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STRUCTURAL RESPONSE TO MOVING PROJECTILE MASS
BY THE FINITE ELEMENT METHOD

JULY 1975



BENET WEAPONS LABORATORY
WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189

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This work establishes the feasibility of handling problems of projectile/bore interaction via the method of finite elements. The problem treated herein involves the analytical determination of the time-variant matrix coefficients for the structure equations of motion in a manner consistent with the finite element displacement method. The resultant equations of motion are solved numerically and results compare favorably with those reported by experimentalists in the literature. For this purpose, the otherwise general (see other side)														

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procedure is applied to the problem of predicting the response of a simply supported beam to a concentrated mass in motion at constant velocity. The technique can be easily extended to gun tube geometries by merely altering the boundary conditions, increasing the number of finite elements and allowing their dimensions to approximate the tube of interest. Similarly, the feature of variable projectile velocity can easily be included.

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STRUCTURAL RESPONSE TO MOVING PROJECTILE MASS
BY THE FINITE ELEMENT METHOD

T. E. SIMKINS

JULY 1975



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SUMMARY

The results displayed in this report demonstrate the feasibility of handling moving mass problems by the finite element displacement method. There is no apparent reason why the method could not be applied to gun tube geometries - should engineering fund allocations be made available for this purpose. (Nearly all of the work contained in this report was funded through basic in-house research allocations.)

In the interest of improved efficiency, more detailed study should be given to the choice and use of integration algorithms. In the IBM version of Hamming's method (employed in this work), the user is allowed certain freedom to choose error weights. This choice can affect accuracy and run times and therefore should be arrived at on a rational basis.

ACKNOWLEDGMENT

It is estimated that without the help of computer scientist, David Priest, this work would have been slowed considerably. The author extends his deep appreciation for Mr. Priest's aid.

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INTRODUCTION

Within the U.S. Army there has always been an interest in the ballistic response of the gun tube and projectile during firing. This interest has several justifications - not the least of which are the effects of tube motion on round accuracy and the reactions induced between the interior bore surface and the projectile - the latter having to do with the structural integrity of the projectile casing and the wear caused by excessive projectile/bore interface pressures.

During 1974 this laboratory reported a NASTRAN finite element analysis of the axisymmetric response of an unconstrained M113 gun tube to the ballistic pressure/time loading corresponding to a 'zone three' charge. Also computed in detail was the transverse response to the curvature-induced 'Bourdon' load. Less detailed calculations alluded to the potential response which could be expected from the tube as a result of the moving concentrated weight of the projectile along the bore surface. Finally, hope was expressed for future progress in handling the various aspects of the moving projectile mass.

The Importance of the Moving Mass Problem

In more conventional applications the difference in the response of a structure to a moving mass and to the weight of this mass is not very great. However, if the moving mass is an appreciable fraction of the total structural mass of the problem - or if the velocity of the mass is very large, an unconventional structural problem is defined in which the difference may be quite pronounced. There are at least two illustra-

tions which come to mind: (1) - a railroad train crossing a long trestle¹ (in which the moving mass compares to the mass of the trestle) and (2) - a projectile traversing a long tube (in which uncommonly high velocities are evident). We are professionally interested in the latter as a knowledge of the response of a tube to a moving mass leads immediately to a good estimate of the resultant interface pressure between the projectile and the bore surface - an important first step toward the ultimate understanding of the causes of bore and muzzle wear problems and proper design of projectile casings and rotating bands. The importance of the structural response of the tube in altering the round trajectory is also apparent. Thus treatment of the moving mass problem has a wide base of military justification.

State of the Art in Moving Mass Problems

Although the method of computation reported herein is immediately applicable to a finite element beam model of a gun tube, it is first necessary to assess the accuracy of the method by applying it to a problem which has received prior treatment and reported in the literature. The most common base for comparison has been the experimental work of Ayre, Jacobsen, and Hsu² - presumably because of the lack of

(1) Stokes, Sir George G., "Discussion of a Differential Equation Relating to the Breaking of Railway Bridges," Trans Cambridge Phil Soc, 8, p. 707, 1849

(2) Ayre, R.S., Jacobsen, L.S., and Hsu, C.S., "Transverse Vibration of One and of Two Span Beams Under the Action of a Moving Mass Load," Proc. of First National Congress on Applied Mechanics, June 1951

any simple analytical solution with which to compare results. (The analytical solution of Schallenkamp is not convenient for general application - this will be mentioned further on in the report)

This report is therefore concerned with predicting the response of a uniform, simply supported beam while subjected to a concentrated mass moving along its length at constant velocity under the influence of gravity. The method employed, however, is immediately applicable to time variant mass velocities as well as other boundary conditions and variable beam cross section.

Before commencing with the details of the moving mass problem it may be worthwhile to point out the theoretical differences between problems in which masses are in motion and those which involve only moving forces.

To begin with, moving loads* are but special cases of time and space variant forcing functions $f(x,t)$ for one dimensional structures such as beams. The customary beam equation of forced motion can be written³:

$$M\ddot{w} + Lw = f(x,t) \quad \dots (1)$$

where M and L are operators:

$$M = \rho A; \quad L = (EI \partial^2 / \partial x^2)$$

(3) Tong, K., Theory of Mechanical Vibrations, J. Wiley & Sons, 1960
p. 300

* The term 'load' is intended to be general and to represent applied loads such as moving forces and/or masses

Solving the homogeneous form of equation (1) ($f(x,t) = 0$), i.e., the free vibration problem-leads to the eigenfunctions $r_n(x)$ and the eigenvalues ω_n with which any forced motion problem ($f(x,t) \neq 0$) can be solved.

Transforming to modal coordinates $p_i(t)$ defined in the expansion:

$$w(x,t) = \sum_1^{\infty} p_i(t) r_i(x) \quad \dots(2)$$

leads to an infinite number of differential equations which are uncoupled:

$$\text{i.e.} \quad m_{ii}(\ddot{p}_i + \omega_i^2 p_i) = \int_0^L f(x,t) r_i(x) dx \quad \dots(3)$$

$$\text{where} \quad m_{ii} = \int_0^L r_i^2 M(x) dx$$

Now if $f(x,t)$ is a moving concentrated force, i.e.,

$$f(x,t) = -P_0 \delta(x-vt)$$

then (3) becomes:

$$m_{ii}(\ddot{p}_i + \omega_i^2 p_i) = -P_0 r_i(vt) \quad \dots(4)$$

The solution to (4) can be written immediately in terms of the eigenfunctions and eigenvalues of the free vibration problem.

For a moving mass, however,

$$f(x,t) = -m_p \ddot{w} \delta(x-vt)$$

When substituted in (3), the right hand side becomes:

$$-m_p \int_{-\infty}^{\infty} \frac{\partial^2 w}{\partial t^2} r_i(x) \delta(x-vt) dx = \left. \frac{\partial^2 w}{\partial t^2} r_i(x) \right|_{x=vt} (-m_p)$$

$$\text{but } \left. \frac{\partial^2 w}{\partial t^2} \right|_{x=vt} = \ddot{w}(vt) + 2\dot{w}'(vt) + v^2 w''(vt) ; \quad ' = \partial/\partial x$$

$$\text{using (2): } \left. \frac{\partial^2 w}{\partial t^2} \right|_{x=vt} = \sum_j \{ r_j(vt) p_j + 2v r_j'(vt) p_j + v^2 r_j''(vt) p_j \}$$

then (3) becomes:

$$m_{ii} (\ddot{p}_i + \omega_i^2 p_i) = -m_p \sum_j \{ r_j(vt) p_j + 2v r_j'(vt) p_j + v^2 r_j''(vt) p_j \} \dots (5)$$

In contrast to (4), equations (5) are not uncoupled. In fact all of the variables p_i , $i = 1, \dots$, appear in each of the infinite number of differential equations. To make matters worse, each variable has a time-variant coefficient. To date the only exact mathematical treatment appears to be that due to Schallenkamp⁴ involving a triple infinite series equation for unknown Fourier coefficients.

Thus the whole concept of natural frequencies and modes of vibration loses its value in quantitative determination of the response of a system with time variant properties - of which the moving mass is but a special case. For every location of the mass along the beam we have a new infinity of eigenvalue solutions. With an infinite num-

(4) Schallenkamp, A., "Schwingungen von Trägern bei bewegten Lasten," Ingenieur-Archiv, v.8, 1937, pp. 182-198

ber of locations for the mass to occupy, we thus have a double infinity of eigenfunctions and eigenvalues.

Equation (1) can, however, be solved numerically - regardless of the space and time dependent material properties and the forcing term. In what follows we eliminate the space variable through the finite element process. The time variable is handled by a numerical integration procedure of common variety (predictor-corrector). In essence therefore, we approximate our continuum description (1), by a finite number of ordinary differential equations with time dependent coefficients.

Continuum Description of the Moving Mass Problem

In previous work ⁷ a derivation was given for the equation of forced transverse motion of a beam model of the M113 gun tube:

$$\begin{aligned} (EIy'')'' = & -k(x,t)y'' + (\rho A \ddot{X}_0(t) + \rho A g \sin \alpha)y''(x-l) + (\rho A \ddot{X}_0(t) + \\ & + \rho A g \sin \alpha)y' - m_p(\ddot{y} + 2v\dot{y}' + g \cos \alpha + v^2 y'')\delta(x-vt) - \rho g A \cos \alpha - \rho A \ddot{y} \\ & \dots(6) \end{aligned}$$

This equation assumes that A, the beam cross section is uniform and that v - the velocity of the moving mass m_p is constant. In that we will be applying the equation to beam elements of uniform and equal cross section the former assumption is consistent. This restriction is not necessary to the generality of the method. The assumption of

(7) Simkins, T.E., Pflegl, G. and Scanlon, R., "Dynamic Response of the M113 Gun Tube to Travelling Ballistic Pressure and Data Smoothing as Applied to XM15C Acceleration Data," Watervliet Arsenal Technical Report WVT-TR-75015, April 1975

constant velocity is motivated by our desire to compare results with the work of Ayre, Jacobsen and Hsu. In (6) the term $k(x,t)y''$ is the so called 'Bourdon' force. Terms in $\ddot{X}_0$ represent transverse forces induced by axial recoil acceleration when beam curvature and slope are non-zero. Terms in ρAg are due to the beam weight. None of these forces are of interest in this report which deals exclusively with forces induced by the moving mass:

$$\text{i.e.} \quad f(x,t) = -m_p(\ddot{y} + 2v\dot{y}' + g + v^2y'')\delta(x-vt) \quad \dots(7)$$

where α , the tube elevation angle, has been made equal to zero. Thus the special version of equation (6) we will be concerned with in this report:

$$Ely'''' + \rho A \ddot{y} = -m_p(\ddot{y} + 2v\dot{y}' + g + v^2y'')\delta(x-vt) \quad \dots(8)$$

where E, I, ρ , and A assume constant values. The left hand side of (8) is recognized as originating from the simplest of beam theory, i.e., where the entire transverse deflection of the beam is assumed to be due to bending moment only. The right side of (8) therefore represents the totality of applied loads, the first term corresponding to the inertia of m_p ; the second is a 'Coriolis' type load; the third is due to the gravitational force on m_p (its weight) and the fourth is the 'centrifugal' force due to m_p following the beam curvature. The Dirac - delta function δ specifies that each of these forces acts in a concentrated fashion at the location $x = vt$ along the beam.

As written equation (8) is a continuum description of the problem.

Using the method of finite elements the space variable, x , will be discretized - resulting in a set of ordinary differential equations with time as the independent variable.

THE FINITE ELEMENT DISPLACEMENT METHOD

The basic procedure in finite element procedures is to consider the structure of interest as being composed of elements connected together at adjacent attachment points - called nodes or grid points. For the case at hand, the elements will be considered as short beams connected end to end to form the longer beam structure of interest. One then seeks to relate (at least approximately) the displacement at any point interior to an element solely in terms of certain generalized displacements assumed at its attachments.

Figure 1a shows the beam structure of interest (corresponding to equation (8)) broken down into three shorter beam segments or elements. The generalized displacements at the points of attachment consist of one translation and one rotation. (We could define more). It is obvious that when adjacent elements are connected the element displacements at each point of attachment must agree, i.e., must be continuous.

$$\text{e.g.} \quad u_3^1 = u_2^2, \text{ etc.}$$

This continuity requirement between element displacements therefore reduces the number of generalized grid point displacements. For the three beam elements shown in fig 1(a), the number of independent displacements is thus reduced from twelve to eight upon connection of the elements as shown in figure 1(b).

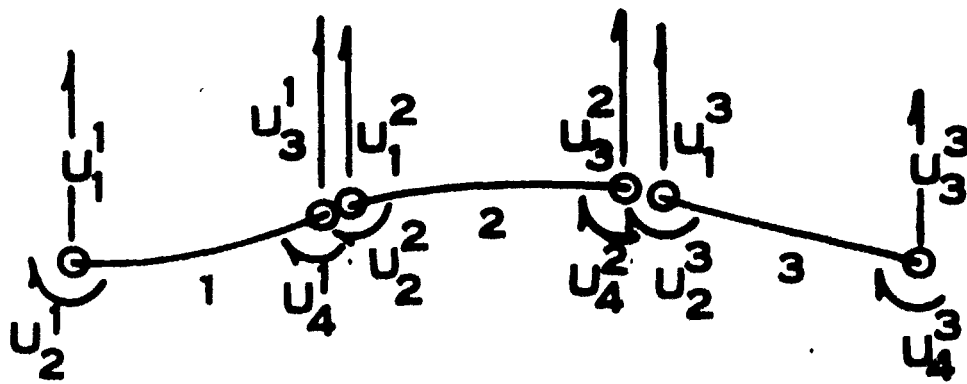


Figure 1(a). Finite Element Breakdown of Beam Structure.

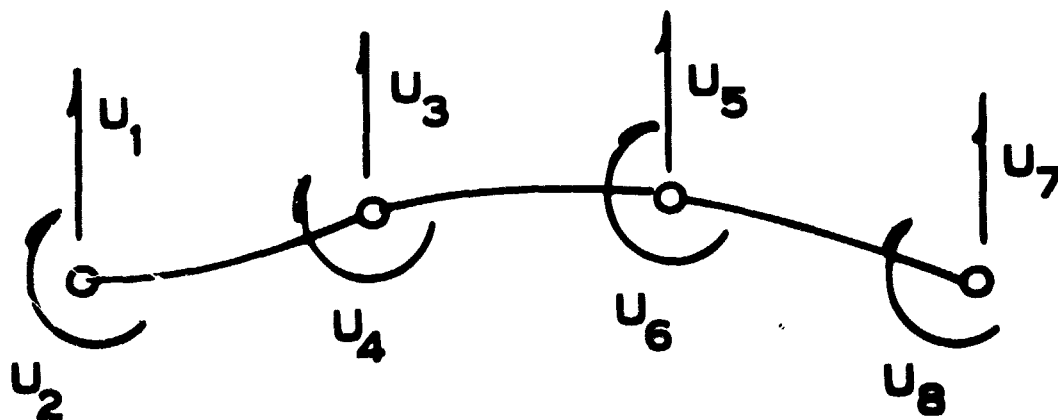


Figure 1(b). Interconnection of Beam Elements to Form Beam Structure.

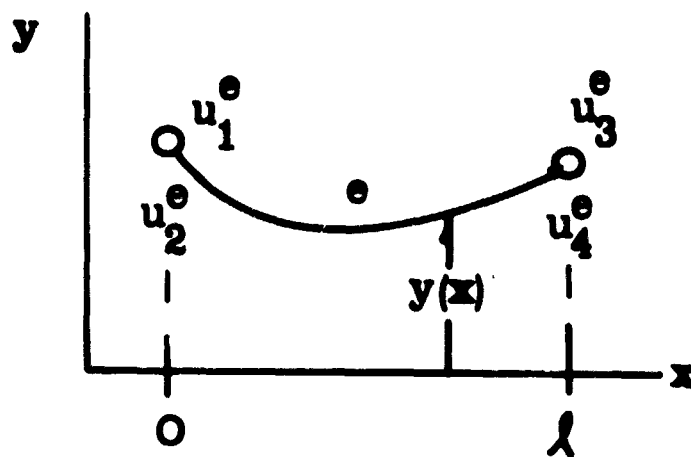


Figure 1(c). A General Beam Element in Deformed Configuration.

Letting $y(x,t)$ represent the transverse displacement of the beam continuum between the end points of a given element (fig 1(c)) the procedure is then to approximate $y(x,t)$ as a linear function of the generalized element displacements u_i^e . (The assumption of linearity will lead to linear differential equations in the u_i^e) Since there are four u_i^e per element - all of which are as yet arbitrary - we can try a polynomial expression with four arbitrary constants, i.e. a cubic:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 = a_i x^i$$

where, in the latter notation we intend that a sum be performed over repeated index i , with i ranging from 0 to 3.

The a_i can easily be determined from the four conditions:

$$y(0,t) = u_1^e ; \quad y'(0,t) = u_2^e ;$$

$$y(l,t) = u_3^e ; \quad y'(l,t) = u_4^e ;$$

The result can be expressed as the vector product:

$$y(x,t) = \underline{a(x)} \underline{u^e} ; \quad 0 \leq x \leq l \quad \dots (9)$$

explicitly,

$$y(x,t) = \left\{ \begin{array}{l} 1 - 3\xi^2 + 2\xi^3 \\ l(\xi - 2\xi^2 + \xi^3) \\ 3\xi^2 - 2\xi^3 \\ (\xi^3 - \xi^2)l \end{array} \right\} \begin{array}{c} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{array}$$

where $\xi = x/l$

Equation (9) constitutes a formal discretization of the continuum in that all interior displacement information has been referred to the end point, or element, displacements. The goal then is to determine these displacements $u_i^e(t)$.

When subjected to sets of applied forces and constraints (boundary conditions), the beam element responds according to the laws of mechanics from which we are free to choose any one of several in which to formulate the equations of motion of the element. It is very convenient to employ the principle of virtual work for dynamic loading⁵ - which states that in a virtual displacement $\delta y(x)$, of the beam element from its instantaneous state of equilibrium, the increment in strain energy, i.e., the virtual strain energy, is equal to the sum of the virtual work done by all the forces including the inertia loads.

$$\text{i.e.} \quad \delta U = \delta W - \int_0^L \delta y \ddot{y} A dx \quad \dots(10)$$

where δU represents the virtual elastic strain energy resulting from the virtual work δW of the applied forces and the virtual work of the inertia forces. (ρ is the material density and A the beam cross-sectional area.) In general the elastic strain energy due to a virtual displacement can be written:

$$\delta U = \int_V \sigma \delta \epsilon dx dy dz \quad \dots(11)$$

where σ is the induced stress due to virtual strain $\delta \epsilon$.

(5) Przemieniecki, J.S., Theory of Matrix Structural Analysis, McGraw Hill, 1968, p. 267

The virtual work due to applied forces $f(x,t)$ per unit length:

$$\delta W = \int_0^l f(x,t) \delta y dx$$

Thus (10) becomes:

$$\int_V \sigma \delta \epsilon dx dy dz = \int_0^l f(x,t) \delta y dx - \int_0^l \rho \ddot{y} \delta y A dx \quad \dots (12)$$

Hooke's Law specifies:

$$\sigma = E \epsilon$$

In beam theory, $E =$ Young's modulus, and $\epsilon = hy''$, h being the distance from the beam neutral axis to the fiber in which σ is being defined. Substituting in (12), the left term becomes:

$$\int_0^l \int_A E h^2 y'' \delta y'' dx dy dz = EI \int_0^l y'' \delta y'' dx$$

so that (12) becomes:

$$EI \int_0^l y'' \delta y'' dx = \int_0^l f(x,t) \delta y dx - \int_0^l \rho \ddot{y} \delta y A dx \quad \dots (13)$$

Making use of the approximation (9), i.e., $y = \underline{a}(x) \underline{u}^e$;

$$\delta y = \underline{a}(x) \delta \underline{u}^e \quad \text{and} \quad \delta y'' = \underline{a}''(x) \delta \underline{u}^e$$

Substituting these expressions in (13):

$$EI \int_0^l \delta \underline{u}^e \underline{a}'' \underline{a}'' \underline{u}^e dx = \int_0^l \delta \underline{u}^e \underline{a} f(x,t) dx - \rho A \int_0^l \delta \underline{u}^e \underline{a} \ddot{\underline{u}}^e dx$$

Since the virtual displacements $\underline{u}^e$ are arbitrary:

$$\left\{ \rho A \int_0^l \bar{a} \underline{a} dx \right\} \ddot{\underline{u}}^e + \left\{ EI \int_0^l \bar{a}'' \underline{a}'' dx \right\} \underline{u}^e = \int_0^l \bar{a} f(x,t) dx \quad \dots (14)$$

(a bar over a quantity denotes its transpose)

The nxn matrix coefficient expressions ($n = 4$ for the problem at hand) of $\ddot{\underline{u}}$ and $\underline{u}$ deserve to be called $\underline{m}^e$ and $\underline{k}^e$ respectively and the right hand term is the force vector $\underline{f}^e$ whose elements replace the distributed and applied forces present in the continuum problem. These forces are considered as being applied to the ends of the element e . The matrices $\underline{m}^e$ and $\underline{k}^e$ have been evaluated many times in the literature⁶ and will simply be repeated here for beam elements of four degrees of freedom.

$$\underline{k}^e = EI/l^3 \cdot \begin{bmatrix} 12 & & & \\ 6l & 4l^2 & \text{symm} & \\ -12 & -6l & 12 & \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\underline{m}^e = \rho A l / 420 \cdot \begin{bmatrix} 156 & & & \\ 22l & 4l^2 & \text{symm} & \\ 54 & 13l & 156 & \\ -18l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

(6) Przemieniecki, J.S., Theory of Matrix Structural Analysis, McGraw Hill, 1968, p. 81, 297

To form the force vector $\underline{f}^e$, the expression $f(x,t)$ from equation (7) is substituted into the right side of equation (14).

$$\text{thus } \underline{f}^e = - \int_0^l \underline{\bar{a}} m_p (\ddot{y} + 2v\dot{y}' + g + v^2 y'') \delta(x-vt) dx$$

substituting the relation $y(x) = \underline{au}^e$:

$$\begin{aligned} \underline{f}^e &= -m_p \int_0^l \underline{\bar{a}}(x) \{ \underline{a}\ddot{u}^e + 2v\underline{\bar{a}}'\dot{u}^e + g + v^2 \underline{\bar{a}}''u^e \} \delta(x-vt) dx \\ &= -m_p \{ c_1(t)\underline{u}^e + c_2(t)\dot{u}^e + c_3(t)u^e \} - m_p \underline{g}\underline{\bar{a}}(vt) \\ &\quad \dots 0 \leq vt \leq l \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where

$$\begin{aligned} c_1(t) &= \underline{\bar{a}}(vt)\underline{a}(vt) \\ c_2(t) &= 2v\underline{\bar{a}}'(vt)\underline{a}'(vt) \\ c_3(t) &= v^2 \underline{\bar{a}}(vt)\underline{a}''(vt) \end{aligned}$$

Equation (14) can then be written:

$$\begin{aligned} (\underline{m}^e + m_p c_1(t))\ddot{u}^e + m_p c_2(t)\dot{u}^e + (\underline{k}^e + m_p c_3(t))u^e &= -m_p \underline{g}\underline{\bar{a}}(vt) \\ \dots 0 \leq vt \leq l \end{aligned}$$

or simply,

$$\underline{u}^e(t)\ddot{u}^e + \underline{\gamma}(t)\dot{u}^e + \underline{\kappa}(t)u^e = \underline{\phi}^e(t) \quad \dots (15)$$

Outside the interval $0 < vt \leq l$, the time variant coefficients $c_i(t)$ and $a(vt)$ must be replaced by zeros. This can be accomplished by nullifying the m_p factor outside this interval.

Equation (15) represents a set of n ordinary differential equations with time dependent coefficients. They are the differential equations of motion for any beam element of density ρ and section modulus EI as well as cross-sectional area A . All of these element properties may differ from element to element.

STRUCTURE EQUATIONS OF MOTION

The equations of motion for the combined structure, i.e.,

$$\underline{M}(t)\ddot{\underline{U}} + \underline{C}(t)\dot{\underline{U}} + \underline{K}(t)\underline{U} = \underline{F}(t) \quad \dots(16)$$

are formed as follows.

Each term of equation (15) constitutes a force - ψ_i^0 , at the element attachment points, $i = 1$ thru n . When all elements are joined the resultant force at the connections (grid points) is the sum of the individual forces at the attachments. For example, the inertia forces at the right end of the first element (cf. fig 1) are to be added to those at the left end of the second element to yield the total inertia force at the connection grid point.

i.e., the inertia forces acting on the first element - element #1 in figure 1 - in the u_1 , u_2 , u_3 , and u_4 directions are the vector components:

$$\psi_i^1 = \mu_{ij}^1 \ddot{u}_j$$

Summation over repeated subscripts is intended and i & j range from 1 to n . Similarly for element #2:

$$\psi_i^2 = \mu_{ij}^2 \ddot{u}_j^2$$

Upon joining these two elements, ψ_3^1 combines with the force ψ_1^2 and ψ_4^1 with ψ_2^2 so that the resulting forces on the two-beam substructure are, for the case $n = 4$:

$$\begin{bmatrix} \psi_1^1 \\ \psi_2^1 \\ \psi_3^1 + \psi_1^2 \\ \psi_4^1 + \psi_2^2 \\ \psi_3^2 \\ \psi_4^2 \end{bmatrix} = \begin{bmatrix} \mu_{11}^1 & \mu_{12}^1 & \mu_{13}^1 & \mu_{14}^1 & 0 & 0 & 0 & 0 \\ \mu_{21}^1 & \mu_{22}^1 & \mu_{23}^1 & \mu_{24}^1 & 0 & 0 & 0 & 0 \\ \mu_{31}^1 & \mu_{32}^1 & \mu_{33}^1 & \mu_{34}^1 & \mu_{11}^2 & \mu_{12}^2 & \mu_{13}^2 & \mu_{14}^2 \\ \mu_{41}^1 & \mu_{42}^1 & \mu_{43}^1 & \mu_{44}^1 & \mu_{21}^2 & \mu_{22}^2 & \mu_{23}^2 & \mu_{24}^2 \\ 0 & 0 & 0 & 0 & \mu_{31}^2 & \mu_{32}^2 & \mu_{33}^2 & \mu_{34}^2 \\ 0 & 0 & 0 & 0 & \mu_{41}^2 & \mu_{42}^2 & \mu_{43}^2 & \mu_{44}^2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1^1 \\ \ddot{u}_2^1 \\ \ddot{u}_3^1 \\ \ddot{u}_4^1 \\ \ddot{u}_1^2 \\ \ddot{u}_2^2 \\ \ddot{u}_3^2 \\ \ddot{u}_4^2 \end{bmatrix}$$

Enforcing the equality of the displacements, velocities, and accelerations of the attachments, i.e.

$$\ddot{u}_3^1 = \ddot{u}_1^2 = \ddot{U}_3 \quad \text{and} \quad \ddot{u}_4^1 = \ddot{u}_2^2 = \ddot{U}_4$$

where the upper case letter denotes that $\ddot{U}_3$, $\ddot{U}_4$ are grid point accelerations in conformance with figure 1(b).

Hence in terms of grid point notation:

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix} = \begin{bmatrix} \mu_{11}^1 & \mu_{12}^1 & \mu_{13}^1 & \mu_{14}^1 & 0 & 0 \\ \mu_{21}^1 & \mu_{22}^1 & \mu_{23}^1 & \mu_{24}^1 & 0 & 0 \\ \mu_{31}^1 & \mu_{32}^1 & (\mu_{33}^1 + \mu_{11}^2) & (\mu_{34}^1 + \mu_{12}^2) & \mu_{13}^2 & \mu_{14}^2 \\ \mu_{41}^1 & \mu_{42}^1 & (\mu_{43}^1 + \mu_{21}^2) & (\mu_{44}^1 + \mu_{22}^2) & \mu_{23}^2 & \mu_{24}^2 \\ 0 & 0 & \mu_{31}^2 & \mu_{32}^2 & \mu_{33}^2 & \mu_{34}^2 \\ 0 & 0 & \mu_{41}^2 & \mu_{42}^2 & \mu_{43}^2 & \mu_{44}^2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{bmatrix}$$

$\equiv \underline{M}(t)\ddot{\underline{U}}(t)$... the inertia forces acting on the grid points of a 2-element substructure. The other forces in the equations of motion for this structure are formed by similar superposition. Thus the structure equations of motion are formed by overlapping and summing the element matrices wherever a grid point connection is made. For an N - element beam there will be N-1 such overlaps (shown schematically in figure 2). Each overlap will contain n/2 entries from the lower right corner of the nxn matrix corresponding to the element to the left of the grid point - and n/2 entries from the upper left corner of the matrix corresponding to the element to the right of this grid point. These overlapping elements are to be summed.

It is to be noticed from equation (15) that in general, the coefficient matrices $\underline{u}^e$, $\underline{y}^e$, and $\underline{\kappa}^e$ consist of a constant part (null in the case of $\underline{y}^e$) and a time variant part which derives from the moving mass m_p . These time variant elements are null except when t is such that m_p is located within the length of a particular beam element. Only then are the time-variant components of the corresponding element matrix finite. Thus in figure 2, one conceives of a conventional matrix of constant coefficient multipliers of the acceleration, velocity, and displacement terms plus a time variant set of components which propagate in a band along the diagonal of each structure coefficient matrix as the moving mass traverses the beam in time. Thus at any instant only n of the structure equations of motion possess time variant coefficients - n being the number of element displacements (degrees of freedom) considered for each beam element - four for the case at hand. Thus the prospect of solving the full set of equations numerically - without incurring extraordinarily long computer run times - would appear to be good. For example it is not uncommon to solve via computer, a fifty degree of freedom transient problem in conventional structure dynamics via the finite element technique - where all of the coefficients are constant in time. It should therefore involve only a moderate increase in computation time to allow four of these equations to take on time-variant coefficients as in our problem of the moving concentrated mass. Roughly speaking one might expect that each time-variant matrix element will create additional computation no greater than that caused by adding another degree of freedom to a conventional constant coefficient problem. Thus a 50 degree of freedom problem - e.g.,

twenty-four connected beam elements subjected to a moving concentrated mass - could be solved with a computation time not in excess of a ninety-eight degree of freedom problem in which all the matrix coefficients are constant. ($\underline{u}^e(t)$, $\underline{y}^e(t)$, and $\underline{\kappa}^e(t)$ each comprise sixteen time dependent components).

Boundary Conditions

A great convenience of the finite element procedure - as compared say, to the Ritz or Galerkin procedures - is that all ambiguity is removed in choosing the boundary conditions to enforce. (This is due to the particular stage of deduction at which the finite element idealization is invoked in a variational procedure*). In practice all one has to do is mimic physical reality. For example, a beam with hinge (simple) supports at each end requires that the corresponding displacements vanish; e.g., in a three-element beam model, $U_1 = U_7 = 0$. Similarly a clamped cantilevered beam would insist that $U_1 = U_2 = 0$. Instead of specification of particular zero values, however, it is more efficient to merely delete the corresponding rows and columns from the coefficient matrices $\underline{M}$, $\underline{C}$, and $\underline{K}$ of equation (16). Thus for the case of a three-element hinged-hinged beam, we simply delete the first and the seventh rows and columns from these matrices. Similarly we delete the corresponding elements from any force vector $\underline{F}$, appearing on the right hand side of this equation.

*Conversations with Dr. Gary Anderson, Applied Mathematics and Mechanics Div, Benet Weapons Laboratory, Watervliet Arsenal, Watervliet, N.Y.

PROBLEM STATEMENT AND SOLUTION

For the purpose of comparing results achieved by finite elements with those appearing in the literature, the problem to be solved is one of three beam elements connected end to end to form a simply supported beam having uniform cross section and material properties. A moving mass, m_p , is assumed to traverse the beam from left to right at a constant velocity v - see figure 3.

Deleting the first and seventh rows and columns from the coefficient matrices $\underline{M}$, $\underline{C}$ and $\underline{K}$ along with the first and seventh components of the vector $\underline{F}$ of equation (16) results in a six degree of freedom problem - that is, six coupled equations of forced motion. These equations will be solved numerically with m_p and its velocity v , serving as parameters.

In reality we have no clairvoyance to guide the choice of values for m_p and v except, of course, to repeat those used in the literature so that a comparison may be made. It appears, however, that the values chosen by Ayre, et al, were not completely arbitrary in that certain values of v will cause resonant (secular) behavior in the moving force problem. From equation (3) we can verify that resonance will indeed occur for any value $v = v^*$, such that $m_{ii}(v^*) = 0$. Although applicable only to moving force problems, it is intuitively plausible that extraordinary behavior in the moving mass problem might occur for values of velocity not far removed from these values v^* . Having no 'closed form' analytical solution with which to anticipate points of singular - or otherwise interesting - behavior, one

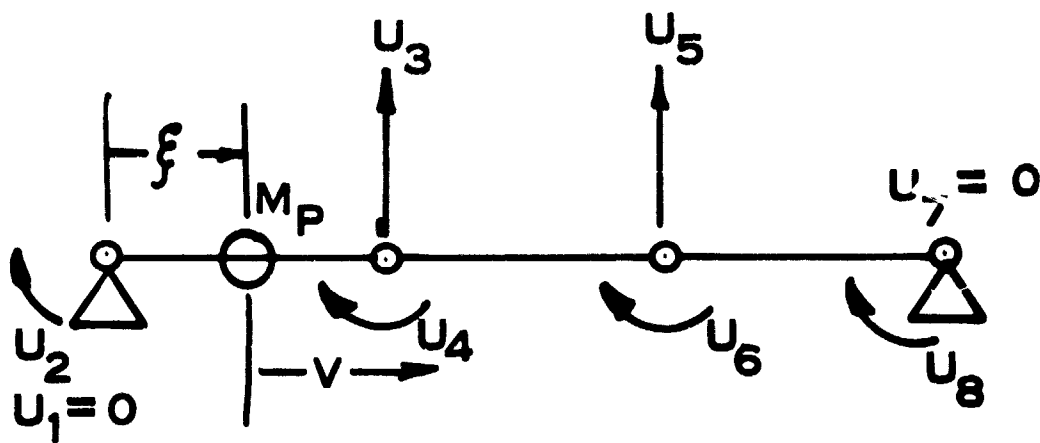


Figure 3. Finite Element Model for Moving Mass Problem.

can do no better than to at least allow the velocity of the mass to range through the lowest v^* which will cause m_{ii} to vanish. This apparently was the reasoning behind the choice of velocities investigated by Ayre, et al, who performed experiments in which $\dot{v}$ was chosen as $v^*/4$, $v^*/2$, and v^* for the moving force and $v^*/4$ and $v^*/2$ for the moving mass - v^* being the first value of v to cause m_{ii} to vanish, i.e., the first resonant or 'critical' velocity of the moving force solution. (Maintaining contact between the beam and mass for velocities higher than $v^*/2$ was apparently impossible.)

For the simply supported beam being considered:

$$m_{ii}(v) = \alpha_i^2 v^2 - \omega_i^2 \quad \dots (17)$$

where $\alpha_i = i\pi/L$, L being the overall length of the combined beam structure and $\omega_i = \alpha_i^2 \sqrt{EI/\rho A}$ ⁸

Table 1 lists the material constants and dimensions employed for the three identical beam elements used in this work. From these values one calculates from (17) for $i=1$:

$$v^* = 899.13 \text{ in/sec}$$

We intend to examine numerical solutions for the moving force and the moving mass throughout the range $0 < v \leq v^*$.

Two primary references will be used as basis' of comparison:

- (i) The exact solution for the moving force solution⁸ which for

(8) Nowacki, W. Dynamics of Elastic Systems, Chapman & Hall Ltd, London, 1963, p. 136

TABLE 1. MATERIAL PROPERTIES AND NOMENCLATURE

	Material	aluminum
ρ	Density	$3.14 \times 10^{-4} \text{ lbsec}^2/\text{in}^4$
L	Overall Beam Length	32
l	Length of each element	120.0 in
ρAL	Beam Mass	
m_p	Moving Mass	0.0, $\rho AL/4$, $\rho AL/2$
E	Young's Modulus	$1.0 \times 10^7 \text{ psi}$
A	Beam Cross Sect. Area	3I
I	Beam Area Moment of Inertia (fixed by choice of ω_1 below)	
h	Beam Thickness	2.0 in.
$\omega_1/2\pi$	Beam Fundamental Frequency	1.25 hz
v	Mass Velocity	100., $v^*/4$, $v^*/2$, v^*
v^*	Fundamental Resonant Velocity	899.13 in/sec

the case of a moving concentrated downward force $-m_p \delta(x-vt)$ is :

$$y(x,t) = -2m_p g / \rho A L \sum_1^{\infty} \sin \alpha_n x (\alpha_n v \sin \omega_n t - \omega_n \sin \alpha_n vt) / \omega_n (\alpha_n^2 v^2 - \omega_n^2) \dots (18)$$

The finite element solutions for the moving mass problem (Equation (16) after imposing support constraints) with $m_p = 0$ will be compared with computations of (18) above.

(ii) The experimental work of Ayre, Jacobsen and Hsu². The mass velocities employed in this work were quasi-static $v \dot{=} 0$, and $v = v^*/4$, $v^*/2$, and v^* . The moving mass values chosen were $m_p = 0$, and $m_p = \rho A L/4$ and $\rho A L/2$ where $\rho A L$ is the total beam mass.

Equation (16) is solved using Hamming's modified predictor-corrector method which uses fourth order Runge-Kutta method suggested by Ralston¹⁰ for adjustment of the initial increment and for computation of starting values. The method is taken directly from the IBM Scientific Subroutine Package for the IBM System #360, re: Programmer's Manual # H20-0205 available this laboratory. The method was found to be about four times faster than using the Runge-Kutta method throughout the entire problem. In general, run times in the order of 20 - 30 minutes on the IBM model 44 computer.

(2) Ayre, R.S., Jacobsen, L.S., and Hsu, C.S., "Transverse Vibration of One and of Two Span Beams Under the Action of a Moving Mass Load," Proc. of First National Congress on Applied Mechanics, June 1951

(10) Ralston and Wilf, Mathematical Methods for Digital Computers, Wiley and Sons, New York, London, 1960, pp. 95-109

Results - Moving Force

Figures 4 through 7 show the transverse displacement of the beam at the grid point locations $x = L/3, 2L/3$ as computed either by the finite element method or by evaluation of the exact solution (18) - any discrepancy between the results being too small to be discerned even in plots of this scale. This is substantially less error than appears in other treatments yielding approximate results.^{4,9}

Using the relationship (9), i.e., $y(x,t) = \underline{a}(x)\underline{u}^{\circ}(t)$ with $x = vt$, gives the displacement directly beneath the moving load to be compared with the results of Ayre, et al, who recorded displacement information exclusively at this location. Figures 8 - a,b,c show these comparisons and it is obvious that the agreement with experiment is much better in figures b and c than in a. Actually, Ayre and his co-workers experienced considerable experimental difficulty when the force was translated at $v^*/4$. Quoting from their publication² in which the authors remark on their disagreement with Schallenkamp's theoretical solution for the moving force (evidently a three term approximation of expression (18)):
..."The agreement is generally good [except at $v^*/4$] where it has been found that comparatively small errors in velocity may result in marked differences in the shape of the trajectory." ... In 8a the theoretical curve used by Ayre as a basis of comparison has been included.

(2) Ayre, R.S., Jacobsen, L.S., and Hsu, C.S., "Transverse Vibration of One and of Two Span Beams Under the Action of a Moving Mass Load," Proc. of First National Congress on Applied Mechanics, June 1951

(4) Schallenkamp, A., "Schwingungen von Trägern bei bewegten Lasten," Ingenieur-Archiv, v.8, 1937, pp. 182-198

(9) Hutton, D.V., and Counts, J., "Deflections of a Beam Carrying a Moving Mass," Trans. ASME, Sept, 1974, p. 803

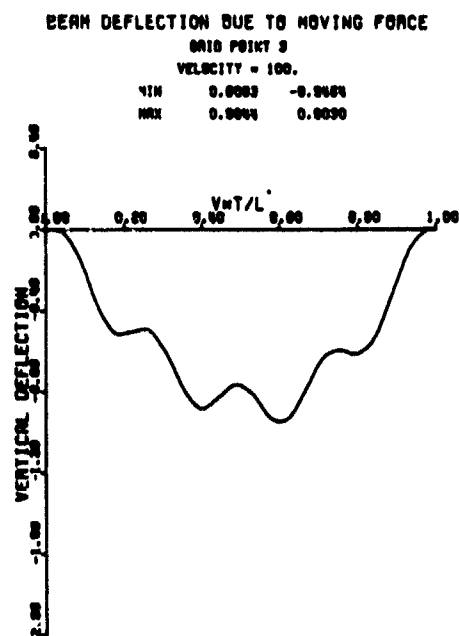
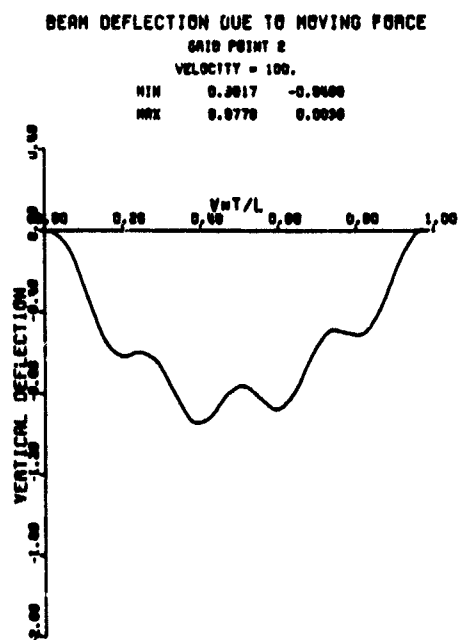


Figure 4

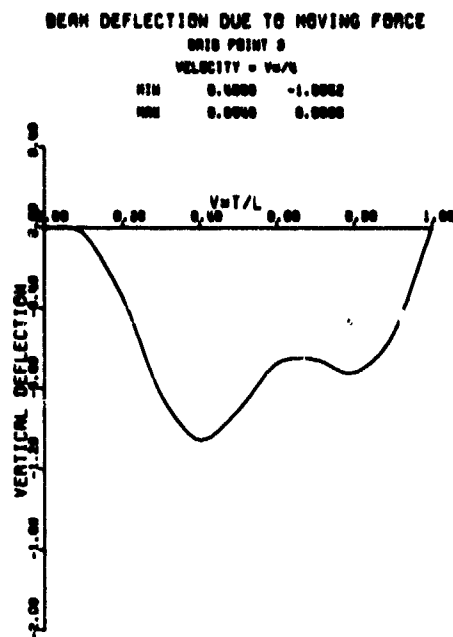
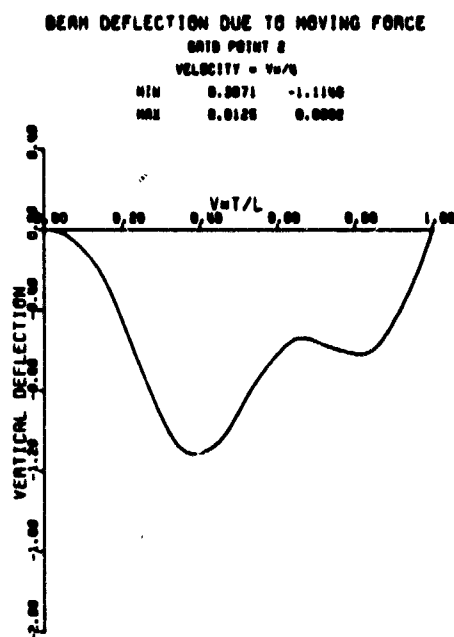


Figure 5

Transverse Motion at Grid Points in Response to Moving Force

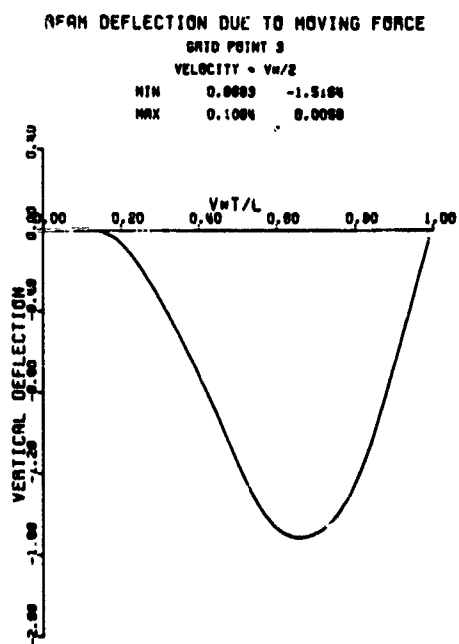
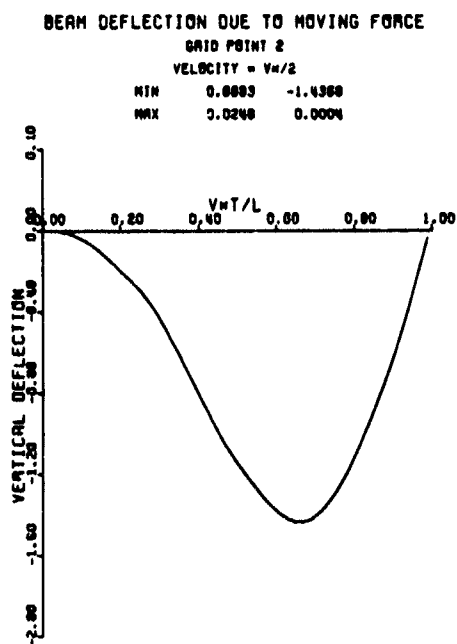


Figure 6

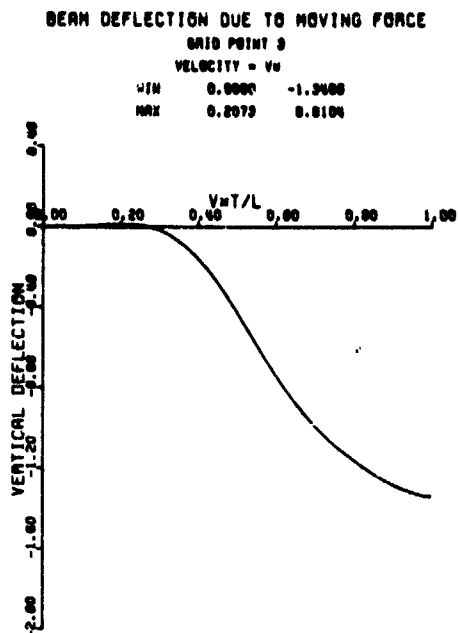
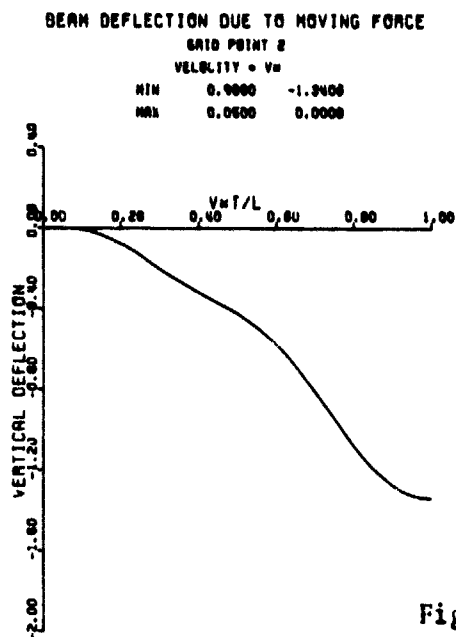


Figure 7

Transverse Motion at Grid Points in Response to Moving Force

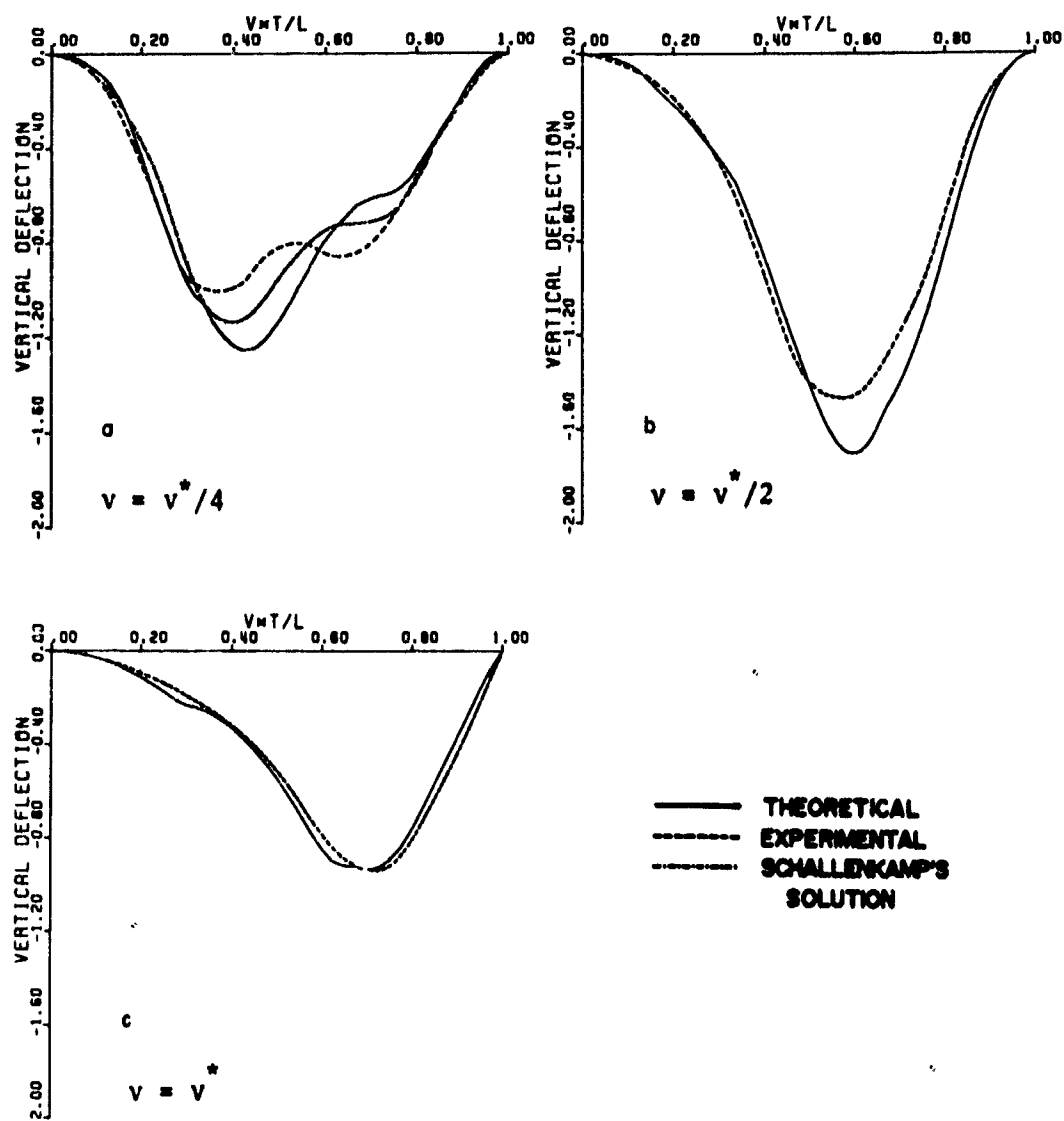


Figure 8. Transverse Motion at $x = vt$ in Response to Moving Force.

The exceptional agreement* (using only three finite beam elements) of the finite element results for the moving force gives confidence in extending the technique to the moving mass problem - for which no simple solution as (18) exists.

Results - Moving Mass

Figures 9 through 14 show the transverse displacement of the beam at the grid point locations $x = L/3, 2L/3$ as computed by the finite element procedure for various cases of velocity and mass values. Again use will be made of relationship (9) to convert this information to displacement beneath the moving mass so that a direct comparison with the experimental results of Ayre, Jacobsen, and Hsu can be made. Figure 15 - a,b,c shows this comparison to be quite good - excellent agreement occurring at the grid point locations. A closer look shows, however, that the slopes of the curves generated by the finite element analysis are discontinuous at the grid point locations. The reason for this is that the displacement approximation (9) is built from cubic polynomials which are not continuous in the second derivative at the grid point connections. Indeed, only continuity in y and y' were demanded in constructing these polynomials.

The discontinuities of $y''(x)$ at grid points might not be serious if it was not for a 'force' in the continuum equation of motion (8)... $m_p v^2 y''(x)$. Thus at higher velocities jump discontinuities in y'' will cause increasingly powerful disturbances which are nonphysical in character. This is apt to be especially influential in armament

*Note: All displacements are normalized with respect to the displacement which occurs at midspan due to a static load at this point.

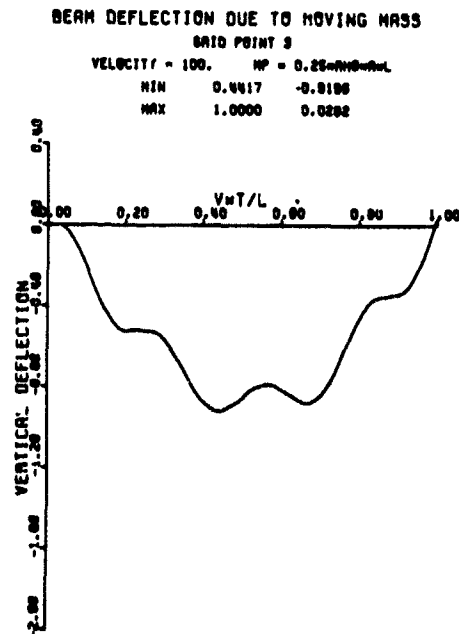
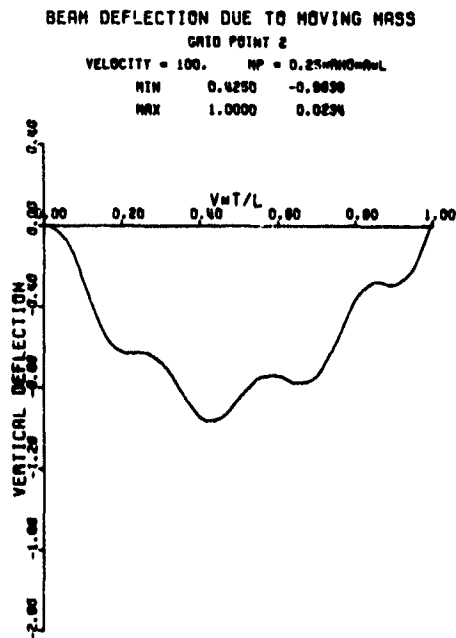


Figure 9

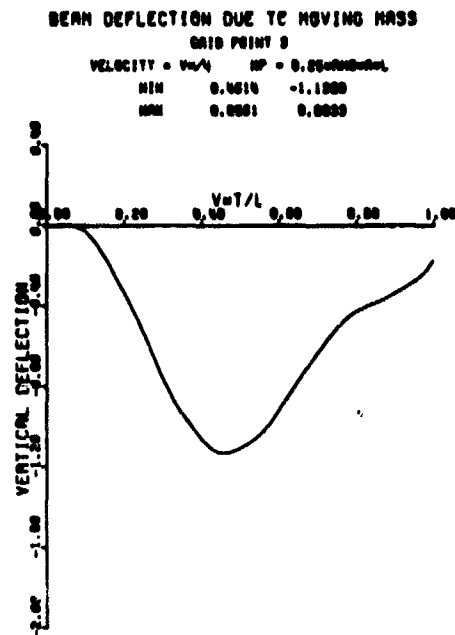
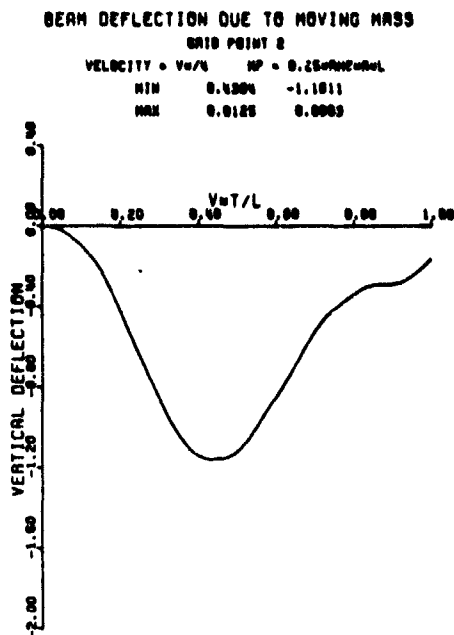


Figure 10

Transverse Motion at Grid Points in Response to Moving Mass

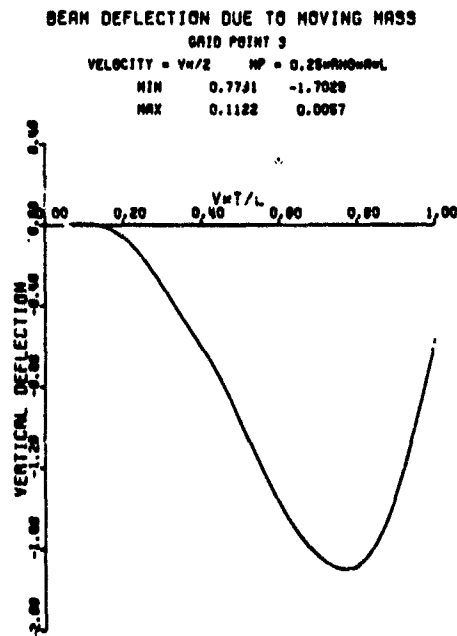
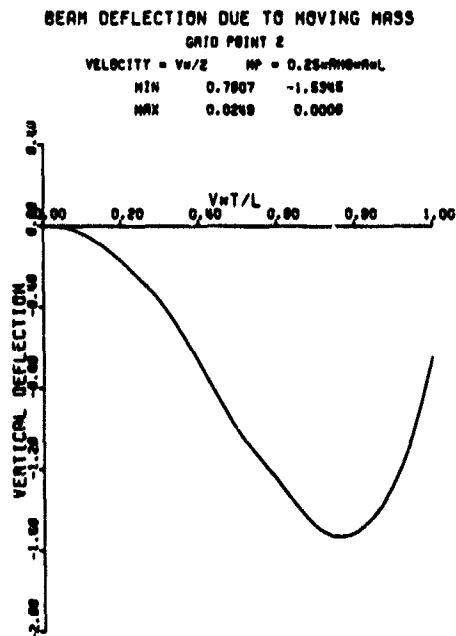


Figure 11

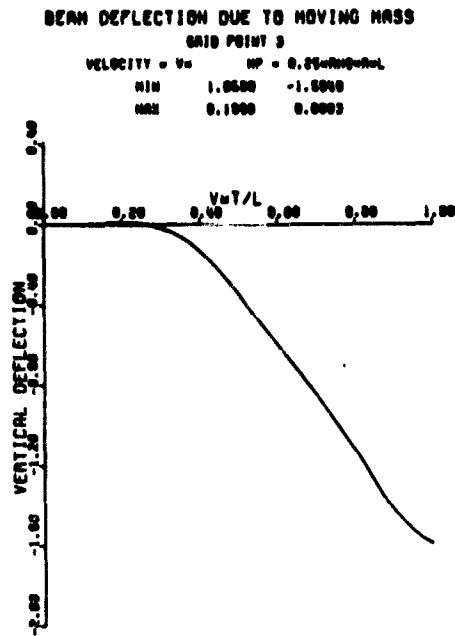
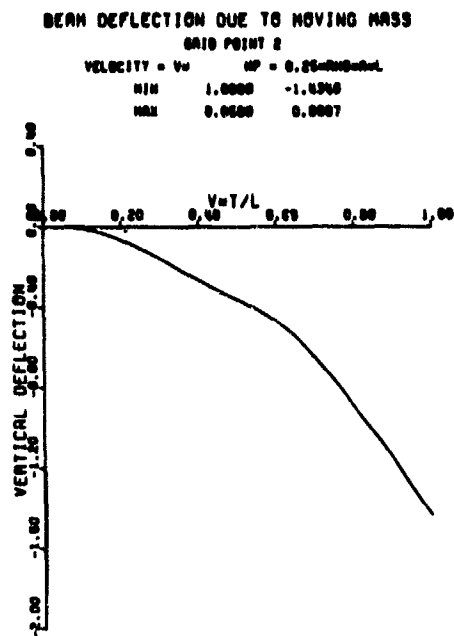


Figure 12

Transverse Motion at Grid Points in Response to Moving Mass

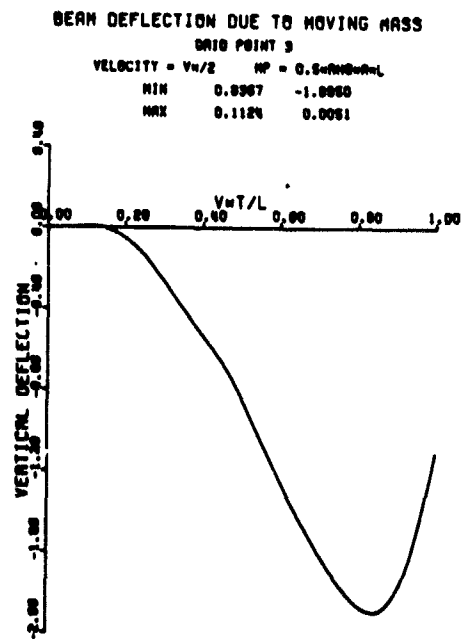
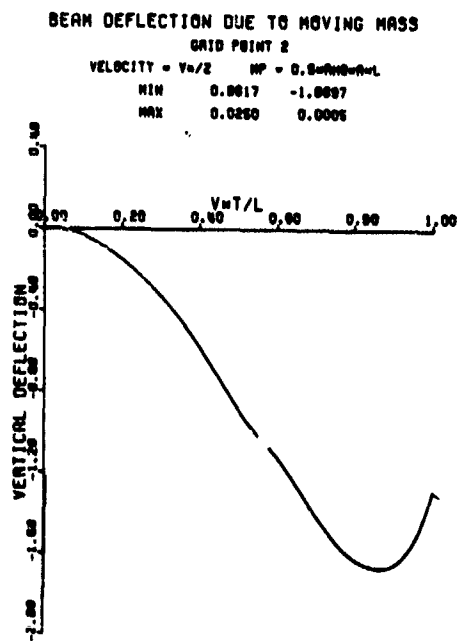


Figure 13

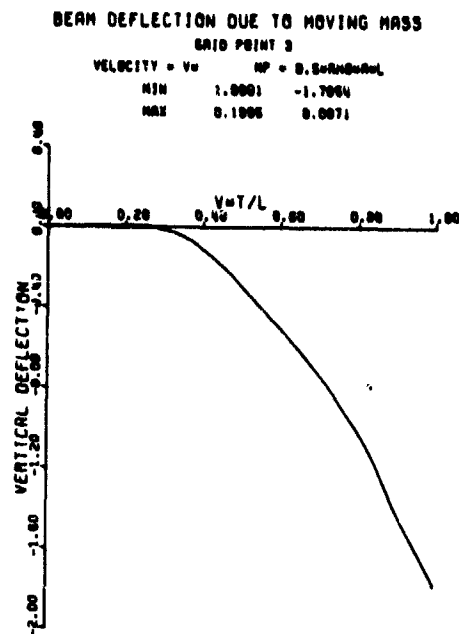
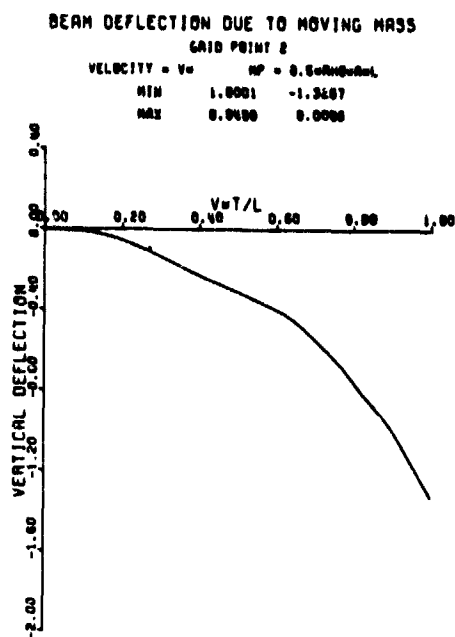


Figure 14

Transverse Motion at Grid Points in Response to Moving Mass

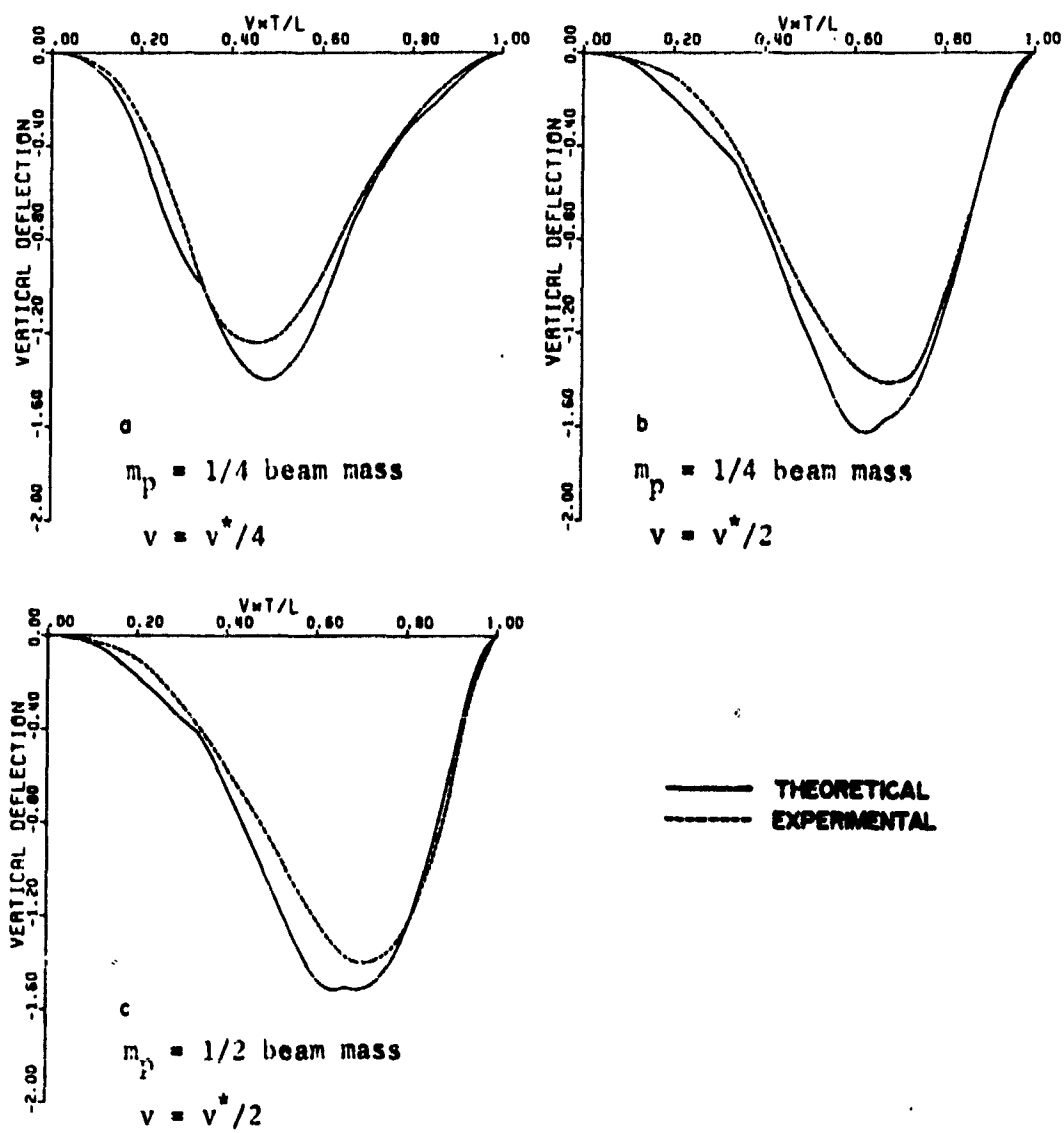


Figure 15. Transverse Motion of Moving Mass.

applications involving velocities considerably larger than v^* .

One approach toward rectification of this problem requires the formation of higher-ordered polynomials which are continuous in the second derivative across the grid points of the structure. Higher polynomials have additional coefficients and, as in the previous derivation, these coefficients are equal in number to the element degrees of freedom, i.e., the generalized displacements u_i^e .

The establishment of polynomial displacement functions proceeds in the same manner as before, with the added conditions:

$$y''(0) = u_3^e \quad ; \quad y''(L) = u_6^e$$

The result is that $y(x)$ will be approximated interior to an element by:

$$y(x) = \hat{a}(x) \underline{u}^e ;$$

where $\hat{a}$ and $\underline{u}^e$ are now vectors with six components instead of four.

The new (*) element matrices are calculated according to equation (14). The multiplication and integration was accomplished analytically by a computer program called MANIP which was written in the FORMAC language - giving analytical expressions as output. Similarly, new expressions for c_1 , c_2 , and c_3 for use in equation (15) were formed using this program. $\hat{k}^e$ and $\hat{m}^e$ compose the constant elements of the matrices $\hat{\underline{\mu}}^e$ and $\hat{\underline{\nu}}^e$ and are found to be:

$$\hat{k} = \frac{EI}{7l^3} \cdot \begin{bmatrix} 120 & 60l & 3l^2 & -120 & 60l & -3l^2 \\ & (192/5)l^2 & (11/5)l^3 & -60l & (108/5)l^2 & -4/5l^3 \\ & & (3/5)l^4 & -3l^2 & (4/5)l^3 & l^4/10 \\ & & & 120 & -60l & 3l^2 \\ & \text{symm} & & & (192/5)l^2 & -\frac{11l^3}{5} \\ & & & & & 3l^4/5 \end{bmatrix}$$

$$\hat{m}^e = \frac{\rho A l}{231} \cdot \begin{bmatrix} 181/2 & (311/20)l & (281/240)l^2 & 25 & -151l/20 & \frac{181l^2}{240} \\ & 52l^2/15 & 23l^3/80 & 151l/20 & -133l^2/60 & \frac{13l^3}{60} \\ & & l^4/40 & \frac{181l^2}{240} & -13l^3/60 & \frac{l^4}{48} \\ & \text{symm} & & 181/2 & -311l/20 & \frac{281l^2}{240} \\ & & & & (52/15)l^2 & -\frac{23l^3}{80} \\ & & & & & \frac{l^4}{40} \end{bmatrix}$$

Figure 16 shows the improved results obtained through the use of the quintic polynomial. Comparison should be made with the results obtained via the cubic polynomial as shown in figure 15(a). The price paid, however, is increased computation time due to the additional degrees of freedom induced through the use of the quintic. One also faces increasing amount of ill-conditioning in the matrix equations of motion due to the widely differing magnitudes induced in the coefficient matrices and their corresponding output variables. Future work will concentrate on these problems and the use of less time-consuming integration algorithms.

BEAM DEFLECTION UNDER A MOVING LOAD

$$\text{VELOCITY} = V \times / 4 \quad \text{MP} = 0.25 \times \text{RHO} \times A \times L$$

MIN 0.4739 -1.3598

MAX 1.0001 0.0000

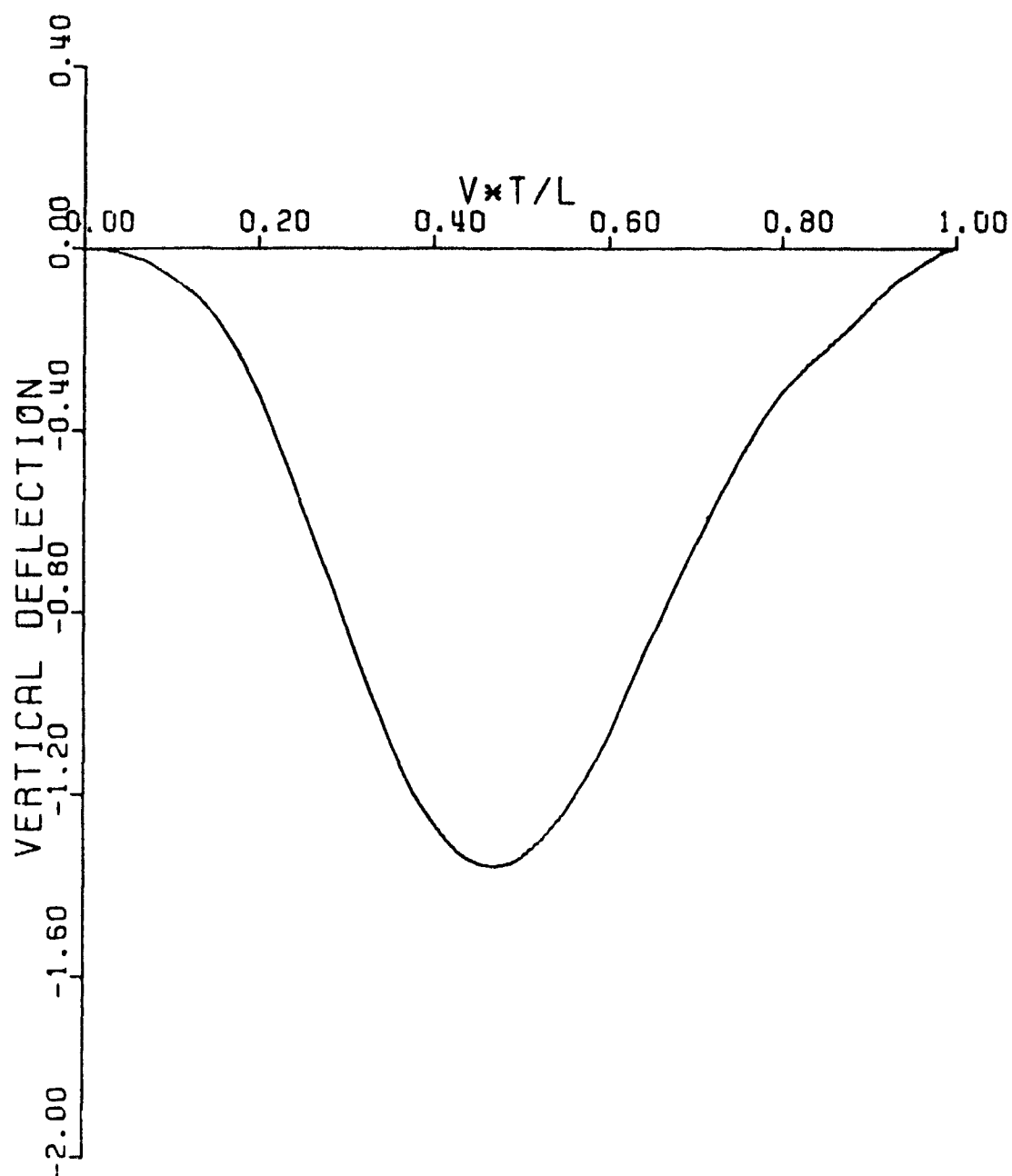


Figure 16. Improved Results Using Quintic Shape Function.

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