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STABILITY THEORY FOR DIFFERENCE EQUATIONS

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The article was written for undergraduate teachers of mathematics but it should also serve as a good introduction for engineers and scientists to the latest results in the theory of the stability of dynamical systems.

Abstract

This article is designed to give through the study of difference equation (discrete dynamical systems) a view of and an introduction to the general theory of the stability of dynamical systems in its most modern aspect. Much of what is presented here is known, although not perhaps as well known as it should be, and there are some things that are new. One of these has to do with a connectedness property of the positive limit sets of the solutions of difference equations which provides a means through the use of Liapunov functions of establishing the existence of equilibrium points (fixed points) and oscillations (periodic points). Another is the generalization of the usual concept of a vector Liapunov function, and this leads to a possible method of designing control systems where the measure of the error or the performance criterion is a vector rather than a scalar. Applications of the theory are illustrated by simple examples.

The article was written for undergraduate teachers of mathematics but it should also serve as a good introduction for engineers and scientists to the latest results in the theory of the stability of dynamical systems.

STABILITY THEORY FOR DIFFERENCE EQUATIONS

1. Introduction

Applications in the Soviet Union of Liapunov's direct method to nonlinear problems in the design of control systems in the late 1940's (see [1,29]) brought about a renewed interest in his theory of stability. The original theory dealt with systems described by ordinary differential equations. An elementary introduction to this theory with some applications is given in [25]. During the past fifteen years there has been a rather rapid development of this theory within, and far beyond, ordinary differential equations. The purpose of this article is to provide an introduction to this theory and to these more recent developments. However, even for ordinary differential equations, this requires an understanding of some of the basic properties of solutions (their existence and uniqueness, continuous dependence on initial conditions, maximal domains of definition, etc.). None of these difficulties with basic properties are encountered for difference equations, and it turns out to be possible to present through the study of difference equations almost all of the basic features of the general theory, as it stands today, in an elementary and concise manner. This is our objective, and we assume only a good introduction to algebra and analysis in m-dimensional Euclidean space R^m (finite dimensional vector spaces and convergence and continuity with respect to

Euclidean distance). See, for instance, the Appendix in [12], and [17], Chapters 3 and 5. A good introduction to classical Liapunov theory for difference equations is [22], and [20] goes beyond classical theory and gives some applications to numerical analysis. We go beyond the developments to be found in these two references, and give a number of results that are new for difference equations.

The finite calculus (the calculus of finite differences) and difference equations have been studied as long as the continuous (infinitesimal) calculus and date back to Brook Taylor (1717) and Jacob Stirling (1730). The first treatise was written by L. Euler in 1755. The finite calculus has always had many important applications and was at one time considered to be a prerequisite to infinitesimal calculus. Today, outside of numerical analysis (see [33,34]) and engineering (system analysis) (see [8,36]), the finite calculus and difference equations are seldom studied systematically. Elementary textbooks on finite mathematics barely mention the finite calculus and finite difference equations. We learn much more about derivatives, integrals, and Laplace transforms than we do about finite differences, sums, and what engineers call z-transforms. Difference equations are important and significant mathematical models for real phenomena and systems in the physical and nonphysical sciences. After all, the observational data we have for real systems is often discrete, which is why engineers are interested in what they call "sampled-data systems".

In order to make the presentation more complete and more suitable for independent study or an undergraduate seminar, we have included a large number of exercises and references.

2. Difference Equations

Let J denote the set of all integers and J_+ the set of all nonnegative integers. R^m is real m-dimensional Euclidean space; if

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in R^m, \ ||x|| = (x_1^2+x_2^2+\cdots+x_m^2)^{1/2}$$

(the Euclidean length of the vector x). For a sequence of vectors x^j, $x^j \to y$ as $j \to \infty$ means $||x^j-y|| \to 0$ as $j \to \infty$. We will also use x to denote a function on J_+ to R^m ($x\colon J_+ \to R^m$). Thus we allow the usual and convenient ambiguity where x may denote either a function or a vector (point).

The <u>first difference</u> $\dot{x}$ of a function x is defined by $\dot{x}(n) = x(n+1) - x(n)$. Corresponding to the fundamental theorem of calculus, we have $\sum_{k=j}^{n} \dot{x}(k) = x(n+1) - x(j)$ and, if $y(n+1) = \sum_{k=j}^{n} x(k)$, $\dot{y}(n) = x(n)$. We use x' to denote, the function defined by $x'(n) = x(n+1)$ so that $x' = \dot{x} + x$.

A function $T\colon R^m \to R^m$ is continuous (on R^m) if $x^j \to y$ implies $T(x^j) \to T(y)$. We do not need to do so now but later will always assume that T is continuous. If x is a vector, $T(x)$ is a vector and is the value of the function at x. If x is a function, the product Tx will denote the composition of functions — $(Tx)(n) = T(x(n))$. Thus the difference equation

$$x(n+1) = T(x(n))$$

can be written

$$x' = Tx. \tag{2.1}$$

The solution to the initial value problem

$$x' = Tx, \quad x(0) = x^0 \tag{2.2}$$

is $x(n) = T^n(x^0)$, where T^n is the n^{th}-iterate of $T - T^0 = I$, the identity function $(Ix = x)$, and $T^n = TT^{n-1}$. The solution is defined on J_+. There are no difficult questions about the existence and uniqueness of the solutions. Also, it is clear that the solutions are continuous with respect to the initial condition (state) x^0 if T is continuous. Unlike ordinary differential equations, the existence and uniqueness is only in the forward direction of time $(n \geq 0)$. Equation (2.2) is simply an algorithm defining a function x on J_+.

Let $g: R^m \to R$. Then

$$u(n+m) = g(u(n), u(n+1), \ldots, u(n+m-1)) \tag{2.3}$$

is an m^{th}-order difference equation. The state $u(n+1)$ of the system at time $n+1$ depends on its state at times $n, n-1, \ldots, n-m+1$ — i.e., upon this portion of its past history. Note that if we define $x_1(n) = u(n)$, $x_2(n) = x_1'(n) = u(n+1)$, $\ldots$, $x_m(n) = x_{m-1}'(n) = u(n+m-1)$, then (2.3) is equivalent to the following system of m first-order difference equations:

$$x_1' = x_2$$
$$x_2' = x_3$$
$$\vdots$$
$$x_{m-1}' = x_m$$
$$x_m' = g(x_1, x_2, \ldots, x_m) = g(x)$$

or

$$x' = Tx$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad T(x) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_m \\ g(x) \end{pmatrix}.$$

For instance, the 3rd-order linear equation

$$u''' + a_2 u'' + a_1 u' + a_0 u = 0$$

is equivalent to

$$x' = Ax,$$

where $x_1 = u,\ x_2 = u',\ x_3 = u'',\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix},$$

a 3×3 matrix. The initial conditions $u(0) = u_0$, $u'(0) = u_0'$, $u''(0) = u_0''$ correspond to

$$x(0) = \begin{bmatrix} u_0 \\ u_0' \\ u_0'' \end{bmatrix} = x^0.$$

The solution of this initial value problem is

$$x(n) = A^n x^0.$$

We are often interested in obtaining information about the asymptotic behavior of solutions; i.e., what happens to solutions for large n? For instance, in the above example, when does $A^n \to 0$ as $n \to \infty$ ($A^n = (a_{ij}^{(n)})$ and $A^n \to 0$ means $a_{ij}^{(n)} \to 0$ for all i,j or, equivalently, $A^n x \to 0$ for each x).

3. <u>Positive Limit Sets of Solutions</u>. <u>Invariance</u>.

We now adopt the following notations: For S a subset of R^m, $\bar{S}$ is the closure of M. If $x \in R^m$ and S is a closed set of R^m, then $\rho(x,S) = \min\{||x-y||;\ y \in S\}$, the <u>distance of</u> x <u>from</u> S. $T^n(x^0) \to S$ as $n \to \infty$ means that $\rho(T^n(x^0),S) \to 0$ as $n \to \infty$.

Thus, if we can locate a closed set that a solution $T^n(x^0)$ approaches as $n \to \infty$, we have obtained asymptotic information about the solution. We shall show that Birkhoff's positive limit set (ω-limit set) of a solution (introduced in [6]) is the smallest closed set that $T^n(x^0)$ approaches as $n \to \infty$. In this sense, locating a solution's positive limit set is the "best" asymptotic information we can hope to obtain.

Our objective is to learn that the role of Liapunov functions is to help locate positive limit sets of solutions. To that end, we proceed to define limit sets and to study their basic properties.

3.1. Definition (Birkhoff). A point y is a positive limit point of $T^n x$ if there is a sequence of integers n_i such that $n_i \to \infty$ and $T^{n_i} x \to y$ as $i \to \infty$. The positive limit set $\Omega(x)$ of $T^n x$ is the set of all its positive limit points.

3.2. Exercise. Show that an alternate definition for $\Omega(x)$ is

$$\Omega(x) = \bigcap_{j=0}^{\infty} \overline{\bigcup_{j=n}^{\infty} T^n x}.$$

3.3. Exercise. Let S_n be a sequence of closed (compact) nonincreasing ($S_{n+1} \subset S_n$) sets in R^m. Show that $\bigcap_{j=0}^{\infty} S_n$ is closed (nonempty and compact).

3.4. Definition. Relative to (2.1), or to T, a set $H \subset R^m$ is said to be positively (negatively) invariant if $T(H) \subset H$ $(H \subset T(H))$. H is said to be invariant if $T(H) = H$; i.e., it is both positively and negatively invariant.

Now to have an interesting theory we must put a smoothness condition on T. All that we need is to assume that T is continuous, and we do so from this point on. As a model for the real world, at least, we expect continuity.

For ordinary differential equations whose solutions are continuous curves, one is able to conclude that positive limit sets of bounded solutions are connected. Here we cannot expect this. The concept of connectedness has to be related to invariance.

3.5. Definition. A closed invariant set H is said to be invariantly connected if it is not the union of two nonempty disjoint invariant closed sets.

3.6. Definition. A solution $T^n x$ is said to be periodic (or cyclic) if for some $k > 0$, $T^k x = x$. The least such k is called the period of the solution or the order of the cycle. If $k = 1$, x is a fixed point of T or an equilibrium state of (2.1).

Note here that, unlike ordinary differential equations, a solution can reach an equilibrium state in finite time.

Periodic phenomena (oscillations) are of theoretical and practical interest, and the next exercise shows a relationship between periodicity and invariant connectedness.

3.7. Exercise. Show that: A finite set (a finite number of points) is invariantly connected if and only if it is a periodic motion. (Hint: Any permutation can be written as a product of disjoint cycles. See, for instance, [7], p. 140, Theorem 11.)

3.8. Exercise. Show that:

(a) The closure of a positively invariant set is positively invariant.

(b) The closure of a bounded invariant set is invariant.

As the above exercise suggests, the closure of an unbounded invariant set may not be invariant.

3.9. Definition. $T_n x$ (defined for all $n \in J$) is called an extension of the solution $T^n x$ to J if $T_0 x = x$ and $T(T_n x) = T_{n+1} x$ for all $n \in J$.

Note that $T_n x = T^n x$ for all $n \in J_+$, and that, if x is in an invariant set H, then $T^n x$ has an extension on J, which may or may not be unique.

3.10. Exercise. Show that a set H is invariant if and only if each motion starting in H has an extension on J that is in H for all n.

An invariant set H can have an extension on J from a point in H that is not in H. However, the above exercise says it always has at least one that does not leave H.

3.11. Exercise. Let E be a set in R^m, and let M be the largest (by inclusion) invariant set in E. Show that:

(a) M is the union of all extended solutions that remain in E for all n.

(b) $x \in M$ if and only if there is an extended solution from M in E for all n.

(c) If E is compact (bounded and closed), M is compact.

At last we are in a position to establish the basic properties of positive limit sets. The most interesting case is when $T^n(x)$ is bounded for all $n \geq 0$. Such motions are often said to be <u>positively stable in the sense of Lagrange</u>.

3.12. <u>Theorem</u>. Every positive limit set is closed and positively invariant. If $T^n(x)$ is bounded for all $n \in J_+$, then $\Omega(x)$ is nonempty, compact, invariant, invariantly connected, and is the smallest closed set that $T^n x$ approaches as $n \to \infty$.

<u>Proof</u>. It is easily seen that the complement of $\Omega(x)$ is open, and hence $\Omega(x)$ is closed (see also Exercises 3.2 and 3.3). If $q \in \Omega(x)$, then there is a sequence n_i such that $n_i \to \infty$ and $T^{n_i}x \to q$ as $i \to \infty$. By continuity of T, $T(T^{n_i}(x)) = T^{n_i+1}(x) \to T(q)$, and $T(q) \in \Omega(x)$. Hence $T(\Omega(x)) \subset \Omega(x)$, and $\Omega(x)$ is positively invariant.

If $T^n(x)$ is bounded, it contains a convergent subsequence and $\Omega(x)$ is nonempty. Also $\Omega(x) \subset \overline{\bigcup_{n=0}^{\infty} T^n(x)}$, and is, therefore, bounded and hence compact. If $q \in \Omega(x)$, we now have the existence of a sequence n_i such that $n_i \to \infty$, $T^{n_i}x \to q$ and $T^{n_i-1}x \to p$ as $i \to \infty$. Then $T(T^{n_i-1}x) = T^{n_i}(x) \to T(p) = q$. Therefore, $\Omega(x) \subset T(\Omega(x))$, and $\Omega(x)$ is invariant when $T^n(x)$ is bounded.

We will show next that $T^n(x) \to \Omega(x)$ as $n \to \infty$ if $T^n(x)$ is bounded. Clearly $\rho(T^n(x),\Omega(x))$ is bounded. If $\rho(T^n(x),\Omega(x))$ does not approach 0 as $n \to \infty$, there is a subsequence n_i such that $n_i \to \infty$, $T^{n_i}(x)$ converges and $\rho(T^{n_i}(x),\Omega(x))$ does not converge to 0. Clearly this leads to a contradiction, since the limit of $T^{n_i}(x)$ is in $\Omega(x)$, and $T^n(x) \to \Omega(x)$ as $n \to \infty$. If $T^n(x) \to E$ and E is closed, then it is easy to see that $\Omega(x) \subset E$. Hence $\Omega(x)$ is the smallest closed set that $T^n(x)$ approaches.

It remains to show that if $T^n(x)$ is bounded, then $\Omega(x)$ is invariantly connected. Assume $\Omega(x)$ is the union of two disjoint closed nonempty invariant sets Ω_1 and Ω_2. Since $\Omega(x)$ is compact, so are Ω_1 and Ω_2. There then exist disjoint open sets U_1 and U_2 such that $\Omega_1 \subset U_1$ and $\Omega_2 \subset U_2$. Also, since T is uniformly continuous on Ω_1, there is an open set V_1 such that $\Omega_1 \subset V_1$ and $T(V_1) \subset U_1$. Since $\Omega(x)$ is the smallest closed set that $T^n(x)$ approaches $T^n(x)$ must intersect both V_1 and U_2 are infinite number of times. But this implies the existence of a convergent subsequence $T^{n_i}x$ that is not in either V_1 or U_2. This contradiction, since $T^{n_i}(x)$ has a convergent subsequence, shows that $\Omega(x)$ is invariantly connected and completes the proof.

3.13. Exercise. Show that: If K is compact and positively invariant, then $\bigcap_{n=0}^{\infty} T^n(K)$ is nonempty, compact and invariant, and is obviously the largest invariant set in K.
(Suggestion: This can be proved, and more is learned, by first studying the properties of

$$\Omega(H) = \bigcap_{j=0}^{\infty} \overline{\bigcup_{n=j}^{\infty} T^n(H)}.$$

Note that $y \in \Omega(H)$ if and only if there are sequences $n_j \in J_+$ and $y^j \in H$ such that $T^{n_j}y^j \to y$ and $n_j \to \infty$ as $j \to \infty$.)

4. Lipaunov Functions. An Extension of Liapunov's Direct Methods

What we will do here is to so define the concept of a Liapunov function that we obtain, exploiting the invariance property of positive limit sets, a result which relates Liapunov functions to the location of positive limit sets. Because of this, the principal result is called an "invariance principle". Our definition is much less restrictive than Liapunov's and greatly extends his direct method. (His method is called "direct" because it does not depend on a knowledge of solutions.) This was first done for autonomous ordinary differential equations in [24] (see also [26]), and today has been extended to infinite dimensional dynamical systems (difference-differential equations, functional differential equations, certain types of partial differential equations and evolutionary equations (for example, see [16]). It also has been extended, to nonautonomous systems (references are given later). It is an elementary result but one with many important applications.

Let $V: R^m \to R$. Relative to (2.1) (or to T) define

$$\dot{V}(x) = V(T(x)) - V(x).$$

If $x(n)$ is a solution of (2.1),

$$\dot{V}(x(n)) = V(x(n+1)) - V(x(n));$$

$\dot{V}(x) \leq 0$ means that V is nonincreasing along solutions.

4.1. Definition. Let G be any set in R^m. We say that V is a Liapunov function of (2.1) on G if (i) V is continuous on R^m, and (ii) $\dot{V}(x) \leq 0$ for all $x \in G$.

For V a Liapunov function of (1.1) on G, we define

$$E = \{x;\ \dot{V}(x) = 0,\ x \in \bar{G}\}$$

($\bar{G}$ is the closure of G). We use M to denote the largest invariant set in E (see Exercise 3.11). $V^{-1}(c) = \{x;\ V(x) = c,\ x \in R^m\}$, a level surface.

4.2. Theorem (Invariance Principle). If (i) V is a Liapunov function of (2.1) on G, and (ii) $x(n)$ is a solution of (2.1) bounded and in G for all $n \geq 0$, then there is a number c

such that

$$x(n) \to M \cap V^{-1}(c).$$

<u>Proof</u>. The proof now is quite easy. By our assumptions $V(x(n))$ is nondecreasing with n and is bounded from below. (Hence, $V(x(n)) \to c$ as $n \to \infty$. If $p \in \Omega(x(0))$, then there is a sequence n_i such that $n_i \to \infty$ and $x(n_i) \to p$. By continuity of V, $V(x(n_i)) \to V(p) = c$ and $\Omega(x(0)) \subset V^{-1}(c)$. Since $\Omega(x(0))$ is invariant, $V(T^n(p)) = c$ and $\dot{V}(p) = 0$. Therefore, $\Omega(x(0)) \subset E$, and hence in M. This completes the proof.

The difficulty in applications is to find "good" Liapunov functions — ones that make M as small as possible. For instance, a constant function is always a Liapunov function but gives no information.

Let us look at a simple example which illustrates how the result is applied. Consider the 2-dimensional system

$$x(n+1) = \frac{ay(n)}{1+x^2(n)}$$
$$y(n+1) = \frac{bx(n)}{1+y^2(n)}$$

or

$$x' = \frac{ay}{1+x^2} \qquad (4.1)$$
$$y' = \frac{bx}{1+y^2}$$

Take $V(x,y) = x^2 + y^2$. Then

$$\dot{V}(x,y) = \left(\frac{b^2}{(1+y^2)^2} - 1\right)x^2 + \left(\frac{a^2}{(1+x^2)^2} - 1\right)y^2.$$

Case 1. $a^2 < 1$, $b^2 < 1$. Now

$$\dot{V} \le (b^2-1)x^2 + (a^2-1)y^2,$$

and V is a Liapunov function of (4.1) on R^2. Here $M = E = \{(0,0)\}$, and since every solution is clearly bounded, we have by Theorem 4.2 that every solution approaches the origin as $n \to \infty$ (the origin is a global attractor and later, as we shall explain, we can conclude in this case that the origin is globally asymptotically stable). This is Liapunov's classical case — $V(x)$ and $-\dot{V}(x)$ are positive definite.

Case 2. $a^2 \le 1$, $b^2 \le 1$ and $a^2 + b^2 < 2$. We may assume that $a^2 < 1$ and $b^2 = 1$. V is still a Liapunov function of (4.2) on R^2 but $-\dot{V}$ is not positive definite. In fact, $\dot{V} \le (a^2-1)y^2$, and E is the x-axis ($y = 0$). However, since $T(x,0) = (0,bx)$, we see that again M is just the origin. The conclusion is the same as in Case 1.

Case 3. $a^2 = b^2 = 1$. V is still a Liapunov function of (4.2) and all solutions are still bounded. Here $E = M$ is the union of the two coordinate axes, and by Theorem 4.2 we know that each solution approaches $\{(c,0), (0,c), (-c,0), (0,-c)\}$ — the intersection of E with the circle $x^2 + y^2 = c^2$. There are two subcases.

(i) $ab = 1$. Then $T(c,0) = (0,bc)$, $T^2(c,0) = T(abc,0) = (c,0)$. Since positive limit sets are invariantly connected, every solution approaches one of these periodic motions — the origin or a periodic motion of period 2 (see Exercise 3.7).

(ii) $ab = -1$. Here $T(c,0) = (0,bc)$, $T^2(c,0) = (abc,0) = (-c,0)$, $T^3(c,0) = (0,-bc)$, and $T^4(c,0) = (-abc,0) = (c,0)$. If $c \neq 0$, these are periodic motions of period 4. As in (i) above, each solution approaches the origin or one of these periodic motions of period 4.

<u>Case 4</u>. $a^2 > 1$, $b^2 > 1$, Let $B_\delta = \{(x,y);\ x^2 + y^2 < \delta^2\}$. For $x \in B_\delta$ and δ sufficiently small

$$\dot{V} \geq \left(\frac{b^2}{1+\delta^2} - 1\right)x^2 + \left(\frac{a^2}{1+\delta^2} - 1\right)y^2 > 0,$$

and $-\dot{V}$ is a Liapunov function of (4.2) on B_δ for δ sufficiently small, and $E = M = \{(0,0)\}$. No solution starting at a point in B_δ other than the origin can approach the origin from within B_δ (its distance from the origin is increasing) and $T(x,y) = (0,0)$ implies $x = y = 0$. Therefore each such solution must leave B_δ by Theorem 4.2 (instability) and, since no solution can jump to the origin in finite time except the trivial solution, there is no nontrivial solution that can approach the origin as $n \to \infty$.

This is called strong instability (see Section 5). There is not much more information that can be obtained from this particular Liapunov function. The only cases that could be handled by Liapunov's classical direct method are $a^2 < 1$ and $b^2 < 1$, and $a^2 > 1$ and $b^2 > 1$.

5. Stability and Instability

The original definition by Liapunov (see [30]) for stability was for solutions (motions) and can be viewed as an asymptotic continuity with respect to initial conditions. We will restrict ourselves to stability of equilibrium states or positively invariant sets (equilibrium sets).

5.1. Definition. A set H is said to be stable if given a neighborhood U of H (an open set containing $\bar{H}$), there is a neighborhood W of H such that $T^n(W) \subset U$ for all $n \in J_+$ $(T(W) = \{T(x);\ x \in W\})$.

If H is bounded, a complete neighborhood system of H is given by the spheres $B_\varepsilon(H) = \{x;\ \rho(x,H) < \varepsilon\}$.

5.2. Exercise. Show that: If H is stable, then $\bar{H}$ is positively invariant. In particular, if H is a point, it is an equilibrium point.

For H a set in R^m define $\hat{H}$ as follows: $z \in \hat{H}$ if there is a sequence $x_i \in R^m$ and a sequence $n_i \in J_+$ such that

$x_i \to y \in \bar{H}$ and $T^{n_i}(x_i) \to z$. In topological dynamics, the set of all such z for which n_i is bounded is called the <u>prolongation</u> of H (if H is a point x, $\hat{H}$ is called a prolongation of the solution $T^n x$). The set of all such z for which $n_i \to \infty$ is called the <u>prolongation limit set</u> of H. Note that $H \subset \hat{H}$.

5.3. <u>Lemma</u>. (a) Let H be a compact positively invariant set. Then H is stable if and only if $\hat{H} = H$.

(b) Let H be a closed invariant set contained in an open and bounded positively invariant set G. Then $\hat{H}$ is invariant.

<u>Proof</u>. (a) It is clear that, if there is a $z \in \hat{H}$ that is not in H, then H is not stable. Suppose H is not stable. Then for some bounded neighborhood U of H there exists a sequence x^i such that $x^i \to y \in H$ as $i \to \infty$ and each motion $T^n x^i$ eventually leaves U. Let n_i be the smallest integer with the property that $T^{n_i} x^i$ is not in U. Since T is continuous, $T(U)$ is bounded and so is the sequence $T^{n_i} x^i$. It, therefore, has a convergent subsequence which converges to a point not in U, and hence H not stable implies $\hat{H} \neq H$. This completes the proof.

(b) The proof is about the same as the proof of invariance in Theorem 3.12.

The stability concept of greatest practical interest is "asymptotic stability". We will see why in Section 9. We now define this type of stability.

5.4. <u>Definition</u>. A set H is an <u>attractor</u> if there is a neighborhood U of $\bar{H}$ such that $x \in U$ implies $T^n x \to \bar{H}$ as $n \to \infty$. It is a <u>global attractor</u> if $T^n x \to H$ for all $x \in R^m$. If H is both stable and an attractor, then H is said to be <u>asymptotically stable</u>. If H is stable and is a global attractor, H is said to be <u>globally asymptotically stable</u>. <u>Unstable</u> means not stable. If H is neither stable nor an attractor, we will say that H is <u>strongly unstable</u>.

5.5. <u>Exercise</u>. Given a set H its inverse image $T^{-1}(H) = \{y;\ T(y) \in H\}$. A set is said to be <u>inversely invariant</u> if $T^{-1}(H) = H$. Show that:

(a) A set H is inversely invariant if and only if H is positively invariant and $T^{-1}(H) \subset H$.

(b) A set H is negatively invariant if and only if $T^{-1}(x)$ intersects H for each $x \in H$.

For ordinary differential equations inverse invariance and invariance are identical (there is uniqueness of solutions in both directions of time).

5.6. <u>Exercise</u>. <u>The region of attraction</u> $\mathcal{R}(H)$ of a set H is the set of all x such that $T^n x \to H$ as $n \to \infty$. The boundary of H is denoted by ∂H and its complement by

$\mathscr{C}H$ ($\partial H = \bar{H} \cap \overline{\mathscr{C}H}$). Show that: If H is asymptotically stable, then

(a) $\mathscr{R}(H)$ is open

(b) $\mathscr{R}(H)$, ∂H, and $\mathscr{C}\mathscr{R}(H)$ are inversely invariant.

We now have as an almost immediate consequence of Theorem 4.2 and Lemma 5.3:

5.7. <u>Theorem</u>. Let G be a bounded open positively invariant set. If (i) V is a Liapunov function of (2.1) on G, (ii) $M \subset G$, then M is an attractor and $G \subset \mathscr{R}(M)$. If, in addition, (iii) V is constant on M, then M is asymptotically stable (globally asymptotically stable relative to G).

<u>Proof</u>. The first part of the theorem is an immediate consequence of Theorem 4.2. E is closed and in $\bar{G}$ and hence is compact. Therefore, M is compact (Exercise 3.11). Assume now that (iii) holds. Then by Lemma 5.3 M not stable implies $\hat{M} \neq M$. However, it is not difficult to see that $V(x) = c$ for $x \in \hat{M}$ and since $\hat{M}$ is invariant, $\hat{M} \subset E$. This contradicts the definition of M, which is the largest invariant set in E and completes the proof.

Note that, if M is a single point, then (iii) is automatically satisfied. For example, we can now conclude global asymptotic stability of (4.1) if $a^2 \leq 1$, $b^2 \leq 1$, and $a^2 + b^2 < 2$. This is also true if M is a finite invariantly connected set (see Exercise 3.7). Note also that if no solution outside M in G can reach M in finite time (i.e., M is also inversely invariant), then condition (iii) can be replaced by: (iii)' M is constant on the boundary of M. Another point of interest in applications is that the size of G gives information as to the "extent" of asymptotic stability.

5.8. Exercise. Show that: If (i), (ii), and (iii) above are satisfied and M is the largest positively invariant set in E, then $V(x) > c$ for each $x \in G - M$, where c is the value of V on c (i.e., $V(x) - c$ is positive definite relative to M).

It turns out in applications that the largest positively invariant set in E is usually the same as the largest invariant set in E (examples to the contrary are quite artificial) and "good" Liapunov functions will usually be positive definite. However, except for quadratic forms, where there are computable criteria (see [21]), positive definiteness may be difficult to establish, even when M is a point. Theorem 5.7 tells us that it is not necessary to verify that V is positive definite with respect to M. Rather, Theorem 5.7 plus Exercise 5.8 gives a sufficient condition for positive definiteness $(c = 0)$ (when M is a single point, this is a sufficient condition for the existence of a local minimum of V).

5.9. Exercise. (The analog of Liapunov's first two theorems on stability). Let 0 be an equilibrium point of (2.1), and let N be a neighborhood of the origin. $V: R^n \to R$ is said to be positive definite if $V(x) \geq 0$ on N and $V(x) = 0$ and $x \in N$ implies $x = 0$ (the origin is an isolated minimum of V). Show that: If (i) V positive definite and (ii) $\dot{V}$ is a Liapunov function of (2.1) on N, then the origin is stable. If, in addition, $\dot{V}$ is negative definite (i.e., $-V$ is positive definite), the origin is asymptotically stable. Suggestion. You may assume N is bounded. Let $2m_0$ be the minimum of V on the boundary of N. Let $G = \{x; V(x) < m_0; x \in N\}$. Argue that G is positively invariant and apply Theorem 5.7.

5.10. Exercise. Generalize the result in Exercise 5.9 for the stability of compact positively invariant sets.

We could develop specialized criteria for instability but prefer to place emphasis on the use of Theorem 4.2. This was illustrated in our discussion of (4.11). We look at another example which is a bit more interesting. The technique we will use generalizes that given by Ĉetaev for ordinary differential equations. He wanted to show for conservative dynamical systems that, if an equilibrium is not a minimum of the potential energy, then it is not stable (see [25], p. 56). Lagrange had enunciated the converse — at a minimum the equilibrium is stable — and Liapunov proved it. A Liapunov function is a generalized energy function. Consider

$$x' = f(x,y)$$
$$y' = g(x,y), \; f(0,0) = g(0,0) = 0. \qquad (5.1)$$

Let $V = -xy$. Then $\dot{V} = xy - f(x,y)g(x,y)$. Assume that $f(x,y)g(x,y) > xy$ for $xy > 0$.

Let $G = \{(x,y);\ xy > 0,\ x^2 + y^2 < \delta^2\}$. Note that $xy > 0$ is positively invariant and that V is a Liapunov function on G. Since V vanishes on $x = 0$, $y = 0$, and no solution starting in G can remain in G and approach these axes ($\dot{V} < 0$ in G), no solution starting in G can remain in G for all $n > 0$ (this would contradict Theorem 4.2). Also when a solution leaves G it goes outside $x^2 + y^2 < \delta^2$. Hence the origin is not stable. Since $xy > 0$ is positively invariant, the origin is not an attractor, and it is strongly unstable — δ is a measure of its instability.

6. Liapunov Functions

Everyone who has worked with Liapunov functions knows that two Liapunov functions are better than one, and except for notational convenience there is very little gained by the usual concept of a vector Liapunov function. The term vector Liapunov function seems to have been first used by Bellman [5]. For an application where again the terminology is used and other references see [14]. We will generalize the usual concept of a vector Liapunov function. The idea comes from a construction of a Liapunov function by the economist Arrow *et al* in [2], and this has been much exploited by economists in studying stability in competitive analysis. They, however, seem not yet to be acquainted with the extensions of Liapunov's direct method and the use of an invariance principle. For applications of stability theory in economics see [2,3,18,19]. An interesting difference equation model for the control of unemployment and inflation is given in [40].

For $x \in R^m$, $x > 0$ means $x_i > 0$ and $x \geq 0$ means $x_i \geq 0$, $i = 1,\ldots,m$. Let $v: R^m \to R^q$ and define $\dot{v}(x) = v(T(x)) - v(x)$. All definitions and result translate exactly, and we have the exact analog of Theorem 4.2. Now, of course, if v is a vector Liapunov function — in this the usual sense — then each v_i is a scalar Liapunov function for each i and so is $V = \sum_{i=1}^{q} v_i$. The set M for v is the same as the set M for V, but there may be a difference between $M \cap v^{-1}(c)$ and $M \cap V^{-1}(c)$. So there may be some information gained by the use of the vector Liapunov function v but this has not been demonstrated by a significant example. In any case what we want to do is to go to something more general.

For a vector function $w: R^m \to R^q$, define $W(x) = \max_i w_i(x)$. If w is continuous, then so is W. Define $\overset{\circ}{w}(x) = w(T(x)) - W(x)u$, where $u_i = 1$ for $i = 1,2,\ldots,q$ $(\overset{\circ}{w}_i(x) = w_i(T(x)) - W(x))$.

6.1. <u>Definition</u>. We will say that w is a <u>vector Liapunov function</u> of $x' = T(x)$ on G if (i) w is continuous and (ii) $\overset{\circ}{w}(x) \leq 0$ for all $x \in G$.

We define $E = \{x;\ \overset{\circ}{w}(x) \not< 0,\ x \in \bar{G}\}$; i.e., $x \in E$ if $x \in \bar{G}$ and $\overset{\circ}{w}_i(x) = 0$ for some $i = 1,\ldots,q$. M is the largest invariant set in E. Note that

$$\overset{\circ}{w}(x) = \dot{w}(x) + w(x) - W(x)u$$

so that $\overset{\circ}{w}(x) \leq \dot{w}(x)$ for all x, and requiring $\overset{\circ}{w} \leq 0$ is not as strong as $\dot{w} \leq 0$. In fact, w can be a vector Liapunov function in our sense, and yet it may be that no component of w is a scalar Liapunov function. However, $\dot{W}(x) = \operatorname{Max}_i \overset{\circ}{w}_i(x)$ so that, if w is a vector Liapunov function, then W is a scalar Liapunov function. This is just another, and a good way, for constructing a scalar Liapunov function from a number of scalar functions. This idea would seem to have natural applications to problems in control where you wish always to be sure that at each time you reduce the largest component of a vector measure w of the error in control (or performance). The use of a vector Liapunov function of this type to design a control that does this may be an idea that is worth exploring. We have immediately from Theorem 4.2

6.2. <u>Corollary</u>. If (i) w is a vector Liapunov function of (2.1) on G and (ii) a solution $x(n)$ of (2.1) is in G and bounded for all $n \geq 0$, then, for some c, $x(n) \to M \cap W^{-1}(c)$ as $n \to \infty$.

We illustrate the use of this result in the next section. For an application to the study of an epidemic model see [28].

7. Linear Difference Equations

Let $A = (a_{ij})$ be a real $m \times m$ matrix. $A^{\tau} = (a_{ji})$, the transpose of A. $A^n = AA^{n-1}$, $A^0 = I$. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of A, $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ (the spectrum of A) and $r(A) = \operatorname{Max}_i |\lambda_i|$ (the spectral radius).

The general linear difference equation of dimension m is

$$x' = Ax. \tag{7.1}$$

The solution satisfying $x(0) = x^0$ is $A^n x^0$. The columns of A are the principal solutions of (7.1). If v^i is an eigenvector of A with eigenvalue λ_i, then $c_i \lambda_i^n v^i$ is a solution of (9.1). If the eigenvalues of A are distinct, then the general solution of (7.1) is

$$x(n) = c_1 \lambda_1^n v^1 + \cdots + c_n \lambda_q^n v^q.$$

Thus, if $r(A) \geq 1$, there is always a solution that does not approach the origin. If $r(A) > 1$, there are unbounded solutions.

7.1. An algorithm for computing A^n from its eigenvalues.

This algorithm is the analog of Putzer's algorithm in [35] for computing e^{At}.

We look for a representation of A^n in the form

$$A^n = \sum_{j=1}^{m} w_j(n) Q_{j-1} \tag{7.2}$$

where

$$Q_j = (A-\lambda_j I) Q_{j-1}, \; Q_0 = I. \tag{7.3}$$

$Q_m = 0$ by the Hamilton-Cayley Theorem (every matrix satisfies its characteristic equation). It is just this fact that suggests the form of the representation (7.2).

The initial condition $A^0 = I$ is satisfied by taking $w_1(0) = 1,\; w_2(0) = \cdots = w_m(0) = 0$. We want $A \sum_{j=1}^{m} w_n(n) Q_{j-1} = \sum_{j=1}^{m} w_j(n+1) Q_{j-1}$ or, since $AQ_{j-1} = Q_j + \lambda_j Q_{j-1}$,

$$\sum_{j=1}^{m} w_j(n)(Q_j + \lambda Q_{j-1}) = \sum_{j=1}^{m} w_j(n+1) Q_{j-1}.$$

Thus, (7.2) holds if

$$\begin{aligned} & w_1(n+1) = \lambda_1 w_1(n),\; w_1(0) = 1 \quad (w_1(n) = \lambda_1^n) \\ & w_j(n+1) = \lambda_j w_j(n) + w_{j-1}(n),\; w_j(0) = 0,\; j = 2 \cdots m. \end{aligned} \tag{7.4}$$

Equations (7.3) and (7.4) are algorithms for computing the Q_j and the $w_j(n)$ in terms of the eigenvalues of A (i.e., for computing A^n if we know or have computed the eigenvalues of A).

By way of illustration let us use the algorithm to find the solution of

$$y''' - 3y'' + 3y' - y = 0; \; y''(0) = y_0'', \; y'(0) = y_0', \; y(0) = y_0.$$

This 3rd-order equation is equivalent to $x' = Ax$ where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

$$\phi(\lambda) = \det(A-\lambda I) = -(\lambda-1)^3 \text{ and } \lambda_1 = \lambda_2 = \lambda_3 = 1.$$

$$Q_0 = I, \; Q_1 = A - I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -3 & 2 \end{pmatrix}$$

$$Q_2 = (A-I)^2 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$$

Solving (7.4) directly, or by using Exercise 7.3, we obtain $w_1(n) = 1$, $w_2(n) = n$, $w_3(x) = \frac{1}{2} n(n-1)$. Hence

$$A^n = I + n(A-I) + \frac{1}{2} n(n-1)(A-I)^2$$

$$= \begin{pmatrix} \frac{1}{2}(n-1)(n-2) & -n(n-2) & \frac{1}{2} n(n-1) \\ \frac{1}{2} n(n-1) & -(n+1)(n-1) & \frac{1}{2}(n+1)n \\ \frac{1}{2}(n+1)n & -(n+2)n & \frac{1}{2}(n+2)(n+1) \end{pmatrix}.$$

The solution $y(n)$ is the first component of $A^n x^0$. This gives $y(n) = \frac{1}{2}(n-1)(n-2)y_0 - n(n-2)y_0' + \frac{1}{2}n(n-1)y_0''$.

7.2. Exercise. (a) Show that: If $r(A) = r_0 < \beta$ then

$$|w_j(n)| \le \frac{\beta^n}{(\beta - r_0)^{j-1}}, \quad j = 1,\dots,m.$$

(b) if $\alpha < r(A)$ establish a lower bound for $|w_j(n)|$.

From Exercise 7.2 we see that, if $|r(A)| < 1$, then $w_j(n) \to 0$ as $n \to \infty$, and hence $A^n \to 0$ as $n \to \infty$. We already know when $r(A) \ge 1$ that A^n does not approach 0 as $n \to \infty$. $A^n \to 0$ corresponds to the global asymptotic stability of (7.1), and hence we see that (7.1) is globally asymptotically stable if and only $r(A) < 1$. In this case we will say A is stable. For computational criteria that the roots of a polynomial lie in the unit circle see [21]. Although we have assumed, and continue to do so, that A is real, nothing we have done so far uses this assumption.

7.3. Exercise. Show that

$$w_1(n) = \lambda_1^n$$

$$w_j(n+1) = \sum_{k=0}^{n} \lambda_j^{n-k} w_{j-1}(k), \quad j = 2,\dots,m.$$

7.4. Exercise. If the eigenvalues of A are distinct, show that

$$w_1(n) = \lambda_1^n$$

$$w_j(n) = \sum_{i=1}^{j} c_{ij}\lambda_i^n$$

where

$$c_{ij}^{-1} = \prod_{k=1,k\neq i}^{j} (\lambda_i - \lambda_k).$$

7.5. <u>Exercise (Variation of constants formula)</u>. Show that the solution of the initial problem $(f: J_+ \to R^m)$

$$x' = Ax + f(n),\ x(0) = x^0$$

is

$$x(n+1) = A^{n+1}x^0 + \sum_{k=0}^{n} A^{n-k}f(k).$$

7.6. <u>Exercise</u>. Show that the solution of

$$y(n+m) + a_1 y(n+m-1) + \cdots + a_m y(n) = g(n) \qquad (7.5)$$

satisfying $y(0) = y(1) = \cdots = y(m-1) = 0$ is

$$y(n+1) = \sum_{k=0}^{n} w(n-k)g(k)$$

where w is the m^{th} principal solution of the homogeneous equation $(g(n) \equiv 0)$; that is, w is the solution of

$$y(n+m) + a_1 y(n+m-1) + \cdots + a_m y(n) = 0$$

satisfying

$$y(0) = \cdots = y(n-2) = 0, \quad y(m-1) = 1.$$

Since $A^{k+1} - A^k = A^k(A-I)$, we see that

$$A^{n+1} - I = (A-I)\sum_{k=0}^{n} A^k.$$

If $A^{n+1} \to 0$ $(r(A) < 1)$, we obtain

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k. \tag{7.6}$$

$R(\lambda) = (\lambda I-A)^{-1}$ is called the resolvent of A. We see, therefore, if $|\lambda| > r(A)$ that

$$R(\lambda) = (\lambda I-A)^{-1} = \sum_{k=0}^{\infty} \lambda^{-(k+1)}A^k. \tag{7.7}$$

It is also of interest to know when it is true that each solution of (7.1) approaches a point — which, of course, must be an equilibrium point. Assume that $A^n x \to y = f(x)$ for each x $(\Omega(x) = \{f(x)\})$. Since $f(x)$ is linear, $f(x) = Bx$ for some matrix B, and the question we are asking is: when does A^n converge? We know, if $r(A) < 1$, that $A^n \to 0$ as $n \to \infty$, and, if $r(A) > 1$, the sequence is unbounded. This leaves the case $r(A) = 1$. Assume that $A^n \to B$ as $n \to \infty$ (i.e., $A^n x \to Bx$ for each x). Hence, the point Bx is an equilibrium point of

$\dot{x} = Ax$ (a fixed point of A) and $ABx = Bx$. Since $B \neq 0$, a necessary condition for $A^n \to B$ when $r(A) = 1$ is that $\lambda = 1$ be an eigenvalue of A and be the only eigenvalue on the unit circle. Now $A^n \to B$ implies A^n is bounded, and we see from equation 7.7 that $(\lambda-1)R(\lambda)$ is bounded for $\lambda > 1$. Hence $\lambda = 1$ is a simple pole of the resolvent (this means that 1 is a simple root of the minimal polynomial). The converse of this is true (see, for instance, [37], Chapter I), and hence

7.7. <u>Theorem</u>. $A^n \to 0$ if and only if $r(A) < 1$. If $r(A) > 1$, A^n is unbounded and does not converge. If $r(A) = 1$, then A^n converges if and only if $\lambda = 1$ is a simple pole of the resolvent of A and is the only eigenvalue of A on the unit circle.

This can also be seen from the algorithm for A^n or from the Jordan canonical form for A.

7.8. <u>Exercise</u>. Show that: If $A^n \to B$ as $n \to \infty$, then $AB = BA = B$ and $B^2 = B$.

7.9. <u>Exercise</u>. When is it true that each solution of (7.1) is bounded?

8. <u>Stability of Linear Systems</u>

To illustrate further the application of Liapunov's direct method and the use of Liapunov functions, we will study a bit more the question of the stability of

$$\dot{x} = Ax. \tag{8.1}$$

We shall say, in place of "the origin for (8.1) is asymptotically stable", simply, "(8.1) is asymptotically stable" or A is stable. Also, for linear systems asymptotic stability is always global, and the adjective is not needed. We know that A is stable if and only if $r(A) < 1$. The next criterion is the analog of the one given orginally by Liapunov for the real parts of all the eigenvalues of A to be negative ($e^{At} \to 0$ as $t \to \infty$).

Let $V(x) = x^{T}Bx$ where B is positive definite. Then, with respect to (8.1), $\dot{V}(x) = x^{T}(A^{T}BA-B)x$. Hence, if $A^{T}BA - B$ is negative definite, (8.1) is asymptotically stable by Exercise 5.9, and A is stable. Conversely, suppose that A is stable, and consider the equation

$$A^{T}BA - B = -C. \tag{8.2}$$

If it has a solution, then

$$-\sum_{k=0}^{n} (A^{T})^{k}CA^{k} = (A^{T})^{n+1}BA^{n+1} - B.$$

Letting $n \to \infty$, we see that the solution must be

$$B = \sum_{k=0}^{n} (A^{T})^{k}CA^{k}.$$

It is easily verified that this is a solution and that, if C is positive definite, it is positive definite.

Hence, we have shown

8.1. Theorem. If there are positive definite matrices B and C satisfying (8.2), then A is stable. Conversely, if A is stable, then given C, (8.2) has a unique solution B. If C is positive definite, B is positive definite.

This result plays an important role in the theory of linear discrete control systems. It is also a converse theorem. If (8.1) is asymptotically stable, there is a positive definite quadratic Liapunov function V with $-\dot{V}$ positive definite. (An easy proof of the first statement in Exercise 8.3 is obtained using this result.)

Consider the nonlinear difference equation

$$x' = Ax + f(n,x). \tag{8.3}$$

Assume that $f(n,x)$ is $o(x)$ uniformly with respect to $n \geq 0$. This means that given $\varepsilon > 0$ there is a $\delta > 0$ such that $||f(n,x)|| < \varepsilon||x||$ for all $||x|| < \delta$ and all $n \geq 0$. Near the origin $f(n,x)$ should not have much effect on stability. To some extent this is true and that is the next exercise.

8.3. Exercise (Stability by the linear approximation). Consider the nonlinear difference equation (8.3) where $f(n,x)$ is $o(x)$ as described above. Show that: If the linear approximation $x' = Ax$ is asymptotically stable $(r(A) < 1)$, then so is (8.3). If $r(A) > 1$, then (8.3) is unstable.

The last statement in the result above is more difficult to prove. The critical cases are when $r(A) = 1$. Then one must take into account the nonlinearities. This result on the linear approximation is useful in applications. It was, up until 1950, the way most control systems were designed. But it must be kept in mind that it gives no information about the size of the region of asymptotic stability. It is purely a local result. The region of asymptotic stability may be so small relative to a given application that from a practical point of view the origin is unstable. Also the origin of the nonlinear system can be unstable but a very small neighborhood of the origin could be an attractor and from a practical point of view it could be stable. There could be a stable periodic oscillation about the origin so small that its effect could be negligible. One of the advantages of Liapunov's direct method versus deciding stability on the basis of the linear approximation is that it takes the nonlinearities into account and can yield information about the extent of the stability or instability.

Nonnegative matrices, $A = (a_{ij})$ are those for which $a_{ij} \geq 0$ $(A \geq 0)$. They arise naturally in many applications, and have been studied extensively for a long time. (See [4,11]). For instance, for the linear difference equation $x' = Ax$ where the state variables are naturally nonnegative (populations, prices, number of particles, etc.) the matrix A will be nonnegative since $\overline{R^m_+} = \{x; x \geq 0, x \in R^m\}$ must be positively invariant.

8.4. Exercise. Show that:

(a) $\overline{R^m}$ is positively invariant for $x' = Ax$ if and only if $A \geq 0$

(b) $R^m_+ = \{x;\ x > 0,\ x \in R^m\}$ is positively invariant for $x' = Ax$ if and only if $A \geq 0$ and no row of A is zero.

A good illustration of the use of our results is the proof of part of the following theorem.

8.5. Theorem. If $A \geq 0$, the following are equivalent:

(1) $|\lambda(A)| < 1$

(2) $(A-I)^{-1} \leq 0$

(3) There is a $c > 0$ such that $Ac < c$.

(4) $(A^T-I)D + D(A-I)$ is negative definite for some positive definite diagonal matrix D.

(5) The real parts of the eigenvalues of $A - I$ are all negative.

(6) The principal minors of $I - A$ are all positive.

Proof. We shall prove the equivalence of the first three statements. (1) ==> (2) follows from Equation (7.6). To see that (2) ==> (3) take $c = (I-A)^{-1}b$ for any $b > 0$. Then, since $(I-A)^{-1} \geq 0$ and is nonsingular, $c > 0$. Assume (3) and let $w_i(x) = \frac{|x_i|}{c_i}$. We will show that w is a vector Liapunov function. Here $W(x) = \underset{i}{\text{Max}} \frac{|x_i|}{c_i}$

$$w_i(T(x)) = \frac{|(Ax)_i|}{c_i} \leq \frac{1}{c_i} \sum_{k=1}^{m} a_{ik} c_k \frac{|x_k|}{c_k} \leq W(x) \frac{(Ac)_i}{c_i}$$

$$< W(x), \; x \neq 0.$$

Hence, $\overset{\circ}{w}(x) < 0$ for $x \neq 0$ and $w(x)$ is a vector Liapunov function. Since $W(x) \to \infty$ as $||x|| \to \infty$, every solution is bounded and approaches M, which is simply the origin. Therefore, (3) ==> (1), and we have shown the equivalence of (1), (2), and (3).

It is known that (3), (4) and (5) are equivalent. A rather detailed discussion of such equivalences can be found in [11]. What we have seen is that our stability theory gives simple proofs of some significant and nontrivial results. For applications of results of this type to economics see [19].

8.6. <u>Exercise</u>. Define $|A| = (|a_{ij}|)$. With the vector Liapunov function w used in the above proof, show that: $|A|$ stable implies A is stable, and hence that $r(A) \leq r(|A|)$.

8.7. Exercise. Show that:

(a) $|Ax| \le |A|\cdot|x|$, $(|x|_i = |x_i|)$

(b) If $|T(x)| \le B|x|$ for all x and B is stable, then $T^n x \to 0$ for all x.

9. Stability Under Perturbations

The converse theorems of Liapunov theory are important theoretically — too important not at least to mention. They guarantee the existence of Liapunov functions if there is asymptotic stability, as did our earlier result for linear systems (Theorem 8.1). The general results are nonconstructive and are of no help in finding Liapunov functions, but, as we shall see, they do enable us to answer an important practical question.

A proof of the following converse theorem can be found in [15]. Here we consider again the general difference equation

$$\dot{x} = T(x), \quad T(0) = 0. \tag{9.1}$$

The system has an equilibrium which we locate at the origin. T is said to be lipschitz continuous near the origin if for some $L > 0$ and $r > 0$

$$||T(x) - T(y)|| \le L||x-y||$$

for all $||x|| < r$ and $||y|| < r$.

9.1. Theorem. If T is lipschitz continuous near the origin and the origin is asymptotically stable, there exists a positive definite $V: R^m \to R$ which is lipschitz continuous near the origin with $\dot{V}$ negative definite.

Now our definitions of stability and asymptotic stable, which are Liapunov's, are in terms of perturbations of initial conditions. In the real world, where nothing is known exactly, a system is being constantly perturbed and a better model for the perturbations is .

$$x' = T(x) + P(n,x), \tag{9.2}$$

where $P(n,x)$ is unknown but hopefully not too large. Now simple stability of the origin for the unperturbed system (9.1) is too fragile to expect it to imply a stability of the perturbed system (9.2). Not so with asymptotic stability, and this is why asymptotic stability is of practical interest. We will now describe this stability under pertrubations. It is called "strong" because originally (and this was for ordinary differential equations) only stability under perturbations was proved (for instance, this is all that is proved in [15] for difference equations). The stronger result was first obtained within the context of topological dynamics (see [38]).

9.2. <u>Definition</u>. Let $x^*(n,x^0)$ denote the solution of (9.2) satisfying $x^*(0,x^0) = x^0$. The origin is said to be <u>stable under perturbations</u> if for some ε_0 and each $\varepsilon > 0$ there exist $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ such that $||\dot{x}|| < \delta_1(\varepsilon)$ and $||P(n,y)|| < \delta_2(\varepsilon)$ for all $n \geq 0$ and all $||y|| < \varepsilon_0$ imply $||x^*(n,x^0)|| < \varepsilon$ for all $n \geq 0$. If, in addition, there is a δ_0, an r_0, and an $N(\varepsilon)$ such that $||x|| < \delta_0$ and $||P(n,y)|| \leq \delta_2(\varepsilon)$ for all $n \geq 0$ and all $||y|| \leq r_0$ imply $||x^*(n,x)|| < \varepsilon$ for all $n \geq N(\varepsilon)$, the origin is said to be strongly stable under perturbations.

The following theorem can then be proved using the above converse theorem (a proof can be found in [28]).

9.3. Theorem. If T is lipschitz continuous in a neighborhood of the origin, then the origin is strongly stable under perturbations if and only if it is asymptotically stable.

A rather recent and significant development for ordinary differential equations is the introduction of skew-product flows (see [39]) and their use to establish invariance properties for the positive limit sets of solutions of nonautonomous equations (see [9,10,27]). This then gives for a large class of nonautonomous ordinary differential equations $(\frac{dx}{dt} = f(t,x))$ an invariance principle and a stability theory much like what we have developed here for autonomous systems. The same can be done for nonautonomous difference equations $x' = T(n,x)$ (see [28]). This is a new tool for the analysis of stability of nonautonomous systems that has not yet been exploited. Unfortunately, this goes beyond the scope of this article.

Although this theory of difference equations has been presented in a concise and sophisticated language, much of it can be made easily accessible to undergraduates, particularly, if one is willing to study first dimensions 2 and 3. Some of it would make a good introduction to the study of ordinary differential equations. Linear difference equations also provide a nontrivial application within an elementary course on linear algebra.

At this point the author wishes to confess that this has been the first time he has looked systematically at the subject of difference equations. What he has done is to do everything by analogy with the more highly developed theory for ordinary differential equations and difference-differential equations (functional differential equations). It has been surprising to him to discover new results and to find at this level so interesting a theory and so much of practical significance. He is now an advocate for considering difference equations as a prerequisite to the study of differential equations, control and stability theory, and the theory of systems. Interesting phenomena modeled by difference equations are not too difficult to find, computations are easy, and it is good introductory applied mathematics.

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