PAPER P-1080

A GENERALIZED STOCHASTIC LANCHESTER ATTRITION PROCESS

Alan F. Karr

September 1975

INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION

IDA Log No. HQ 74-16780
Copy 63 of 125 copies

The work reported in the publication was conducted under IDA's Independent Research Program. Its publication does not imply endorsement by the Department of Defense or any other government agency, nor should the contents be construed as reflecting the official position of any government agency.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER P-1080	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Generalized Stochastic Lanchester Attrition Process		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Alan F. Karr		8. CONTRACT OR GRANT NUMBER(s) IDA Independent Research Program
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Defense Analyses 400 Army-Navy Drive Arlington, Virginia 22202		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE July 1975
		13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document is unclassified and suitable for public release.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

stochastic attrition process, Lanchester attrition process, Lanchester square combat, combat simulation model

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A mixed-mode, heterogeneous, stochastic Lanchester attrition process is described, which is based on a four-category classification of weapons. Comments are made concerning the distinction between square-law and linear-law combat. Potential applicability of the model in combat simulations is also discussed.

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

PAPER P-1080

A GENERALIZED STOCHASTIC LANCHESTER ATTRITION PROCESS

Alan F. Karr

September 1975

INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION
400 Army-Navy Drive, Arlington, Virginia 22202

IDA Independent Research Program

CONTENTS

TABLE

PREFACE

In this paper we present a stochastic Lanchester-type mathematical model of the attrition suffered in a combat between two heterogeneous forces. The model is based on a four-category classification of weapons originally noted in Reference [5, pp. 5-6, p. 17] and includes as special cases all the attrition models treated there. We also give comments concerning the possible application of the model in computerized combat simulations such as IDAGAM I (Reference [1]).

The author is indebted to Drs. Lowell Bruce Anderson, Jerome Bracken, and Jerry Blankenship of IDA for many helpful discussions during the preparation of this paper. Their suggestions have contributed substantially to its content.

Chapter I

SQUARE-LAW AND LINEAR-LAW COMBAT: THE DISTINCTION

The analysis leading to the generalized stochastic Lanchester attrition process described below is relevant to the previously rather obscure distinction between Lanchester square-law and Lanchester linear-law combat, to which we now address ourselves. Lanchester (Reference [6]) himself sought to distinguish the two by whether the numerically superior side is able to make full use of its superiority (square law) or not (linear law). This is one interpretation of the distinction as we have come to understand it, at least in the case of homogeneous forces. For heterogeneous forces, however, the problem is considerably more subtle; we feel that the explanation to be presented below is the most cogent yet constructed.

The first distinction among weapons is based on the qualitative nature of the rates of engagement initiation (i.e., rates at which shots are fired). A particular type of weapon is said to have *square-law engagement initiation* if the mean total rate at which it engages (i.e., fires upon) opposition weapons is independent of the numerical size of the opposition force (as well as of its precise structure). In other words, the mean engagement rate depends only on the type of weapon. Physical assumptions compatible with this behavior are discussed in Reference [5, pp. 29-40]. The possibility that the rates at which particular types of opposition weapons are engaged may depend on the numbers of various kinds of opposition weapons that are present is not inconsistent with our earlier statement, which concerned only the mean total rate of engagement.

On the other hand, a type of weapon is said to possess *linear-law engagement initiation* if the mean total rate at which it engages elements of a target force $y = (y_1, \ldots, y_n)$--assuming that the opposition has n different types of weapons--is of the form $\sum_{j=1}^{n} \alpha_j y_j$, where the α_j are nonnegative constants. A word of warning is in order here concerning our usage of the term "mean rate." We intend that it be interpreted in the sense of the infinitesimal generator of a continuous time Markov process (i.e., the instantaneous rate at which the event in question tends to occur, given the current state of the process). We refer the reader to Reference [5] for further details. A physical interpretation of linear-law engagement initiation can be found in Reference [5, pp. 41-48, 101-122]. In particular, the rate at which opposition weapons of any given type are engaged is directly proportional to the number of weapons of that type currently surviving.

It is also true, however, that for square-law engagement the rate of engagement *of a particular type* j of opposition weapon is proportional to the number of weapons of that type present. But in this case the constant of proportionality is of the form β/y^*, where y^* is the total number of opposition weapons present, whereas in the linear-law case the proportionality factor is simply a constant α_j. Note that β is independent of the type j of the target weapon, although both β and the α_j, of course, depend on the type of the engaging weapon. Table 1 clarifies this rather important distinction: y_j is the number of opposition type-j weapons present; y^* is the total number of opposition weapons present.

Table 1. ENGAGEMENT RATES FOR LANCHESTER PROCESSES

Engagement Initiation	Type-j Weapons	All Opposition Weapons
Linear-Law	$\alpha_j y_j$	$\sum_{j=1}^{n} \alpha_j y_j$
Square-Law	$\beta \frac{y_j}{y^*}$	β

From this analysis, the first--and possibly most important--conclusion to be drawn is that the square-law/linear-law distinction *applies not to the combat as a whole but to individual types of weapons*. In other words, while a particular type of weapon may be said to have square- or linear-law engagement initiation, the combat itself cannot be said to possess either property. Even if all weapons present belong to one of the two classes (in this case, one would have essentially Process S3 or Process L3--see Reference [5]), it is still inaccurate to say that the combat itself is of that type. The distinction simply is not of that nature; it applies only to individual types of weapons.

It then becomes quite important to determine whether a particular type of weapon possesses square- or linear-law engagement initiation. We believe that Lanchester's original differentiation is fairly close to the truth. A weapon type has square-law engagement initiation if all weapons of that type are able simultaneously to bring their fire to bear on the opposition. Two ways in which this simultaneity can be envisioned are that (1) shots are fired, at a rate determined only by the shooting weapon, at an area in which the opposition is known (or thought) to be located; or (2) weapons are mobile and push forward in such a way as to maintain a rate of contacts with enemy weapons that is independent of the number of enemy weapons present. In Reference [5], the former interpretation is adopted; the latter is due to L. B. Anderson.

On the other hand, linear-law engagement initiation arises when weapons of the type under consideration must engage opposition weapons essentially on a one-on-one basis. Probably the easiest way to envision this form of engagement is that of Reference [5]: each given opposition weapon requires an exponentially distributed random time to detect, different opposition weapons are detected independently, and an engagement occurs if and only if an opposition weapon is detected. (See Process L3 in Reference [5, pp. 47-48, 115-122] for details and further interpretations; Processes L1 and L2 of Reference [5] are also relevant.)

The problem of deciding whether a particular type of weapon has square- or linear-law engagement initiation seems to us, in terms of attrition modeling, crucial and difficult. The implications in terms of computed levels of attrition and FEBA movement are likely to be substantial--as is confirmed by experiments with simplified homogeneous models (essentially discretized versions of Lanchester's original differential equations). Hence this classification should not be undertaken lightly or carelessly, as it may have overwhelming effects on the outputs of combat simulation models. For certain types of weapons--e.g., artillery (square law) or small arms (linear law)--the choice seems fairly clear. But for some other types (e.g., tanks), the choice is not at all clear. For example, it appears that to which category a tank belongs may be the result of tactical decisions by the two sides, may change during the course of a battle, and is more properly an output of the attrition model than a prescribed input. We cannot refute these criticisms except by noting that no attrition model yet devised correctly addresses any of the difficulties raised. The possibility that engagement initiation is in its qualitative nature the result of tactical decisions is particularly interesting, however; the weaker side would seek to create linear-law engagement initiation, and the stronger side would desire square-law initiation.

We hope that these remarks have clarified the square-law/linear-law distinction and that they will serve as a stimulus to further research in this important area.

Chapter II

THE MATHEMATICAL MODEL

A unique characteristic of the generalized Lanchester attrition model presented here is a qualitative classification of weapons into four categories, based on two distinctions. The first of these characterizes weapons as to the qualitative nature of the rate at which they initiate engagements, and has been discussed at length above.

The second classification characterizes weapons as *single-kill* or *multiple-kill*, depending on whether one shot can kill either at most one or possibly more than one opposition weapon. An artillery piece is an example of a multiple-kill weapon, while antitank weapon exemplifies the notion of a single-kill weapon. It seems that whether a weapon be single-kill or multiple-kill should depend on the type of target (an artillery piece fired at infantrymen is clearly multiple-kill, but single-kill when fired at tanks); the mathematical structure we describe here is, in fact, sufficiently general to represent this phenomenon--but only with somewhat of a loss of clarity. (For further details, we refer the reader to comments in Chapter III.)

According to our scheme of classification, there are thus four qualitative classes of weapons:

- SS - Weapons with Square-law engagement initiation and Single kill per shot.
- LS - Weapons with Linear-law engagement initiation and Single kill per shot.
- SM - Weapons with Square-law engagement initiation and Multiple kills per shot.

- LM - Weapons with Linear-law engagement initiation and Multiple kills per shot.

The classical term "area fire" seems best represented by SM weapons; small arms point fire involves weapons of class LS. The interested reader can devise his own interpretations for all four categories.

For each class of weapon, certain input parameters will be required for the attrition process to be discussed below; our new process generalizes the processes presented in Reference [5], from which the qualitative nature of the input parameters is determined. In particular, SS weapons behave as described in the Processes S3 and S3a of Reference [5], LS weapons as in Process L3, SM weapons as in Process A1 (suitably generalized), and LM weapons as in the linear-law analogue of Process A1 (which does not appear in Reference [5]). For the statement and interpretations of these specialized attrition processes, the reader is referred to Reference [5].

Here are the assumptions and notations for the new, generalized Lanchester attrition process. The combat is bilateral, involving two heterogeneous forces (Red and Blue); weapons are not distinguished except by the qualitative classes given above and, within each class, only by numerical values of parameters.

Assumptions

(1) The Blue force consists of M_1 types of weapons of the class SS, M_2 weapon types of class LS, M_3 weapon types of the class SM, and M_4 weapon types of the class LM. Hence, Blue has altogether $M = M_1 + M_2 + M_3 + M_4$ *types* of weapons. The analogous numbers for the Red side are N_1, N_2, N_3, N_4, and $N = N_1 + N_2 + N_3 + N_4$.

The notation used below is of the following form: The letter "i" is used as a generic index for Blue weapon types.

Types $1,\ldots,M_1$ are the SS weapons; $M_1+1,\ldots,M_1+M_2$ are LS weapons; weapon types $M_1+M_2+1,\ldots,M_1+M_2+M_3$ are of class SM; and the remaining types belong to the class LM. Certain parameters below are defined only for Blue weapons belonging only to one class (e.g., if that class is SM, the relevant index i ranges over $1,\ldots,M_3$; when all Blue weapon types are considered, the index i ranges over $1,\ldots,$ M. For Red, a similar situation holds for the index j).

Blue forces are denoted by vectors $x \in \underset{\sim}{N}^M$, where $\underset{\sim}{N} = \{0,1,2,\ldots\}$; x_i is the number of type-i weapons ($i=1,\ldots,M$) currently present. (Red forces are analogously denoted by vectors $y \in \underset{\sim}{N}^N$.)

(2) Times between engagements initiated by a surviving Blue SS type-i weapon are independent and identically exponentially distributed with mean $1/r_B(i)$, $i=1,\ldots,M_1$.

(3) When a Blue SS type-i weapon initiates an engagement, it attacks exactly one Red weapon (chosen from *all* Red weapons currently surviving, according to a uniform distribution) independently of the past history of the process.

(4) The conditional probability that a Blue SS type-i weapon kills a Red type-j weapon, given an attack on that weapon, is $p_B(i,j)$ for $i=1,\ldots,M_1$ and $j=1,\ldots,N$.

(5) Assumptions (2), (3), and (4) hold for Red SS weapons with parameters $r_B(j)$, $j=1,\ldots,N_1$ and $p_R(j,i)$, $j=1,\ldots,N_1$; $i=1,\ldots,M$.

(6) The time required for a particular surviving Blue LS type-i weapon to detect a particular surviving Red type-j weapon is exponentially distributed with mean $1/d_B(i,j)$ for $i=1,\ldots,M_2$ and $j=1,\ldots,N$. A Blue LS weapon detects different Red weapons independently of one another and can engage an opposition weapon only after detecting it.

(7) A Blue LS weapon engages every Red weapon it detects. The conditional probability that a Blue LS type-i weapon kills a Red type-j weapon, given detection and attack, is $k_B(i,j)$ for $i=1,\ldots,M_2$; $j=1,\ldots,N$.

(8) Red LS weapons satisfy Assumptions (6) and (7) with mean detection times $1/d_R(j,i)$ and kill probabilities $k_R(j,i)$, defined for $j=1,\ldots,N_2$ and $i=1,\ldots,M$.

(9) Times between engagements initiated by a surviving Blue SM type-i weapon ($i=1,\ldots,M_3$) are independent and identically exponentially distributed with mean $1/r_B^*(i)$.

(10) Given that a Blue SM type-i weapon initiates an engagement against a currently surviving Red force y (recall the notation introduced in Assumption (1), above) the probability that the surviving target force has the composition z is $\varphi_B(i,y;z)$. Symbolically,

$$\varphi_B(i,y;z) = P\{\text{surviving Red force is } z \mid \text{engagement initiated by Blue SM type-i weapon against Red force with composition } y\}.$$

Here, $y,z \in \underset{\sim}{N}^M$.

(11) Red SM weapons obey Assumptions (9) and (10), with mean engagement rates $r_R^*(j)$, $j=1,\ldots,N_3$ and kill distributions $\varphi_R(j,x;w)$ defined for $j=1,\ldots,N_3$; $x,w \in \underset{\sim}{N}^M$.

(12) The time required for a particular surviving Blue LM type-i weapon ($i=1,\ldots,M_4$) to detect a particular surviving Red type-j weapon ($j=1,\ldots,N$) is exponentially distributed with mean $1/d_B^*(i,j)$. A Blue LM weapon detects different Red weapons independently of one another.

(13) At the instant of each detection, a Blue LM weapon initiates an engagement.

(14) Each engagement is independent of the previous history of the attrition process. If a Blue LM type-i weapon engages a Red force of composition y after having detected a Red type-j weapon ($j=1,\ldots,N$), the probability that z is the Red force surviving the engagement is denoted by $\mu_B(i,j,y;z)$.

(15) Red weapons of the class (not type!) LM satisfy Assumptions (12), (13), and (14) with parameters $d_R^*(j,i)$ and $\mu_R(j,i,x;w)$ defined for $j=1,\ldots,N_4$, $i=1,\ldots,M$, and $x,w \in \underset{\sim}{N}^M$.

(16) The detection, engagement, and kill processes of all weapons are mutually independent.

Remarks on the Assumptions

The spirit and meaning of the assumptions is that of Reference [5]. In particular, the terminology used--though suggestive and frequently (we feel) the most plausible interpretation--need not be adhered to exactly. Especially, the apparent dichotomy between square- and linear-law engagement initiations--on the basis of no detections or individual detection--can be interpreted differently. Further comments can be found in Reference [5, p.39]; Chapter I is also relevant in this context.

The derivation of the kill distributions φ_B, φ_R, μ_B, and μ_R is a problem that we have not yet considered in any depth. In any application of our model to computerized simulations, this would be the problem most in need of attention. Here we have indicated abstractly what functional dependences seem plausible and, hence, those that we feel can safely be ignored. For example, the kill distributions do not depend on the structure of the force to which a shooting weapon belongs, though in principle they could. Indeed, a reasonable way of describing such dependence would allow representation of the synergistic effects of weapons on the same side. Similarly, the kill distributions μ_B and μ_R for LM weapons indicate that the distribution of weapons killed can depend on the particular (type of) weapon first detected; differing ammunition or tactics used against different detected weapons can thus be modeled. (Further comments appear in Chapter III, below, where we also give some examples.)

To avoid unnecessary proliferation of notation, certain parameters are denoted by an asterisk for multiple-kill weapons and no asterisk for single-kill weapons.

All "engagements" occur instantaneously, with ensuing total loss of contact. This is admittedly an unrealistic feature of the model (though no other theater-level models seem to have successfully addressed the difficulty either). One way of including binary engagements with exponentially distributed lengths is discussed in Process L2 (of Reference [5]).

Results

We can now describe and characterize the stochastic attrition process engendered by Assumptions (1)-(16) above. Let $E = \underset{\sim}{N}^{M+N}$ consist of states denoted by $\alpha = (x,y)$; α is to be thought of as a possible pair of surviving forces at some instant--with x corresponding to the Blue force and y the Red. As a measurable space, E is assumed to be endowed with the σ-algebra of all its subsets. The sample space for our attrition process is the family Ω of functions from $\underset{\sim}{R}_+$ to E, which are right-continuous and have limits from the left with respect to the discrete topology on E. The coordinate (vector-valued) stochastic process $\big((B_t,R_t)\big)_{t\geq 0}$ (here $B_t:\Omega \to \underset{\sim}{N}^M$ and $R_t:\Omega \to \underset{\sim}{N}^N$ for each t) has the usual interpretation: B_t is the surviving Blue force at time t; R_t, the Red force at time t. We further define

$$\underline{\underline{F}}_t = \sigma\big((B_u,R_u);\ 0 \leq u \leq t\big) ,$$

which is the history of the attrition process until time t, and

$$\underline{\underline{F}} = \sigma\big((B_u,R_u);\ u \geq 0\big) ,$$

which is the entire history of the process. For each $\alpha \in E$, we denote by P^α the probability law on $(\Omega,\underline{\underline{F}})$ of the attrition process governed by Assumptions (1)-(16), subject to the initial condition

$$P^\alpha\{(B_0,R_0)=\alpha\} = 1 .$$

Our notation and terminology concerning regular Markov processes are those of References [2], [3], and [4] (to all of which the interested reader is referred for background material; also, Reference [5] contains a rather lengthy discussion of the role of Markov processes in attrition models of Lanchester type.)

Here is our main result:

THEOREM. Subject to Assumptions (1)-(16) listed above, the stochastic process

$$(\underset{\sim}{B},\underset{\sim}{R}) = \left(\Omega,\underset{=}{F},\underset{=}{F}_t,\ (B_t,R_t),P^\alpha\right) \tag{1}$$

is a regular Markov process with state space E. The infinitesimal generator A of the process is of the following form: consider two states $\alpha = (x,y)$ and $\alpha' = (x',y')$ such that $y' = y$, $x_i' = x_i - 1$ for some i and $x_k' = x_k$ for all $k \neq i$ (the new state α' is reached from α by the destruction of *exactly one* Blue type-i weapon). Then

$$\begin{aligned} A(\alpha,\alpha') = {} & \frac{x_i}{\sum_{k=1}^{M} x_k} \sum_{j=1}^{N_1} r_R(j)p_R(j,i)y_j \\ & + x_i \sum_{j=1}^{N_2} d_R(j,i)k_R(j,i)y_{N_1+j} \\ & + \sum_{j=1}^{N_3} r_R^*(j)\varphi_R(j,x;x')y_{N_1+N_2+j} \\ & + \sum_{k=1}^{M} x_k \sum_{j=1}^{N_4} d_R^*(j,k)\mu_R(j,k,x;x')y_{N_1+N_2+N_3+j} \ . \end{aligned} \tag{2}$$

For any other state $\alpha' = (x',y')$ with $y' = y$, $x_i' \leq x_i$ for all i, and $\sum_i (x_i - x_i') \geq 2$ (this corresponds to the simultaneous destruction of more than one Blue weapon and can be effected only by Red weapons of classes SM and LM), we have

$$A(\alpha\ \alpha') = \sum_{j=1}^{N_3} r_R^*(j)\varphi_R(j,x;x')y_{N_1+N_2+j}$$

$$+ \sum_{k=1}^{M} x_k \sum_{j=1}^{N_4} d_R^*(j,k)\mu_R(j,k,x;x')y_{N_1+N_2+N_3+j} \ . \qquad (3)$$

Similarly, for $\alpha' = (x',y')$ such that $x' = x$, $y_j' = y_j - 1$ and $y_\ell' = y_\ell$ for $\ell \neq j$,

$$A(\alpha,\alpha') = \frac{y_j}{\sum_{\ell=1}^{N} y_\ell} \sum_{i=1}^{M_1} r_B(i)p_B(i,j)x_i$$

$$+ y_j \sum_{i=1}^{M_2} d_B(i,j)k_B(i,j)x_{M_1+i}$$

$$+ \sum_{i=1}^{M_3} r_B^*(i)\varphi_B(i,y;y')x_{M_1+M_2+i}$$

$$+ \sum_{\ell=1}^{N} y_\ell \sum_{i=1}^{M_4} d_B^*(i,\ell)\mu_B(i,\ell,y;y')x_{M_1+M_2+M_3+i} \ ,$$

while for any other state $\alpha' = (x',y')$ such that $x' = x, y_j' \leq y_j$ for all j and $\sum_j (y_j - y_j') \geq 2$, we have

$$A(\alpha,\alpha') = \sum_{i=1}^{M_3} r_B^*(i)\varphi_B(i,y;y')x_{M_1+M_2+i}$$

$$+ \sum_{\ell=1}^{N} y_\ell \sum_{i=1}^{M_4} d_B^*(i,\ell)\mu_B(i,\ell,y;y')x_{M_1+M_2+M_3+i} \ .$$

Moreover, for all $\alpha' \neq \alpha$ and not of the forms above, we have

$$A(\alpha,\alpha') = 0$$

and, finally,

$$A(\alpha,\alpha) = - \sum_{\alpha' \neq \alpha} A(\alpha,\alpha') \ . \qquad \square$$

We omit the proof of the Theorem, which one constructs by using the results and by the methods of the appendix of Reference [5]. The expressions for the generator A have probabilistic interpretations and are written in what we feel is the most revealing form. Consider, for example, the term $A(\alpha,\alpha')$ of Equation (2). Here the first summand (of four) on the right is the (instantaneous) rate at which Blue type-i weapons are being killed by Red SS weapons when the two forces have compositions x and y, respectively, and the second term is a similar rate arising from Red LS weapons. For kills caused by single-shot weapons, "rate of kill" and "rate of kill one at a time" are the same notion. This is not so for multiple-shot weapons. Hence, the third summand on the right-hand side of Equation (2) must be interpreted as the instantaneous rate at which Blue type-i weapons are being killed *exactly one at a time* by Red SM weapons. The interpretation of the fourth summand is then analogous; note that the kill of a type-i weapon can, in principle, arise from the detection of any type of weapon. Hence $A(\alpha,\alpha')$ is--when the structure of the two forces is $\alpha = (x,y)$--the instantaneous rate at which Blue type-i weapons are being destroyed precisely one at a time. We emphasize that single kills can, in general, arise from multiple-shot weapons as indicated by the presence of the third and fourth summands in Equation (2).

For the term of A given by Equation (3), only multiple-kill weapons need be considered, and interpretations are similar to those given above.

We next list, for purposes of reference and completeness, some consequences of the Theorem.

COROLLARY. The jump function λ of the process $(\underset{\sim}{B},\underset{\sim}{R})$ is given by

$$\lambda(\alpha) = - A(\alpha,\alpha) , \tag{4a}$$

and the transition matrix P of the imbedded Markov chain is given by

$$P(\alpha,\alpha') = \frac{A(\alpha,\alpha')}{\lambda(\alpha)} \tag{4b}$$

for $\alpha' \neq \alpha$, and

$$P(\alpha,\alpha) = 0 \; . \tag{4c}$$

[For the sake of simplicity, we do not write these expressions in full; they may be so written, from the Theorem, in a straightforward manner.]

COROLLARIES. (a) For the Markov process $(\tilde{B},\tilde{R})$, all states of the form $(x,0)$ with $x \in \tilde{N}^M$ or of the form $(0,y)$ with $y \in \tilde{N}^N$ are absorbing; all other states are stable and transient;

(b) if $\alpha \neq (0,0)$, then

$$P^{\alpha}\{(B_t,R_t) = (0,0) \text{ for some } t\} = 0 \; ; \tag{5}$$

(c) for any α, with P^{α} - probability 1 each component of the sample paths $t \to B_t$ and $t \to R_t$ is nonincreasing. (There is no provision for reinforcements; one indication of how this problem might be handled is given in the appendix of Reference [5, pp. 73-79]).

Chapter III

POTENTIAL APPLICATIONS

We discuss in this chapter the potential applications of the attrition model derived in Chapter II as the assessment mechanism in a theater-level computerized combat simulation such as IDAGAM I (Reference [1]). There are three principal problems to be considered: implementation of the attrition model on a computer, derivation of the qualitative form of the kill distributions φ_B, φ_R, μ_B, and μ_R for multiple-shot weapons, and selection of the exact values of input parameters. We shall discuss mostly the first two problems--with only brief consideration of the third problem and of other, minor problems.

Within an iterative computerized simulation like IDAGAM I, a most difficult problem is the proper handling of expected values of random variables. At present, there is no model that justifiably does so in this context. Consider the attrition model presented in Chapter II. When the initial forces are (the deterministic point) (x,y), what is the expected attrition to a given type of weapon in a unit interval of time? Denote by $(P_t)_{t>0}$ the transition function of the survivor process $(\underset{\sim}{B},\underset{\sim}{R})$; i.e.,

$$P_t(\alpha,\alpha') = P^{\alpha}\{(B_t,R_t) = \alpha'\}$$

for all $\alpha,\alpha' \in E$. The expected number of Blue type-i weapons surviving at the end of one time period is then

$$E^{(x,y)}[B_1(i)] = \sum_{k=1}^{x_i} k \, P^{(x,y)}\{B_1(i)=k\}$$

$$= \sum_{k=1}^{x_i} k \sum_{\alpha'=(x',y'):x_i'=k} P_1\big((x,y),\alpha'\big) \ ,$$

when initial conditions are (x,y). From this expression, the expected attrition is easily computed. Hence, proper expected attritions for deterministic initial conditions can be computed once the transition function is known--and, in fact, once the Markov matrix P_1 is known.

Computation of the transition function from the generator A, however, is not a simple matter in practice. The relevant expression is

$$P_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \ . \tag{6}$$

Uniformly good finite approximations may be possible, but they would entail enormous requirements for storage and computation time for forces of theater- (or even sector-) level magnitude. The only approximation involving no such difficulties is the first-order approximation given by

$$E^{(x,y)}[B_1(i)] \sim \sum_{k=1}^{x_i} k \sum_{(x',y'):x_i'=k} A\big((x,y);(x',y')\big) \ , \tag{7}$$

which is quite feasible for implementation. Note that, in addition, we have

$$P^{(x,y)}\{B_1(i) = k\} \sim \sum_{(x',y'):x_i'=k} A\big((x,y);(x',y')\big) \ , \tag{8}$$

which gives the approximate probability distribution of the number of surviving Blue type-i weapons. Joint distributions of number of surviving weapons may be similarly approximated.

IDAGAM I and similar models employ an iterative method of calculation for representing the evolution of time. In such a scheme, the outputs of the attrition calculation for one time period constitute a portion of the inputs for the computation corresponding to the next time period. In particular, if a deterministic equation of the form of Equation (7)--or even a correct evaluation of expected numbers of survivors--is used to compute numbers of survivors, then the inputs for the calculation of the second period are the expected number of survivors from the first; and, therefore, the result of that calculation will not, in general, be the expected number of survivors at the end of the second period. What has happened, of course, is that the random number of survivors at the end of the first period was replaced by its expected value for use as an initial condition for the second period. Within the current structure of IDAGAM, there is no way to circumvent this difficulty. However, simulation experiments with simple homogeneous models indicate that the errors are rather small provided that the probability of one side or the other is annihilated be sufficiently small.

Rather drastic alterations to IDAGAM might allow the program to carry from one period to the next the joint distribution of the numbers of surviving weapons, in which case a version of Equation (8) is the relevant approximation and no further approximations or simplifications are required.

In a somewhat different setting, the generator A contains sufficient data to perform Monte Carlo simulations designed to improve the approximations in Equations (7) and (8) without performing the computations required to obtain Equation (6) or one of the approximating partial sums. This possibility appears worthy of further investigation.

The storage and bookkeeping problems associated with implementation of Equation (7) as an attrition equation would be significant, though probably not overwhelming. Most of the difficulties would arise from the large size of the generator matrix A and the large number of kill distributions required to be stored for possible use in computations.

We discuss next the problem of derivation of the form of the kill distributions for multiple-shot weapons. First of all, each type of weapon must be placed in one of the four classes of weapons defined in Chapter II. Then, for each multiple-kill weapon, the form of the kill distributions must be determined. This is evidently an arduous and lengthy task, and there is no clear conception as to how one should proceed. We offer here only some plausible examples and interpretive comments:

(1) We first note how single-shot weapons can be looked upon as special cases of multiple-shot weapons. Consider a Blue SM type-i weapon and the associated kill distribution $\varphi_B(i,\cdot;\cdot)$. If for each y the probability measure $\varphi_B(i,y;\cdot)$ is of the form

$$\varphi_B(i,y;y^{(j)}) = \frac{y_j}{\sum_{\ell=1}^{N} y_\ell}\, p_B'(i,j) ,$$

$$y_k^{(j)} = \begin{cases} y_k & k \neq j \\ y_j - 1 & k = j \end{cases}$$

and $0 \leq p_B'(i,j) \leq 1$, then the weapon becomes a SS weapon satisfying Assumptions (2)-(4), provided that $\varphi_B(i,y;y') = 0$ for y' neither of the form above nor equal to y.

(2) More generally suppose that, whenever y' is such that $y_\ell' = y_\ell$ for all $\ell \neq j$, one has

$$\varphi_B(i,y;y') = \frac{y_j}{\sum_{\ell=1}^{N} y_\ell}\, \varphi_B'(i,j,y_j;y_j') ,$$

where $\varphi_B'(i,j,y_j;y_j')$ is the probability that, if the weapon in question engages a type-j target when there are y_j such targets present, then y_j' of them survive.

In this case, the target type is chosen by the uniform fire allocation of Assumption (3), but possibly (though not necessarily) more than one of the class of target weapons can be destroyed. This structure admits, therefore, the situation in which a weapon is single-kill against some kinds of targets but multiple-kill against others--answering the potential criticism raised in the second paragraph of Chapter II.

(3) The binomial distributions of the Process A1 of Reference [5] could be used for the kill distributions to φ'_B of the previous example.

(4) For weapons of Class LM, it might be reasonable to allow only weapons of the same type as that detected to be attacked and killed.

Further work on deriving reasonably simple kill distributions based on rigorous but plausible hypotheses is an aspect of this model in great need of further research.

Gathering and interpretation of data could also prove troublesome. Firing rates, for example, must be averaged to account for periods of time in which no interaction occurs, which may possibly preclude use of existing data. This problem, however, is irrelevant to the internal consistency and plausibility of the attrition model. Statistical verification of the hypotheses of the model would be useful, but is almost surely impossible; statistical estimation of parameter values, however, can probably be carried out.

REFERENCES

[1] Anderson, L. B., J. Bracken, J. G. Healy, M. J. Hutzler, and E. P. Kerlin. *IDA Ground-Air Model I (IDAGAM I)*. IDA Report R-199. Arlington, Va.: Institute for Defense Analyses, May 1974.

[2] Blumenthal, R. M., and R. K. Getoor. *Markov Processes and Potential Theory*. New York: Academic Press, 1968.

[3] Çinlar, E. *Introduction to Stochastic Processes*. Englewood Cliffs, N.J.: Prentice-Hall, 1975.

[4] Freedman, D. *Markov Chains*. San Francisco, Calif.: Holden-Day, 1971.

[5] Karr, A. F. *Stochastic Attrition Models of Lanchester Type*. IDA Paper P-1030. Arlington, Va.: Institute for Defense Analyses, 1974.

[6] Lanchester, F. W. *Aircraft in Warfare: The Dawn of the Fourth Arm*. London: Constable and Co., 1916.