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GENERAL STATISTICAL PROCEDURES:
PARAMETER ESTIMATION USING
WEIBULL DISTRIBUTION, RELIABILITY
TEST OF HYPOTHESIS, AND
COMPUTATION OF EXPECTED
NUMBER OF RENEWALS

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AUGUST 1975

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SYSTEMS ANALYSIS OFFICE
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available to reject the hypothesis that the required durability had been obtained. The subsystem durability was then used to compute replacement requirements over the system lifetime.

CONTENTS

	<u>Page</u>
INTRODUCTION.....	5
PROBLEMS.....	5
PARAMETER ESTIMATION.....	6
AN ILLUSTRATIVE EXAMPLE.....	10
TEST OF HYPOTHESIS.....	12
A PROCEDURE FOR COMPUTING EXPECTED NUMBER OF RENEWALS	18
LITERATURE CITED.....	25
DISTRIBUTION LIST.....	27

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INTRODUCTION

This addendum details the statistical procedures utilized in preparing the report, Decision Risk Analysis for XM204, 105MM Howitzer, Towed Reliability/Durability Requirements (PAA-TR-1-73).¹ A computer program was developed for this study to simulate DT/OT II testing, analysis of test data, the decision made based on this analysis, and the funding implications of this decision.

A two-parameter Weibull family was assumed to describe durability failures. The test results were used to estimate the two parameters for each subsystem: carriage, recoil system, tube, and breech. This was followed by a test of hypothesis which tested whether sufficient information was available to reject the hypothesis that the required durability had been obtained. The subsystem durability was then used to compute replacement requirements over the system lifetime.

PROBLEMS

The first statistical problem, parameter estimation, was unusual in that a large number of subsystems would survive testing. This occurred as the carriage was a long-lived subsystem and the test truncation point was near the expected carriage life. Also, other failed subsystems would be replaced until one of the following occurred:

- a. The carriage failed.
- b. The truncation point was obtained.
- c. No additional back-up subsystems were available.

Thus, for each subsystem the test data would consist of rounds-to-failure

¹Mazza, Thomas N. and Banash, Robert C., PAA-TR-1-73. Decision Risk Analysis for XM204, 105MM Howitzer, Towed Reliability/Durability Requirements, Systems Analysis Division, Plans and Analysis Directorate, US Army Weapons Command, April 1973.

and random truncation points. Parameter estimation was performed by maximizing the log likelihood equations.

The next problem was to use these data in a test of hypothesis. This was accomplished by using the large-sample distribution properties of maximum likelihood estimators, i.e., that the parameters have a joint normal distribution. The α -percentile confidence interval is approximately ellipsoidal; the α -percentile durability boundary was obtained by maximizing over this ellipse.

The third problem was solving for the number of renewals in a specified time interval. This was accomplished, initially, by the method of Lomnicki² and later (to save computer time), by taking averages.

The purpose of this addendum is to detail these procedures.

PARAMETER ESTIMATION

The form of the density function for the two-parameter Weibull distribution used in this study is:

$$f(t) = \lambda \alpha t^{\alpha-1} e^{-\lambda t^\alpha} \quad t \geq 0; \alpha > 0; \lambda > 0. \quad (1)$$

Consider the following subsystem-life test. M-like systems are placed on test and each system is composed on one critical subsystem and several different noncritical subsystems. A total system configuration is required to conduct the test; however, with respect to probability of failure, each subsystem is assumed to be independent. As each non-critical subsystem failure occurs, the failure time is noted, and the failed subsystem

²Lomnicki, Z. A., A Note on the Weibull Process. Biometrika 53, 375-381 (1966).

is replaced with an identical new subsystem. Each of the M systems continue with the test, with failed noncritical subsystems being replaced, until either a predetermined system truncation point (for example, when a stated number of rounds have been fired) is reached or until a critical subsystem failure occurs. This test differs from the well-known Type I and Type II censoring in that the subsystem truncation points are random variables as is the number of each type of noncritical subsystems put on test.

Let N be the total number of a particular noncritical subsystem put on test. Let n of these subsystems fail and their failure times be observed. The remaining $m = N - n$ subsystems are removed from test at the truncation points $T_1, T_2, T_3, \dots, T_m$. Then the logarithmic likelihood function $\ln L(\alpha, \lambda)$, based on the above sample where (1) is the applicable failure density function, is given by

$$\ln L(\alpha, \lambda) = n \ln \alpha + n \ln \lambda + (\alpha - 1) \sum_{i=1}^n \ln y_i - \lambda \sum_{i=1}^n y_i^\alpha - \lambda \sum_{j=1}^m x_j^\alpha \quad (2)$$

where

y_i = an observed failure time,

and

x_j = a system truncation point.

Then

$$\ln L(\alpha, \lambda) = n \ln \alpha + n \ln \lambda + (\alpha - 1) \sum_{i=1}^n \ln t_i - \lambda \sum_{i=1}^N t_i^\alpha \quad (3)$$

where

$$t_i = y_i \quad i \leq n$$

$$= x_i \quad n < i \leq N.$$

This function yields the following likelihood equations

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln t_i - \lambda \sum_{i=1}^N t_i^\alpha \ln t_i = 0, \quad (4)$$

and

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^N t_i^\alpha = 0. \quad (5)$$

Solving (5) in terms of λ and substituting into (4) yields

$$\frac{n}{\alpha} + \sum_{i=1}^n \ln t_i - \frac{n}{\sum_{i=1}^N t_i^\alpha} \cdot \sum_{i=1}^N t_i^\alpha \ln t_i = 0. \quad (6)$$

Equation (6) can be solved for by a standard iterative procedure but by writing (6) in the form

$$K(\alpha) = \frac{\sum_{i=1}^N t_i^\alpha \ln t_i}{\sum_{i=1}^N t_i^\alpha} - \frac{1}{\alpha} - \frac{\sum_{i=1}^n \ln t_i}{n} = 0, \quad (7)$$

it has been shown³ that if we find a $K(\alpha_1) < 0$ and a $K(\alpha_2) > 0$ and α_1 and α_2 are within a sufficiently narrow interval such that $\alpha_1 < \hat{\alpha} < \alpha_2$, a linear interpolation will yield the required value. While this method will eventually yield an answer, it would appear that the rate of convergence can be immensely improved by utilizing basic search techniques. In addition, the search space can be reduced by choosing an initial α_1 that is relatively close to $\hat{\alpha}$. Dubey⁴ has suggested that the moment estimators be utilized for the initial value and Cohen³ developed a table of values relating α and the sample coefficient of variation. Tables relating the coefficient of variation to α require external storage areas (disks/tapes/drums) or, if internal to a computer program, take up precious core storage. Therefore, the following equation is utilized which yields an initial estimator based on the coefficient of variation which is within 0.1% of similar table values.

$$\begin{aligned} \alpha_1 = & .64364 + .18035 \operatorname{csch}(z) - .0317523z + .000684128z^2 \\ & - .00129259 \operatorname{csch}^2(z) + .619344(e^{-z}) + .534717(e^{-z})^4 \\ & + .54047(e^{-z})^7 \end{aligned} \quad (8)$$

where

$$z = \frac{\frac{s^2}{2}}{\bar{x}} = cv = \text{coefficient of variation.}$$

With this initial estimator, a "direct search with acceleration" is utilized to obtain the maximum likelihood estimate (MLE) of α or $\hat{\alpha}$.

³Cohen, A. C., Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples. *Technometrics* 7 (4), 579-588 (1965).

⁴Dubey, S. A., On Some Statistical Inferences for Weibull Laws, *Naval Research Logistics Quarterly* 13, 227-251 (1966).

It has been shown⁵ that $\frac{\hat{\alpha}}{\alpha}$ is distributed independently of α and λ and has the same distribution as $\hat{\alpha}$. Therefore, the bias in $\hat{\alpha}$ is independent of the true value of α and λ and is only dependent on the sample size n . A table of unbiasing factors $B(n)$ was developed so that $E[B(n)\hat{\alpha}] = \alpha$. The values in this table are approximated by the following equation:

$$B(N) = .06541717052 - .00006172777867n + .997393979 \tanh^{10}(n) \\ - 220.5624312 \tanh^7(\ln n/n) - 1.86171021(\ln n)(\tanh(\ln n/n^2)) \quad (9) \\ + .01424021769(\ln n) \tanh^{10}(n) - 39302536740000 \cdot \tanh^8(\ln n/n^3)$$

Once the MLE of α is obtained from the search of equation (7), the appropriate unbiasing factor equation (9) is applied before solving for λ , using equation (5).

AN ILLUSTRATIVE EXAMPLE

Ten systems are placed on test and the truncation point, T_i , for each system is given (in rounds) below.

T_1	-	35,142	T_6	-	76,958
T_2	-	43,839	T_7	-	78,962
T_3	-	45,540	T_8	-	79,864
T_4	-	61,471	T_9	-	91,190
T_5	-	66,681	T_{10}	-	95,202

During the course of the test, 23 failures/replacement (Y_i) of sub-system z occurred at the following time:

⁵Thoman, D. R., Bain, L. J., and Antle, C. E., Inferences on the Parameters of the Weibull Distribution. Technometrics 11 (3), 445-460 (1969).

y ₁ - 33,748	y ₉ - 36,102	y ₁₇ - 14,326
y ₂ - 24,898	y ₁₀ - 6,822	y ₁₈ - 27,959
y ₃ - 10,376	y ₁₁ - 28,824	y ₁₉ - 10,128
y ₄ - 27,186	y ₁₂ - 37,452	y ₂₀ - 31,864
y ₅ - 34,377	y ₁₃ - 1,170	y ₂₁ - 9,366
y ₆ - 13,591	y ₁₄ - 39,886	y ₂₂ - 21,978
y ₇ - 878	y ₁₅ - 10,212	y ₂₃ - 12,924
y ₈ - 41,524	y ₁₆ - 24,029	

and at the time of each system truncation, the following subsystem times (x_i) were observed:

x ₁ - 1,395	x ₆ - 40,855
x ₂ - 8,564	x ₇ - 5,863
x ₃ - 18,354	x ₈ - 38,806
x ₄ - 12,624	x ₉ - 14,662
x ₅ - 25,157	x ₁₀ - 8,940

This sample for subsystem z is from a population in which $\alpha = 1.364$ and $\lambda = .00000102336$. This data can be summarized as:

$$n=23, N=33, \sum_{i=1}^{23} t_i = 499618, \sum_{i=1}^{23} t_i^2 = 14396873864.$$

It follows that $\bar{x} = 21722.52$, $s^2 = 154083086.858$, and the coefficient of variation $cv = .3265$. Then from (8) the initial estimate of $\alpha = 1.81$. To obtain the maximum likelihood estimate, α is varied in equation (7) until $|K(\alpha)|$ is minimized. This results in $\hat{\alpha} = 1.6421$ and applying the unbiasing factor for a sample size of 33, $\hat{\alpha} = 1.5731$.

TEST OF HYPOTHESIS

Consider the following failure density function:

$$f(t; \alpha, \lambda) = \alpha \lambda t^{\alpha-1} e^{-\lambda t^\alpha} \quad \alpha > 0; \lambda > 0; t > 0. \quad (10)$$

When the resulting sample likelihood function of a life test can be written as

$$L = \prod_{i=1}^n \left[\alpha \lambda t_i^{\alpha-1} e^{-\lambda t_i^\alpha} \right] e^{-\lambda \sum_{j=n+1}^N t_j^\alpha} \quad (11)$$

then, this function yields the following partial derivatives of the logarithmic likelihood equations:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln t_i - \lambda \sum_{i=1}^N t_i^\alpha \ln t_i = 0, \quad (12)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^N t_i^\alpha = 0, \quad (13)$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \lambda \sum_{i=1}^N t_i^\alpha (\ln t_i)^2 = 0, \quad (14)$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{-n}{\lambda^2} = 0, \quad (15)$$

$$\frac{\partial \ln L}{\partial \alpha \partial \lambda} = - \sum_{i=1}^N t_i^\alpha \ln t_i = 0. \quad (16)$$

The maximum likelihood estimate (MLE) $\hat{\alpha}$ and $\hat{\lambda}$ can be obtained by solving (13) in terms of λ and substituting into (12). Having obtained the MLE for the parameters of the Weibull law, it is then possible to determine confidence limits for meaningful parametric functions such as durability and reliability.

Durability can be defined as the probability that a randomly selected item from an infinite lot will continue to perform satisfactorily without a durability failure beyond t_0 and is given by

$$\text{Prob}[T \geq t_0] = \int_{t_0}^{\infty} f_T(t) dt = \exp(-\lambda t_0^{\alpha}) . \quad (17)$$

If the parameters α and λ are known, the above problem is completely solved and the exact answer is given by (17); however, usually these parameters are unknown. It is well known that the large sample MLE is approximately normally distributed about the true parameter value as a mean for large samples. This is a powerful tool and will be used to establish confidence limits on durability when the true parameters α and λ are unknown. Assuming the above, it follows that⁶

$$\begin{bmatrix} \hat{\alpha} - \alpha \\ \hat{\lambda} - \lambda \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \hat{\alpha} - \alpha \\ \hat{\lambda} - \lambda \end{bmatrix} \approx \chi^2(2) , \quad (18)$$

where

$$C_{11} = -E \left[\frac{\partial^2 \ln L}{\partial \alpha^2} \right] , \quad (19)$$

⁶Mood, A. M., Introduction to the Theory of Statistics. McGraw-Hill Book Co., Inc., New York, New York, 1950.

$$C_{22} = -E \left[\frac{\partial^2 \ln L}{\partial \lambda^2} \right], \quad (20)$$

and

$$C_{12} = C_{21} = -E \left[\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \right]. \quad (21)$$

From (14) and (19)

$$C_{11} = E \left[\frac{n}{\alpha^2} + \lambda \sum_{i=1}^N t_i^\alpha (\ln t_i)^2 \right]. \quad (22)$$

Examine

$$I = \int_0^\infty t^\alpha (\ln t)^2 \alpha \lambda t^{\alpha-1} e^{-\lambda t} dt$$

and let

$$\begin{aligned} \omega &= t^\alpha, \\ d\omega &= \alpha t^{\alpha-1} dt, \\ \ln \omega &= \alpha \ln t, \\ \ln t &= \frac{1}{\alpha} \ln \omega, \\ (\ln t)^2 &= \frac{1}{\alpha^2} (\ln \omega)^2. \end{aligned}$$

then

$$I = \frac{\lambda}{\alpha^2} \int_0^\infty \omega (\ln \omega)^2 e^{-\lambda \omega} d\omega.$$

From page 578 of referenced literature⁷, equation 4.358(2)

⁷ Gradshteyn, I.S., and Ryzhik, I.M., Table of Integrals, Series, and Products. Academic Press, New York, New York. 1965.

$$\int_0^{\infty} x^{v-1} e^{\mu x} (\ln x)^2 dx = \Gamma(v)/\mu^v \{ [\psi(v) - \ln(\mu)]^2 + \zeta(2, v-1) \}$$

$$\mu > 0, v > 0$$

now let

$$x = \omega,$$

$$v = 2,$$

and

$$\mu = \lambda,$$

then

$$\begin{aligned} I &= \frac{\lambda}{\alpha^2} \int_0^{\infty} \omega (\ln \omega)^2 e^{-\lambda \omega} d\omega = \lambda \Gamma(2) / (\alpha^2 \lambda^2) \{ [\psi(2) - \ln \lambda]^2 + \zeta(2, 1) \} \\ &= \frac{\Gamma(2)}{\lambda \alpha^2} \{ [\psi(2) + \ln \frac{1}{\lambda}]^2 + \zeta(2, 1) \} \end{aligned}$$

From page 1073 of referenced literature⁸, equation 9.521(1) the Riemann's Zeta Function

$$\zeta(z, q) = \sum_{N=0}^{\infty} \frac{1}{(q+N)^z} \quad z > 1$$

$$\therefore \zeta(2, 1) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

From page 260 of referenced literature⁸, equation 6.4.10 the Polygamma Function

$$\psi^N(z) = (-1)^{N+1} N! \sum_{k=0}^{\infty} (z+k)^{-N-1} \quad (z \neq 0, -1, -2, \dots)$$

⁸ Abramowitz, M., and Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards. US Government Printing Office, Washington, DC. 1970

$$\therefore \psi^1(1) = (-1)^1 1! \sum_{k=0}^{\infty} (1+k)^{-2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots ,$$

therefore,

$$\zeta(2,1) = \psi^1(1)$$

and

$$I = \frac{\Gamma(2)}{\lambda \alpha^2} \{ [\psi(2) + \ln \frac{1}{\lambda}]^2 + \psi^1(1) \} .$$

Equation (22) can now be written as

$$C_{11} = \frac{n}{\alpha^2} + \frac{N}{\alpha^2} \{ [\psi(2) + \ln \frac{1}{\lambda}]^2 + \psi^1(1) \} \quad (23)$$

where

$$\psi(2) = .4227843351 ,$$

and

$$\psi^1(1) = 1.6449340668 .$$

From (16) and (21)

$$C_{12} = E \left[\sum_{i=1}^N t_i^{\alpha} \ln t_i \right] \quad (24)$$

Examine

$$I = \int_0^{\infty} t^{\alpha} \ln t \alpha \lambda t^{\alpha-1} e^{-\lambda t^{\alpha}} dt$$

and let

$$\omega = t^{\alpha} ,$$

$$d\omega = \alpha t^{\alpha-1} dt ,$$

$$\ln \omega = \alpha \ln t ,$$

$$\ln t = \ln \omega / \alpha ,$$

then

$$I = \frac{\lambda}{\alpha} \int_0^{\infty} \omega \ln \omega e^{-\lambda \omega} d\omega .$$

From page 576 of referenced literature⁷, equation 4.352(1)

$$\int_0^{\infty} x^{v-1} e^{-\mu x} \ln x dx = \frac{1}{\mu^v} \Gamma(v) [\psi(v) - \ln(\mu)] \quad u > 0, v > 0$$

let

$$v = 2 ,$$

$$\mu = \lambda ,$$

$$x = \omega ,$$

then

$$\begin{aligned} I &= \frac{\lambda}{\alpha} \int_0^{\infty} \omega \ln \omega e^{-\lambda \omega} d\omega = \left(\frac{\lambda}{\alpha}\right) \left(\frac{1}{\lambda^2}\right) \Gamma(2) [\psi(2) - \ln \lambda] \\ &= \frac{1}{\alpha \lambda} [\psi(2) + \ln \frac{1}{\lambda}] . \end{aligned}$$

Equation (24) can now be written

$$C_{12} = \frac{N}{\lambda \alpha} [\psi(2) + \ln \frac{1}{\lambda}] , \quad (25)$$

and it follows that

$$C_{22} = \frac{N}{\lambda^2} . \quad (26)$$

As just shown, the $C_{i,j}$ matrix is the inverse asymptotic variance-covariance matrix of $(\hat{\lambda}, \hat{\alpha})$ and is obtained by taking the negatives of the expected values of the second order derivatives of logarithms of the likelihood functions. Using (18), we can now obtain the appropriate confidence limits for the true probabilities. In the case of reliability

⁷Loc. Cit.

or durability, we want to insure that the true reliability or durability is above some minimum value R_m or D_m at a desired confidence level. From (18), then,

$$C_{11}(\hat{\alpha}-\alpha)^2 + 2C_{12}(\hat{\lambda}-\lambda)(\hat{\alpha}-\alpha) + C_{22}(\hat{\lambda}-\lambda)^2 \approx \chi^2(2) . \quad (27)$$

The above equation describes the boundary of a confidence region in the parametric space (α, λ) . It is approximately chi-squared distributed with two degrees of freedom and will serve to determine an ellipsoidal confidence region in the (α, λ) space. In order to obtain the upper confidence limit, all that is required is to search the parameter space, (27), varying α and λ until the durability function (17) is maximized.

A PROCEDURE FOR COMPUTING EXPECTED NUMBER OF RENEWALS

Let $X_1, X_2, X_3, \dots$ be a renewal process, that is, a sequence of independent, nonnegative and identically distributed random variables which are not all zero with probability one, with the probability density function $f(X)$ and the distribution function $F(X)$. $S_k = X_1 + X_2 + \dots + X_k$ is interpreted in renewal theory as the time up to the k th renewal and the probability that $S_k < x$ is given by the k -fold convolution of $F(x)$

$$F_K(x) = \int_0^x F_{K-1}(x-t) dF(t),$$

where

$$F_0(x) = 1.$$

The primary purpose of this procedure is to calculate N_t which is defined to be the maximum suffix K for which $S_{\overleftarrow{k}} \leq t$, subject to the

convention $N_t = 0$, if $X_1 > t$. In this application, the X_1 represents successive lifetimes of the object being renewed, and N_t is the number of renewals made by time t , subject to the original object having been installed at time 0, i.e., N_t is the number of renewals in $(0, t)$.

The usual procedure for determining various functions of Renewal Theory is to find the Laplace Transforms of these functions and then revert to the time domain. However, in the case of the Weibull distribution, this approach is not convenient and Smith and Leadbetter⁹ have developed an expansion of the Renewal Function into a power series of t^α where α is the Weibull shape parameter. White¹⁰ demonstrated a procedure for evaluating the higher moments and cumulants of the number of renewals N_t and computed the mean $M(t)$ and the standard deviation, $(\text{Var } N_t)^{1/2}$, of N_t . Lomnicki² developed a method to evaluate the distribution of the number of renewals which is defined by the family of functions $W_k(t)$ where

$$W_k(t) = F_k(t) - F_{k+1}(t) \quad (k=0,1,\dots)$$

and is the probability of exactly k renewals in $(0, t)$.

In order to be able to evaluate more of the various forms of the renewal functions in the case of the Weibull renewal process, the basic method developed by Lomnicki was used.

As shown by Lomnicki, if we assume that $W_k(t)$ is represented by the unique series

$$W_k(t) = \sum_{s=k}^{\infty} a_k(s) P_s(t^\alpha)$$

⁹Smith, W. L., and Leadbetter, M. R., On the Renewal Function for the Weibull Distribution. Technometrics 5(3), 393-396 (1963).

¹⁰White, J. S., Weibull Renewal Analysis. Third Annual Aerospace Reliability and Maintainability Conference. pp 639-657. Society of Automotive Engineers, Inc. New York, New York. 1964.

²Loc. Cit.

where

$$P_k(t) = e^{-t} \frac{t^k}{k!} \quad (k=0,1,2,\dots) ,$$

then

$$W_k(t) = \sum_{s=k}^{\infty} P_s(t^a) \sum_{p=k}^s (-1)^{p+k} \binom{s}{p} \frac{b_k(p)}{\gamma(p)} ,$$

and we have

$$a_k(s) = \sum_{p=k}^s (-1)^{p+k} \binom{s}{p} \frac{b_k(p)}{\gamma(p)} \quad (k=0,1,\dots; s=k,k+1,\dots)$$

where

$$\gamma(r) = \Gamma(ar+1)/\Gamma(r+1) \quad (r=0,1,\dots)$$

and

$$b_0(s) = \gamma(s). \quad (s=0,1,\dots)$$

$$b(s) = \sum_{r=k}^{s-1} b_k(r) \gamma(s-r) \quad (k=0,1,\dots; s=k,k+1,k+2,\dots) .$$

The expected number of renewals $M(t) = E[N(t)]$ which is the renewal function is given by

$$M(t) = \sum_{n=0}^{\infty} n P[N(t)=n]$$

$$M(t) = \sum_{n=0}^{\infty} n W_n(t) .$$

Since

$$W_1(t) + 2W_2(t) + 3W_3(t) + \dots$$

is equal to

$$F_1(t) - F_2(t) + 2F_2(t) - 2F_3(t) + 3F_3(t) - 3F_4(t) + \dots ,$$

then

$$\sum_{n=0}^{\infty} n W_n(t) = \sum_{n=0}^{\infty} F_n(t)$$

and

$$M(t) = \sum_{n=0}^{\infty} F_n(t),$$

where $F_n(t)$ is the convolution of the cumulative distribution function $F(t)$ of the first renewal time and the $n-1$ subsequent renewal-time distributions and is the distribution of S_n , the time of the n th renewal.

Now,

$$F_k(t) = W_k(t) + F_{k+1}(t)$$

$$= \dots + W_{k+1}(t) + F_{k+3}(t)$$

$$F_k(t) = \sum_{r=k}^{\infty} W_r(t) \quad (k=1,2,3\dots)$$

or

$$F_k(t) = 1 - \sum_{r=0}^{k-1} W_r(t),$$

so we can now express $F_k(t)$ as

$$F_k(t) = \sum_{r=k}^{\infty} \sum_{s=r}^{\infty} a_r(s) P_s(t^\alpha):$$

Substituting the Poisson cumulative function

$$D_s(t) = \sum_{r=s}^{\infty} e^{-t} \frac{t^r}{r!}$$

means that

$$F_k(t) = \sum_{s=k}^{\infty} \beta_k(s) D_s(t^\alpha)$$

where

$$\beta_k(k) = a_k(k) ,$$

and

$$\beta_k(s) = \sum_{r=k}^s a_r(s) - \sum_{r=k}^{s-1} a_r(s-1) \quad (s > k) ,$$

so that

$$M(t) = \sum_{k=1}^{\infty} F_k(t) = \sum_{s=1}^{\infty} D_s(t^\alpha) \sum_{k=1}^s \beta_k(s) ,$$

or

$$M(t) = \sum_{k=1}^{\infty} C(s) D_s(t^\alpha) ,$$

where

$$C(s) = \sum_{k=1}^s \beta_k(s) \quad . \quad (s=1,2,\dots) .$$

We can now evaluate $M_k(t)$, the Renewal Function, $W_k(t)$; the probability of exactly k renewals; and $F_k(t)$, the distribution of time of the k th renewal. These functions are very useful in evaluating models of reliability, inventory, and queueing process.

$$M_k(t) = E\left(\binom{N(t)}{k}\right) = \sum_{n=0}^{\infty} \binom{n}{k} (F_n(t) - F_{n+1}(t)) .$$

The moments of $N(t)$ may be derived from $M_k(t)$ by the relationship

$$E(N(t)^k) = \sum_{n=1}^k t_{k,n} M_n(t) n!$$

where $t_{k,n}$ are the Stirling numbers of the second kind and represent the number of ways of partitioning a set of k elements into n nonempty subsets.

In particular

$$E\{N(t)\} = M_1(t) ,$$

and

$$E\{N(t)^2\} = M_1(t) + 2M_2(t) .$$

Or, if desired, the moments of $N(t)$ may be derived from the definition of expected values.

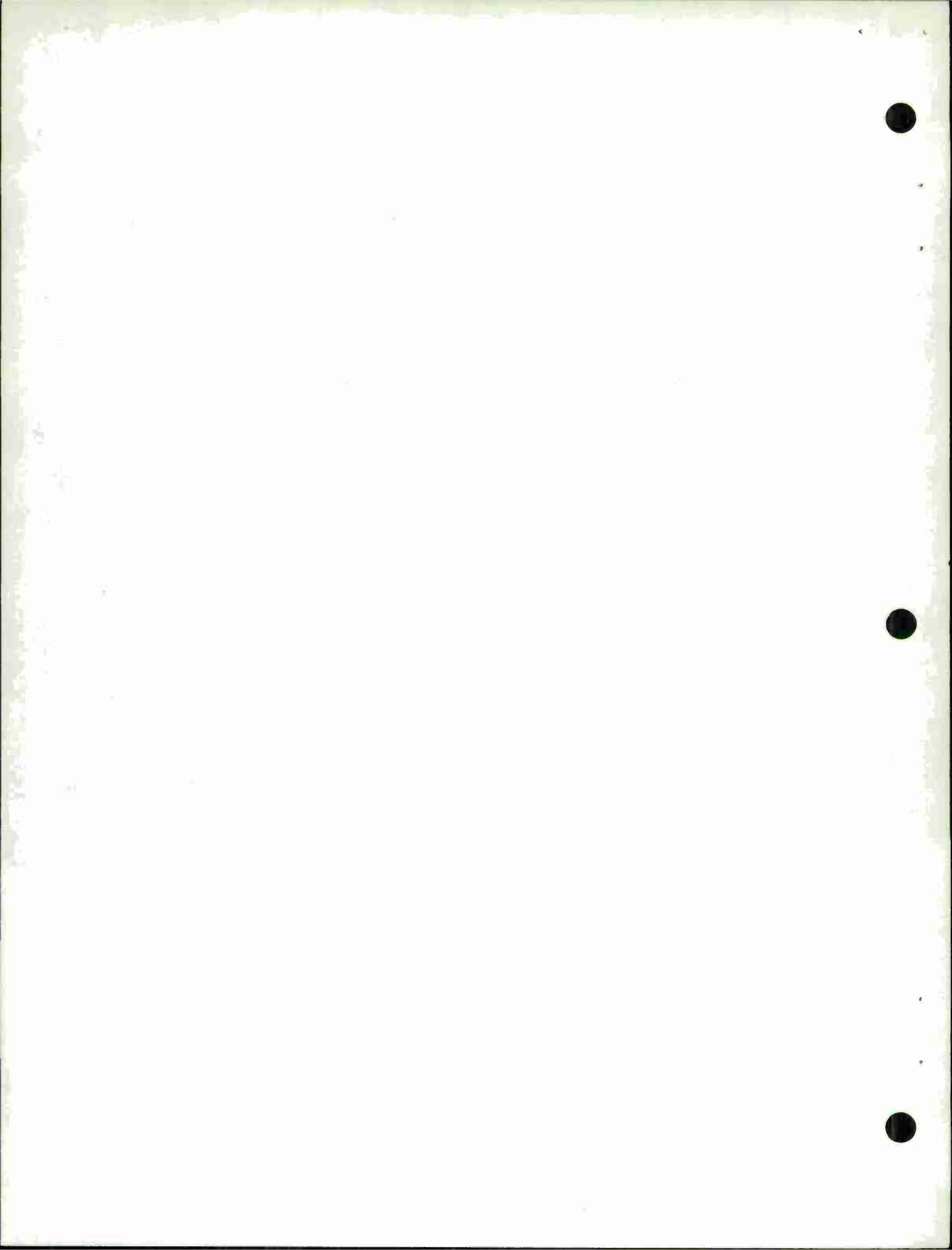
$$E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K \text{Prob}(N(t)=n) ,$$

$$E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K \{F_n(t) - F_{n+1}(t)\} ,$$

$$E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K W_n(t) ,$$

$$E\{N(t)^K\} = \sum_{n=1}^{\infty} n^{K-(n-1)} F_n(t) .$$

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