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OPTIMAL DESIGN OF SABOTS

K. Matsui, et al

Iowa University

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Recently, a number of useful numerical algorithms have been developed for optimal design of mechanical systems. In most cases, domain of optimization of the mechanical system is assumed to be fixed. The objective of the present study is to develop an optimization technique in which the domain of the optimization problem may vary. In other words, for a continuous system, an optimization technique is sought that will allow the shape of the boundary of the system to be determined under various constraint conditions.

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Report No. 16

OPTIMAL DESIGN OF SABOTS

by

K. MATSUI and J.S. ARORA

Division of Materials Engineering
College of Engineering
The University of Iowa
Iowa City, Iowa 52242

A Final Work Report
For Contract No. DAAA-09-74- M-2021

Project Title: MINIMUM WEIGHT SABOT DESIGN

March, 1975

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TABLE OF CONTENTS

1. Definition of Problem

A sabot may be viewed as a device that transmits an internal ballistic pressure to a projectile and accelerates it down a barrel. After the exit from the barrel, the sabot will separate from the projectile, which flies to its target. The effectiveness of the sabot will be judged by the accuracy of projectile flight and the maximum kinetic energy imparted on the projectile. The proposed study is only concerned with the latter factor.

Assuming the total kinetic energy available is fixed, the kinetic energy of the projectile be maximized when that of sabot is minimized. This will be accomplished by minimization of sabot weight. Currently a massive solid sabot of revolution is used. By making use of optimization theory of Ref. [1], the optimal profile of a sabot will be sought, which will maximize its weight. Also of interest is a new type of light weight sabot construction that will be discussed in the following section.

2. Preliminary Study on Sabot Optimization

During the summer of 1974, a preliminary study of sabot design optimization was conducted under CAD-E support from the Army Armaments Command. Working papers from this work are in Enclosure 1. Preliminary findings are summarized.

Among all structures, it has been known that a shell type structure has a great load carrying capacity for the amount of material used. It is conceivable that the best design will be achieved in the area of shell material that is reinforced by modern high strength materials, such as carbon fibers. The system concept considered first was a multilayer shell of revolution as shown in Fig. 1. The bore pressure is transmitted from the outer layer to inner layers by elastic material that is emplaced between two adjacent layers, and to the projectile at one end of each layer.

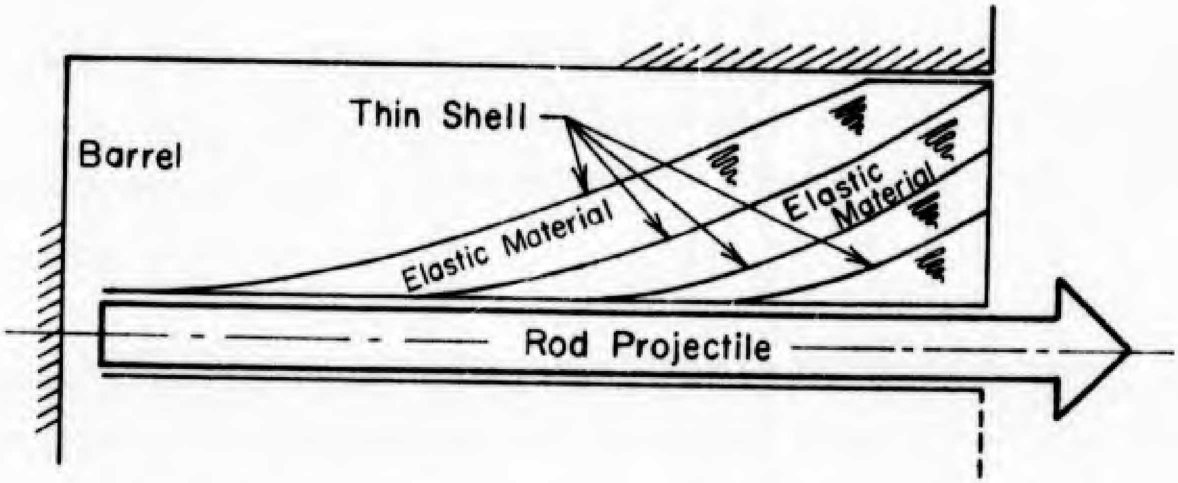

Fig. 1. A Sabot of Multilayer Shell of Revolution

A feasibility study was carried out on a simple model, consisting of a single layer of shell resting on an elastic foundation (Fig. 2). It was found that there is an extremely high compression stress at the forward end. This results in an excessively thick shell. It was finally concluded that this

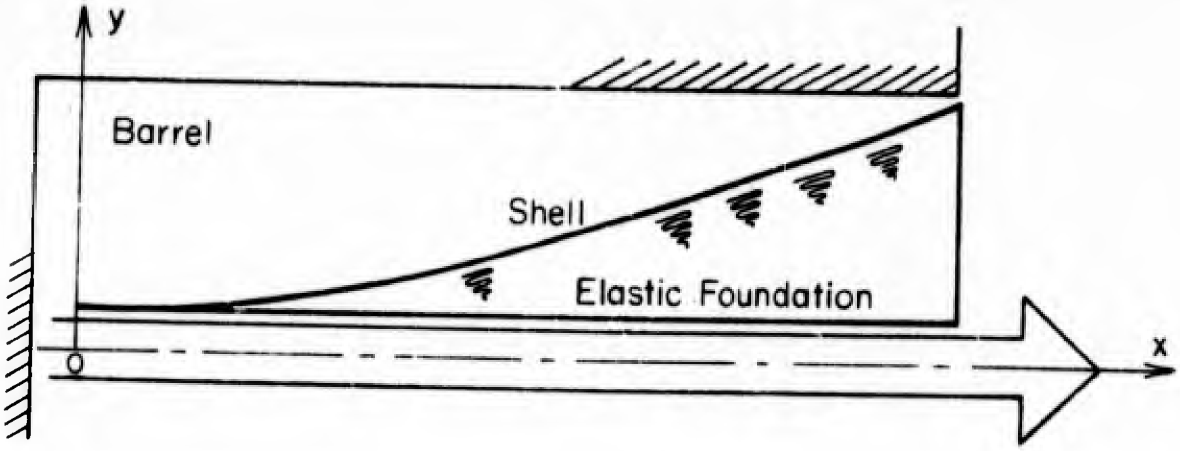

Fig. 2. A Single Layer Shell Used for a Feasibility Study

design concept was not feasible. For details, see Sections 16.1 to 16.6 of Enclosure 1.

A different type of structure has been studied, which consists of an outer shell and a number of plates extending from the projectile in a radial direction. Bore pressure will be transmitted from the outer shell to the shear plate and in turn to the projectile, through the plates, as shown schematically in Fig. 3. In this case, compressive stress in the shell is expected to reduce with an increase in number of plates, because the plates will prevent the shell from reducing its radius. Another advantage of this design is that there is a postbuckling equilibrium condition of the shell that results in tensile stress, rather than the compressive tangential stress that negated the previous concept. Since stress in all directions will be tensile, a high tensile strength material can be effectively utilized for the outer shell. The postbuckled shape is shown in Fig. 3. In this design concept, the shape of the shear plates plays a very important role. Shear plate optimization has been studied and looks very promising. For details, see Sections 16.7 to 16.10 of Enclosure 1.

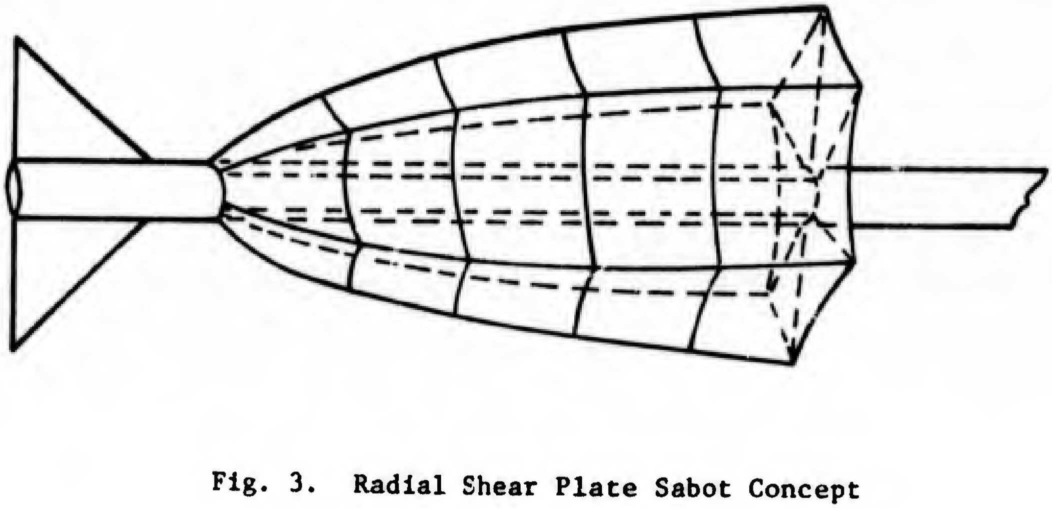

Fig. 3. Radial Shear Plate Sabot Concept

3. Formulation of Optimal Design Problem for a Solid Sabot of Revolution

A solid sabot of revolution is assumed to have a geometry that is generated by rotation of a cross section, shown in Fig. 4, around the x_1-axis.

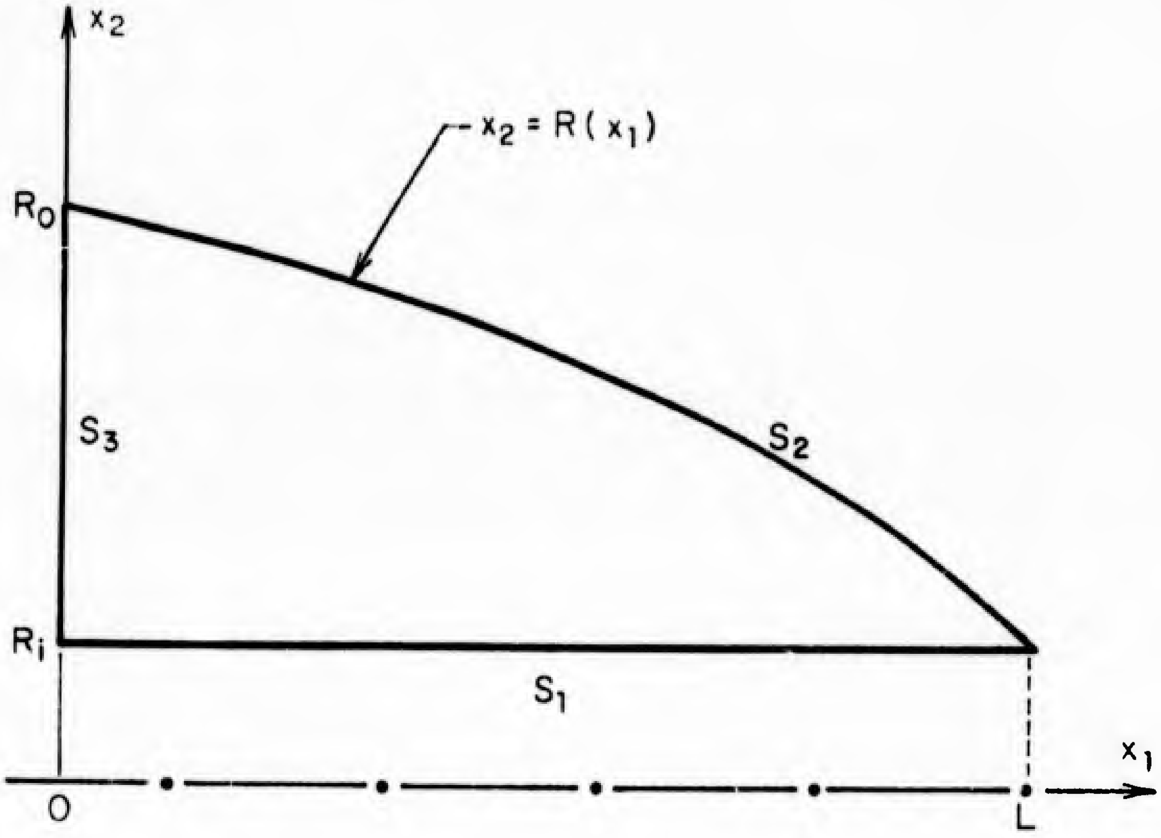

Fig. 4. Section of Solid of Revolution

Using Reissner's variational principle [2], the governing equations can be obtained as the following first-order partial differential equations.

$$- \frac{\partial z_3}{\partial x_1} - \frac{1}{x_2} \frac{\partial}{\partial x_2} (x_2 z_5) = F_{x_1}$$

$$- \frac{1}{x_2} \frac{\partial}{\partial x_2} (x_2 z_4) - \frac{\partial z_5}{\partial x_1} + \frac{\nu}{x_2} (z_3 + z_4) + E \frac{z_2}{x_2^2} = F_{x_2}$$

$$\frac{\partial z_1}{\partial x_1} + \nu \frac{z_2}{x_2} + \frac{1 + \nu}{E} \{(1 - \nu) z_3 - \nu z_4\} = 0 \tag{1a}$$

$$\frac{\partial z_2}{\partial x_2} + \nu \frac{z_2}{x_2} + \frac{1 + \nu}{E} \{(1 - \nu) z_4 - \nu z_3\} = 0$$

$$\frac{\partial z_1}{\partial x_2} + \frac{\partial z_2}{\partial x_1} - \frac{z_5}{G} = 0$$

$$z_3(0, x_2) = 0$$

$$z_5(0, x_2) = 0$$

$$z_1(x_1, R_1) = 0$$

$$z_2(x_1, R_1) = 0 \tag{1b}$$

$$z_3(x_1, R(x_1)) \cos(x_1, n) + z_5(x_1, R(x_1)) \cos(x_2, n) = f_{x_1}$$

$$z_4(x_1, R(x_1)) \cos(x_2, n) + z_5(x_1, R(x_1)) \cos(x_1, n) = f_{x_2}$$

where

z_1 = displacement in the axial direction

z_2 = displacement in the radial direction

z_3 = normal stress in the axial direction σ_x

z_4 = normal stress in the radial direction σ_r

z_5 = shearing stress τ_{rx}

n = normal to the boundary

ν = Poisson's ratio

E = Young's modules

G = shear modules

F_{x_1} = body force in the axial direction

F_{x_2} = body force in the radial direction

f_{x_1} = boundary traction in the axial direction

f_{x_2} = boundary traction in the radial direction

The boundaries S_1 and S_2 in Fig. 4 are fixed, but S_3 has to be designed. The boundary can be expressed in terms of slope v as

$$\frac{dR(x_1)}{dx_1} = v(x_1) \tag{2a}$$

$$\begin{aligned} R(0) &= R_o \\ R(L) &= R_1 \end{aligned} \tag{2b}$$

Since Eq. (2a) is a first-order differential equation, one of Eqs. (2b) has to be treated as a constraint condition.

The objective is to minimize the weight of sabot, which is equivalent to minimizing its volume, namely:

$$J = 2\pi \int_0^L \int_{R_i}^{R(x_1)} x_2 \, dx_2 \, dx_1 \tag{3}$$

The minimization will be carried out subject to stress constraints in the material. Von Mises yield criterion is used, namely:

$$\omega = (z_3 + z_4)^2 + 3(z_3 - z_4)^2 + 12z_5^2 - 4\sigma_{yp}^2 \leq 0 \quad , \tag{4}$$

where σ_{yp} is yield stress. Equation (4) is a state variable constraint [1] and can most effectively be expressed by an equivalent functional constraint form.

$$\psi = 2\pi \int_0^L \int_{R_i}^{R(x_1)} \{\omega + |\omega|\} \, x_2 \, dx_2 \, dx_1 = 0 \tag{5}$$

Introducing operator notation, the above optimal design problem can be summarized as follows: Minimize

$$J = 2\pi \int_0^L \int_{R_i}^{R(x_1)} x_2 \, dx_2 \, dx_1 \tag{6}$$

subjected to constraints

$$Lz = 0 \tag{7a}$$

$$B(v)z = q(v, R) \tag{7b}$$

$$\frac{dR(x_1)}{dx_1} = v(x_1) \tag{8a}$$

$$R(0) = R_o \tag{8b}$$

$$\psi_1 = R(L) - R_1 = 0 \tag{9}$$

$$\psi_2 = 2\pi \int_0^L \int_0^{R(x_1)} F(z)\ x_2\ dx_2\ dx_1 = 0 \tag{10}$$

It should be noted that the domain of integration varies with the variable $R(x_1)$

To obtain an optimal solution, one needs to know the effect of a small change in design variables. In this problem, v is the only design variable. Using first-order variational notation from the calculus of variations [3], one can write

$$\delta J = 2\pi \int_0^L R(x_1)\ \delta R\ dx_1 \tag{11}$$

$$L\delta z = 0 \tag{12a}$$

$$B(v)\delta z + (B(v)z)_v \delta v = \frac{\partial q}{\partial v}\ \delta v + \frac{\partial q}{\partial R}\ \partial R \tag{12b}$$

$$\frac{d\delta R}{dx_1} = \delta v \tag{13a}$$

$$\delta R(0) = 0 \tag{13b}$$

$$\delta\psi_1 = \delta R(L) = 0 \tag{14}$$

$$\delta\psi_2 = 2\pi \int_0^L \int_0^{R(x_1)} \frac{\partial F}{\partial z} \delta z \; x_2 \; dx_2 \; dx_1 + 2\pi \int_0^L F(z) \; R(x_1) \delta R \; dx_1 \qquad (15)$$

Equations (11), (14), and (15) contain the state variables δz and δR. These quantities can be removed in the following manner. Let

$$\frac{d\lambda_R}{dx_1} = - R(x_1)$$

$$\lambda_R(L) = 0$$

Then

$$\frac{d\lambda_R}{dx_1} \delta R = - R(x_1) \delta R$$

Integrating by parts, one obtains

$$\int_0^L - \lambda_R \frac{d\delta R}{dx_1} dx_1 + \lambda_R \delta R \Big|_0^L = \int_0^L - R\delta R \; dx_1$$

So,

$$\int_0^L \lambda_R \; \delta v \; dx_1 = \int_0^L R\delta R \; dx_1$$

Substitution of the above into Eq. (11) yields

$$\delta J = 2\pi \int_0^L \lambda_R \delta v \; dx_1 \qquad (16)$$

Also,

$$\delta\psi_1 = \delta R(L) = \int_0^L \delta v \, dx_1 = 0 \tag{17}$$

To remove the state variable dependency from Eq. (15), one can use Eqs. (12a) and (12b) and write

$$\int_0^L \int_{R_1}^{R(x_1)} \left\{ \lambda^T L\delta x - \delta z^T L\lambda \right\} x_2 \, dx_2 \, dx_1 = \int_S \lambda^T B(v)\delta z \, ds \quad , \tag{18}$$

where L is a self-adjoint operator [4]. S is a boundary and consists of S_1, S_2, and S_3 as shown in Fig. 4. Define λ to satisfy the equation

$$L\lambda = \frac{\partial F^T}{\partial z} \tag{19}$$

Then, substituting from Eqs. (18) and (19) into Eq. (15), one obtains

$$\delta\psi_2 = 2\pi \int_0^L F(z) \, R(x_1) \, dx_1$$

$$- 2\pi \int_s \left\{ \lambda^T \frac{\partial q}{\partial v} \delta v + \lambda^T \frac{\partial q}{\partial R} \delta R - \lambda^T (B(v)z)_v \, dv \right\} ds$$

The above equation can be arranged in the following form

$$\delta\psi_2 = 2\pi \int_0^L P_1(z, R) \, \delta R \, dx_1 + 2\pi \int_0^L P_2 \, \delta v \, dx_1 \tag{20}$$

δR dependency can be eliminated by using an equation adjoint to Eqs. (8a) and (8b)

$$\frac{d\lambda_p}{dx_1} = - P_1 \tag{21}$$

$$\lambda_p(L) = 0$$

Multiplying δR on the both sides of Eq. (21) and integrating by parts, one will obtain

$$\int_0^L \lambda_p \, \delta v \, dx_1 = \int_0^L P_1 \, \delta R \, dx_1$$

Then, Eq. (20) can be given in terms of only the design variable δv as

$$\delta\psi_2 = 2\pi \int_0^L (\lambda_p + Q) \, \delta v \, dx_1 \tag{22}$$

After the effect of small changes in design variables is found, the optimization algorithm in Chapter 9 of [1] is applied. In the algorithm, one needs to know displacements and stresses. These quantities can be obtained by either solving the state equations (1a) and (1b) directly, or by using finite element methods. The adjoint Eq. (19) can be solved in the same manner, wherein the right-hand side can be treated as body force or initial stress in the system.

An algorithm for optimization of the sabot of revolution will be developed and applied to test cases in which stress analysis can be performed approximately by variational methods or with finite element stress analysis codes, if such codes can be provided by the government.

Variations in enforcement of the interface condition between the sabot and rod will also be studied. A formulation will be developed that attempts to optimally distribute the normal and shear stress along the rod so that a friction bond can be employed to accelerate the rod.

The shell shear plate approach outlined in Section 2 will also be studied. Realistic optimal designs, using fiber reinforced materials, will be calculated and compared with results obtained for the solid sabot of revolution.

4. References

[1] Computer Aided Design of Mechanical Systems, AMC Design Handbook, AMCP 706-192, HQ AMC, Washington, D.C., 1973.

[2] Reissner, E., "On a Variational Theorem in Elasticity," J. of Math. and Phys. 29, July 1950, pp. 90-95.

[3] Gelfand, M. and Fomin, S. W., Calculus of Variations, Prentice Hall, Englewood Cliffs, N. J., 1963.

[4] Lanczos, C., Linear Differential Operators, Van Nostrand, London, 1961.

Enclosure 1 to Report Number 16

OPTIMAL DESIGN OF SABOTS

Submitted to

U.S. Army Armament Command
AMSAR-RDT
Rock Island Arsenal
Rock Island, Il 61201

Chapter 16: Optimal Design of a Sabot for Antitank Artillery.

16 - 1 Objective

A sabot transmit interior ballistic pressure to thrust a rod that accelerates the rod in the gun tube. The effectiveness of the sabot will be judged by the accuracy of flight of rod and the kinetic energy imparted to the rod. This study is only concerned with the latter factor.

Assuming the total kinetic energy available is fixed, an improvement in design will be achieved through minimization of sabot weight. In this way, the kinetic energy of the penetrator is maximized.

16 - 2 Design of a Sabot

Among all structures that might be used for a sabot a shell type structure has great potential, particularly if modern high strength fiber materials such as carbon fibers, can be used.

The system concept considered here consists of multilayers (See Fig 1) of shells of revolution. Adjacent shells are spaced by an elastic material so that a traction of the outer pressure will be transmitted to inner shells. Each shell is reinforced by high strength material in a direction of generating curve, i.e. the axial direction of the tube.

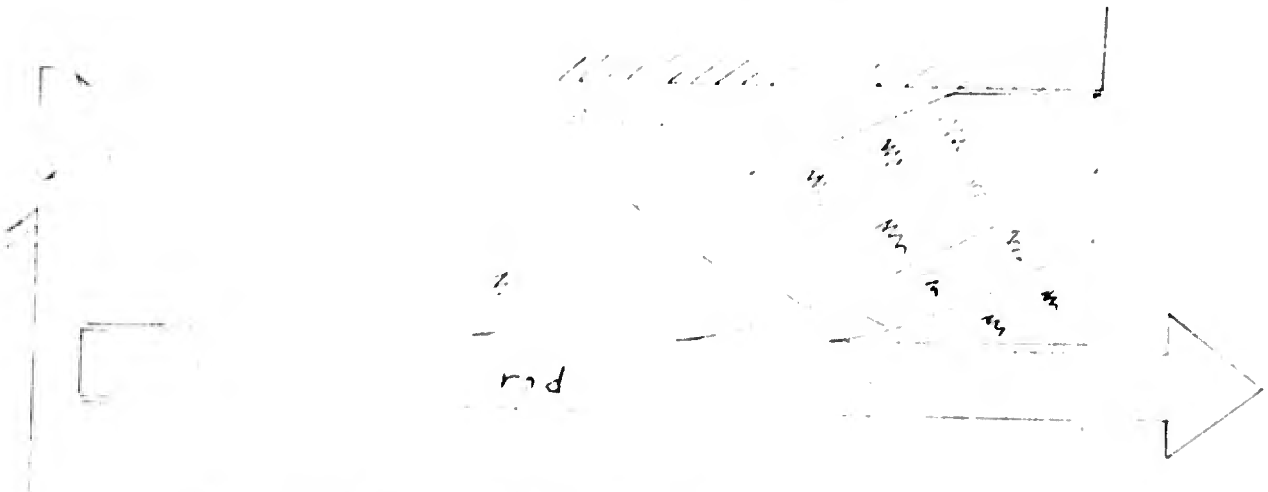

Fig. 1. A Sabot of Multilayer Shell of Revolution

16 - 3 A Simplified Model

Feasibility will be studied first on a much simplified model, consisting of a single shell resting on an elastic foundation as shown in Fig 2. The shell material will be taken as aluminum, with carbon fibers imbeded in the axial direction.

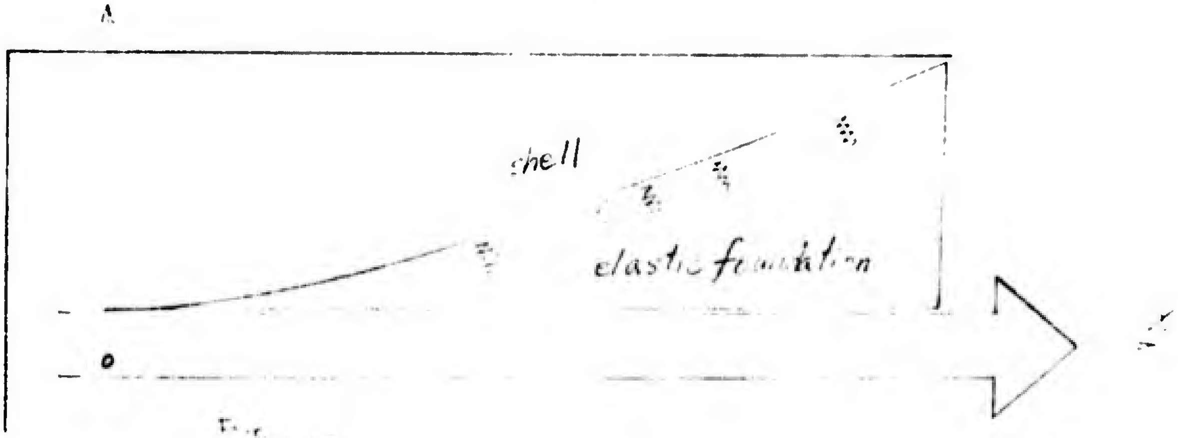

Fig 2

Analysis of the structure will initially use thin shell membrane theory with an anisotropic shell construction. Preliminary calculations will determine if the shell configuration can be chosen so that compressive tangential stress can be kept to a minimum with a thin shell.

16 - 4 Basic Structural Equations and Constraints

Equilibrium. For a thin shell with rotational symmetry, the equations of equilibrium are (16-1,2):

$$R_0 \frac{dN_1}{dx} + \frac{dR_0}{dx} N_1 - \frac{dR}{dx} N_2 = 0 \qquad (16\text{-}1)$$

$$-\frac{N_1 \frac{d^2R_0}{dx^2}}{\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{3/2}} + \frac{N_2}{R_0\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} - q + kw = 0 \qquad (16\text{-}2)$$

where

N_1 is the stress resultant in axial direction

N_2 is the stress resultant in circumferential direction

$R_0(x)$ is the Radius of shell; a design variable

q is the distributed force normal to the shell surface outward positive

$k(x)$ is the Elastic constant of the foundation

Strain - StressResultant. Stress-Strain relations for the anisotropic thin shell are (   ):

$$\varepsilon_1 = \frac{1}{E_1 h} (N_1 - \nu_{21} N_2) \qquad (16\text{-}3)$$

$$\varepsilon_2 = \frac{1}{E_2 h} (N_2 - \nu_{12} N_1) \qquad (16\text{-}4)$$

and

$$\gamma_{12} = 0 , \qquad (16\text{-}5)$$

where

$\overset{\varepsilon_1}{E_1}$ and $\overset{\varepsilon_2}{E_2}$ are Strain in axial and corcumferential directions, respectively

E_1 and E_2 are Youngs' modulus in axial and circumferential directions, respectively

$\overset{\nu_{12}}{V_{12}}$ and $\overset{\nu_{21}}{V_{21}}$ are Poison's ratios

h (x) is the thickness of shell; a design variable.

Strain-displacement. Strain displacement relations for the shell are (   ):

$$\varepsilon_1 = \frac{1}{\left\{1 + \left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} \frac{du}{dx} - \frac{w \frac{d^2 R_0}{dx^2}}{\left\{1 + \left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} \qquad (16\text{-}6)$$

$$\varepsilon_2 = \frac{u \frac{dR_0}{dx}}{R_0 \left\{1 + \left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} + \frac{w}{R_0 \left\{1 + \left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} \qquad (16\text{-}7)$$

and

$$\gamma_{12} = 0 \qquad (16\text{-}8)$$

where

$\overset{u}{U}$ and $\overset{w}{W}$ are displacements in the axial and radial directions, respectively. Equating for E_1 and E_2 from Eqs.(3) and (4) into Eqs. (6) and (7) yields

$$\frac{1}{E_1 h}(N_1 - \nu_{21} N_2) = \frac{\frac{du}{dx}}{\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} - \frac{w\frac{d^2R_0}{dx^2}}{\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} \qquad (16\text{-}9)$$

and

$$\frac{1}{E_2 h}(N_2 - \nu_{12} N_1) = \frac{u\frac{dR_0}{dx}}{R_0\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} + \frac{w}{R_0\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} \qquad (16\text{-}10)$$

Solving Eqs. (2) and (4) for N_2 and W yields

$$N_2 = \nu_{12} N_1 + \frac{E_2 h u \frac{dR_0}{dx}}{R_0\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} + \frac{E_2 h R_0\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}}{E_2 h + kR_0^2\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}} \qquad (16\text{-}11)$$

and

$$\times\left[\frac{\frac{d^2R_0}{dx^2}N_1}{\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{3/2}} - \frac{\nu_{12}N_1}{R_0\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} - \frac{E_2 h u\frac{dR_0}{dx}}{R_0^2\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}} + q\right]$$

$$w = \frac{R_0^2\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}}{E_2 h + kR_0^2\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}}\left[\frac{\frac{d^2R_0}{dx^2}N_1}{\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{3/2}} - \frac{\nu_{12}N_1}{R_0\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}}\right.$$

$$\left. - \frac{E_2 h u\frac{dR_0}{dx}}{R_0^2\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}} + q\right] \qquad (16\text{-}12)$$

Boundary Conditions and constraints are

$$N_1(0) = T \qquad (16\text{-}13)$$

$$u(L) = 0 \qquad (16\text{-}14)$$

$$R_0(0) = R_r \qquad (16\text{-}15)$$

$$R_0'(0) = 0 \qquad (16\text{-}16)$$

$$R_0(L) = R_b \qquad (16\text{-}17)$$

$$N_1 - N_{1a} \leq 0 \qquad (16\text{-}18)$$

$$N_2 - N_{2a} \leq 0 \qquad (16\text{-}19)$$

$$R_r - R_0 \leq 0 \qquad (16\text{-}20)$$

$$R_0 - R_b \leq 0 \qquad (16\text{-}21)$$

$$h_{min} - h \leq 0 \qquad (16\text{-}22)$$

where

T is a specified shear force on the rod

R_r Rr is the radius of rod

R_b Rb is the radius of barrel

N_{1a} and N_{2a} are allowable stress resultants in the axial and circumferential directions, respectively.

hmin is the specified minimum thickness of the shell.

Eq. (13) guarantees sufficient thrust on the rod. Eqs (15), (16), (17), (20), and (21) are geometrical constraints on the sabot middle surface. Eqs (18) and (19) are constraints on the allowable stress resultant. Eq (22) prevents the thickness of the shell from becoming zero. It should be noted here that membrane theory does not allow imposition of displacement or sheer constraints in the radial direction.

In the concept being studied, high strength carbon fiber is imbeded in the axial direction of the shell. Youngs' modulas and allowable stress resultant are

$$E_1 = E_{Al}\left(1 - \frac{A_{GF}}{2\pi R_r h_1}\right) + E_{GF} \cdot \frac{A_{GF}}{2\pi R_r h_1} \qquad (16\text{-}23)$$

and

$$N_{1a} = \sigma_{AL}\left(1 - \frac{A_{GF}}{2\pi R_r h_1}\right) + \sigma_{GF} \frac{A_{GF}}{2\pi R_r h_1} \qquad (16\text{-}24)$$

where

E_{Al} is Youngs modulas of aluminum

E_{GF} is Youngs modules of carbon fibers

A_{GF} is the arear of cross section of the carbon fiber

σ_{AL} is the allowable stress of aluminum

σ_{GF} is the allowable stress of the carbon fibers

To be consistent with notations in Chapt. 8, the following replacement will be made ·

$$\left.\begin{aligned} x &\rightarrow t \\ N_1 &\rightarrow x_1 \\ N_2 &\rightarrow x_2 \\ R_0 &\rightarrow x_3 \\ \frac{dR_0}{dx} &\rightarrow x_4 \\ \frac{d^2R_0}{dx^2} &\rightarrow u_1 \\ h &\rightarrow u_2 \\ A_{G.T.} &\rightarrow b \end{aligned}\right\} \tag{16-25}$$

16 - 5 <u>Steepest decendent Method for solving the optimal Design problem</u>

If one examines the state equations it is clear that a closed form solution is not impossible. The steepest decendent method can be used in adving optimal design problems of this complexity, and will be adopted for solution of this problem. The design problem will now be written in the standard form of section 8.2.1 of (Handbook) to allow orderly implementation of the steepest descent method.

The object function is the weight of the shell

$$J = \int_0^L x_3 u_2 \, dt = \int_0^L f_0 \, dt \tag{16-26}$$

The state differential equations are

$$\begin{aligned} \frac{dx_1}{dt} = f_1 = \frac{x_4}{x_3} \Big[(\nu_{12} - 1) x_1 + \frac{x_4 E_2 u_2}{x_3 (1 + x_4^2)^{1/2}} \\ + \frac{E_2 u_2 x_3 (1 + x_4^2)^{1/2}}{E_2 u_2 + k x_3^2 (1 + x_4^2)} \Big\{ \frac{x_1 u_1}{(1 + x_4^2)^{3/2}} - \frac{\nu_{12} x_1}{x_3 (1 + x_4^2)^{1/2}} \\ - \frac{E_2 u_2 x_2 x_4}{x_3^2 (1 + x_4^2)} + q \Big\} \Big] \end{aligned} \tag{16-27}$$

$$\frac{dx_2}{dt} = f_2 = \frac{x_1}{E_1 u_2}(1+x_4^2)^{1/2}(u_2 - u_1 \nu_{12}) - \frac{\nu_{21}}{E_1 x_3} E_2 u_2 x_2 x_4$$
$$+ \frac{x_3\{x_3 u_1 E_1 - E_2 \nu_{21}(1+x_4^2)\}}{E_2\{E_2 u_2 + k x_3^2 (1+x_4^2)\}} \left\{ \frac{x_1 u_1}{(1+x_4^2)^{1/2}} - \frac{\nu_{12} x_1}{x_3 (1+x_4^2)^{1/2}} \right.$$
$$\left. - \frac{E_2 u_2 x_4 x_2}{x_3^2 (1+x_4^2)} + g \right\} \qquad (16\text{-}28)$$

$$\frac{dx_3}{dt} = f_3 = x_4 \qquad (16\text{-}29)$$

$$\frac{dx_4}{dt} = f_4 = u_1 \qquad (16\text{-}30)$$

The boundary conditions are

$$\theta_1 = x_1(0) - T = 0 \qquad (16\text{-}31)$$

$$\theta_2 = x_2(L) = 0 \qquad (16\text{-}32)$$

$$\theta_3 = x_3(0) - R_r = 0 \qquad (16\text{-}33)$$

$$\theta_4 = x_4(0) = 0 \qquad (16\text{-}34)$$

Finally, the pointwise performance constraints are

$$\omega_1 = x_1 - (\sigma_{G1} - \sigma_{AL})\frac{b}{2\pi x_3} - \sigma_{AL} u_2 \leq 0 \qquad (16\text{-}35)$$

$$\omega_2 = \nu_{12} x_1 + \frac{E_2 u_2 x_2 x_4}{x_3(1+x_4^2)^{1/2}} + \frac{E_2 u_2 x_3 (1+x_4^2)^{1/2}}{E_2 u_2 + k x_3^2 (1+x_4^2)} \times$$
$$\left\{ \frac{x_1 u_1}{(1+x_4^2)^{1/2}} - \frac{\nu_{12} x_1}{x_3 (1+x_4^2)^{1/2}} - \frac{E_2 u_2 x_2 x_4}{x_3^2 (1+x_4^2)} + g \right\} - N_{20} \leq 0 \qquad (16\text{-}36)$$

$$\omega_3 = R_r - x_3(t) \leq 0 \qquad (16\text{-}37)$$

$$\omega_4 = x_3(t) - R_b \leq 0 \qquad (16\text{-}38)$$

$$\phi_1 = h_{min} - u_2(t) \leq 0 \qquad (16\text{-}39)$$

The constrains $\overset{\omega_1}{W_1}$ through $\overset{\omega_4}{W_4}$ must be transformed into functional constraints in order to employ the steepest de$\overset{c}{s}$ant methods Define

$$\psi_1 = \int_0^L \{\omega_1 + |\omega_1|\} dt \equiv \int_0^L L_1 \, dt = 0 \qquad (16\text{-}40)$$

$$\psi_2 = \int_0^L \{\omega_2 + |\omega_2|\} dt \equiv \int_0^L L_2 \, dt = 0 \qquad (16\text{-}41)$$

$$\psi_3 = \int_0^L \{\omega_3 + |\omega_3|\} dt \equiv \int_0^L L_3 \, dt = 0 \qquad (16\text{-}42)$$

$$\psi_4 = \int_0^L \{\omega_4 + |\omega_4|\} dt \equiv \int_0^L L_4 \, dt = 0 \qquad (16\text{-}43)$$

This is the full set of equations that defines the sabot optinal design problem.

A numerical method is first developed to solve the boundary - value problem for shell performance. In the boundary - value problem, the only initial value missing is $X_2(0)$. Defining the parameter ξ as this missing initial value, one has the full set of initial conditions:

$$\left.\begin{aligned} x_1(0) &= T \\ x_2(0) &= \xi \\ x_3(0) &= R_r \\ x_4(0) &= 0 \end{aligned}\right\} \qquad (16\text{-}44)$$

For a given value of , one can easily integrate the differential equations

$$\frac{dx}{dt} = f(x, u, b, t)$$

from $t=0$ to $t=L$, and obtain the terminal values

$$x(L;\xi) = [\eta_1, \eta_2, \eta_3, \eta_4]^T.$$

The terminal boundary condition ~~X~~ $x_2(L)=0=\eta(\xi)$ is just an algebraic equation for ξ. Many numerical methods can be used to solve the problem effectively. Newton's method for algebraic equations will be employed here.

The iterative solution by Newton's method proceeds by improving estimates according to the formula

$$\xi^n=\xi^{n-1}-\left[\frac{\partial\eta}{\partial\xi}(\xi^{n-1})\right]^{-1}\left[\eta(\xi^{n-1})-0\right], \tag{16-45}$$

where ξ^i is the i th iteration solution. To implement the ~~alberithm~~ algorithm, $\left[\frac{\partial\eta}{\partial\xi}(\xi^i)\right]^{-1}$ must be computed.

Note that

$$\frac{d}{dt}\left(\frac{\partial x}{\partial\xi}\right)=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\xi} \tag{16-46}$$

Since, at t=0, the values of x_1, x_3, x_4 are given

~~but for X_2~~ $\frac{\partial x_1}{\partial\xi}(0)=0$, $\frac{\partial x_3}{\partial\xi}(0)=0$, $\frac{\partial x_4}{\partial\xi}(0)=0$

but for x_2

$$\frac{\partial x_2}{\partial\xi}(0)=1$$

For given $\overset{u}{U}$,and b, equation (46) can be integrated with the initial values given above to obtain

$$\frac{\partial\eta}{\partial\xi}=\frac{\partial x}{\partial\xi}(L) \tag{16-47}$$

which is than used in the iterative formula (45).

Next, the adjoint equations must be defined and solved. They are

$$\frac{d\lambda}{dt}=-\frac{\partial f}{\partial x}^T\lambda+\frac{\partial f_0}{\partial x}^T \tag{16-48}$$

The boundary conditions of SX are, from (44)

$$\left.\begin{aligned} \delta x_1(0) &= 0 \qquad & \delta x_2(L) &= 0 \\ \delta x_3(0) &= 0 \qquad & \delta x_4(0) &= 0 \end{aligned}\right\} \tag{16-49}$$

The boundary conditions on are defined so that

$$\lambda^T \delta x(0) - \lambda^T(L)\,\delta x(L) = 0 \tag{8-13 Handbook}$$

or written in explicity form

$$\lambda_2(0)\delta x_2(0) - \lambda_1(L)\delta x_1(L) - \lambda_3(L)\delta x_3(L) - \lambda_4(L)\delta x_4(L) = 0$$

for all δx satisfying Eq. (a-5). (16-50)

Since $\delta x_2(0)$, $\delta x_1(L)$, $\delta x_3(L)$ and $\delta x_4(L)$ are not specified in the problem, they can assume any value, this implys that

$$\left.\begin{aligned} \lambda_2(0) &= 0 \\ \lambda_1(L) &= 0 \\ \lambda_3(L) &= 0 \\ \lambda_4(L) &= 0 \end{aligned}\right\} \tag{16-51}$$

Equation (48), together with boundary conditions of EQ. (51), is a linear, in homogenious boundary value problem for λ . In order to use superposition in solving this boundary value problem, put

$$\lambda_1(L) = 0,\quad \lambda_3(L) = 0,\quad \lambda_4(L) = 0,\quad \text{and}\quad \lambda_2(L) = 1$$

and integrate the homoglnious differential equation.

$$\frac{d\lambda}{dt} = -\frac{\partial f}{\partial x}^T \lambda \tag{16-52}$$

backwards. Denate this solution as $\lambda^1(t)$.

Next, set $\lambda_i(L)=0$, for i=1.2.3.4., and integrate (a-4) backwards. Denate this solution as $\lambda^2(t)$.

Now, $$\lambda = \alpha\lambda^1 + \lambda^2 \tag{16-53}$$

where α is any constant, is a solution to (48) satifying boundary conditions

$$\lambda_1(L)=0, \quad \lambda_3(L)=0, \quad \lambda_4(L)=0$$

and the initial condition

$$\lambda_2(0)=\lambda_2^1(0)+\alpha\lambda_2^2(0)$$

The constant can now be determined to satisfy the initial condition $\lambda_2(0)=0$. solving for α one obtains

$$\alpha = -\frac{\lambda_2^1(0)}{\lambda_2^2(0)}$$

This value of α is substituted into Eq. (53) to obtain the solution of the adjoint boundary - value problem.

The adjoint equations corresponding to L_1, L_2, L_3, L_4, and L_5 are solved in essentially the same way as eq. (48), except that the inhomogenious part is replaced by $\frac{\partial L_i}{\partial x}$. The boundary conditions for each of the adjoint equations are exactly the same as (51). Hence the backwords integration of (52) should be done only once, and the result used for solution of all other adjoint equations.

Except those calculations discussed above, the numerical method for solving the sabot design problem poses no difficulty. All the necessary differentiations of functions required to implement the steepest descent algorithm of Section 8.2 of (   ) are given in Appendix ___.

16-6 Further Simplification of the Model

In current optimization techniques, it is often very important to make a good estimate of initial design variables so that the iterative optimization method employed will not diverge or ~~eonouge~~ converge to an eronoius design. For this purpose, a simple shell is chosen and statically analyzed. The generating curve of the shell is $R_0(x) = \frac{3}{8} + (\frac{9}{8})(\frac{x}{8})^p, \quad 2 < p < 8$

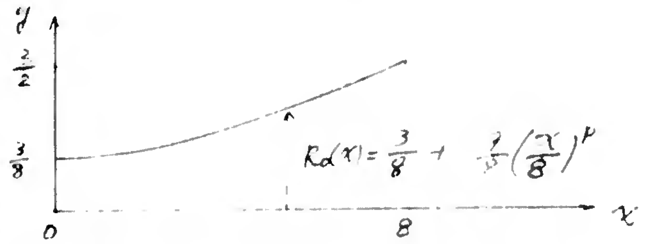

Fig 3

R_0 is so chosen that the curve passes through (0, 3/8) and (8, 3/2) and has zero slope at (0, 3/8).

Using membrane shell theory, the stress resultants N_x and N_θ are computed for p = 2 ∿ 8 and the thickness of ~~eto~~ the shell is evaluated at x = 8, for each value of P. The resulults are shown in Table 1.

Table 1

P	max N_x lb/in	max N_θ lb/in	t (in)	t/R_0
2	0.63281+E05	-0.47058+E05	0.471 0.314	0.313 0.212
3	"	-0.49907+E05	0.499 0.333	0.332 0.220
4	"	-0.54151+E05	0.542 0.361	0.361 0.239
5	"	-0.60090+E05	0.601 0.401	0.400 0.266
6	"	-0.68221+E05	0.682 0.455	0.454 0.301
7	"	-0.79280+E05	0.793 0.529	0.527 0.350
8	"	-0.94288+E05	0.943 0.629	0.628 0.419

Col. 1 The power of P in $R_0 =$ ~~/ƀ/ƀ/ƚ/ƀ/ƀ/ƚ/~~ $\frac{3}{8} + \frac{9}{8}(\frac{\overset{x}{\cancel{8}}}{8})^P$.

Col 2. Max of N_x, which occurs at x = 0.

Col 3. Max of N_θ (in a absolute balue sensa), which occurs ar x = 8.

Col 4. The thickness of shell is determined by $| \max N\theta / \sigma_Y^{AL} |$, where σ_Y^{AL} is the yielding strength of material. Two different values for σ_Y^{AL} are

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used. They are 100,000 psi and 150,000 psi. The upper no. is for 100,000 psi and the lower no. is for 150,000 psi.

Col 5. The ratio of thickness over the radius of shell is also computed.

All numbers in the table are computed for an equally distributed load 30,000 psi parpendicular to the shell surface.

As the power p in the generating curve increases; max N_θ. increases in absolute value, also must the thickness of shell. The thickness of shell is always greater than 0.1, which is beyond the applicable range of thin shell theory. To make thin shell theory applicable, either the yielding strength of the material must be drastically increased, or external load (30,000 psi for this analysis) has to be cut down by more than 50%. Neither of these are possible. Hence, the assumed geometry of the shell is NOT proper.

Appendix

Equilibrium equations of the shell are

$$R_0 \frac{dN_x}{dx} + N_\theta \frac{dR_0}{dx} - N_2 \frac{dR_0}{dx} = 0$$

$$-\frac{N_1 \frac{d^2R_0}{dx^2}}{\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{3/2}} + \frac{N_2}{R_0\left\{1+\left(\frac{dR_0}{dx}\right)^2\right\}^{1/2}} - q = 0$$

The boundary condition is

$$N_x(0) = T.$$

The Solution of the above equations is

$$N_x(x) = \frac{\left\{1+\left(\frac{dR_0}{dx}(x)\right)^2\right\}^{1/2}}{R_0}\left\{\frac{q}{2}\left(R_0^2(x)-R_0^2(0)\right) + TR_0(0)\right\}$$

$$N_\theta(x) = q\,R_0(x)\left\{1+\left(\frac{dR_0}{dx}(x)\right)^2\right\}^{1/2} + \frac{\frac{d^2R_0}{dx^2}(x)}{\left\{1+\left(\frac{dR_0}{dx}(x)\right)^2\right\}^{1/2}}\left\{\frac{q}{2}\left(R_0^2(x)-R_0^2(0)\right) + TR_0(0)\right\},$$

where $R_0(x)$ is the generating curve and is defined between 0 and some positive no. Also derivative of R_0 at $x = 0$ has to be zero.

16.7 A Radial Shear Plate Array Sabot Concept

Having arrived at a negative conclusion regarding the practicability of a sequence of composite sourcal shells, totally different concept of construction is considered. A radial array of shear plates, as shown in Fig. 4, is covered by a circumferentially fiber reinforced shell that transmits bore pressure to the shear plates- It is presumed that this shell will reach a postbuckled equilibrium condition and will not fail. There assumptions will be checked later, if the concept appears to have merrit.

The problem reduces to design of a shear plate, as shown in Fig. 5. Displacements u and v along x axis are zero. The edge along the y axis is free from tractions. Along y = R(X) the traction, the magnitude of which is proportional to R(X), is acting in a direction normal to the boundary.

Optimal design in this section concerns minimization of the weight of a she~~e~~ar plate under certain constraints (geometrical, physical, etc). Generalized plane stress theory* is used for the analysis: i.e. The quantities such as stresses, strains, and displacements are average value over the thickness of a plate. It is assumed here that the theory can be applied without significant error to a plate of variable thickness, when its variation is small.

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* "Elasticity in Engineering Mechanics" by A.P. Bores: Prentice Hall, pp. 135-136

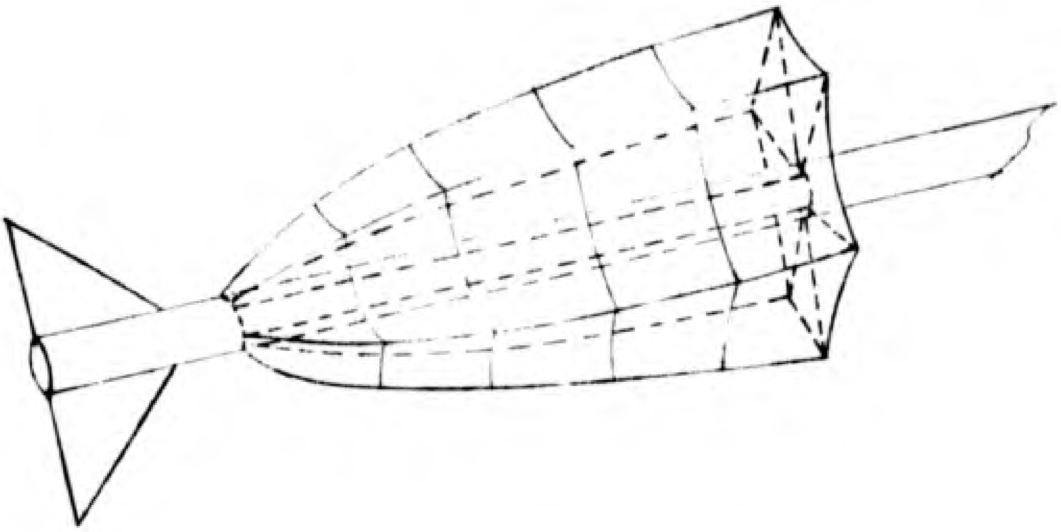


Figure 4. A Radial Shear Plate Sabot Concept

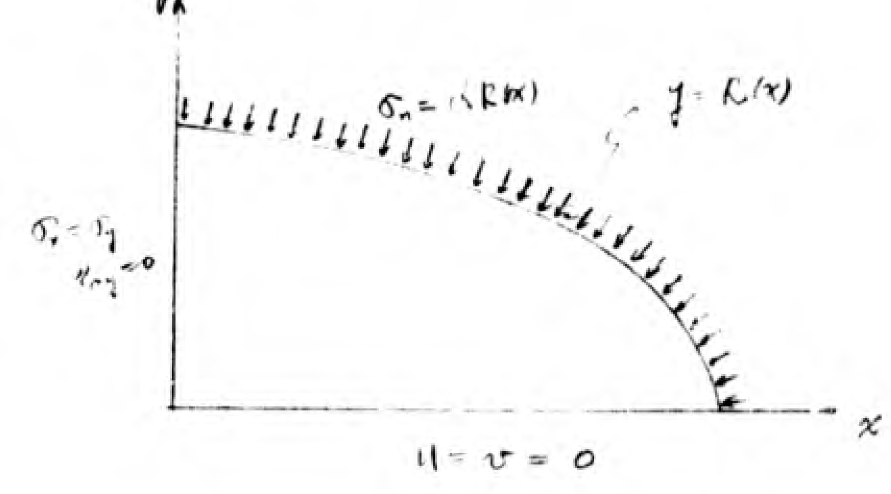


Fig. 5 Shear Plate

16-8 The derivation of the governing eqs.

The constitutive equations for plane stress are

$$\left.\begin{aligned} \sigma_x &= \Lambda(\varepsilon_x + \varepsilon_y) + 2G\varepsilon_x \\ \sigma &= \Lambda(\varepsilon_x + \varepsilon_y) + 2G\varepsilon_y \\ \tau_{xy} &= G\gamma_{xy} \end{aligned}\right\} \quad (1)$$

where

$$\Lambda = \frac{2\lambda G}{(\lambda + 2G)}$$

λ = Rame Constant

G = Shear Modulus

The strain displacement relationships are

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \quad (2)$$

where u and v are displacement in x - and y-direction respectively.

Strain energy U for the system is

$$U = \frac{1}{2}\int_\Omega \sigma_{ij}\, e_{ij} \cdot d\Omega = \frac{1}{2}\int_\Omega \{\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\}\, d\Omega$$

$$= \frac{1}{2}\int_\Omega \left[\{\Lambda(\varepsilon_x + \varepsilon_y) + 2G\varepsilon_x\}\varepsilon_x + \{\Lambda(\varepsilon_x + \varepsilon_y + 2G\varepsilon_y\}\varepsilon_y + G\gamma_{xy}\gamma_{xy}\right] d\Omega$$

$$= \frac{1}{2}\int_\Omega \left[\{\Lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2G\frac{\partial u}{\partial x}\}\frac{\partial u}{\partial x} + \{\Lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2G\frac{\partial v}{\partial y}\}\frac{\partial v}{\partial y}\right.$$

$$\left. + G(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2\right] d\Omega$$

Work done by the external forces is

$$W = \int_S (f_x u + f_y v)\, dS$$

The total potential energy V is

$$V = U - W$$

$$= \frac{1}{2}\int_\Omega \Big[\Big\{\Lambda\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + 2G\frac{\partial u}{\partial x}\Big\}\frac{\partial u}{\partial x} + \Big\{\Lambda\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + 2G\frac{\partial v}{\partial y}\Big\}\frac{\partial v}{\partial y}$$

$$+ G\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2\Big]d\Omega - \int_S (f_x u + f_y v)\, dS \qquad (3)$$

Introducing the explicit expressions for the volume ~~and~~ surface integration. and integrating over the thickness.

$$V = \int_0^L\int_0^{R(x)} \Big[\Big\{\Lambda h\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + 2Gh\frac{\partial u}{\partial x}\Big\}\frac{\partial u}{\partial x} + \Big\{\Lambda h\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big)$$

$$+ 2G\frac{\partial v}{\partial y}\Big\}\frac{\partial v}{\partial y} + G\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)^2\Big]dy\,dx - \int_S (\bar{f}_x u + \bar{f}_y v)\, ds,$$

where $\bar{f}_x = f_x h$, $\bar{f}_y = f_y h$, and s is the boundary of the plate in Fig 5.

Note that u and v are average quanties across the thickness, hence they are independent of [illegible].

Setting $\delta V = 0$

$$= \int_S \Big[\Big[\Big\{\Lambda h\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + 2Gh\frac{\partial u}{\partial x}\Big\}\cos(x,n) + Gh\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)\cos(y,n)$$

$$- \bar{f}_x\Big]\delta u + \Big[\Big\{\Lambda h\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + 2Gh\frac{\partial v}{\partial y}\Big\}\cos(y,n) + Gh\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)\cos(x,n)$$

$$- \bar{f}_y\Big]\delta v\Big]ds - \int_0^L\int_0^{R(x)} \Big[\Big\{(\Lambda + 2G)\frac{\partial}{\partial x}\Big(h\frac{\partial u}{\partial x}\Big) + G\frac{\partial}{\partial y}\Big(h\frac{\partial u}{\partial y}\Big) + \Lambda\frac{\partial}{\partial x}\Big(h\frac{\partial v}{\partial y}\Big)$$

$$+ G\frac{\partial}{\partial y}\Big(h\frac{\partial v}{\partial x}\Big)\Big\}\delta u + \Big\{\Lambda\frac{\partial}{\partial y}\Big(h\frac{\partial u}{\partial x}\Big) + G\frac{\partial}{\partial x}\Big(h\frac{\partial u}{\partial y}\Big) + G\frac{\partial}{\partial x}\Big(h\frac{\partial v}{\partial x}\Big)$$

$$+ (\Lambda + 2G)\frac{\partial}{\partial y}\Big(h\frac{\partial v}{\partial y}\Big)\Big\}\delta v\Big]dy\,dx$$

Since δu and δv are independent, the following set of equations are obtained.

Governing equations:

$$\left.\begin{aligned}&(\Lambda+2G)\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)+G\frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial y}\right)+\Lambda\frac{\partial}{\partial x}\left(h\frac{\partial v}{\partial y}\right)+G\frac{\partial}{\partial y}\left(h\frac{\partial v}{\partial x}\right)=0\\&\Lambda\frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial x}\right)+G\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial y}\right)+G\frac{\partial}{\partial x}\left(h\frac{\partial v}{\partial x}\right)+(\Lambda+2G)\frac{\partial}{\partial y}\left(h\frac{\partial v}{\partial y}\right)=0\end{aligned}\right\}\quad(4a)$$

Boundary conditions:

$$\left.\begin{aligned}&y=0\\&\qquad u(x,0)=0\\&\qquad v(x,0)=0\\&x=0\\&\qquad(\Lambda+2G)h\frac{\partial u}{\partial x}+Gh\frac{\partial v}{\partial y}=0\\&\qquad G\left(h\frac{\partial u}{\partial y}+h\frac{\partial v}{\partial x}\right)=0\\&y=k(x)\\&\qquad\left\{(\Lambda+2G)h\frac{\partial u}{\partial x}+\Lambda h\frac{\partial v}{\partial y}\right\}\cos(x,n)+Gh\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos(y,n)=\bar{f}_x\\&\qquad\left\{(\Lambda+2G)h\frac{\partial v}{\partial y}+\Lambda h\frac{\partial u}{\partial x}\right\}\cos(y,n)+Gh\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos(x,n)=\bar{f}_y\end{aligned}\right\}\quad(4b)$$

Eqs 4a and 4b can be written in operator notation as

$$\underline{A}\,\underline{u}=\underline{0}\qquad(4c)$$

$$\underline{B}\,\underline{u}=\underline{f}\qquad(4d)$$

$\underline{u}_0$

Let $\underline{U}_0$ satisfy the boundary condition (4d), but not neccessarily equation (4c). Introducing a new unknown ũ, such that

$$\underline{\tilde{u}}=\underline{u}-\underline{u}_0\qquad(5)$$

the nonhomogeneous boundary-value problem can be re-written as

$$\underline{A}\,\underline{\tilde{u}}=-\underline{A}\,\underline{u}_0\qquad(6a)$$

and

$$\underline{B}\,\underline{\tilde{u}}=\underline{0}\qquad(6b)$$

If A is a positive definite and bounded below operator, the solution of the above equations minimizes the following functional [Mikhlin]:

$$F_1(\tilde{\underline{u}}) = (\underline{A}\tilde{\underline{u}}, \tilde{\underline{u}}) - 2(\tilde{\underline{u}}, -\underline{A}\underline{u}_0) \qquad (7)$$

Inserting (5) ~~and~~ into (7)

$$F_1(\tilde{\underline{u}}) = (\underline{A}\underline{u} - \underline{A}\underline{u}_0, \underline{u} - \underline{u}_0) - 2(\underline{u} - \underline{u}_0, -\underline{A}\underline{u}_0)$$

$$= (\underline{A}\underline{u}, \underline{u}) + (\underline{u}, \underline{A}\underline{u}) - (\underline{A}\underline{u}, \underline{u}_0) - (\underline{A}\underline{u}_0, \underline{u}_0)$$

$$= (\underline{A}\underline{u}, \underline{u}) - (\underline{u}, \underline{f}) + (\underline{u}_0, \underline{f}) - (\underline{A}\underline{u}_0, \underline{u}_0)$$

In the above functional, the last two terms on the right hand side are constant. Hence the solution of the given problem is equivalent to finding a function that minimizes.

$$\Phi(u) = (\underline{A}\underline{u}, \underline{u}) - (\underline{u}, \underline{f})$$

$$= \int_0^L \int_0^{B(x)} \Big[(\Lambda+2G)h\left(\frac{\partial u}{\partial x}\right)^2 + Gh\left(\frac{\partial u}{\partial y}\right)^2 + 2\Lambda h\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)$$

$$+ 2Gh\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) + G\left(\frac{\partial v}{\partial x}\right)^2 + (\Lambda+2G)\left(\frac{\partial v}{\partial y}\right)^2 \Big] dy\, dx$$

$$- 2\int_S (u\bar{f}_x + v\bar{f}_y)\, ds \qquad (8)$$

For $\Phi(u)$ to be minimum, $\delta\Phi(u) = 0$. Then

$$\delta\Phi(u) = 0 = 2\int_0^L \int_0^{B(x)} h\Big[(\Lambda+2G)\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right) + G\left(\frac{\partial u}{\partial y}\right)\delta\left(\frac{\partial u}{\partial y}\right)$$

$$+ \Lambda\left(\frac{\partial v}{\partial y}\right)\delta\left(\frac{\partial u}{\partial x}\right) + \Lambda\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial v}{\partial y}\right) + G\left(\frac{\partial v}{\partial x}\right)\delta\left(\frac{\partial u}{\partial y}\right)$$

$$+ G\left(\frac{\partial u}{\partial y}\right)\delta\left(\frac{\partial v}{\partial x}\right) + G\left(\frac{\partial v}{\partial x}\right)\delta\left(\frac{\partial v}{\partial x}\right) + (\Lambda+2G)\left(\frac{\partial v}{\partial y}\right)\delta\left(\frac{\partial v}{\partial y}\right) \Big] dy\, dx$$

$$- 2\int_S (\delta u\, \bar{f}_x + \delta v\, \bar{f}_y)\, ds \qquad (9)$$

Since the operator A is positive definite and bounded below, we can construct the minimizing sequence with Ritz method. An appropriate choice for the coordinate functions is as follows:

$$\left.\begin{aligned} u &= y\,(a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \\ &\qquad + a_7 x^3 + a_8 x^2 y + a_9 x y^2 + a_{10} y^3 + \cdots\ ) \\ v &= y\,(a_{n+1} + a_{n+2} x + a_{n+3} y + a_{n+4} x^2 + a_{n+5} xy + a_{n+6} y^2 \\ &\qquad + a_{n+7} x^3 + a_{n+8} x^2 y + a_{n+9} x y^2 + a_{n+10} y^3 + \cdots\ ) \end{aligned}\right\} \quad (10)$$

Substitution of (10) into (9) yields

$$\begin{aligned} \delta\Phi(u) = 0 - 2\int_0^L\!\!\int_0^{R(x)} h\Big[&(\Lambda+2G)\left(\frac{\partial u}{\partial x}\right)\sum_{i=1}^{n}\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial x}\right)\delta a_i + G\left(\frac{\partial u}{\partial y}\right)\sum_{i=1}^{n}\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right)\delta a_i \\ &+ \Lambda\left(\frac{\partial v}{\partial y}\right)\sum_{i=1}^{n}\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right)\delta a_i + \Lambda\left(\frac{\partial u}{\partial x}\right)\sum_{i=n+1}^{2n}\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial y}\right)\delta a_i + G\left(\frac{\partial v}{\partial x}\right)\sum_{i=1}^{n}\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right)\delta a_i \\ &+ G\left(\frac{\partial u}{\partial y}\right)\sum_{i=n+1}^{2n}\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial x}\right)\delta a_i + G\left(\frac{\partial v}{\partial x}\right)\sum_{i=n+1}^{2n}\left(\frac{\partial v}{\partial x}\right)\delta a_i + (\Lambda+2G)\left(\frac{\partial v}{\partial y}\right)\sum_{i=n+1}^{2n}\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial x}\right)\delta a_i\Big]\,dy\,dx \\ &- 2\int_S\left(\bar{f}_x\sum_{i=1}^{n}\frac{\partial u}{\partial a_i}\delta a_i + \bar{f}_y\sum_{i=n+1}^{2n}\frac{\partial v}{\partial a_i}\delta a_i\right)ds \\ = 2\sum_{i=1}^{n}\delta a_i\Big[\int_0^L\!\!\int_0^{R(x)} h\Big\{ &(\Lambda+2G)\left(\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial x}\right) + G\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right) + \Lambda\left(\frac{\partial v}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial x}\right) \\ &+ G\left(\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right)\Big\}\,dy\,dx - \int_S \bar{f}_x\frac{\partial u}{\partial a_i}\,ds\Big] + 2\sum_{i=n+1}^{2n}\delta a_i\Big[\int_0^L\!\!\int_0^{R(x)} h\Big\{\Lambda\left(\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial y}\right) \\ &+ G\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial x}\right) + G\left(\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial x}\right) + (\Lambda+2G)\left(\frac{\partial v}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial y}\right)\Big\}\,dy\,dx \\ &- \int_S \bar{f}_y\frac{\partial v}{\partial a_i}\,ds\Big] \end{aligned}$$

Since the a_i's are linearly independent, the coefficients have to be zero, i.e.

$$\int_0^L\int_0^{E(x)} h_i\Big\{(\Lambda+2G)\left(\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial x}\right)+G\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right)+\Lambda\left(\frac{\partial v}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial x}\right)$$

$$+G\left(\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right)\Big\}\,dy\,dx=\int_S \bar{f}_x\frac{\partial u}{\partial a_i}\,dS \qquad i=1,2,\quad n \tag{11a}$$

$$\int_0^L\int_0^{E(x)} h_i\Big\{\Lambda\left(\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial a_j}\left(\frac{\partial v}{\partial y}\right)+G\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial a_j}\left(\frac{\partial v}{\partial x}\right)+G\left(\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial x_j}\left(\frac{\partial v}{\partial x}\right)$$

$$+(\Lambda+2G)\left(\frac{\partial v}{\partial y}\right)\frac{\partial}{\partial a_j}\left(\frac{\partial v}{\partial y}\right)\Big\}\,dy\,dx=\int_S \bar{f}_y\frac{\partial v}{\partial a_j}\,dS \qquad j=n+1,n+2,\quad 2n \tag{11b}$$

Eqs. (11a) and (11b) yeild a set of 2n linear equations in the a_i (i = 1,~ 2n), with constant coefficients. Hence, one can solve for the a_i's.

16-9 Formulation of the Optimal Design Problem (O.D.P.) for a Shear Plate with Variable Thickness

To utilize the steepest descent technique of [handbook] the state equations (4) must be written in first order form.

Let

$$
\left.\begin{aligned}
z_1 &= u \\
z_2 &= v \\
z_3 &= (\Lambda+2G)h\frac{\partial u}{\partial x} + \Lambda h\frac{\partial v}{\partial y} \\
z_4 &= \Lambda h\frac{\partial u}{\partial x} + (\Lambda+2G)h\frac{\partial v}{\partial y} \\
z_5 &= G\left(h\frac{\partial u}{\partial y} + h\frac{\partial v}{\partial x}\right) \\
x_1 &= x\,, \quad x_2 = y\,, \quad u = h
\end{aligned}\right\} \quad (12)
$$

Then eq. 4 can be expressed in the following first order, self-adjoint form:

Governing equation

$$
\left.\begin{aligned}
-\frac{\partial z_3}{\partial x_1} - \frac{\partial z_5}{\partial x_2} &= 0 \\
-\frac{\partial z_4}{\partial x_2} - \frac{\partial z_5}{\partial x_1} &= 0 \\
\frac{\partial z_1}{\partial x_1} &= \frac{\Lambda+2G}{4G(\Lambda+G)u}\,z_3 - \frac{\Lambda}{4G(\Lambda+G)u}\,z_4 \\
\frac{\partial z_2}{\partial x_2} &= \frac{\Lambda+2G}{4G(\Lambda+G)u}\,z_4 - \frac{\Lambda}{4G(\Lambda+G)u}\,z_3 \\
\frac{\partial z_1}{\partial x_2} + \frac{\partial z_2}{\partial x_1} &= \frac{z_5}{Gh}
\end{aligned}\right\} \quad (13a)
$$

Boundary conditions

$$
\left.\begin{aligned}
x_2 = 0 \qquad & z_1(x_1, 0) = 0 \\
& z_2(x_1, 0) = 0 \\
x_1 = 0 \qquad & z_3(0, x_2) = 0 \\
& z_4(0, x_2) = 0
\end{aligned}\right\} \quad (13b)
$$

$$\left.\begin{aligned} x_2 &= R(x_1) \\ &z_3\cos(x_1,n) + z_5\cos(x_2,n) - \bar{f}_{x_1} = 0 \\ &z_4\cos(x_2,n) + z_5\cos(x_1,n) - \bar{f}_{x_2} = 0 \end{aligned}\right\}$$

Note: The reduction of the system of 2^{nd} order partial differential eqs. to a system of 1st order is not unique. The 1st order equations so obtained might have multiple solutions, although the original 2nd order eqs have a unique solution.

The O.D.P. will be defined in the following manner:

Cost function

$$J = \int_0^L \int_0^{R(x_1)} u(x_1, x_2)\, dx_2\, dx_1 \qquad (14)$$

Constraints

$$\left.\begin{aligned} &-\frac{\partial z_3}{\partial x_1} - \frac{\partial z_5}{\partial x_2} = 0 \\ &-\frac{\partial z_4}{\partial x_2} - \frac{\partial z_5}{\partial x_1} = 0 \\ &\frac{\partial z_1}{\partial x_1} = \frac{\Lambda+2G}{4G(\Lambda+G)u} z_3 - \frac{\Lambda}{4G(\Lambda+G)u} z_4 \\ &\frac{\partial z_2}{\partial x_2} = \frac{\Lambda+2G}{4G(\Lambda+G)u} z_4 - \frac{\Lambda}{4G(\Lambda+G)u} z_3 \\ &\frac{\partial z_1}{\partial x_2} + \frac{\partial z_2}{\partial x_1} = \frac{z_5}{Gu} \\ &z_1(x_1, 0) = 0 \\ &z_2(x_1, 0) = 0 \\ &z_3(0, x_2) = 0 \\ &z_5(0, x_2) = 0 \\ &z_3(x_1, R(x_1))\cos(x_1,n) + z_5(x_1, R(x_1))\cos(x_2,n) = \bar{f}_x \\ &z_4(x_1, R(x_1))\cos(x_2,n) + z_5(x_1, R(x_1))\cos(x_1,n) = \bar{f}_y \end{aligned}\right\} \qquad (15)$$

$$\left.\begin{aligned} &z_3 - \sigma_{max}\, u \le 0 \\ &z_4 - \sigma_{max}\, u \le 0 \\ &z_5 - z_{max}\, u \le 0 \qquad \text{for } x_1, x_2 \in \Omega \end{aligned}\right\} \qquad (16)$$

$$h_{min} - u \le 0 \qquad \text{for } x_1, x_2 \in \Omega \qquad (17)$$

Comparing notation with Chapter 4 of [handbook],

$$f_0 = u$$

$$L(u,b)[z] = \begin{bmatrix} 0 & 0 & -\frac{\partial}{\partial x_1} & 0 & -\frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_1} & 0 & -\frac{\Lambda+2G}{4G(\Lambda+G)u} & \frac{\Lambda}{4G(\Lambda+G)u} & 0 \\ 0 & \frac{\partial}{\partial x_2} & \frac{\Lambda}{4G(\Lambda+G)u} & -\frac{\Lambda+2G}{4G(\Lambda+G)u} & 0 \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 & 0 & -\frac{1}{Gu} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}$$

$$Q(x,u,b) = [0, 0, 0, 0, 0]^T$$

$$B(v,b)[z] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \qquad \text{at } x_2 = 0$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \qquad \text{at } x_1 = 0$$

$$= \begin{bmatrix} 0 & 0 & \cos(x_1,n) & 0 & \cos(x_2,n) \\ 0 & 0 & 0 & \cos(x_2,n) & \cos(x_1,n) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \qquad \text{at } x_2 = R(x_1)$$

$$q = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{at } x_2 = 0$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{at } x_1 = 0$$

$$= \begin{bmatrix} \bar{f}_x \\ \bar{f}_y \end{bmatrix} \qquad \text{at } x_2 = R(x_1)$$

Since eqs (16) depend on , U this type of constraints has to be modified as follows

$$\omega_1 = Z_3 - \sigma_{max}\, U$$

$$\omega_2 = Z_4 - \sigma_{max}\, U$$

$$\omega_3 = Z_5 - \tau_{max}\, U$$

and let

$$L_1 = \omega_1 + |\omega_1|$$

$$L_2 = \omega_2 + |\omega_2|$$

$$L_3 = \omega_3 + |\omega_3|$$

Further

$$\psi_1 = \int_0^L \int_0^{R(x_1)} L_1 \, dx_2 \, dx_1$$

$$\psi_2 = \int_0^L \int_0^{R(x_1)} L_2 \, dx_2 \, dx_1$$

$$\psi_3 = \int_0^L \int_0^{R(x)} L_3 \, dx_2 \, dx_1$$

The last constraint eq (17) will be

$$\phi_1 = h_{min} - U \leq 0$$

The quantities corresponding to 8.4.2 in [handbook] (the effect of small changes in design variables and parameters) are given in the following

$$L[u^{(0)}, b^{(0)}][\delta z] = \begin{bmatrix} 0 & 0 & -\frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\partial}{\partial x_2} & 0 \\ \frac{\partial}{\partial x_1} & 0 & -\frac{\Lambda+2G}{4G(\Lambda+G)u} & \frac{\Lambda}{4G(\Lambda+G)u} & 0 \\ 0 & \frac{\partial}{\partial x_2} & \frac{\Lambda}{4G(\Lambda+G)u} & -\frac{\Lambda+2G}{4G(\Lambda+G)u} & 0 \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 & 0 & -\frac{1}{Gu} \end{bmatrix} \begin{bmatrix} \delta z_1 \\ \delta z_2 \\ \delta z_3 \\ \delta z_4 \\ \delta z_5 \end{bmatrix}$$

$$\Delta_u L[u^{(0)}, b^{(0)}][z] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Lambda+2G}{4G(\Lambda+G)u^2} & -\frac{\Lambda}{4G(\Lambda+G)u^2} & 0 \\ 0 & 0 & -\frac{\Lambda}{4G(\Lambda+G)u^2} & \frac{\Lambda+2G}{4G(\Lambda+G)u^2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{Gu^2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}$$

$$\Delta_b L[u, b][z] = [\,0\,]$$

$$\frac{\partial L_1}{\partial z} = [0, 0, 1, 0, 0] \text{ if } \omega_1 > 0, \quad \frac{\partial L_1}{\partial z} = [\,0\,] \text{ if } \omega_1 \leq 0$$

$$\frac{\partial L_2}{\partial z} = [0, 0, 0, 1, 0] \text{ if } \omega_2 > 0, \quad \frac{\partial L_2}{\partial z} = [\,0\,] \text{ if } \omega_2 \leq 0$$

$$\frac{\partial L_3}{\partial z} = [0, 0, 0, 0, 1] \text{ if } \omega_3 > 0, \quad \frac{\partial L_3}{\partial z} = [\,0\,] \text{ if } \omega_3 \leq 0$$

$$\frac{\partial L_1}{\partial u} = -\sigma_{max} \text{ if } \omega_1 > 0, \quad \frac{\partial L_1}{\partial u} = 0 \text{ if } \omega_1 \leq 0$$

$$\frac{\partial L_2}{\partial u} = -\sigma_{max} \text{ if } \omega_2 > 0, \quad \frac{\partial L_2}{\partial u} = 0, \text{ if } \omega_2 \leq 0$$

$$\frac{\partial L_3}{\partial u} = -\tau_{max} \text{ if } \omega_2 > 0, \quad \frac{\partial L_3}{\partial u} = 0 \text{ if } \omega_3 \leq 0$$

Eq (8-149) in [handbook] becomes

$$\iint_{\Omega} \{\lambda^T L(u,b)\{\delta z\} - \delta z^T L^*(u,b)\{\lambda\}\} d\Omega$$

$$= 0 = \int_{\Gamma} A\{\lambda\}^T C\{\delta z\}\, dz$$

$$= \int_{\Gamma} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & \cos(x_1,n) & 0 & \cos(x_2,n) \\ 0 & 0 & 0 & \cos(x_2,n) & \cos(x_1,n) \\ -\cos(x_1,n) & 0 & 0 & 0 & 0 \\ 0 & -\cos(x_2,n) & 0 & 0 & 0 \\ -\cos(x_2,n) & -\cos(x_1,n) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta z_1 \\ \delta z_2 \\ \delta z_3 \\ \delta z_4 \\ \delta z_5 \end{bmatrix} d\Gamma$$

The integrand takes the following form at each boundary:

At $x_2 = 0$, $x_1 = x_1$

$$\{\lambda\}^T \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \{\delta z\}$$

At $x_1 = 0$, $x_2 = x_2$

$$\{\lambda\}^T \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \{\delta z\}$$

At $x_2 = R(x_1)$

$$\{\lambda\}^T \begin{bmatrix} 0 & 0 & \cos(x_1,n) & 0 & \cos(x_2,n) \\ 0 & 0 & 0 & \cos(x_2,n) & \cos(x_1,n) \\ -\cos(x_1,n) & 0 & 0 & 0 & 0 \\ 0 & -\cos(x_2,n) & 0 & 0 & 0 \\ -\cos(x_2,n) & -\cos(x_1,n) & 0 & 0 & 0 \end{bmatrix} \{\delta z\}$$

Now from (8-152) in [handbook] one can obtain the boundary condition for λ's

$$\frac{\partial g}{\partial z}\delta z - A[\lambda]^T C[\delta z]$$

In the present O.D.P. none of the equations contain the term corresponding to $\frac{\partial g}{\partial z}\delta z$, so the boundary conditions for the λ's can be determined by setting $A[\lambda]^T C[\delta z] = 0$

By so doing,

At $x_2 = 0, \quad x_1 = x_1$

$$\lambda_1(x_1, 0) = 0$$
$$\lambda_2(x_1, 0) = 0$$

At $x_1 = 0, \quad x_2 = x_2$

$$\lambda_3(0, x_2) = 0$$
$$\lambda_5(0, x_2) = 0$$

At $x_2 = R(x_1)$

$$\lambda_3 \cos(x_1, n) + \lambda_5 \cos(x_2, n) = 0$$
$$\lambda_4 \cos(x_2, n) + \lambda_1 \cos(x_1, n) = 0$$

After all this calculation has been completed, the following quantites ~~above~~ have to be computed to remove the state variable dependency from the O.D.P.

$$\Lambda^{\psi_1}(x) = -\sigma_{max} - \begin{bmatrix} 0 \\ 0 \\ \frac{\Lambda + 2G}{4G(\Lambda + G)} \frac{z_3}{u^2} - \frac{\Lambda}{4G(\Lambda + G)} \frac{z_4}{u^2} \\ -\frac{\Lambda}{4G(\Lambda + G)} \frac{z_1}{u^2} + \frac{\Lambda + 2G}{4G(\Lambda + G)} \frac{z_4}{u^2} \\ \frac{z_5}{Gu^2} \end{bmatrix}$$

$$\Lambda^{\psi_2}(x) = -\sigma_{max} - \begin{bmatrix} 0 \\ 0 \\ \frac{\Lambda+2G}{4G(\Lambda+G)}\frac{z_3}{u^2} - \frac{\Lambda}{4G(\Lambda+G)}\frac{z_4}{u^2} \\ -\frac{\Lambda}{4G(\Lambda+G)}\frac{z_3}{u^2} + \frac{\Lambda+2G}{4G(\Lambda+G)}\frac{z_4}{u^2} \\ \frac{z_5}{Gu^2} \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}$$

$$\Lambda^{\psi_1}(x) = -\tau_{max} - \begin{bmatrix} 0 \\ 0 \\ \frac{\Lambda+2G}{4G(\Lambda+G)}\frac{z_3}{u^2} - \frac{\Lambda}{4G(\Lambda+G)}\frac{z_4}{u^2} \\ -\frac{\Lambda}{4G(\Lambda+G)}\frac{z_3}{u^2} + \frac{\Lambda+2G}{4G(\Lambda+G)}\frac{z_4}{u^2} \\ \frac{z_5}{Gu^2} \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix},$$

$$\Pi^{\psi_1}(x) = 0, \quad \Pi^{\psi_2}(x) = 0, \quad \Pi^{\psi_3}(x) = 0$$

After all these quanties are determined the steepest descent algorithm may be implemented.

16-10 Optimal Design of Shear Plate with a Constant Thickness

The objective of this section is to find the shape of a segment of the boundary curve of a plate, such that the weight of the plate is minimized under certain constraint conditions. Here the thickness is assumed to be constant. The governing equations (4a) and the boundary conditions are simplified and take the following form:

Governing equations;

$$(\Lambda+2G)\frac{\partial^2 u}{\partial x^2} + G\frac{\partial^2 u}{\partial y^2} + (\Lambda+G)\frac{\partial^2 v}{\partial x \partial y} = 0$$

$$(\Lambda+G)\frac{\partial^2 u}{\partial x \partial y} + G\frac{\partial^2 v}{\partial x^2} + (\Lambda+2G)\frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{(18a)}$$

and boundary conditions;

$y = 0$:

$$u(x, 0) = 0.$$

$$v(x, 0) = 0$$

$x = 0$:

$$(\Lambda+2G)\frac{\partial u}{\partial x} + G\frac{\partial v}{\partial y} = 0 \qquad \text{(18b)}$$

$$G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$y = R(x)$

$$\left\{(\Lambda+2G)\frac{\partial u}{\partial x} + \Lambda\frac{\partial v}{\partial y}\right\}\cos(x,n) + G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos(y,n) = f_x$$

$$\left\{(\Lambda+2G)\frac{\partial v}{\partial y} + \Lambda\frac{\partial u}{\partial x}\right\}\cos(y,n) + G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos(x,n) = f_y$$

The above can be written in operator notation as

$$\underline{A}\,\underline{u} = 0$$

$$\underline{B}\,\underline{u} = f$$

The operator $\underline{A}$ is found to be positive definite and bounded below [Mikhlin]. Hence, the Ritz method can be used and eq. (10) are an appropriate choice of coordinate functions. Furthermore, the shape of the boundary, R(x), will be assumed in the form

$$R(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_m x^m$$

Since R(x) has to go through (0, R_o),

$$R(x) = R_0 + C_1 x + C_2 x^2 + \cdots + C_m x^m \tag{19}$$

In this case, the minimizing functional corresponding to eq. (8) is

$$\begin{aligned} \Phi(u) = b \int_0^L \int_0^{R(x)} \Big\{ & (\Lambda+2G)\left(\frac{\partial u}{\partial x}\right)^2 + G\left(\frac{\partial u}{\partial y}\right)^2 + 2\Lambda\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) \\ & + 2G\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) + G\left(\frac{\partial v}{\partial x}\right)^2 + (\Lambda+2G)\left(\frac{\partial v}{\partial y}\right)^2 \Big\}\, dy\, dx \\ & - 2\int_S (u f_x + v f_y)\, ds \end{aligned}$$

Then the variation of Φ(u) becomes

$$\delta\Phi(u) = 0 = 2b\int_0^L\int_0^{R(x)}\Big[(\Lambda+2G)\Big(\frac{\partial u}{\partial x}\Big)\delta\Big(\frac{\partial u}{\partial x}\Big) + G\Big(\frac{\partial u}{\partial y}\Big)\delta\Big(\frac{\partial u}{\partial y}\Big)$$

$$+\Lambda\Big(\frac{\partial v}{\partial y}\Big)\delta\Big(\frac{\partial u}{\partial x}\Big) + \Lambda\Big(\frac{\partial u}{\partial x}\Big)\delta\Big(\frac{\partial v}{\partial y}\Big) + G\Big(\frac{\partial v}{\partial x}\Big)\delta\Big(\frac{\partial u}{\partial y}\Big)$$

$$+G\Big(\frac{\partial u}{\partial y}\Big)\delta\Big(\frac{\partial v}{\partial x}\Big) + G\Big(\frac{\partial v}{\partial x}\Big)\delta\Big(\frac{\partial v}{\partial x}\Big) + (\Lambda+2G)\Big(\frac{\partial v}{\partial y}\Big)\delta\Big(\frac{\partial v}{\partial y}\Big)\Big]$$

$$-2\int_S(\delta u f_x + \delta v f_y)\,ds \tag{20}$$

Substituting eqs. (10) into the above, one can obtain a set of linear equations corresponding to eqs. (11).

$$\int_0^L\int_0^{R(x)} b\Big\{(\Lambda+2G)\Big(\frac{\partial u}{\partial x}\Big)\frac{\partial}{\partial a_i}\Big(\frac{\partial u}{\partial x}\Big) + G\Big(\frac{\partial u}{\partial y}\Big)\frac{\partial}{\partial a_i}\Big(\frac{\partial u}{\partial y}\Big) + \Lambda\Big(\frac{\partial v}{\partial y}\Big)\frac{\partial}{\partial a_i}\Big(\frac{\partial u}{\partial x}\Big)$$

$$+ G\Big(\frac{\partial v}{\partial x}\Big)\frac{\partial}{\partial a_i}\Big(\frac{\partial u}{\partial y}\Big)\Big\}\,dy\,dx = \int_S f_x\frac{\partial u}{\partial a_i}\,ds$$

$$i = 1, \sim, n \tag{21a}$$

$$\int_0^L\int_0^{R(x)} b\Big\{\Lambda\Big(\frac{\partial u}{\partial x}\Big)\frac{\partial}{\partial a_j}\Big(\frac{\partial v}{\partial y}\Big) + G\Big(\frac{\partial u}{\partial y}\Big)\frac{\partial}{\partial a_j}\Big(\frac{\partial v}{\partial x}\Big) + G\Big(\frac{\partial v}{\partial x}\Big)\frac{\partial}{\partial a_j}\Big(\frac{\partial v}{\partial x}\Big)$$

$$+ (\Lambda+2G)\Big(\frac{\partial v}{\partial y}\Big)\frac{\partial}{\partial a_j}\Big(\frac{\partial v}{\partial y}\Big)\Big\}\,dy\,dx = \int_S f_y\frac{\partial v}{\partial a_j}\,ds$$

$$j = n+1, \sim, 2n \tag{21b}$$

Along the x and y axes, either traction or displacement is prescribed to be zero. Hence, the integration over those boundaries does not give a rise to the right-hand sides in eqs. (21). Only the third boundary, along y = R(x), contributes to the boundary integration. After the integration is carried out, eqs. (21) become

$$b \sum g_{\alpha i}(c_1, c_2, \ldots, c_m)\, i_i - g_\alpha^0(c_1, c_2, \ldots, c_m) = 0 \tag{22}$$

$$\alpha = 1, 2, \ldots, 2n$$

Other constraints to be considered are stress and geometrical constraints. For stress constraint, Von Mises yield criterion is used. The principal stresses in a plate have to satisfy the following inequality:

$$\left(\frac{\sigma_1}{\sigma_{yp}}\right)^2 - \left(\frac{\sigma_1}{\sigma_{yp}} \frac{\sigma_2}{\sigma_{yp}}\right) + \left(\frac{\sigma_2}{\sigma_{yp}}\right)^2 \leq 1. \tag{23}$$

where σ_1 and σ_2 are principal stresses, and σ_{yp} is yielding stress of the plate. The stresses σ_1 and $\sigma_2 (\sigma_1 \geq \sigma_2)$ can be written in terms of σ_x, σ_y and σ_{xy}.

$$\begin{aligned} \sigma_1 &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \sigma_2 &= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \end{aligned} \tag{24}$$

The stresses, σ_x, σ_y, and σ_{xy} are related to displacements by

$$\sigma_x = (\Lambda + 2G)\frac{\partial u}{\partial x} + \Lambda\frac{\partial v}{\partial y}$$

$$\sigma_y = \Lambda\frac{\partial u}{\partial x} + (\Lambda + 2G)\frac{\partial v}{\partial y}$$

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{25}$$

Substitution of eqs. (24) into (23) yields

$$(\sigma_x + \sigma_y)^2 + 3(\sigma_x - \sigma_y)^2 + 12\tau_{xy}^2 \leq 4\sigma_{yp}^2 \tag{26}$$

Further, substituting eqs. (25) into eq. (26), one can express the stress constraint in terms of displacement

$$\{4(\Lambda+G)^2 + 12G^2\}\left(\frac{\partial u}{\partial x}\right)^2 + 2\{4(\Lambda+G)^2 - 12G^2\}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)$$
$$+ \{4(\Lambda+G)^2 + 12G^2\}\left(\frac{\partial v}{\partial y}\right)^2 + 12G^2\left(\frac{\partial u}{\partial y}\right)^2 + 24G^2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)$$
$$+ 12G^2\left(\frac{\partial v}{\partial x}\right)^2 \leq 4\sigma_{yp}^2 \tag{27}$$

For convenience, eq. (27) will be expressed by

$$\omega(x_1, x_2, a) \leq 0 \tag{27a}$$

$$x_1 = x, \quad x_2 = y$$

Geometrically, the plate thickness b has to be greater than a minimum allowable thickness b^o,

$$b \geq b_o \tag{28}$$

and R(x) has to satisfy,

$$R(0) = R_o$$

$$R(L) = 0 = R_o + C_1 L + C_2 L^2 + \dots + C_m L^m \tag{29}$$

Since eq. (19) for R(x) already is chosen to satisfy the first of eqs. (29), only the second equation arises as a constraint.

The optimal design of this section will be defined as follows

Minimize

$$J = \int_0^L \left(R_o + C_1 x_1 + C_2 x_1^2 + \dots + C_m x_1^m \right) dx_1$$

$$= R_o L + \frac{C_1 L^2}{2} + \frac{C_2 L^3}{3} + \dots + \frac{C_m L^{m+1}}{m+1} \tag{30}$$

Subject to constraint conditions

$$b \sum_{i=1}^{2n} g_{\alpha i}(c_1, c_2, \dots, c_m) a_i - g_\alpha^o(c_1, c_2, \dots, c_m) = 0 \tag{31a}$$

$$\omega(x_1, x_2, a's) \leq 0 \tag{31b}$$

$$R_0 + c_1 L + c_2 L^2 + \cdots + c_m L^m = 0 \tag{31c}$$

$$b^\circ - b \leq 0 \tag{31d}$$

In the above b and the c_i's are design parameters, and the a_j's are state parameters. The constraint condition, eq. (31b) has to be modified as follows:

$$\int_0^L \int_0^{\ell(x_1)} \{\omega(x_1, x_2, a's) + |\omega(x_1, x_2, a's)|\} \, dx_2 \, dx_1 = 0 \tag{32}$$

The effect of a small change in design parameters on eqs. (30), (31) and (32) is following:

From eq. (30)

$$\delta J = \frac{L^2}{2} \delta c_1 + \frac{L^3}{3} \delta c_2 + \cdots + \frac{L^{m+1}}{m+1} \delta c_m \tag{33}$$

From eq. (31a)

$$\sum_{j=1}^{m} \left(b \sum_{i=1}^{2n} \frac{\partial g_{\alpha i}}{\partial c_j} \delta a_i - \frac{\partial g_\alpha^\circ}{\partial c_j} \right) \delta c_j + \delta b \sum_{i=1}^{2n} g_{\alpha i} a_i$$

$$+ \, b \sum_{i=1}^{2n} g_{\alpha i} \, \delta a_i = 0 \tag{34a}$$

From eq. (31c)

$$L\delta c_1 + L^2 \delta c_2 + \qquad + L^m \delta c_m = 0. \qquad (34c)$$

From eq. (31d)

$$-\delta b \leq 0 \qquad (34d)$$

From eq. (31b) and (27)

$$\delta\omega(x_1, x_2, a_i's) = z_1\{4(\Lambda+G)^2 + 12G^2\}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \delta u}{\partial x}\right)$$

$$+ z_1\{4(\Lambda+G)^2 - 12G^2\}\left(\frac{\partial \delta u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + z_1\{4(\Lambda+G)^2 - 12G^2\}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \delta v}{\partial y}\right)$$

$$+ z_1\{4(\Lambda+G)^2 + 12G^2\}\left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial \delta v}{\partial y}\right) + 24G^2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \delta u}{\partial x}\right)$$

$$+ 24G^2\left(\frac{\partial \delta u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) + 24G^2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial \delta v}{\partial x}\right)$$

$$+ 24G^2\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial \delta v}{\partial x}\right)$$

$$= z\sum_{i=1}^{n} \delta a_i \Big[\{4(\Lambda+G)^2 + 12G^2\}\left(\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial x}\right)$$

$$+ \{4(\Lambda+G)^2 - 12G^2\}\left(\frac{\partial v}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial x}\right) + 12G^2\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right)$$

$$+ 12G^2\left(\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial u}{\partial y}\right)\Big] + z\sum_{i=n+1}^{2n}\Big[\{4(\Lambda+G)^2 - 12G^2\}\left(\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial y}\right)$$

$$+\Big[\{4(\lambda+G)^2+12G^2\}\left(\frac{\partial v}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial y}\right)+12G^2\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial x}\right)$$

$$+12G^2\left(\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial a_i}\left(\frac{\partial v}{\partial x}\right)\Big]$$

$$=\sum_{i=1}^{2n} h_i(x_1,x_2)\,\delta a_i$$

Now from eq. (32)

$$\int_0^L\int_0^{R(x)}\{\delta\omega+\mathrm{sign}(\omega)\,\delta\omega\}\,dx_2\,dx_1=0$$

$$=\int_0^L\int_0^{R(x)}\left\{\sum_{i=1}^{2n} h_i(x_1,x_2)\,\delta a_i+\mathrm{sign}(\omega)\sum_{i=1}^{2n} h_i(x_1,x_2)\,\delta a_i\right\}dx_2\,dx_1$$

$$=\int_0^L\int_0^{R(x)}\left\{\sum_{i=1}^{2n} h_i(x_1,x_2)+\mathrm{sign}(\omega)\sum h_i(x_1,x_2)\right\}\delta a_i\,dx_2\,dx_1$$

(32e)

16-11 Solid Sabot of Revolution

Formulation of the symmetric elasticity problem.

Strain energy is

$$U = \int_{\Omega} \frac{1}{2} \sigma_{ij} e_{ij} \, d\Omega \tag{35}$$

Kinetic Energy is

$$K = \int_{\Omega} \frac{1}{2} \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} \, d\Omega \tag{36}$$

Work done by external and body forces is

$$W = \int_{\Omega} X_i u_i \, d\Omega + \int_{S} F_i u_i \, dS \tag{37}$$

Hamilton's principle states

$$\delta \int_{t^0}^{t'} (U - K - W) \, dt = 0 \tag{38}$$

$$\delta u_i(t^0) = \delta u_i(t') = 0$$

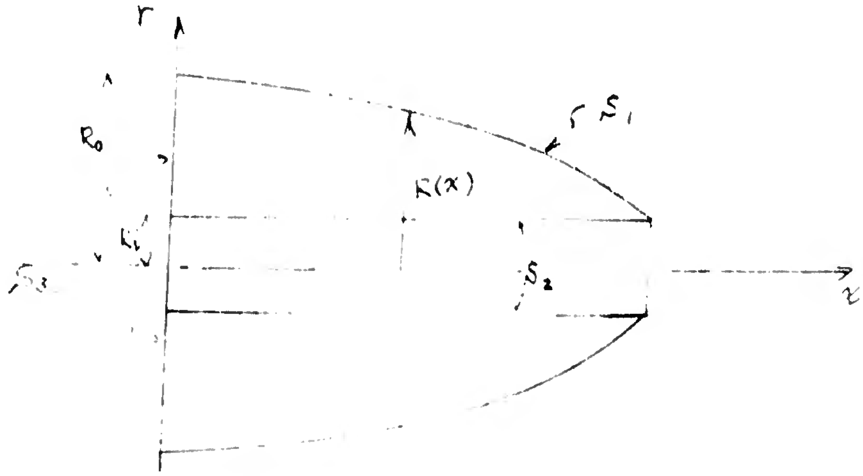


Fig. 6. Cross Section of Solid Sabot in x-r Plane

Substituting (35), (36), (37) into (38) and integrating by parts, one obtains

$$
-\int_{t^0}^{t'}\int_{\Omega}\left\{(\lambda+G)\nabla(\nabla\cdot\bar{u})+G\nabla^2\bar{u}-\rho\frac{\partial^2\bar{u}}{\partial t^2}\right\}\cdot\delta\bar{u}\,d\Omega\,dt
$$

$$
+\int_{t^0}^{t'}\int_{S}\left\{\lambda(\nabla\cdot\bar{u})\bar{\nu}\cdot\delta\bar{u}+G\bar{\nu}\cdot\nabla\bar{u}\cdot\delta\bar{u}+G\,\delta\bar{u}\cdot\nabla\bar{u}\cdot\bar{\nu}-\bar{F}\cdot\delta\bar{u}\right\}dS\,dt
$$

$$
=0. \tag{39}
$$

In the axially symmetric case, Eq. (39) becomes

$$
-\int_{t^0}^{t'}\int_{\Omega}\left\{(\lambda+2G)\frac{\partial^2 w}{\partial r^2}+\frac{\lambda+2G}{r}\frac{\partial w}{\partial r}-\frac{\lambda+2G}{r^2}w+G\frac{\partial^2 w}{\partial x^2}\right.
$$

$$
+(\lambda+G)\frac{\partial^2 u}{\partial x\,\partial r}\Big\}\delta w+\Big\{(\lambda+G)\frac{\partial^2 w}{\partial r\,\partial x}+\frac{\lambda+G}{r}\frac{\partial w}{\partial x}+G\frac{\partial^2 u}{\partial r^2}
$$

$$
\left.+(\lambda+2G)\frac{\partial^2 u}{\partial x^2}+\frac{G}{r}\frac{\partial u}{\partial r}\right\}\delta u\,d\Omega\,dt
$$

$$
-\int_{t^0}^{t'}\int_{S_1}\Big[\Big\{F_x^1-\lambda\Big(\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial u}{\partial x}\Big)\nu_x-G\Big(\frac{\partial u}{\partial r}\nu_r+\frac{\partial w}{\partial x}\nu_r+2\frac{\partial u}{\partial x}\nu_x\Big)\Big\}\delta u_1
$$

$$
+\Big\{F_r^1-\lambda\Big(\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial u}{\partial x}\Big)\nu_r-G\Big(2\frac{\partial w}{\partial r}\nu_r+\frac{\partial w}{\partial x}\nu_x+\frac{\partial u}{\partial r}\nu_x\Big)\delta w_1\Big\}\Big]dS_1
$$

$$
-\int_{t^0}^{t'}\int_{S_2}\Big[\Big\{F_x^2+G\Big(\frac{\partial u}{\partial r}+\frac{\partial w}{\partial x}\Big)\Big\}\delta u_2+\Big\{F_r^2+\lambda\Big(\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial u}{\partial x}\Big)
$$

$$
+2G\frac{\partial w}{\partial r}\Big\}\delta w_2\Big]dS_2-\int_{t^0}^{t'}\int_{S_3}\Big[\Big\{F_x^3+\lambda\Big(\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial u}{\partial x}\Big)
$$

$$
+2G\frac{\partial u}{\partial x}\Big\}\delta u_3+\Big\{F_r^3+G\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial r}\Big)\Big\}\delta w_3\Big]dS_3
$$

$$
=0.
$$

Since δu and δ are arbitrary, the governing equations can be obtained by setting their coefficients equal to zero

$$\begin{gathered}(\lambda+2G)\frac{\partial^2 w}{\partial r^2}+(\lambda+2G)\frac{1}{r}\frac{\partial w}{\partial r}-(\lambda+2G)\frac{1}{r^2}w+G\frac{\partial^2 w}{\partial x^2}\\+(\lambda+G)\frac{\partial^2 u}{\partial x\,\partial r}+X_r-\rho\frac{\partial^2 w}{\partial t^2}=0\\(\lambda+G)\frac{\partial^2 w}{\partial r\,\partial x}+(\lambda+G)\frac{1}{r}\frac{\partial w}{\partial x}+G\frac{\partial^2 u}{\partial r^2}+(\lambda+2G)\frac{\partial^2 u}{\partial x^2}\\+G\frac{1}{r}\frac{\partial u}{\partial r}+X_x-\rho\frac{\partial^2 u}{\partial t^2}=0\end{gathered}\tag{40a}$$

Boundary conditions are:

On S_1, where tractions are prescribed,

$$\begin{gathered}F_r^1-\lambda\left(\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial u}{\partial x}\right)\nu_r-G\left(2\frac{\partial w}{\partial r}\nu_r+\frac{\partial w}{\partial x}\nu_x+\frac{\partial u}{\partial r}\nu_x\right)=0\\F_x^1-\lambda\left(\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial u}{\partial x}\right)\nu_x-G\left(\frac{\partial u}{\partial r}\nu_r+\frac{\partial w}{\partial r}\nu_r+2\frac{\partial u}{\partial x}\nu_x\right)=0\end{gathered}\tag{40b}$$

On S_2, where displacements are prescribed,

$$w_2=w^0$$

$$u_2=u^0$$

On S_3, where tractions are all zero.

$$G\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial r}\right)=0$$

$$\lambda\left(\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial u}{\partial x}\right)+2G\frac{\partial u}{\partial x}=0$$

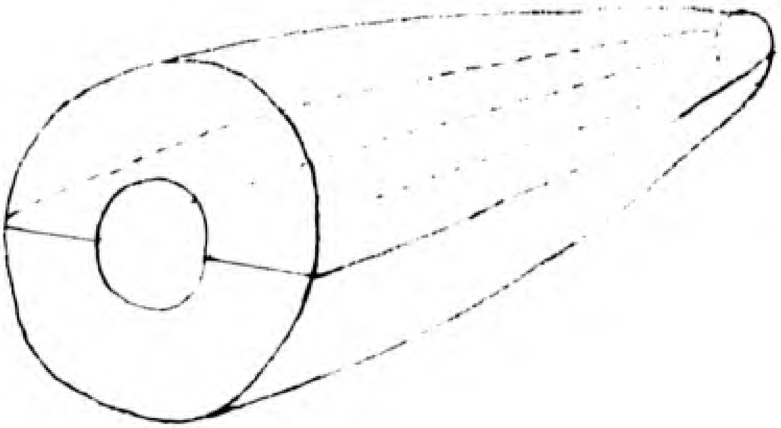


Fig. 7. Solid Sabot

If a sabot consists of more than two setments and if they don't contact each other under loading, λ in Eqs. (40a) and (40b) has to be replaced by

$$\Lambda = \frac{2\lambda G}{\lambda + 2G} .$$

16-12 Optimal Design Problem with Varying Boundary

In previous sections all governing equations were obtained from the principle of stationary total potential energy, which result in a set of second order partial differential equations. Sometimes it is more desirable to write the state equations in first order, self adjoint form. This will be most conveniently done by using Reissner's variational principle.* With the use of this variational principle, the governing equations for the shear plate and the solid sabot of revolution, with quasi static loading, can be written in the following manner.

a) Shear plate

$$-\frac{\partial z_3}{\partial x_1} - \frac{\partial z_5}{\partial x_2} = 0$$

$$-\frac{\partial z_4}{\partial x_2} - \frac{\partial z_5}{\partial x_1} = 0$$

$$\frac{\partial z_1}{\partial x_1} - \frac{1}{Ed} z_3 + \frac{\nu}{Ed} z_4 = 0 \qquad (41a)$$

$$\frac{\partial z_2}{\partial x_2} - \frac{1}{Ed} z_4 + \frac{\nu}{Ed} z_3 = 0$$

$$\frac{\partial z_1}{\partial x_2} + \frac{\partial z_2}{\partial x_1} - \frac{z_5}{Gd} = 0$$

$$z_1(x_1, R_i) = 0$$

$$z_2(x_1, R_i) = 0$$

$$z_3(0, x_2) = 0 \qquad (41b)$$

$$z_5(0, x_2) = 0$$

$$z_3(x_1, R(x_1)) \cos(x_1, n) + z_5(x_1, R(x_1)) \cos(x_2, n) = f_{x_1}$$

$$z_4(x_1, R(x_1)) \cos(x_2, n) + z_5(x_1, R(x_1)) \cos(x_1, n) = f_{x_2}$$

---

*Reissner, E., "On a Variational Theorem in Elasticity"

where

E = Young's modulus

G = Shear modulus

ν = Poisson's ratio

z_1 = The displacement in the x_1 direction

z_2 = The displacement in the x_2 direction

z_3 = The resultant force in the x_1 direction on a cross section normal to the x_1 axis

z_4 = The resultant force in the x_2 direction on a cross section normal to the x_2 axis

z_5 = The resultant force in the x_2 direction on a cross section normal to the x_2 (x_1) axis

The resultant force is defined as stress times the thickness of the plate, i.e.,

$$z_3 = \sigma_{x_1} u \,, \qquad z_4 = \sigma_{x_2} u \,, \qquad z_5 = \tau_{x_1 x_2} u \,.$$

where u is thickness of plate.

When the shear plate thickness is not constant, u is a function of x_1 and x_2 and will be treated as a design variable.

b) A solid sabot of revolution

$$-\frac{\partial z_3}{\partial x_1} - \frac{1}{x_2}\frac{\partial}{\partial x_2}(x_2 z_5) = F_{x_1}$$

$$-\frac{1}{x_2}\frac{\partial}{\partial x_2}(x_2 z_4) - \frac{\partial z_5}{\partial x_1} + \frac{\nu}{x_2}(z_3 + z_4) + E\frac{z_2}{x_2^2} = F_{x_2}$$

$$\frac{\partial z_1}{\partial x_1} + \nu\frac{z_2}{x_2} + \frac{1+\nu}{E}\{(1-\nu)z_3 - \nu z_4\} = 0 \qquad (42a)$$

$$\frac{\partial z_2}{\partial x_2} + \nu\frac{z_2}{x_2} + \frac{1+\nu}{E}\{(1-\nu)z_4 - \nu z_3\} = 0$$

$$\frac{\partial z_1}{\partial x_2} + \frac{\partial z_2}{\partial x_1} - \frac{z_5}{G} = 0$$

$$z_1(x_1, R_i) = 0$$

$$z_2(x_1, R_i) = 0$$

$$z_3(0, x_2) = 0$$

$$z_5(0, x_2) = 0$$

$$z_3(x_1, R(x_1))\cos(x_1, n) + z_5(x_1, R(x_1))\cos(x_2, n) = f_{x_1}$$

$$z_4(x_1, R(x_1))\cos(x_2, n) + z_5(x_1, R(x_1))\cos(x_1, n) = f_{x_2}$$

where x_1 and x_2 represent cylindrical coordinates in the axial and radial directions, respectively. Since the system is axisymmetric, the circumferential coordinate does not appear in the above equations.

The following physical significance is ascribed to the variables of the problem:

z_1 = Displacement in the axial direction

z_2 = Displacement in the radial direction

z_3 = Stress in the axial direction (σ_x)

z_4 = Stress in the radial direction (σ_r)

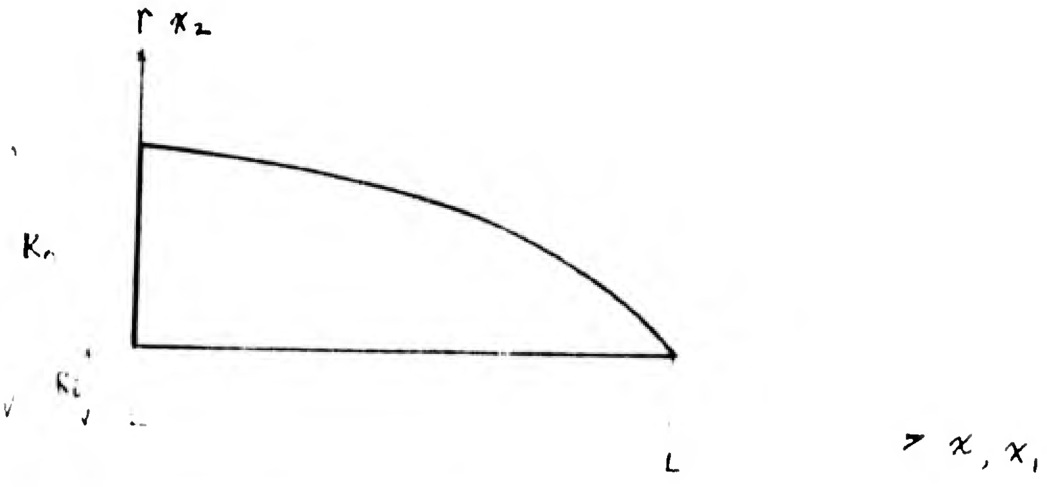
z_5 = Shearing stress τ_{xy}

ν = Poisson's ratio

E = Young's modulus

G = Shear modulus

F_{x_1}, F_{x_2} = Body forces in the axial and radial directions respectively.



One wishes to minimize the weight of shear plate or sabot under certain constraint conditions. Hence, the optimal problem can be written in the following manner

Minimize

$$J = \int_0^L \int_{R_i}^{R(x_1)} u \, dx_2 \, dx_1 \tag{43}$$

subject to:

$$\left.\begin{aligned} L(u)\, z &= [illegible] \\ B(v)\, z &= g(v, K) \end{aligned}\right\} \tag{44}$$

$$\left.\begin{aligned} \frac{dR(x_1)}{dx_1} &= v \\ R(0) &= R_0 \\ R(L) &= R_1 \end{aligned}\right\} \tag{45}$$

$$\Psi = \int_0^L \int_0^{R(x_1)} F(u, z) \, dx_2 \, dx_1 \tag{46}$$

The above optimal design problem is for a shear plate. The problem for a solid sabot differs slightly and will be treated explicitly below. Equations (44) represent the relationship between displacements and forces in the plate. Equations (45) control the design of the boundary, in which one of the boundary conditions should be treated as a constraint equation. Equations (46) is a typical functional constraint. For example, a stress constraint will be given in this manner. To compute the sensitivity coefficient, one needs to know the effect of small changes in design. Using calculus of variations and keeping the boundary change in mind this will be one as follows:

$$\delta J = \int_0^L \int_0^{R(x_1)} \overline{\delta u}\, dx_2\, dx_1 + \int_0^L u\, \overline{\delta R}\, dx_1 \tag{47}$$

$$\begin{aligned} L(u)\overline{\delta z} + (L(u)z)_u \overline{\delta u} &= \frac{\partial f}{\partial u}\overline{\delta u} \\ B(v)\overline{\delta z} + (B(v)z)_v \overline{\delta v} &= \frac{\partial g}{\partial v}\overline{\delta v} + \frac{\partial g}{\partial R}\overline{\delta R} \end{aligned} \tag{48}$$

$$\begin{aligned} \frac{d}{dx_1}\overline{\delta R}(x_1) &= \overline{\delta v} \\ \overline{\delta R}(0) &= 0 \\ \overline{\delta R}(L) &= 0 \end{aligned} \tag{49}$$

$$\begin{aligned} \delta \psi = &\int_0^L \int_0^{R(x_1)} \left(\frac{\partial F}{\partial u}\overline{\delta u} + \frac{\partial F}{\partial z}\overline{\delta z} \right) dx_2\, dx_1 \\ &+ \int_0^L F\, \overline{\delta R}\, dx_1 \end{aligned} \tag{50}$$

The bars over the variation quantities are used to indicate net changes, which should be distinguished from total changes of variables.* Equations (47) and (50) are constraints on the variations in the state variables z and R. These quantities can be removed in this following manner. Let

$$\frac{d\lambda_R}{dx_1} = -u(x_1, R(x_1))$$

Then

$$\frac{d\lambda_R}{dx_1}\overline{\delta R}(x_1) = -u\,\overline{\delta R}(x_1)$$

Integrating by parts, one obtains

$$\int_0^L -\lambda_R \frac{d}{dx_1}\overline{\delta R}(x_1)\,dx_1 + \lambda_R \overline{\delta R}_1 \Big|_0^L = -\int_0^L u\,\overline{\delta R}\,dx_1 ,$$

so

$$\int_0^L \lambda_R \overline{\delta v}\,dx_1 = \int_0^L u\,\overline{\delta R}\,dx_1$$

Substituting the above into Eq. (47) yields

$$\delta J = \int_0^L \int_{R_i}^{R(x_1)} \overline{\delta u}\,dx_2\,dx_1 + \int_0^L \lambda_R \overline{\delta R}\,dx_1 \tag{51}$$

If the L(u) is a self adjoint operator, then

$$\int_0^L \int_{R_i}^{R(x_1)} \{\lambda^T L(u)\overline{\delta z} - \overline{\delta z}^T L(u)\lambda\}\,d\Omega = \int_S \lambda^T B(v)\,\delta z^T\,dS \tag{52}$$

---

* Gelgand & Fomin, "Calculus of Variation" Sec. 37, pp. 168-179.

Define λ to satisfy the equation

$$L(u)\lambda = \frac{\partial F}{\partial z}^T$$

Then, using Eq. (52) and (48), Eq. (50) can be written as

$$\delta\psi = \int_0^L\int_{R_i}^{R(x_1)} \left(\frac{\partial F}{\partial u}\,\overline{\delta u} - \lambda^T(L(u)z)_u + \frac{\partial Q}{\partial u}\right)\overline{\delta u}\; dx_2\, dx_1$$

$$+ \int_0^L F\,\overline{\delta R}\; dx_1 + \int_S \left\{\lambda^T(B(v))_v\,\overline{\delta v} - \lambda^T\frac{\partial g}{\partial R}\,\delta R + \frac{\partial g}{\partial v}\,\overline{\delta v}\right\} dS$$

$$= \int_0^L\int_{R_i}^{R(x_1)} \left(\frac{\partial F}{\partial u}\,\overline{\delta u} - \lambda^T(L(u)z)_u + \frac{\partial Q}{\partial u}\right)\overline{\delta u}\; dx_2\, dx_1$$

$$+ \int_S \left\{\left(\lambda^T(B(v)z)_v + \frac{\partial g}{\partial v}\right)\overline{\delta v} + \left(F - \lambda^T\frac{\partial g}{\partial R}\right)\overline{\delta R}\right\} dS$$

$$= \int_0^L\int_{R_i}^{R(x_1)} \left(\frac{\partial F}{\partial u}\,\overline{\delta u} - \lambda^T(L(u)z)_u + \frac{\partial Q}{\partial u}\right)\overline{\delta u}\; dx_2\, dx_1$$

$$+ \int_0^L \left\{\lambda^T(B(v)z)_v + \frac{\partial g}{\partial v} + \lambda_F\right\}\overline{\delta v}\; dx_1 \tag{53}$$

where λ_F is a solution of

$$\frac{d\lambda_F}{dx_1} = -\left(F - \lambda^T\frac{\partial g}{\partial R}\right)$$

$$\lambda_F(0) = 0.$$

At this stage, sensitivity coefficients can be computed and the optimal design approach of Chapter 8 of the Handbook implemented for direct numerical calculation.

In the case of a solid sabot of revolution, cost function is given by

$$J = 2\pi \int_0^L \int_{R_i}^{L(x_1)} x_2 \, dx_2 \, dx_1 \tag{54}$$

State equations are, in an operator form

$$\begin{aligned} L\, z &= Q \\ B(v)\, z &= g(v, R) \end{aligned} \tag{55}$$

$$\begin{aligned} \frac{dR}{dx_1} &= v \\ R(0) &= R_0 \\ R(L) &= R_i \end{aligned} \tag{56}$$

Functional constraint equation is

$$\psi = 2\pi \int_0^L \int_{r_i}^{L(x_1)} F(z)\, x_2 \, dx_2 \, dx_1 \tag{57}$$

It is seen from the above set of equations that v is the only design variable. Hence the effect of small change in design can be written in a simple manner.

$$\delta J = 2\pi \int_0^L R(x_1)\,\delta R\, dx_1 \tag{58}$$

$$L\,\delta z = 0$$

$$B(v)\,\delta z + (B(v)\,z)_{,v}\,\delta v = \frac{\partial \mathcal{G}}{\partial v}\,\delta v + \frac{\partial \mathcal{G}}{\partial R}\,\delta R \tag{59}$$

$$\frac{d\,\delta R}{dx_1} = v$$

$$\delta R(0) = 0 \tag{60}$$

$$\delta R(L) = 0$$

$$\delta \psi = 2\pi \int_0^L \int_{R_i}^{R(x_1)} \frac{\partial F}{\partial z}\,\delta z\, x_2\, dx_2\, dx_1 \tag{61}$$

To remove the state variable dependency from Eq. (58), set

$$\frac{d\lambda_R}{dx_1} = -R(x_1)$$

Then

$$\frac{d\lambda_R}{dx_1}\,\delta R = -R\,\delta R.$$