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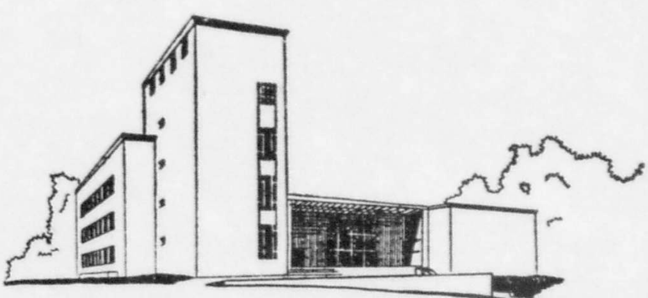

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MINIMAL INEQUALITIES

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Abstract

We provide a characterization of minimal inequalities for bounded mixed integer programs, in terms of subadditive functions. The condition on the columns of the integer-constrained variables is analogous to that obtained earlier for the group problem, and we also determine the condition on the columns of the continuous variables.

Key Words

1. Integer programming.
2. Cutting-planes.
3. Subadditivity.
4. Group problem.

MINIMAL INEQUALITIES

by R. G. Jeroslow

The main results of this paper are characterizations of the form of minimal inequalities for bounded mixed-integer programs (Theorems 8 and 9), extending the kind of characterization of minimal inequalities found earlier for the group problem [5], [6], [9], to the integer program prior to the group relaxation. We recover the previous group-theoretic characterization for the columns of integer-constrained variables, and determine the proper characterization for the columns of the continuous variables.

The plan of the paper is as follows.

In Section 1, we present some basic new results in the subadditive theory of cutting-planes for integer programs prior to relaxation. We also obtain a reformulation of our result on the form of valid cutting-planes [7], [8], in terms of a new duality theorem (Theorem 5).

It is in Section 2 that we state and prove our characterization of minimal inequalities, and we also state the relationship between minimal inequalities and the usual polyhedral concept of a supporting hyperplane (Corollary 10). As is known from [5], [6], a characterization of minimal inequalities yields one for the extreme valid inequalities, as the extreme inequalities among those minimal.

The extension of this subadditive theory from the group problem, where it originates, to the integer program prior to relaxation, requires the use of subadditive functions on real space. The previously discussed subadditive functions on the group are isomorphic to that class of subadditive functions on real space, which have unit periods in every co-ordinate direction (see e.g. [8, Lemma 11]).

As one would expect from this latter fact, the subadditive functions in real space can have many characteristics not exhibited by those on the group, and these often translate into characteristics of the cutting-planes obtained from the functions. For instance, negative intercepts in a cut (properties of cuts in [1], [2]) are not possible when periodic subadditive functions are used.

Section 1: Results on Duality and the Form of Valid Cuts

The starting point for the subadditive approach is the study of the value function of a mixed-integer program, as it depends upon the right-hand-side. This value function is the function G defined by:

$$\begin{aligned} G(v) = \inf \ & \sum_{a \in A} \pi_a t_a + \sum_{b \in B} \sigma_b r_b \\ \text{(MIP)} \qquad \text{subject to} \ & \sum_{a \in A} m_a t_a + \sum_{b \in B} p_b r_b = v \\ & t_a \geq 0 \text{ and integer}, \ a \in A \\ & r_b \geq 0, \ b \in B. \end{aligned}$$

Here A and B are index sets, m_a and p_b are vectors in a finite-dimensional real space R^p, and π_a and σ_b are scalars.

Applications below sometimes require infinite A and B, so we permit this in all results which follow, unless mention to the contrary occurs. This generality has been frequently used in [6],

[9]. For infinite A and B, only finitely many of the t_a, r_b may be nonzero, so that all sums written are finite sums. Many of the results below are true with m_a , p_b from infinite vector spaces over the reals, but we shall have no need of this kind of generality.

Before we proceed to subadditive functions, we discuss their domains. A monoid $M \subseteq R^p$ is any subset of R^p such that $0 \in M$ and M is closed under addition: $v, w \in M$ implies $v + w \in M$. If $M \neq \{0\}$, M is infinite. Note that, if a monoid is closed under subtraction, it is a group.

A subadditive function F is a function $F: M \longrightarrow R \cup \{-\infty\}$, with M a monoid, such that for $v, w \in M$ we have $F(v + w) \leq F(v) + F(w)$. In what follows we also require $F(0) \leq 0$. It is easy to show that, if $F(0) < 0$, then F is identically $-\infty$ (see, e.g., [7]). Our conventions on $-\infty$ are the usual (e.g., $r + -\infty = -\infty$, and $r \cdot (-\infty) = -\infty$ for $r > 0$), but we also have the convention $(-\infty) \cdot 0 = 0 \cdot (-\infty) = 0$.

In what follows, we shall use the definitions:

$$S(v) = \left\{ (t,r) \;\middle|\; \begin{array}{l} t = (t_a) \text{ and } r = (r_b) \text{ are (possibly infinite)} \\ \text{vectors, only finitely non-zero, and} \\ \text{non-negative, with } t_a \text{ integer, } a \in A, \\ \text{such that} \\ \qquad v = \sum_{a \in A} m_a t_a + \sum_{b \in B} p_b r_b \end{array} \right\}$$

$$(1) \qquad M(A,B) = \left\{ v \;\middle|\; S(v) \neq \emptyset \right\}$$

$$V(t,r,\pi,\sigma) = \sum_{a \in A} \pi_a t_a + \sum_{b \in B} \sigma_b r_b ,$$

the latter notation with $\pi = (\pi_a)$ and $\sigma = (\sigma_b)$ as (possibly infinite) vectors. Note that M(A,B) is a monoid.

Lemma 1: The value function G of (MIP) is subadditive on M(A,B).

Proof: From (MIP), we have

$$(2) \qquad G(v) = \inf \left\{ V(t,r,\pi,\sigma) \;\middle|\; (t,r) \in S(v) \right\} .$$

Therefore, we if we can prove

$$(3) \qquad S(v + w) \supseteq S(v) + S(w) \quad , \text{ for } v, w \in M(A,B),$$

with vector addition co-ordinatewise, we would have

$$\begin{aligned} G(v + w) &\leq \inf \left\{ V(t + t', r + r', \pi, \sigma) \;\middle|\; (t,r) \in S(v),\ (t',r') \in S(w) \right\}, \\ &= \inf \left\{ V(t,r,\pi,\sigma) \;\middle|\; (t,r) \in S(v) \right\} \\ &\qquad + \inf \left\{ V(t',r',\pi,\sigma) \;\middle|\; (t',r') \in S(w) \right\} \\ &= G(v) + G(w) \end{aligned}$$

and the proof will be complete.

However, if $(t,r) \in S(v)$ and $(t',r') \in S(w)$, then clearly $(t + t', r + r') \geq 0$, $t + t'$ is integer, and

$$\sum_{a \in A} m_a(t_a + t_a') + \sum_{b \in B} p_b(r_b + r_b') = v + w$$

since all sums are finite, so $(t + t', r + r') \in S(v + w)$, and (3) is established. Q.E.D.

A concept first introduced by Gomory and Johnson [6], and systematically studied by Johnson for its value in cutting-plane theory, is that of the upper directional derivative (at zero) in the direction of v, specifically:

$$(4) \qquad \overline{F}(v) = \limsup_{\delta \to 0+} F(\delta v)/\delta \quad .$$

For our purposes, $\overline{F}(v)$ exists as long as the limsup is not $+\infty$. Gomory and Johnson have shown that, if $\overline{F}(v)$ is finite, the lim sup of (4) and the ordinary limit co-incide [6]. For (4) to be written, F must be defined on the entire ray $\{\delta v \mid \delta \geq 0\}$.

A function F defined on a cone is conical if F is subadditive, and also $F(\lambda v) \leq \lambda F(v)$, $\lambda \geq 0$. This last condition actually forces $F(\lambda v) = \lambda F(v)$. It is known that the set of all v such that (4) exists is a cone, and on this cone $\overline{F}$ is a conical function (see [8] or [9]).

Essentially, the directional derivative F is needed to treat the continuous non-basics r_b, $b \in B$. More precisely, the inequality $F(m_a t_a) \leq F(m_a)t_a$ follows from the integer nature of $t_a \geq 0$, but for continous $r_b \geq 0$, the relevant inequality is $F(p_b r_b) \leq \overline{F}(p_b)r_b$; both inequalities are needed in the proofs below.

A simple heuristic calculation shows why upper derivatives must arise for continuous nonbasics. For a continuous $\lambda \geq 0$, taking $\lambda = p/q$ to be rational, subadditivity alone yields

$$F(\lambda v) = F((pv)/q) \leq p\,F(v/q)$$

(5) $$= \left(\frac{p}{q}\right)\left(F(v/q)/\ (1/q)\right)$$

$$\simeq \lambda\,\bar{F}(v),$$

since q can be taken very large, so that $\delta = 1/q$ is very small. The rigorous proof is given in [8]

__Lemma 2__: The value function G of (MIP) satisfies all the conditions

(6) $$G(m_a) \leq \pi_a \ , \quad a \in A$$
$$\bar{G}(p_b) \leq \sigma_b \ , \quad b \in B.$$

Moreover, if H is any other subadditive function on M(A,B) satisfying these conditions, then $G(v) \geq H(v)$ for all $v \in M(A,B)$.

__Proof__: Note that, fixing $c \in A$, and setting $t_c = 1, t_a = 0$ for $a \neq c \in A$, $r_b = 0$ for $b \in B$, we obtain $v = m_a$ on the right-hand-side in MIP, so $G(m_c) \leq 1 \cdot \pi_c = \pi_c$, and $c \in A$ was arbitrary. Similarly, fixing $c \in B$ and setting $t_a = 0$, $a \in A$, $r_c = \delta$, $r_b = 0$ for $b \neq c \in B$, we obtain $v = \delta\,p_c$ in (MIP), so $G(\delta\,p_c) \leq \delta \cdot \sigma_c$, which implies $\bar{G}(p_c) \leq \sigma_c$, and $c \in B$ was arbitrary.

Next, let H be any other subadditive function satisfying these conditions. If $(t,r) \in S(v)$, by subadditivity

$$H(v) \leq H\left(\sum_{a \in A} t_a m_a\right) + H\left(\sum_{b \in B} r_b p_b\right)$$

(7) $$\leq \sum_{a \in A} t_a H(m_a) + \sum_{b \in B} r_b \bar{H}(p_b)$$

$$\leq \sum_{a \in A} t_a \pi_a + \sum_{b \in B} r_b \sigma_b \ .$$

By the definition of G, G(v) is the infimum over all sums of the type last written, and the above computation shows that H(v) is a lower bound for all these sums. Hence, $H(v) \leq G(v)$. Q.E.D.

Theorem 3: Suppose that A and B are finite, and each m_a, $a \in A$, and p_b, $b \in B$, are vectors of rationals.

Then there is a finite-valued subadditive function meeting the conditions (6) if and only if there is no solution to the conditions

(8a) $$\sum_{a \in A} t_a \pi_a + \sum_{b \in B} r_b \sigma_b < 0$$

(8b) $$\sum_{a \in A} t_a m_a + \sum_{b \in B} r_b p_b = 0$$

$$t_a \text{ , } r_b \geq 0 \text{ , } a \in A, \; b \in B$$

On the other hand, if there is a solution to (8), then any solution to the conditions (6) is identically $- \infty$.

Proof: Suppose there is a finite-valued subadditive function meeting (6). Then G defined in MIP also meets (6) and is finite-valued.

If (8) has a solution, it has a rational solution, and multiplication by a suitably large positive integer gives a solution to (8) with t_a integral, $a \in A$. Then multiplication by larger integers shows that G(0) is arbitrarily small, so $G(0) = - \infty$, a contradiction. Also $G(v) \leq G(v) - \infty = - \infty$ for all v.

For the converse, suppose that no solution to (8) exists.

By [10, Theorem 3.9] , K = conv (S(v)), the convex span of S(v), is polyhedral, hence either $G(v) = -\infty$, or G(v) is attained at an extreme point of K. But $G(v) = -\infty$ is not possible, or (8a) holds for a vector (t,r) which is a direction of infinity of K. However, directions of infinity of K also solve (8b), and no such solution exists. Q.E.D.

<u>Corollary 4:</u> Let $A_1 \subseteq A$ and $B_1 \subseteq B$. Then there exists a subadditive function F satisfying the conditions

$$F(m_a) \leq \pi_a,\ a \in A \setminus A_1$$
$$F(m_a) = \pi_a,\ a \in A_1$$
$$\bar{F}(p_b) \leq \sigma_b,\ b \in B \setminus B_1 \qquad (9)$$
$$\bar{F}(p_b) = \sigma_b,\ b \in B_1$$

if and only if the function G defined by MIP satisfies these conditions.

<u>Proof</u>: Suppose a function F exists satisfying (9). Since $F(m_a) \leq \pi_a$, $a \in A$ and $\bar{F}(p_b) \leq \sigma_b$, $b \in B$, by Theorem 2 we have $G(v) \geq F(v)$ for all $v \in M(A,B)$. But then for all $a \in A_1$, by (6), we have $\pi_a \geq G(m_a) \geq F(m_a) = \pi_a$, and hence $G(m_a) = \pi_a$. Similarly, $\bar{G}(p_b) = \sigma_b$, $b \in B_1$. The "if" part of the corollary is immediate.

Q.E.D.

Note that, by Theorem 3, if A and B are finite and any one of the quantities π_a, $a \in A_1$, or σ_b, $b \in B_1$ are finite in Corollary 4, then a solution exists to (9) if and only if some finite-valued solution exists (since G will be finite valued).

<u>Theorem 5:</u>

The (possibly infinite) consistent mixed-integer program in equality format

$$\text{(P)} \quad \begin{aligned} &\inf \sum_{a\in A} \pi_a t_a + \sum_{b\in B} \sigma_b r_b \\ &\text{subject to} \sum_{a\in A} m_a t_a + \sum_{b\in B} p_b r_b = d \\ &t_a \geq 0 \text{ and integer, } a\in A \\ &r_b \geq 0,\ b\in B \end{aligned}$$

and the subadditive program

$$\text{(D)} \quad \begin{aligned} &\max F(d) \\ &\text{subject to } F(m_a) \leq \pi_a,\ a\in A \\ &\bar{F}(p_b) \leq \sigma_b,\ b\in B \\ &F \text{ subadditive, } F(0) \leq 0, \\ &F \text{ defined on } M(A,B), \end{aligned}$$

bear the following primal-dual relations to each other:

1) For any pair of solutions, one to (P) and one to (D), the inequality

$$\sum_{a\in A} \pi_a t_a + \sum_{b\in B} \sigma_b r_b \geq F(d) \tag{10}$$

is valid.

2) The optimal values of (P) and (D) are the same.

<u>Proof</u>:

1) By the subadditivity of F, plus the remarks just prior to

Lemma 2, and the fact that only finitely many of t_a, r_b are non-zero, (10) holds due to a computation similar to (7).

2) This part follows from (10).

3) Clearly, this part is true if and only if the function G of (MIP) has G(d) = the optimum value to (P) = π_0 (say), since G(d) is the maximum value of F(d) for all F satisfying these constraints. But G(d) = π_0 follows from the definition of G. Q.E.D.

If we view (P) as the program describing how far we may move a hyperplane, whose co-efficients are those of the criterion function (π,σ) of (P), toward the convex span of the feasible solutions to the constraints of (P), then (D) and (10) can be understood as the general form of a valid cut for these constraints. This theme is elaborated further in the next result.

Here we only wish to note that this general form (D), (10) remains valid when the constraints of (P) are replaced by inequality constraints, as in

$$(P)' \qquad \sum_{a \in A} m_a t_a + \sum_{b \in B} p_b r_b \leq d$$

$$t_a \geq 0 \text{ and integer, } a \in A$$

$$r_b \geq 0, \ b \in B,$$

provided that we only consider cuts with non-zero coefficients for the structural variables (t,r).

In fact, defining the value function G to be the optimum with inequality constraints (P)', all the previous work goes through with the remark that G is monotone non-increasing in the right-hand-side d. For if $d' \geq d$, then there are more solutions to the constraints of (P)' for d', from which $G(d') \leq G(d)$ follows.

Since G is monotone non-increasing, we may reason from (P)' that

$$G(d) \leq G\Big(\sum_{a \in A} m_a t_a + \sum_{b \in B} p_b r_b \Big)$$

$$\leq G\Big(\sum_{a \in A} m_a t_a \Big) + G\Big(\sum_{b \in B} p_b r_b \Big) ,$$

and from this point the reasoning of (7) can be applied to obtain the previous results.

For the inequality format (P)', a dual program (D)' is still valid, provided that it requires F to be monotone non-increasing.

This remark, which applies here and again in Theorem 7 below, is worth making, because otherwise one would be tempted to use

$$\inf \left\{ F(d - z) \;\middle|\; z \geq 0 \right\}$$

on the right in (10), an unnecessary complication. On the other hand, although (10) theoretically allows for all valid cuts in the structural variables, advantages may still accrue to strengthening these cuts by considering negative co-efficients for the slacks which are introduced.

It is of interest to note, that only rarely can the optimum in the dual (D) of Theorem 5 be attained by a convex function F. This is the content of our next result. In this result, K denotes the cone generated by M(A,B). The linear programming relaxation of P is obtained when the constraints that t_a be integer are dropped.

<u>Theorem 6</u>:

Suppose that (P) is consistent and bounded.

1) The optimum to (D), when only convex subadditive functions F on K are considered, is equal to the optimum to (D) when only linear functions are considered, provided that d is in the relative interior of K.

2) For d in the relative interior to K, the optimum to (D) is attained by a convex function, if and only if the value of (P) and of its linear programming relaxation agree.

<u>Proof</u>: Note that 1) implies 2), since (D) is the usual linear programming dual to the relaxation of (P), when F is a linear function. We prove only 1).

First, note that if F is convex and subadditive on K, then for any $v \in K$ and any $\theta > 0$, we have $F(\theta v) \leq \theta F(v)$. In fact, if $0 \leq \theta \leq 1$, then $F(\theta v) = F(\theta v + (1 - \theta) \cdot 0) \leq \theta F(v) + (1 - \theta) \cdot F(0) \leq \theta F(v)$. Then if $\theta > 1$, let p be so large a positive integer that $\theta / p \leq 1$, and note from the previous that $F(\theta v) = F(p (\theta / p) v) \leq p F((\theta / p) v) \leq \theta F(v)$.

From $F(\theta v) \lesssim \theta v$ and [8], we have $F(\theta v) = \theta F(v)$ for $\theta > 0$. This in turn gives $\bar{F}(v) = F(v)$.

Next, if d is in the relative interior of K, there is a non-vertical supporting hyperplane to the epigraph of F at the point (d, F(d)), hence a vector λ with

$$F(v) \geq F(d) + \lambda (v - d)$$

for all $v \epsilon K$. Chosing $v = \theta d$ in the above, and using $F(\theta d) = \theta F(d)$, we obtain the inequality

$$\lambda d(1 - \theta) \geq F(d) (1 - \theta)$$

which is valid for all $\theta > 0$. Picking $\theta = 1/2$, we obtain $\lambda d \geq F(d)$, and picking $\theta = 2$, we obtain $\lambda d \leq F(d)$. This gives $\lambda d = F(d)$, showing that the linear function $H(v) = \lambda v$, which is subadditive, gives the same value H(d) in (P) as does F(d).

Finally, using $\lambda d = F(d)$ we obtain $F(v) \geq \lambda v$ for all $v \epsilon K$, in particular $\pi_a \geq F(m_a) \geq \lambda m_a$ for all $a \epsilon A$, and also $\sigma_b \geq \overline{F}(p_b) = F(p_b) \geq \lambda p_b$ for all $b \epsilon B$. Since $\overline{H} = H$, this shows that the linear function H also satisfies the constraints of (D).

Hence, the optimum to (D) over convex functions is the same as over linear functions . Q.E.D.

In a sense, Theorem 6 concerns the irreducibility of the non-convexity of integer programming, and states the primary barrier to the use of purely convex methods in the resolution of integer programs. Of course, such a result is to be expected. The essentially subadditive, non-convex formulations which attempt to close the "duality gap" of (D) for convex F, usually involve some implicit or explicit enumeration--this point can be made precise, though we chose not to do so here.

We shall say that one inequality

(11) $$\sum_{a\in A} \pi_a t_a + \sum_{b\in B} \rho_b r_b \geq \pi_0 \qquad (\pi_0 \in R)$$

is a <u>weakening</u> of another in primed notation, if we have

$$\pi_0 \leq \pi_0' \text{ and } \pi_a \geq \pi_a', \; a\in A, \text{ and } \rho_b \geq \rho_b', \; b\in B .$$

Conversely, we call the latter inequality a <u>strengthening</u> of the former. Clearly, a weakening is implied by its strengthening plus the nonnegativity conditions $t_a, r_b \geq 0$, $a\in A$, $b\in B$.

We say that a valid inequality (11) is <u>minimal</u> if it has no strict strengthening (i.e., no strengthening other than itself).

<u>Theorem 7</u>:

Suppose that a vector $z \in R^p$ is constrained by the relations

(12) $$z = \sum_{a\in A} m_a t_a + \sum_{b\in B} p_b r_b$$

$$t_a \geq 0 \text{ and integer, } a\in A$$

$$r_b \geq 0, \; b\in B ,$$

and also by the requirement

(13) $$z \in T$$

sor some set T. Let us define, for F subadditive on M(A,B)

(14) $$\beta = \inf \left\{ F(z - q) \mid z \in T \text{ and } z - q \in M(A,B) \right\} .$$

Then the cut

(15) $$\sum_{a\in A} F(m_a) t_a + \sum_{b\in B} \ddot{F}(p_b) r_b \geq \beta$$

is valid whenever the indicated derivatives exist, and conversely, any valid cut (11) is a weakening of a cut of the form (15) for F subadditive in M(A,B) .

<u>Proof:</u> The validity of (15) follows from

$$\beta \leq F(z - q) = F(\sum_{a \in A} m_a t_a + \sum_{b \in B} p_b r_b),$$

$$\leq \sum_{a \in A} F(m_a) t_a + \sum_{b \in B} \overline{F}(p_b) r_b ,$$

the detailed computation paralleling (7).

For the converse, take G as defined by (MIP). G is subadditive by Theorem 1 and (15), with G in place of F, has $G(m_a) \leq \pi_a$, $a \in A$, and $\overline{G}(p_b) \leq \sigma_b$, $b \in B$, by Theorem 2. It remains only to show that $\pi_0 \leq \beta$.

However, if $\pi_0 > \beta$, then there exists $z \in T$ with $z - q \in M(A,B)$, such that $G(z - q) < \pi_0$. Then there exists $(t,r) \in S(z - q)$ with $V(t,r, \pi, \sigma) < \pi_0$. But this last inequality contradicts the validity of (11). Q.E.D.

Theorem 7 is a generalization of our result Theorem 1.7 of [7] (which is repeated as Theorem 4 of [8]), on the form of a valid cut. Earlier we only treated $A \cup B$ finite, in which case (12) can be viewed as the current Dual Simplex Tableau for a mixed integer program, and T will usually be $T = \{ z \geq 0 \mid z_i$ integer, $i \in I_1 \}$, where I_1 is a subset of $I = \{1,\ldots,p\}$. However, the earlier proof goes over substantially unchanged; we supply the proof here only for the sake of completeness.

For the group relaxation, which can be obtained by understanding all equalities above in the group (i.e. modulo unity), both the forward and converse of Theorem 7 was obtained by Johnson for A =

the unit interval, modulo unity; and several other results on the form of cuts for the group are in [9]. Both the forward and converse was obtained even earlier by Gomory and Johnson, for the mixed-integer group relaxation of one row (p=1) [6] and for the pure-integer group relaxation [5].

We shall also call the value function G of (MIP), the <u>subadditive envelope</u> of the quantities π_a, σ_b, at the points m_a, p_b. Clearly, for any subadditive function F, indexing all the points of its domain by a set A = B, and putting $\pi_a = F(m_a)$, and also $\sigma_b = \overline{F}(p_b)$, when the derivatives exist, the value function G of (MIP) is F. Briefly put: a subadditive function is its own envelope.

Section 2: Minimal Inequalities

We shall say that the constraints of (P) are <u>bounded</u>, if there exist bounds T_a, $a \in A$, and R_b, $b \in B$, such that $(t,r) \in S(d)$ implies $0 \leq t_a \leq T_a$, $a \in A$, and $0 \leq r_b \leq R_b$, $b \in B$. We now seek a characterization of minimal inequalities for (P) bounded.

Before we state our result on minimal inequalities, it is worth remarking, that a finite-valued minimal inequality need not necessarily exist as a weakening of any valid finite-valued inequality. For instance, with constraints $t_1 + r_1 = 1/2$, $t_1, r_1 \geq 0$ and t_1 integer, we have $t_1 = 0$ in all solutions, so the coefficient π_1 of t_1 in any valid cut (11) can always be lowered indefinitely, and in any minimal cut $\pi_1 = -\infty$ is necessary.

However, one shows very simply that this difficulty is the only kind which can exist. After lowering to $-\infty$ any $\pi_a, a\epsilon A$, such that $(t,r) \epsilon S(d)$ implies $t_a = 0$, and lowering to $-\infty$ any σ_b, $b\epsilon B$, such that $(t,r) \epsilon S(d)$ implies $r_b = 0$, we can construct a minimal inequality as a strengthening of any valid inequality (11) as follows.

For $\pi_a \neq -\infty$, since there is $(t,r) \epsilon S(d)$ with $t_a > 0$, clearly π_a cannot be lowered indefinitely. Let π'_a be the greatest lower bound on π_a. Then by sending $\pi_a \searrow \pi'_a$, without changing other co-efficients in (11), we always have a valid cut; so the cut with π'_a replacing π_a is valid. The procedure for lowering a $\sigma_b \neq -\infty$ is the same. The procedure must be repeated for all $a\epsilon A$ and $b\epsilon B$; clearly, for $A \cup B$ uncountable, a transfinite process is necessary. The final cut arrived at will depend on the sequence of indices $a\epsilon A$ and $b\epsilon B$ chosen, but the final cut will be minimal.

It is a corollary of the easy remarks above, that if there is $(t,r) \epsilon S(d)$ such that $t_a > 0$ for each $a\epsilon A$, and similarly for each $b\epsilon B$, then any valid inequality (11) has finite coefficients π_a, σ_b, and any valid inequality is the weakening of some finite-valued minimal inequality. (Note however: S(d) can lie in a co-ordinate plane $t_a = 0$ even if the continuous relaxation of (P) does not).

<u>Theorem 8</u>:

Suppose that the constraints of (P) are bounded.

Then in order for (11) to be a minimal inequality, it is both necessary and sufficient that the subadditive envelope G of π_a, $a \epsilon A$, and σ_b, $b \epsilon B$, satisfy the following conditions:

1) $G(m_a) = \pi_a$, $a \epsilon A$; $\bar{G}(p_b) = \sigma_b$, $b \epsilon B$; and $G(d) = \pi_0$.

2) For all $c \epsilon A$, if $t_c = 0$ for all $(t,r) \epsilon S(d)$, then $\pi_c = -\infty$; otherwise $\pi_c > -\infty$, $(d - m_c) \epsilon M(A,B)$, and

(16) $$G(m_c) + G(d - m_c) = G(d).$$

3) For all $c \epsilon B$, if $r_c = 0$ for all $(t,r) \epsilon S(d)$, then $\sigma_c = -\infty$; otherwise $\sigma_c > -\infty$, $d - \delta p_c \epsilon M(A,B)$ for all $\delta > 0$ sufficiently small, and

(17) $$\bar{G}(p_c) = \lim \left\{ (G(d) - G(d - \delta p_c))/\delta \,\middle|\, \delta \to 0+ \right\}.$$

<u>Proof</u>: We begin with some easy remarks of a general nature, which we will use throughout the proof without mention.

First, if we encounter any expression

$$v + w = d$$

with the full expressions for v,w given by

$$v = \sum_{a \epsilon A} m_a t_a + \sum_{b \epsilon B} p_b r_b$$

$$w = \sum_{a \epsilon A} m_a t'_a + \sum_{b \epsilon B} p_b r'_b$$

$t_a, r_b, t'_a, r'_b \geq 0$ and all t_a, t'_a integer, then with G as in (MIP), we may conclude that

$$(d - v) \in M(A,B) \quad ,$$

$$G(d - v) \leq \sum_{a \in A} \pi_a t'_a + \sum_{b \in B} \sigma_b r'_b \quad ,$$

and $$G(v) \leq \sum_{a \in A} \pi_a t_a + \sum_{b \in B} {}_b r_b \quad .$$

Furthermore, assuming v is among $\left\{ p_b \middle| b \in B \right\}$, if for some $\theta > 0$, $d - \theta v \in M(A,B)$, then for all δ in the range $0 \leq \delta \leq \theta$, $(d - \delta v) \in M(A,B)$. In fact, for some $w \in M(A,B)$ we have

$$w = d - \theta v,$$

and hence

$$(\theta - \delta)v + w = d - \delta v,$$

and the left-hand-side is in M(A,B) since $(\theta - \delta) \geq 0$, and v is a p_b.

Now we show that minimality implies 1), 2), 3).

As regards 1), with G as defined in (MIP), we have $G(m_a) \leq \pi_a$, $\overline{G}(p_b) \leq \sigma_b$, $G(d) \geq \pi_0$. If strictly inequality holds in any of these relations, then the cut (15) with F replaced by G, and β by G(d), is valid, by the argument for (15), and it is a strict strengthening of (11). But (11) is minimal, so equalities hold in these relations.

As regards 2), our remarks prior to the theorem allow us to consider only the case that $\pi_c > -\infty$ and that there is $(t,r) \in S(d)$ with $t_c > 0$. Hence, $t_c \geq 1$, so there is an expression $m_c + w = d$ of the type discussed above, and we conclude that $(d - m_c) \in M(A,B)$.

For the equation (16), by subadditivity $G(m_c) + G(d - m_c) \geq G(d)$, so we need only prove the reverse inequality next.

However, if $\alpha = G(m_c) + G(d - m_c) - G(d) > 0$, then by lowering only π_c to $\pi_c - \alpha/T_c$ (bounds T_a, R_b are assumed), whenever $(t,r) \in S(d)$ and $0 < t_c \leq T_c$, we have, since $t_c \geq 1$,

$$V(t,r,\pi' , \sigma) \geq V(t,r, \pi,\alpha) - (\alpha /T_c)T_c$$

$$\geq G(m_c) + G(d - m_c) - \alpha \geq G(d) = \pi_0 ,$$

where π' arises from π only by the change in π_c. For $t_c = 0$, $V(t,r,\pi', \sigma) = V(t,r,\pi,\sigma) \geq G(d) = \pi_0$ is immediate. Hence for all $(t,r) \in S(d)$,

$$V(t,r,\pi' , \sigma) \geq \pi_0.$$

This contradicts the minimality of (11), and so $\alpha = 0$, as required for (16).

We now establish 3).

As we did for 2), we may restrict our attention to the case that $\sigma_c > -\infty$ and there exists some $(t,r,) \in S(d)$ with $r_c > 0$. Hence we get an expression of the form $r_c p_c + w = d$, so for every δ in the range $0 \leq \delta \leq r_c$, we have $(d - \delta r_c) \in M(A,B)$.

By subadditivity, for any δ with $0 \leq \delta \leq r_c$, we have

$$G(\delta p_c) \geq G(d) - G(d - \delta p_c) .$$

Hence, dividing by $\delta > 0$ on both sides, we have

$$\bar{G}(p_c) \geq \limsup \left\{(G(d) - G(d - \delta\, p_c))/\delta \mid \delta \to 0+\right\}.$$

We next show that

$$\text{(18)} \qquad \bar{G}(p_c) \leq \liminf \left\{(G(d) - G(d - \delta\, p_c))/\delta \mid \delta \to 0+\right\}$$

to establish (17). Let us study the function

$$\text{(19)} \qquad H(\delta) = -\pi_0 + \inf\left\{V(t,r,\pi,\sigma) \mid (t,r)\in S(d) \text{ and } r_c \geq \delta\right\}$$

for $\delta > 0$. Clearly $H(\delta) \geq 0$. We distinguish two cases.

<u>Case 1</u>: For some $\delta > 0$, $H(\delta) = 0$.

Then there is a sequence of values δ_n in the interval $[\delta, R_b]$ such that, for each n, there is $(t^n,r^n) \in S(d)$ with $r_c^n = \delta_n$ and

$$V(t^n,r^n,\pi,\sigma) \leq \pi_0 + 1/2^n.$$

Now we can assume without loss of generality that this sequence δ_n has a limit $2\tau > 0$, and that for all n, $\delta_n \geq \tau$. Then from the above, for each n we have $\delta_n p_c + w^n = d$ with $V(t^n,r^n,\pi,\sigma)$ the value of this solution, or, more perspicuously, for each n

$$\tau\, p_c + ((\delta_n - \tau)p_c + w^n) = d$$

and

$$G(\tau\, p_c) + G(d - \tau\, p_c) \leq \pi_0 + 1/2^n = G(d) + 1/2^n .$$

From subadditivity, since n is arbitrary, we have

$$\text{(20)} \qquad G(\tau\, p_c) + G(d - \tau\, p_c) = G(d).$$

Moreover, we see that our above argument would establish (20) for any δ in the range $0 \leq \delta \leq \tau$.

Rearranging (20) with $G(d - \delta\, p_c)$ on the right, dividing both sides by $\delta > 0$ and taking limits we obtain (17).

Case 2: For every $\delta > 0$, $H(\delta) > 0$.

In this case, we first wish to establish that

(21) $$\liminf \left\{ H(\delta) / \delta \,\middle|\, \delta \to 0+ \right\} = 0 .$$

In fact, if (21) failed and the liminf were $2a > 0$, then we replace only σ_c by $\sigma_c - \min(a, H(\theta)/R_c) < \sigma_c$, where $\theta > 0$ is sufficiently small so that $H(\delta)/\delta \geq a$ for all δ in the range $0 < \delta \leq \theta$. Then for $(t,r) \in S(d)$, we have

$$\begin{aligned} V(t,r,\pi,\sigma') &= V(t,r,\pi,\sigma) - r_c \min(a, H(\theta)/R_c) \\ &\geq H(r_c) + \pi_0 - r_c \min(a, H(\theta)/R_c) \\ &\geq \begin{cases} ar_c + \pi_0 - ar_c = \pi_0, & \text{if } 0 \leq r_c \leq \theta \\ H(\theta) + \pi_0 - R_c \min(a, H(\theta)/R_c) \geq \pi_0, & \text{if } \theta \leq r_c \leq R_c, \end{cases} \end{aligned}$$

since $H(\delta)$ is increasing in δ . This would contradict the minimality of (11), and hence establishes (21).

Next, note that the definition (19) of H is equivalent to

$$\begin{aligned} H(\delta) &= -\pi_0 + G(d - \delta p_c) + \delta\, \sigma_c \\ &= -G(d) + G(d - \delta p_c) + \delta \bar{G}(p_c), \end{aligned}$$

Using this and (21) we immediately obtain

$$\begin{aligned} G(p_c) &= \liminf \left\{ (G(d) - G(d - \delta p_c))/\delta + H(\delta)/\delta \,\middle|\, \delta \to 0+ \right\} \\ &= \liminf \left\{ (G(d) - G(d - \delta p_c))/\delta \,\middle|\, \delta \to 0+ \right\} , \end{aligned}$$

yielding (18), and have (17).

For the converse, suppose that (11) is a valid cut and that 1), 2), and 3) hold. We show that (11) is minimal.

If a strict strengthening of the G-cut (11) were possible, then the subadditive envelope G' of this strict strengthening would also be a strict strengthening of (11). Hence $G'(m_a) \leq G(m_a)$ for all $a \in A$, $\bar{G}'(p_b) \leq \bar{G}(p_b)$ for all $b \in B$, and $G'(d) \geq G(d) = \pi_0$. But from the definition (MIP) of the subadditive envelope G' of the strict strengthening, $G'(v) \leq G(v)$ for all $v \in M(A,B)$, since G, being subadditive, is its own envelope and G' satisfies the conditions (6) of Lemma 2. Hence $G'(d) = G(d)$ and π_0 cannot be improved.

If, for any $a \in A$, we have $G'(m_a) < G(m_a)$, then we have from (16)

$$G'(m_a) + G'(d - m_a) < G(m_a) + G(d - m_a) = G(d) = G'(d),$$

which contradicts subadditivity. Therefore, since we hypothesize a strict strengthening, for some $b \in B$, $\bar{G}'(p_b) < \bar{G}(p_b)$ must hold. From (17), for all sufficiently small $\delta > 0$, we have

$$G'(\delta p_b)/\delta < (G(d) - G(d - \delta p_b))/\delta \quad ,$$

which, after multiplication by $\delta > 0$, and transposition of $G(d - \delta p_b)$ to the left, is a clear contradiction to the subadditivity of G'. Q.E.D.

<u>Theorem 9:</u>

Suppose that the constraints of (P) are bounded and that $A \cup B$ is finite.

Then a necessary and sufficient condition for (11) to be a minimal inequality, is that the subadditive envelope G satisfies 1), 2) of Theorem 8 plus the condition:

3)' For all $c \in B$, if $r_c = 0$ for all $(t,r) \in S(d)$, then $\sigma_c = -\infty$; otherwise, $\sigma_c > -\infty$, $d - \delta p_c \in M(A,B)$ for all sufficiently small $\delta > 0$, and also

(22) $$G(\delta p_c) + G(d - \delta p_c) = G(d)$$

for all sufficiently small $\delta > 0$.

Moreover, if (11) is minimal, then:

4) For every $c \in A$, if $\pi_c > -\infty$ then there exists $(t,r) \in S(d)$ with $t_c \geq 1$, such that $V(t,r,\pi,\sigma) = \pi_0$.

5) For every $c \in B$, if $\sigma_c > -\infty$ then there exists $(t,r) \in S(d)$ with $r_c > 0$, such that

$$V(t,r,\pi,\sigma) = \pi_0 .$$

<u>Proof</u>:

Note that 3)' follows from 5); we prove 4) and 5).

Returning to the proof of (16) in Theorem 8, we see that, for any $\alpha > 0$, there exists $(t,r) \in S(d)$ with $t_c \geq 1$, such that $V(t,r,\pi,\sigma) \leq \pi_0 + \alpha$. Hence, $\inf \{ V(t,r,\pi,\sigma) \mid (t,r) \in S(d) \text{ and } t_c \geq 1 \} = \pi_0$. By our hypothesis, the set $\{ (t,r) \mid (t,r) \in S(d) \text{ and } t_c \geq 1 \}$ is compact, so the infimum is attained, and this gives 4).

To establish 5), we return to the proof of (17) of Theorem 8.

We note that it suffices to prove that $H(\delta) = 0$ for some $\delta > 0$ where H is the function of (19). Then we obtain some $\tau > 0$ such that $\inf\left\{V(t,r,\pi,\sigma) \mid (t,r)\in S(d) \text{ and } r_c \geq \tau\right\} = \pi_0$, and by compactness, this infimum will be attained, giving 5).

In the proof of Theorem 8, we noted that, if $H(\delta) > 0$ for all $\delta > 0$, then (21) holds. Let us suppose, for the sake of contradiction $H(\delta) > 0$ for all $\delta > 0$.

There are only finitely many t for which there exists r with $(t,r)\in S(d)$, and for each of these t the program

$$
(P_{\delta,t})\quad
\begin{aligned}
&\inf \sum_{a\in A} \pi_a t_a + \sum_{b\in B} \sigma_b r_b \\
&\text{subject to } \sum_{a\in A} m_a t_a + \sum_{b\in B} p_b r_b = d \\
&\qquad r_c \geq \delta \\
&\qquad r_b \geq 0 \,,\ b \in B
\end{aligned}
$$

is linear. Hence, for each t, by the results of parametric linear programming, the criterion value to $(P_{\delta,t})$ is piecewise-linear in $\delta > 0$. Therefore, $H(\delta)$ is also piecewise-linear in δ, since it derives from the best criterion value in $(P_{\delta,t})$ for all these finitely many t. But a piecewise linear, nonnegative function $H(\delta)$ with $H(0) = 0$, has positive derivative at the origin if it is positive for all $\delta > 0$, contradicting (21). Q.E.D.

A few simple remarks, which are clear from the proof of Theorem 8, are of some interest. First, the converse direction does not require the boundedness hypothesis. Second, if one is concerned only whether some one specific π_c, for $c \in A$, cannot be reduced, then the hypothesis of a bound T_c on t_c alone is sufficient to make this question equivalent to whether or not (16) holds. A similar remark holds regarding (17), if one is concerned whether or not one specific σ_c, for $c \in B$, cannot be reduced.

The inequality (16)was proven for the group relaxation with $B = \emptyset$, and A a group, in [5]. Extensions of this result for various cases and groups A have been established by Gomory and Johnson [6], and by Johnson [9], in the mixed-integer group relaxation.

C. E. Blair [3] has extended the characterization of the necessity conditions of Theorem 9, to the case of $A \cup B$ finite, when all m_a, p_b are rational vectors.

It is worthwhile if we make a few remarks which relate Gomory's concept of "minimal inequality" to the usual concept of supporting hyperplane.

Clearly, a minimal inequality must be a supporting hyperplane, or π_0 would be increased in (11). On the other hand, not every supporting hyperplane gives a minimal inequality; e.g., for the group relaxation, the facet $t_1 \geq 0$ is not a minimal inequality, since it can be "improved" to the minimal inequality $0 \cdot t_1 \geq 0$.

<u>Corollary 10</u>:

Assume that (P) is bounded $A \cup B$ is finite, for every $a \in A$ there is $(t,r) \in S(d)$ with $t_a > 0$, and for every $b \in B$ there is $(t,r) \in S(d)$ with $r_b > 0$.

The inequality (11) is minimal if and only if it is a supporting hyperplane to the convex hull CH of the feasible solutions S(d) to (P) at a positive point, i.e., if there exists $(t,r) \in CH$, $(t,r) > 0$, such that $V(t,r,\pi,\sigma) = \pi_0$.

<u>Proof</u>: Clearly, a supporting hyperplane at a positive point $(t,r) > 0$, $(t,r) \in CH$, is minimal, since no $\pi_a, a \in A$, and no σ_b, $b \in B$, can be reduced without reducing $V(t,r,\pi,\sigma) = \pi_0$; and similarly π_0 cannot be increased.

Next, suppose (11) is minimal. Then by Theorem 9, for every $a \in A$ we find $(t^a, r^a) \in S(d)$ with $t_a^a \geq 1$ and $V(t^a, r^a, \pi,\sigma) = \pi_0$, and for every $b \in B$, there is $(t^b, r^b) \in S(d)$ with $r_b^b > 0$, such that $V(t^b, r^b, \pi,\sigma) = \pi_0$. It is then easily checked that the point (t,r) given by

$$(t,r) = \Big(\sum_{a \in A} (t^a, r^a) + \sum_{b \in B} (t^b, r^b) \Big) / n,$$

where n is the size of $A \cup B$, is a strictly positive point of CH, and $V(t,r,\pi,\sigma) = \pi_0$. Q.E.D.

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Carnegie-Mellon University
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We provide a characterization of minimal inequalities for bounded mixed integer programs is given in terms of subadditive functions. The condition on the columns of the integer-constrained variables is analogous to that obtained earlier for the group problem, and we also determines the condition on the columns of the continuous variables.

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