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TREATMENT CONTRASTS IN PAIRED COMPARISONS
I. BASIC PROCEDURES WITH APPLICATION TO FACTORIALS[1]

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SUMMARY

A simple procedure for consideration of specified treatment contrasts or sets of contrasts in a paired comparisons experiment is developed. General likelihood estimation and likelihood ratio tests are given. Specified treatment comparisons as appropriate in a particular experiment may be made. The procedure may be used for consideration of factor effects and interactions when the treatments in paired comparisons are factorial treatment combinations. An example is given of a taste preference experiment on coffee with factors, brew strength, roast color and brand, each at two levels. Results are summarized in an analysis of chi-square table very analogous to the typical analysis of variance summary.

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1. INTRODUCTION

Bradley and Terry (1952) presented a model and a method of analysis for paired comparisons generalized slightly by Dykstra (1960). In the basic experiment, t treatments, $T_1, \ldots, T_t$, are compared with n_{ij} comparisons of T_i and T_j, $n_{ij} \geq 0$, $i < j$, $i,j = 1, \ldots, t$. Some of the comparison sizes n_{ij} may be zero but linkage of comparisons is required in the sense that there must not be any subset of the treatments for which no treatment is compared with any treatment of the complementary subset. The model postulates the existence of treatment parameters, π_i for T_i, $\pi_i \geq 0$,

$$\sum_i \pi_i = 1 \quad , \tag{1.1}$$

such that the probability of selection of T_i from the pair (T_i,T_j), $j \neq i$, is

$$P(T_i > T_j) = \pi_i/(\pi_i + \pi_j) \quad . \tag{1.2}$$

Likelihood methods were used. On the assumption of independence of selections, the likelihood function is

$$L(\underline{\pi}) = \prod_i \pi_i^{a_i} \Big/ \prod_{i<j} (\pi_i + \pi_j)^{n_{ij}} \quad , \tag{1.3}$$

where a_i is the total number of selections of T_i in the entire experiment, $i = 1, \ldots, t$, $\sum_i a_i = \sum_{i<j} n_{ij}$, and $\underline{\pi}' = (\pi_1, \ldots, \pi_t)$. When $L(\underline{\pi})$ is maximized subject to (1.1) as a constraint and if p_i is the likelihood estimator of π_i (and $\underline{p}$ of $\underline{\pi}$), the likelihood equations are

$$\frac{a_i}{p_i} - \sum_j{}' \frac{n_{ij}}{p_i + p_j} = 0 \ , \quad i = 1, \ldots, t \quad , \tag{1.4}$$

$$\sum_i p_i = 1 \quad , \tag{1.5}$$

where $\sum_j{}'$ represents a sum with $j \neq i$. Bradley (1975) has summarized the various bases for the model and results associated with it, giving an extensive bibliography.

In applications, situations arise in which special comparisons or contrasts among treatments are of interest. Abelson and Bradley (1954) considered the 2×2 factorial and El-Helbawy (1974) the 2^m factorial. This paper considers orthogonal treatment contrasts more generally and the results may be applied to any set of factorial treatment combinations. The factorial models attempted earlier have been modified and simplified although numerical results in applications are closely similar.

David (1963, Section 4.1) noted that the basic method is unchanged if (1.1) is replaced by other scale-determining constraints and suggested

$$\sum_i \ln \pi_i = 0 \quad . \tag{1.6}$$

Bradley (1953), in comparing the method with that of Thurstone (1927), suggested that $\ln \pi_i$ plays the role of a location parameter for T_i. Treatment contrasts will be specified as linear contrasts among

$$\gamma_i = \ln \pi_i \ , \quad i = 1, \ldots, t \quad . \tag{1.7}$$

2. LIKELIHOOD ESTIMATION

Let $\underline{B}_m$ be an $m \times t$ matrix, $0 \leq m \leq (t - 1)$, with zero-sum, orthonormal rows. Let $\underline{\gamma}(\underline{\pi})$ and $\underline{\gamma}(\underline{p})$ be the column vectors with i-th elements γ_i in (1.7) and $\ln p_i$, $i = 1, \ldots, t$; let $\underline{\Gamma}$ be an m-element column vector with elements Γ_k, $k = 1, \ldots, m$, to be used as Lagrange multipliers; and let $\underline{0}_m$

and $\underline{1}_t$ be respectively column vectors of m zero elements and t unit elements. The general estimation problem is to find estimators p_i of π_i, $i = 1, \ldots, t$, or $\underline{p}$ of $\underline{\pi}$, maximizing $L(\underline{\pi})$ in (1.3) subject to the constraints,

$$\begin{bmatrix} \underline{1}_t' \\ \underline{B}_m \end{bmatrix} \underline{\gamma}(\underline{\pi}) = \underline{0}_{m+1} \tag{2.1}$$

incorporating (1.6). Thus, using the method of Lagrange multipliers, we maximize

$$Q(\underline{\pi}) = \ell n\, L(\underline{\pi}) + \underline{\gamma}'(\underline{\pi})[\underline{1}_t, \underline{B}_m'] \begin{bmatrix} \Gamma_0 \\ \underline{\Gamma} \end{bmatrix} \tag{2.2}$$

subject to (2.1). Throughout this work we assume that $p_i > 0$, $i = 1, \ldots, t$; Ford (1957), in showing convergence of an iterative solution to (1.4) and (1.5) assured this by the following Ford Condition: In every possible partition of the t treatments into two non-empty subsets, some treatment in the second subset has been preferred at least once to some treatment in the first subset.

Let

$$E_i(\underline{p}) = a_i - \sum_j{}' \frac{n_{ij} p_i}{p_i + p_j}, \quad i = 1, \ldots, t \;, \tag{2.3}$$

and let $\underline{E}(\underline{p})$ be the t-element column vector with elements in (2.3). The equations necessary for maximization of $Q(\underline{\pi})$ reduce to

$$\underline{E}(\underline{p}) + \underline{B}_m'\underline{\Gamma} = \underline{0}_t \;, \tag{2.4}$$

and

$$\underline{B}_m \underline{\gamma}(\underline{p}) = \underline{0}_m \tag{2.5}$$

together with the scale-determining equation,

$$\sum_i \ln p_i = 0 \tag{2.6}$$

or (1.5) as desired; it is easy to show that $\Gamma_0 = 0$. Multiplication on the left of (2.4) by $\underline{B}_m$ shows that $\underline{\Gamma} = -\underline{B}_m\underline{E}(\underline{p})$ and that (2.4) reduces to

$$(\underline{I}_t - \underline{B}_m'\underline{B}_m)\underline{E}(\underline{p}) = \underline{0}_t \quad , \tag{2.7}$$

where $\underline{I}_t$ is the $t \times t$ identity matrix. Let

$$\underline{D} = \underline{I}_t - \underline{B}_m'\underline{B}_m \tag{2.8}$$

with typical element D_{ij}. Then (2.7) may be rewritten

$$\frac{a_i}{p_i} - \phi_i(\underline{p}) = 0 \ , \quad i = 1, \ldots, t \quad , \tag{2.9}$$

where

$$\phi_i(\underline{p}) = \sum_j{}' \frac{n_{ij}}{p_i + p_j} - \frac{1}{p_i}\sum_j{}' E_j(\underline{p}) \frac{D_{ij}}{D_{ii}} \quad . \tag{2.10}$$

Note that $D_{ii} > 0$, $m = 1, \ldots, t - 2$. When $m = (t - 1)$, a deterministic solution follows from (2.1) alone - each $\gamma_i = 0$, $\pi_i = 1$ or, with scale adjustment, each $\pi_i = 1/t$. When $m = 0$, $\underline{B}_m$ is non-existent and (2.4) or (2.7) reduces to (1.4) or we may use (2.9) with $D_{ij} = 0$, $i \neq j$, $D_{ii} = 1$, $i,j = 1, \ldots, t$, in (2.10). An iterative solution to (2.9), (2.5), and (2.6) is considered now.

3. ITERATIVE SOLUTION OF LIKELIHOOD EQUATIONS

Let $\underline{p}^{(0)}$ and $\underline{p}^{(r)}$ satisfying (2.5) and (2.6) be initial and r-th approximations to $\underline{p}$, the solution of (2.9), (2.5), and (2.6). That $\underline{p}^{(0)}$ exists is clear because $\underline{p}^{(0)} = \underline{1}_t$ meets the requirements. The means of obtaining $\underline{p}^{(r+1)}$ from $\underline{p}^{(r)}$ are described.

Think of $r = t(J - 1) + j - 1$, $J = 1,2,3, \ldots,$ $j = 1, \ldots, t$; J indicates a cycle of iterations and j, the step in the J-th cycle. To initiate the calculation of $\underline{p}^{(r+1)}$ from $\underline{p}^{(r)}$, calculate $\phi_j(\underline{p}^{(r)})$ from (2.10). The iteration is done by calculation of

$$p_i^{(r+1)}(k) = \exp[\ell n\, p_i^{(r)} + (\tfrac{1}{2})^k \Delta_j^{(r)}(D_{ij} - \tfrac{1}{t})] , \quad i = 1, \ldots, t , \tag{3.1}$$

for successive values of k, $k = 0,1,2, \ldots,$ where

$$\Delta_j^{(r)} = \begin{cases} \ell n[a_j/p_j^{(r)} \ \ \phi_j(\underline{p}^{(r)})] & \text{if } \phi_j(\underline{p}^{(r)}) \geq 0 \\ 1 \ \text{otherwise.} & \end{cases} \tag{3.2}$$

When $a_j/p_j^{(r)} = \phi_j(\underline{p}^{(r)})$, $\underline{p}^{(r+1)} = \underline{p}^{(r)}$; if this occurs for t successive values of r, $\underline{p}^{(r)} = \underline{p}$. When $a_j/p_j^{(r)} \neq \phi_j(\underline{p}^{(r)})$, compute $\underline{p}^{(r+1)}(k)$ and $L[\underline{p}^{(r+1)}(k)]$ for $k = 0,1,2, \ldots$ until a value k^* of k is found for which $L[\underline{p}^{(r+1)}(k^*)] > L(\underline{p}^{(r)})$. Then

$$\underline{p}^{(r+1)} = \underline{p}^{(r+1)}(k^*) \ .$$

El-Helbawy and Bradley (1975) show that k^* exists and examine the convergence of this iterative process. The procedure is easy to program on a computer.

Note that $\underline{p}^{(r+1)}$ satisfies conditions (2.5) and (2.6) if $\underline{p}^{(r)}$ does. This is demonstrated easily for $\underline{p}^{(r+1)}(k)$ for all k through use of (3.1), the orthonormality of the rows of $\underset{\sim}{B}_m$, and the fact that $\underset{\sim}{B}_m$ has zero-sum rows. Note also that if $m = 0$, $D_{ij} = 0$, $i \neq j$, $D_{ii} = 1$, $i,j = 1, \ldots, t$, $k = 0$, and then $\phi_j(\underset{\sim}{p}^{(r)}) > 0$ and

$$p_j^{(r+1)} = \frac{a_j}{\phi_j(\underset{\sim}{p}^{(r)})} c_{r+1} \, , \quad p_i^{(r+1)} = p_i^{(r)} c_{r+1} \, , \quad i \neq j$$

where

$$c_{r+1} = \left[\frac{a_j}{p_j^{(r)} \phi_j(\underset{\sim}{p}^{(r)})} \right]^{-1/t}$$

is such that $\prod_i p_i^{(r+1)} = 1$. Then the iterative procedure is equivalent to the one suggested by Bradley and Terry (1952) and shown to converge by Ford (1957).

4. INFERENCE AND FACTORIALS

Likelihood estimation and likelihood ratio tests were proposed by Bradley and Terry in consideration of the basic model for paired comparisons used here. In extending those results, we can formulate a single test procedure that may be used to test particular treatment contrasts or for the analysis of factorial treatment combinations.

Consider two matrices $\underset{\sim}{B}_m$ and $\underset{\sim}{B}_{m+n} = \begin{bmatrix} \underset{\sim}{B}_m \\ \underset{\sim}{B}_n \end{bmatrix}$ where $0 \leq m < m + n \leq (t - 1)$, $\underset{\sim}{B}_m$ defined as before and $\underset{\sim}{B}_{m+n}$ having zero-sum, orthonormal rows. We assume that $\underset{\sim}{B}_m \underset{\sim}{\gamma}(\underset{\sim}{\pi}) = \underset{\sim}{0}_m$ and test the hypothesis,

$$H_0: \ \underset{\sim}{B}_n \underset{\sim}{\gamma}(\underset{\sim}{\pi}) = \underset{\sim}{0}_n \ , \qquad (4.1)$$

against the alternative,

$$H_a: \underline{B}_n \underline{\gamma}(\underline{\pi}) \neq \underline{0}_n \quad . \tag{4.2}$$

Let $\underline{\pi} = \underline{p}_0$ maximize $L(\underline{\pi})$ given $\underline{B}_{m+n}\underline{\gamma}(\underline{\pi}) = \underline{0}_{m+n}$ and (1.6) and let $\underline{\pi} = \underline{p}_a$ maximize $L(\underline{\pi})$ given $\underline{B}_m\underline{\gamma}(\underline{\pi}) = \underline{0}_m$ and (1.6); both $\underline{p}_0$ and $\underline{p}_a$ may be obtained through use of the likelihood estimation procedure given in Sections 2 and 3. The likelihood ratio statistic for testing H_0 versus H_a given $\underline{B}_m\underline{\gamma}(\underline{\pi}) = \underline{0}_m$ is

$$-2 \ln \lambda = -2[\ln L(\underline{p}_0) - \ln L(\underline{p}_a)] \tag{4.3}$$

which, for large values of the n_{ij}, has the central chi-square distribution with n degrees of freedom as its limiting distribution under H_0. When $m = 0$ and $n = (t - 1)$, the original chi-square test with $(t - 1)$ degrees of freedom for treatment equality is obtained. The test based on (4.3) may be used to partition that chi-square with $(t - 1)$ degrees of freedom into several independent and additive chi-squares for properly chosen and sequenced orthonormal sets of contrast hypothesesand designated assumed null contrasts.

An important and natural application of the technique developed is to experiments wherein the treatments constitute a full or fractional set of factorial treatment combinations. Tests of factorial effects, main effects and interactions, may be made directly based on (4.3) and suitably chosen rows for $\underline{B}_m$ and $\underline{B}_{m+n}$. Alternatively, the paired comparisons model may be reparameterized to introduce factorial parameters.

Consider a q-factor mixed factorial for which factor s has b_s levels, $b_s \geq 2$, $s = 1, \ldots, q$ and let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_q)$ be a vector designating the levels of the q factors for a particular treatment combination. Now replace T_i by $T_{\underline{\alpha}}$ and π_i by $\pi_{\underline{\alpha}}$; if a full factorial is considered, $t = \prod_{s=1}^{q} b_s$. Factorial parameters are introduced multiplicatively; take

$$\pi_{\underline{\alpha}} = \prod_{i_1} \pi_{\alpha_{i_1}}^{i_1} \prod_{i_1<i_2} \pi_{\alpha_{i_1}\alpha_{i_2}}^{i_1 i_2} \cdots \prod_{i_1<\ldots<i_r} \pi_{\alpha_{i_1}\cdots\alpha_{i_r}}^{i_1\cdots i_r} \cdots \pi_{\alpha_1\cdots\alpha_q}^{1\cdots q} \qquad (4.4)$$

for all $\underline{\alpha}$ where $1 \le i_1 < \ldots < i_r \le q$, $r = 1, \ldots, q$, and $\pi_{\alpha_{i_1}\cdots\alpha_{i_r}}^{i_1\cdots i_r}$ represents the r-factor interaction (main effect if $r = 1$) of factors $i_1, \ldots, i_r$ at levels $\alpha_{i_1}, \ldots, \alpha_{i_r}$. The new factorial parameters are $\prod_{s=1}^{q} (1 + b_s) - 1$ in number and must be subject to $[\prod_{s=1}^{q} (1 + b_s) - \prod_{s=1}^{q} b_s]$ functionally independent constraints. But (4.4) is linear in the logarithms of parameters. The system is exactly analogous to that of the analysis of variance (anova) of, say, a randomized complete block design with the same factorial treatment combinations. The anova constraints on anova factorial parameters apply to the $\ell n\, \pi_{\alpha_{i_1}\cdots\alpha_{i_r}}^{i_1\cdots i_r}$. If $p_{\alpha_{i_1}\cdots\alpha_{i_r}}^{i_1\cdots i_r}$ is the likelihood estimator of $\pi_{\alpha_{i_1}\cdots\alpha_{i_r}}^{i_1\cdots i_r}$, $\ell n\, p_{\alpha_{i_1}\cdots\alpha_{i_r}}^{i_1\cdots i_r}$ is the same linear function of the $\ell n\, p_{\underline{\alpha}}$ as is the anova estimator of the corresponding anova interaction parameter of the anova treatment effects estimators. Known algorithms for the analysis of factorial experiments may be applied in the analysis of factorial treatment combinations in paired comparisons. An hypothesis of no r-factor interaction among factors $i_1, \ldots, i_r$ is formulated by the specification that the usual $\prod_{s=1}^{r} (b_{i_s} - 1)$ interaction contrasts among the $\ell n\, \pi_{\alpha_{i_1}\cdots\alpha_{i_r}}^{i_1\cdots i_r}$ are zero, a specification equivalent to the requirement that each $\ln \pi^{i_1 \ldots i_r}_{\alpha_{i_1} \ldots \alpha_{i_r}} = 0$ in view of the parameter constraints or that $\pi^{i_1 \cdots i_r}_{\alpha_{i_1} \cdots \alpha_{i_r}} = 1$ for $\alpha_{i_s} = 1, \ldots, b_{i_s}$, $s = 1, \ldots, r$.

It has been seen that likelihood estimators $\underline{p}$ of parameters $\underline{\pi}$ under constraints (2.1) may be found for various choices of $\underline{B}_m$. In each such situation, multivariate normal distributions may be specified for the large-sample distributions of $\sqrt{N}\,(\underline{p} - \underline{\pi})$ or of $\sqrt{N}\,[\underline{\gamma}(\underline{p}) - \underline{\gamma}(\underline{\pi})]$ with means zero and determined variance-covariance matrices where $n_{ij} = N\lambda_{ijN}$, $\lambda_{ijN} \to \lambda_{ij}$ as $N = \sum_{i<j} n_{ij} \to \infty$. Large-sample variances and covariances for specified treatment contrasts may be found and power functions for local alternatives for tests based on (4.3) may be specified. While these results are needed in some aspects of data analysis for paired comparisons, they will be given by the authors in a subsequent paper.

5. EXAMPLE - A 2^3 FACTORIAL

Brew strength, roast color, and coffee brand were the three factors, each at two levels, in an example of the use of paired comparisons with factorial treatment combinations in consumer preference testing. Twenty-six preference judgments were obtained on each of the 28 possible treatment comparisons. The data are summarized in Table 1. The three-element vectors $\underline{\alpha}$ and $\underline{\beta}$ designate the treatment combinations.

TABLE 1 - Preference Data in Coffee Testing[1]

Preferred Treatment $T_{\underline{\alpha}}$	Treatment Not Preferred $T_{\underline{\beta}}$								Total Preferences
$\underline{\alpha}$	$\underline{\beta}$: 000	001	010	011	100	101	110	111	$a_{\underline{\alpha}}$.
000	0	15	15	16	19	14	19	16	114
001	11	0	10	15	15	14	15	12	92
010	11	16	0	15	15	14	18	15	104
011	10	11	11	0	14	11	15	13	85
100	7	11	11	12	0	9	14	13	77
101	12	12	12	15	17	0	16	18	102
110	7	11	8	11	12	10	0	12	71
111	10	14	11	13	13	8	14	0	83

[1] Data provided through the courtesy of Mavis B. Carroll and John C. Heimlich and of the General Foods Corporation.

Consider the matrix,

$$\underline{B}_7 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} . \tag{5.1}$$

The overall test of treatment preference equality, equivalent to that of Bradley and Terry (1952), is obtained from (4.3) with $\underline{B}_n$ in (4.1) equal to $\underline{B}_7$ in (5.1) and $m = 0$. Under the null hypothesis (4.1), each $\pi_{\underline{\alpha}} = 1$ and $L(\underline{p}_0)$ in (4.3) is $L(\underline{1}_8)$ - the scale constraint (1.6) is used instead of (1.1). In the analysis summary of Table 2, $-2\ \ell n\ L(\underline{1}_8)$ is given in the first row. Under the alternative hypothesis (4.2), $m = 0$ and $\underline{B}_0$ does not exist; $L(\underline{\pi})$ is maximized subject only to (1.6) with the resultant values of $\underline{p}_a$ and $-2\ \ell n\ L(\underline{p}_a)$ given in row 2 of Table 2. The chi-square statistic of (4.3) is, by subtraction, 29.58 with 7 degrees of freedom. This value is shown in the totals rows of analysis of chi-square Tables 3 and 4.

The total chi-square may be partitioned in various ways, two of which are shown in Tables 3 and 4. We give details on Table 3 and leave the reader to check Table 4. Hypothesize no three-factor interaction. Then $\underline{B}_n = \underline{B}(7)$, the last row in (5.1), $m = 0$, B_m does not exist and $\underline{B}_{m+n} = \underline{B}_1(7)$. Values of $\underline{p}$ and $-2\ \ell n\ L(\underline{p})$ under H_0 are given in row 3 of Table 2. The chi-square statistic of (4.3), now with 1 degree of freedom, is the difference of entries $-2\ \ell n\ L(\underline{p})$ in rows 2 and 3 of Table 2, the value being 0.63. Now assume no three-factor interaction and hypothesize no two-factor interactions; $\underline{B}_m = \underline{B}_1(7)$, $\underline{B}_n = \underline{B}_3(4,5,6)$, $\underline{B}_{m+n} = \underline{B}_4(4,5,6,7)$, the arguments of $\underline{B}$ indicating the rows chosen from (5.1). The required chi-squared statistic from (4.3) has 3 degrees of freedom and the value 15.34 from rows 3 and 4 of Table 2. This chi-square may be partitioned in a number of ways and we choose only one of them. Chi-square statistics, each with 1 degree of freedom, are calculated for the following hypotheses and assumptions: (i) No brand, roast color interaction; no three-factor interaction assumed, (ii) No brand, brew strength interaction, no three-factor interaction and no brand, roast color interaction assumed, (iii) no roast

TABLE 2 - Parameter Estimates $\underline{p}$ and $-2\ell nL(\underline{p})$ for Various Constraints $\underline{P}_{n+n} = \underline{0}_{n+n}$

Row	$\underline{B}_{n+n}$	p_{000}	p_{001}	p_{010}	p_{011}	p_{100}	p_{101}	p_{110}	p_{111}	$-2\ell nL(\underline{p})$
1	$\underline{B}_7$ in (5.1)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1009.222
2	Non-existent	1.5739	1.0206	1.2915	0.8897	0.7592	1.2408	0.6728	0.8548	979.645
3	$\underline{B}_1(7)$ (See text)	1.5146	1.0599	1.3417	0.8552	0.7903	1.1931	0.6471	0.8899	980.275
4	$\underline{B}_4(4,5,6,7)$	1.3002	1.2753	1.0598	1.0395	0.9620	0.9435	0.7841	0.7691	995.611
5	$\underline{B}_2(6,7)$	1.5499	1.0345	1.3111	0.8751	0.8093	1.1675	0.6315	0.9110	980.499
6	$\underline{B}_3(5,6,7)$	1.2724	1.2500	1.0806	1.0616	0.9805	0.9633	0.7685	0.7550	995.457
7	$\underline{B}_5(3,4,5,6,7)$	1.2847	1.2847	1.0479	1.0479	0.9543	0.9543	0.7784	0.7784	995.652
8	$\underline{B}_6(2,3,4,5,6,7)$	1.1611	1.1611	1.1611	1.1611	0.8613	0.8613	0.8613	0.8613	999.946
9	$\underline{B}_1(1)$	1.3463	0.8778	1.1074	0.7642	0.8849	1.4426	0.7871	0.9953	989.114
10	$\underline{B}_2(1,2)$	1.2145	0.7921	1.2261	0.8479	0.8000	1.2995	0.8729	1.1020	993.443
11	$\underline{B}_3(1,2,3)$	1.2033	0.7995	1.2149	0.8556	0.7914	1.3135	0.8643	1.1131	993.482
12	$\underline{B}_6(1,2,3,4,5,6)$	1.0384	0.9630	0.9630	1.0384	0.9630	1.0384	1.0384	0.9630	1008.606
13	$\underline{B}_4(1,2,3,4)$	1.2253	0.8161	1.1923	0.8387	0.7767	1.2875	0.8813	1.1347	993.638
14	$\underline{B}_5(1,2,3,4,5)$	1.0143	0.9859	0.9859	1.0143	0.9400	1.0638	1.0638	0.9400	1008.365

color, brew strength interaction; no three-factor interaction, no brand, roast color interaction, and no brand, brew strength interaction assumed. The matrices $\underline{B}_m$, $\underline{B}_n$ and $\underline{B}_{m+n}$ respectively are (i) $\underline{B}_1(7)$, $\underline{B}_1(6)$, $\underline{B}_2(6,7)$, (ii) $\underline{B}_2(6,7)$, $\underline{B}_1(5)$, $\underline{B}_3(5,6,7)$, (iii) $\underline{B}_3(5,6,7)$, $\underline{B}_1(4)$, $\underline{B}_4(4,5,6,7)$. Pertinent row pairs in Table 2 are (i) 3,5, (ii) 5,6, (iii) 6,4 and the chi-square values are 0.22, 14.96, 0.15. Three remaining degrees of freedom are for tests on main effects. The following hypotheses and assumptions are used: (iv) No brand effect, no interactions assumed; (v) No roast color effect; no brand effect and no interactions assumed; (vi) No brew strength effect; no roast color effect, no brand effect and no interactions assumed. The matrices $\underline{B}_m$, $\underline{B}_n$ and $\underline{B}_{m+n}$, the row pairs of Table 2 and the corresponding values of chi-square, each with 1 degree of freedom, are as follows: (iv) $\underline{B}_4(4,5,6,7)$, $\underline{B}_1(3)$, $\underline{B}_5(3,4,5,6,7)$, 4, 7, 0.04; (v) $\underline{B}_5(3,4,5,6,7)$, $\underline{B}_1(2)$, $\underline{B}_6(2,3,4,5,6,7)$, 7, 8, 4.29; (vi) $\underline{B}_6(2,3,4,5,6,7)$, $\underline{B}_1(1)$, $\underline{B}_7$, 8, 1, 9.28. These results are given in Table 3. The alternative analysis of Table 4 is based on rows 1, 2, 9-14 of Table 2.

The analyses of Tables 3 and 4 are remarkably similar. While our purpose has been to illustrate a technique, it is clear from both analyses that brew strength and roast color have main effects while brand interacts with brew strength even though it has no main effect.

Factorial parameters may be estimated. We illustrate using the full factorial model and $\underline{p}$ from row 2 of Table 2. Because of the constraints on the factorial parameters and the one degree of freedom available for the estimation of each main effect or interaction, each main effect or interaction may be obtained from the vector $\frac{1}{8}\underline{B}_7\underline{\gamma}(\underline{p})$. Thus

TABLE 3 - An Analysis of Chi Square for the Coffee Data

Hypothesis Tested[1]	Conditions Assumed	Degrees of Freedom	Chi Square
No F_1 effect	No F_2, F_3, No interactions	1	9.28
No F_2 effect	No F_3, No interactions	1	4.29
No F_3 effect	No interactions	1	0.04
No F_1F_2, F_1F_3, F_2F_3 interactions	No $F_1F_2F_3$ interaction	3	15.34
No F_2F_3 interaction	No $F_1F_2F_3$ interaction	1	0.22
No F_1F_3 interaction	No F_2F_3, $F_1F_2F_3$ interactions	1	14.96
No F_1F_2 interaction	No F_1F_3, F_2F_3, $F_1F_2F_3$ interactions	1	0.15
No $F_1F_2F_3$ interaction	None	1	0.63
No treatment effects	None	7	29.58

[1] F_1 is brew strength, F_2 is roast color, F_3 is brand.

TABLE 4 - An Alternative Analysis of Chi Square for the Coffee Data

Hypothesis Tested[1]	Conditions Assumed	Degrees of Freedom	Chi Square
No F_1 effect	None	1	9.47
No F_2 effect	No F_1 effect	1	4.33
No F_3 effect	No F_1, F_2 effects	1	0.04
No F_1F_2, F_1F_3, F_2F_3 interactions	No main effects	3	15.12
No F_1F_2 interaction	No main effects	1	0.16
No F_1F_3 interaction	No main effects, No F_1F_2 interaction	1	14.73
No F_2F_3 interaction	No main effects, No F_1F_2, F_1F_3 interactions	1	0.24
No $F_1F_2F_3$ interaction	No main effects, No two-factor interactions	1	0.62
No treatment effects	None	7	29.58

[1] F_1 is brew strength, F_2 is roast color, F_3 is brand.

$$\ln p_0^1 = 0.1534\ ,\quad \ln p_0^2 = 0.1038\ ,\quad \ln p_0^3 = 0.0096\ ,\quad \ln p_{00}^{12} = -0.0196\ ,$$

$$\ln p_{00}^{13} = 0.1924\ ,\quad \ln p_{00}^{23} = -0.0238\ ,\quad \ln p_{000}^{123} = 0.0393\ ,$$

and

$$p_0^1 = 1.1658\ ,\quad p_0^2 = 1.1094\ ,\quad p_0^3 = 1.0097\ ,\quad p_{00}^{12} = 0.9806\ ,$$

$$p_{00}^{13} = 1.2122\ ,\quad p_{00}^{23} = 0.9765\ ,\quad p_{000}^{123} = 1.0401\ .$$

These estimates are listed in the same order as the pertinent rows in $\underline{B}_7$. The remaining factorial parameter estimates are identical to or the reciprocal of those shown for the same effect or interaction depending on whether the sum of the subscripts is even or odd. For example, $p_1^1 = (1.1658)^{-1} = 0.8578$, $p_{11}^{12} = 0.9806$, $p_{001}^{123} = (1.0401)^{-1} = 0.9614$.

It is seen that the analysis for the factorial treatment combinations in paired comparisons is very similar to the analysis of variance for factorials. Interpretations are made in the usual way, particularly when the logarithms of factorial parameter estimates are considered.

7. CONCLUDING REMARKS

The techniques of this paper provide means for much new flexibility in the analysis of paired comparisons experiments. Factorial treatment combinations may be used. Since specified treatment contrasts may be used generally, fractional factorials are available also. Special treatment contrasts may have interest in special experimental situations.

It has been noted that convergence of the iteration process given for solution of likelihood equations, asymptotic variances and covariances for treatment contrasts, and large-sample properties of test procedures will be given in subsequent papers. The similarity of analyses in Tables 3 and 4 suggests that conditions assumed for a test may not be constraining and that tests in the two tables may be asymptotically equivalent. This possibility will be investigated.

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20. ABSTRACT

A simple procedure for consideration of specified treatment contrasts or sets of contrasts in a paired comparisons experiment is developed. General likelihood estimation and likelihood ratio tests are given. Specified treatment comparisons as appropriate in a particular experiment may be made. The procedure may be used for consideration of factor effects and interactions when the treatments in paired comparisons are factorial treatment combinations. An example is given of a taste preference experiment on coffee with factors, brew strength, roast color and brand, each at two levels. Results are summarized in an analysis of chi square table very analogous to the typical analysis of variance summary.