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STEADY AND UNSTEADY FLOW IN STRAIGHT RECTANGULAR DUCTS

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by V. O'BRIEN

THE JOHNS HOPKINS UNIVERSITY ▪ APPLIED PHYSICS LABORATORY

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A new solution for steady parallel flow in a rectangular duct is given that is valid for any aspect ratio a/b (where a and b are orthogonal half-widths of the duct), in contrast to older formulas that are restricted to certain ranges of a/b. New solutions for oscillatory flow in rectangular ducts are also given that replace a set of invalid published solutions. They provide calibration constants for relating local and average values of the unsteady velocity flow measurements in any rectangular channel.

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CONTENTS

ILLUSTRATIONS

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1. INTRODUCTION

Fully developed laminar flow in pipes of any uniform cross section can be handled analytically because there is only one velocity component — that parallel to the walls. Published solutions for the flow in a rectangular duct driven by a steady pressure gradient dp/dz (z being the downstream direction) can be found (Refs. 1 and 2) which satisfy viscous boundary conditions on walls at $x = \pm a$, $y = \pm b$ (Fig. 1). Unfortunately, these solutions differ and neither reflects the symmetry expected from purely physical considerations. Each of these solutions appears to be based on a different planar flow solution and is limited to a range of rectangular aspect ratios (a/b). These limitations are not mentioned by the authors or others who quote them.

On the other hand, one can start afresh with the exact solution in an elliptical duct whose cross section is given by $x^2/a^2 + y^2/b^2 = 1$, which is valid for any fineness ratio, a/b. In Section 2 this solution is modified to satisfy the nonslip boundary condition on the corresponding rectangular duct with aspect ratio a/b. The closed-form velocity formula shown is exact, symmetric in form, and fulfills limit constraints. Some new series identities that follow from comparing the various steady solutions are given in the Appendix.

Ref. 1. R. Berker, "Intégration des Équations du Mouvement d'un Fluide Visqueux Incompressible," Handbuch der Physik, Vol. VIII/2, Springer-Verlag, Berlin, 1963, p. 70.

Ref. 2. L. S. Han, "Hydrodynamic Entrance Lengths for Incompressible Laminar Flow in Rectangular Ducts," J. Appl. Mech., Vol. 27 (Trans. ASME, Vol. 82), 1960, p. 403.

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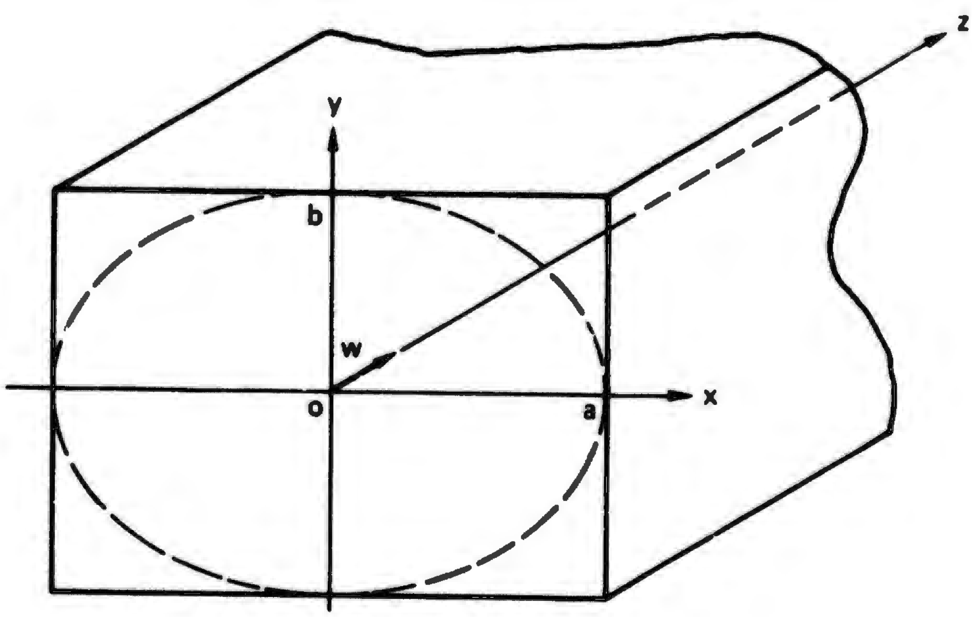

Fig. 1 Geometry of the Flows (Rectangular Duct)

Published solutions for oscillatory fully developed flow can be found (Ref. 3). Upon inspection, these are found to be invalid. A valid set of oscillatory fully developed solutions in rectangular ducts is presented in Section 3. Thanks to the linearity of the flow equation, the oscillatory solutions form the basis for any periodic time-dependent flow. After the pressure gradient is Fourier decomposed, harmonic component solutions can be summed (including, of course, the steady flow). The solutions should be useful for calibrations in unsteady flow in rectangular ducts.

Finally, the approximate development of parallel viscous flow from the entrance of a rectangular duct is discussed in Section 4. The analysis is based on linearized forms of the momentum equation that have appeared in the literature. The formulas are rather involved and no useful calculations have been performed.

Ref. 3. D. G. Drake, "On the Flow in a Channel Due To a Periodic Pressure Gradient," Quart. J. Mech. Appl. Math., Vol. 18, 1965, p. 1.

2. STEADY FULLY DEVELOPED FLOW

Assuming incompressible, Newtonian fluid the momentum equation for the single velocity component (w) of parallel fully developed flow is simply

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{dp}{dz} + \nu \nabla^2 w . \qquad (1)$$

For steady flow ($\partial w/\partial t = 0$) the parabolic plane Poiseuille flow solution w(y) is well known (Ref. 4):

$$\frac{w(y)}{w_0} = \left(1 - \frac{y^2}{b^2}\right), \qquad (2)$$

where $w_0 = \left|\frac{b^2}{2\mu}\frac{dp}{dz}\right|$ represents the given pressure gradient. By adding a perturbation solution w^* which satisfies $\nabla^2 w^* = 0$, and constraining the perturbed solution to satisfy $w = 0$ on $x = \pm a$ (for $|y| < b$), one can duplicate the rectangular duct solution cited in Berker (Ref. 1). This apparently is only intended for $a \geq b$ although no restriction has been noted (see Table 1 of Ref. 1).

Alternatively, for $b \geq a$ one can start with the complementary planar solution w(x) and duplicate the formula given by Han (Ref. 2) for his $\beta = 0$. Although Han shows a figure with $a > b$, he only uses the solution for $0 < a/b \leq 1$ (see Table 1 in Ref. 2). There is no warning against its use for $a > b$.

Although these formulas are correct and useful (each in its own range of a/b) a single formula was sought to be valid for all values of a/b from 0 to ∞. The new solution also shows a symmetry of form that is inherent in

Ref. 4. H. L. Dryden, F. D. Murnaghan, and H. Bateman, Hydrodynamics, Dover, N. Y., 1956, pp. 181 and 184.

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the physical situation but lacking in the published "one-sided" formulas.

The analysis starts with the steady parallel flow solution in an elliptic pipe of any fineness ratio a/b (after Ref. 4):

$$\frac{w}{w_1} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} , \qquad (3)$$

where $w_1 = \dfrac{a^2 b^2}{a^2 + b^2} \dfrac{1}{2\mu} \left| \dfrac{dp}{dz} \right|$.

This is modified by adding a perturbation solution to satisfy the nonslip boundary conditions on the flat walls

$$x = \pm a \qquad (|y| < b)$$

and

$$y = \pm b \qquad (|x| < a) .$$

Let

$$\frac{w}{w_1} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{w^*}{w_1}(x, y) , \qquad (4)$$

where

(i) $\quad x = \pm a : \quad \dfrac{w}{w_1} = 0 = - \dfrac{y^2}{b^2} + \dfrac{w^*}{w_1} (\pm a, y)$,

(ii) $\quad y = \pm b : \quad \dfrac{w}{w_1} = 0 = - \dfrac{x^2}{a^2} + \dfrac{w^*}{w_1} (x, \pm b)$.

Ref. 4. H. L. Dryden, F. D. Murnaghan, and H. Bateman, Hydrodynamics, Dover, New York, 1956, p. 181.

Physical symmetry conditions restrict the form of the (harmonic) perturbation solution to

$$\frac{w^*}{w_1} = \sum_n \left(A_n \cosh p_n x \cos p_n y + B_n \cosh q_n y \cos q_n x + C_n\right). \qquad (5)$$

With the use of series expansions (see Ref. 5)

(a) $$\frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = 1$$

(b) $$\frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos (2n+1)\theta = 1 - \frac{4}{\pi^2}\,\theta^2 \qquad \left(|\theta| < \frac{\pi}{2}\right),$$

where $\theta = \frac{\pi}{2}\frac{y}{b}$ or $\frac{\pi}{2}\frac{x}{a}$,

the coefficients are found to be

$$A_n = -\frac{32}{\pi^3} \frac{(-1)^n}{(2n+1)^3} \frac{1}{\cosh p_n a},$$

$$B_n = -\frac{32}{\pi^3} \frac{(-1)^n}{(2n+1)^3} \frac{1}{\cosh q_n b},$$

$$C_n = \frac{32}{\pi^3} \frac{(-1)^n}{(2n+1)^3},$$

where $p_n = \frac{2n+1}{3}\frac{\pi}{b}$; $q_n = \frac{2n+1}{2}\frac{\pi}{a} = p_n \frac{b}{a}$.

Ref. 5. L. B. W. Jolley, Summation of Series, Dover, New York, 1961, pp. 16, 64, and 100.

Thus Eq. (5) becomes

$$\frac{w^*}{w_1} = \frac{32}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left(1 - \frac{\cosh p_n x}{\cosh p_n a} \cos p_n y - \frac{\cosh q_n y}{\cosh q_n b} \cos q_n x\right), \quad (6)$$

and the total "two-sided" solution (Eq. (4)) can be written as

$$\frac{w(x,y)}{w_1} = 2 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left(\frac{\cosh p_n x}{\cosh p_n a} \cos p_n y + \frac{\cosh q_n y}{\cosh q_n b} \cos q_n x\right) . \quad (7)$$

The symmetry is most apparent when a = b (i.e., a square duct). Then $p_n = q_n$ and $p_n a = q_n b \equiv P_n$ so the similar centerline velocity profiles are respectively

$$\frac{w}{w_1}(x,0) = 2 - \left(\frac{x}{a}\right)^2 - \sum_n \frac{32}{\pi^3} \frac{(-1)^n}{(2n+1)^3} \left(\frac{\cosh p_n x}{\cosh P_n} + \frac{\cos p_n x}{\cosh P_n}\right) , \quad (8a)$$

$$\frac{w}{w_1}(0,y) = 2 - \left(\frac{y}{b}\right)^2 - \sum_n \frac{32}{\pi^3} \frac{(-1)^n}{(2n+1)^3} \left(\frac{\cos p_n y}{\cosh P_n} + \frac{\cosh p_n y}{\cosh P_n}\right) . \quad (8b)$$

This symmetry is lacking in the older "one-sided" velocity formulas (see the Appendix).

The normalized average velocity over the section is

$$\frac{\langle w \rangle}{w_1} = \frac{1}{ab} \int_0^a dx \int_0^b dy \, \frac{w(x,y)}{w_1}$$

$$= \frac{4}{3} - \frac{128}{\pi^5} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} \left[\frac{b}{a} \tanh\left(\frac{2n+1}{2} \pi \frac{a}{b}\right) + \frac{a}{b} \tanh\left(\frac{2n+1}{2} \pi \frac{b}{a}\right)\right] \quad (9)$$

This formula is symmetric in a and b. The minimum value occurs at a/b = 1, i.e., when the hydraulic radius is smallest. The analytic expression for $\langle w \rangle$ at this ratio is the same at that given by Berker; also see Table 1 for values at other ratios. The total flux or discharge (Q) is equal to $\langle w \rangle$ 4ab.

The approach to a limiting very flat rectangular section is checked by allowing a/b to approach ∞.

$$\frac{\langle w \rangle}{w_1} \rightarrow \frac{\langle w \rangle}{w_0} \rightarrow 4\left(\frac{1}{3} - \frac{16}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}\right) = 4\left(\frac{1}{3} - \frac{1}{6}\right) = \frac{2}{3} , \tag{10}$$

again using Jolley (Ref. 5). This is the same value obtained by integrating Eq. (2).

The variation of center velocity relative to the mean velocity $(w(0,0)/\langle w \rangle)$ is a quantity used to calibrate local velocity measurements against average velocity (volume flow) measurements. It varies with duct shape (a/b) as shown in Table 1. This is illustrated in Fig. 2 along with the values for all ellipses and the planar asymptotic limits. The range of the older formulas is indicated. Note that the corresponding values calculated by Han for a/b $\le$ 1 fall on the curve.

Table 1

Typical Values for Steady Flow in Rectangular Ducts

$\frac{a}{b}$ or $\frac{b}{a}$	$\frac{w_1}{w_0}$	$\frac{w(0,0)}{w_1}$	$\frac{\langle w\rangle}{w_1}$	$\frac{w(0,0)}{\langle w\rangle}$	$\frac{\langle w\rangle}{w_0}$
1	0.5	1.17874	0.562313	2.09624	0.281157
2	0.8	1.13872	0.571708	1.99178	0.457366
3	0.9	1.09051	0.585153	1.86363	0.526638
5	0.961538	1.03917	0.605939	1.71497	0.582634
8	0.984615	1.01562	0.623742	1.62826	0.614146
10	0.99099	1.01000	0.630897	1.60089	0.624650
∞	1.0	1.0	0.66667	1.5	0.66667

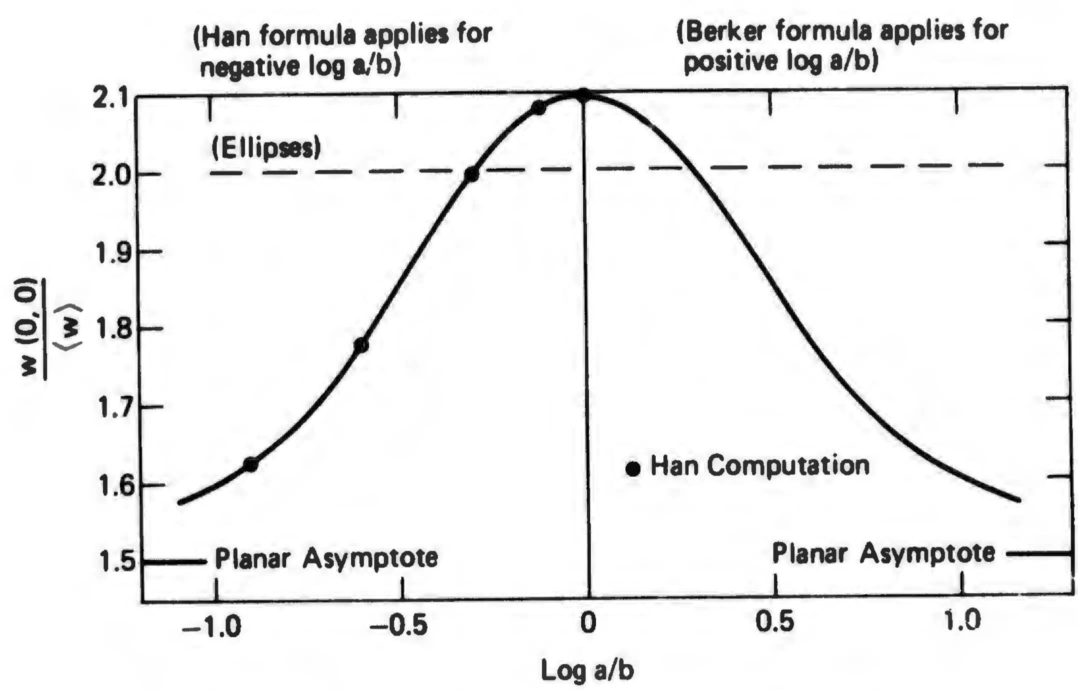

Fig. 2 Ratio of Center Velocity to Mean Velocity for Steady Rectangular Duct Flows

3. FULLY DEVELOPED OSCILLATORY FLOW

For oscillatory flow Eq. (1) becomes

$$\frac{\partial w}{\partial t} = - K \cos \omega t + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) , \qquad (11)$$

where $K\rho \cos \omega t$ is the assumed periodic pressure gradient. Let

$$w \equiv \text{real part of } [f(x,y)e^{i\omega t}] , \qquad (12a)$$

and

$$\frac{\partial w}{\partial t} = \text{real part of } [i\omega \, f(x,y)e^{i\omega t}] . \qquad (12b)$$

Assume that b is the shorter of the duct dimensions. Normalize by w_0, scale the lengths $x/b = X$, $y/b = Y$, and factor out $e^{i\omega t}$, so that the nondimensional form of Eq. (11) becomes

$$(i\omega b^2/\nu) \, F = - K' + \nabla^2 F , \qquad \left(\frac{a}{b} \geq 1\right)^* \qquad (13)$$

where $F \equiv f/w_0$ and $K' \equiv Kb^2/w_0\nu$ is a nondimensional constant. If we define $\eta^2 \equiv \omega b^2/\nu$ (a "Stokes number"), the equation to be solved becomes

$$(\nabla^2 - i\eta^2) \, F = K' . \qquad (14)$$

A particular solution is given by

$$F_0 = \frac{K'}{-i\eta^2} = \frac{K'}{\eta^2} e^{i\pi/2} . \qquad (\eta \neq 0) \qquad (15)$$

---

*Note this restriction.

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This corresponds to a uniform oscillating "plug" velocity that establishes the scale of the velocity solution. The remaining portion of the solution F(x,y) can satisfy

$$(\nabla^2 - i\eta^2)\, F = 0 \,. \tag{16}$$

Assume separable solutions

$$F_1(X,Y) = H_1(X)\, G_1(Y) \,,$$

which from Eq. (16) gives

$$\frac{1}{G_1}\frac{\partial^2 G_1}{\partial X^2} - i\eta^2 = -\frac{1}{H_1}\frac{\partial^2 H_1}{\partial Y^2} = p^2 \,, \tag{17a}$$

where p^2 is a separation constant and

$$G_1 = \begin{cases} \sinh \sqrt{p^2 + i\eta^2}\; X \\ \cosh \sqrt{p^2 + i\eta^2}\; X \end{cases} , \qquad H_1 = \begin{cases} \sin pY \\ \cos pY \,. \end{cases} \tag{17b}$$

Similarly if the separation constant were $-q^2$ and

$$\frac{1}{G_2}\frac{\partial^2 G_2}{\partial X^2} = -q^2 = -\frac{1}{H_2}\frac{\partial^2 H_2}{\partial Y^2} + i\eta^2 \,, \tag{18a}$$

this leads to

$$F_2 = G_2 \cdot H_2 = \begin{pmatrix} \sin qX \\ \cos qX \end{pmatrix} \begin{pmatrix} \sinh \sqrt{q^2 + i\eta^2}\; Y \\ \cosh \sqrt{q^2 + i\eta^2}\; Y \end{pmatrix} . \tag{18b}$$

Because by physical symmetry requirements F must be even in X and Y, the most general solution for F is

$$F = F_0 + \sum_p \alpha_p \cosh\sqrt{p^2 + i\eta^2}\, X \cos pY + \sum_q \beta_q \cos q X \cosh\sqrt{q^2 + i\eta^2}\, Y , \quad (19)$$

which must satisfy nonslip on the stationary walls (X = a/b; Y = 1) or F = 0 there.

(i) On X = a/b: Take $q_n = \frac{2n+1}{2} \pi \frac{b}{a}$ and for Y < 1 expand F_0 in odd cosine series so the zero velocity condition can be written as

$$0 = \sum_{n=0}^{\infty} \cos p_n Y \left[F_0 \frac{4}{\pi} \frac{(-1)^n}{(2n+1)} + \alpha_n \cosh\sqrt{p_n^2 + i\eta^2}\, \frac{a}{b} \right],$$

where $p_n = \frac{2n+1}{2} \pi$.

$$\text{So } \alpha_n = \frac{4F_0}{\pi} \frac{(-1)^{n+1}}{(2n+1)} \frac{1}{\cosh \gamma_n \frac{a}{b}}, \quad (20)$$

where $\gamma_n \equiv p_n^2 + i\eta^2$ (complex numbers).

(ii) Similarly on Y = 1:

$$0 = \sum_{n=0}^{\infty} \cos q_n X \left[\frac{4F_0}{\pi} \frac{(-1)^n}{(2n+1)} + \beta_n \cosh\sqrt{q_n^2 + i\eta^2} \right],$$

or

$$\beta_n = \frac{4F_0}{\pi} \frac{(-1)^{n+1}}{(2n+1)} \frac{1}{\cosh \mu_n}, \quad (21)$$

where $\mu_n = \sqrt{q_n^2 + i\eta^2}$ (complex numbers).

So the oscillatory solution is

$$F = F_0\left[1 + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)}\left(\frac{\cosh\gamma_n X\cos p_n Y}{\cosh\gamma_n \frac{a}{b}} + \frac{\cos\alpha_n X\cosh\mu_n Y}{\cosh\mu_n}\right)\right] \tag{22}$$

The centerline (X = 0) velocity profile is

$$\frac{F(0,Y)}{F_0} = 1 + \frac{4}{\pi}\sum_n\frac{(-1)^{n+1}}{(2n+1)}\left(\frac{\cos p_n Y}{\cosh\gamma_n \frac{a}{b}} + \frac{\cosh\mu_n Y}{\cosh\mu_n}\right), \tag{23}$$

and center velocity becomes

$$\frac{F(0,0)}{F_0} = 1 + \frac{4}{\pi}\sum_n\frac{(-1)^{n+1}}{(2n+1)}\left(\frac{1}{\cosh\gamma_n \frac{a}{b}} + \frac{1}{\cosh\mu_n}\right). \tag{24}$$

As a/b approaches ∞ the centerline formulas of Eq. (23) should reduce to the well-known plane oscillatory solution (Ref. 6). In the limit,

$$\frac{F(0,Y)}{F_0} \to 1 + \frac{4}{\pi}\sum_n\frac{(-1)^{n+1}}{(2n+1)}\left(\frac{\cosh kY}{\cosh k}\right)$$

$$= 1 - \frac{\cosh kY}{\cosh k}, \tag{25}$$

where $k = \sqrt{i\eta^2}$.

So the oscillatory solution does have the proper asymptotic behavior.

Ref. 6. N. Rott, Theory of Laminar Flows (F. K. Moore, Ed.), Princeton University Press, Princeton, New Jersey, 1964, p. 401.

The average velocity is given by

$$\frac{\langle F\rangle}{F_0} = \frac{1}{(a/b)} \int_0^{a/b} dX \int_0^1 dY\, F(X,Y)$$

$$= \frac{b}{a}\left\{\frac{a}{b} + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)}\left[\frac{1}{\gamma_n p_n}\frac{\sinh(\gamma_n a/b)\sin p_n}{\cosh(\gamma_n a/b)}\right.\right.$$

$$\left.\left. + \frac{1}{\mu_n q_n}\frac{\sinh\mu_n \sin(q_n a/b)}{\cosh\mu_n}\right]\right\}$$

$$= \left\{1 + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)}{(2n+1)}\left[\frac{b/a}{\gamma_n p_n}\tanh(\gamma_n a/b) + \frac{1}{\mu_n p_n}\tanh\mu_n\right]\right\}, \quad (26)$$

so that

$$\frac{F(0,0)}{\langle F\rangle} = \frac{1 + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)}\left[\frac{1}{\cosh(\gamma_n a/b)} + \frac{1}{\cosh\mu_n}\right]}{1 + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)}{(2n+1)^2}\left[\frac{b}{a}\frac{\tanh(\gamma_n a/b)}{(\gamma_n)} + \frac{\tanh\mu_n}{\mu_n}\right]}, \quad (27)$$

where

$$\gamma_n = \sqrt{\left(\frac{2n+1}{2}\right)^2 \pi^2 + i\eta^2} = \sqrt{\frac{(2n+1)^2}{4}\pi^2 + \frac{i\omega b^2}{\nu}}$$

$$\mu_n = \sqrt{\left(\frac{2n+1}{2}\right)^2 \pi^2\left(\frac{b}{a}\right)^2 + i\eta^2} = \sqrt{\frac{(2n+1)^2}{4}\pi^2\left(\frac{b}{a}\right)^2 + \frac{i\omega b^2}{\nu}}$$

As η approaches ∞, γ_n and μ_n also approach ∞ for all values of n, and the ratio $F(0,0)/\langle F\rangle$ approaches 1.

This reflects the fact that the rapidly changing velocities are concentrated more and more to a thin layer at the wall as η increases, so the velocity profile becomes more plug-like as the value of η becomes higher. The relative area occupied by this thin, oscillatory, phase-shifting layer will be a maximum at a/b = 1 (i.e., a square duct). For this case the ratio of center-to-mean velocity as a function of η reduces to

$$\left.\frac{F(0,0)}{\langle F\rangle}\right|_{\frac{a}{b}=1} = \frac{1 - \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh T_n}}{1 - \frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{\tanh T_n}{T_n}} \tag{28}$$

where $T_n = \sqrt{\left(\frac{2n+1}{2}\right)^2 \pi^2 + i\eta^2}$ (see Table 2).

To be complete one should also show that in the limit as ω approaches 0, the steady flow solution is recovered. It is easiest to work with Eq. (28) for the square duct. Recall that F_0 is inversely proportional to $N = \eta^2$, which becomes infinite as N approaches 0. The value of the ratio $F(0,0)/\langle F\rangle$ becomes indeterminate as N approaches 0 (see the Appendix), for both numerator and denominator are of order N. The limit of the right-hand side of Eq. (28) as N approaches 0 must be determined by L'Hopital's rule; it is

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{\tanh\left(\frac{2n+1}{2}\pi\right)}{\cosh\left(\frac{2n+1}{2}\pi\right)}}{\sum_{n=0}^{\infty}\frac{4}{\pi^2(2n+1)^4}\left[\operatorname{sech}^2\left(\frac{2n+1}{2}\pi\right) - \frac{2}{\pi(2n+1)}\tanh\left(\frac{2n+1}{2}\pi\right)\right]} = 2.096.$$

(Compare this with Table 1.)

Table 2

Velocity Amplitude Ratios for Oscillatory Flow in Rectangular Ducts

$$\left|\frac{w(0,0)}{\langle w\rangle}\right|$$

Aspect Ratio, a/b	Stokes Number, $\eta = \sqrt{\frac{\omega}{\nu} b^2}$			
	1.0	3.0	10.0	∞
1.0*	2.094	1.944	1.149	1.0
10*	1.593	1.460	1.079	1.0
∞	(1.499)†	(1.420)†	(1.070)†	1.0

*n.b., the values were obtained from truncated series which converge rather slowly, so that the last digit is uncertain.

†Evaluated by closed form planar formulas (Ref. 6).

It should be noted that the oscillatory solution given here differs from the asymmetric formula presented by Drake (Ref. 7). There has been no confirmation of his theory by other investigators. His limiting form for quasi-steady flow does not reduce to the similar steady one-sided solutions of either Refs. 1 or 2, except at $x = 0$, contrary to the implication in the published paper. Moreover, if one looks at the centerline ($x = 0$) profiles as a/b approaches ∞, the resulting expression of Drake should reduce to the oscillatory planar flow (Eq. (25)), which it does not.

Asymptotic expressions for w at large values of η could be written out to correct the relations given by Drake. (Qualitatively, his conclusions are valid.) Let it suffice to say that for large η as a/b approaches ∞, the correct asymptotic center-to-mean velocity ratio is given by

$$\frac{w(0,0)}{\langle w\rangle} \xrightarrow[\frac{a}{b},\eta\to\infty]{} \left(1 - \frac{\sqrt{2}}{\eta}\right)^{-1} e^{-i\left[\arc\tan\left(\frac{\eta}{\eta\sqrt{2}}\right) - \frac{\pi}{4}\right]} .$$

See Fig. 3 for amplitude and phase of $\lim\limits_{a/b\to\infty} w(0,0)/\langle w\rangle$ versus η. The insignificant differences between centerline profiles for plane oscillatory flow and those for $a/b = 22$ have been shown experimentally for several values of η (Ref. 7).

Note that the velocity formulas given here up to Eq. (26) are not quite symmetric in (a, b) and they are not valid for all a/b ratios because b was used as a normalizing length and w_0 (proportional to b^2) was the normalizing velocity. In the oscillatory flow the narrow dimension of the duct is more important in establishing the diffusion of

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Ref. 7. V. O'Brien and F. E. Logan, "Velocity Overshoot within the Boundary Layer in Laminar Pulsating Flow," Phys. Fl., Vol. 9, 1966, p. 214.

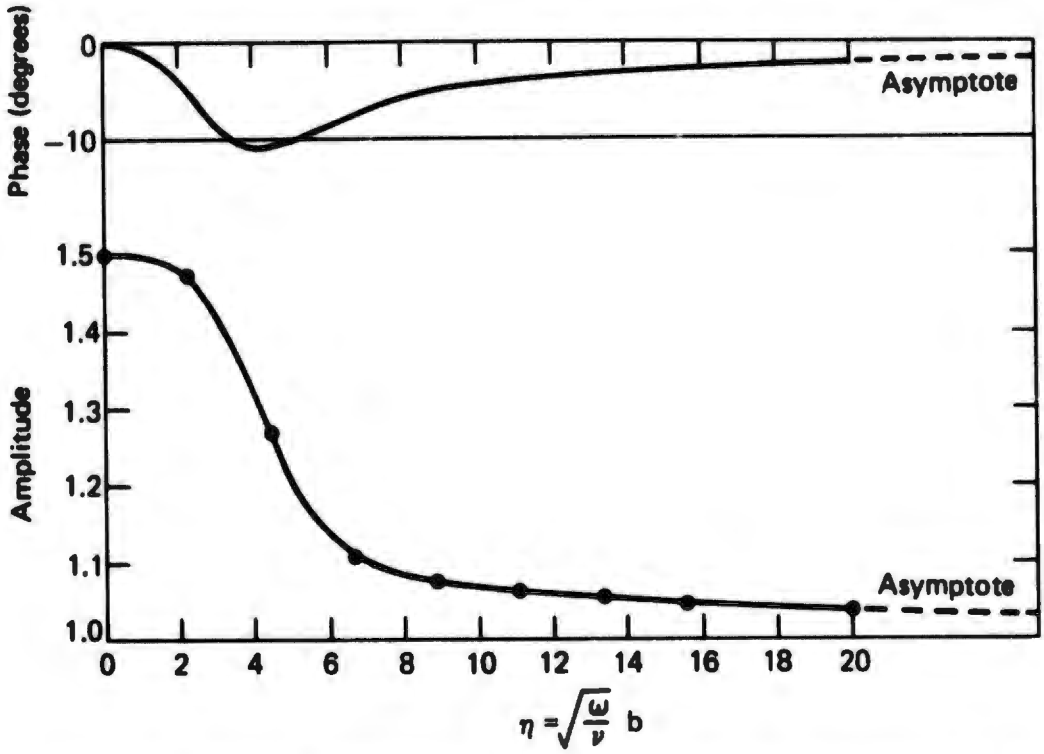

Fig. 3 Center-To-Mean Velocity Ratio for Plane Oscillatory Flow (after Ref. 6)

vorticity from the walls to the central part of the duct. We could renormalize by length a if it were smaller than b, and then ratio b/a would appear in similar equations valid for b/a > 1. Alternatively, one could rotate axes by 90° so that y traverses the narrow section and use the present formulas for a/b ≥ 1.

However, in Eq. (27) the normalizing velocity has cancelled out and it can be rewritten in a symmetric form valid for all values of a/b ($\omega \neq 0$) as

$$\frac{F(0,0)}{\langle F\rangle} = \frac{1 - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{1}{\cosh \xi_n} + \frac{1}{\cosh \mu_n}\right)}{1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\left(\frac{\tanh \xi_n}{\xi_n} + \frac{\tanh \mu_n}{\mu_n}\right)}, \qquad \text{(all a/b)} \tag{29}$$

where

$$\xi_n = \frac{a}{b}\gamma_n = \sqrt{\left(\frac{2n+1}{2}\right)^2 \pi^2 \left(\frac{a}{b}\right)^2 + \frac{i\omega a^2}{\nu}},$$

$$\mu_n = \sqrt{\left(\frac{2n+1}{2}\right)^2 \pi^2 \left(\frac{b}{a}\right)^2 + \frac{i\omega b^2}{\nu}}.$$

This form more clearly reflects the fact that there are <u>three</u> lengths in the oscillatory rectangular duct flow problem: a, b, and a "viscous diffusion length" $\left(\sqrt{\nu/\omega}\right)$.

4. DEVELOPING FLOWS IN A RECTANGULAR DUCT

When a viscous fluid enters a straight duct of any shape it takes some distance to reach a parallel fully developed flow. Theoretically the distance is infinite but, for practical purposes, the flow is regarded as fully developed if the centerline flow is within 1% of the analytic parallel value. It is often assumed the flow at the entrance has a flat uniform profile and the actual measured distribution may be very close to that. The downstream development of the flow is governed by the nonlinear Navier-Stokes equation but various approximation schemes have been used to reduce the analysis to a linear mathematical problem (e.g., Ref. 2).

Langhaar (Ref. 8) replaced the nonlinear convective terms of the steady momentum equation with a term proportional to w where it was assumed that the constant of proportionality (β^2) depends on downstream distance (z). This reduces the flow equation to

$$(\nabla^2 - \beta^2)\, w = \frac{1}{\mu}\frac{\partial p}{\partial z} = K' \quad \text{(a constant)} . \tag{29}$$

This is the equation considered by Han (Ref. 2) for steady developing flow in a rectangular duct. He gave a "one-sided" analytic solution of Eq. (29) and used it to calculate the entrance length and the excess pressure drop for ducts of six aspect ratios. The calculations were compared to other approximations for developing flow and (limited) experimental data. The results are certainly qualitatively correct.

Ref. 8. H. L. Langhaar, "Steady Flow in the Transition Length of a Straight Tube," J. Appl. Mech., Vol. 9 (Trans. ASME, Vol. 64), 1942, p. A-55.

It should be noted that except for the substitution of β^2 for $(i\eta^2)$ in Eq. (14), Han's equation for steady developing flow has already been discussed, in principle, in Section 3. This indicates that symmetric developing formulas could be written down immediately by the obvious substitutions

$$\gamma_n = \sqrt{p_n^2 + i\eta^2} \rightarrow \sqrt{p_n^2 + (\beta b)^2} \quad ,$$

$$\mu_n = \sqrt{q_n^2 + i\eta^2} \rightarrow \sqrt{q_n^2 + (\beta b)^2} \quad , \qquad \text{etc.}$$

Because of the difference in form one cannot readily make comparisons with Han's solution except in the limiting case $a/b \rightarrow \infty$. See Han's Eq. (19), which is equivalent to Eq. (25) here.

It is obvious that pulsatile flow development is also influenced by the shape of the duct. For rectangular ducts this could be studied approximately by enlarging the results of Section 3 on oscillatory flow to include the effects of a convective parameter $\beta(z)$. In effect, this would combine the approximate method of Han with the fully-developed oscillatory flow equations.

The full three-dimensional w-component momentum equation is

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w \quad ,$$

in the absence of any other forces. In developing flow analysis it is customary to neglect the first two convective terms $u\frac{\partial w}{\partial x}$ and $v\frac{\partial w}{\partial x}$ on the assumption that u and v are small compared to w. The pressure gradient is assumed to be, at most, a function of z. Now make the further hypothesis that $|w\frac{\partial w}{\partial z}|$ is proportional to w (i.e.,

$\beta^2 w$, where β is a function of z alone), following Langhaar and Han. Also neglect $\partial^2 w/\partial z^2$ with respect to $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$. The normalized linear w equation then takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 b^2 - i\eta^2\right) F = K$$

(a constant depending on z), where w/w_0 is the real part of $[F(x,y,z)\, e^{i\omega t}]$.

As β approaches 0 this becomes the equation for fully developed oscillatory flow (Section 3). As β approaches ∞ (equivalent to moving upstream) the flow becomes more plug-like, approaching a uniform entrance flow. The nonslip boundary conditions are the same, so after Eq. (22), the approximate developing oscillatory solution is (with w/w_0 as the real part $F(x,y,z)\, e^{i\omega t}$)

$$F(x,y,z) = F_0 \left[1 + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)} \cdot \left(\frac{\cosh \gamma_n X \cos p_n Y}{\cosh \gamma_n a/b} + \frac{\cos q_n X \cosh \mu_n Y}{\cosh \mu_n}\right)\right] ,$$

where

$$F_0 = \frac{-K(z)}{\beta^2 b^2 + i\eta^2} ,$$

$$\gamma_n = \sqrt{p_n^2 + i\eta^2 + \beta^2 b^2} , \quad \text{and}$$

$$\mu_n = \sqrt{q_n^2 + i\eta^2 + \beta^2 b^2} .$$

This solution varies with z through $\beta(z)$ and K(z), which are still unknown. Based on the approximate momentum equation, however, there is an integral expression relating the center velocity w(0, 0) and the average velocity $\langle w \rangle$ obtained by integrating across the section (after Han). Thus

$\beta(z)$ can be determined indirectly from a number of calculations for a range of β values. The pressure gradient at each β can be evaluated and, by integrating downstream, the total pressure drop over the entrance section can be evaluated. Of course, β will be a function of a/b and η and thus can only be determined numerically by considerable computation, equivalent at least to all the work of Han for each pair values of $(a/b, \eta)$. Even though it would be a landmark analytic calculation of a three-dimensional unsteady flow, the magnitude of the task is so great that a finite-difference calculation of the same mathematical problem would probably be faster. Moreover, the nonlinear term does not have to be approximated in the numerical scheme so a more accurate solution can be obtained within the limit of the truncation error.

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Appendix

SOME SERIES IDENTITIES

The steady rectangular duct flow solutions of Section 2 and the previous published solutions (Refs. 1 and 2) are all solutions of the same linear differential equation, they satisfy the same boundary conditions for a square duct, and they are valid for a/b = 1. Let us look at the corresponding velocity profiles in the vertical center plane, x = 0. With $p_n = (2n + 1)(\pi)/2$ and $|Y| < 1$

$$w(0, Y) = w_0 \left[1 - Y^2 + \sum_{n=0}^{\infty} \frac{32}{\pi^2} \frac{(-1)^{n+1}}{(2n+1)^3} \frac{\cos p_n Y}{\cosh p_n} \right]. \quad \text{(Ref. 1)}$$

$$w(0, Y) = w_0 \left[1 - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\cosh p_n Y}{\cosh p_n} \right]. \quad \text{(Ref. 2)}$$

$$w(0, Y) = \frac{w_0}{2} \left[2 - Y^2 - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left(\frac{\cos p_n Y + \cosh p_n Y}{\cosh p_n} \right) \right]. \quad \text{(Eq. (7))}$$

These must all be representations of the same function, for in the well-posed viscous flow problem the solution is unique. Thus for $|Y| < 1$,

$$Y^2 = \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\cosh p_n Y - \cos p_n Y}{\cosh p_n}. \quad \text{(A-1)}$$

Differentiating results in

$$Y = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\sinh p_n Y + \sin p_n Y}{\cosh p_n}. \quad \text{(A-2)}$$

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$$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{\cosh p_n Y + \cos p_n Y}{\cosh p_n} \quad . \quad * \qquad \text{(A-3)}$$

Integrating Eq. (A-1) results in

$$Y^3 = \frac{3.2^6}{\pi^4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} \left(\frac{\sinh p_n Y - \sin p_n Y}{\cosh p_n} \right) . \qquad \text{(A-4)}$$

Evaluating the series at $|Y| = 1$, we can extend the range to $|Y| \leq 1$ for all these expressions.

Some of these relations are useful in the oscillatory flow analysis as N approaches 0. Evaluating Eq. (A-3) at $Y = 0$ results in

$$\frac{\pi}{8} = \sum_{n=0} \frac{(-1)^n}{2n+1} \frac{1}{\cosh p_n} \quad . \qquad \text{(A-5)}$$

*This was verified analytically by E. P. Gray of APL via Laplace transforms.

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