

STATISTICAL INFERENCE FOR MULTIPLY  
TRUNCATED POWER SERIES DISTRIBUTIONS

BY

T. CACOULLOS

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THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS  
STANFORD UNIVERSITY  
STANFORD, CALIFORNIA



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# 1. INTRODUCTION AND SUMMARY

This paper is concerned with the problem of best estimation, i.e., minimum variance unbiased estimation (mvue), of certain functions of the parameter  $\theta$  and the truncation points of a power series distribution (PSD) truncated on the left at several known or unknown truncation points. More specifically, given  $k$  independent random samples  $X_{ij}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, k$ , the  $i$ th sample from the PSD with probability function

$$(1.1) \quad p_i(x) = P[X_{ij} = x] = \frac{a_i(x) \theta^x}{f_i(\theta, r_i)}, \quad t = r_i, r_i+1, \dots, N_i$$

where

$$(1.2) \quad f_i(\theta, r_i) = \sum_{t=r_i}^{N_i} a_i(x) \theta^x, \quad (N_i \text{ finite or infinite}),$$

it is desired to estimate some function of  $\theta$ , including  $p_i(x)$  itself; mvue are also provided for the truncation points  $r_i, \dots, r_k$ .

Estimability conditions for certain functions of the parameter  $\theta$  of a one-parameter PSD with probability function

$$(1.3) \quad P[X = x] = \frac{a(x)\theta^x}{f(\theta)}, \quad x \in T \quad (\theta > 0)$$

where  $T \subset I_0$ ,  $I_0$  the set of non-negative integers, were given by Patil [1963]. Thus it was shown that, on the basis of a random sample of size

$n \geq 1$  from (1.3),  $\theta$  is estimable, i.e., it has an unbiased estimator if, and only if,

$$(1.4) \quad n[T] = S_{\min(n[T])} ,$$

where

$$n[T] = T_1 + T_2 + \dots + T_n \text{ with } T_i = T, (i = 1, \dots, n)$$

and  $S_a$  denotes the tail set of  $a$ , that is,

$$S_a = \{x : x \in I_0 \text{ and } x \geq a\} .$$

In particular, (1.4) implies that  $\theta$  is not estimable whenever  $T$  is a finite set, e.g.,  $\theta = p/q$  in the binomial case. On the other hand,  $\theta$  is estimable if

$$T = S_r, \quad r \in I_0$$

where  $r$  denotes the truncation point, assumed known. This situation includes the Poisson, the logarithmic series and the negative binomial distributions truncated on the left at a known point. In fact, Roy and Mitra [1957] provide the mvue of  $\theta^\alpha$  for any integer  $\alpha > 0$  in the form

$$(1.5) \quad \hat{\theta}_\alpha = \frac{A(x-\alpha, n, r)}{A(x, n, r)}$$

where

$$(1.6) \quad A(x, n, r) = \sum a(x_1) \dots a(x_n)$$

the summation extending over all  $n$ -tuples  $(x_1, \dots, x_n)$  of integers  $x_i \geq r$  with  $x_1 + \dots + x_n = x$ .

The corresponding case when the truncation point  $r$  is unknown was studied by Joshi [1972], who showed that on the basis of a random sample  $x_1, \dots, x_n$  from (1.3) the mvue of  $r$  is

$$(1.7) \quad \hat{r}(y, x) = y - \frac{A(x, n, y+1)}{A(x, n, y) - A(x, n, y+1)}$$

where

$$x = x_1 + \dots + x_n \quad \text{and} \quad y = \min(x_1, \dots, x_n) .$$

Interestingly enough, though  $\theta$  is not estimable in the case of a finite-range distribution, it was shown that the probability function itself (hence also the corresponding distribution function) of a PSD truncated on the left is estimable, both when the truncation point  $r$  is known, Patil [1963], or unknown, Joshi [1972].

The main difficulty, reflected even in the treatment of some special cases of truncation away from zero ( $r=1$ ), Tate and Goen [1958], Cacoullos [1961], Patil and Bildikar [1966], Cacoullos and Charalambides [1972a], lies in deriving an explicit or computationally attractive expression for the distribution of the sufficient statistics, the sum of

See also [1972].

the observations; equivalently, to get around the evaluation of expressions such as  $A(x, n, r)$  in (1.6).

The above difficulty was resolved by employing exponential generating function (egf) techniques, which emerge quite naturally in the treatment of the aforementioned distribution problem, the  $n$ -fold convolution of a PSD truncated on the left at  $r$  ( $r \geq 1$ ). Indeed, Charalambides [1972], [1973a], [1973b] and Cacoullos and Charalambides [1972b] show how the egf technique provides a unified approach for a solution to the problem of mvue of certain functions of  $\theta$  in (1.3), both when  $r$  is known or unknown. At the same time, the treatment of truncated versions of classical families of PSD's leads to natural extensions of certain kinds of numbers; the Poisson family is associated with generalized Stirling numbers of the second kind, whereas the logarithmic series distribution with Stirling numbers of the first kind (the well known Stirling numbers correspond to the simple case of truncation away from zero, i.e.,  $r=1$ ); finally, the binomial and negative binomial cases led to the introduction, Cacoullos and Charalambides [1972a], and extension, Charalambides [1972], [1973a], of a new kind of numbers, called C-numbers.

The more general situation of truncation at several known truncation points, that is, (1.1) with  $a_i(x) = a(x)$ ,  $i = 1, \dots, k$ , motivated the introduction of certain multiparameter Stirling and C-numbers, Cacoullos [1973a] and [1973b]. The mvue of  $\theta^\alpha$  and the probability function itself were given in terms of these multiparameter numbers, Cacoullos [1973c].

Here we consider the relevant distribution (Section 2) and estimation problems (Section 4) in the more general case of (1.1); especially, when

the truncation points  $r_1, \dots, r_k$  are unknown,  $a_i(x) = a(x)$  and  $N_i = \infty$ ,  $i = 1, \dots, k$ . When  $\rho = (r_1, \dots, r_k)$  is known, the mvue of  $p_i(x)$  is also considered. In Section 3 we review the multiparameter Stirling numbers and, in particular, the C-numbers emerging out of the more general set up of different  $a_i(x)$  for the binomial and negative binomial distributions. Some of the results of this Section are based on properties of the multiparameter Stirling and C-numbers, studied by Cacoullos [1973b]. In Section 5, we exhibit several properties of conditional distributions for PSD's and indicate how they can be employed to provide confidence intervals for certain functions, in particular products and quotients of the parameters. Applications, including confidence interval estimation of attribute-failure reliability models for multi-component systems, are exhibited in Section 6.

## 2. CONVOLUTIONS OF LEFT-TRUNCATED PSD'S

Let  $X_{ij}$ ,  $i=1, \dots, k$   $j=1, \dots, n_i$  be independently distributed with p.f.'s

$$(2.1) \quad p_i(x) = P[X_{ij} = x] = \frac{a_i(x)}{f_i(\theta, r_i)} \theta^x, \quad \begin{array}{l} x = r_i, r_i+1, \dots, N_i \\ j = 1, \dots, n_i \end{array}$$

where  $a_i(x)$  is independent of the parameter  $\theta$  and the series function

$$f_i(\theta, r_i) = \sum_{x=r_i}^{N_i} a_i(x) \theta^x$$

converges in some interval  $0 < \theta < \rho_i$ ;  $N_i$  may be finite or infinite ( $i = 1, \dots, k$ ). When the truncation points  $r_i$  are known, the k-sample sum

$$X = \sum_{i=1}^k \sum_{j=1}^{n_i} X_{ij}$$

is a complete sufficient statistic for  $\theta$ , since a PSD belongs to the exponential family of distributions. It is easily shown that (cf. Cacoullos and Charalambides, [1972])

Proposition 2.1. The p.f. of  $X$  is given by

$$(2.2) \quad P[X=x] = \frac{c(x; \underline{v}, \underline{\rho})}{g_{\underline{v}}(\theta, \underline{\rho})} \frac{\theta^x}{x!} \quad x = m, m+1, \dots, M,$$

where we set  $\underline{v} = (n_1, \dots, n_k)$ ,  $\underline{\rho} = (r_1, \dots, r_k)$ ,

$$(2.3) \quad m = n_1 r_1 + \dots + n_k r_k \equiv (\underline{v}, \underline{\rho}), \quad M = n_1 N_1 + \dots + n_k N_k,$$

and  $g_{\underline{v}}(\theta, \underline{\rho})$  is the egf of the numbers  $c(x; \underline{v}, \underline{\rho})$  given by

$$(2.4) \quad g_{\underline{v}}(t, \underline{\rho}) = \sum_{x=m}^M c(x; \underline{\rho}, \underline{v}) \frac{t^x}{x!} = \prod_{i=1}^k \frac{1}{n_i!} [f_i(t, r_i)]^{n_i};$$

moreover, the numbers  $c(x; \underline{v}, \underline{\rho})$  have the representation

$$(2.5) \quad c(x; \underline{v}, \underline{\rho}) = \frac{x!}{n_1! \dots n_k!} \sum_x \prod_{i=1}^k \prod_{j=1}^{n_i} a_i(x_{ij})$$

the summation extending over all n-tuples  $(n = n_1 + \dots + n_k)$  of integers  $x_{ij}$  such that

$$x_{ij} \geq r_i \quad \sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij} = x \quad .$$

Suppose now that, in addition to  $\theta$ , the truncation vector  $\rho$  is unknown. Then by the results of Fraser [1952], a complete sufficient statistic for  $(\theta, \rho)$  is  $(X, Z) = (X, Z_1, \dots, Z_k)$  where

$$Z_i = \min_{1 \leq j \leq n_i} \{X_{ij}\} \quad i = 1, \dots, k \quad .$$

For our purposes, we require the following

Definition 2.1. Let  $G(\underline{u})$  be a real-valued function with domain the set of lattice points  $\underline{u} = (u_1, \dots, u_k)$  with integer coordinates. The kth-order partial difference of  $G(\underline{u})$ , to be denoted by  $\Delta_{\underline{u}}$ , is defined by

$$\Delta_{\underline{u}} G(\underline{u}) = (\Delta_{u_k} \Delta_{u_{k-1}} \dots \Delta_{u_1}) G(\underline{u})$$

where  $\Delta_{u_i}$  denotes the usual forward difference operating on  $u_i$ , that is,

$$(2.6) \quad \Delta_{u_i} G(\underline{u}) = G(\underline{u} + \underline{e}_i) - G(\underline{u}) \quad ,$$

$\underline{e}_i$  denoting the  $i$ th coordinate axis unit vector  $(0, \dots, 0, 1, 0, \dots, 0)$ .

The following lemma, of interest in itself, will be used in deriving the p.f. of the sufficient statistic  $(X, \underline{Z})$ .

Lemma 2.1. Let  $Q(\underline{u})$  denote the tail probability function of an integer-valued random vector  $\underline{U} = (U_1, \dots, U_k)$ , that is,

$$Q(\underline{u}) = P[U_1 \geq u_1, \dots, U_k \geq u_k] \equiv P[\underline{U} \geq \underline{u}] .$$

Then

$$(2.7) \quad P[\underline{U} = \underline{u}] = (-1)^k \Delta_{\underline{u}} Q(\underline{u}) .$$

Proof: For  $k = 2$ , we have

$$P[U_1 = u_1, U_2 = u_2] = Q(u_1, u_2) - Q(u_1 + 1, u_2) - Q(u_1, u_2 + 1) + Q(u_1 + 1, u_2 + 1) ,$$

which, in view of (2.6), gives (2.7). For  $k > 2$  the proof is analogous (cf. the  $k$ th order partial difference  $\Delta_{\underline{h}} F(\underline{u})$  of a  $k$ -variate continuous distribution function  $F(\underline{u})$  which is used in the definition of the density at  $\underline{u}$  by letting  $\underline{h} = (h_1, \dots, h_k) \rightarrow 0$ ).

We can now state

Theorem 2.1. The distribution of the sufficient statistic  $(X, \underline{Z})$  is a PSD

$$(2.8) \quad P[X=x, \underline{Z}=\underline{z}] = \frac{(-1)^k \Delta_{\underline{z}} c(x; \underline{z}, \underline{v})}{g_{\underline{v}}(\theta, \underline{\rho})} \frac{\theta^x}{x!}, \quad (\underline{v}, \underline{y}) \leq x \leq M, \quad \underline{z} \geq \underline{\rho}$$

where  $g_{\underline{v}}(\theta, \underline{\rho})$  and  $c(x; \underline{z}, \underline{\rho})$  are defined by (2.3) and (2.4).

Proof: Setting

$$(2.9) \quad Q(\underline{z}; x) = P[\underline{Z} \geq \underline{z}, X=x] ,$$

we obtain by Lemma 2.1

$$(2.10) \quad P[X=x, \underline{Z}=\underline{z}] = (-1)^k \Delta_{\underline{z}} Q(\underline{z}; x) .$$

However, we have

$$\begin{aligned} Q(\underline{z}; x) &= P[\underline{Z} \geq \underline{z}, X=x] = P[X_{ij} \geq z_i, X=x, i=1, \dots, k, j=1, \dots, n_i] \\ &= \frac{\theta^x}{\prod_{i=1}^k [f_i(\theta, r_i)]^{n_i}} \sum_{\underline{z}} \prod_{i=1}^k \prod_{j=1}^{n_i} a_i(x_{ij}) \end{aligned}$$

where the summation extends over all  $n$ -tuples of integers  $x_{ij}$  with

$$x_{ij} \geq z_i \quad \text{and} \quad \sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij} = x .$$

Thus, by (2.3) and (2.4),  $Q(\underline{z}; x)$  can be written as

$$(2.11) \quad Q(\underline{z}; x) = \frac{c(x, \underline{z}, \underline{v})}{g_{\underline{v}}(\theta, \underline{\rho})} \frac{\theta^x}{x!} .$$

Hence, as  $\Delta_{\underline{z}}$  operates only on  $\underline{z}$ , we have

$$\Delta_{\underline{z}} Q(\underline{z}; x) = \frac{\Delta_{\underline{z}} c(x, \underline{z}, \underline{v})}{g_{\underline{v}}(\theta, \underline{\rho})} \frac{\theta^x}{x!} ,$$

and (2.8) follows from (2.10).

### 3. MULTIPARAMETER STIRLING AND C-NUMBERS

It was already mentioned in Section 1 that the present approach to the distribution theory relating to mvue problems of a left-truncated PSD was motivated by the results of the simple case of a Poisson distribution truncated away from zero, which leads in a natural way to the well-known Stirling numbers of the second kind  $S(x,n)$ , with egf

$$g(t) = \sum_{x=n}^{\infty} S(x,n) \frac{t^x}{x!} = \frac{1}{n!} (e^t - 1)^n .$$

In the present notation, this corresponds to the Poisson series function

$$f_1(\lambda, 1) \equiv f(\lambda, 1) = e^\lambda - 1 \quad (k=1) .$$

The general case of truncation of a PSD distribution on the left at several points  $r_1, \dots, r_k$  motivated (Cacoullos, 1973) the definition of multiparameter numbers; their egf is always of the general form (2.4).

Of special interest are the following cases:

a) Poisson: Here  $\theta$  is the Poisson parameter  $\lambda$  and the PSD series functions are

$$f_i(\lambda, r_i) = f(\lambda, r_i) = e^\lambda - \sum_{j=0}^{r_i-1} \frac{\lambda^j}{j!} \quad i = 1, \dots, k .$$

The corresponding egf defines the multiparameter Stirling numbers of the second kind  $S_k(x; p, v)$ ; for  $k = 1$  they reduce to the generalized Stirling numbers of the second kind  $S(x, r_i, n_i)$  considered by Charalambides (1972); for  $k = 1$  and  $r_i = 2$  they give the associated Stirling numbers of the second kind (see Riordan [1958], p. 77).

Recurrence relations can be obtained by using the difference-differential equation satisfied by the egf  $G_{\underline{\nu}}(t, \underline{\rho})$ , say, of the numbers  $S(x; \underline{\rho}, \underline{\nu})$ :

$$\frac{d}{du} G_{\underline{\nu}}(t, \underline{\rho}) = n G_{\underline{\nu}}(t, \underline{\rho}) + \sum_{i=1}^k \frac{t^{r_i-1}}{(r_i-1)!} G_{\underline{\nu}-\underline{e}_i}(t, \underline{\rho})$$

where, as before,  $n = n_1 + \dots + n_k$ . In the sequel, we will require the  $(x, \underline{\nu})$ -wise recurrence:

$$(3.1) \quad S(x+1; \underline{\rho}, \underline{\nu}) = n S(x; \underline{\rho}, \underline{\nu}) + \sum_{i=1}^k \binom{x}{r_i-1} S(x-r_i+1; \underline{\rho}, \underline{\nu}-\underline{e}_i).$$

b) Logarithmic series: Here we have

$$f(\theta, r_i) = -\log(1-\theta) - \sum_{j=1}^{r_i-1} \frac{\theta^j}{j} \quad i = 1, \dots, k \quad (r_i \geq 1)$$

where  $0 < \theta < 1$ . The corresponding egf,  $h_{\underline{\nu}}(t, \underline{\rho})$ , say, defines the multiparameter signless Stirling numbers of the first kind,  $|s(x; \underline{\rho}, \underline{\nu})|$ ; the numbers  $s(x; \underline{\rho}, \underline{\nu})$ , the multiparameter analogues of the usual Stirling numbers of the first kind  $s(x; n)$ , are generated by the basic series function  $\log(1+\theta)$ , instead of  $-\log(1-\theta)$ , and their egf is

$$h_{\underline{\nu}}^*(t, \underline{\rho}) = \sum_{x=m}^{\infty} s(x; \underline{\rho}, \underline{\nu}) \frac{t^x}{x!} = \prod_{i=1}^{n_i} \frac{1}{n_i!} \left[ \log(1+t) - \sum_{j=1}^{r_i-1} (-1)^{j-1} \frac{t^j}{j} \right]^{n_i}.$$

The corresponding expansions (representations) of  $S(x; \underline{\rho}, \underline{\nu})$  and  $|s(x; \underline{\rho}, \underline{\nu})|$  (cf. (2.4)) show that

$$s(x; \underline{\rho}, \underline{\nu}) = (-1)^{n-x} |s(x; \underline{\rho}, \underline{\nu})|.$$

For  $k = 1$ ,  $r_1 = 1$ ,  $s(x, 1, n_1)$  and  $|s(x, 1, n_1)|$  reduce to the usual Stirling and signless (positive) Stirling numbers of the first kind, respectively.

Recurrence relations for  $s(x; \rho, \underline{v})$  and  $|s(x; \rho, \underline{v})|$  can be obtained from the corresponding difference-differential equations satisfied by their egf's  $h_{\underline{v}}^*(t, \rho)$  and  $h_{\underline{v}}(t, \rho)$ , respectively:

$$(1+t) \frac{d}{dt} h_{\underline{v}}^*(t, \rho) = \sum_{i=1}^k (-1)^{r_i-1} t^{r_i-1} h_{\underline{v}-\underline{e}_i}^*(t, \rho)$$

$$(1-t) \frac{d}{dt} h_{\underline{v}}(t, \rho) = \sum_{i=1}^k t^{r_i-1} h_{\underline{v}-\underline{e}_i}(t, \rho)$$

Thus we obtain the recurrences:

$$s(x+1; \rho, \underline{v}) + x s(x; \rho, \underline{v}) = \sum_{i=1}^k (-1)^{r_i-1} (x)_{r_i-1} s(x-r_i+1; \rho, \underline{v}-\underline{e}_i), \quad (3.2)$$

$$|s(x+1; \rho, \underline{v})| = x |s(x; \rho, \underline{v})| + \sum_{i=1}^k (x)_{r_i-1} |s(x-r_i+1; \rho, \underline{v}-\underline{e}_i)|.$$

c) Binomial: Now we have

$$f_i(\theta, r_i) = (1+\theta)^{s_i} - \sum_{j=0}^{r_i-1} \binom{s_i}{j} \theta^j \quad i = 1, \dots, k \quad \theta = p/q$$

(the  $s_i$  denoting positive integers). The corresponding egf  $H_{\underline{v}}(t, \rho)$  defines the multiparameter C-numbers  $C(x; \rho, \underline{v}, \underline{\sigma})$  with  $\underline{\sigma} = (s_1, \dots, s_k)$ . The special simple case of  $k = 1$  with  $s_1 = s$ ,  $n_1 = n$ ,  $r_1 = 1$ , motivated the definition of the C-numbers  $C_{s,n}^x$  (Cacoullos and

Charalambides [1972a]); in the present notation, we have

$$(3.2a) \quad C(x; l, n, s) = s^n C_{s, n}^x = \frac{1}{n!} \sum_{j=1}^n (-1)^{n-j} \binom{n}{j} (sj)_x.$$

This explicit expression was obtained by elementary occupancy-model arguments in obtaining the  $n$ -fold convolution of a binomial distribution (with parameters  $s$  and  $p = \theta/(1+\theta)$ ) truncated away from zero (also referred to as positive binomial).

Recurrence relations for the  $C$ -numbers can be obtained from the difference-differential equation (of Cacoullos, 1973)

$$(3.3) \quad (1+t) \frac{d}{dt} H_{\underline{v}}(t, \underline{\rho}, \underline{\sigma}) = (\underline{v}, \underline{\sigma}) H_{\underline{v}}(t, \underline{\rho}, \underline{\sigma}) + \sum_{i=1}^k \frac{(s_i)_{r_i}}{(r_i-1)!} t^{r_i-1} H_{\underline{v}-\underline{e}_i}(t, \underline{\rho}, \underline{\sigma}).$$

Thus an  $(x, \underline{v})$ -wise recurrence is

$$(3.4) \quad C(x+1; \underline{\rho}, \underline{v}, \underline{\sigma}) = [(\underline{\sigma}, \underline{v}) - x] C(x; \underline{\rho}, \underline{v}, \underline{\sigma}) + \sum_{i=1}^k \binom{x}{r_i-1} (s_i)_{r_i} C(x-r_i+1; \underline{\rho}, \underline{v}-\underline{e}_i, \underline{\sigma}).$$

It should be noted that

$$C(x; \underline{\rho}, \underline{v}, \underline{\sigma}) = 0 \quad \text{if} \quad x < m \equiv (\underline{\rho}, \underline{v})$$

and the  $C$ -numbers are positive integers for  $m \leq x \leq (\underline{v}, \underline{\sigma})$ .

d) Negative Binomial: Here we have

$$f_i(\theta, r_i) = (1-\theta)^{-s_i} - \sum_{j=0}^{r_i-1} \binom{-s_i}{j} (-1)^j \theta^j \quad i = 1, \dots, k$$

( $s_i > 0$  and  $\theta \equiv p$ ; for the Pascal distribution the  $s_i$  are integers).

The corresponding egf defines the signless C-numbers

$|C(x; \rho, \nu, -\sigma)| = (-1)^x C(x; \rho, \nu, -\sigma)$ ; the numbers  $C(x; \rho, \nu, -\sigma)$  are defined by the egf  $H_\nu(u; \rho, -\sigma)$ , that is, as in c) with  $\sigma$  replaced by  $-\sigma$ , so that  $x$  has an infinite range of values

$$x \geq (\rho, \nu).$$

It can be shown, e.g. by using the analogues of (3.3) and (3.4), that

$$(3.5) \quad |C(x+1; \rho, \nu, -\sigma)| = [(\sigma, \nu) + x] |C(x; \rho, \nu - e_1, -\sigma)| \\ + \sum_{i=1}^k \binom{x}{r_i-1} (s_i + r_i - 1)_{r_i} |C(x - r_i + 1; \rho, \nu - e_i, -\sigma)|.$$

It should be observed that in the special case  $k = 1$ ,  $r_1 = 1$ ,  $n_1 = n$ ,  $s_1 = s$ ,

$$C(x; 1, n, -s) = \frac{(-1)^x}{n!} \sum_{j=1}^n (-1)^{n-j} \binom{n}{j} (sj+x-1)_x,$$

since in (3.2a)  $(sj)_x$  must be replaced by  $(-sj)_x = (-1)^x (sj+x-1)_x$ .

For additional recurrence relations and combinatorial interpretations of the multiparameter Stirling and C-numbers, we refer to Cacoullos (1973b).

#### 4. MINIMUM VARIANCE UNBIASED ESTIMATION

For the construction of the mvue of a parametric function  $\phi(\theta, \underline{\rho})$  it suffices to find an unbiased estimator  $\hat{\phi}(x, \underline{z})$  which is a function of the sufficient statistic  $(X, \underline{Z})$ , whose distribution we considered in Section 2. Extending the results of Patil [1963] and Joshi [1972] concerning the case  $k = 1$ , or, by a close examination of the condition of unbiasedness, as obtained by using the p.f. (2.8),

$$(4.1) \quad \sum_{x \geq m} \sum_{\underline{z} \geq \underline{\rho}} (-1)^k \phi(x, \underline{z}) \Delta_{\underline{z}} c(x; \underline{z}, \underline{v}) \frac{\theta^x}{x!} = g_{\underline{v}}(\theta, \underline{\rho}) \phi(\theta, \underline{\rho})$$

we get the following

Theorem 4.1. On the basis of the  $k$  independent samples  $X_{ij}$ ,  $i=1, \dots, k$ ,  $j=1, \dots, n_i$  from (2.1), a parametric function  $(\theta, \underline{\rho})$  has a mvue if, and only if, for every  $\underline{\rho} = (r_1, \dots, r_k)$  and every  $\theta$ , (a) the function  $\phi(\theta, \underline{\rho}) g_{\underline{v}}(\theta, \underline{\rho})$  admits a power series expansion in  $\theta$ , say

$$(4.2) \quad \phi(\theta, \underline{\rho}) g_{\underline{v}}(\theta, \underline{\rho}) = \sum_{x \in T} c^*(x; \underline{\rho}, \underline{v}) \frac{\theta^x}{x!},$$

and (b) for each  $\underline{\rho}$  the index set  $T$  of  $\phi(\theta, \underline{\rho}) g_{\underline{v}}(\theta, \underline{\rho})$  is a subset of the index set of the series function  $g_{\underline{v}}(\theta, \underline{\rho})$ , i.e. (see (2.2) and (2.3)),

$$(4.3) \quad T \subset \{m, m+1, \dots, M\}.$$

If  $\hat{\phi}(x, \underline{z})$  exists, then it is given by

$$(4.4) \quad \hat{\phi}(x, \underline{z}) = \frac{\Delta_{\underline{z}} c^*(x; \underline{z}, \underline{v})}{\Delta_{\underline{z}} c(x; \underline{z}, \underline{v})}.$$

Proof: We have from (4.1), (4.2) and (2.4) for each  $\underline{\rho}$

$$\sum_{\underline{x} > \underline{m}} \sum_{\underline{z} \geq \underline{\rho}} (-1)^k \phi(\underline{x}, \underline{z}) \Delta_{\underline{z}} C(\underline{x}; \underline{z}, \underline{v}) \theta^{\underline{x}} = \sum_{\underline{x} \in T} C^*(\underline{x}; \underline{\rho}, \underline{v}) \theta^{\underline{x}},$$

which will hold for all  $\theta$  if, and only if,

$$(4.5) \quad \sum_{\underline{z} \geq \underline{\rho}} (-1)^k \phi(\underline{x}, \underline{z}) \Delta_{\underline{z}} C(\underline{x}; \underline{z}, \underline{v}) = C^*(\underline{x}; \underline{\rho}, \underline{v}).$$

Applying the same argument involved in (2.9) and (2.10) to the function

$$Q^*(\underline{\rho}) = \sum_{\underline{z} \geq \underline{\rho}} (-1)^k \phi(\underline{x}, \underline{z}) \Delta_{\underline{z}} C(\underline{x}; \underline{z}, \underline{v})$$

we get from (4.5)

$$\Delta_{\underline{\rho}} Q^*(\underline{\rho}) = \phi(\underline{x}, \underline{\rho}) \Delta_{\underline{\rho}} C(\underline{x}; \underline{\rho}, \underline{v}) = \Delta_{\underline{\rho}} C^*(\underline{x}; \underline{\rho}, \underline{v}).$$

Since this holds for every  $\underline{\rho}$ , (4.4) readily follows.

From the preceding theorem, we have

Corollary 4.1. The truncation point  $\underline{\rho} = (r_1, \dots, r_k)$ , as well as

$\underline{p}^{(\alpha)} = (r_1^\alpha, \dots, r_k^\alpha)$ ,  $\alpha > 0$  integer, is always mvu estimable; the mvue of  $\underline{p}^{(\alpha)}$  is

$$(4.6) \quad \hat{\underline{\rho}}_{\alpha}^{\wedge}(\underline{x}, \underline{z}) = \frac{\Delta_{\underline{z}} [z_i^\alpha c(\underline{x}; \underline{z}, \underline{v})]}{\Delta_{\underline{z}} c(\underline{x}; \underline{z}, \underline{v})}$$

$$= z_i^\alpha + \frac{\Delta_{(\underline{i})} c(\underline{x}; \underline{z} + \underline{e}_{\underline{i}}, \underline{v})}{\Delta_{\underline{z}} c(\underline{x}; \underline{z}, \underline{v})} \sum_{j=0}^{\alpha-1} \binom{\alpha}{j} z_i^j, \quad i = 1, \dots, k$$

where  $\Delta_{(i)}$  denotes the  $(k-1)$ -order partial difference operator  $\prod_{j \neq i} \Delta z_j$ .

It should be noted that the second expression in (4.4) gives the "correction" term for the corresponding maximum likelihood estimate  $z_i^\alpha$  of  $r_i^\alpha$  to yield an unbiased estimate of  $r_i^\alpha$ .

Corollary 4.2 The probability functions  $p_i(r_i+j)$  in (1.1) are mvu estimable and the mvue of  $p_i(r_i+j)$  is given by

$$(4.7) \quad \hat{p}_i(r_i+j; x, \tilde{z}) = a_i(z_i+j) \frac{\Delta_{\tilde{z}}[(x)_{z_i+j} c(x-z_i-j, \tilde{z}, \tilde{v}-e_i)]}{n_i \Delta_{\tilde{z}} c(x; \tilde{z}, \tilde{v})} \quad \begin{matrix} j \geq 0 \\ i=1, \dots, k \end{matrix};$$

hence the corresponding distribution functions are also mvu estimable.

As in the case of known truncation points, it is observed that condition (4.4) is not satisfied if all the  $N_i$  in (1.1) are finite and the parameter to be estimated is  $\theta$ . However, we have the following

Corollary 4.3. For any integer  $\alpha > 0$ ,  $\theta^\alpha$  is mvu estimable if at least one of the  $N_i$  in (1.1) is infinite; that is, at least one of the distributions  $p_i(x)$  has range equal to the tail set of  $r_i$ . Then the mvue of  $\theta^\alpha$  is given by

$$(4.8) \quad \hat{\theta}_\alpha(x, \tilde{z}) = (x)_\alpha \frac{\Delta_{\tilde{z}} c(x-\alpha; \tilde{z}, \tilde{v})}{\Delta_{\tilde{z}} c(x; \tilde{z}, \tilde{v})} \quad \text{for } \tilde{y} \geq \rho, \quad x \geq m.$$

In particular, the mvue of  $\theta$  is

$$(4.9) \quad \hat{\theta}(x, \tilde{z}) = x \frac{\Delta_{\tilde{z}} c(x-\alpha; \tilde{z}, \tilde{v})}{\Delta_{\tilde{z}} c(x; \tilde{z}, \tilde{v})}.$$

For comparison purposes, we give the corresponding estimates of  $\theta^\alpha$  and  $p_i(x)$  obtained by Cacoullos [1973c] when the truncation points  $r_1, \dots, r_k$  are known. They are

$$(4.10) \quad \hat{\theta}_\alpha(x) = (x)_\alpha \frac{c(x-\alpha; \underline{\rho}, \underline{\nu})}{c(x; \underline{\rho}, \underline{\nu})} \quad x \geq m + \alpha$$

$$(4.11) \quad \hat{p}_i(r_i + j; x) = a_i(r_i + j) \frac{c(x - r_i - j; \underline{\rho}, \underline{\nu} - e_i)}{c(x; \underline{\rho}, \underline{\nu})} \frac{(x)_{r_i + j}}{n_i} \quad \begin{array}{l} x \geq m + j \\ j \geq 0 \end{array}$$

for  $i = 1, \dots, k$ .

It is interesting to observe that the role of  $c(x; \underline{\rho}, \underline{\nu})$  when  $\underline{\rho}$  is known is played by  $(-1)^k \Delta_{\underline{z}} c(x; \underline{z}, \underline{\nu})$  when  $\underline{\rho}$  is estimated. The mvue of the estimable parametric functions in specific situations are obtained from the preceding general expressions by replacing the numbers  $c(x; \underline{z}, \underline{\nu})$  by their analogues in each case: in the Poisson case,  $c(x; \underline{z}, \underline{\nu}) = S(x; \underline{z}, \underline{\nu})$  (the multiparameter Stirling number of the second kind); in the logarithmic series distribution,  $c(x; \underline{z}, \underline{\nu}) = |S(x; \underline{z}, \underline{\nu})|$  (the multiparameter signless Stirling numbers of the first kind); in the binomial case,  $c(x; \underline{z}, \underline{\nu}) = C(x; \underline{z}, \underline{\nu}, \underline{\sigma})$  (the multiparameter C-numbers); in the negative binomial case,  $c(x; \underline{z}, \underline{\nu}) = |C(x; \underline{z}, \underline{\nu}, -\underline{\sigma})|$  (the multiparameter signless C-numbers).

In certain cases, the mvue obtain a more illuminating form by using some of the recurrence relations given in Section 3. Thus the mvue of the Poisson parameter  $\lambda$  can be written, by using the recurrence (3.1), as follows:

$$\hat{\lambda}(x) = \frac{x}{n} \left[ 1 - \sum_{i=1}^k \binom{x-1}{r_i-1} \frac{S(x-r_i; \rho, \underline{v}-\underline{e}_i)}{S(x; \rho, \underline{v})} \right] = c_1 \hat{\lambda}_0(x)$$

when  $\rho$  is known, and

$$\hat{\lambda}(x, \underline{z}) = \frac{x}{n} \left[ 1 - \sum_{i=1}^k \binom{x-1}{r_i-1} \frac{\beta(x-r_i; \underline{z}, \underline{v}-\underline{e}_i)}{\beta(x; \underline{z}, \underline{v})} \right] = c_2 \hat{\lambda}_0(x),$$

when  $\rho$  is estimated, where we set  $\beta(x; \underline{z}, \underline{v}) = (-1)^k \Delta_{\underline{z}} S(x; \underline{z}, \underline{v})$  and  $\hat{\lambda}_0(x) = x/n = \bar{x}$  denotes the usual mvue of  $\lambda$  under no truncation, i.e.,  $\rho = (0, 0, \dots, 0)$ ;  $c_1$  and  $c_2$  may be regarded as the correction factors by which  $\hat{\lambda}_0(x)$  must be multiplied to give the corresponding estimate of  $\lambda$  under truncation on the left at a known or unknown point  $\rho$ , respectively. It should be noted that  $0 \leq c_1 \leq 1$ ,  $0 \leq c_2 \leq 1$ . Similarly, in the negative binomial case the mvue of  $\theta$  (the usual  $p$ ) can be written, by using the recurrence (3.5), as follows:

$$\hat{\theta}(x) = \frac{x}{(\underline{\sigma}, \underline{v}) + x - 1} \left[ 1 - \sum_{i=1}^k \binom{x-1}{r_i-1} (S_i + r_i - 1)_{r_i} \frac{|C(x-r_i; \rho, \underline{v}-\underline{e}_i, -\underline{\sigma})|}{|C(x; \rho, \underline{v}, -\underline{\sigma})|} \right]$$

when  $\rho$  is known, and a similar expression with  $|C(x; \rho, \underline{v}, -\underline{\sigma})|$  replaced by  $\Delta_{\underline{z}} |C(x; \underline{z}, \underline{v}, -\underline{\sigma})|$  when  $\rho$  is estimated. In both cases,  $\hat{\theta}$  takes the form

$$\hat{\theta} = c \hat{\theta}_0 = c \frac{x}{(\underline{\sigma}, \underline{v}) + x - 1}$$

where  $\hat{\theta}_0$  is the usual mvue of  $\theta$  in the absence of truncation and  $c$ ,  $0 \leq c \leq 1$ , the corresponding correction factor.

## 5. SOME CONDITIONAL-DISTRIBUTION PROPERTIES OF A PSD

In the present section, we state certain properties of conditional distributions of PSD's; some of these properties have been used for confidence interval estimation of the reliability of multicomponent attribute-failure models in certain special cases such as the negative binomial (Hwang and Buehler [1973]) and the Poisson distributions (Harris [1971]); apparently, however, the general underlying property for PSD's has not been explicitly stated. The following property is basically a consequence of the exponential structure of a PSD.

Proposition 5.1. Let  $X_1, \dots, X_s$  and  $Y_1, \dots, Y_k$  be independently distributed each according to a PSD and let

$$(5.1) \quad \begin{aligned} P[X_i=x] &= \frac{a_i(x)}{A_i(\lambda_i)} \lambda_i^x & x \in S_i & \quad i = 1, \dots, s, \\ P[Y_j=y] &= \frac{b_j(y)}{B_j(\mu_j)} \mu_j^y & y \in T_j & \quad j = 1, \dots, k, \end{aligned}$$

where the sets  $S_i$  and  $T_j$  are subsets of the set  $I_0$  of non-negative integers and each of the series functions

$$A_i(\lambda_i) = \sum_{x \in S_i} a_i(x) \lambda_i^x, \quad B_j(\mu_j) = \sum_{y \in T_j} b_j(y) \mu_j^y$$

converges in some interval:  $0 < \lambda_i < L_i$ ,  $0 < \mu_j < M_j$  ( $i=1, \dots, s$ ,  $j=1, \dots, k$ ).

Define

$$(5.2) \quad \begin{aligned} U_i &= X_i - X_1 & i &= 2, \dots, s, & \underline{U} &= (U_2, \dots, U_s) \\ V_j &= Y_j + X_1 & j &= 1, \dots, k, & \underline{V} &= (V_1, \dots, V_k). \end{aligned}$$

Then the conditional distribution of  $X_1$  given  $\underline{U} = (u_2, \dots, u_s) = \underline{u}$  and  $\underline{V} = (v_1, \dots, v_k) = \underline{v}$  is a one-parameter PSD with parameter

$$\theta = \prod_{i=1}^s \lambda_i / \prod_{j=1}^k \mu_j$$

and probability function

$$(5.3) \quad p_{\theta}(x; \underline{u}, \underline{v}) = \frac{d(x; \underline{u}, \underline{v})}{D(\theta; \underline{u}, \underline{v})} \theta^x$$

with series function

$$D(\theta; \underline{u}, \underline{v}) = \sum_{x \in T} d(x; \underline{u}, \underline{v}) \theta^x$$

$$d(x; \underline{u}, \underline{v}) = a_1(x) \prod_{i=2}^s a_i(u_i + x) \prod_{j=1}^k b_j(v_j - x)$$

where the range  $T$  of  $x$  is determined by the ranges  $S_i$ ,  $T_j$  and the values of the conditioning variables  $\underline{u}$  and  $\underline{v}$ .

The proof is straightforward and is therefore omitted. It is observed that the conditional distribution (5.3) depends only on the parameter  $\theta$ ,

not on the individual  $\lambda_i$ 's and  $\mu_j$ 's. The disappearance of all nuisance parameters is guaranteed by the general result of Lehmann and Scheffé (see Lehmann, 1959, Ch. 2, Lemma 8) concerning exponential families. It follows, furthermore, that the theory of uniformly most powerful unbiased tests and corresponding uniformly most accurate unbiased confidence intervals (Lehmann, 1959, Ch. 4) can be applied for inferences about  $\theta$ .

Another conditional distribution arises in relation to making inferences about  $\theta$  in the presence of unknown truncation points playing the role of nuisance parameters. Since in applications, especially in reliability of attribute-failure models (Hwang and Buehler [1973], Harris [1971]),

$$\lambda = \lambda_1, \dots, \lambda_s,$$

we are going to consider conditional distributions given only  $\underline{U}$ , i.e. with conditioning variables differences of  $X_i$ 's; in the preceding Proposition 5.1, this would lead to a PSD with p.f. (of  $X_1$  given  $\underline{U}=\underline{u}$ )

$$(5.4) \quad p_{\lambda}(x; \underline{u}) = \frac{d(x; \underline{u})}{D(\lambda, \underline{u})} \lambda^x$$

where

$$d(x; \underline{u}) = a_1(x) \prod_{i=2}^s a_i(u_i + x), \quad D(\lambda; \underline{u}) = \sum_x d(x; \underline{u}) \lambda^x.$$

Proposition 5.2. Let  $X_{ij}$   $i=1, \dots, s$ ,  $j=1, \dots, n_i$  be independently distributed with p.f.'s

$$P[X_{ij}=x] = \frac{a_i(x)}{f_i(\theta_i, r_i)} \lambda_i^x \quad \begin{matrix} x = r_i, r_i+1, \dots, N_i \\ i = 1, \dots, s \end{matrix}$$

Then the statistics

$$X_i = \sum_{j=1}^{n_i} X_{ij} \quad W_i = \min_j \{X_{ij}\} \quad i = 1, \dots, s$$

define a complete sufficient statistic for the parameter set

$\{\theta_1, \dots, \theta_s, r_1, \dots, r_s\}$  and the conditional distribution of  $X_1$  given

$\tilde{W} = \tilde{w} = (w_1, \dots, w_s)$  and  $\tilde{U} = \tilde{u}$  ( $\tilde{U}$  defined as in (5.2)) is a one-parameter PSD with parameter  $\lambda = \lambda_1, \dots, \lambda_s$  and p.f.

$$(5.5) \quad p(x; \tilde{u}, \tilde{w}) = \frac{c^*(x; \tilde{u}, \tilde{w})}{C^*(\lambda; \tilde{u}, \tilde{w})} \cdot \frac{\lambda^x}{x!}$$

where  $\max[W_1, \max_{2 \leq i \leq s} (W_i - U_i)] \leq x \leq \min\{N_1, N_2 - u_2, \dots, N_s - u_s\}$ , and

$$c^*(x; \tilde{u}, \tilde{v}) = \Delta_{w_1} c_1(x; w_1, n_1) \prod_{i=2}^s \Delta_{w_i} c_i(x + u_i; w_i, n_i) \frac{1}{(u_1 + x)!},$$

$$C^*(\lambda; \tilde{u}, \tilde{w}) = \sum_x C^*(x; \tilde{u}, \tilde{w}) \frac{\lambda^x}{x!}.$$

Proof. We have

$$P[X_1=x, \tilde{U}=\tilde{u}, \tilde{W}=\tilde{w}] = P[X_1=x, W_1=w_1] \prod_{i=2}^s P[X_i=u_i+x, W_i=w_i]$$

since the pairs  $(X_1, W_1), \dots, (X_s, W_s)$  are independently distributed. Now

applying Theorem 2.1 with  $k=1$  for each  $(X_i, W_i)$ , we obtain

$$P[X_1=x, U=u, W=w] = (-1)^s \frac{\Delta_{w_1} c_1(x; w_1, n_1)}{g_{n_1}(\lambda_1, r_1)} \frac{\lambda_1^x}{x!} \prod_{i=2}^s \frac{\Delta_{w_i} c_i(u_i+x; w_i, n_i)}{g_{n_i}(\lambda_i, r_i)} \frac{\lambda_i^{u_i+x}}{(u_i+x)!}$$

with  $\Delta_{w_i}$  operating only on  $w_i$  ( $i=1, \dots, s$ ). Similarly, by Theorem 2.1, the joint distribution of  $U$  and  $W$  is

$$P[U=u, W=w] = \sum_k (-1)^s \frac{\Delta c_1(k; w_1, n_1)}{g_1(\lambda_1, r_1)} \frac{\theta_1^k}{k!} \prod_{i=2}^s \frac{\Delta c_i(u_i+k; w_i, n_i)}{g_i(\lambda_i, r_i)} \frac{\theta_i^{u_i+k}}{(u_i+k)!}$$

so that (5.5) follows.

## 6. SOME APPLICATIONS

Here we indicate some applications of the results given in the preceding sections. The first kind of applications concern the mvu estimation of parameters involved in a PSD truncated on the left. This calls for the theory developed in Sections 2, 3, and especially, 4. The second kind relates to testing hypotheses and constructing confidence intervals for products and/or quotients of a number of parameters, each associated with a distribution, usually, of the same family. This was already briefly discussed in the preceding section.

As regards the first kind of applications, there are many practical situations in which sampling is naturally restricted to truncated PSD's, especially the classical ones: Poisson, binomial and negative binomial, with typical truncation away from zero. For example, a multiply truncated Poisson may arise, e.g., in estimating the accident rate  $\lambda$  on the basis

of reports from several sources (localities, factories, etc.) where source  $i$  provides relevant information only if at least certain number  $r_i$  of accidents occur;  $r_i$  may be fixed and known or unspecified. Other examples in which left-truncated Poisson, as well as binomial or negative binomial distributions, may arise can be found in the statistical literature and some of the references given in this article.

We give an application in the area of reliability of multi-component systems. It is assumed that each of the  $m$ , say, components of a complex system (a missile, an aircraft, a computer, etc.) can only either perform or fail so that the system reliability in this so-called attribute failure model can be expressed as a function of Bernoulli parameters. Consider the reliability of a "k-out-of-m system" (see, e.g., Birnbaum et al. [1961]) which consists of  $m$  components and will function properly if at least  $k$  out of  $m$  components function, where  $1 \leq k \leq m$ ;  $k = 1$  corresponds to a parallel system and  $k = m$  to a series system. The reliability of such a system with  $m$  independent identical components is given by the tail binomial probability function

$$(6.1) \quad R_k = \sum_{i=k}^m \binom{m}{i} p^i (1-p)^{m-i}$$

where  $p$  is the reliability of each component. Suppose observations (of the number of failing components) are available for such k-out-of-m systems; then the mvu estimation of  $R_k$  provides an application of the theory of Section 4 (see (4.11)). It should be observed that if we were simply interested in a confidence interval for  $R_k = R_k(p)$ , then it would

be possible to convert a confidence interval for  $p$  to a corresponding one for  $R_k(p)$ , which is an increasing function of  $p$ .

Moreover, in certain situations it may be assumed that data for such  $k$ -out-of- $m$  systems are available only under the condition that at least  $r$ ,  $1 \leq r < k$ , of the  $m$  components function in all instances. Then the corresponding (conditional) reliability of the system is given in terms of  $R_k$  by

$$R_r^* = \frac{R_k}{R_r} \quad 1 \leq r < k \leq m.$$

Thus in effect we arrive at a truncated binomial model, where we sample from a left-truncated probability function

$$P[X=x] = \frac{\binom{m}{x} p^x (1-p)^{m-x}}{R_r} \quad x = r, \dots, m.$$

Then in estimating  $R_r^*$  we can use the mvue based on (4.11) or (4.7), depending on whether  $r$  is known or unknown.

With respect to tests of hypotheses and confidence intervals for products of parameters of PSD's, we recall the application of Harris [1971] to products of Poisson parameters; in an attribute failure model it is assumed that component  $i$  fails with probability  $p_i$ ,  $i = 1, \dots, k$  and hence the reliability of a parallel or series system can be studied in terms of the product  $p_1 p_2 \dots p_m$ ; if component  $i$  is tested separately  $n_i$  times, then for "large"  $n_i$  and  $p_i$  "small", the problem can be reformulated in terms of the product  $\lambda = \lambda_1 \lambda_2 \dots \lambda_k$  where  $\lambda_i = n_i p_i$  is

the parameter of the  $i$ th Poisson distribution which approximates the  $i$ th binomial  $(n_i, p_i)$ ,  $i = 1, \dots, m$ . The theory of Section 5 for constructing uniformly most powerful unbiased tests and corresponding uniformly most accurate unbiased confidence intervals applies to such products of Poisson parameters even when the underlying Poisson distributions are left-truncated at different known or unknown truncation points. When the truncation points  $r_1, \dots, r_k$  are known Proposition 5.1 applies, and when they are unknown Proposition 5.2 applies; note, however, that the latter case requires taking more than one observation from each of the  $k$  Poisson distributions.

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20. ABSTRACT

Convolutions of one-parameter power series distributions (PSD) truncated on the left at several known or unknown points are studied via exponential generating functions. The special cases of the logarithmic series, the Poisson and the binomial and negative binomial distributions lead to multiparameter Stirling numbers of the first and second type and C-numbers respectively. Minimum variance unbiased estimators are found for certain functions of the parameters, including the probability functions themselves. Some conditional distribution properties are given and it is indicated how they can be used in confidence interval estimation of the reliability of multicomponent attribute failure models.

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