$ (1 2 3 31 33)
- 53 $DEPEND((x \text{ gen } f) \text{ cc } sq, f1) \supset AXIOM(f1)$ (1 2 3 31 33)
- 54 $\forall f1. (DEPEND((x \text{ gen } f) \text{ cc } sq, f1) \supset AXIOM(f1))$ (1 2 3 31 33)
- 55 $(x \text{ gen } f) = scar((x \text{ gen } f) \text{ cc } sq)$ (2)
- 56 $PROOFTREE((x \text{ gen } f) \text{ cc } sq) \wedge ((x \text{ gen } f) = scar((x \text{ gen } f) \text{ cc } sq) \wedge \forall f1. (DEPEND((x \text{ gen } f) \text{ cc } sq, f1) \supset AXIOM(f1)))$ (1 2 3 31 33)
- 57 $BEW(x \text{ gen } f) = \exists sq. (PROOFTREE(sq) \wedge ((x \text{ gen } f) = scar(sq) \wedge \forall f1. (DEPEND(sq, f1) \supset AXIOM(f1))))$

- 58 $\exists sq.(PROOFTREE(sq) \wedge ((x \text{ gen } f) = \text{scar}(sq) \wedge \forall f1.(DEPEND(sq, f1) \Rightarrow AXIOM(f1))))$ (1 2 3 31)
- 59 $BEW(x \text{ gen } f)$ (1 2 3 31)
- 60 $BEW(f) \supset BEW(x \text{ gen } f)$ (1 2 3)
- 61 $BEW(x \text{ gen } f) = BEW(f)$ (1 2 3)
- 62 $\forall x f.(BEW(x \text{ gen } f) = BEW(f))$ (1 2 3)
- 63 $BEW(x1 \text{ gen } (x2 \text{ gen } f)) = BEW(x2 \text{ gen } f)$ (1 2 3)
- 64 $BEW(x2 \text{ gen } f) = BEW(f)$ (1 2 3)
- 65 $BEW(x1 \text{ gen } f) = BEW(f)$ (1 2 3)
- 66 $BEW(x2 \text{ gen } (x1 \text{ gen } f)) = BEW(x1 \text{ gen } f)$ (1 2 3)
- 67 $BEW(x1 \text{ gen } (x2 \text{ gen } f)) \supset BEW(x2 \text{ gen } (x1 \text{ gen } f))$ (1 2 3)
- 68 $\forall x1 x2 f.(BEW(x1 \text{ gen } (x2 \text{ gen } f)) \supset BEW(x2 \text{ gen } (x1 \text{ gen } f)))$ (1 2 3)

5.5 FOL commands for the main lemma in the earlier axiomatization

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LABEL HPT; ASSUME INDVAR(x) ^ FORM(f1);
LABEL TH1; ASSUME  $\forall x1 x2 f1 f2.((INDVAR(x1) \wedge INDVAR(x2) \wedge FORM(f1) \wedge FORM(f2) \wedge$ 
       $SBT(x1,x2,f1,f2)) \supset SBV(x1,x2,f1,f2));$ 
LABEL TH2; ASSUME  $\forall x.(INDVAR(x) \supset TERM(x));$ 
LABEL TH3; ASSUME  $\forall x.(FORM(x) \supset STRING(x));$ 
 $\forall e$  TH1, x,x,f1,sbt(x,x,f1);
 $\forall e$  TH2, x;
 $\forall e$  TH3, f1;
 $\forall e$  TH3, sbt(x,x,f1);
 $\forall e$  SUBSTDF3 x,x, f1, sbt(x,x,f1);
 $\forall e$  SUBSTDF4 x,x, f1;
 $\forall e$  SUBDEFO x, x,f1,sbt(x,x,f1);
tauteq  $-\#2\#2,1:-;$ 
 $\forall e$  -,n;
 $\forall e$  FREEVO, x, n, f1;
 $\forall e$  FREEVO, x, n, sbt(x,x,f1);
 $\forall e$  SUBDEF1 n, f1, sbt(x,x,f1);
tauteq INTEGER(n)  $\supset ((n \text{ gl } f1) = (n \text{ gl } sbt(x,x,f1)))$  1:-;
 $\forall i$  -,n;
 $\forall e$  EQS, f1,sbt(x,x,f1);
taut  $-\#2\#2,1:-;$ 
 $\supset i,1:-;$ 
 $\forall i$  -,x,f1-f;

```

5.6 Printout of the proof of the main lemma in the second axiomatization

- 1 $\text{INDVAR}(x) \wedge \text{FORM}(f1)$ (1) ASSUME
- 2 $\forall x1 x2 f1 f2. ((\text{INDVAR}(x1) \wedge (\text{INDVAR}(x2) \wedge (\text{FORM}(f1) \wedge (\text{FORM}(f2) \wedge \text{SBT}(x1, x2, f1, f2)))))) \Rightarrow \text{SBV}(x1, x2, f1, f2))$ (2) ASSUME
- 3 $\forall x. (\text{INDVAR}(x) \Rightarrow \text{TERM}(x))$ (3) ASSUME
- 4 $\forall x. (\text{FORM}(x) \Rightarrow \text{STRING}(x))$ (4) ASSUME
- 5 $(\text{INDVAR}(x) \wedge (\text{INDVAR}(x) \wedge (\text{FORM}(f1) \wedge (\text{FORM}(\text{sbt}(x, x, f1)) \wedge \text{SBT}(x, x, f1, \text{sbt}(x, x, f1)))))) \Rightarrow \text{SBV}(x, x, f1, \text{sbt}(x, x, f1))$ (2) $\forall E$ 2 $x, x, f1, \text{sbt}(x, x, f1)$
- 6 $\text{INDVAR}(x) \Rightarrow \text{TERM}(x)$ (3) $\forall E$ 3 x
- 7 $\text{FORM}(f1) \Rightarrow \text{STRING}(f1)$ (4) $\forall E$ 4 $f1$
- 8 $\text{FORM}(\text{sbt}(x, x, f1)) \Rightarrow \text{STRING}(\text{sbt}(x, x, f1))$ (4) $\forall E$ 4 $\text{sbt}(x, x, f1)$
- 9 $(\text{INDVAR}(x) \wedge (\text{TERM}(x) \wedge (\text{FORM}(f1) \wedge \text{FORM}(\text{sbt}(x, x, f1)))) \Rightarrow (\text{SBT}(x, x, f1, \text{sbt}(x, x, f1)) \Rightarrow \text{sbt}(x, x, f1) = \text{sbt}(x, x, f1))$ $\forall E$ SUBSTDF3 $x, x, f1, \text{sbt}(x, x, f1)$
- 10 $(\text{INDVAR}(x) \wedge (\text{TERM}(x) \wedge \text{FORM}(f1))) \Rightarrow \text{FORM}(\text{sbt}(x, x, f1))$ $\forall E$ SUBSTDF4 $x, x, f1$
- 11 $\text{SBV}(x, x, f1, \text{sbt}(x, x, f1)) ((\text{INDVAR}(x) \wedge (\text{INDVAR}(x) \wedge (\text{FORM}(f1) \wedge \text{FORM}(\text{sbt}(x, x, f1)))) \Rightarrow \forall n. (\text{INTEGER}(n) \Rightarrow ((\neg \text{INDVAR}(n \text{ gl } f1) \Rightarrow (n \text{ gl } f1) = (n \text{ gl } \text{sbt}(x, x, f1))) \wedge (\text{INDVAR}(n \text{ gl } f1) \Rightarrow ((\text{FRN}(x, n, f1) \Rightarrow \text{FRN}(x, n, \text{sbt}(x, x, f1))) \wedge (\neg \text{FRN}(x, n, f1) \Rightarrow \text{INVARV}(n, f1, \text{sbt}(x, x, f1)))))))$ $\forall E$ SUBDEFO $x, x, f1, \text{sbt}(x, x, f1)$
- 12 $\forall n. (\text{INTEGER}(n) \Rightarrow ((\neg \text{INDVAR}(n \text{ gl } f1) \Rightarrow (n \text{ gl } f1) = (n \text{ gl } \text{sbt}(x, x, f1))) \wedge (\text{INDVAR}(n \text{ gl } f1) \Rightarrow ((\text{FRN}(x, n, f1) \Rightarrow \text{FRN}(x, n, \text{sbt}(x, x, f1))) \wedge (\neg \text{FRN}(x, n, f1) \Rightarrow \text{INVARV}(n, f1, \text{sbt}(x, x, f1)))))))$ (1 2 3 4) 1 : 11
- 13 $\text{INTEGER}(n) \Rightarrow ((\neg \text{INDVAR}(n \text{ gl } f1) \Rightarrow (n \text{ gl } f1) = (n \text{ gl } \text{sbt}(x, x, f1))) \wedge (\text{INDVAR}(n \text{ gl } f1) \Rightarrow ((\text{FRN}(x, n, f1) \Rightarrow \text{FRN}(x, n, \text{sbt}(x, x, f1))) \wedge (\neg \text{FRN}(x, n, f1) \Rightarrow \text{INVARV}(n, f1, \text{sbt}(x, x, f1)))))))$ (1 2 3 4) $\forall E$ 12 n
- 14 $\text{FRN}(x, n, f1) = (x = (n \text{ gl } f1) \wedge \neg \text{GEB}(x, n, f1))$ $\forall E$ FREEVO $x, n, f1$
- 15 $\text{FRN}(x, n, \text{sbt}(x, x, f1)) = (x = (n \text{ gl } \text{sbt}(x, x, f1)) \wedge \neg \text{GEB}(x, n, \text{sbt}(x, x, f1)))$ $\forall E$ FREEVO $x, n, \text{sbt}(x, x, f1)$
- 16 $\text{INVARV}(n, f1, \text{sbt}(x, x, f1)) = (\text{INTEGER}(n) \wedge (\text{FORM}(f1) \wedge (\text{FORM}(\text{sbt}(x, x, f1)) \wedge ((\text{GEB}(n \text{ gl } \text{sbt}(x, x, f1), n, \text{sbt}(x, x, f1)) = \text{GEB}(n \text{ gl } f1, n, f1) \wedge ((\text{FRN}(n \text{ gl } \text{sbt}(x, x, f1), n, \text{sbt}(x, x, f1)) = \text{FRN}(n \text{ gl } f1, n, f1) \wedge (n \text{ gl } \text{sbt}(x, x, f1) = n \text{ gl } f1)))))))$ $\forall E$ SUBDEF1 $n, f1, \text{sbt}(x, x, f1)$
- 17 $\text{INTEGER}(n) \Rightarrow (n \text{ gl } f1) = (n \text{ gl } \text{sbt}(x, x, f1))$ (1 2 3 4) 1 : 16
- 18 $\forall n. (\text{INTEGER}(n) \Rightarrow (n \text{ gl } f1) = (n \text{ gl } \text{sbt}(x, x, f1)))$ (1 2 3 4) $\forall I$ 17 $n \leftarrow n$
- 19 $(\text{STRING}(f1) \wedge \text{STRING}(\text{sbt}(x, x, f1))) \Rightarrow (\forall n. (\text{INTEGER}(n) \Rightarrow (n \text{ gl } f1) = (n \text{ gl } \text{sbt}(x, x, f1)))) \Rightarrow f1 = \text{sbt}(x, x, f1)$ $\forall E$ EQS $f1, \text{sbt}(x, x, f1)$

20 $f1 = sbl(x, x, f1)$ (1 2 3 4) 1 : 19
 21 $(INDVAR(x) \wedge FORM(f1)) \Rightarrow f1 = sbl(x, x, f1)$ (2 3 4) \Rightarrow 1 20
 22 $\forall x f. ((INDVAR(x) \wedge FORM(f)) \Rightarrow f = sbl(x, x, f))$ (2 3 4) \forall 21 $x \leftarrow f \ f1 \leftarrow x$

5.7 FOL commands in the earlier axiomatization

LABEL FIRSTLEMMA;
 ASSUME $\forall x f. ((INDVAR(x) \wedge FORM(f)) \Rightarrow sbl(x, x, f) = f)$;

 LABEL THEON1;
 ASSUME $\forall s sq. ((STRING(s) \wedge SEQUENCE(sq)) \Rightarrow scar(s \ cc \ sq) = s)$;
 LABEL THEON2;
 ASSUME $\forall s sq. ((STRING(s) \wedge SEQUENCE(sq)) \Rightarrow scdr(s \ cc \ sq) = sq)$;
 LABEL TH1;
 ASSUME $\forall x f. ((INDVAR(x) \wedge FORM(f)) \Rightarrow FORM(x \ gen \ f))$;
 LABEL TH2;
 ASSUME $\forall f. (FORM(f) \Rightarrow STRING(f))$;
 LABEL TH3;
 ASSUME $\forall f sq. ((FORM(f) \wedge SEQUENCE(sq)) \Rightarrow SEQUENCE(f \ cc \ sq))$;
 LABEL TH4;
 ASSUME $\forall x. (INDVAR(x) \Rightarrow TERM(x))$;
 LABEL TH5;
 ASSUME $\forall pf. (PROOFTREE(pf) \Rightarrow SEQUENCE(pf))$;

Proof of the Lemma $BEW(x \ gen \ f) \Rightarrow BEW(f)$ Under the Assumption: $INDVAR(x) \wedge FORM(f)$

LABEL HPTT;
 ASSUME $INDVAR(x) \wedge FORM(f)$;
 LABEL HPT;
 ASSUME $BEW(x \ gen \ f)$;

LABEL THTAUT;
 \forall FIRSTLEMMA x, f ;

 \forall PROVABLE $x \ gen \ f$;
 \forall TH1 x, f ;
 TAUT ---: #2*2, HPTT:-;
 \forall TH2, f;
 \forall TH3, f, sq;
 \forall TH4, x;
 \forall TH5, sq;
 LABEL HPAUX;
 \exists -----, sq ;

\forall GENRULO $f \ cc \ sq, sq, x, x$;
 LABEL THN1;
 \forall THEON1 f, sq ;
 \forall THEON2 f, sq ;
 TAUTEQ ---: #2*2*2*2*2*2*1[f1 ← f], 1:-;

UNIFY ----:#2#2#2#2#2#2 , -;
 TAUTEQ -----:#1 , 1:-;

∇e PROOF f cc sq ;
 LABEL GENE1;
 TAUTEQ PROOFTREE(sq)∧INDVAR(x)∧TERM(x)∧(GENI(f cc sq,sq,x,x) ∇ --: ∇
 EXI(f cc sq,sq,x,x)) 1:-;
 UNIFY --:#2#2#2#1 , - ;
 LABEL PROOFTR;
 TAUT ----:#1, 1:-;

∧e HPAUX :#2#2;
 ∇e - , f1;

∇e DEPEND f cc sq, sq,f1;
 ∧E GENE1:#2;
 UNIFY --:#2#2#2#2#2, - ;

TAUTEQ DEPEND(f cc sq,f1) ⊃ AXIOM (f1) ,1:-;
 ∇i -,f1←f1;
 TAUTEQ f=scar(f cc sq) 1:-;
 ∧i PROOFTR, - , -- ;
 LABEL USEFUL;
 ∇e PROVABLE f;
 UNIFY --:#2#2 ,--;
 TAUT --:#1,1:-;
 LABEL C!TH1;
 ⊃I HPT,-;

Proof of the Lemma BEW(f) ⊃ BEW(x gen f) Under the Assumption: INDVAR(x) ∧ FORM(f)

LABEL HPT1;
 ASSUME BEW(f);

TAUT USEFUL:#2 , -,HPT1,USEFUL;
 ∧E --:#2
 ∃e - ,sq;

∧e --:#2#2;
 ∇e - , f1;
 ∇e GENRUL2 x,sq;
 ∇e THEORY x,f1;
 TAUTEQ --:#2#1#2#1[f←f1] ,HPTT,HPT1:-;
 ∇i - ,f1←f1;
 TAUT ----:#1 ,HPTT,HPT1:-;

∇e GENRUL1 ((x gen f) cc sq) , sq ,x,x ;
 LABEL THN2;
 ∇e THEON1 x gen f , sq ;
 ∇e THEON2 x gen f , sq ;
 ∇E TH1 x ,f;
 ∇E TH2 x gen f;
 ∇E TH5 sq;
 TAUTEQ -----:#2#2#2#1[f1 ← f] ,HPTT, THTAUT,HPT1:-;

UNIFY -----: #2#2#2 , - ;
 VE TH3, x gen f, sq;
 TAUTEQ -----: #1, HPTT, THTAUT, HPTT1 :-;

VE PROOF (x gen f) cc sq .
 VE TH4, x;
 LABEL GEN1;
 TAUTEQ PROOFTREE(sq) ^ INDVAR(x) ^ TERM(x) ^ (---: v GENE((x gen f) cc sq, sq, x, x) v
 EXI((x gen f) cc sq, sq, x, x)) HPTT, HPTT1 :-;
 UNIFY ---: #2#2#2#1 , - ;

LABEL PROOFTR1;
 TAUT -----: #1, HPTT1 :-, THTAUT, HPTT;

VE DEPEND (x gen f) cc sq, sq, f1;
 ^E GEN1: #2;
 Ii - , x ← f1 OCC 2 5 8 11;
 Ii - , x ← x1 OCC 1 3 5 7;

TAUTEQ DEPEND((x gen f) cc sq, f1) ⇒ AXIOM (f1) , THTAUT, HPTT, HPTT1 :-;
 Vi -, f1 ← f1;
 TAUTEQ x gen f = scar((x gen f) cc sq), HPTT, HPTT1 :-;
 Ai PROOFTR1, - , -- ;
 VE PROVABLE x gen f;
 UNIFY -: #2#2 , --;
 TAUT -: #1, THTAUT, HPTT1 :-;
 LABEL C2TH1;
 ⇒ HPTT1, -;
 ⇒ C1TH1, C2TH1;
 LABEL THGEN;
 ⇒ HPTT, -;
 Vi -, x, f;
 VE TH1 x1, x2 gen f;
 VE TH1 x2, f;
 VE TH1 x1, f;
 VE TH1 x2, x1 gen f;
 VE TH1, x1, f;
 VE TH1, x2, f;
 TAUT (INDVAR(x1) ^ (INDVAR(x2) ^ FORM(f))) ⇒ (BEW(x1 gen (x2 gen f)) ⇒
 BEW(x2 gen (x1 gen f))), THGEN :-;
 Vi -, x1, x2, f;

5.6 Printout of the proof in the earlier axiomatization

- 1 $\forall x f. ((\text{INDVAR}(x) \wedge \text{FORM}(f)) \Rightarrow \text{sb}(x, x, f) = f)$ (1) ASSUME
- 2 $\forall s \text{sq}. ((\text{STRING}(s) \wedge \text{SEQUENCE}(\text{sq})) \Rightarrow \text{scar}(s \text{ cc } \text{sq}) = s)$ (2) ASSUME
- 3 $\forall s \text{sq}. ((\text{STRING}(s) \wedge \text{SEQUENCE}(\text{sq})) \Rightarrow \text{scdr}(s \text{ cc } \text{sq}) = \text{sq})$ (3) ASSUME

- 4 $\forall x f. ((\text{INDVAR}(x) \wedge \text{FORM}(f)) \Rightarrow \text{FORM}(x \text{ gen } f))$ (4) ASSUME
- 5 $\forall f. (\text{FORM}(f) \Rightarrow \text{STRING}(f))$ (5) ASSUME
- 6 $\forall f \text{ sq}. ((\text{FORM}(f) \wedge \text{SEQUENCE}(\text{sq})) \Rightarrow \text{SEQUENCE}(f \text{ cc } \text{sq}))$ (6) ASSUME
- 7 $\forall x. (\text{INDVAR}(x) \Rightarrow \text{TERM}(x))$ (7) ASSUME
- 8 $\forall pf. (\text{PROOFTREE}(pf) \Rightarrow \text{SEQUENCE}(pf))$ (8) ASSUME
- 9 $\text{INDVAR}(x) \wedge \text{FORM}(f)$ (9) ASSUME
- 10 $\text{BEW}(x \text{ gen } f)$ (10) ASSUME
- 11 $(\text{INDVAR}(x) \wedge \text{FORM}(f)) \Rightarrow \text{sbl}(x, x, f) = f$ (1) $\forall e$ 1 x, f
- 12 $\text{BEW}(x \text{ gen } f) = (\text{FORM}(x \text{ gen } f) \wedge \exists \text{sq} (\text{PROOFTREE}(\text{sq}) \wedge ((x \text{ gen } f) = \text{scar}(\text{sq}) \wedge \forall f1. (\text{DEPEND}(\text{sq}, f1) \Rightarrow \text{AXIOM}(f1)))))) \vee \text{PROVABLE } x \text{ gen } f$
- 13 $(\text{INDVAR}(x) \wedge \text{FORM}(f)) \Rightarrow \text{FORM}(x \text{ gen } f)$ (4) $\forall e$ 4 x, f
- 14 $\exists \text{sq}. (\text{PROOFTREE}(\text{sq}) \wedge ((x \text{ gen } f) = \text{scar}(\text{sq}) \wedge \forall f1. (\text{DEPEND}(\text{sq}, f1) \Rightarrow \text{AXIOM}(f1))))$ (1 4 9 10) 9 : 13
- 15 $\text{FORM}(f) \Rightarrow \text{STRING}(f)$ (5) $\forall e$ 5 f
- 16 $(\text{FORM}(f) \wedge \text{SEQUENCE}(\text{sq})) \Rightarrow \text{SEQUENCE}(f \text{ cc } \text{sq})$ (6) $\forall e$ 6 f, sq
- 17 $\text{INDVAR}(x) \Rightarrow \text{TERM}(x)$ (7) $\forall e$ 7 x
- 18 $\text{PROOFTREE}(\text{sq}) \Rightarrow \text{SEQUENCE}(\text{sq})$ (8) $\forall e$ 8 sq
- 19 $\text{PROOFTREE}(\text{sq}) \wedge ((x \text{ gen } f) = \text{scar}(\text{sq}) \wedge \forall f1. (\text{DEPEND}(\text{sq}, f1) \Rightarrow \text{AXIOM}(f1)))$ (19) ASSUME
- 20 $\text{GENE}(f \text{ cc } \text{sq}, \text{sq}, x, x) = (\text{SEQUENCE}(f \text{ cc } \text{sq}) \wedge (\text{INDVAR}(x) \wedge (\text{TERM}(x) \wedge (\text{schr}(f \text{ cc } \text{sq}) = \text{sq} \wedge (\text{PROOFTREE}(\text{sq}) \wedge \exists f1. (\text{FORM}(f1) \wedge (\text{scar}(\text{sq}) = (x \text{ gen } f1) \wedge \text{scar}(f \text{ cc } \text{sq}) = \text{sbl}(x, x, f1)))))))))) \vee \text{GENRULO } f \text{ cc } \text{sq}, \text{sq}, x, x$
- 21 $(\text{STRING}(f) \wedge \text{SEQUENCE}(\text{sq})) \Rightarrow \text{scar}(f \text{ cc } \text{sq}) = f$ (2) $\forall e$ 2 f, sq
- 22 $(\text{STRING}(f) \wedge \text{SEQUENCE}(\text{sq})) \Rightarrow \text{schr}(f \text{ cc } \text{sq}) = \text{sq}$ (3) $\forall e$ 3 f, sq
- 23 $\text{FORM}(f) \wedge (\text{scar}(\text{sq}) = (x \text{ gen } f) \wedge \text{scar}(f \text{ cc } \text{sq}) = \text{sbl}(x, x, f))$ (1 2 3 4 5 6 7 8 9 10 19) 1 : 22
- 24 $\exists f1. (\text{FORM}(f1) \wedge (\text{scar}(\text{sq}) = (x \text{ gen } f1) \wedge \text{scar}(f \text{ cc } \text{sq}) = \text{sbl}(x, x, f1)))$ (1 2 3 4 5 6 7 8 9 10 19) UNIFY 23
- 25 $\text{GENE}(f \text{ cc } \text{sq}, \text{sq}, x, x)$ (1 2 3 4 5 6 7 8 9 10 19) 1 : 24
- 26 $\text{PROOFTREE}(f \text{ cc } \text{sq}) \vee ((\text{SEQUENCE}(f \text{ cc } \text{sq}) \wedge \text{FORM}(f \text{ cc } \text{sq})) \vee (\exists pf. (\text{PROOFTREE}(pf) \wedge (\text{ORI}(f \text{ cc } \text{sq}, pf) \vee (\text{ANDE}(f \text{ cc } \text{sq}, pf) \vee (\text{FALSEE}(f \text{ cc } \text{sq}, pf) \vee (\text{NOTI}(f \text{ cc } \text{sq}, pf) \vee (\text{NOTE}(f \text{ cc } \text{sq}, pf) \vee (\text{IMPLI}(f \text{ cc } \text{sq}, pf)))))))))) \vee (\exists pf \ x \ 1 (\text{PROOFTREE}(pf) \wedge (\text{INDVAR}(x) \wedge (\text{TERM}(t) \wedge (\text{GENI}(f \text{ cc } \text{sq}, pf, x, t) \vee (\text{GENE}(f \text{ cc } \text{sq}, pf, x, t) \vee (\text{EXI}(f \text{ cc } \text{sq}, pf, x, t)))))) \vee (\exists pf1 \ pf2. (\text{PROOFTREE}(pf1) \wedge (\text{PROOFTREE}(pf2) \wedge (\text{ANDI}(f \text{ cc } \text{sq}, pf1, pf2) \vee$

- (FALSE!(f cc sq,pf1,pf2)∨IMPLE(f cc sq,pf1,pf2))))∨(∃pf1 pf2 x1 x2.(PROOFTREE(pf1)∧
 (PROOFTREE(pf2)∧(INDVAR(x1)∧(INDVAR(x2)∧EXE(f cc sq,pf1,pf2,x1,x2))))∨∃pf1 pf2 pf3.
 (PROOFTREE(pf1)∧(PROOFTREE(pf2)∧(PROOFTREE(pf3)∧ORE(f cc sq,pf1,pf2,pf3))))))))
 VE PROOF f cc sq
- 27 PROOFTREE(sq)∧(INDVAR(x)∧(TERM(x)∧(GENI(f cc sq,sq,x)∨(GENE(f cc sq,sq,x)∨
 EXI(f cc sq,sq,x)))))) (1 2 3 4 5 6 7 8 9 10 19) 1 : 26
- 28 ∃pf x t.(PROOFTREE(pf)∧(INDVAR(x)∧(TERM(t)∧(GENI(f cc sq,pf,x,t)∨(GENE(f cc sq,pf,x,t)∨
 EXI(f cc sq,pf,x,t)))))) (1 2 3 4 5 6 7 8 9 10 19) UNIFY 27
- 29 PROOFTREE(f cc sq) (1 2 3 4 5 6 7 8 9 10 19) 1 : 28
- 30 ∀f1.(DEPEND(sq,f1)⇒AXIOM(f1)) (19) ∧E 19 :#2#2
- 31 DEPEND(sq,f1)⇒AXIOM(f1) (19) VE 30 f1
- 32 ((PROOFTREE(f cc sq)∧(PROOFTREE(sq)∧sq=sdcr(f cc sq)))⇒(DEPEND(f cc sq,f1)=DEPEND(sq,f1)))=
- (ORI(f cc sq,sq)∨(ANDE(f cc sq,sq)∨(FALSEE(f cc sq,sq)∨(∃f.(FORM(f)∧((NOTID(f cc sq,sq,f)∨
 (NOTED(f cc sq,sq,f)∨IMPLID(f cc sq,sq,f))∧f≠f1))∨∃x t.(INDVAR(x)∧(TERM(t)∧
 (GENI(f cc sq,sq,x,t)∨(GENE(f cc sq,sq,x,t)∨EXI(f cc sq,sq,x,t))))))))))
 VE DEPEND f cc sq , sq , f1
- 33 INDVAR(x)∧(TERM(x)∧(GENI(f cc sq,sq,x)∨(GENE(f cc sq,sq,x)∨EXI(f cc sq,sq,x))))
 (1 2 3 4 5 6 7 8 9 10 19) ∧E 27 :#2
- 34 ∃x t.(INDVAR(x)∧(TERM(t)∧(GENI(f cc sq,sq,x,t)∨(GENE(f cc sq,sq,x,t)∨EXI(f cc sq,sq,x,t))))))
 (1 2 3 4 5 6 7 8 9 10 19) UNIFY 33
- 35 DEPEND(f cc sq,f1)⇒AXIOM(f1) (1 2 3 4 5 6 7 8 9 10 19) 1 : 34
- 36 ∀f1.(DEPEND(f cc sq,f1)⇒AXIOM(f1)) (1 2 3 4 5 6 7 8 9 10 19) VI 35 f1 ← f1
- 37 f=scar(f cc sq) (1 2 3 4 5 6 7 8 9 10 19) 1 : 36
- 38 PROOFTREE(f cc sq)∧(f=scar(f cc sq)∧∀f1.(DEPEND(f cc sq,f1)⇒AXIOM(f1)))
 (1 2 3 4 5 6 7 8 9 10 19) ∧I (29 (37 36))
- 39 BEW(f)=(FORM(f)∧∃sq.(PROOFTREE(sq)∧(f=scar(sq)∧∀f1.(DEPEND(sq,f1)⇒AXIOM(f1))))))
 VE PROVABLE f
- 40 ∃sq.(PROOFTREE(sq)∧(f=scar(sq)∧∀f1.(DEPEND(sq,f1)⇒AXIOM(f1)))) (1 2 3 4 5 6 7 8 9 10) UNIFY 38
- 41 BEW(f) (1 2 3 4 5 6 7 8 9 10) 9 , 39 , 40
- 42 BEW(x gen f)⇒BEW(f) (1 2 3 4 5 6 7 8 9) ⇒I 10 41
- 43 BEW(f) (43) ASSUME
- 44 FORM(f)∧∃sq.(PROOFTREE(sq)∧(f=scar(sq)∧∀f1.(DEPEND(sq,f1)⇒AXIOM(f1)))) (43) 43 , 43 , 39
- 45 ∃sq.(PROOFTREE(sq)∧(f=scar(sq)∧∀f1.(DEPEND(sq,f1)⇒AXIOM(f1)))) (43) ∧E 44 :#2

- 46 PROOFTREE(sq) \wedge (f=scar(sq) \wedge \forall f1.(DEPEND(sq,f1) \supset AXIOM(f1))) (46) ASSUME
- 47 \forall f1.(DEPEND(sq,f1) \supset AXIOM(f1)) (46) \wedge E 46 :#2#2
- 48 DEPEND(sq,f1) \supset AXIOM(f1) (46) \forall E 47 f1
- 49 APGENI(x,sq)=((INDVAR(x) \wedge \forall f.(DEPEND(sq,f) \supset FR(x,f)) \wedge PROOFTREE(sq)) \forall E GENRUL2 x , sq
- 50 AXIOM(f1) \supset (FR(x,f1) \wedge FORM(f1)) \forall E THEORY x , f1
- 51 DEPEND(sq,f1) \supset FR(x,f1) (1 2 3 4 5 6 7 8 9 43 46) 9 , 43 : 50
- 52 \forall f1.(DEPEND(sq,f1) \supset FR(x,f1)) (1 2 3 4 5 6 7 8 9 43 46) \forall I 51 f1 \leftarrow f1
- 53 APGENI(x,sq) (1 2 3 4 5 6 7 8 9 43 46) 9 , 43 : 52
- 54 GENI((x gen f) cc sq,sq,x,x) (SEQUENCE((x gen f) cc sq) \wedge (INDVAR(x) \wedge (INDVAR(x) \wedge (scdr((x gen f) cc sq)=sq \wedge (PROOFTREE(sq) \wedge \exists f1.(FORM(f1) \wedge (scar((x gen f) cc sq)=
(x gen f1) \wedge (scar(sq)=sbt(x,x,f1) \wedge APGENI(x,sq)))))))))) \forall E GENRUL1 (x gen f) cc sq , sq , x , x
- 55 (STRING(x gen f) \wedge SEQUENCE(sq)) \supset scar((x gen f) cc sq)=(x gen f)(2) \forall E 2 x gen f , sq
- 56 (STRING(x gen f) \wedge SEQUENCE(sq)) \supset scdr((x gen f) cc sq)=sq (3) \forall E 3 x gen f , sq
- 57 (INDVAR(x) \wedge FORM(f)) \supset FORM(x gen f) (4) \forall E 4 x , f
- 58 FORM(x gen f) \supset STRING(x gen f) (5) \forall E 5 x gen f
- 59 PROOFTREE(sq) \supset SEQUENCE(sq) (8) \forall E 8 sq
- 60 FORM(f) \wedge (scar((x gen f) cc sq)=(x gen f) \wedge (scar(sq)=sbt(x,x,f) \wedge APGENI(x,sq)))
(1 2 3 4 5 6 7 8 9 43 46) 11 , 43 : 59 , 9
- 61 \exists f1.(FORM(f1) \wedge (scar((x gen f) cc sq)=(x gen f1) \wedge (scar(sq)=sbt(x,x,f1) \wedge
APGENI(x,sq)))) (1 2 3 4 5 6 7 8 9 43 46) UNIFY 60
- 62 (FORM(x gen f) \wedge SEQUENCE(sq)) \supset SEQUENCE((x gen f) cc sq) (6) \forall E 6 x gen f , sq
- 63 GENI((x gen f) cc sq,sq,x,x) (1 2 3 4 5 6 7 8 9 43 46) 9 , 11 , 43 : 62
- 64 PROOFTREE((x gen f) cc sq) ((SEQUENCE((x gen f) cc sq) \wedge FORM((x gen f) cc sq)) \vee
(\exists pf.(PROOFTREE(pf) \wedge (ORI((x gen f) cc sq,pf) \vee (ANDE((x gen f) cc sq,pf) \vee
(FALSEE((x gen f) cc sq,pf)) \vee (NOTI((x gen f) cc sq,pf)) \vee (NOTE((x gen f) cc sq,pf)) \vee
IMPLI((x gen f) cc sq,pf)))))) \vee (\exists pf x1 t.(PROOFTREE(pf) \wedge (INDVAR(x1) \wedge (TERM(t) \wedge
(GENI((x gen f) cc sq,pf,x1,t) \vee (GENE((x gen f) cc sq,pf,x1,t) \vee EXI((x gen f)
cc sq,pf,x1,t)))))) \vee (\exists pf1 pf2.(PROOFTREE(pf1) \wedge (PROOFTREE(pf2) \wedge (ANDI((x gen f)
cc sq,pf1,pf2) \vee (FALSEI((x gen f) cc sq,pf1,pf2)) \vee IMPLE((x gen f) cc sq,pf1,pf2)))))) \vee
(\exists pf1 pf2 x1 x2.(PROOFTREE(pf1) \wedge (PROOFTREE(pf2) \wedge (INDVAR(x1) \wedge (INDVAR(x2) \wedge EXE(
x gen f) cc sq,pf1,pf2,x1,x2)))))) \vee (\exists pf1 pf2 pf3.(PROOFTREE(pf1) \wedge (PROOFTREE(pf2) \wedge
(PROOFTREE(pf3) \wedge ORE((x gen f) cc sq,pf1,pf2,pf3))))))))) \forall E PROOF (x gen f) cc sq
- 65 INDVAR(x) \supset TERM(x) (7) \forall E 7 x

- 66 PROOFTREE(sq)^(INDVAR(x)^(TERM(x)^(GENI((x gen f) cc sq,sq,x,x)v(GENE((x gen f) cc sq,sq,x,x)vEXI((x gen f) cc sq,sq,x,x)))) (1 2 3 4 5 6 7 8 9 43 46) 9,43 : 65
- 67 $\exists pf \ x1 \ t. (PROOFTREE(pf) \wedge (INDVAR(x1) \wedge (TERM(t) \wedge (GENI((x gen f) cc sq,pf,x1,t) \vee (GENE((x gen f) cc sq,pf,x1,t) \vee EXI((x gen f) cc sq,pf,x1,t))))))$
(1 2 3 4 5 6 7 8 9 43 46) UNIFY 66
- 68 PROOFTREE((x gen f) cc sq) (1 2 3 4 5 6 7 8 9 43 46) 43 : 67 , 11 , 9
- 69 ((PROOFTREE((x gen f) cc sq)^(PROOFTREE(sq)^(sq=scdr((x gen f) cc sq)))=(DEPEND((x gen f) cc sq,fl)=DEPEND(sq,fl)))=(ORI((x gen f) cc sq,sq)^(ANDE((x gen f) cc sq,sq)^(FALSEE((x gen f) cc sq,sq)^($\exists f. (FORM(f) \wedge (NOTID((x gen f) cc sq,sq,t) \vee (NOTED((x gen f) cc sq,sq,f) \vee IMPLID((x gen f) cc sq,sq,f))) \wedge f \neq fl)) \vee \exists x1 \ t. (INDVAR(x1) \wedge (TERM(t) \wedge (GENI((x gen f) cc sq,sq,x1,t) \vee (GENE((x gen f) cc sq,sq,x1,t) \vee EXI((x gen f) cc sq,sq,x1,t))))))$ VE DEPEND (x gen f) cc sq , sq , fl
- 70 INDVAR(x)^(TERM(x)^(GENI((x gen f) cc sq,sq,x,x)v(GENE((x gen f) cc sq,sq,x,x)vEXI((x gen f) cc sq,sq,x,x)))) (1 2 3 4 5 6 7 8 9 43 46) $\wedge E$ 66 :#2
- 71 $\exists f. (INDVAR(x) \wedge (TERM(t) \wedge (GENI((x gen f) cc sq,sq,x,t) \vee (GENE((x gen f) cc sq,sq,x,t) \vee EXI((x gen f) cc sq,sq,x,t))))$ (1 2 3 4 5 6 7 8 9 43 46) 70 x \leftarrow f OCC
- 72 $\exists x1 \ t. (INDVAR(x1) \wedge (TERM(t) \wedge (GENI((x gen f) cc sq,sq,x1,t) \vee (GENE((x gen f) cc sq,sq,x1,t) \vee EXI((x gen f) cc sq,sq,x1,t))))$ (1 2 3 4 5 6 7 8 9 43 46) 71 x \leftarrow x1 OCC
- 73 DEPEND((x gen f) cc sq,fl) \supset AXIOM(fl) (1 2 3 4 5 6 7 8 9 43 46) 11 , 9 , 43 : 72
- 74 $\forall f1. (DEPEND((x gen f) cc sq,fl) \supset AXIOM(fl))$ (1 2 3 4 5 6 7 8 9 43 46) $\forall I$ 73 fl \leftarrow f1
- 75 (x gen f)=scar((x gen f) cc sq) (1 2 3 4 5 6 7 8 9 43 46) 9 , 43 : 74
- 76 PROOFTREE((x gen f) cc sq)^(x gen f)=scar((x gen f) cc sq) $\wedge \forall f1. (DEPEND((x gen f) cc sq,fl) \supset AXIOM(fl))$ (1 2 3 4 5 6 7 8 9 43 46) $\wedge I$ (68 (75 74))
- 77 BEW(x gen f) \rightarrow (FORM(x gen f) $\wedge \exists sq. (PROOFTREE(sq) \wedge ((x gen f)=scar(sq) \wedge \forall f1. (DEPEND(sq,fl) \supset AXIOM(fl))))$ VE PROVABLE x gen f
- 78 $\exists sq. (PROOFTREE(sq) \wedge ((x gen f)=scar(sq) \wedge \forall f1. (DEPEND(sq,fl) \supset AXIOM(fl))))$
(1 2 3 4 5 6 7 8 9 10 19 43 46) UNIFY 76
- 79 BEW(x gen f) (1 2 3 4 5 6 7 8 9 43) 11 , 9 , 43 : 78
- 80 BEW(f) \supset BEW(x gen f) (1 2 3 4 5 6 7 8 9) $\supset I$ 43 79
- 81 BEW(x gen f) \rightarrow BEW(f) (1 2 3 4 5 6 7 8 9) $\supset I$ 42 80
- 82 (INDVAR(x) \wedge FORM(f)) \supset (BEW(x gen f) \rightarrow BEW(f)) (1 2 3 4 5 6 7 8) $\supset I$ 9 81
- 83 $\forall x \ f. ((INDVAR(x) \wedge FORM(f)) \supset (BEW(x gen f) \rightarrow BEW(f)))$ (1 2 3 4 5 6 7 8) $\forall I$ 82 x , f
- 84 (INDVAR(x1) \wedge FORM(x2 gen f)) \supset (BEW(x1 gen (x2 gen f)) \rightarrow BEW(x2 gen f))
(1 2 3 4 5 6 7 8) VE 83 x1 , x2 gen f

85 $(\text{INDVAR}(x_2) \wedge \text{FORM}(f)) \Rightarrow (\text{BEW}(x_2 \text{ gen } f) = \text{BEW}(f))$ (1 2 3 4 5 6 7 8) $\forall E$ 83 x_2, f

86 $(\text{INDVAR}(x_1) \wedge \text{FORM}(f)) \Rightarrow (\text{BEW}(x_1 \text{ gen } f) = \text{BEW}(f))$ (1 2 3 4 5 6 7 8) $\forall E$ 83 x_1, f

87 $(\text{INDVAR}(x_2) \wedge \text{FORM}(x_1 \text{ gen } f)) \Rightarrow (\text{BEW}(x_2 \text{ gen } (x_1 \text{ gen } f)) = \text{BEW}(x_1 \text{ gen } f))$
 (1 2 3 4 5 6 7 8) $\forall E$ 83 $x_2, x_1 \text{ gen } f$

88 $(\text{INDVAR}(x_1) \wedge \text{FORM}(f)) \Rightarrow \text{FORM}(x_1 \text{ gen } f)$ (4) $\forall E$ 4 x_1, f

89 $(\text{INDVAR}(x_2) \wedge \text{FORM}(f)) \Rightarrow \text{FORM}(x_2 \text{ gen } f)$ (4) $\forall E$ 4 x_2, f

90 $(\text{INDVAR}(x_1) \wedge (\text{INDVAR}(x_2) \wedge \text{FORM}(f))) \Rightarrow (\text{BEW}(x_1 \text{ gen } (x_2 \text{ gen } f)) = \text{BEW}(x_2 \text{ gen } (x_1 \text{ gen } f)))$
 (1 2 3 4 5 6 7 8) 84 : 89

91 $\forall x_1 x_2 f. ((\text{INDVAR}(x_1) \wedge (\text{INDVAR}(x_2) \wedge \text{FORM}(f))) \Rightarrow (\text{BEW}(x_1 \text{ gen } (x_2 \text{ gen } f)) = \text{BEW}(x_2 \text{ gen } (x_1 \text{ gen } f))))$ (1 2 3 4 5 6 7 8) $\forall I$ 90 x_1, x_2, f

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