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TURBULENT WAKE BEHIND A SELF-PROPELLED BODY

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Toshi Kubota

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13. ABSTRACT

The wake behind a self-propelled body is studied for laminar and turbulent cases. For the laminar wake, asymptotic solutions are obtained with and without swirl. For the turbulent wake with the eddy-viscosity model, the solution is obtained from the laminar flow solution by transformation that reduces the turbulent-flow equation to the equation for laminar flow. With the mixing-length model for the turbulent shear stress, the far-wake solution becomes that of non-linear eigenvalue problem. These two models yield results that do not agree with experimental results. The far-wake solution is formulated based on a two-equation model for turbulent shear--turbulent energy and dissipation--with an additional assumption of negligible turbulence production from the mean flow.

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Laminar Flow						

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1. Introduction

A self-propelled body should not leave any wake under practically unrealizable ideal conditions in which the momentum defect created by the viscous effect on the body is cancelled at every point in the flow by the propulsive device. The propulsive devices in real life, however, are far from this ideal condition. What propellers and jets accomplish is to compensate the total drag by the total thrust, and therefore in the flow immediately behind the body some regions have excess momentum and other regions have momentum defect. This nonuniform distribution of momentum is smoothed out as it moves downstream from the body.

The asymptotic laminar wake behind a non-lifting body was examined theoretically by Tollmien,(1) Goldstein,(2) and Stewartson(3) as well as many other investigators. It has been established by these investigations that the asymptotic expansion for large distances contains a series that is uniquely determined by the drag coefficient C_D and in addition series that depend not only on C_D but also on the details of the initial velocity profile. Behind a self-propelled body, the part of the wake determined by C_D vanishes identically. Birkhoff(4) showed that the remaining part is determined by the second moments of the velocity profile, $\int_{-\infty}^{\infty} (u_\infty - u_e) y^2 \, dy$, in two-dimensional flows and the velocity "defect" decays as $x^{-3/2}$ (x is the distance from the body) in contrast to the drag-wake decay of $x^{-\frac{1}{2}}$.

The turbulent wake of a self-propelled body, however, appears to behave quite differently from the turbulent drag-wake. In the latter flow the decaying mean velocity defect and the turbulence motion reach some sort of equilibrium condition beyond some distance from the body. The mean value of turbulence velocities becomes proportional to the mean velocity defect and the turbulence scale length (integral scale or dissipation length) proportional to the mean-velocity wake thickness. Consequently, semiempirical turbulence shear stress models, eddy viscosity and mixing length, give adequate description of the wake.

On the other hand, in the wake behind a self-propelled body, the turbulence, which is injected from the boundary layer into the wake and also generated by the propulsion device, decays much slower than the mean velocity "defect" that decays much faster than in the drag-wake on account of near-zero total momentum flux, and because of this disparity in the decay rates the equilibrium between the mean flow and the turbulence does not seem to exist in the self-propelled body wake.

If an eddy viscosity coefficient is assumed constant across the wake, Birkhoff showed that the second moment of velocity defect is constant also for momentumless turbulent wake. Then it follows that, if we assume eddy viscosity proportional to the velocity defect and wake width, the wake grows like $x^{1/5}$ and the mean-velocity "defect" decays as $x^{-4/5}$ in axisymmetric flow. With Prandtl's mixing-length model for for turbulent viscosity, no simple integral theorem can be derived as in laminar flows and the wake-growth exponent must be obtained from a non-linear eigenvalue problem of ordinary differential equation for velocity distribution. We carried out the solution and the results differ only slightly from the constant eddy-viscosity results.

Compared with experimental results (Naudascher,[5] and Ginevskii, et al.[6]) these predictions agree with the measurements fairly well as far as the wake growth is concerned. The mean velocity decay, however, is predicted much too slow compared with the measured decay (Naudascher shows the velocity decay roughly as x^{-2}). In laminar flow, on the other hand, the theory predicts the velocity decay as x^{-2}. It appears that in turbulent momentumless wakes turbulence decays slower than the mean velocity which is subjected to more or less constant (in streamwise direction as well) turbulent viscosity.

In order to describe nonequilibrium turbulent shear flows, the next hierarchy of turbulence modelling employs differential equations for turbulence energy, turbulence scale (or dissipation scale), etc. in addition to the equations for mean velocities.

The existing mathematical models for turbulence in shear flows may be classified in the following manner:

a) Algebraic models define turbulent viscosity coefficient

$$\nu_t = -\overline{u'v'}/(\partial u/\partial y)$$

a-1) Mixing length model (Prandtl[7])

$$\nu_t = \ell_m^2 \left| \frac{\partial u}{\partial y} \right|$$

$$\ell_m = 0.07\,\delta \sim 0.13\,\delta \quad \text{for jets and wakes}$$

a-2) Eddy-viscosity model (Prandtl)

$$\nu_t = |u_{C.L.} - u_\infty|\,\ell$$

$$\ell \propto \delta$$

b) One-equation models introduce one additional differential equation for turbulence quantity.

b-1) Prandtl[8]-Kolmogoror[9] model assumes

$$\nu_t = e^{\frac{1}{2}}\,\ell$$

where e = time-averaged turbulence energy

ℓ = specified length scale $\propto \delta$

The growth or decay of the turbulence energy is governed by a model equation

$$\frac{De}{Dt} = \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_e}\frac{\partial e}{\partial y}\right) + \nu_t\left(\frac{\partial u}{\partial y}\right) - C_D\frac{e^{\frac{3}{2}}}{\ell}$$

σ_e = turbulence energy Prandtl number ≈ 1

C_D = dissipation constant ≈ 0.09

b-2) Turbulent-stress model (Bradshaw[10])

$\tau = a\rho e$ (a, constant ≈ 0.3)

A model equation for τ is

$$\frac{D}{Dt}\left(\frac{\tau}{a}\right) = \tau\frac{\partial u}{\partial y} - \tau\left(\frac{\tau}{\rho}\right)^{\frac{1}{2}} G - \rho\left(\frac{\tau}{\rho}\right)^{\frac{3}{2}}/L$$

where G and L are empirical functions of y/δ.

Application of this modek is limited to boundary layers without velocity maxima.

b-3) Nee-Kovasznay model[11]

A model equation for turbulent viscosity

$$\frac{D\nu_t}{Dt} = \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_\nu}\frac{\partial \nu_t}{\partial y}\right) + \left[A - C\frac{\nu_t}{u_e^2}\frac{du_e}{dx}\right]\nu_t\left|\frac{\partial u}{\partial y}\right| - B\left(\frac{\nu_t}{\ell}\right)^2$$

Empirical constants introduced are approximately A = 0.1, B = C = σ_ν=1. ℓ is an empirical function of y/δ.

c) Two-equation models introduce an equation for the length scale ℓ (or equivalent) in addition to the turbulence energy equation. The number of specific models increases rapidly, but typical ones are:

c-1) Turbulent vorticity model (Kolmogorov, Saffman[12])

The transport equation for the length scale ℓ is derived from a model equation for turbulence vorticity $\omega = e^{\frac{1}{2}}/\ell$

$$\frac{D\omega^2}{Dt} = \alpha\omega^2 \frac{\partial u}{\partial y} - \beta\omega^3 + \frac{\partial}{\partial y}\left(\sigma \nu_t \frac{\partial \omega^2}{\partial y}\right)$$

$0.15 < \beta < 0.20, \quad 1.7 < \beta < 2, \quad \sigma = \frac{1}{2}$

c-2) Dissipation Model (Harlow and Nakayama[13])

The scale-length equation is derived from a model equation for turbulence dissipation $\varepsilon = e^{\frac{3}{2}}/\ell$

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) + C_1\, e\left(\frac{\partial u}{\partial y}\right)^2 - C_2\,\frac{\varepsilon^2}{e}$$

$\sigma_\varepsilon \doteq 1.3, \; \sigma_1 \doteq 1.45, \; C_2 \doteq 0.19$

d) Multi-equation models introduce model equations for many, if not all, turbulence velocity correlations $\overline{u_i' u_j'}$. (Donaldson,[14] Rotta,[15] Hanjalic-Launder[16]). For example, Hanjalic-Launder model for boundary layer type flows is

$$\frac{D\tau}{Dt} = -2.8\left(\frac{\varepsilon}{e}\,\tau - 0.07\,\rho e\,\frac{\partial u}{\partial y}\right) + 0.08\,\frac{\partial}{\partial y}\left(\frac{e^2}{\varepsilon}\frac{\partial \tau}{\partial y}\right)$$

$$\frac{De}{Dt} = \frac{\tau}{\rho}\frac{\partial u}{\partial y} - \varepsilon + 0.064\,\frac{\partial}{\partial y}\left(\frac{e^2}{\varepsilon}\frac{\partial e}{\partial y}\right)$$

$$\frac{D\varepsilon}{Dt} = -1.45\,\frac{\tau}{\rho e}\frac{\partial u}{\partial y} - 2.0\,\frac{\varepsilon^2}{e} + 0.065\,\frac{\partial}{\partial y}\left(\frac{e^2}{\varepsilon}\frac{\partial \varepsilon}{\partial y}\right)$$

Unfortunately the applicability of these mathematical models for untried situations has to be tested by comparing the predictions with measurements. Given initial and boundary conditions, one can numerically integrate these partial differential equations and compare the results with measurements (this approach may be called experiments on a computer). A more satisfactory and efficient approach is to look for similarity solution so that the governing equations reduce to ordinary differential equations. Even in overall non-equilibrium turbulent wakes, some, if not all quantities must reach an equilibrium stage in a far-wake where other quantities decay to an insignificant level.

If we consider the equations for turbulence energy and scale, these can be reduced to the following form in a region where the production due to mean-velocity gradient is decreased to negligible level (in two-dimensional flow):

$$\frac{d}{d\eta}\left(f^{\frac{1}{2}}\, g\, \frac{df}{d\eta}\right) + m\,\eta\,\frac{df}{d\eta} - 2(m-1)f - \frac{f^{\frac{3}{2}}}{g} = 0$$

$$\frac{d}{d\eta}\left(f^{\frac{1}{2}}\, g\, \frac{dg}{d\eta}\right) + m\,\eta\,\frac{dg}{d\eta} - 3mg + f^{\frac{1}{2}} = 0$$

where $\eta \sim y/x^m$, turbulence energy $e \sim x^{2(m-1)} f(\eta)$, scale $\ell \sim x^m g(\eta)$. The boundary conditions are: $f' = g' = 0$ at $\eta = 0$ from symmetry and $f = f' = g' = 0$ at the wake edge $\eta = \eta_e$. This again constitutes an eigenvalue problem for m and η_e (since the turbulent viscosity $\sim f^{\frac{1}{2}} g$ vanishes at the edge, the wake has a well-defined edge). (The solution for this problem has yet to be carried out.) This solution presumably describes the far-wake behind a self-propelled body. Closer to the body we have to take into account the decay of mean-velocity and the turbulence production, which together with the initial condition will determine the magnitudes of turbulence energy and scale in the far-wake.

For a given situation, one can numerically integrate the governing differential equations to predict the wake behavior. The approach outlined above, however, is more suitable for delineating the similarity parameters and the sensitivity of solutions to the empirically determined constants invariably incorporated in any turbulence model equations.

2. Laminar Flow

2.1. Laminar Wake without Swirl

Boundary-layer approximation gives, without pressure gradient or swirl,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{1}{r^j} \frac{\partial}{\partial r} \left(r^j \nu \frac{\partial u}{\partial r} \right) \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{1}{r^j} \frac{\partial r^j v}{\partial r} = 0 \tag{2}$$

where $\begin{cases} j = 0 & \text{2-D} \\ j = 1 & \text{Axi-symmetric} \end{cases}$

The condition for zero drag is

$$D = \int_r^{r_o} u(u - U_o) r^j \, dr = 0 \tag{3}$$

If we define

$$u_d = u - U_o , \tag{4}$$

where U_o is the freestream velocity, Eqs. (1) ~ (3) become

$$U_o \frac{\partial u_d}{\partial x} - \frac{1}{r^j} \frac{\partial}{\partial r} \left(r^j \nu \frac{\partial u_d}{\partial r} \right) = -u_d \frac{\partial u_d}{\partial x} - v \frac{\partial u_d}{\partial r} \tag{1'}$$

$$v = -\frac{1}{r^j} \int_o^r r^j \frac{\partial u_d}{\partial x} \, dr \tag{2'}$$

$$U_o \int_o^{r_o} u_d \, r^j \, dr = -\int_o^{r_o} u_d^2 \, r^j \, dr \tag{3'}$$

The boundary conditions are

$$\left. \begin{array}{l} u_d \to 0 \\ \dfrac{\partial u_d}{\partial r} \to 0 \end{array} \right\} \text{as } r \to \infty \tag{5}$$

$$\frac{\partial u_d}{\partial r} = 0 \quad \text{at} \quad r = 0$$

We look for the solution of the form:

$$\frac{1}{U_o} u_d = \sum_{n=1}^{\infty} A_n x^{-k_n} f_n(\eta) \tag{6}$$

where

$$\eta = \frac{r}{b(x)} \tag{7}$$

We expand b(x) as

$$b(x) = \sum_{n=1}^{\infty} B_n x^{m_n} \tag{8}$$

2.2. First-Order Term

When higher order terms of u_d are neglected, Eq. (1') and Eq. (2') yield the following linear equations

$$U_o \frac{\partial u_d}{\partial x} - \frac{1}{r^j} \frac{\partial}{\partial r}\left(r^j \nu \frac{\partial u_d}{\partial r}\right) = 0 \tag{9}$$

$$\int_o^{r_o} u_d r^j dr = 0 \tag{10}$$

First-order terms of Eqs. (6) and (8) are

$$\frac{1}{U_o} u_d = A x^{-k} f(\eta) ; \quad \eta = \frac{r}{b(x)} \tag{11}$$

$$b(x) = B x^m \tag{12}$$

When Eqs. (11) and (12) are substituted into Eq. (9), we require for the similarity;

$$B = 2\sqrt{\frac{\nu}{U_o}} , \quad m = \tfrac{1}{2}$$

Hence the jet width is given by

$$b = 2\sqrt{\frac{\nu}{U_o}}\, x^{\frac{1}{2}} \tag{13}$$

and the equation for the velocity profile $f(\eta)$ is given by

$$f''(\eta) + (2\eta + j\,\eta^{-1})f'(\eta) + 4k\, f(\eta) = 0 \tag{14}$$

subject to the boundary conditions

$$\left.\begin{array}{l} f \to 0 \\ f' \to 0 \end{array}\right\} \text{ as } \eta \to \infty$$

$$\left.\begin{array}{l} f = 1 \\ f' = 0 \end{array}\right\} \text{ at } \eta = 0$$

The solutions are given by

Two-dimensional case:

$$f(\eta) = H_{2k-1}(\eta)e^{-\eta^2} \tag{15}$$

where H_n is the Hermite Polynomial.

Axi-symmetric case:

$$f(\eta) = L_{(k-1)}(\eta^2)e^{-\eta^2} \tag{16}$$

where L_n is the Laguerre Polynomial.

From Eq. (10),

$$\int_o^\infty f\,\eta^j\, d\eta = 0 \tag{17}$$

We multiply Eq. (14) by η^{2n+j} (n; integer), integrate from 0 to ∞, and get:

$$(2n+1+j-2k)\int_o^\infty f\,\eta^{2n+j}\, d\eta = n(2n-1+j)\int_o^\infty f\,\eta^{2n-2+j}\, d\eta \tag{18}$$

n = 0;

Eq. (18) becomes

$$(-2k+1+j) \int_0^\infty f \eta^j \, d\eta = 0 \tag{19}$$

Hence for $k \neq \frac{j+1}{2}$ corresponds to the ordinary wake $D \neq 0$)

n = 1;

Eqs. (18) and (19) give

$$(2n+1+j-2k) \int_0^\infty f \eta^{2+j} \, d\eta = 0 \tag{20}$$

For the non-zero second moment of velocity $\int_0^\infty f \eta^{2+j} \, d\eta \neq 0$, we get

$$k = \frac{j+3}{2} \tag{21}$$

In summary, the first order solution is given by

$$b(x) \simeq x^{\frac{1}{2}}$$

$$u_{d\mathbb{C}} \simeq x^{-(j+3)/2}$$

$$\frac{u_d}{u_{d\mathbb{C}}} \simeq \begin{cases} H_2(\eta) e^{-\eta^2} & \text{(two-dimensional flow)} \\ L_1(\eta^2) e^{-\eta^2} & \text{(axisymmetric flow)} \end{cases}$$

From the continuity equation

$$v_{\eta = \infty} = 0$$

2.3. Higher-Order Terms

When Eq. (7) and (8) are substituted in Eqs. (1') and (2'), with $m_n = \frac{1}{2}$ and $b = 2(\nu x/U_o)^{\frac{1}{2}}$ determined from the similarity condition, the equation becomes

$$\sum_{n=1} A_n x^{-k_n-1} [f''(\eta) + (\eta + j\eta^{-1}) f'(\eta) + 2k f(\eta)]$$

$$= \sum_{n=1} A_n x^{-k_n} f_n(\eta) \cdot \sum_{n=1} A_n x^{-k_n-1} [k_n f_n(\eta) + \tfrac{1}{2} \eta f_n'(\eta)]$$

$$- \sum_{n=1} A_n x^{-k_n-\frac{1}{2}} \eta^{-j} [(k_n - \tfrac{1}{2} j - \tfrac{1}{2}) \int_0^\eta \eta^j f_n(\eta) d\eta + \tfrac{1}{2} \eta^{j+1} f(\eta)] \cdot \sum_{n=1} A_n x^{-k_n-\frac{1}{2}} f_n'(\eta) \tag{22}$$

Equating the same power of x, we get non-homogeneous linear equations to be satisfied by $f_n(\eta)$.

The solutions are as follows:

Two-dimensional flow:

$$\frac{1}{U_o} u_d = A_1 x^{-\frac{3}{2}} H_2(\eta) e^{-\eta^2} + A_2 x^{-\frac{5}{2}} H_4(\eta) e^{-\eta^2} + A_3 x^{-\frac{7}{2}} H_6(\eta) e^{-\eta^2} + \cdots$$

$$- \frac{1}{8} A_1^2 x^{-3} e^{-2\eta^2} [4 H_o(\eta) + H_2(\eta)] + \cdots \tag{23}$$

Axi-symmetric flow:

$$\frac{1}{U_o} u_d = A_1 x^{-2} L_1(\eta^2) e^{-\eta^2} + A_2 x^{-3} L_2(\eta^2) e^{-\eta^2} + A_1 x^{-4} L_3(\eta^2) e^{-\eta^2}$$

$$+ \cdots$$

$$+ \tfrac{1}{2} A_1^2 x^{-4} [f_n \text{ of } \eta] \tag{24}$$

Thw coefficients A_1, A_2, $\cdots$ are to be determined by expanding the given velocity profile at a certain initial stage in Hermite or Laguerre Polynomials. For example, for the two-dimensional solution, we have

$$A_1 = \frac{1}{2U_o} \pi^{-\frac{1}{2}} x_o^{\frac{3}{2}} \int_o^\infty u_d(x_o, \eta) H_2(\eta) d\eta \, .$$

2.4. Swirling Flow

For laminar axisymmetric flow with swirl velocity the boundary-layer approximation gives

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}\right) \quad (25)$$

$$\frac{w^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r} \quad (26)$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial r} + \frac{vw}{r} = \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - \frac{w}{r^2}\right) \quad (27)$$

$$\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial r v}{\partial r} = 0 \quad (28)$$

Assuming $u_d (= u - U_o)$, v, w << U_o, Eq. (27) becomes

$$U_o \frac{\partial w}{\partial x} = \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - \frac{w}{r^2}\right) \quad (29)$$

Angular momentum is given by

$$M = 2\pi \int_0^\infty \rho\, u\, w\, r^2\, dr \simeq 2\pi\, \rho\, U_o \int_0^\infty w\, r^2\, dr \quad (30)$$

We assume the form

$$w = x^{-j} f(\eta) \quad \text{where} \quad \eta = \frac{r}{B_x^m} \quad (31)$$

Substituting Eq. (31) into Eq. (29), we require for similarity,

$$m = \tfrac{1}{2}$$

$$B^2 U_o / \nu = \text{const.}$$

Since B is arbitrary, we take the constant to be 4 for convenience. Then Eq. (29) becomes

$$f''(\eta) + (2\eta + \frac{1}{\eta}) f'(\eta) + 2(j - \frac{1}{\eta^2}) f(\eta) = 0 \quad (32)$$

From Eq. (30) we have

$$M \sim \int_0^\infty f(\eta)\eta^2\, d\eta$$

Hence we multiply Eq. (32) by η^2 and integrate to get

$$(2j-3) \int_0^\infty f \, \eta^2 \, d\eta = 0$$

Thus, we obtain

$$j = \frac{3}{2} \quad \text{for} \quad M \neq 0$$

$$j \neq \frac{3}{2} \quad \text{for} \quad M = 0$$

From the condition that $f(0)$ is finite, we find that

$$j = \frac{3}{2} + n, \qquad n \text{ integer}$$

Hence,

$$f(\eta) = \eta \, L_n^{(1)} (\eta^2) e^{-\eta^2}$$

where $L_n^{(1)}$ is Generalized Laguerre Polynomials.

The lowest power of x for M = 0 gives

$$w = c \, x^{-\frac{5}{2}} [\eta - \tfrac{1}{2} \eta^3] e^{-\eta^2}$$

Substituting into Eq. (26) we get the pressure as

$$p - p_\infty = - \frac{c^2}{32} \rho \, x^{-5} (5 - 6 \eta^2 + 2 \eta^4) e^{-2\eta^2}$$

When this is substituted into Eq. (25) for $u_d = u - U_o$, it is found that the solution obtained in the section 2.1 (neglecting the pressure gradient due to swirl motion) is valid up to x^{-5} term.

3. Turbulent Flow

3.1 Eddy-Viscosity Model

In the eddy-viscosity model, we use the same expression for the shear stress as in the laminar case except that the viscosity coefficient is replaced by an eddy viscosity reflecting increased momentum exchange due to turbulent motion. The eddy viscosity is no longer a fluid property like the molecular viscosity, but depends on the flow field characteristics.

Prandtl's hypothesis states

$$\epsilon = \kappa \, b(x) \, u_d(x, 0) \tag{33}$$

where κ is a constant. Since ϵ is a function of x only in this model, the turbulent flow solution is obtained from the corresponding laminar solution by the following transformation:

$$\left.\begin{aligned} \xi &= \frac{1}{\nu} \int_0^x \epsilon(x) dx \\ U(\xi, r) &= u(x, r) \\ V(\xi, r) &= \frac{\nu}{\epsilon(x)} v(x, r) \end{aligned}\right\} \tag{34}$$

After the transformation the equations for U and V are identical to those for laminar flow. Hence

$$b(x) = 2 \frac{\nu}{U_o} \xi_*^{\frac{1}{2}} , \qquad \xi_* = U_o \xi / \nu \tag{35}$$

and u_d is of the form:

$$\frac{1}{U_o} u_d(x, r) \simeq A \, \xi_*^{-\frac{1}{2}(3+j)} \left[f_o(\eta) + \frac{a_1}{\xi_*} f_1(\eta) + \cdots \right] \tag{36}$$

where $f_n(\eta)$ are Hermite or Laguerre polynomials normalized to be 1 at $\eta = 0$. Therefore

$$\frac{dx}{d\xi_*} = \frac{\nu^2}{U_o \epsilon(x)} \simeq \frac{1}{2\kappa A} \frac{\nu}{U_o} \xi_*^{\frac{1}{2}(2+j)} \left(1 - \frac{a_1}{\xi_*} - \cdots\right)$$

Thus

$$2\kappa A U_o x/\nu \simeq \frac{2}{4+j} \xi_*^{\frac{1}{2}(4+j)} \left(1 - \frac{4+j}{2+j} \frac{a_1}{\xi_*} + \cdots\right)$$

Inverting the asymptotic series, we obtain

$$\xi_* \simeq x_*^{2/(4+j)} \left[1 + \frac{2}{2+j} a_1 x_*^{-2/(4+j)} + \cdots\right] \qquad (37)$$

$$x_* \equiv (4+j) \kappa A U_o x/\nu$$

Hence

$$b(x) \simeq 2 \frac{\nu}{U_o} x_*^{1/(4+j)} \left[1 + \frac{1}{2+j} a_1 x_*^{-2/(4+j)} + \cdots\right] \qquad (38)$$

$$\frac{1}{U_o} u_d \simeq A x_*^{-(3+j)/(4+j)} \left\{f_o(\eta) + a_1 \left[f_1(\eta) - \frac{3+j}{2+j} f_o(\eta)\right] x_*^{-2/(4+j)} + \cdots\right\} \qquad (39)$$

3.2. Prandtl's Mixing Length Theory

Prandtl's eddy viscosity is independent of r at each cross section. In order to examine the effect of the eddy viscosity that varies in the radial direction, Prandtl's mixing length theory will be applied to the momentumless wake.

Prandtl's mixing length theory assumes

$$\frac{\tau}{\rho} = \ell^2 \left|\frac{\partial u}{\partial r}\right| \frac{\partial u}{\partial r} \qquad (40)$$

We take the mixing length proportional to the jet width

$$\ell = c_2 \, b(x)$$

and look for the similarity in the form

$$u_d = u - U_o = U_o A x^{-k} f(\zeta) \qquad (42)$$

where $\zeta = \dfrac{r}{b(x)} = \dfrac{r}{Bx^m}$

After we substitute u_d from (42) into the linearized momentum equation

$$U_o \frac{\partial u_d}{\partial x} - \frac{1}{r^j} \frac{\partial}{\partial r} (r^j \frac{\tau}{\rho}) = 0 \qquad \begin{cases} j = 0 & \text{(2-dimensional)} \\ j = 1 & \text{(axisymmetric)} \end{cases}$$

we get the condition for similarity

$$m + k = 1 , \tag{43}$$

and if we choose

$$B = C_2^2 A/m \tag{44}$$

the equation for f becomes

$$2f'f'' + j \zeta^{-1} f'^2 + \mathrm{sgn}(f') [\zeta f' + \alpha f] = 0 \tag{45}$$

where $\alpha = \frac{k}{m}$

Boundary conditions are as follows

$$\begin{cases} f(0) = 1 & f'(0) = 0 \\ f(\zeta_e) = 0 & f'(\zeta_e) = 0 \end{cases} \tag{46}$$

where ζ_e denotes the value at the edge of the jet.

If we multiply Eq. (45) by ζ^j and integrate, we obtain

$$[f'^2 \zeta^j]_{\zeta=\zeta_e} + [f \zeta^{1+j}]_{\zeta=\zeta_e} + (\alpha-1-j) \int_0^{\zeta_e} f \zeta^j \, d\zeta = 0$$

For the momentumless wake the last integral vanishes, and therefore, we require, if ζ_e should be infinity,

$$\begin{cases} [f'^2 \zeta^j]_{\zeta=\zeta_e} = 0 \\ [f \zeta^{1+j}]_{\zeta=\zeta_e} = 0 \\ \alpha \neq j + 1 \end{cases} \tag{47}$$

Eq. (45) is a second-order nonlinear ordinary differential equation and we have four boundary conditions (46) to be satisfied. Thus the problem is a nonlinear eigenvalue problem in which α and ζ_e are the eigenvalues to be determined.

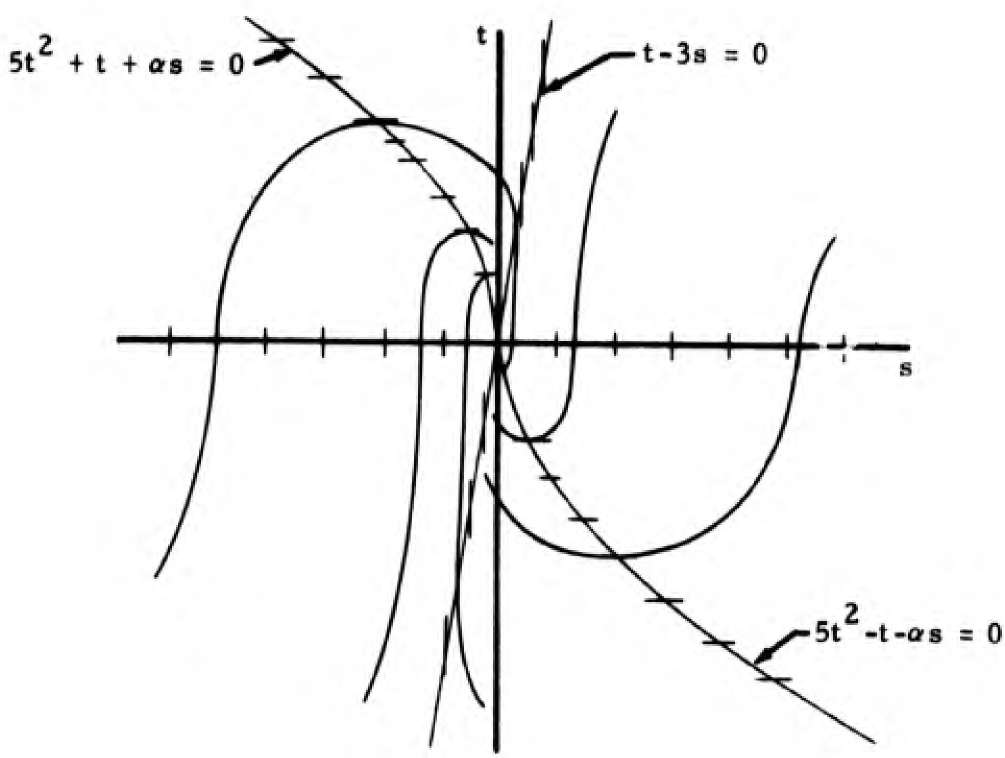
In order to study the behavior of the solution of the nonlinear equation (45), we make the following change of variables:

$$\begin{cases} s = \dfrac{f}{\zeta^3} \\ t = \dfrac{f'}{\zeta^2} \end{cases} \tag{48}$$

and then Eq. (45) becomes

$$\frac{dt}{ds} = -\frac{5t^2 + \mathrm{sgn}(t)\cdot(t+\alpha s)}{2t(t-3s)} \tag{49}$$

Integral curves are shown in the following sketch.



From the boundary condition (46), we find $t = -\infty$, $s = +\infty$, $\zeta = 0$, and $t = 0$, $s = 0$ at $\zeta = \zeta_c$.

Around the singular point at the origin, we assume first a series expansion as

$$s = a_1 t + a_2 t^2 + a_4 t^4 + \cdots +$$

Substituting into Eq. (49) and determining the coefficients, we get

$$s = -t^2 + \frac{2}{3}(2-\alpha)t^3 + \frac{1}{6}(2-\alpha)(3\alpha-\eta)t^4 + \cdots \quad (50)$$

or

$$s = -\frac{1}{\alpha}t + \frac{2\alpha+1}{\alpha}t^2 + \frac{2(2\alpha+1)(2\alpha+3)}{\alpha}t^3 + \cdots \quad (51)$$

Further, when we solve for the deviation $\sigma(t)$ from the above regular solutions

$$s = a_1 t + a_2 t^2 + \cdots + \sigma(t)$$

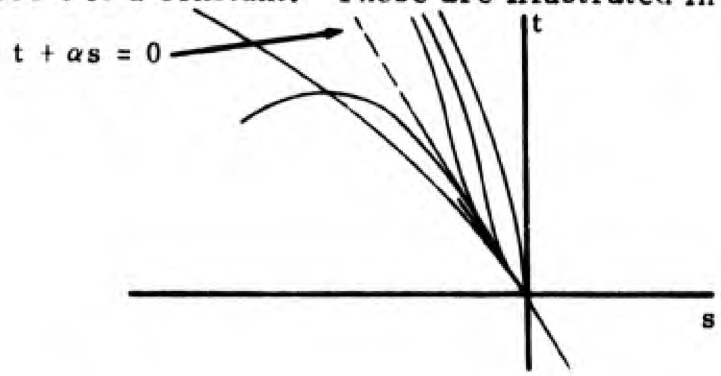
we get $\sigma(t) \doteq c\, e^{(2\alpha+6)t}$ for Eq. (50), and $\sigma(t) \doteq c\, e^{-\frac{1}{(2\alpha+6)}\frac{1}{t}}$ for Eq. (51). Therefore, the integral curves passing through the origin have the form either

$$s = -t^2 + \frac{2}{3}(2-\alpha)t^3 + \frac{1}{6}(2-\alpha)(3\alpha-\eta)t^4 + \cdots$$

or

$$s = -\frac{1}{\alpha}t + \frac{2\alpha+1}{\alpha}t^2 + \frac{2(2\alpha+1)(2\alpha+3)}{\alpha}t^3 + \cdots + c\, e^{-\frac{1}{2\alpha+6}\frac{1}{t}}$$

where c is a constant. These are illustrated in the following figure:



In terms of $f(\zeta)$, we see that

(1) Case $s \sim -\frac{1}{\alpha} t$: We obtain from the definition of s and t,

$$\frac{f'}{f} \sim -\alpha \frac{1}{\zeta}$$

Therefore

$$f \sim \zeta^{-\alpha}$$

Substituting into Eq. (40), we get

$$f'' \zeta^{j} \to 0, \quad f \zeta^{1+j} \to 0 \quad \text{as} \quad \zeta \to \infty .$$

Therefore, the drag is zero when $\alpha > 1+j$.

(2) Case $s \sim -t^2$: We obtain

$$f' \sim -\zeta^{\frac{1}{2}} f^{\frac{1}{2}}, \quad f^{\frac{1}{2}} \sim -\frac{1}{3}(\zeta^{\frac{3}{2}} + \text{const.})$$

Therefore $f = 0$, $f' = 0$ for a finite ζ_e, and the condition of the zero drag is $\alpha \neq 1+j$.

For $\alpha = j+1$, the solution for $D \neq 0$ is obtained by Schlichting(17) for two-dimensional wake, and Swain(17) for circular wake. For both cases the velocity distribution is given by $f = (1 - \frac{1}{3}\zeta^{\frac{1}{2}})^2$.

A numerical integration of the ordinary differential equation was employed to solve the nonlinear eigenvalue problem. In order to avoid two-parameter iteration, the solution for the outer part was obtained in the st-plane and the solution in the inner part was carried out in the $f\zeta$-plane. For the inner part where $f' < 0$, we have from (45)

$$\frac{df}{d\zeta} = -\left(\frac{g}{\zeta}\right)^{\frac{1}{2}}$$

$$\frac{dg}{d\zeta} = \zeta\left[-(\zeta g)^{\frac{1}{2}} + df\right]$$

We integrate the equations from $\zeta = 0$, where $f(0) = 1$, $g(0) = 0$, till we get $f' = 0$. Then the numerical integration of Eq. (49) starting at $t = s = 0$ along $s \sim -t^2$ is carried out, and two solutions are matched on the s-axis, where $t = 0$ and hence $f' = 0$. By this shooting method α was obtained after iterations as

$$\alpha = \frac{k}{m} \doteq 5.538$$

This given the main characteristic of the momentumless wake as follows:

	Mixing length	Eddy viscosity	Naudascher
$u_{d\mathrm{C\!\!\!L}}$	$x^{-0.847}$	$x^{-0.8}$	x^{-2}
b	$x^{0.153}$	$x^{0.2}$	$x^{.29}$
τ_{max}	$x^{-1.694}$	$x^{-1.6}$	x^{-2}

The integral curve and the velocity profiles are given in Fig. 1 and 2. The comparison with the experiment by Ginevskii[6] et al. shows that both the eddy viscosity and the mixing length theory underpredict the decay of the centerline velocity except in the initial region of the experiment.

3.3. Two-Equation Model

It is clear from the previous analysis that either the mixing-length model or the eddy-viscosity model is not capable of explaining Naudascher's experiment.

The two-equation models are attempts to incorporate the physics of turbulent flow into the description of turbulent shear flows. In these models, the turbulent shear stress is modelled as in Prandtl's eddy viscosity model,

$$\frac{\tau}{\rho} = \nu_t \frac{\partial u}{\partial y}$$

The eddy viscosity ν_t is expressed, from the dimensional consideration, as

$$\nu_t = e^{\frac{1}{2}} \ell$$

where e is the mean energy of turbulent motion, and ℓ is a scale associated with the turbulence. Instead of assuming the proportionality between $e^{\frac{1}{2}} \ell$ and $(\Delta u)b$, model equations for e and ℓ are introduced.

A model equation for the turbulence energy for axisymmetric flow with boundary-layer type approximation, is

$$\frac{De}{Dt} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\nu_t}{\sigma_e} \frac{\partial e}{\partial r} \right) + \nu_t \left(\frac{\partial u}{\partial r} \right)^2 - C_D \frac{e^{\frac{3}{2}}}{\ell} \tag{52}$$

where σ_e, C_D are constants. On the right-hand side, the first term represents the diffusion of turbulence energy, the second term the production of turbulence energy by the mean flow shear, and the third term the dissipation of turbulence energy (direct viscous dissipation is not included in the above form). The turbulence scale length ℓ is determined indirectly from the turbulent dissipation ϵ defined by

$$\epsilon = e^{\frac{3}{2}} / \ell \tag{53}$$

for which a model equation may be written as[13]

$$\frac{D\epsilon}{Dt} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial r} \right) + c_1 \, e \left(\frac{\partial u}{\partial r} \right)^2 - c_2 \frac{\epsilon^2}{e} \tag{54}$$

where σ_ϵ, c_1 and c_2 are constants determined from some experiments. If we integrate (52) across the wake, we obtain

$$\frac{d}{dx} \int_0^b e r dr = \int_0^b \nu_t \left(\frac{\partial u}{\partial r} \right)^2 r dr - C_D \int_0^b \frac{e^{\frac{3}{2}}}{\ell} r dr$$

According to Naudascher's experiments on momentumless wake,

$$e_{max}/U^2 \propto (x/D)^{-1.6}$$

$$b/D \propto (x/D)^{0.3}$$

$$u_d/U \propto (x/D)^{-2}$$

$$\overline{-u'v'}/U^2 = \nu_t \frac{\partial u}{\partial r}/U^2 \propto (x/D)^{-2.1}$$

for $x/D \geqslant 10$. Hence if it is assumed that $\ell \propto b$, we obtain

$$\frac{d}{dx}\int_0^b e r dr \propto x^{-2.0}$$

$$\int_0^b \nu_t \left(\frac{\partial u}{\partial r}\right)^2 r dr \propto x^{-3.8}$$

$$\int_0^b \frac{e^{\frac{3}{2}}}{\ell} r dr \propto x^{-2.1}$$

Thus, at large distances from the body, the dissipation outweighs the production.

In the far-wake the governing equations may be approximated by

$$U\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu_t r \frac{\partial u}{\partial r}\right) \tag{55a}$$

$$U\frac{\partial e}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\nu_t}{\sigma_e} r \frac{\partial e}{\partial r}\right) + \nu_t\left(\frac{\partial u}{\partial r}\right)^2 - c_D\,\epsilon \tag{55b}$$

$$U\frac{\partial \epsilon}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\nu_t}{\sigma_\epsilon} r \frac{\partial \epsilon}{\partial r}\right) + v_1 e\left(\frac{\partial u}{\partial r}\right)^2 - c_2\frac{\epsilon^2}{e} \tag{55c}$$

and

$$\nu_t = e^2/\epsilon$$

Suppose that the velocity defect, the integrated turbulent energy and dissipation entering the far-wake region are u_o, E_o, and ϵ_o, respectively, and we introduce the following dimensionless variables:

$$\bar{x} = (\epsilon_o/U E_o)x, \quad \bar{r} = (\epsilon_o^{\frac{1}{2}}/E_o^{3/4})r$$

$$\bar{e} = (E_o^{\frac{1}{2}}/\epsilon_o)e, \quad \bar{\epsilon} = (E_o^{\frac{3}{2}}/\epsilon_o^2)\epsilon$$

$$\bar{u} = u/u_o$$

Then Eqs. (55) become (omitting the bar from the dimensionless variables)

$$\frac{\partial e}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^2 r}{\sigma_e \epsilon}\frac{\partial e}{\partial r}\right) - c_D\,\epsilon + \Pi\,\frac{e^2}{\epsilon}\left(\frac{\partial u}{\partial r}\right) \tag{57a}$$

$$\frac{\partial \epsilon}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^2 r}{\sigma_\epsilon \epsilon}\frac{\partial \epsilon}{\partial r}\right) - c_2\,\frac{\epsilon^2}{e} + \Pi\, c_1\, e\left(\frac{\partial u}{\partial r}\right)^2 \tag{57b}$$

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^2 r}{\epsilon}\frac{\partial u}{\partial r}\right) \tag{57c}$$

where

$$\Pi = u_o^2\, E_o^{\frac{1}{2}}/\epsilon_o \tag{58}$$

When the production is insignificant compared to the diffusion and dissipation terms, we may put $\Pi = 0$, and the wake growth, the decay of e and ϵ are determined by Eqs. (57a, b). Then, Eq. (57c) governs the velocity decay in that wake.

In the $\Pi = 0$ wake, we look for solutions of the form:

$$e = x^{-p}\, g(\eta), \quad \epsilon = x^{-g}\, h(\eta); \quad \eta = r\, x^{-m} \tag{59}$$

Substituting these forms into Eqs. (57 a, b), we find

$$p = 2(1-m), \quad q = 3-2m \tag{60}$$

for the similarity condition, and the equations for g and h become

$$\frac{1}{\sigma_e \eta}\frac{d}{d\eta}\left(\eta\,\frac{g^2}{h}\frac{dg}{d\eta}\right) + m\,\eta\,\frac{dg}{d\eta} + 2(1-m)g - c_D\, h = 0 \tag{61a}$$

$$\frac{1}{\sigma_\epsilon \eta}\frac{d}{d\eta}\left(\eta\,\frac{g^2}{h}\frac{dh}{d\eta}\right) + m\,\eta\,\frac{dh}{d\eta} + (3-2m)h - c_2\,\frac{h^2}{g} = 0 \tag{61b}$$

with the boundary conditions

$$\frac{dg}{d\eta} = \frac{dh}{d\eta} = 0 \qquad \text{at} \qquad \eta = 0 \tag{62a}$$

$$g,\ h \to 0 \qquad \text{for large } \eta \tag{62b}$$

Eqs. (61) and the boundary conditions (62) constitute a nonlinear eigenvalue problem.

Let

$$g_o = g(0), \qquad h_o = h(0)$$

$$z = \eta(m\sigma_e h_o)^{\frac{1}{2}}/g_o$$

$$g(\eta) = g_o\, w_1(z), \quad h(\eta = h_o\, w_2(z)$$

Then Eqs. (61) and (62) become

$$\frac{d}{dz}\left(z\,\frac{w_1^2}{w_2}\,\frac{dw_1}{dz}\right) + z\left[z\,\frac{dw_1}{dz} + 2\left(\frac{1}{m} - 1\right)w_1 - c_D\,\lambda\,w_2\right] = 0 \tag{63a}$$

$$\frac{d}{dz}\left(z\,\frac{w_1^2}{w_2}\,\frac{dw_2}{dz}\right) + dz\left[z\,\frac{dw_2}{dz} + \left(\frac{3}{m} - 2\right)w_2 - c_2\,\lambda\,\frac{w_2^2}{w_1}\right] = 0 \tag{63b}$$

$$\left.\begin{aligned} &w_1 = w_2 = 1, \quad \frac{dw_1}{dz} = \frac{dw_2}{dz} = 0 \quad \text{at} \quad z = 0 \\ &w_1 = w_2 = 0 \qquad \text{for large } z \end{aligned}\right\} \tag{64}$$

where

$$\lambda = h_o/mg_o, \quad \alpha = \sigma_\epsilon/\sigma_e$$

Thus two eigenvalues, m and λ, have to be determined to satisfy the boundary conditions.

After these eigenvalues are determined, Eq. (57c) has to be solved to determine the velocity defect.

Let

$$u = u_c x^{-n} f(z) , \qquad f(0) = 1 .$$

Then Eq. (57c) becomes

$$\frac{d}{dz}\left(z \frac{w_1^2}{w_2} \frac{df}{dz}\right) + \frac{z}{\sigma_e}\left(z \frac{df}{dz} + \frac{n}{m} f\right) = 0 \tag{65}$$

$$f = 1, \quad \frac{df}{dz} = 0 \quad \text{at} \quad z = 0$$

$$f = 0 \quad \text{for large } z$$

In addition, f has to satisfy the zero-momentum condition

$$\int_o^{\infty} f\, z\, dz = 0 \tag{66}$$

The problem for f is also an eigenvalue problem, which determines the value of n.

A crude estimate for m and λ can be made by integral method. By integrating Eqs. (63) from 0 to ∞, we obtain

$$\begin{cases} 2(\frac{1}{m} - 2) \int_o^{\infty} z\, w_1 dz - c_D \lambda \int_o^{\infty} z\, w_n\, dz = 0 \\ (\frac{3}{m} - 4) \int_o^{\infty} z\, w_2\, dz - c_2 \lambda \int_o^{\infty} z \frac{w_2^2}{w_1}\, dz = 0 \end{cases}$$

Assume

$$w_1 = e^{-z^2}, \qquad w_2 = e^{-\frac{3}{2} z^2}$$

Then

$$\begin{cases} (\frac{1}{m} - 2) - \frac{1}{3} C_D \lambda = 0 \\ (\frac{1}{m} - \frac{4}{3}) - \frac{1}{4} C_2 \lambda = 0 \end{cases}$$

Hence

$$\begin{cases} m = (\frac{1}{4} C_2 - \frac{1}{3} C_D)/(\frac{1}{2} C_2 - \frac{4}{9} C_D) \\ \lambda = 2/(\frac{3}{4} C_2 - C_D) \end{cases}$$

For $C_2 = 0.18$, $C_D = 0.09$ (Spalding [18]), we obtain

$$m = 0.3, \qquad \lambda = 22 \tag{56(}$$

The estimated value of m agrees very well with Naudascher's experimental result -- $m = 0.29$ for $10 \leqslant (x-x_o)/D \leqslant 50$. The corresponding rate of turbulence energy decay is, from Eq. (60),

$$p = 1.4$$

while Naudascher's measurements indicate that this value is close to 1.57.

4. Conclusion

We have examined turbulent shear flow in the momentumless wake using the assumptions of (1) eddy-viscosity, (2) mixing-length, and (3) turbulence energy and dissipation model.

The eddy-viscosity and mixing-length models were incapable of describint the momentumless-wake flow. We attribute this to the non-equilibrium between the mean flow and the turbulence in this kind of wake.

The two-equation model was further simplified by neglecting productions by mean-flow shear in turbulence equations. A nonlinear eigenvalue problem was formulated for describing the far-field solution for the momentumless wake. A crude estimate of eigenvalues gives the wake growth and turbulence decay rate that agree fairly well with Naudascher's experiment.

In order to complete the solution, further work has to be carried out on the numerical solution of the nonlinear eigenvalue problem and on the extension of asymptotic solution to higher-order terms so that the connection to the near-field solution can be accomplished.

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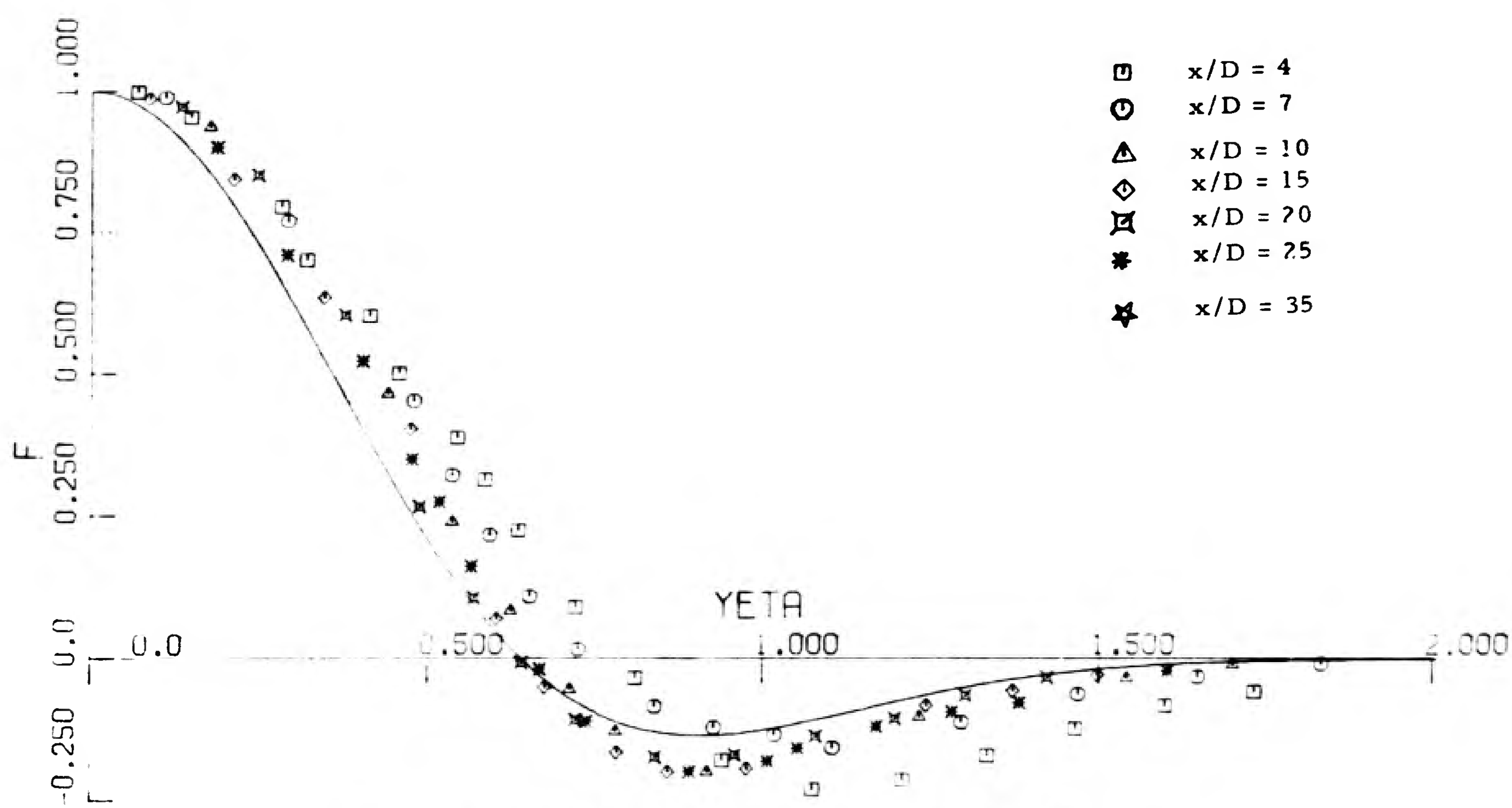

Fig. 1-a Eddy-Viscosity Model--Velocity Profile

(η at u_{min} matched between experiment and theory)

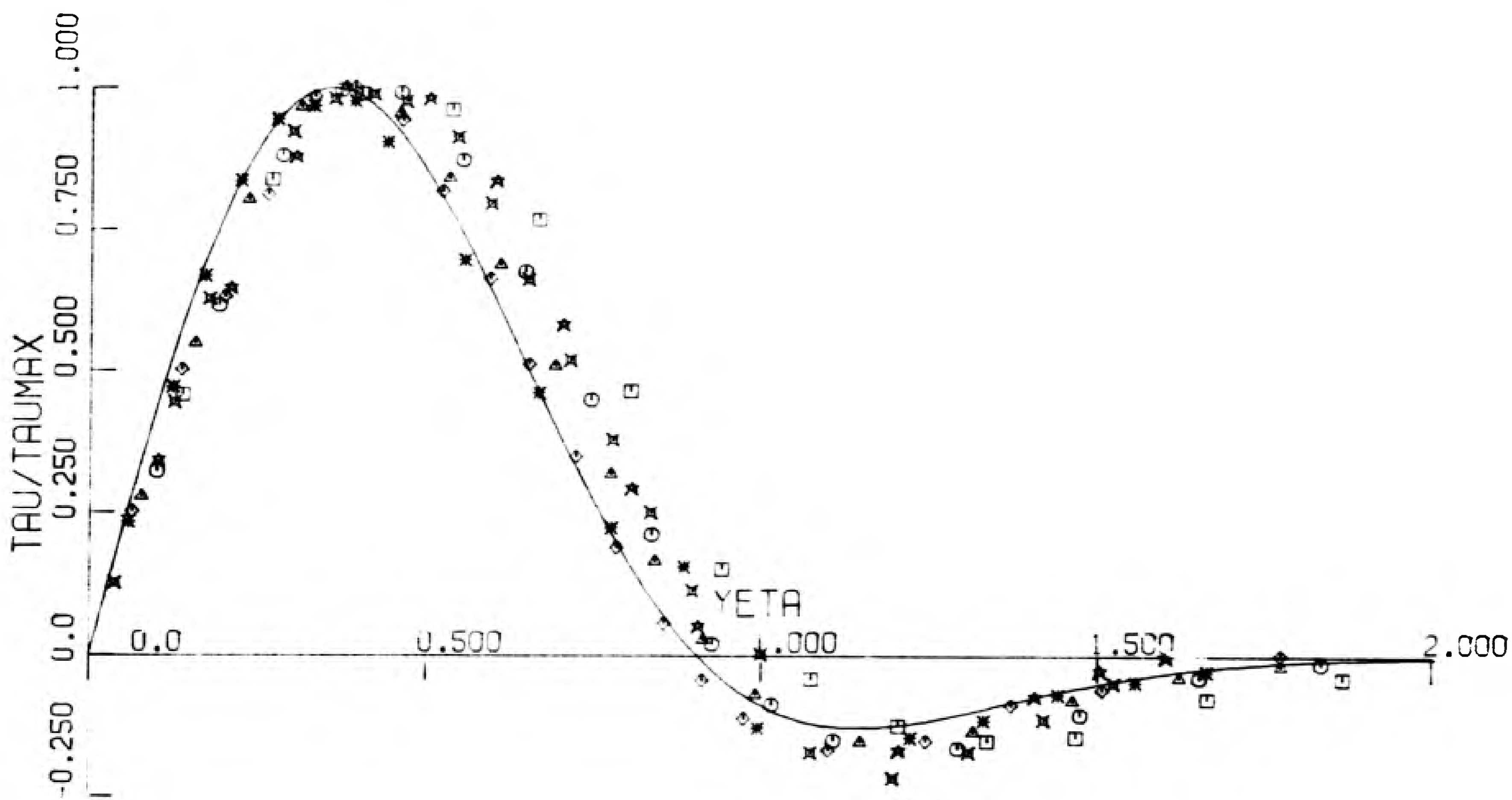

Fig. 1-b Eddy-Viscosity Model--Shear Stress

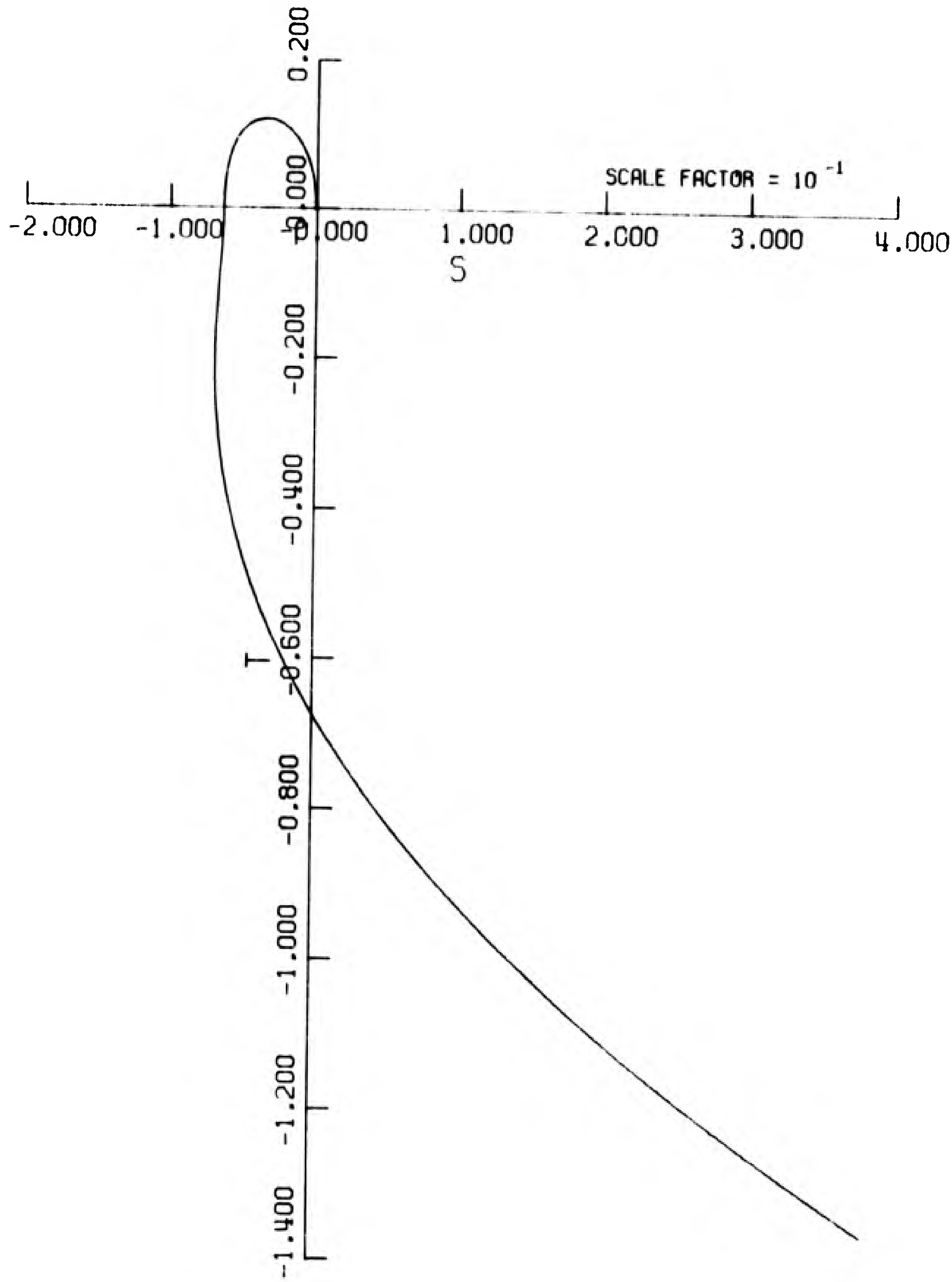

Fig. 2-a Mixing-Length Model--Integral Curve in st-Plane

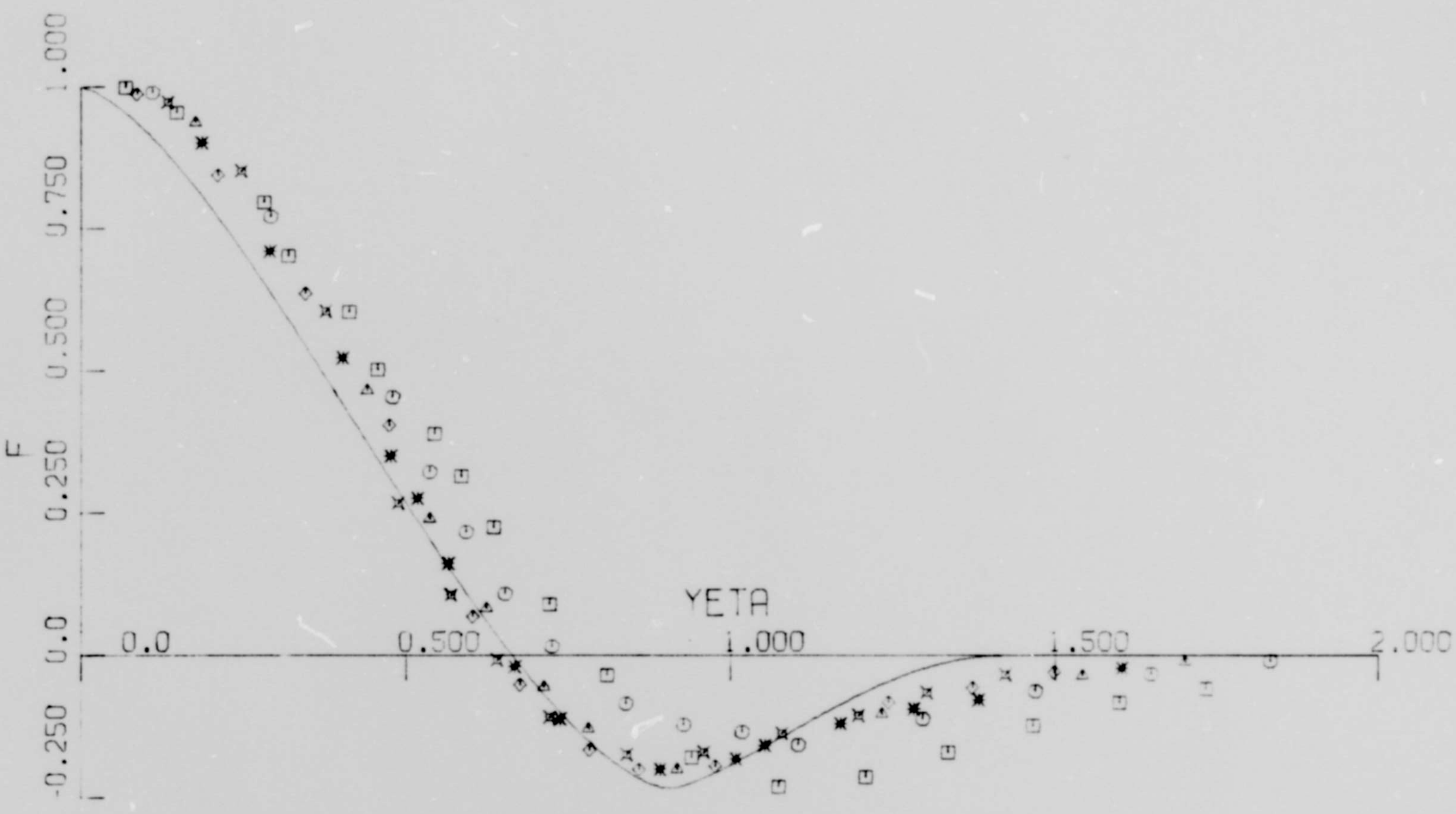

Fig 2-b Mixing-Length Model--Velocity Profile
(See Fig. 1-a for legend; η at u_{min} matched between experiment and theory)

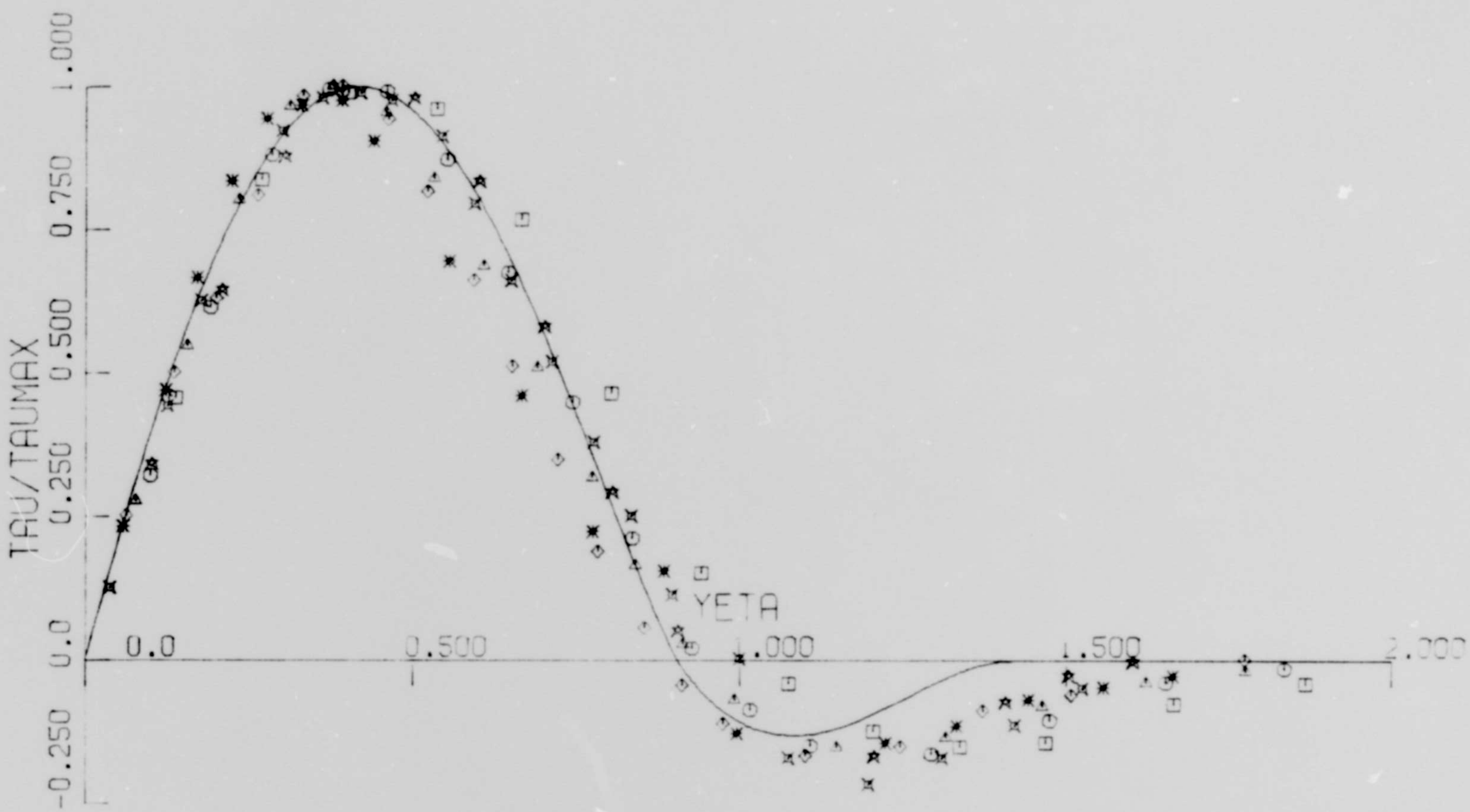

Fig. 2-c Mixing-Length Model--Shear Stress