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VEHICLE ROUTING PROBLEMS: FORMULATIONS AND HEURISTIC
SOLUTION TECHNIQUES

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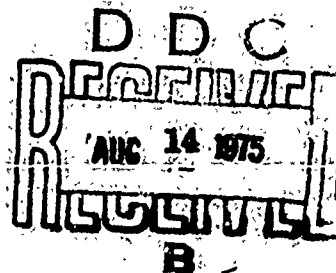
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**VEHICLE ROUTING PROBLEMS:
FORMULATIONS AND HEURISTIC SOLUTION
TECHNIQUES**

by
BRUCE L. GOLDEN

**Technical Report No. 113
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FOREWORD

The Operations Research Center at the Massachusetts Institute of Technology is an interdepartmental activity devoted to graduate education and research in the field of operations research. The work of the Center is supported, in part, by government contracts and industrial grants-in-aid. The work reported herein was supported (in part) by the Office of Naval Research under Contract N00014-75-C-0556.

John D.C. Little
Director

ABSTRACT

An essential element of the newspaper logistics system is the allocation and routing of vehicles for the purpose of delivering newspapers on a daily basis. In this paper, we present various vehicle routing problems. Formulations defining the mathematical models are discussed in conjunction with several widely-used heuristic solution techniques. The focus is on providing a unified framework for these very difficult combinatorial programming problems.

I. FORMULATIONS

Introduction

Logistics is concerned with the effective management of the total flow of goods, from the acquisition of raw materials to the delivery of finished products to the final customer. An essential element of the newspaper logistics system is the allocation and routing of vehicles for the purpose of delivering newspapers on a daily basis. In this report, we present various vehicle routing problems. Formulations defining the mathematical models are discussed in conjunction with several widely-used heuristic solution techniques. The intention is to complement the survey paper by Gabbay [12] which presented an overview of vehicle routing problems by focusing upon node-routing and branch-routing components. In addition, while these difficult combinatorial problems are frequently discussed verbally in the literature, precise mathematical formulations are not readily available (Pierce [29] gives formulations in his special survey paper which are neither linear nor integer programs). We hope to correct this situation.

Typically, the newspaper vehicle routing problem is of the node-routing variety. Vehicle routing problems, sometimes referred to as truck-dispatching problems, are almost always encountered by complex organizations, and reliable procedures for dealing with them are of primary importance. This is especially true in the newspaper industry where yesterday's product is worthless today. Recently, in response to higher vehicle costs due to increased oil prices and rising truck drivers' salaries, these issues have been receiving more and more attention.

There may be several hundred delivery points in and around a city, each with specific demands (in quantity of newspapers). In addition, there may be delivery deadlines or earliest delivery time constraints. The fleet of trucks may contain different types of trucks with different capacities. There may be several afternoon editions of a newspaper. In the future, it is conceivable that each edition may have several distinguishable products. That is, each edition may well represent a multi-product newspaper. There may be a common base of news coverage in all the newspapers of a particular edition, but one product may emphasize sports, another finance, another international news, another leisure and the arts, and so on. Several newspapers with widespread distribution already are publishing regional issues to distinguish customers by geography.

The objectives involve minimizing the number of vehicles required in the fleet, minimizing travel time by vehicles, increasing circulation, and generally providing efficient service in order to deliver newspapers from the press to the streets and ultimately to the people of a community as quickly and cost-effectively as possible. All vehicles depart from the central depot, make a tour of a subset of the demand nodes, and return to the central depot. We have been referring to the specific newspaper vehicle routing problem. However, it should be clear that all vehicle routing problems are more or less the same. As an illustration we mention the application of vehicle routing techniques to municipal waste collection by Beltrami and Bodin [5], where instead of making a delivery we perform a pickup at each collection point. Another such example is the routing of school buses. Operationally the problems may seem different, but theoretically they can be thought of as equivalent.

Proposed techniques for solving problems of this sort have fallen into two classes - those which solve the problem optimally by branch and bound techniques [11], [29], and those which solve the problem heuristically [9], [10], [14], [16], [18], [24], [30], [34], [35], [38]. In a loose sense, heuristic algorithms represent sets of rules which produce good solutions to given combinatorial programming problems, but not necessarily the best possible (optimal) solutions. Since the optimal algorithms are viable only for very small problems, we prefer to concentrate in this paper on the study of several heuristic algorithms.

The Traveling Salesman Problem

In this section we discuss the ubiquitous Traveling Salesman Problem; for a very thorough overview see Bellmore and Nemhauser [4]. Suppose we are given the matrix of pairwise costs or distances c_{ij} between node i and node j for the n nodes $1, 2, \dots, n$. We assume $c_{ii} = \infty$ for $i = 1, 2, \dots, n$. The problem is to form a tour of the n nodes beginning and ending at node 1 (which we refer to as the origin) in such a fashion that the minimum total cost or distance tour results. We present the integer programming formulation below.

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \end{array} \quad (1.1)$$

$$\begin{array}{ll} \text{subject to} & \sum_{i=1}^n x_{ij} = 1, \quad j = 1, \dots, n \end{array} \quad (1.2)$$

$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, n \quad (1.3)$$

$$\sum_{i \in Q} \sum_{j \in \bar{Q}} x_{ij} \geq 1, \text{ every } Q \subset V, Q \neq \emptyset \quad (1.4)$$

$$x_{ij} = 0, 1, \quad i, j = 1, \dots, n \quad (1.5)$$

where n = the number of nodes in the network

V = the set of nodes

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is in the tour} \\ 0 & \text{otherwise} \end{cases}$$

c_{ij} = the cost on arc (i, j)

Q = a subset of the nodes

$\bar{Q} = V - Q.$

Note that there are $2^n - 2$ subtour constraints. The Traveling Salesman Problem has probably received more attention than any other problem in the Operations Research literature.

Equation (1.1) states that we minimize total cost. Equations (1.2) and (1.3) represent the fact that there must be one arc into and out of every node. Equations (1.4) are the subtour-breaking constraints. Alternative, and rather ingenious subtour-breaking constraints were proposed by Miller, Tucker, and Zemlin [27]. If we replace (1.4) by $y_i - y_j + nx_{ij} \leq n - 1$, for $2 \leq i \neq j \leq n$ where the variables y_i are arbitrary real numbers, we reduce the number of constraints from $2^n - 2$ to $n^2 - 3n + 2$.

The simplest vehicle routing problem occurs when the capacity of a vehicle exceeds the total quantity demanded at the n nodes and there are no time constraints, in which case we have a Traveling Salesman Problem. For optimal and very powerful heuristic approaches to the Traveling Salesman

Problem see Held and Karp [19], [20], Lin [25], Lin and Kernighan [25], and Christofides and Eilon [7].

The Multiple Traveling Salesmen Problem

The Multiple Traveling Salesmen Problem (MTSP) is a generalization of the Traveling Salesman Problem (TSP) and comes closer to accomodating more real-world problems; here there is a need to account for more than one salesman. Multiple Traveling Salesmen Problems arise in many sorts of scheduling and sequencing applications. For example, the framework could be used to develop the basic route structure for a pickup or delivery service (perhaps a school bus or rural bus service); it has proved to be an appropriate model for the problem of bank messenger scheduling, where a crew of messengers picks up deposits at branch banks and returns them to the central office for processing [32].

Given m salesmen and n nodes in a network the MTSP is to find m sub-tours (each of which includes the origin) such that every node (except origin) is visited exactly once by exactly one salesman, so that the total distance traveled by all m salesmen is minimum. A MTSP formulation is displayed below.

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \end{array} \quad (2.1)$$

$$\begin{array}{ll} \text{subject to} & \sum_{i=1}^n x_{ij} = b_j = \begin{cases} m & \text{if } j = 1 \\ 1 & \text{if } j = 2, 3, \dots, n \end{cases} \end{array} \quad (2.2)$$

$$\sum_{j=1}^n x_{ij} = a_i = \begin{cases} m & \text{if } i = 1 \\ 1 & \text{if } i = 2, 3, \dots, n \end{cases} \quad (2.3)$$

$$\sum_{i \in Q} \sum_{j \in Q} x_{ij} \geq 1, \text{ every } Q \subset V, Q \neq \emptyset \quad (2.4)$$

$$x_{ij} = 0, 1, \quad i, j = 1, \dots, n \quad (2.5)$$

Svestka and Huckfeldt present constraints for the MTSP which resemble the Miller-Tucker-Zemlin formulation of the TSP [32], and apply a subtour elimination type branch and bound procedure using Bellmore and Malone branching to obtain the optimal solution; mean run time for 55 city problems is one minute. Three different papers published in 1973 and 1974 independently derived equivalent TSP formulations of the MTSP [3], [28], [32]. We describe this now with the motivation that the m-salesmen problem is no more difficult to solve than its one-salesman counterpart.

In equations (1.2) and (1.3) we notice assignment problem constraints. Equations (2.2) and (2.3) typify more general transportation constraints (a transportation problem with right-hand-side values of unity becomes an assignment problem if the number of supply nodes and demand nodes is the same). An approach for solving the TSP and the MTSP is to first solve the associated assignment problem or transportation problem. If, as a result, the appropriate equations (1.4) or (2.4) are satisfied, terminate. Otherwise, a subset of the violated constraints are implicitly introduced into the problem.

If we decompose node 1 (the origin) into nodes 1, 2, ..., m representing the origins for the m salesmen, we now have an expanded network of m+n-1 nodes with the augmented cost matrix displayed below.

	1	2... m-1	m
1			
2			
...			
...			
m-1			
m			
	infinity	Copy of row 1	
	copy of column 1	Original Cost Matrix C	

Figure I. Augmented Cost Matrix C^N .

We have transformed the original MTSP to the following integer programming problem:

$$\text{minimize} \quad \sum_{i=1}^{m+n-1} \sum_{j=1}^{m+n-1} c_{ij}^N x_{ij} \quad (3.1)$$

$$\text{subject to} \quad \sum_{i=1}^{m+n-1} x_{ij} = 1, \quad j = 1, \dots, m+n-1 \quad (3.2)$$

$$\sum_{j=1}^{m+n-1} x_{ij} = 1, \quad i = 1, \dots, m+n-1 \quad (3.3)$$

$$\sum_{i \in Q} \sum_{j \in \bar{Q}} x_{ij} \geq 1, \quad \text{every } Q \subset V^N, Q \neq \emptyset \quad (3.4)$$

$$x_{ij} = 0, 1, \quad i, j = 1, \dots, m+n-1 \quad (3.5)$$

where the coefficients and variables refer to the expanded network; V^N is the set of nodes in the expanded network. Thus, we have shown that the MTSP can be solved by solving the standard TSP on an expanded network where the number of nodes is increased by $m-1$; m is the number of salesmen available.

For example, consider a four-city problem with two salesmen. Originally we have nodes 1,2,3, and 4 with node 1 as the origin or home base. Our expanded network is comprised of nodes 1,2,3,4, and 5 where nodes 1 and 2 stand for the home city and nodes 3,4, and 5 correspond to the original cities 2,3, and 4 respectively. In Figure II, we see that tour 13245 in the expanded network is equivalent in the two-salesmen interpretation to subtours 12 and 134.

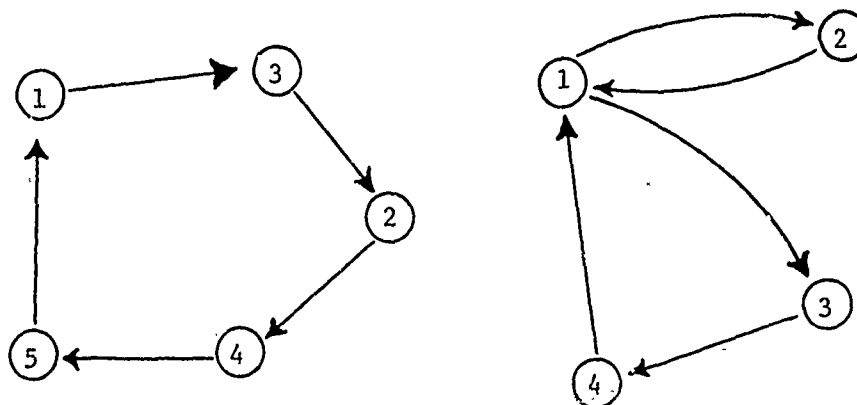


Figure II. A sample tour for a four-city two-salesmen problem in expanded and original network.

We notice that $c_{ij} = \infty$ for $i \in \{1,2,\dots, m\}$ and $j \in \{1,2,\dots, m\}$ in the augmented cost matrix; this insures that exactly m tours will be formed. There is potential benefit from treating $c_{ij} = \lambda$ for $i \neq j$ and $i,j \in \{1,2,\dots, m\}$, in other words by treating these costs as being equal to a parameter λ which we can vary. For instance, if we set $\lambda = -\infty$ then the TSP on the extended network will yield the minimum cost set of tours with a minimum number of tours since there is a premium placed on returning to the central depot

directly from the central depot. As a consequence, we minimize the number of vehicles required, albeit at the expense of total travel time. If on the other hand $\lambda = 0$, then we minimize total travel time subject to the constraint that there are at most m tours. In any real-world application it might be desirable to solve this converted TSP for a range of values for the parameter λ and select the most attractive routing strategy from among these candidates. This bridge between the two objectives of minimizing total travel time and minimizing the number of tours required was suggested recently by Christofides in his excellent survey [8].

If distance measures on the arcs satisfy the triangle inequality, then for $\lambda \leq 0$ the optimal strategy will entail only one tour. In any case, for $\lambda = -\infty$, only one tour will result.

The Truck Dispatching Problem

The truck dispatching problem was first considered by Dantzig and Ramser [10] who developed a heuristic approach using linear programming ideas and aggregation of nodes. The problem is to obtain a set of delivery routes from a central depot to the various demand points each of which has known requirements, which minimizes the total distance covered by the entire fleet. Trucks have capacities and maximum route time constraints. All trucks start and finish at the central depot. We will formulate this now and refer to it as the generic vehicle routing problem. As far as the author knows, complete and accurate linear-integer programming formulations for this subset of combinatorial programming problems do not exist in the literature.

$$\text{minimize } z = \sum_{i=1}^{NN} \sum_{j=1}^{NN} \sum_{k=1}^{NV} t_{ij} \ell_{ijk} \quad (4.1)$$

$$\text{subject to } \sum_{i=1}^{NN} \sum_{k=1}^{NV} \ell_{ijk} = 1, \quad j = 2, \dots, NN \quad (4.2)$$

$$\sum_{j=1}^{NN} \sum_{k=1}^{NV} \ell_{ijk} = 1, \quad i = 2, \dots, NN \quad (4.3)$$

$$\sum_{i=1}^{NN} \ell_{ipk} - \sum_{j=1}^{NN} \ell_{pjk} = 0, \quad k = 1, \dots, NV \quad (4.4)$$

$$p = 1, \dots, NN$$

$$p_k - \sum_{i=1}^{NN} \{Q_i \sum_{j=1}^{NN} \ell_{ijk}\} \geq 0, \quad k = 1, \dots, NV \quad (4.5)$$

$$T_k - \sum_{i=1}^{NN} \{t_i \sum_{j=1}^{NN} \ell_{ijk}\} + \sum_{i=1}^{NN} \sum_{j=1}^{NN} t_{ij} \ell_{ijk} \geq 0, \quad k = 1, \dots, NV \quad (4.6)$$

$$\sum_{j=2}^{NN} \ell_{1jk} \leq 1, \quad k = 1, \dots, NV \quad (4.7)$$

$$\sum_{i=2}^{NN} \ell_{i1k} \leq 1, \quad k = 1, \dots, NV \quad (4.8)$$

$$u_i - u_j + \sum_{k=1}^{NV} \ell_{ijk} \leq NN - 1, \quad 2 \leq i \neq j \leq NN \quad (4.9)$$

$$\ell_{ijk} \in \{0,1\} \quad \text{for all } i,j,k \quad (4.10)$$

where $NN = \#$ of nodes

$NV = \#$ of vehicles

$u_i =$ arbitrary real numbers which satisfy constraints (4.9)

$p_k =$ capacity of vehicle k

$T_k =$ maximum time allowed for route of vehicle k

Q_i = demand at node i ($Q_1 = 0$)

t_i = time required to deliver or collect at node i ($t_1 = 0$)

t_{ij} = travel time from node i to node j ($t_{ii} = \infty$)

$l_{ijk} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is traversed by truck } k \\ 0 & \text{otherwise.} \end{cases}$

Equation (4.1) states that total travel time is to be minimized. Equations (4.2) and (4.3) insure that each demand node is served by some truck and only one truck. Route continuity is represented by equations (4.4), i.e., if a truck enters a demand node, it must exit from that node. Equations (4.5) are the truck capacity constraints; similarly, equations (4.6) are the total elapsed route time constraints. For instance, a newspaper delivery truck may be restricted from spending more than one hour on a tour in order that the maximum time interval from press to street be made as short as possible. Equations (4.7) and (4.8) make certain that truck availability is not exceeded. Finally, the subtour-breaking constraints (4.9) here are of the Miller-Tucker-Zemlin variety. Since (4.2) and (4.4) imply (4.3), and (4.4) and (4.7) imply (4.8), from now on we consider the generic model to include (4.1) - (4.10) excluding (4.3) and (4.8), which are redundant. We assume that $\max_{1 \leq i \leq NN} Q_i < \min_{1 \leq k \leq NN} P_k$. That is, the demand at each node does not exceed the capacity of any truck.

In our generic model we make the additional assumption that when a demand node is serviced, its requirements are satisfied. In other words, one visit is sufficient. A mixed integer programming heterogeneous fleet problem formulation was given in 1957 by Garvin [13] in which this assumption

is relaxed. This is the earliest paper in the literature concerned with vehicle routing problems as discussed here, and deals with an application in the oil industry. It does not, however, provide for maximum route time constraints. In addition, the number of variables is much greater than in the previous model. We present Garvin's formulation below.

$$\text{minimize} \quad \sum_{i,j,s} c_{ijs} x_{ijs} \quad (5.1)$$

$$\text{subject to} \quad \sum_i x_{ijs} = \sum_u x_{jus} \quad \text{for all } j,s \quad (5.2)$$

$$\sum_k y_{ijk} \leq \sum_s w_s x_{ijs} \quad \text{for all } i,j, i \neq j \quad (5.3)$$

$$\sum_i y_{ijk} = \sum_u y_{juk} \quad \text{for all } j,k, j \neq k \quad (5.4)$$

$$\sum_i y_{ikk} = v_k \quad \text{for all } k \quad (5.5)$$

$$\sum_{k,j} y_{ljk} = \sum_k v_k \quad (5.6)$$

$$x_{ijs} \in \{0,1,2,\dots\} \quad (5.7)$$

$$y_{ijk} \geq 0 \quad (5.8)$$

where c_{ijs} is the cost incurred by a carrier of type s in traveling from destination D_i to D_j , x_{ijs} is the number of carriers of type s which proceed from D_i to D_j , w_s is the capacity of a carrier of type s , v_i is the delivery

quantity at D_i , y_{ijk} is the quantity carried from D_i to D_j destined for D_k , and node 1 is the origin. NT is the number of types of vehicles.

In (5.1) we minimize the total travel cost. Equations (5.2) state that the number of carriers into and out of node j is the same. The quantity shipped from node i to node j destined for node k is less than the capacity of vehicles from node i to node j (equations (5.3)). Equations (5.4) state that the quantity shipped from node i to node j destined for node k ($k \neq j$) is equal to the quantity out of node j destined for node k . In (5.5) we find that the quantity from node i to node k destined for node k is equal to the demand at node k . Finally, constraints (5.6) tell that the quantity sent out from the origin is equal to the total demand.

As a unifying model, we prefer the primarily binary integer programming formulation (4) which has fewer variables and more types of constraints than (5). Both formulations have on the order of NN^2 constraints. Model (4) has structure which is more closely related to the fundamental Traveling Salesman Problem formulation (especially the Miller-Tucker-Zemlin formulation).

The Multicommodity Vehicle Routing Problem

We now generalize model (4) to the multicommodity case. In other words, there are now several different types of products which we must route simultaneously over a network in order to satisfy whatever demands may exist at the delivery points for the various products. We present the formulation of the Multicommodity Vehicle Routing Problem, an extension to our generic model, below.

$$\text{minimize } z = \sum_{i=1}^{NN} \sum_{j=1}^{NN} \sum_{k=1}^{NV} t_{ij} \ell_{ijk} \quad (6.1)$$

$$\text{subject to } \sum_{i=1}^{NN} \sum_{k=1}^{NV} \ell_{ijk} \geq 1, \quad j = 2, \dots, NN \quad (6.2)$$

$$\sum_{i=1}^{NN} \ell_{ipk} - \sum_{j=1}^{NN} \ell_{pjk} = 0, \quad k = 1, \dots, NV \quad (6.3)$$

$$p = 1, \dots, NN$$

$$p_k - \sum_{C=1}^{NC} \sum_{i=1}^{NN} \{Q_{iC} \sum_{j=1}^{NN} y_{ijk}^{(C)}\} \geq 0, \quad k = 1, \dots, NV \quad (6.4)$$

$$T_k - \sum_{i=1}^{NN} \{t_i \sum_{j=1}^{NN} \ell_{ijk}\} + \sum_{i=1}^{NN} \sum_{j=1}^{NN} t_{ij} \ell_{ijk} \geq 0, \quad k = 1, \dots, NV \quad (6.5)$$

$$\sum_{i=1}^{NN} \sum_{k=1}^{NV} y_{ijk}^{(C)} = S_{jC}, \quad j = 2, \dots, NN \quad (6.6)$$

$$C = 1, \dots, NC$$

$$\ell_{ijk} \geq \frac{1}{NC} \cdot \sum_{C=1}^{NC} y_{ijk}^{(C)}, \quad i = 1, \dots, NN \quad (6.7)$$

$$j = 1, \dots, NN$$

$$\sum_{j=2}^{NN} \ell_{1jk} \leq 1, \quad k = 1, \dots, NV \quad (6.8)$$

$$k = 1, \dots, NV$$

$$u_i - u_j + NN \sum_{k=1}^{NV} \ell_{ijk} \leq NN - 1, \quad 2 \leq i \neq j \leq NN \quad (6.9)$$

$$\ell_{ijk}, y_{ijk}^{(C)} \in \{0,1\} \quad (6.10)$$

where NC = # of commodities

$$S_{iC} = \begin{cases} 1 & \text{if demand for commodity C at node i} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ijk}^{(C)} = \begin{cases} 1 & \text{if commodity } C \text{ is carried on truck } k \text{ between nodes } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

Q_{iC} = demand for commodity C at node i .

Some comments regarding this formulation are in order. The model (6) is analogous to the one-product generic model in most respects. Equations (6.6) reflect that demands for commodities are satisfied. We assume that one truck load is sufficient to satisfy a commodity requirement at a demand node, and that

$$\max_{\substack{1 \leq i \leq NN \\ 1 \leq C \leq NC}} Q_{iC} < \min_{1 \leq k \leq NV} p_k.$$

Now suppose in model (6) that $NC = 1$. Then $y_{ijk}^{(1)} = x_{ijk}$, equations (6.7) become redundant, and we obtain model (4). Further suppose that the truck capacity constraints and maximum route time constraints are relaxed. We obtain the system (4.1), (4.2), (4.4), (4.7), (4.9), (4.10) which is a MTSP of the type where $\lambda = 0$. In other words, at most NV salesmen are used. Finally suppose that $NV = 1$. Then we have precisely the Miller-Tucker-Zemlin formulation of the Traveling Salesman Problem mentioned previously in this paper.

Extensions

We can incorporate timing restrictions into the truck dispatching model, however, these constraints will be nonlinear in nature. If we define a_j as the arrival time at node j then delivery deadlines and earliest delivery time constraints can be represented by the following equations:

$$a_j = \sum_k \sum_i (a_i + t_i + t_{ij}) l_{ijk}, \quad j = 1, \dots, NN$$

$$a_1 = 0$$

$$\underline{a}_j \leq a_j \leq \bar{a}_j, \quad j = 2, \dots, NN.$$

Newspapers must confront the additional complication that there is zero supply when the workday begins. As the day progresses, newspapers come off the press as some vehicles are in the process of making deliveries. The supply, therefore, is being produced in the same time interval that the routing must take place, and the newspapers must be delivered.

This complication can be included in our formulation through the multicommodity framework. That is, if we divide the workday into periods (as input we are given the amount of production in each period), we consider the production in period C to be commodity C, and define a_j^C as the arrival time at node j with commodity C (or edition C). a_1^C is specified as input, indicating that commodity C is available for distribution at the end of period C. These are our production constraints.

Concluding Comments on Formulations

Hopefully, the formulations discussed here provide some kind of unified basis for viewing these vehicle routing problems (1),(2),(4), and (6) as one sub-class of combinatorial programming problems. These linear-integer programming problems, though complex, are still too simplistic to account for all the real-world constraints encountered in practice. We saw how nonlinear constraints can handle earliest delivery times and delivery deadlines.

Alternatively, we might include these constraints explicitly by optimizing over permissible permutations of stops where variables specify the orderings in the tours. For example, Pierce [29] expresses the TSP as a minimization problem over possible permutations. Given nodes $1, 2, \dots, n$ where 1 is the origin, we seek an ordering of the stops $2, \dots, n$ such as to minimize

$$Z_{\theta} = \sum_{k=2}^{n+1} C(i_{k-1}, i_k) \text{ where } i_1 = i_{n+1} = 1 \text{ and } \theta = (i_1, i_2, \dots, i_n, i_{n+1}) \text{ is an}$$

augmented permutation of the integers 2 through n and $C(i_j, i_k)$ is the cost on arc (i_j, i_k) . Pierce formulates various extensions to the TSP in a similar manner. Perhaps the fact that the linear-integer programming formulations are only crude approximations to reality partially explains why little attention has been given to them in the literature. In any case, we feel they provide a point of departure from which good heuristics can be applied to obtain good feasible solutions to the real-world problems. In addition, they give insight into related scheduling projects such as demand responsive "Dial-A-Ride" transportation systems [36], and multiple depot vehicle dispatch problems [15], [33], [37].

II. HEURISTIC SOLUTION TECHNIQUES

Background

In section II of this report rather than summarizing all the various papers which have appeared in the literature on methods of solving vehicle scheduling problems, we will discuss in depth three heuristics which have been used on real-world problems. For more of a literature review we recommend Christofides [8].

Exact route enumeration methods exist [11],[29], however, Christofides claims that the largest vehicle routing problem of any complexity that has been solved exactly involved only 23 customers. A logical non-optimal approach involves heuristic algorithms based on the Traveling Salesman Problem (or n salesmen problem). The strategy would be to solve the TSP or MTSP first and then make modifications in order to satisfy capacity and other restrictions. However, the combinatorial simplicity of these problems, such as model (2), in comparison with the additional more realistic constraints which must be confronted sooner or later make this approach, we feel, not as attractive as others. The three vehicle routing heuristic methods which we will discuss are found in Clarke and Wright [9], Tyagi [35], and Gillett and Miller [16].

The Clarke-Wright Algorithm

Undoubtedly, the Clarke-Wright "savings" method is the most widely used and cited truck dispatching heuristic algorithm. Since the publication of their paper "Scheduling of Vehicles from a Central Depot to a Number of Delivery Points" in 1964, IBM has developed VSPX, a computer code designed

to handle complex vehicle routing problems. VSPX is an implementation of the Clarke-Wright approach which considers a multitude of constraint options [22]. Beltrami and Bodin [5] have recently employed a slight modification to treat the routing of garbage trucks. Christofides and Eilon found from 10 test problems that tours produced from the "savings" method averaged only 3.2 percent longer than the optimal tours [6]. Of course, one must remember that because the number of alternative routes and combinations of vehicles are growing exponentially we can obtain optimal tours only for very small problems. The method we are about to outline is remarkably simple and can be performed by hand in many cases.

The Clarke-Wright algorithm is an "exchange" algorithm in the sense that at each step one set of tours is exchanged for a better set of tours. Initially, we suppose that every two demand points i and j are supplied individually from two trucks (refer to Figure III below).

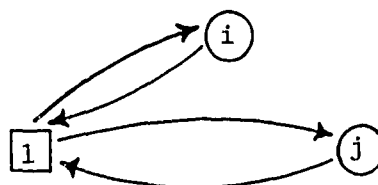


Figure III. Initial Setup.

Now if instead of two trucks, we used only one, then we would experience a savings in travel time of $(2d_{1i} + 2d_{1j}) - (d_{1i} + d_{1j} + d_{ij}) = d_{1i} + d_{1j} - d_{ij}$ (see Figure IV below). We assume symmetric travel times.

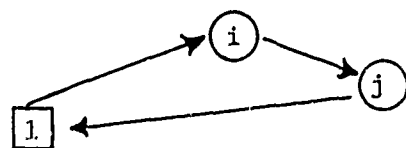


Figure IV. Nodes i and j have been linked.

For every possible pair of demand points i and j there is a corresponding savings s_{ij} . We order these savings from greatest to least. Starting from the top of the list we link nodes i and j on a single truck route where s_{ij} represents the current maximum savings and the following conditions are satisfied:

- (1) nodes i and j are not already on the same truck run;
- (2) neither i nor j are interior to an existing tour;
- (3) truck availability is not exceeded (we assume that the number of trucks with smallest capacity is unlimited);
- (4) truck capacity is not exceeded;
- (5) maximum tour length is not exceeded.

We now cross s_{ij} off the list and continue until no further link between nodes can be made that will decrease total travel time, at which point we have the solution. A modification involves extending one route as far as constraints will allow and then starting another route. Yellow [38] has made this version especially appealing by giving a procedure for generating the largest-savings link out from an end-node of the tour under consideration.

The Tyagi Algorithm

In "A Practical Method for the Truck Dispatching Problem," Tyagi presents a method which he claims makes best possible use of truck capacity, and is geared towards medium to large size problems [35]. It should be pointed out that Tyagi's paper (at least its English translation) is miserably written and the discussion of the algorithm is inconsistent from one section to the next.

Suppose that a large number of trucks are available, each of capacity C . The case where different vehicles have different capacities requires almost no additional computational effort, but we prefer to consider the case where all capacities are the same, in order to facilitate description of the algorithm. Also assume that the maximum route length is very large. The demand points are grouped in the following very straightforward fashion. Starting with node 2 (node 1 is the central depot) we find its nearest neighbor, say node k , subject to the restriction that $Q_2 + Q_k \leq C$. We next find the nearest neighbor to node k , say node j , such that $Q_2 + Q_k + Q_j \leq C$ and continue until adding a nearest neighbor will result in a tour exceeding truck capacity. Next we build another tour, until all demand points are served. Rules of thumb are indicated to minimize the frequency of a group consisting of only one delivery point, especially, in the case where the delivery is small or the distance from the central depot to this point is more than half the distance from the farthest point to the central depot.

Tyagi states that the number, N , of trips desired can be determined first, i.e., $N = \left\lceil \sum_{i=1}^n Q_i / C \right\rceil$ where Q_i is the demand at node i and $\lceil x \rceil$ is the ceiling of x (least integer function: $\min_{k \geq x} k$). This can only be true under the condition that we may require two tours to satisfy a particular demand at an end-node of a tour. The splitting of a delivery is mentioned nowhere in his paper. In any case, having grouped the delivery points into m tours, the truck dispatching problem reduces to m Traveling Salesman Problems.

Although we have outlined the essence of his work, Tyagi develops his method into a more flexible algorithm. In contrast to the exchange-type algorithm, this approach is an example of a "buildup" algorithm. Tyagi's

algorithm (1968) is very similar to the Karg and Thompson [23] heuristic algorithm for solution of the TSP (1964); it is well-suited for both hand and computer solution.

The Gillett-Miller Algorithm

In a recent paper by Gillett and Miller [16] an efficient buildup algorithm for handling up to about 250 nodes was introduced. In the previously discussed algorithms, input included a travel time (or distance) matrix. Here we require rectangular coordinates for each demand point, from which we may calculate polar coordinates. We select a "seed" node randomly. With central depot as the pivot we start sweeping (clockwise or counterclockwise) the ray from the central depot to the seed. Demand nodes are added to a route as they are swept. If the polar coordinate indicating angle is ordered for the demand points from smallest to largest (with seed's angle 0) we enlarge routes as we increase the angle until capacity restricts us from enlarging a route by including an additional demand node. This demand point becomes the seed for the following route. Once we have the routes we can apply TSP algorithms such as the Lin-Kernighan heuristic to improve tours and obtain significantly better results. In addition, we can vary the seed and select the best solution.

Shortcomings of Algorithms

The most apparent aspect of the three algorithms presented is their simplicity and computational efficiency. Each has its drawbacks, however. The Clarke-Wright algorithm is initiated with an infeasible solution since the number of trucks available is generally less than the number of demand

points. Consequently, there is no guarantee that the final solution will be feasible. Tyagi's algorithm may require that two deliveries be made to one point. Also it may be necessary to form a tour consisting of a small number of deliveries of total quantity much less than C . The Gillett-Miller algorithm is not valid unless the travel time is euclidean and obeys the triangle inequality; in many instances this is not the case.

Similarities Between Algorithms

Though these algorithms are obviously dissimilar in certain respects we can, in some sense, place them in an integrated context. That is, they differ primarily in the order in which nodes are linked and tours are formed. Suppose we introduce a utility function $u_{ij} = f(d_{li}, d_{lj}, d_{ij})$ which states that the utility of joining nodes i and j on a tour is a function of the travel times d_{li} , d_{lj} , and d_{ij} . Then each algorithm becomes an implementation of a unified algorithm which links nodes i and j when feasible where u_{ij} is the current maximal utility. For each algorithm there is a particular utility function of the form $u_{ij} = f(d_{li}, d_{lj}, d_{ij})$. In the Clarke-Wright algorithm we use $u_{ij} = d_{li} + d_{lj} - d_{ij}$. Gaskell's Π method (not discussed in this paper) uses $u_{ij} = d_{li} + d_{lj} - 2d_{ij}$. A utility function of the form $u_{ij} = d_{li} + d_{lj} - \gamma d_{ij}$ is such that as we increase γ ($\gamma \geq 0$), greater emphasis is placed on the distance between points i and j rather than their position relative to the central depot. We refer to γ as the route shape parameter. The utility function $u_{ij} = -d_{ij}$ is sufficient for Tyagi's algorithm (alternatively $u_{ij} = d_{li} + d_{lj} - M d_{ij}$ where M is relatively large). With respect to the Gillett-Miller sweep algorithm, suppose P_i and P_j represent the rectangular

coordinates of nodes i and j where the central depot is the origin. From vector analysis we know that $\cos \Theta = A \cdot B / \|A\| \|B\|$; in other words the angle between two vectors A and B is given by the inverse cosine of the vector product $A \cdot B$ divided by the product of the norms of A and B . Since we seek the point which makes the smallest angle with the present point, for inclusion in a route we can write $u_{ij} = \frac{-d_{li}d_{lj}}{P_i \cdot P_j}$ where $P_i \cdot P_j$ is a vector product.

Conclusions

Yellow's modification of the Clarke-Wright procedure appears computationally to be the most powerful vehicle routing program (based on computational experience mentioned in the literature); problems of 200 nodes have been solved in less than a minute and a problem with 1000 nodes was solved in five minutes. However, there is no computational experience offered for Tyagi's algorithm.

This report has presented the vehicle routing problem, discussed various integer programming formulations, and studied several heuristic approaches for solution of the problem. The focus has been on unifying previous work in order to gain insight into this important sub-class of combinatorial problems. Hopefully this insight will result in better analysis of complex logistics and transportation systems, in particular with respect to newspaper distribution.

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References

1. Agin, N., and Cullen, D., "An Algorithm for Transportation Routing and Vehicle Loading," MATHEMATICA, INC., June 1974.
2. Altman, S., Bhagat, N., and Bodin, L., "Extension of the Clarke and Wright Algorithm for Routing Garbage Trucks," Presented at the 18th TIMS Int. Conf. (Washington, D.C.) March 1971.
3. Bellmore, M., and Hong, S., "Transformation of Multisalesmen Problem to the Standard Traveling Salesman Problem," JACM, Vol. 21, No. 3, July 1974, pp. 500-504.
4. Bellmore, M., and Nemhauser, G., "The Traveling Salesman Problem: A Survey," Operations Research (1968), Vol. 16, pp. 538-558.
5. Beltrami, E., and Bodin, L., "Networks and Vehicle Routing for Municipal Waste Collection," Networks, Vol. 4, No. 1, 1974, pp. 65-94.
6. Christofides, W., and Eilon, S., "An Algorithm for the Vehicle Dispatching Problem," Operational Research Quarterly, Vol. 20, 1969, p. 309.
7. Christofides, N., and Eilon, S., "Algorithms for Large Scale TSP's," Operational Research Quarterly, Vol. 23, 1972, p. 511.
8. Christofides, N., "The Vehicle Routing Problem," July 1974, NATO Conference on Combinatorial Optimization, Paris.
9. Clarke, G., and Wright, J., "Scheduling of Vehicles from a Central Depot to a Number of Delivery Points," Operations Research, Vol. 12, No. 4, 1964, pp. 568-581.
10. Dantzig, R., and Ramser, J., "The Truck Dispatching Problem," Management Science, October 1959, pp. 81-91.
11. Eilon, S., Watson-Gandy, C., and Christofides, N., Distribution Management, Griffin, London, 1971.
12. Gabbay, H., "An Overview of Vehicular Scheduling Problems," M.I.T. Technical Report, Sept. 1974.
13. Garvin, W., Crandall, H., John, J., and Spellman, R., "Applications of Linear Programming in the Oil Industry," Management Science, Vol. 3, No. 4, July 1957, pp. 407-430.
14. Gaskell, T., "Bases for Vehicle Fleet Scheduling," Operational Research Quarterly, Vol. 18, 1967, p. 281.

15. Gillett, B., and Johnson, J., "Sweep Algorithm for the Multiple Depot Vehicle Dispatch Problem," presented at the ORSA/TIMS meeting San Juan, Puerto Rico, Oct. 1974.
16. Gillett, B., and Miller, L., "A Heuristic Algorithm for the Vehicle Dispatch Problem," Operations Research, Vol. 22, 1974, p. 340.
17. Hausman, W., and Gilmour, P., "A Multi-Period Truck Delivery Problem," Transportation Research, Vol. 1, 1967, pp. 349-357.
18. Hays, R., "The Delivery Problem," Carnegie Inst. of Tech., Management Science Research Report No. 106, 1967.
19. Held, M., and Karp, R., "The Traveling-Salesman Problem and Minimum Spanning Trees," Operations Research, Vol. 18, No. 6, 1970, pp. 1138-1162.
20. Held, M., and Karp, R., "The Traveling-Salesman Problem and Minimum Spanning Trees, Part II," Mathematical Programming, Vol. 1, No. 1, 1971, pp. 6-25.
21. Hudson, J., Grossman, D., and Marks, D., "Analysis Models for Solid Waste Collection," M.I.T. Civil Engineering Report, Sept. 1973.
22. IBM Corp., "System 360/Vehicle Scheduling Program Application Description-VSP," Report H20-0464, White Plains, NY, 1968.
23. Karg, L., and Thompson, G., "A Heuristic Approach to Solving Traveling Salesman Problems," Management Science, Vol. 10, pp. 225-248.
24. Knowles, K., "The Use of a Heuristic Tree Search Algorithm for Vehicle Routing and Scheduling," Lecture at OR Conference, Exeter, England, 1967.
25. Lin, S., "Computer Solutions of the TSP," Bell Systems Technical Journal, Vol. 44, 1965, p. 2245.
26. Lin, S., and Kernighan, B., "An Effective Heuristic Algorithm for the TSP," Operations Research, Vol. 21, 1973, p. 498.
27. Miller, C., Tucker, A., and Zemlin, R., "Integer Programming Formulation of Traveling Salesman Problems," JACM, 7, (1960), pp. 326-329.
28. Orloff, C., "Routing a Fleet of M Vehicles to/from a Central Facility," Networks, Vol. 4, pp. 147-162.
29. Pierce, J., "Direct Search Algorithms for Truck-Dispatching Problems, Part I," Transportation Research, Vol. 3, pp. 1-42.
30. Russell, R., "Efficient Truck Routing for Industrial Refuse Collection," presented at the ORSA/TIMS meeting San Juan, Puerto Rico, Oct. 1974.

31. Scott, A., Combinatorial Programming, Spatial Analysis and Planning Methuen & Co. Ltd., London, 1971.
32. Svestka, J., and Huckfeldt, V., "Computational Experience with an M-Salesmen Traveling Salesman Algorithm," Management Science, Vol. 19, No. 7, 1973, pp. 790-799.
33. Tillman, F., and Cain, T., "An Upper Bounding Algorithm for the Single and Multiple Terminal Delivery Problem," Management Science, Vol. 18, No. 11, 1972, pp. 664-682.
34. Tillman, F., and Cochran, H., "A Heuristic Approach for Solving the Delivery Problem" J. Indust. Eng., Vol. 19, 1968, p. 354.
35. Tyagi, M., "A Practical Method for the Truck Dispatching Problem," Journal of the Operations Research Society of Japan, Vol. 10, pp. 76-92.
36. Wilson, N., Sussman, J., Wong, H., and Higonnet, T., "Scheduling Algorithms for a Dial-A-Ride System," Report USL TR-70-13, Urban Systems Laboratory, M.I.T., March 1971.
37. Wren, A., and Holliday, A., "Computer Scheduling of Vehicles from One or More Depots to a Number of Delivery Points," Operational Research Quarterly, 23, pp. 333-344 (1972).
38. Yellow, P., "A Computational Modification to the Savings Method of Vehicle Scheduling," Operational Research Quarterly, Vol. 21, 1970, p. 281.