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IMPLICATIONS OF VORTEX THEORY FOR FIREBALL MOTION

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Mission Research Corporation

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Mission Research Corporation
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A theory of low altitude fireball rise is developed based on the current state of the art of Vortex theory. Indefensible assumptions are required and are identified. All effects which are believed to be important for the rise and stabilization are included. These are: (1) release of fireball dissociation energy (2) release of water vapor condensation energy (3) effect of nonadiabatic lapse rate (4) wind shear (5) ground shock interaction (6) atmospheric blast reaction ("shotgun effect"), and (7) buoyancy effect. The theory includes the entire

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rise and stabilization but does not attempt to describe the early or very late phases in detail. It is hoped that this theory will provide an adequate framework to identify specific weaknesses which can be addressed individually.

PREFACE

Many members of the MRC professional staff contributed valuable suggestions and criticisms, particularly Dr. Ball who triggered the effort, and Dr. McCartor who made some special calculations on bubble rise.

TABLE OF CONTENTS

IMPLICATIONS OF VORTEX THEORY FOR FIREBALL MOTION

1.0 INTRODUCTION

This is an interim report which collects the author's present ideas on the state of vortex theory and such implications as can be drawn regarding low altitude fireball rise and development. The contents of the following discussion are about half original and half arguments from a variety of sources covering 100 years of research, recast as to language and emphasis. It is likely that the reader will find the discussions somewhat confusing and unsatisfactory. That is because neither the author nor anyone else understands vortex motion.

The collection of this material is believed to be worth publication despite the unsatisfactory status of the theory because there are a few fairly firm facts, not all of which are generally known, and a larger number of fairly strong inferences which can be drawn and which should be helpful to theorists and modelers of fireball and atmospheric "thermal" motion. It is the author's hope that this report can provide a helpful starting point for future research on low altitude fireballs.

The following sections proceed systematically from a general statement of the low altitude fireball rise problem through a definition of the concepts to be used, development of some standard vortex theory results, and a series of extensions of the theory toward the case of interest. It has been necessary to make some indefensible assumptions, relying on intuition and reasoning by analogy. It is hoped that enough of the results are reasonable to allow identification and correction of the unreasonable ones in future work.

2.0 GENERAL CHARACTERISTICS OF BUBBLE RISE MODELS

Whenever the pressure equilibrium radius of a fireball exists and is small compared to the atmospheric scale height, it is traditional to treat the rise and stabilization of the fireball by a "bubble rise" model.

A number of such models have been developed and can be found in the literature.

Assumptions common to all of the models are:

1. Bubble pressure is equal to local ambient pressure.
2. The bubble entrains ambient air mass at a rate proportional to the rise velocity, some area associated with the bubble, the ambient density, and an "entrainment coefficient", α.

Usually the bubble is assumed to be spherical in shape.

Such models always meet immediately with a limited success, in the sense that a reasonable value of α can be found such that stabilization altitude can be predicted within a few kilometers for a wide range in yield and some range in burst altitude.

Weaknesses in such models are: (1) it is not clear that α should be constant nor what parameters determine it, (2) if dynamic information (e.g., velocity) is required, another set of rather arbitrary assumptions is added, (3) factors believed to be important have never been treated, as real atmospheric temperature, humidity, wind shear profiles, turbulence, and the detailed profile of temperature, density, etc.

The basic reason for the above situation is that one is attempting to extract desired results from a complex multidimensional turbulent hydrodynamic problem without recourse to a detailed solution, and without enough conservation laws.

In particular, one normally has: (1) either momentum or energy conservation, (2) mass conservation, and (3) the condition of pressure equilibrium to determine: (1) density, (2) temperature, (3) volume, and (4) entrainment parameter. Hence one is forced to make an additional assumption. In cases where both momentum and energy conservation are used, a fifth unknown appears, the velocity, so one is still one equation short and must invent two arbitrary relations, one for α and one for drag coefficient, etc. rather than just one (for α).

Some workers have attempted to resolve this dilemma by appealing to conservation of circulation, in particular the Mt. Auburn group has been a proponent of this method of resolution. We will follow their lead but with great trepidation.

The reason for our caution is that almost all useful circulation theorems apply only to incompressible flow in uniform media, certainly not a situation of interest to us. Circulation is defined as

$$\Gamma = \int (\nabla \times \vec{v}) \cdot d\vec{s} = \oint \vec{v} \cdot d\vec{\ell} \qquad (1)$$

and accordingly is a kinematical quantity, not a dynamical one (the density, ρ, does not appear in Equation 1). If ρ is constant and everywhere uniform* the distinction is academic and Γ is a useful quantity, but its utility is suspect whenever ρ is variable.

A rationale exists which allows the "impulse" of a vortex to be used as a physically meaningful quantity.[1] For a spherical vortex the equation is

$$\frac{dP}{dt} = \frac{d}{dt}\left(\frac{3}{2}\,\rho VU\right) \tag{2}$$

where P is impulse, V is volume, U is mean vortex rise velocity and ρ is the density, assumed constant everywhere. The form Equation 2 associates momentum ρVU with the mass of fluid which accompanies the vortex in its motion and takes account of momentum which must be radiated to infinity, $\frac{1}{2}\rho VU$, in order to establish the external flow field.

For the case of a real atmosphere the momentum and energy equations become more mysterious, in that both integrals may receive major contributions from the air far below the region of interest due to the exponential stratification of air density.

The theoretical status of circulation and related quantities associated with real vortex motion is thus in a very unsatisfactory state. However, considerable research has been done recently in applications of such concepts to atmospheric thermals by dynamical meteorologists[2,3] with considerable success. It therefore appears worthwhile to attempt to use the concept of circulation to furnish an extra relation which allows the entrainment parameter, α, to be determined as a function of other bubble parameters, including circulation. If successful, the determination

* This is assumed in almost all vortex theory.

of α then follows from a determination of all of the ways in which circulation can be altered. One hopes that these various ways can be tabulated and successfully attacked one by one.

3.0 DEFINITIONS OF CONCEPTS

3.1 VORTEX

For our purposes a vortex is defined as a mass of air which contains vorticity $(\nabla \times \vec{V} \neq 0)$ and which moves as a unit through the surrounding medium. We will assume it to be defined by a simple surface, as a spheroid, whose position is determined by the condition that material does not cross the surface. A sketch of a vortex is shown in Figure 1.

Note the vortex is not identical to the visible torus, when such exists. The visible torus is included in the vortex. We have shown the surface as closed. For vortices which entrain, the surface is believed to be open at the bottom where material enters. It is likely that in many cases the surface is unstable, resulting in entrainment (and/or loss) of vortex material across the boundary. The usual way of treating this process is to continuously redefine the surface position.[3] Thus the vortex boundary is frequently troublesome to define. This problem does not affect the concept of the vortex being defined as all of the material moving as a unit with the vorticity, nor does it reduce the value of the concept of the surface.

3.2 HILL VORTEX

The only vortex form which is stable and has been described with simple complete equations is Hill's spherical vortex.[4] This vortex moves through a uniform, incompressible medium with constant velocity and no

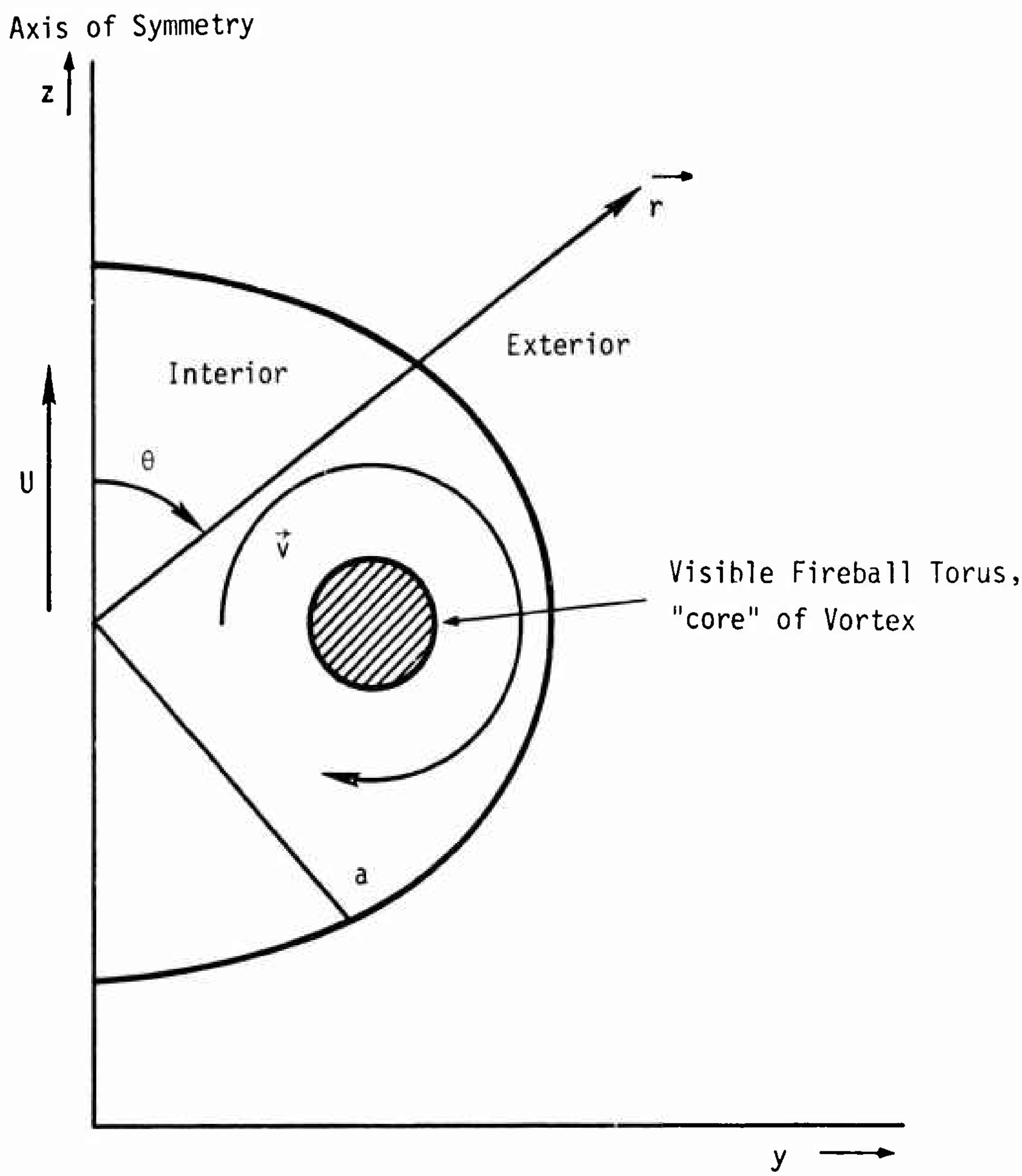


Figure 1. Sketch of vortex.

entrainment and there are no external forces acting. It is far from the situation of interest for fireball motion* but the only description available which is easy to work with.

Inside the Vortex

The vorticity, ω, is given by (see Figure 1),

$$\omega = \frac{15}{2} Ur \sin\theta / a^2 \quad . \tag{3}$$

In the frame moving with the vortex the velocity components are,

$$v_z = \frac{3}{2} U[1 - r^2/a^2 (1 + \sin^2\theta)] \quad , \tag{4}$$

$$v_y = \frac{3}{2} U \sin\theta \cos\theta \; r^2/a^2 \quad , \tag{5}$$

$$v_r = \frac{3}{2} U \cos\theta \; (1 - r^2/a^2) \quad , \tag{6}$$

and

$$v_\theta = \frac{3}{2} U \sin\theta \; (2r^2/a^2 - 1) \quad . \tag{7}$$

Outside the Vortex

$$\omega = 0 \quad . \tag{8}$$

In the frame moving with the vortex

$$v_z = U[1 - (1 - \frac{3}{2} \sin^2\theta) \; a^3/r^3] \quad , \tag{9}$$

---

* A count of the underlined words in the preceeding sentence reveals no less than five assumptions which are invalid for the case of interest.

$$v_y = \frac{3}{2} U \frac{a^3}{r^3} \sin\theta \cos\theta \quad , \qquad (10)$$

$$v_r = - U \cos\theta \; (1 - a^3/r^3) \quad , \qquad (11)$$

and

$$v_\theta = U \sin\theta \; (1 + \frac{1}{2} a^3/r^3) \quad . \qquad (12)$$

Gross Quantities

Mean square radius of vorticity:

$$\langle r_\omega^2 \rangle \equiv \left(\int r^2 d\omega \Big/ \int d\omega \right) = 2a^2/5 \quad . \qquad (13)$$

Impulse:

$$P = 2\pi \rho a^3 U \qquad (14)$$

Energy:

$$E = \frac{10}{7} \pi \rho a^3 U^2 \qquad (15)$$

Circulation:

$$\Gamma = 5 U a \qquad (16)$$

4.0 DIMENSIONAL ANALYSIS FOR RISING VORTEX

We begin with some fairly general results.

4.1 Entrainment Parameter and Radius

Under any of the three assumptions: (a) incompressible flow, (b) pressure equilibrium without small scale mixing, or (c) pressure equilibrium with vortex gas constant, γ , the same as the exterior gas constant, the increase of volume due to entrainment of materials is given by

$$dV = \frac{1}{\rho_a} dM \tag{17}$$

where V is volume, ρ_a is outer material density and M is mass of vortex.

For a spherical vortex we can always write

$$\frac{dM}{dz} = 4\pi a^2 \rho_a \alpha \tag{18}$$

where z is altitude, a is radius, and α is the entrainment parameter, defined by Equation 18.

But by differentiation of the LHS of Equation 17, it follows that for spherical vortices and any of assumptions a-c

$$\frac{da}{dz} = \alpha \tag{19}$$

4.2 Constancy of Buoyant Force in Uniform Medium

The buoyant force operating on the vortex is given by

$$F_B = g\ (\rho_a - \rho)\ \frac{4}{3} \pi a^3 \quad , \tag{20}$$

where g is the acceleration of gravity.

If ρ_a is constant then

$$\frac{dF_B}{dz} = \frac{4}{3}\pi g \left(- a^3 \frac{d\rho}{dz} + 3a^2 (\rho_a - \rho) \frac{da}{dz}\right) , \tag{21}$$

but

$$\frac{d\rho}{dz} = \frac{d}{dz} (M/V) = 3\rho_a \alpha/a - \frac{\rho}{V}\frac{dV}{dz} , \tag{22}$$

where the first term (derivative of M) follows from Equation 18. Expanding the last term in Equation 22 yields

$$\frac{d\rho}{dz} = 3\rho_a \alpha/a - 3\rho\alpha/a . \tag{23}$$

Substitution of Equation 23 into Equation 21 shows the buoyancy force to be constant

$$\frac{dF_B}{dz} = \frac{4}{3}\pi g \, [- 3\alpha a^2(\rho_a - \rho) + 3\alpha a^2(\rho_a - \rho)] = 0 \tag{24}$$

Almost every useful result connected with vortex motion is dependent upon buoyant force being constant, most analyses in fact assume zero buoyancy.

Here we summarize the assumptions which we have made so far, and which underly most literature on the subject of vortex motion:

1. Time similarity. This assumption allows us to characterize the vortex by a single average value of such physical parameters as radius, mass, volume, force, etc.

2. One of (a) incompressibility, (b) pressure equilibrium without small scale mixing, (c) pressure equilibrium with interior and exterior gas constants equal.

3. Spherical shape. This is not an important assumption, the result (Equation 19) is only slightly modified if one allows a more general shape and the result of Equation 24 is unchanged. It is an assumption of convenience.

4.3 Dimensional Analysis

We now add one more assumption which is almost universal in the literature, and is valid for most practical cases. We make the approximation;

4. That in the equations of motion we treat the vortex density, ρ, as if it were equal to the outer density, ρ_a, in all terms where it appears by itself, rather than as the difference $\rho_a - \rho$.

For example, in the impulse equation for a Hill spherical vortex,

$$\frac{d}{dt}\,(2\pi\rho a^3 U) = g(\rho_a - \rho)\,\frac{4}{3}\,\pi a^3 \quad , \tag{25}$$

we are allowed to replace the ρ on the left hand side with ρ_a, which restricts us to consider cases where the average density inside the vortex is only slightly different from ρ_a.

Having done this we can write

$$\frac{d}{dt}\,(a^3 U) = \frac{2}{3}\,g\,(\rho_a - \rho)a^3/\rho_a \equiv F^*, \tag{26}$$

and by Equation 24, the driving force, F^*, is a constant with the dimensions $[L^4 T^{-2}]$.

In searching for a time similar solution for U and a, we can imagine only two independent variables which might be of physical significance, z, and F^* (in reality, this is a matter of taste, we don't propose to defend this statement, we are following the literature here).

Accordingly, we write

$$U = f\ z^{i} (F^*)^{j} \tag{27}$$

and

$$a = h\ z^{k} (F^*)^{\ell} \tag{28}$$

where f and h are dimensionless functions, hopefully constants.

We must have, from Equation 27

$$[L]^{i} [L^4 T^{-2}]^{j} = L/T \tag{29}$$

which requires $j = \frac{1}{2}$ and $i = -1$.

So

$$U = f \sqrt{F^*} / z \quad . \tag{30}$$

From Equation 28

$$[L] = [L]^{k} [L^4 T^{-2}]^{\ell} \tag{31}$$

which requires $\ell = 0$ and $k = 1$.

So

$$a = hz \tag{32}$$

and furthermore, from Equation 19, h must be a constant equal to α.

$$a = \alpha z \quad . \tag{33}$$

These results are more general than indicated. Generalization to a non-spherical or non-Hill vortex only changes the numerical value of α and the factor $\frac{2}{3}$ in Equation 26.

To summarize, we have found for any time similar vortex for which assumptions 1, 2, 3, and 4 hold, the radius is a linear function of z and the circulation, which is proportional to Ua, is constant. These are most important findings.

The above discussion is in no sense a proof of the constancy of circulation or the linear growth of vortices, dimensional analysis can only be a guide to intuition. Such proof as exists for these two findings rests on laboratory data. However, within the limits of error of laboratory data both conclusions are verified, although one could wish the laboratory data were more accurate.

5.0 QUASI-PHYSICS ON VORTEX SHAPE

In this discussion we consider the relationship between energy and impulse in an isolated, stable vortex. We will draw some inferences with regard to the likely shape of an atmospheric vortex, such as a rising fireball or thermal. Our inferences cannot be firm since no general solution exists for such vortices, thus we will rely heavily on intuition. Our procedure will be to begin the discussion with the Hill spherical vortex, the best understood stable vortex, then use intuition to write approximate formulae for other shapes, based on those for the Hill vortex.

We will base our discussion upon the energy of motion of the vortex,

$$\frac{dE}{dz} = F \tag{34}$$

where E is energy, F is the net force on the system, and z is the distance moved. Here we have tacitly assumed a time similar shape in order to be able to identify single characteristic values of E, F, and z.

Our other basic parameter is the impulse of the system,

$$\frac{dP}{dt} = F \tag{35}$$

or

$$\frac{dP}{dz} = F/U \tag{36}$$

where P is impulse and U is the characteristic velocity of the vortex. Recall that the impulse of a vortex system is defined to be the momentum of the material associated with vortex motion plus an amount which must be radiated to infinity whenever the momentum is changed.

Hill Spherical Vortex

For a spherical vortex the amount of momentum lost to infinity is $\frac{1}{2}\rho VU$ where ρ is the exterior density and V is the volume of the sphere. Thus the impulse of a spherical vortex is just

$$P = 2\pi\rho\, a^3 U \quad , \tag{37}$$

where a is the radius of the sphere of material which accompanies the vortex in its motion.

The energy of a Hill spherical vortex is

$$E = \frac{10}{7} \pi\rho a^3 U^2 \quad . \tag{38}$$

This energy is composed of three contributions, (1) the energy in the exterior flow field,

$$E_x = \frac{1}{4} \left(\frac{4}{3} \pi\rho a^3 U^2\right) \quad , \tag{39}$$

consistent with the momentum radiated to infinity, (2) the energy associated with the motion of the center of mass of the sphere,

$$E_{cm} = \frac{1}{2} \left(\frac{4}{3} \pi\rho a^3 U^2\right) \quad , \tag{40}$$

and (3) the energy associated with the circulation of the material in the spherical vortex,

$$E_\Gamma = \frac{3}{7} \pi\rho a^3 U^2 \quad . \tag{41}$$

Of the Equations 37 through 41, only 38 and 41 are peculiar to the Hill vortex, the others depend only on the spherical shape of the vortex (plus, of course, the assumptions of $\rho \approx \rho_a$, incompressibility, and time similarity).

To investigate whether an expanding vortex with flow field instantaneously identical to that of a Hill vortex is an appropriate approximation for vortex motion subject to an external force, we evaluate the energy requirements for a Hill vortex. From Equations 34 and 38,

$$F = \frac{dE}{dz} = \frac{10\pi}{7} \rho a^3 U^2 \left(\frac{3}{a} \frac{da}{dz} + \frac{2}{U} \frac{dU}{dz} \right) . \tag{42}$$

But we know that Ua is a constant so

$$\frac{1}{U} \frac{dU}{dz} = - \frac{1}{a} \frac{da}{dz} , \tag{43}$$

and Equation 42 becomes

$$\frac{dE}{dz} = \frac{10\pi}{7} \rho a^3 U^2 \left(\frac{1}{a} \frac{da}{dz} \right) = \frac{E}{a} \frac{da}{dz} \tag{44}$$

The rate of delivery of energy to the system in order to change the momentum is obtained by substituting Equation 37 into Equation 36

$$F = U \frac{dP}{dz} = 2\pi \rho a^3 U^2 \left(\frac{3}{a} \frac{da}{dz} + \frac{1}{U} \frac{dU}{dz} \right) \tag{45}$$

using Equation 43 again we find

$$U \frac{dP}{dz} = 4\pi \rho a^3 U^2 \left(\frac{1}{a} \frac{da}{dz} \right) \tag{46}$$

Comparison of Equation 46 to 44 shows that much more energy is generated in changing the impulse, by whatever force, than can be absorbed into the flow field of a Hill vortex.

Remembering that we have radiated $\frac{1}{2} MU$ of momentum away to infinity we may argue that all but $\frac{1}{4} MU^2$ of the energy generated in Equation 46 need remain in the system. Thus the energy which must be deposited somewhere in the flow field, E_d, may be as small as

$$\frac{dE_d}{dz} = \frac{11}{3} \pi \rho a^3 U \left(\frac{1}{a} \frac{da}{dz} \right) = \frac{77}{30} E \left(\frac{1}{a} \frac{da}{dz} \right) \tag{47}$$

We need to account for an energy excess of

$$\frac{dE_d}{dz} - \frac{dE}{dz} = \frac{47}{21} \pi\rho a^3 U^2 \left(\frac{1}{a} \frac{da}{dz}\right) = \frac{47}{30} \frac{E}{a} \frac{da}{dz} \quad . \tag{48}$$

There are two obvious ways out of this dilemma. The first is to assume the excess energy goes into turbulence and the second is to assume a flatter shape for the vortex, one which requires more energy. Both of these are plausible and indeed have data to support them. It is known that buoyant vortices are more flat on top than a sphere and in some cases evidence of turbulence is observed.

If all the excess energy went into turbulence it would require more than five times as much energy in turbulence (plus thermal) than in internal circulation energy (47/9 to be exact, from Equations 41 and 48). It is most difficult to imagine that a vortex could maintain as clean and stable a shape as is generally observed in the laboratory and field with five times as much random energy as directed.

Energy Assumption

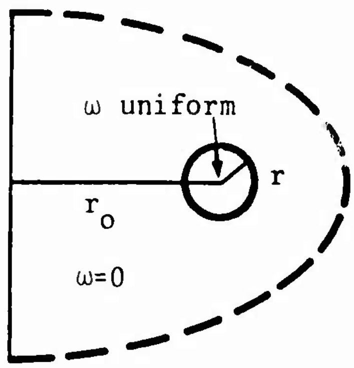
On the basis of intuition, guided by laboratory and field observation we will assume that turbulent energy is usually negligible compared to circulation energy and is at most equal to circulation energy.

Consequences for Vortex Shape

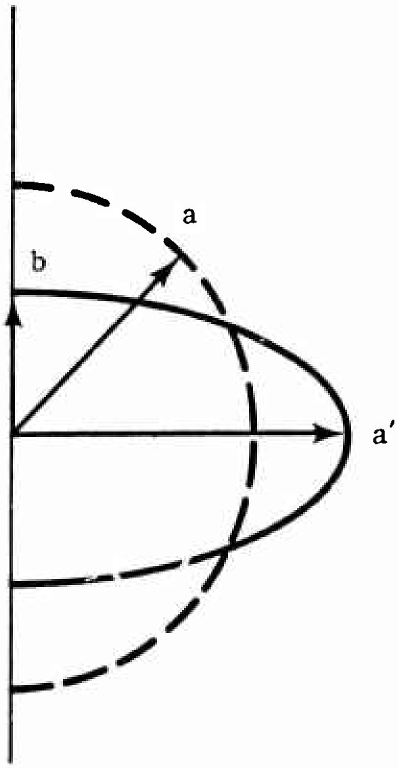
The energy assumption we have adopted implies a shape considerably different from that of a Hill spherical vortex.

No convenient equations exist by which one can accurately study the dependence of energy on vortex shape, at least none such are known to this author.

Two cases are available, the Hill spherical vortex, with a continuous smooth vorticity distribution, and that of a ring of uniform vorticity with cross section small compared to its distance (see figure). Based on the relations on p 241 in Lamb[1] the circulation energy of such a vortex can be many times larger than the energy of translation. We accept this as an indication that a flattened vortex will result in increased circulation energy relative to translational energy and sufficient flattening will in fact increase the circulation energy to any degree we require.



Our next task is to quantitatively estimate the increase of energy with flattening. Since we have no other convenient starting point we will estimate the likely effects beginning with Hill's spherical vortex of radius a. The solid curve represents a flattened spheroidal vortex with horizontal radius a´ and vertical radius b. The volume is to be the same for both cases,



$$\frac{4}{3}\pi b a'^2 = \frac{4}{3}\pi a^3 \qquad (49)$$

and the rise velocity is to be fixed

$$U' = U \qquad (50)$$

as is the mass.

For convenience define

$$f \equiv b/a \quad , \tag{51}$$

then

$$a' = a/\sqrt{f} \quad . \tag{52}$$

We expect the impulse of the new vortex to be about equal to that of the Hill vortex,

$$\begin{aligned} P' &= \text{momentum of moving material + radiated momentum} \\ &\quad \text{proportional to volume} \\ &= MU + \frac{1}{2} MU = 2\pi\rho a^3 U = P \end{aligned} \tag{53}$$

since $ba'^2 = a^3$, from Equation 49.

The energy associated with the center of mass motion is also unchanged

$$E'_{CM} = E_{CM} = \frac{2}{3} \pi\rho a^3 U^2 \tag{54}$$

The energy associated with circulation should be increased due to the necessity for the circulating material to go farther out, in the ratio a'/a, and to have less time to get there, in the ratio b/a. This time factor is required because we must have continuity of tangential velocity at the interface. Thus the internal velocity of circulation should be increased by a factor of roughly

$$\frac{a'a}{a\ b} = a'/b \tag{55}$$

and the corresponding energy increased by the square of this factor, from Equation 41 we write

$$E_\Gamma' = \frac{3}{7}\pi\rho a^3U^2(a'/b)^2 = \frac{3}{7}\pi\rho a^3U^2/f^3 \quad . \tag{56}$$

The energy external to the vortex should also be increased since material near the vortex surface must also flow around the vortex faster in less time. Almost all of the energy external to the Hill vortex resides in material very close to the edge because the velocity falls off as the third power of the distance (thus energy as the sixth power). Perhaps as good a way as any to estimate the new energy is to assign the same relative increase in velocity as that for the internal motion to the same amount of material as was the case for a Hill vortex. This process yields, from Equation 39

$$E_X' = E_X(a'/b)^2 = \frac{\pi}{3}\rho a^3U^2/f^3 \tag{57}$$

The energy of the flattened vortex is

$$E' = E_{CM}' + E_\Gamma' + E_X' = \frac{10}{7}\pi\rho a^3U^2\left[\frac{7}{15} + \frac{8}{15f^3}\right]$$

$$= E(7 + 8/f^3)/15 \tag{58}$$

Comparison of Equation 58 to Equation 47 shows that we require

$$(7 + 8/f^3)/15 = 77/30 \tag{59}$$

if no appreciable amount of energy causes turbulence. For this case

$$f \simeq 0.633 \quad , \quad \text{or } b/a' = 0.506 \tag{60}$$

Note that the ratio b/a' will be the observed ratio of vertical to horizontal radius.

If we take the other extreme of our energy assumption we can assume an amount of energy equal to E'_Γ is in turbulence, then Equation 58 becomes

$$E' = E'_{CM} + 2E'_\Gamma + E'_x = \frac{10}{7}\pi\rho a^3U^2(14 + 25/f^3)/30 \qquad (61)$$

We then require, from Equation 47, that

$$(14 + 25/f^3)/30 = 77/30 \qquad (62)$$

which has the solution

$$f \cong 0.737 \quad , \quad b/a' = 0.642 \qquad (63)$$

Turner[3] on page 188 reports Scorer as having found a value somewhat above our range, $b/a' = 0.716$ from laboratory experiments dropping salt water "thermals" through pure water. In Turner's book a shadowgraph is illustrated which, by this author's measurement, yields

$$0.49 \leq b/a' \leq 0.56 \quad , \qquad (64)$$

in good agreement with our prediction (Turner's shadowgraph shows clear evidence of turbulence , but not extreme turbulence). Similar measurements on Figures 10 and 15 of Reference 5 yield $b/a' = .65$.

Until more definitive data is uncovered (it almost certainly exists), we will conclude that vortices under an external force are not very spherical, but can be expected to be flattened spheroids with ratio of vertical to horizontal radius of about 0.6.

6.0 VORTEX IN UNIFORM MEDIUM

In this section we derive the useful results of Section 4 for a flattened spheroidal vortex.

The Equation 17 is still valid,

$$dV = \frac{1}{\rho_a} \; dM \quad , \tag{65}$$

and we will continue to write the mass entrainment equation as

$$\frac{dM}{dz} = 4\pi a^2 \rho_a \alpha \; , \tag{66}$$

where a will henceforth be understood to mean the horizontal radius. The vertical radius will be written

$$b = f_v a \quad . \tag{67}$$

If we carry out the differentiation of volume with respect to altitude in Equation 65 and use Equations 66 and 67 we find

$$4\pi \; f_v a^2 \; \frac{da}{dz} = \frac{1}{\rho_a} \; 4\pi a^2 \rho_a \; \alpha \tag{68}$$

or

$$\frac{da}{dz} = \alpha / f_v \quad . \tag{69}$$

This is the only effect of aspherical shape on the results of Section 4. In particular, the buoyancy is still constant, as is the quantity*

$$X' \equiv Ua \tag{70}$$

* We use X' in preference to circulation. They differ only by a numerical constant.

If we rewrite Equation 25 the factor f_v cancels and the momentum equation becomes

$$\frac{d}{dz}(a^3U) = \frac{2}{3} g (\rho_a - \rho) a^3/\rho U \qquad (71)$$

where we have written $\frac{d}{dt}$ as $U \frac{d}{dz}$.

Remembering that the quantity X' is constant we can write Equation 71 as

$$\frac{d}{dz} a^2 = \frac{2g(\rho_a - \rho) a^3}{\rho 3X'U} \quad . \qquad (72)$$

Carrying out the indicated differentiation in Equation 72, substituting Equation 69 and rearranging gives

$$\alpha = \frac{g(\rho_a - \rho) a^3 f_v}{3\rho X'^2} \qquad (73)$$

We note that the numerator is proportional to the buoyant force which is constant, $\rho \simeq \rho_a$, and X' is constant. Our result is therefore consistent with our earlier conclusion that α is a constant and contains the surprising information that the entrainment varies inversely as the square of the circulation (or X'), that is, the more circulation present, the less entrainment.

This dependency is said to be verified in laboratory experiments where the assumptions of incompressibility and $\rho \approx \rho_a$ are very good,[3] and the circulation can be varied. We have personally witnessed demonstrations which qualitatively verify Equation 73. In addition, the above treatment is good within experimental error for atmospheric thermals[3] which in nature are weakly buoyant ($\rho \approx \rho_a$) and rise a distance small compared to a scale height.

Encouraged by the above, we will continue to extend the treatment toward the case of interest.

7.0 EXTENSION TO ADIABATIC LAPSE CASE

Our first step will be to make a "plausible" extension to a real atmosphere which has an adiabatic lapse rate. We are relaxing the assumption of incompressible flow but keeping the approximation $\rho \simeq \rho_a$ and arranging a situation where buoyancy is constant.

For convenience, define the density factor to be given by

$$\Delta \equiv \rho_a - \rho \tag{74}$$

Then Equation 71 becomes

$$\frac{d}{dz}(\rho a^3 U) = \frac{2ga^3\Delta}{3U} \quad . \tag{75}$$

The entrainment equation becomes

$$\frac{d}{dz}(\rho a^3) = 3\rho_a a^2\alpha/f_v \quad . \tag{76}$$

The volume equation requires two terms

$$\frac{d}{dz}(a^3) = \frac{\partial}{\partial z} a^3\Big|_{\text{adiabatic}} + \frac{\partial}{\partial z} a^3\Big|_{\text{entrainment}} \tag{77}$$

Under any of the three assumptions, (i) incompressibility,* (ii) no small scale mixing, or (iii) pressure equilibrium where interior and exterior gas constants (γ) are equal, the entrainment term is given

* No longer true for this case.

by

$$\left.\frac{\partial}{\partial z} a^3\right|_{\text{entrainment}} = \frac{1}{\rho_a} \frac{\partial}{\partial z} (\rho a^3) = 3a^3\alpha/f_V \quad , \tag{78}$$

where we have used Equation 76.

The adiabatic term is given by

$$\left.\frac{\partial}{\partial z} a^3\right|_{\text{adiabatic}} = - \frac{a^3}{\gamma p} \frac{dp}{dz} \quad , \tag{79}$$

where p is ambient pressure, assumed equal to vortex mean pressure.

The volume equation is thus

$$\frac{d}{dz} a^3 = - \frac{a^3}{\gamma p} \frac{dp}{dz} + 3a^2\alpha/f_V \quad . \tag{80}$$

The only equation which has a different appearance from the incompressible case is Equation 80. We will first show that the buoyancy term in Equation 75 is constant

$$\frac{d}{dz} (a^3\Delta) = \frac{d}{dz} (\rho_a a^3) - \frac{d}{dz} (\rho a^3)$$

$$= a^3 \frac{d\rho_a}{dz} + \rho_a \frac{da^3}{dz} - \frac{d}{dz} (\rho a^3) \quad . \tag{81}$$

Substituting Equations 76 and 80 we find

$$\frac{d}{dz} (a^3\Delta) = a^3\rho_a \left(\frac{1}{\rho_a} \frac{d\rho_a}{dz} - \frac{1}{\gamma p} \frac{dp}{dz} \right) \quad . \tag{82}$$

But the condition for an adiabatic lapse rate is that the "buoyancy gradient"

$$B \equiv \frac{1}{\rho_a}\frac{d\rho_a}{dz} - \frac{1}{\gamma p}\frac{dp}{dz} = 0 \quad . \tag{83}$$

So we find indeed that the buoyancy is constant

$$\frac{d}{dz}(a^3\Delta) = 0 \quad . \tag{84}$$

We can use this fact to write Equation 75 as

$$\frac{d}{dz}(\rho a^3 U) = \frac{2g\Delta_0 a_0^3}{3U} \quad , \tag{85}$$

where the subscript o indicates initial value.

At this point we have used up Equation 80, having found it to be equivalent to Equation 76, and find ourselves with two equations involving the quantity ρa^3, which look entirely similar to the incompressible case. The dimensional arguments, unconvincing at best, which led us in the analysis of the incompressible case are less useful now since we have an additional dimensioned variable, ρ. We will therefore adopt two methods of reasoning by analogy, one essentially mathematical and one more physical. The results will be identical.

The Mathematical Guess

We will assume that the quantity to be conserved is the product of U and the cube root of its coefficient in Equation 85,

$$X = \rho^{1/3} X' = \rho^{1/3} a U = \text{const.} \tag{86}$$

There is no justification short of experimental comparison for this assumption, as was the case for $X' = \text{const.}$ in incompressible flow. We defend it here only on the basis (1) it seems as reasonable as most alternatives, (2) the consequences of the assumption are simple and seem more reasonable than most other alternatives we have tried.

Rewrite Equation 85, taking into account the constancy of X,

$$\frac{d}{dz} (\rho a^3)^{2/3} = \frac{2g\Delta_0 a_0^3}{3UX} \quad . \tag{87}$$

Carry out the indicated differentiation to find

$$2\rho^{2/3} \; a \; \alpha = \frac{2g\Delta_0 a_0^3 \; f_V}{3UX} \quad , \tag{88}$$

where we have used Equation 76. Solving for α yields

$$\alpha = \frac{g\Delta_0 a_0^3 \; f_V}{3\rho^{1/3} \; X^2} \quad . \tag{89}$$

We find α to be no longer constant but to vary as the inverse cube root of the density.

Substitution of Equation 89 into Equation 76 now allows solution,

$$\frac{d}{dz} (\rho a^3) = \frac{g\Delta_0 a_0^3}{X_a^2} \; (\rho a^3)^{2/3} \quad . \tag{90}$$

Solving Equation 90 yields

$$a = a_0 \left(\frac{\rho_0}{\rho}\right)^{1/3} \left(1 + \frac{g\Delta_0}{3\rho_0 U_0^2} \; z\right) \quad . \tag{91}$$

Substitution of Equation 91 into Equation 85 yields

$$U = U_0\left(1 + \frac{g\Delta_0}{3\rho_0 U_0^2} z\right)^{-1} \tag{92}$$

The above treatment seems to yield reasonably satisfactory results, intuitively, but one is hesitant to base an argument on the assumption that the third moment of the velocity distribution is constant. That is,

$$X^3 = \rho a^3 U^3 \propto MU^3 = \text{const.} \tag{93}$$

While this is fascinating and may be true, one would prefer to derive it rather than assume it.

The Physical Guess

As an alternative way of thinking about the problem we consider the equation which introduces the entrainment parameter, α, and solve it for α in terms of dynamical quantities. Solving Equation 76 we find

$$\alpha \sim \frac{a}{M}\frac{dM}{dz} \quad , \tag{94}$$

where M is the mass associated with the motion. But in the incompressible case we found α could be expressed as, from Equation 73

$$\alpha = \frac{g\Delta_0 a_0^3 f_v}{3\rho a^2 U^2} \sim \frac{a}{E}\frac{dE}{dz} \quad , \tag{95}$$

where we have taken $M \sim \rho a^3$ and $E \sim \rho a^3 U^2$.

We imagine that the forces operating on the fluid require a relation between mass entrainment defined by Equation 94 and energy buildup defined by Equation 95 to balance. Then α may always be given by an expression like Equation 95, whether flow is incompressible or not or indeed even in cases where buoyancy is not constant. Thus we will assume

$$\alpha = f_v \frac{ga\Delta}{3\rho U^2} \quad , \tag{96}$$

Differentiation of the left hand side of Equation 85 and use of Equations 76 and 96 along with 84 yields

$$\frac{d}{dz} (\rho a^3 U) = \frac{g\Delta_0 a_0^3}{U} + \rho a^3 \frac{dU}{dz} = \frac{2g\Delta_0 a_0^3}{3U} \quad , \tag{97}$$

or

$$\rho a^3 \frac{d}{dz} U^2 = - \frac{2g\Delta_0 a_0^3}{3} \tag{98}$$

Substitution of Equation 96 into Equation 76, multiplication by U^2 and making use of Equation 84 yields

$$U^2 \frac{d}{dz} \rho a^3 = g\Delta_0 a_0^3 \tag{99}$$

Addition of Equations 98 and 99 shows

$$\frac{d}{dz} (\rho a^3 U^2) = g\Delta_0 a_0^3/3 \tag{100}$$

so

$$\rho a^3 U^2 = \rho_0 a_0^3 U_0^2 + \frac{1}{3} g\Delta_0 a_0^3 z \tag{101}$$

Solve Equation 101 for U^2, substitute into Equation 99 and rearrange to find

$$\frac{1}{\rho a^3}\frac{d}{dz}(\rho a^3) = \left[\frac{\rho_0 U_0^2}{g\Delta_0} + \frac{1}{3}z\right]^{-1} \quad . \tag{102}$$

which has the solution

$$a = a_0\left(\frac{\rho_0}{\rho}\right)^{1/3}\left(1 + \frac{g\Delta_0}{3\rho_0 U^2}z\right) \quad . \tag{103}$$

Equation 103, derived from physical intuition, is idential to Equation 91 which was based on mathematical intuition. It follows that Equations 92 and 86 are also true under this treatment .

8.0 EXTENSION TO REALISTIC CASE

In the general case of a real fireball vortex in a real atmosphere processes occur which are certain to alter the circulation and χ. The most straightforward approach is to attempt to list all such processes, estimate the effect on χ, then follow through the consequences without altering any other assumption. This will be our approach. We will find additional terms in the equations for χ, α, buoyancy, etc. These will form the set of equations to use in comparison to data.

8.1 Dry-Air Lapse Rate Effect on X

At a particular moment in the history of the vortex it will be entraining air which originally resided at an altitude not equal to the vortex altitude. It will turn out to have originated mostly from below the vortex altitude. The effect on circulation can be calculated via Bjerknes' Theorem:

$$\left.\frac{\partial \Gamma}{\partial t}\right|_L = -\oint \frac{\vec{\nabla} p}{\rho} \cdot d\vec{\ell} \quad , \tag{104}$$

where the partial derivative indicates we are concerned only with the change in circulation due to lapse rate.

If we assume pressure equilibrium, carry the integral up the axis from $-\infty$ to $+\infty$, and complete the circuit in a region where motion is negligible, we find

$$\left.\frac{\partial\Gamma}{\partial t}\right|_L = -\nabla p_a \int_B^T \left(\frac{1}{\rho} - \frac{1}{\rho_a}\right) dz \quad , \tag{105}$$

where the limits include the region of disturbed density (i.e., $\rho \neq \rho_a$). The value of ρ can be determined by using the fact that each air parcel had ambient values ρ_0, p_0, and has expanded adiabatically to pressure equilibrium, p, at the current vortex position.

$$\begin{aligned} \rho &= \rho_0 (p/p_0)^{1/\gamma} \\ &\simeq \left(\rho_a - \delta z' \frac{\partial\rho}{\partial z}\right)\left[p\Big/\left(p - \delta z' \frac{\partial p}{\partial z}\right)\right]^{1/\gamma} \end{aligned} \tag{106}$$

Expanding to first order

$$\frac{1}{\rho} = \frac{1}{\rho_a} + \frac{\delta z' B}{\rho_a} \quad , \tag{107}$$

where, as before,

$$B = \frac{1}{\rho}\frac{\partial\rho}{\partial z} - \frac{1}{\gamma p}\frac{\partial p}{\partial z} \quad , \tag{108}$$

is the buoyancy gradient of the ambient atmosphere and $\delta z'$ is the mean distance below the fireball from which the material along the axis originated. The rate of change of the circulation becomes

$$\left.\frac{d\Gamma}{dz}\right|_L = \frac{1}{U}\left.\frac{\partial\Gamma}{\partial t}\right|_L = -\frac{\nabla p}{U}\int_B^T \frac{\delta z' B d\ell}{\rho} \quad . \tag{109}$$

It is difficult to know what limits to assign to the integral for a realistic case. The upper limit is clearly near and a bit beyond the top of the sphere, call it distance, a, above the center, the lower limit might be as low as $z-\delta z$ since all of the material above the point will eventually be entrained into the vortex and the material below will not. More sophisticated reasoning might lead to a better answer but we will accept these limits for the present

$$\left.\frac{d\Gamma}{dz}\right|_L = -\frac{\nabla p}{U}\int_{z-\delta z}^{z+a} \frac{\delta z' B}{\rho}\, d\ell \tag{110}$$

For simplicity, we will assume $\delta z \ll H$ and that the buoyancy gradient is constant over the interval, thus

$$\left.\frac{d\Gamma}{dz}\right|_L = -\frac{\nabla p B}{U\rho}\int_{z-\delta z}^{z+a} \delta z' d\ell = \frac{gB}{U}\int_{z-\delta z}^{z+a} \delta z' d\ell$$

$$\approx \frac{gB\delta z^2}{2U} \quad . \tag{111}$$

We have also assumed $\delta z' \propto \ell$.

In a later report we will address the question of the value of δz, which depends on the entrainment parameter, α, and the shape of the vortex.

Converting Equation 111 into an equation for X yields

$$\left.\frac{\partial X}{\partial z}\right|_L = \frac{g\rho^{2/3} a\delta z^2}{2CX} \quad B \quad . \tag{112}$$

Our justification for carrying $\rho^{1/3}$ through the derivative is that we have already assumed adjustments in the flow field to occur which cancel density variations for B=0, in our discussion of the adiabatic lapse rate case. The constant C relates Γ to Ua, i.e., $\Gamma = CUa$.

8.2 Effect of Internally Generated Buoyancy

When dissociated fireball atoms combine to form molecules heat energy is released, which increases the buoyancy of the region. Water vapor previously entrained from the ground, lagoon, or along with ambient air can also condense, releasing considerable quantities of heat energy.

These cases appear much more difficult to handle than that just discussed. The conceptual problem arises because release of dissociation or condensation energy may take place partly or wholly in regions remote from the axis. Thus there is no obvious effect upon either ∇p or ρ along the axis and therefore no obvious effect upon the circulation. On the other hand, one's intuition* is repelled by the concept that a sudden release of buoyancy at t=0 will cause the generation of a great deal of circulation whereas a release of buoyancy later will not affect the circulation, provided the region of release is a centimeter or two from the axis, independent of the amount of buoyancy released.

The paradox can be resolved qualitatively be remembering that the vortex moves as a unit, held together by dynamic forces which we may not understand at all well, but which nevertheless succeed in holding real vortices together in the real world. With this point in mind it seems clear that creation of a buoyant element anywhere in the vortex

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*At least this author's intuition.

causes creation of pressure gradients which do not destroy the integrity of the vortex (experimental fact) but which must cause the vortex to accelerate upwards precisely according to the amount of buoyancy released, averaged over the entire vortex. If we could calculate these gradients we would be able to apply Bjerknes' theorem to find the rate of increase of circulation. Since we have not yet succeeded in solving the problem we are forced to use a simple argument which we hope yields the correct dependency but which is likely to be wrong by a numerical factor.

We will assume that only the change in average density due to additional buoyancy is effective in circulation generation (we could reach the same answer using an equivalent argument based on ∇p). The change in density caused by release of energy δE followed by adiabatic expansion is

$$\frac{\nabla\rho}{\rho} = \frac{-1}{\gamma}\frac{\delta E}{E} \tag{113}$$

The residual in Bjerknes' theorm is, since we assume all other factors must cancel,

$$\dot{\Gamma} = - g \int_B^T \frac{\Delta\rho}{\rho}\, dz = g \int_B^T \frac{\delta E}{\gamma E}\, dz \simeq \frac{2ag}{\gamma}\;\frac{\delta E}{E} \tag{114}$$

Now the question arises as to how long the circulation generation is operative. We know that when the vortex was originally generated from a buoyant bubble at rest, circulation increased during roughly the period required for the material to move up the axis, i.e., during the time required for the vortex to turn through a radian or two. Since our mechanism is not essentially different we can expect the appropriate time to be given by

$$\tau \sim a/U \quad . \tag{115}$$

Multiplication of Equation 114 by Equation 115 yields

$$\delta\Gamma \simeq \frac{2a^2 g}{\gamma U}\,\frac{\delta E}{E} \tag{116}$$

or

$$\delta X = \frac{\rho^{1/3}\,\delta\Gamma}{C} \simeq \frac{2a^3\rho^{2/3}\,g}{C\gamma X}\,\frac{\delta E}{E} \tag{117}$$

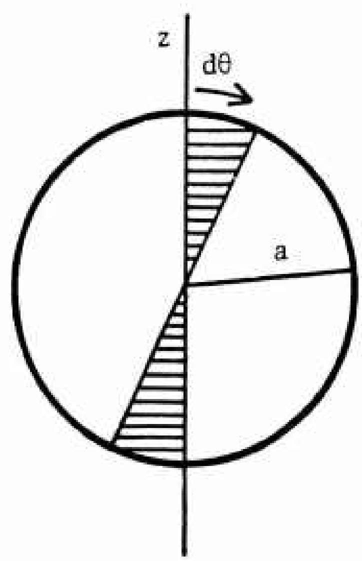
where C is the factor which relates Γ to Ua.

Replacing the δ's with derivatives with respect to z yields

$$\left.\frac{\partial X}{\partial z}\right|_H \simeq \frac{2g\rho^{2/3}a^3}{C\gamma X}\,\frac{1}{E}\,\frac{dE}{dz} \tag{118}$$

8.3 Effect of Wind Shear on X

On the average, the vortex should be transported with the mean horizontal (and vertical) velocity of the local wind. However, vertical gradients in the horizontal wind velocity can be expected to distort the vortex and destroy circulation.



The cross hatched area in the sketch illustrates the effect in question. Vorticity transported across the center-line will have the wrong sign on the new side and will cancel an equal amount of vorticity. Thus if the vorticity were uniformly distributed we would have

$$\delta\Gamma = -\,2\left(\frac{2}{\pi}\right)\frac{\delta\theta}{\pi}\,\Gamma \tag{119}$$

where the factor of 2 arises from the loss of vorticity on one side plus cancellation on the other side and the factor $\frac{2}{\pi}$ arises from an azimuthal average of the cross product of the vorticity vectors ($\langle\cos 2\phi\rangle$). The angle $\delta\theta$ is simply given by

$$\frac{d\theta}{dz} = \frac{1}{U}\left|\frac{d\vec{v}}{dz}\right| = \frac{1}{U}\sqrt{\left(\frac{\partial v_x}{\partial z}\right)^2 + \left(\frac{\partial v_y}{\partial x}\right)^2} \quad . \tag{120}$$

The switch from Γ to X is in this case direct, substituting X for Γ in Equation 119 and X for U in Equation 120 yields

$$\left.\frac{\partial X}{\partial z}\right|_S = -\frac{4\rho^{1/3}a}{\pi^2}\sqrt{\left(\frac{\partial v_x}{\partial z}\right)^2 + \left(\frac{\partial v_y}{\partial z}\right)^2} \quad . \tag{121}$$

We should not be surprised if Equation 121 yields somewhat too large an effect since the vorticity is likely to be concentrated away from the axis.

8.4 Equation for X

If we define

$$\xi \equiv \rho^{1/3} a \quad , \tag{122}$$

After the vortex is formed we anticipate an equation for X which has the form

$$\frac{dX}{dz} = \left.\frac{\partial X}{\partial z}\right|_L + \left.\frac{\partial X}{\partial z}\right|_H + \left.\frac{\partial X}{\partial z}\right|_S$$

$$= \frac{g\xi^2}{CX}\left(\frac{\delta z^2}{2a}B + \frac{2a}{\gamma}\frac{1}{E}\frac{dE}{dz}\right) - \frac{4\xi}{\pi^2}\sqrt{\left(\frac{\partial v_x}{\partial z}\right)^2 + \left(\frac{\partial v_y}{\partial z}\right)^2} \quad . \tag{123}$$

Unfortunately, at the present state of knowledge, each of the terms in Equation 123 may be expected to be in error by a multiplicative factor of order unity (at best).

8.5 Equations for α, ξ

The momentum equation (75) becomes

$$\frac{d}{dz}(\xi^2 X) = \frac{2ga^3\Delta}{3X}\xi \tag{124}$$

and the entrainment equation (76) becomes

$$\frac{d}{dz}\xi = \rho^{1/3}\alpha/f_v \tag{125}$$

If we carry out the indicated differentiation in Equation 124 and substitute Equation 126 we find

$$\alpha = \frac{f_v ga^3\Delta}{3\rho^{1/3}X^2} - \frac{f_v\xi}{2\rho^{1/3}X}\frac{dX}{dz} \tag{126}$$

The first term is the same as we found for incompressible flow and assumed for compressible flow in an adiabatic lapse rate, except that now $a^3\Delta$ may vary. The second term is required by momentum balance provided our assumption for the adiabatic lapse rate case is correct , and time similarity is still a good assumption.

Substitution of Equation 126 into 125 yields the equation for ξ.

$$\frac{d}{dz}\xi = \frac{ga^3\Delta}{3X^2} - \frac{\xi}{2X}\frac{dX}{dz} \quad . \tag{127}$$

8.6 Equation for Mass Discrepancy

Except for the term in the mass discrepancy, Δa^3, the Equations 123 and 127 form the complete set necessary to solve a problem. We now write the equation for the mass discrepancy term,

$$\frac{d}{dz}(a^3\Delta) = \frac{d}{dz}(a^3\rho_a - a^3\rho) = a^3\frac{d\rho_a}{dz} + \rho_a\frac{da^3}{dz} - 3a^3\rho_a\alpha/f_v \qquad (128)$$

the last term is obtained from the entrainment Equation 76.

The derivative of volume, or a^3, has three contributions,

$$\frac{d}{dz}a^3 = \frac{\partial}{\partial z}a^3\Big|_{adiabatic} + \frac{\partial}{\partial z}a^3\Big|_{heat} + \frac{\partial}{\partial z}a^3\Big|_{entrainment} \qquad (129)$$

where

$$\frac{\partial}{\partial z}a^3\Big|_{adiabatic} = -\frac{a^3}{\gamma p}\frac{dp}{dz} \qquad (130)$$

$$\frac{\partial}{\partial z}a^3\Big|_{heat} = \frac{a^3}{\gamma E}\frac{dE}{dz} \qquad (131)$$

$$\frac{\partial}{\partial z}a^3\Big|_{entrainment} = 3a^2\alpha/f_v \quad . \qquad (132)$$

Substitution of Equations 130, 131, and 132 into 128 yields

$$\frac{d}{dz}(a^3\Delta) = \xi^3\left(B + \frac{1}{\gamma E}\frac{dE}{dz}\right) \quad . \qquad (133)$$

This completes the set of equations necessary to solve the vortex motion, given initial conditions on X, ξ, and $a^3\Delta$.

8.7 Removal of the Assumption, $\rho \simeq \rho_a$

The careful reader will note that in the equations defining X (86) and ξ (122) we used the symbol ρ with the understanding that it could be regarded either as mean vortex density or as ambient density, the two being considered near enough to equality.

If one actually applies this theory numerically one must decide which density to use to initialize X and ξ, and do it at zero time when the densities are very far from equal. In practice we have used ρ_a for both of these quantities but one wonders whether the early stages of the calculation, when $\rho \neq \rho_a$ cause a significant effect on the final answer.*

The most straightforward method of answering this question is to remove the approximation $\rho \simeq \rho_a$. We proceed to do this. So far, in practice it has not yielded significantly different results from previous equations embodying the approximation. We present it for completeness.

Define

$$\xi \equiv \rho_a^{1/3} a \quad , \quad X \equiv \rho_a^{1/3} aU \ , \ \beta \equiv a^3\Delta = (\rho_a - \rho)a^3 \ . \tag{134}$$

The entrainment equation (76) becomes

$$\frac{d}{dz}(\xi^3 - \beta) = 3\rho_a^{1/3} \xi^2 \alpha/f_v \quad , \tag{135}$$

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* Beyond a timing error comparable to torus time and an altitude error comparable to the initial value of a.

or

$$\frac{d\xi}{dz} = \rho_a^{1/3} \quad \alpha/f_V + \frac{1}{3\xi^2}\frac{d\beta}{dz} \tag{136}$$

and the impulse equation becomes*

$$\frac{d}{dz}\left[(\rho + \rho_a/2)a^3U\right] = \frac{ga^3\Delta}{U} \tag{137}$$

or

$$\frac{d}{dz}\left[\xi^2X - \frac{2\beta}{3\xi} \quad X\right] = \frac{2g\beta\xi}{3X} \tag{138}$$

Carry out the indicated differentiation in Equation 138 and make use of Equation 135 to solve for α

$$\alpha = \frac{f_V}{\rho_a^{1/3}\left(1+\frac{\beta}{3\xi^3}\right)}\left[\frac{g\beta}{3X^2} - \frac{\xi\left(1 - \frac{2\beta}{3\xi^3}\right)}{2X}\frac{dX}{dz} - \frac{\beta}{9\xi^5}\frac{d\beta}{dz}\right] \tag{139}$$

When $\beta/\xi^3 = \Delta/\rho_a$ is small, the first two terms on the right of Equation 139 yield the previous result, Equation 126. It is not obvious that the third term is negligible but estimates for atmospheric cases indicate it to lie in the range of 10^{-1} to 10^{-3} compared to the first term.

The equation for α (139) and the equation for ξ (136) are the necessary modifications to include the case $\rho \neq \rho_a$. As was previously pointed out, we have not found a case where this change caused a significant numerical effect in any observable quantity.

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* See discussion in Section 5 for the factoring of impulse into interior and exterior terms.

9.0 INITIAL CONDITIONS

We must invent some way to start a calculation. The simplest method is to derive values of a, X, ξ, and β from scaling laws or simple theory and set these values at burst time and altitude. Thus the early motion, presumably up until about torus time, is treated as instantaneous. Clearly this is inadequate if one is interested in early times, and straightforward to correct. However, if one is interested in intermediate or late motion it introduces an error in altitude of at most the order of the initial radius and in time of at most the order of torus time, both small perturbations.

9.1 Free Air Shock Circulation

The circulation generated by reaction forces in giving off a shock by a free air burst in an exponential atmosphere has been estimated.[6] The formula is

$$\Gamma_r' = 4.2 \times 10^5\ (\mathrm{sec}^{-1})\ a_0^2/H \quad , \tag{140}$$

where a_0 is pressure equilibrium fireball radius as usual and H is atmospheric e-folding distance. In terms of X, if we assume a factor of 5 scaling from circulation to Ua, we find

$$X_r' = 8.4 \times 10^4 \rho_a^{1/3}\ a_0^2/H \tag{141}$$

We have placed primes to indicate that these values of Γ and X are not valid for all values of a_0.

(U) The formula (141) results from a first order expansion treatment. The largest expansion parameter reaches a value of 0.1 when $a_0/H = 3.9 \times 10^{-2}$ (or $a_0 \approx 250$ m). For larger pressure equilibrium radii the formula becomes unreliable. We need to invent some way to extend the formula.

A simple method to try is based upon the fact that Equation 141 will yield too large a value for mean rise velocity when $a_0 \sim H$, by a factor of 3 or 4. We might try formula (141) until a_0 is, say twice as big as the value quoted above, then simply hold rise velocity constant for the next factor of 3 or 4 in a_0. That is, we take

$$X_r = \begin{cases} 8.4 \times 10^4 \rho_a^{1/3} \quad a_0^2/H & a_0 < a_1 \\ \\ 8.4 \times 10^4 \rho_a^{1/3} \quad a_0 a_1/H & a_1 \leq a_0 \lesssim 4a_1 \end{cases} \qquad (142)$$

where

$$a_1 \approx 5 \times 10^4 \text{ cm or } \frac{1}{2} \text{ km}.$$

This argument seems fairly reasonable and turns out to give fairly good agreement with data up to the limit $4a_1$ but should be verified (improved).

9.2 Reflected Shock Circulation

No very convincing calculation of the effect on circulation of a shock traversal of a vortex, or even a spherical fireball, exists. We will develop a formula which might be worth trying but which should be viewed with great skepticism.

Brian Murphy has developed a formula for the displacement velocity, V_D, of the edge of a spherical pressure equilibrium bubble struck by a plane shock. The formula, in our units, is

$$V_D = C_a \frac{\Delta}{\rho_a + \rho} f(x) \quad , \tag{143}$$

where

$$x = 1.135 \; z_d \; (p/E_T)^{1/3} \tag{144}$$

C_a is ambient sound speed, z_d(cm) is distance from the center of the image explosion (or real explosion in a multiburst situation), p (dynes) is ambient pressure, E_T (erg) is the total yield of the image explosion, and

$$f(x) = \frac{.0814\, x^{-3} + .2086 x^{-3/2}}{[1 + .096 x^{-3} + .2504 x^{-3/2}]^{1/2}} \quad . \tag{145}$$

To the degree of approximation possible we can assume $f_r V_D$ is given to all of the material inside the fireball, where $f_r \sim 1$. The increment of momentum delivered to the fireball material is then

$$\delta P_r = \frac{4}{3} \pi \rho a^3 f_r V_D \quad . \tag{146}$$

By the time the vortex flow field has readjusted outside as well as inside, this increment in momentum will result in an increment in vortex rise velocity given by momentum conservation,

$$\frac{4}{3}\pi(\rho + \rho_a/2)\ a^3 \delta U_r = \delta P_r \quad , \tag{147}$$

or

$$\delta U_G = \frac{\rho}{\rho+\rho_a/2}\ f_r V_D = f_r C_a \frac{(\rho_a-\rho)\rho}{\left(\rho_a+\rho\right)\left(\rho+\frac{1}{2}\rho_a\right)}\ f(x) \quad . \tag{148}$$

and therefore the reflected shock contribution to X is

$$\delta X_G = f_r \rho_a^{1/3}\ C_a a \frac{(\rho_a-\rho)\rho}{\left(\rho_a+\rho\right)\left(\rho+\frac{1}{2}\rho_a\right)}\ f(x) \quad . \tag{149}$$

The formula (149) has the satisfactory property of going to zero as $\Delta \to 0$ but the unsatisfactory property of going to zero as $\rho/\rho_a \to 0$. In fact δX is a maximum with respect to ρ/ρ_a at $\rho/\rho_a \simeq 0.4$.

Perhaps this is reasonable but it is certainly not obvious.

Another unsatisfactory aspect of Equation 149 is that it diverges for $z_d = 0$. The circulation of a ground burst may indeed differ from that of a free air burst of the same yield but it is not at all obvious whether it is greater or less. That is, the mass and buoyancy are about the same, a strong reaction force due to the ground should tend to add circulation, but the development of the flow field may be severely hampered by the presence of an impenetrable boundary.

For the present we will suggest that $f(x)$ should be held to a finite value or reduced to zero as burst height becomes less than a. Clearly, this is a point which deserves some attention.

Digression to Multibursts

Here we digress to a more general consideration. The ground effect treatment could form the basis for a multiburst shock effect if we generalize our theory slightly. First calculate the value of the shock induced circulation, X_s, according to Equation 149 but with z_d in Equation 144 replaced by r, where r is the distance between the center of the vortex and the new explosion. Let X be a vector. The contribution $\vec{X}_s$ is a vector pointing along the vector $\vec{r}$. Add X_s to X vectorally. Equation 123 becomes the equation for the vertical component of X. Whenever X enters an equation for α, ξ, or β, use $|X|$. The equation for the horizontal component of X is $\dot{X}_H = 0$.

We have not tried this, but presumably comparisons could be made to Reference 8.

9.3 Buoyancy Circulation

Calculations have been carried out for an initially spherical bubble rising under buoyant forces for the extreme case of zero internal density, pressure equilibrium, incompressible, inviscid flow.[6] These calculations yield

$$\Gamma_B \cong 4.5\ a_0 \sqrt{g a_0} \quad , \tag{150}$$

and

$$U_B \cong 0.8 \sqrt{g a_0} \quad . \tag{151}$$

The method of calculation does not converge all the way to torus time. The above numerical values accordingly are extrapolations albeit rather safe ones because Γ and U are approaching constants at the end of the calculation.

The ratio

$$\frac{\Gamma_B}{U_B a_0} = \frac{\Gamma}{U a f_v^{1/3}} = 5/f_v^{1/3} = 5.62 \tag{152}$$

according to Equations 150 and 151. This implies $f_v = .70$ at the end of the calculation (since $\Gamma \sim 5Ua$) and therefore the final, time similar form may have been approached. We cannot make a stronger statement because the ratio Γ/Ua is not, at present, known to sufficient accuracy.

If buoyancy were the only source of early circulation we could write a formula for X at this point. However, it must be true that blast reaction and ground interaction interfere with the mechanism of circulation generation by buoyant forces.

To make this apparent, and to find a way to estimate this interference, we will develop a simplified theory of buoyant circulation.

A simple approximation is to regard the vortex as spherical with constant radius, a_0, until it rises through a distance like a_0 (i.e., until it turns itself through about π radians).

The impulse equation needs to represent contributions from fireball material as well as the exterior flow field,

$$\frac{d}{dt} [(\rho + \rho_a/2) a_0^3 U] = g a_0^3 \Delta \quad . \tag{153}$$

We can treat everything except U and t as constant in Equation 153, then

$$U = \frac{g\Delta}{(\rho + \rho_a/2)} t + U_o \tag{154}$$

where we have introduced an initial value, U_o, to represent vortex velocity imparted by some other mechanism, such as blast reaction or ground interaction.

The rise height is obtained by integrating Equation 154,

$$z = \frac{g\Delta}{(\rho_a + 2\rho)} t^2 + U_o t \quad , \tag{155}$$

The torus presumably forms and cuts off generation of circulation when $z \sim a_0$. The torus time, τ, is obtained by equating z to a_0 in Equation 155 and solving for t,

$$\tau = \frac{(\rho_a + 2\rho)U_o}{2g\Delta} \left(\sqrt{1 + \frac{4ga_0\Delta}{(\rho_a + 2\rho)U_o^2}} - 1 \right) \quad . \tag{156}$$

Limiting behavior of Equation 156 is,

$$\lim_{U_o \to 0} \tau = \sqrt{\frac{(\rho_a + 2\rho)a_0}{g\Delta}} \tag{157}$$

and

$$\lim_{U_o \to \infty} \tau = a_0/U_o \quad . \tag{158}$$

The formula for U at torus time is

$$U = \frac{g\Delta}{(\rho + \rho_a/2)} \tau + U_o \tag{159}$$

and the corresponding value of X is

$$X = \rho_a^{1/3} a_0 U = 2 \frac{\rho_a^{1/3} g a_0 \Delta}{(\rho_a + 2\rho)} \tau + X_r + X_G \quad . \tag{160}$$

We can normalize our buoyant contribution to X by forcing the limit as $U_o \to 0$ to match the results of Reference 6, (Equation 150).

Let

$$X_B = N \frac{\rho_a^{1/3} g a_0 \Delta}{(\rho_a + 2\rho)} \tau \tag{161}$$

and require

$$\lim_{U_o \to 0} X_B = \rho_a^{1/3} \Gamma_B / 5 \tag{162}$$

Substitution of Equation 157 into 161 and Equation 150 into 162 yields

$$N = 4.5/5 = 0.9 \quad , \tag{163}$$

after we take the limit $\rho \to 0$, consistent with Equation 150.

Our formula now reads

$$X_o = X_B + X_r + X_G \quad , \tag{164}$$

where

$$X_B = 0.9 \frac{\rho_a^{1/3} g a_0 \Delta}{(\rho_a + 2\rho)} \tau \tag{165}$$

and τ is given by Equation (156)

The effect of this treatment is to reduce the buoyant contribution to X to some extent whenever some other source of circulation acts promptly, the tendency will be to partially counterbalance effects of other sources on the long term rise history.

We should point out that an attempt at precise treatment of the interaction of these several sources of circulation is formidable. Fortunately, at least the major approximation made here, that both the blast reaction and ground shock occur prior to the time significant circulation has been buoyantly generated, is a good one whenever these effects are significant.

10.0 ENDING THE CALCULATION

One has the intuition that the time similarity assumption must break down after the mass discrepancy, Δa^3, goes negative (i.e., when the fireball becomes more dense than its surroundings). In addition, zero buoyancy is a basic condition for stabilization.

One's first inclination is therefore to stop calculating when Δa^3 equals zero and conclude that the correspondirg altitude is approximately equal to stabilization altitude.

However, if one looks at numerical results he finds that the momentum of the vortex is generally adequate to carry the vortex well past the altitude where the mass discrepancy goes to zero, with any rational entrainment assumption. Further, the combination of entrainment and expansion reduce the vortex density with adequate efficiency that it cannot fall back a very large fraction of the overshoot distance.

Rather than embark on an effort to develop a new theory for this regime we choose, for the first attempt at least, to see how far the present theory can be pushed. Accordingly we will continue calculating until either the mass goes to zero, due to a negative α, or the velocity goes to zero ($X = 0$). Stabilization will be assumed to occur when the mass has been reduced to half its original value, since we are after mean altitude, or when X goes to zero in cases where α does not go negative. For consistency we should reduce X as mass is detrained, we will do this on the basis of cross sectional area (a^2). Perhaps we should also consider the effect of this contribution to dX/dz on α but it is not worth the effort until one has some evidence that the model resembles reality in this regime. If this method were correct we could hope to find not only the debris altitude and extent versus time during the rise, but also the final vertical extent of the stabilized cloud. The lowest point would be where α goes negative and the highest extent where the mass goes to zero. One would be very surprised if nature were so cooperative.

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