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ON SIMULATIONS OF THE STOCHASTIC, HOMOGENEOUS, LANCHESTER SQUARE-LAW ATTRITION PROCESS

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INSTITUTE FOR DEFENSE ANALYSES
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Results of computerized Monte Carlo simulations of a stochastic, homogeneous Lanchester square-law attrition process are described. Topics investigated include expected numbers of survivors at fixed times, iterated calculations of expected numbers of survivors, and probabilities of forcing the opposing side below a prescribed fraction of initial strength before being so reduced, especially in relation to

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PREFACE

The purpose of this paper is to describe the results of some rather extensive computer simulations of a stochastic attrition process which is analogous to the classical "square-law" Lanchester differential equations of combat. We shall consider various measurements of force equality and inequality, scaling and iteration properties of certain expectations, and several different rules for combat termination. Many of the results obtained are in some sense confirmations of heuristic, intuitive beliefs about the attrition process under study while others are, to some extent, rather surprising and counterintuitive.

Chapter I

THE MATHEMATICAL MODEL

The first quantitative description of combat appears to be due to F. W. Lanchester (Reference [6]), a British engineer who suggested that it is the nature of modern warfare that the instantaneous casualty rate on each side be proportional to the current numerical strength of the opposition, provided that the combat situation be such that the numerically superior side is able to bring its full superiority to bear on the opposition.

Analytically, the model proposed by Lanchester is

$$b'(t) = -c_1 r(t) \quad \text{and} \quad r'(t) = -c_2 b(t) , \qquad (1)$$

where $b(t)$ and $r(t)$ are the numbers of surviving Blue and Red combatants, respectively, at time t after the combat begins; and c_1 and c_2 are positive constants, which need not be equal. Each side is homogeneous; and, in order that Equations (1) make sense, the functions b and r must be allowed to assume arbitrary nonnegative values. The solution to these equations is given by

$$\begin{aligned} b(t) &= b(0) \cosh \lambda t - \alpha r(0) \sinh \lambda t \\ r(t) &= r(0) \cosh \lambda t - \alpha^{-1} b(0) \sinh \lambda t , \qquad (2) \end{aligned}$$

where $\lambda = (c_1 c_2)^{1/2}$ and $\alpha = (c_1/c_2)^{1/2}$. λ is a measure of the intensity of the engagement; α, of the relative effectiveness of individual combatants. Let us define

$$\tau = \inf\{t: b(t)=0 \text{ or } r(t)=0\} ;$$

as the time at which one side or the other is annihilated (the solutions to Equations (1) are of interest only until τ; for after τ one function or the other will begin to increase--which has no meaning in the physical situation under study). Then we have the following result:

PROPOSITION. We have

i) $\tau = \infty$ and $\lim_{t\to\infty} b(t) = \lim_{t\to\infty} r(t) = 0$

or

ii) $\tau < \infty$, $b(\tau) = 0$, and $r(\tau) > 0$

or

iii) $\tau < \infty$, $b(\tau) > 0$, and $r(\tau) = 0$

according as $c_1 r(0)^2 = c_2 b(0)^2$, $c_1 r(0)^2 > c_2 b(0)^2$ or $c_1 r(0)^2 < c_2 b(0)^2$, respectively.

Thus the quantities $c_1 r(0)^2$ and $c_2 b(0)^2$ are important measures of relative force strengths. We shall see below that the same is true (in a suitably generalized sense) for the stochastic attrition process analogous to Equations (1).

That process is defined and characterized in Reference [5]; for completeness, we include a brief description here. The assumptions underlying the stochastic homogeneous square-law Lanchester attrition process are the following:

Assumptions

(1) All weapons (i.e., combatants) on each side are identical.

(2) Times between engagements initiated by a surviving Blue weapon are independent and identically exponentially distributed, with expectation $1/e_b$.

(3) When such an engagement occurs, it leads to the destruction of exactly one Red weapon with probability p_b or destruction of no Red weapons (with probability $1 - p_b$), independent of the past history of the process. All engagements occur instantaneously.

(4) Red weapons satisfy Assumptions (2) and (3), with parameters e_r and p_r.

(5) The engagement and kill processes of all weapons are, conditioned on their survival, independent.

In Reference [5] it is shown that one can construct a probability space $(\Omega,\underline{\underline{M}},P)$ and on it a stochastic process $((B_t,R_t))_{t\geq 0}$, the interpretation of which is that B_t is the number of Blue weapons surviving at time t and R_t is the corresponding number of Red weapons, such that the following characterization holds.

THEOREM. Subject to Assumptions (1)-(5) above, the stochastic process $((B_t,T_t))_{t\geq 0}$ is a regular Markov process with state space $E = \underset{\sim}{N} \times \underset{\sim}{N}$ (where $\underset{\sim}{N} = \{0,1,2,...,\}$), jump function λ given by

$$\lambda(i,j) = ie_bp_b + je_rp_r \, , \tag{3}$$

transition kernel P given by

$$P\big((i,j);(i,j-1)\big) = \frac{ie_bp_b}{ie_bp_b + je_rp_r} \, ,$$

$$P\big((i,j);(i-1,j)\big) = \frac{je_rp_r}{ie_bp_b + je_rp_r} \, ;$$

and infinitesimal generator Q given by

$$\begin{aligned} Q\big((i,j);(i,j-1)\big) &= ie_bp_b \\ Q\big((i,j);(i,j)\big) &= -(ie_bp_b + je_rp_r) \\ Q\big((i,j);(i-1,j)\big) &= je_rp_r \, . \end{aligned} \tag{4}$$

These expressions apply only to states (i,j) such that $i > 0$ and $j > 0$. If $i = 0$ or $j = 0$, the state (i,j) is absorbing.

The reader is referred to References [2] and [3] for details concerning regular Markov processes; for purposes of computer simulation, we note the following characterization: When the process $((B_t,R_t))$ enters a state (i,j), it remains there an exponentially distributed length of time, which has expectation $\lambda(i,j)^{-1}$ and is independent of the past history of the process.

It then jumps to a new state according to the distribution $P((i,j);\cdot)$ independent of past history and of the length of the sojourn in (i,j). We denote by $P^{(i,j)}$ the probability law of the Markov process $((B_t,R_t))$ subject to the initial conditions

$$B_0 = i \quad \text{and} \quad R_0 = j \ ;$$

and, by $E^{(i,j)}[\cdot]$ the corresponding expectation.

The research reported here involved Monte Carlo simulations of the process described above; we now outline the procedures used:

- The program requires as inputs the initial numbers of weapons on each side, the engagement rates e_b, e_r, and kill probabilities p_b, p_r.
- The Monte Carlo simulations are performed by using only random numbers that are uniformly distributed on [0,1], on the basis of the independence observation above. The program simulates and records times of changes of state of the process and states entered. For example, suppose that the process enters state (i,j) at time t. Then the time of the next transition is

$$t' = t - \frac{\log x}{\lambda(i,j)} ,$$

  where x is a realization of a random variable that is uniformly distributed on [0,1]. Here we use the well-known fact that, if X is a random variable uniformly distributed on [0,1], then $(-\lambda^{-1} \log X)$ has an exponential distribution with expectation λ^{-1}. To determine the state to be entered at time t′, one takes a second realization x′ from the uniform distribution on [0,1]. The next state is (i,j-1) if $x' \leq P((i,j),(i,j-1))$ and (i-1,j) otherwise.
- Various criteria for termination of each realization are available--namely, annihilation of one side, termination at a fixed time, and termination when one side reaches a prescribed fraction of its original strength.
- Each simulation run produces 50 or 100 realizations of the stochastic process. Outputs available include expected numbers and fractions of survivors (and standard derivations thereof), the probability that each side wins, and the expected duration.

The computer programs are written in BASIC and run on a PDP-11 time-sharing computer.

Chapter II

EXPECTED NUMBERS OF SURVIVORS

To begin, we investigated expected numbers of survivors at ixed times, for different combinations of initial conditions. In the notation of Chapter I, we are here studying the functions

$$t \to E^{(i,j)}[B_t]$$

and

$$t \to E^{(i,j)}[R_t]$$

for various choices of $i=B_0$, $j=R_0$, e_b, e_r, p_b, and p_r. Some representative results appear in Table 1.

The main reason for using simulation to explore properties of these expectations is their analytical intractability. One can define the transition function (P_t) of the process--namely,

$$P_t\big((i,j);\alpha\big) = P^{(i,j)}\{(B_t,R_t)=\alpha\}, \qquad \alpha \in E ;$$

nd it is known that, for each t,

$$P_t = e^{tQ} , \tag{5}$$

iere Q is the infinitesimal generator of the process $\big((B_t,R_t)\big)_{t\geq 0}$. om Equation (5), it follows that

$$\begin{aligned} E^{(i,j)}[B_t] &= \sum_{k=0}^{i} \sum_{\ell=0}^{j} k\, P_t\, \big((i,j);(k,\ell)\big) \\ &= \sum_{k=0}^{i} \sum_{\ell=0}^{j} k \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n\big((i,j);(k,\ell)\big) . \end{aligned}$$

Table 1. EXPECTED NUMBERS OF SURVIVORS

i	j	e_b	e_r	p_b	p_r	t	$E^{(i,j)}[B_t]$	$E^{(i,j)}[R_t]$
200	200	0.1	0.1	0.5	0.5	0.625	194.40	193.78
						1.25	187.28	187.56
						2.5	176.34	176.80
						5.0	155.86	155.68
200	200	0.5	0.5	0.5	0.5	0.5	176.00	176.12
						1.0	155.62	156.44
						2.0	122.10	121.16
						4.0	73.04	73.54
150	100	0.1	0.225	0.5	0.5	1.25	136.38	90.98
						2.5	122.54	82.88
						5.0	103.80	63.46
200	200	0.4	0.5	0.5	0.5	1.25	145.60	155.98
						2.5	101.76	127.22
						5.0	34.84	94.02

We are not able, however, to compute this explicitly. It can also be shown that the functions

$$(i,j,t) \to E^{(i,j)}[B_t] \quad \text{and} \quad (i,j,t) \to E^{(i,j)}[R_t]$$

satisfy convolution-type equations known as Markov renewal equations (cf. Reference [4]), whose solution is obtainable abstractly in terms of a Markov renewal kernel but which we are also unable to obtain explicitly. The difficulty here is essentially the same as that in performing the matrix exponentiation in Equation (5). Hence, we are using simulation to study functions that we have not been able to treat adequately in an analytical matter.

The results (in summarized form in Table 1) are not unexpected. Engagements are intense at the beginning and less so as they progress. For forces with all parameters equal, we nearly have

$$E^{(i,j)}[B_t] = E^{(i,j)}[R_t] \qquad (6)$$

for all t. On theoretical grounds, we would expect that, if $i = j$, $e_b = e_r$, and $p_b = p_r$, then Equation (6) should hold; it can be shown that it does. Over fairly short time intervals (as, e.g., in the first set of results), the expectations appear nearly linear--as shown in Figure 1.

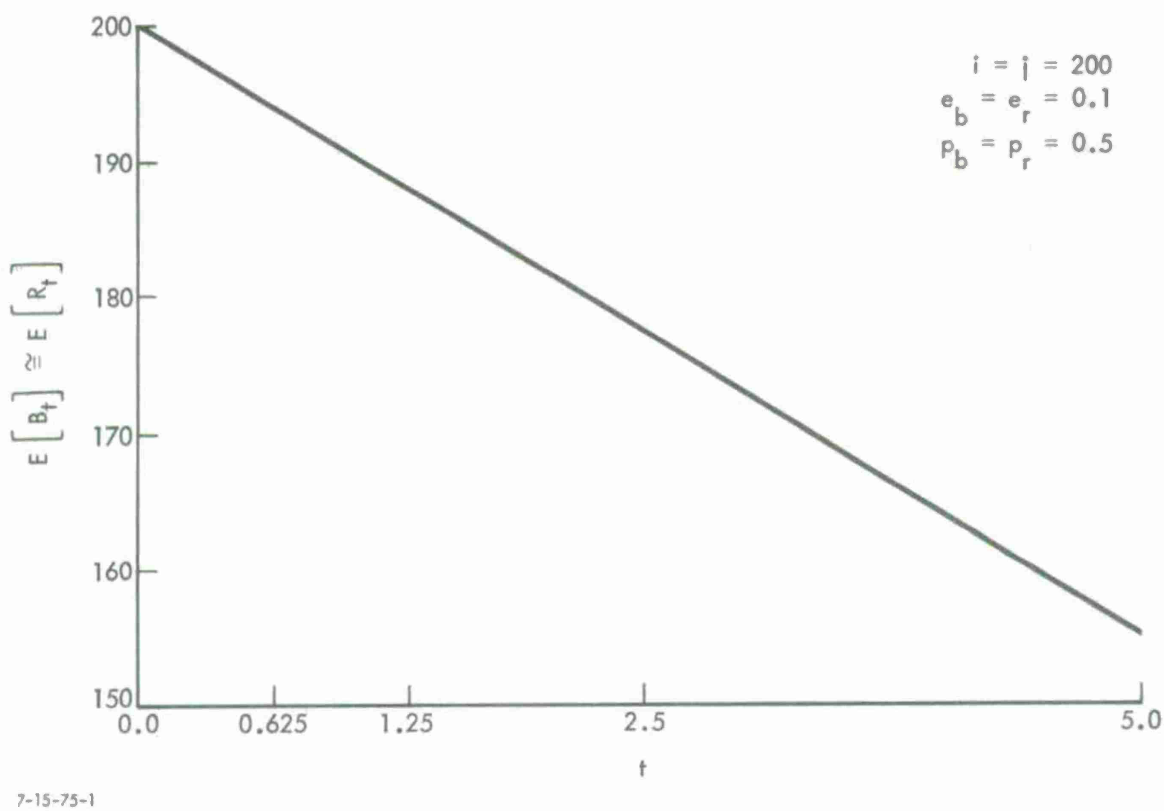

Figure 1. EXPECTED NUMBERS OF SURVIVORS (Engagement Rate = 0.1)

In fact, it is surprising that the functions are, in this case, so near to being linear over such a length of time. For the second set of data, there is less (but still a rather surprising amount) of linearity--as can be seen in Figure 2.

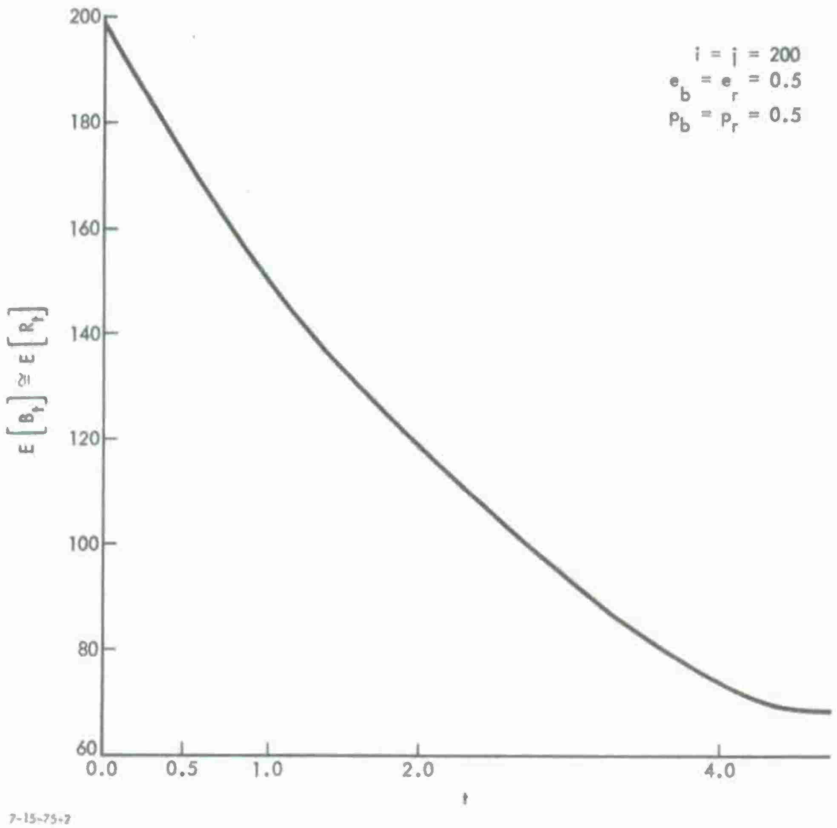

Figure 2. EXPECTED NUMBERS OF SURVIVORS (Engagement Rate = 0.5)

For the situation of Figure 1, we have computed the following approximation:

$$E[B_t] \sim 200(1 - 0.0442t) . \tag{7}$$

If in the classical Lanchester square-law solutions (2) we make the substitutions

$$b(0) = r(0) = 200$$
$$c_1 = c_2 = 0.05$$

(i.e., $c_2 = e_b p_b$ and $c_1 = e_r p_r$, which (cf. Reference [5]) is the proper analogy and make the one-term power series expansions

$$\cosh \lambda t = 1$$
$$\sinh \lambda t = \lambda t \ ,$$

we obtain

$$b(t) \sim 200(1 - 0.05t) \ . \tag{8}$$

The similarity between Equations (7) and (8) is suggestive and intriguing but not, the reader is warned, indicative that Equations (2) are good approximations to the behavior of $E[B_t]$ for large t. Indeed, $E[B_t]$ and $E[R_t]$ do not go to zero in the limit as $t \to \infty$, when all force parameters are equal. That Equations (7) and (8) are similar is a reflection of the effect of similar approximations at small times. This similarity, incidentally, further strengthens the argument that the stochastic attrition process under study is the appropriate analogue of Equations (1).

The third set of data in Table 1 is more meaningful once fractions of survivors are also considered--as we do in Table 2.

Table 2. EXPECTED FRACTION OF SURVIVORS

t	$E^{(i,j)}[B_t]$	$E^{(i,j)}[\frac{B_t}{B_0}]$	$E^{(i,j)}[R_t]$	$E^{(i,j)}[\frac{R_t}{R_0}]$
1.25	136.38	0.909	90.98	0.910
2.5	122.54	0.817	82.88	0.829
5.0	103.80	0.692	63.46	0.685

It is to be noted that in this case the initial conditions i, j, e_b, e_r, p_b, p_r satisfy the relation

$$i^2 e_b p_b = j^2 e_r p_r \ , \tag{9}$$

[i.e., the classical square-law force-equality condition of the Proposition]. In this case, the simulations suggest the following:

CONJECTURE. Suppose that Equation (9) is satisfied. Then for all t,

$$E^{(i,j)}\left[\frac{B_t}{B_0}\right] = E^{(i,j)}\left[\frac{R_t}{R_0}\right] . \qquad (10)$$

If true, Equation (10) would be a significant and useful property of the stochastic attrition process. The reader can easily verify that if in Equations (2) the force equality condition

$$c_1 r(0)^2 = c_2 b(0)^2$$

is satisfied, then

$$\frac{b(t)}{b(0)} = \frac{r(t)}{r(0)}$$

for all t. Hence the Equation (10), if true, further strengthens the analogy between Equations (1) and the stochastic attrition process of the Theorem. It is also possible that Equation (10) may not hold exactly, but only up to some correction involving the probability that one side or the other has been annihilated. As early as Reference [7], it was noted that certain quantities (e.g., moments) from stochastic Lanchester attrition processes satisfy equations similar to the analogous system of differential equations only for times such that the probability of annihilation of either side can be neglected. Except for Markov renewal equations, however, no equation is known which such quantities satisfy exactly.

The last set of data in Table 1 illustrates two forces that are unequal in capability, but initially equal numerically. As expected, the numerical gap between them increases with time.

A conclusion to be drawn from the amount of linearity appearing in Table 1 is that first-order discretized versions of Lanchester equations, such as are used in the IDAGAM I model, are not, for the casualty rates encountered, bad approximations to the corresponding differential equations.

Chapter III

ITERATED CALCULATIONS

Fixed time-step, deterministic simulation models (e.g., IDAGAM I--Reference [1]) perform daily calculations of attrition, using the result of one day's calculation as the input to that of the next day. If the stochastic attrition process of the Theorem were used to calculate attrition in such a context, the numbers of survivors at the end of one day would be--assuming fixed initial conditions $B_0=i$ and $R_0=j$--

$$b_1 = E^{(i,j)}[B_1] \quad \text{and} \quad r_1 = E^{(i,j)}[R_1] ,$$

respectively. (We assume that the time scale is normalized so that a day is the time unit.) An iterative scheme would then compute the numbers of survivors at the end of the *second* day as

$$b_2^* = E^{(b_1,r_1)}[B_1] \quad \text{and} \quad r_2^* = E^{(b_1,r_1)}[R_1] \tag{11}$$

(i.e., this scheme would compute the effect of one day's combat, applied to the *expected* numbers of survivors from the first day). Even though b_1 and r_1 will not be integers in general, it is possible to extend the stochastic attrition process of the Theorem to have state space $[0,\infty) \times [0,\infty)$. In state (x,y), the sojourn is exponentially distributed with parameter

$$\lambda(x,y) = xe_bp_b + ye_rp_r ;$$

and the next state entered is $(x,\max\{y-1,0\})$, with probability $xe_bp_b/\lambda(x,y)$; otherwise, state $(\max\{x-1,0\},y)$ is entered.

In this case, Equations (11) at least make sense; and the stochastic process can be used in iterative calculations. But the *correct* expected numbers of survivors at the end of two days are

$$b_2 = E^{(i,j)}[B_2] \quad \text{and} \quad r_2 = E^{(i,j)}[R_2] \; ; \tag{12}$$

and there is no assurance that

$$b_2 = b_2^* \quad \text{or} \quad r_2 = r_2^* \; .$$

Indeed, our conjecture was that substantial errors would be incurred by using the iterative calculation.

This conjecture, however, has turned out *not* to be true--as Table 3 clearly demonstrates. The numbers appearing in the table are expected numbers of survivors at the times shown.

The agreement among various iterated calculations and the direct calculations is so uniformly good as to be startling--especially since our intuitive belief before performing this particular simulation was to the contrary (namely, that substantial disagreement would result). But no such discrepancy arose, at least in the cases above. We were, however, able to discover rather large discrepancies at times of which the probability that one side or the other has been annihilated is *not* negligible. The nature of the difference between true and iterated expected numbers of survivors was that the iterated calculation overestimates the number of survivors--the reason being that, in the direct calculation, annihilation occurs with positive probability; but, in the iterated calculations, it does not occur (because of the shorter time intervals considered).

We conclude, therefore, that if t is a time such that

$$P^{(i,j)}\{B_t=0 \text{ or } R_t=0\} \sim 0 \tag{13}$$

(i.e., the probability in question is negligible), then it is permissible to use iterated calculations to approximate

Table 3. ITERATED CALCULATIONS OF NUMBERS OF SURVIVORS

$B_0 = R_0 = 200$; $e_b = e_r = 0.1$; $p_b = p_r = 0.5$								
	Iteration Interval							
	0.625		1.25		2.5		5	
Time	Blue	Red	Blue	Red	Blue	Red	Blue	Red
0	200	200	200	200	200	200	200	200
.625	194.4	193.78	--	--	--	--	--	--
1.25	187.72	188.04	187.28	187.56	--	--	--	--
1.875	181.78	182.36	-	--	--	--	--	--
2.5	175.84	177.12	175.8	176.14	176.34	176.8	--	--
3.125	170.58	171.34	--	--	--	--	--	--
3.75	165.74	166.14	164.76	165.02	--	--	--	--
4.375	160.02	160.84	--	--	--	--	--	--
5.0	154.82	156.08	154.94	154.68	155.28	155.84	155.86	155.68
$B_0 = 150$; $R_0 = 100$; $e_b = 0.1$; $e_r = 0.225$; $p_b = p_r = 0.5$								
0	--	--	150	100	150	100	150	100
1.25	--	--	136.38	90.98	--	--	--	--
2.5	--	--	124.76	82.46	122.54	82.88	--	--
3.75	--	--	114.24	74.98	--	--	--	--
5.0	--	--	104.16	67.7	100.48	69.44	103.8	68.46
$B_0 = R_0 = 200$; $e_b = 0.4$; $e_r = 0.5$; $p_b = p_r = 0.5$								
0	--	--	200	200	200	200	200	200
1.25	--	--	145.6	155.98	--	--	--	--
2.5	--	--	103.26	124.34	101.76	127.22	--	--
3.75	--	--	66.76	103.28	--	--	--	--
5.0	--	--	36.28	89.9	34.74	93.64	34.84	94.02
$B_0 = R_0 = 200$; $e_b = e_r = 0.5$; $p_b = p_r = 0.5$								
	0.5		1		2		4	
0	200	200	200	200	200	200	200	200
.5	176	176.12	--	--	--	--	--	--
1.0	156.32	155.9	155.9	155.62	--	--	--	--
1.5	136.98	137.72	--	--	--	--	--	--
2.0	120.92	121.84	120.26	122.6	122.1	121.16	--	--
2.5	106.08	108.02	--	--	--	--	--	--
3.0	93.3	95.84	92.24	96.54	--	--	--	--
3.5	81.58	84.48	--	--	--	--	--	--
4.0	71.56	74.38	72.14	71.62	75.4	72.46	73.04	73.54

$E^{(i,j)}[B_t]$ and $E^{(i,j)}[R_t]$. It follows from the definition of the stochastic attrition process $((B_t,R_t))_{t\geq 0}$ that, for $i>0$ and $j>0$,

$$P^{(i,j)}\{B_t=0 \text{ or } R_t=0\} = 0 \; ; \tag{14}$$

only for $t=0$; and the probability in question is positive (strictly) if $t > 0$, though possibly very small. As a function of t, this probability is, further, continuous and nondecreasing. And it has the property that

$$\lim_{t\to\infty} P^{(i,j)}\{B_t=0 \text{ or } R_t=0\} = 1 \; ;$$

but it also satisfies

$$P^{(i,j)}\{B_t=0 \text{ or } R_t=0\} < 1$$

for every finite t, even though with $P^{(i,j)}$ probability 1,

$$T = \inf\{t\colon B_t=0 \text{ or } R_t=0\} < \infty \; .$$

In order to determine when iterated calculations produce allowable approximations, one then must study the distribution of T with respect to the various probability measures $P^{(i,j)}$. This we plan to do in the future.

Chapter IV

FORCE EQUALITY AND INEQUALITY

In this chapter, we discuss the most interesting discoveries so far concerning the stochastic Lanchester square-law attrition process: the role of the *Lanchester-square force ratio*

$$f = \frac{e_r p_r R_0^2}{e_b p_b B_0^2} \tag{15}$$

as a measure of equality and inequality of forces. As noted in the Proposition, when the corresponding quantity

$$f' = \frac{c_1 r(0)^2}{c_2 b(0)^2}$$

in the deterministic model (1) is equal to 1, the two forces are "equal" in the sense that neither annihilates the other within a finite period of time.

This observation leads one to inquire whether the quantity f plays an analogous role in the stochastic model. To begin, one must decide upon appropriate criteria for equality of forces. It was observed in Table 2 that for parameters and initial conditions satisfying $f=1$, the two forces had nearly equal expected fractions of survivors at three fixed times. These fractions give one measure of force equality. Another measure is equal probability of each side's annihilating the other, a criterion for which we have the sample results of Table 4. (Similar results hold for other cases; we refer the reader also to the discussion below.)

Table 4. COMPARISON OF SQUARE-LAW AND LINEAR-LAW FORCE EQUALITY

B_0	R_0	e_b	e_r	p_b	p_r	P{Blue Wins}	Fractions Surviving	
							Blue	Red
50	75	1	0.444	0.5	0.5	0.5	0.203	0.196
50	75	1	0.67	0.5	0.5	0.0	0.0	0.573

The most important conclusion obtained so far concerns the role of the ratio f of Equation (15) in the following situation: we seek the probability that one side reduces the other to 50 percent of its original strength before it is itself so reduced, as a function of the Lanchester-square force ratio f. Table 5 gives the results of several simulations run in the process of investigating this problem. Here $p_b = p_r = 0.5$, and the probability being computed by the Monte Carlo simulations is

$$\begin{aligned} p &= P^{(i,j)}\{B_t \leq 0.5B_0 \text{ before } R_t \leq 0.5R_0\} \\ &= P^{(i,j)}\{B_t \leq 0.5j \text{ before } R_t \leq 0.5i\} . \end{aligned}$$

Each tabulated value is computed on the basis of 50 realizations.

The results are rather surprising. To a large extent, the probability p of Red's winning is a function only of the Lanchester-square force ratio f and not of the exact values of the parameters B_0, R_0, e_b, e_r, p_b, and p_r. Of course, only fairly limited ranges of the values of the parameters have been investigated. As the numerical scale of the combat (i.e., the initial force levels B_0 and R_0) increases, the graph of the function $p = p(f)$ (as depicted in Figure 3) becomes steeper and nonconstant (i.e., different from 0 and 1) over a smaller range of values of f. In the limit, as $B_0 = R_0 \to \infty$, the function p would become

Table 5. PROBABILITY OF WIN AS A FUNCTION OF FORCE RATIO (Equal Breakpoints)

R_0	B_0	e_r	e_b	$f=\frac{e_r p_r R_0^2}{e_b p_b B_0^2}$	P(Red Wins)
50	50	1	1	1	0.56
50	50	2	1	2	0.98
50	50	1.5	1	1.5	0.94
50	50	1.25	1	1.25	0.8
50	50	1.1	1	1.1	0.74
50	50	1.05	1	1.05	0.54
50	50	3	1	3	1
50	100	1	1	0.25	0
50	100	2	1	0.5	0
50	100	3	1	0.75	0.12
50	100	3.5	1	0.875	0.38
50	100	4	1	1	0.54
50	100	4.5	1	1.125	0.68
50	100	5	1	1.25	0.88
50	100	5.5	1	1.375	0.88
50	100	7	1	1.75	1
50	150	1	1	0.111	0
50	150	2	1	0.222	0
50	150	3	1	0.333	0
50	150	4	1	0.444	0
50	150	5	1	0.555	0
50	150	6	1	0.666	0.06
50	150	7	1	0.777	0.14
50	150	8	1	0.888	0.3
50	150	9	1	1	0.5
50	150	10	1	1.111	0.7
50	150	11	1	1.222	0.84
50	150	12	1	1.333	0.92
50	150	13	1	1.444	0.92

R_0	B_0	e_r	e_b	$f = \frac{e_r p_r R_0^2}{e_b p_b B_0^2}$	P(Red Wins)
100	150	1	1	0.445	0
100	150	2	1	0.89	0.26
100	150	2.25	1	1	0.46
100	150	2.5	1	1.11	0.72
100	150	3	1	1.33	0.94
100	150	4	1	1.78	1
200	200	0.75	1	0.75	0
200	200	0.80	1	0.80	0.08
200	200	0.85	1	0.85	0.18
200	200	0.90	1	0.90	0.25
200	200	0.95	1	0.95	0.34
200	200	1.00	1	1.00	0.51
200	200	1.05	1	1.05	0.68
200	200	1.10	1	1.10	0.79
200	200	1.15	1	1.15	0.8
200	200	1.20	1	1.20	0.87
200	200	1.25	1	1.25	0.91
500	500	0.75	1	0.75	0
500	500	0.80	1	0.80	0
500	500	0.85	1	0.85	0.04
500	500	0.90	1	0.90	0.11
500	500	0.95	1	0.95	0.26
500	500	1.00	1	1.00	0.53
500	500	1.05	1	1.05	0.73
500	500	1.10	1	1.10	0.88
500	500	1.15	1	1.15	0.96

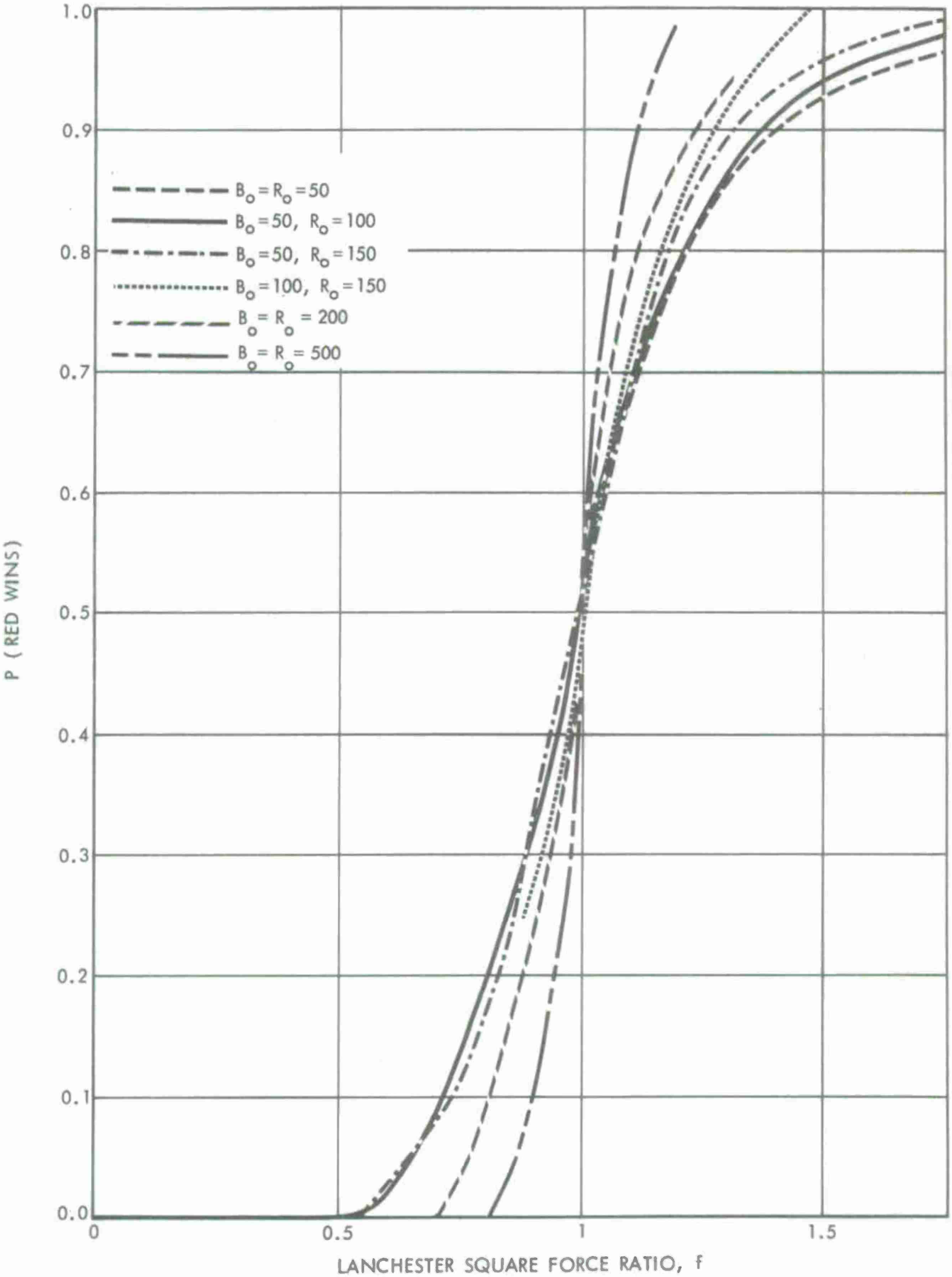

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Figure 3. GRAPHICAL PRESENTATION OF TABLE 5

$$p(f) = \begin{cases} 0 & \text{if } f < 1/2 \ ; \\ 0.5 & \text{if } f = 1/2 \ ; \\ 1 & \text{if } f > 1/2 \ . \end{cases}$$

For the cases shown in Figure 3, nearly all the increase in p as a function of increasing f is concentrated between the values $f = 0.75$ and $f = 1.33 = 1/(0.75)$. In this range, improvements in either quantity (the force strengths B_0 and R_0) or quality (the effectiveness parameters e_b, e_r, p_b, p_r) lead to significant improvements in p--the former more so than the latter. In other ranges, changes in quality or quantity lead to rather small changes in p. These conclusions are of considerable interest in relation to force planning for situations believed to be described by the stochastic Lanchester square-law attrition process--particularly with respect to quality-quantity trade-offs.

An interesting consequence of these results concerns the "force ratio" necessary to ensure a reasonable probability of defeating the other side. Let us consider the case of forces with equal quality. It is frequently asserted that, in this case, a numerical superiority of 3 to 1 is necessary for victory. In terms of the Lanchester-square force ratio f, a 3:1 numerical superiority corresponds to $f = 9$. But the results of Figure 3 indicate that $f = 1.25$ gives 0.9 probability of victory for force levels greater than 200; and this probability corresponds to a numerical force ratio of 1.12, which is only a very slight numerical superiority! Clearly, this result has rather significant implications concerning force planning and procurement. Alternatively, it has significant implications concerning the applicability of Lanchester-type attrition models.

A similar exercise was carried out using an asymmetric termination rule. Red (the attacker) loses if surviving Red strength falls below 79 percent of initial Red strength, while Blue (the defender) is defeated if surviving Blue strength becomes less than 67 percent of initial Blue strength. These numbers (taken from the ATLAS combat model) are intended to represent the differing capabilities of attacker and defender to sustain and absorb casualties. The results of this investigation are shown in Table 6 and are summarized graphically in Figure 4. In this case both kill probabilities are 0.5. Therefore,

$$f = \frac{e_r R_0^2}{e_b B_0^2} \ .$$

The output is the probability that the attacker wins. (The comments made for the case of symmetric breakpoints are qualitatively applicable in this case as well.)

Thus, to achieve 80-percent certainty of victory, the attacker requires a Lanchester-square force ratio of approximately 2. In the case of equal-quality forces, this probability corresponds to a numerical superiority of 1.41 to 1, which is still much less than that usually thought to be required. For higher numbers of forces, the necessary numerical superiority is less. If Blue (defending) forces are 1.5 times as effective as Red forces, then the required numerical superiority is only 1.7--still considerably less than the frequently mentioned factor of 3. Again, the implications on force planning and expenditure levels (or on attrition modeling) are considerable.

An interesting problem is to find f* such that

$$p(f^*) = 1/2$$

(i.e., each side is equally likely to win). From the data, it appears that f* is possibly independent of the absolute parameter levels and equal (approximately) to 1.44. We conjecture that it is in fact true that f* is the same for all parameter levels and is therefore a function only of the breakpoints 0.79 and 0.67. We note that

Table 6. PROBABILITY OF WIN AS A FUNCTION OF FORCE RATIO (Unequal Breakpoints)

B_0	R_0	e_b	e_r	f	P(Attacker Wins)
50	50	1	1.0	1.0	0.16
50	50	1	1.2	1.2	0.25
50	50	1	1.4	1.4	0.45
50	50	1	1.6	1.6	0.58
50	50	1	1.8	1.8	0.81
50	75	1	0.1	0.225	0
50	75	1	0.2	0.45	0
50	75	1	0.3	0.675	0.01
50	75	1	0.4	0.9	0.05
50	75	1	0.5	1.125	0.2
50	75	1	0.6	1.35	0.48
50	75	1	0.7	1.575	0.58
50	75	1	0.8	1.8	0.69
50	75	1	0.9	2.025	0.85
50	100	1	0.1	0.4	0
50	100	1	0.2	0.8	0.02
50	100	1	0.3	1.2	0.29
50	100	1	0.4	1.6	0.63
50	100	1	0.5	2.0	0.8
50	100	1	0.6	2.4	0.9
50	100	1	0.7	2.8	0.98
50	100	1	0.8	3.6	0.99
50	150	1	0.1	0.9	0.03
50	150	1	0.11	0.99	0.05
50	150	1	0.12	1.08	0.12
50	150	1	0.13	1.17	0.19
50	150	1	0.14	1.26	0.34
50	150	1	0.15	1.35	0.39
50	150	1	0.16	1.44	0.45
50	150	1	0.17	1.53	0.55
50	150	1	0.18	1.62	0.63
50	150	1	0.19	1.71	0.61
50	150	1	0.2	1.8	0.71
100	100	1	1.0	1.0	0.01
100	100	1	1.1	1.1	0.19
100	100	1	1.2	1.2	0.23
100	100	1	1.3	1.3	0.32
100	100	1	1.4	1.4	0.4
100	100	1	1.5	1.5	0.55
100	100	1	1.6	1.6	0.62
100	100	1	1.7	1.7	0.72
100	100	1	1.8	1.8	0.73
100	100	1	1.9	1.9	0.83

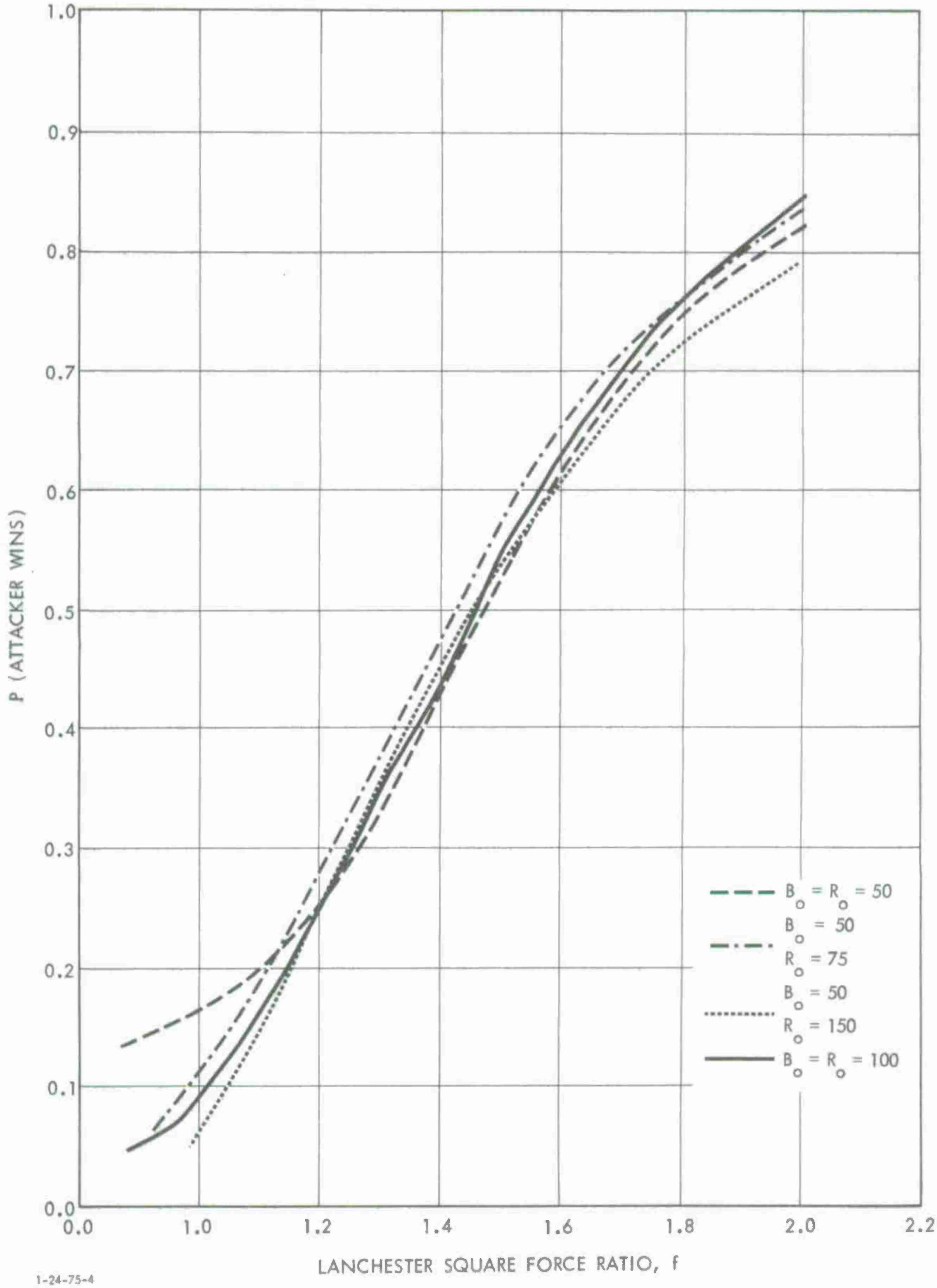

Figure 4. GRAPHICAL PRESENTATION OF TABLE 6

$$\left(\frac{0.79}{0.67}\right)^2 = 1.39 \; ;$$

whether this is purely coincidental we plan to explore further in future investigations. The value of f* is, of course, of considerable practical interest.

We conclude with some representative probability distributions of the terminal state of the process, which appear in Figures 5-8.

In Figure 5, we have data when both breakpoints are 0.5 and the forces are equal numerically and in quality at the beginning of the engagement. It is clear that the winner does not win by very much, having (on the average) about 62 percent of his initial forces left when $R_0 = B_0 = 100$ and 67 percent when $R_0 = B_0 = 200$. For $R_0 = B_0 = 100$, the most likely values of surviving strength are those nearest 50 percent (i.e., the narrowest of victories). However, when $R_0 = B_0 = 200$, the most likely surviving strength is about 60 percent.

Figure 6 contains data with breakpoints of 0.5, $R_0 = B_0 = 200$, and force qualities chosen so that the Lanchester-square force ratio

$$f = \frac{e_r p_r R_0^2}{e_b p_b B_0^2}$$

is equal to 1.1. In this case, the probability of Red's winning is about 0.75 and the most likely fraction of survivors is still about 60 percent. The expected fraction of Blue survivors--given that Blue wins--is, on the other hand, much closer to 0.5. In this case, Blue wins only by small margins, while Red can win by more substantial margins--but never really overwhelmingly.

Figures 7 and 8 have the Red breakpoint equal to 0.79 and Blue breakpoint equal to 0.67; in both cases, $R_0 = B_0 = 100$. For Figure 7, force qualities are chosen so that f = 1 and

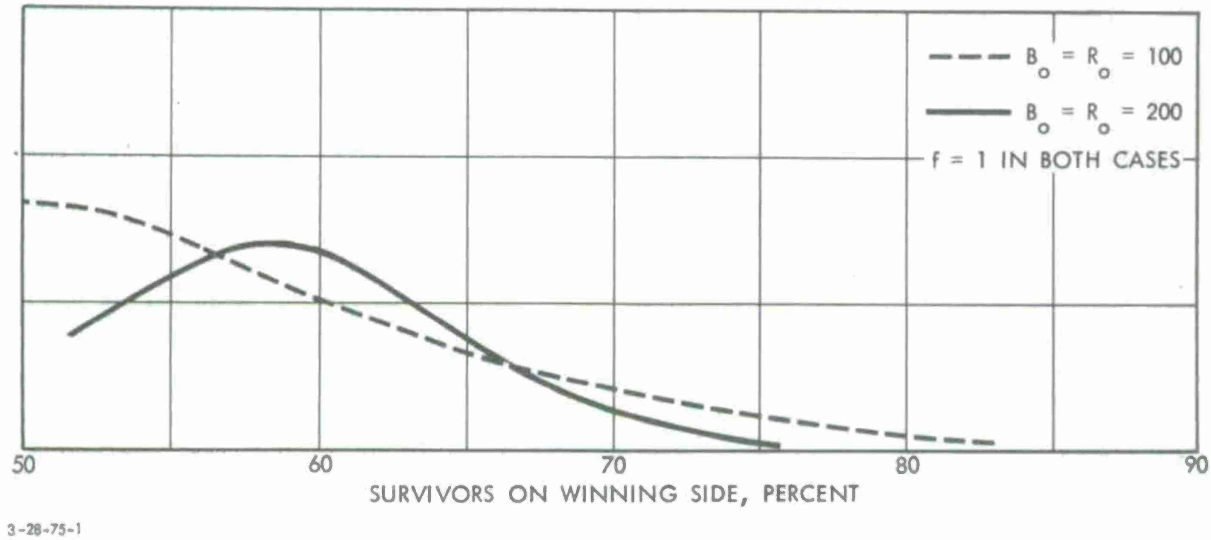

Figure 5. PROBABILITY DISTRIBUTION OF NUMBER OF SURVIVORS (Equal Forces)

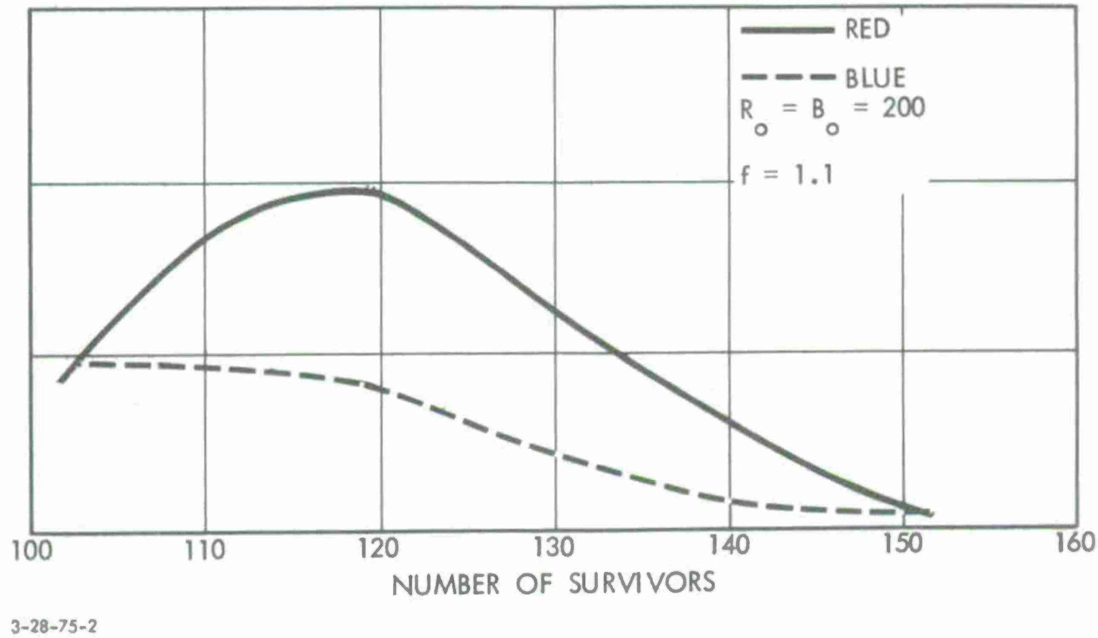

Figure 6. PROBABILITY DISTRIBUTION OF NUMBER OF SURVIVORS (Force Ratio = 1.1)

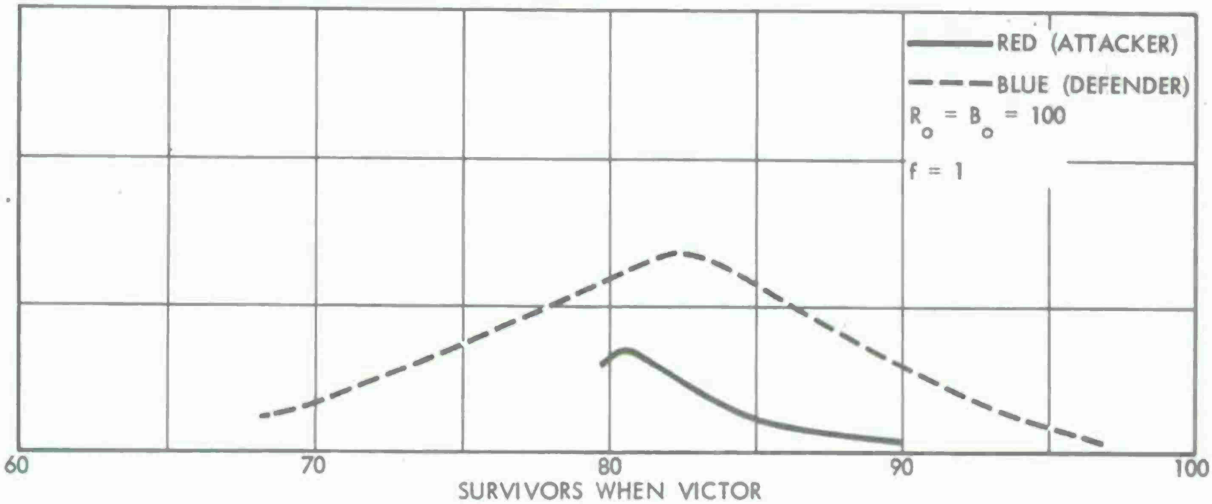

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Figure 7. PROBABILITY DISTRIBUTION OF NUMBER OF SURVIVORS (Unequal Breakpoints and Force Ratio = 1)

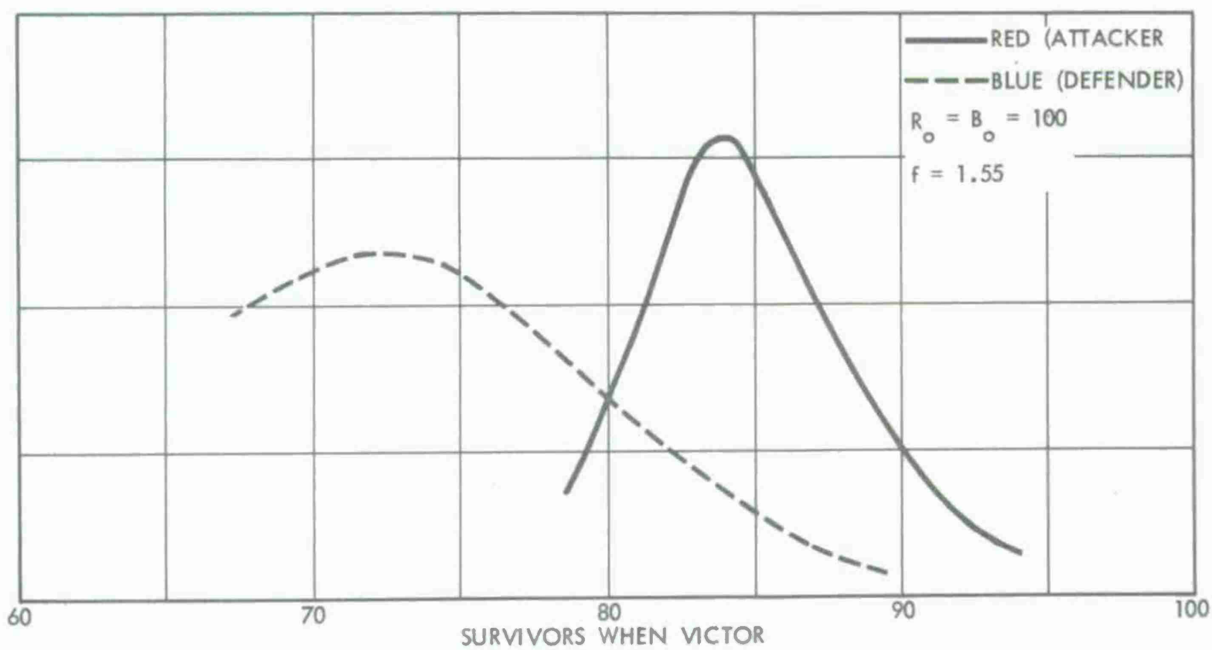

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Figure 8. PROBABILITY DISTRIBUTION OF NUMBER OF SURVIVORS (Unequal Breakpoints and Force Ratio = 1.55)

Blue wins with probability 0.87--with expected value and most likely fraction of survivors both approximately 0.8. Recalling that Red need only be reduced to 0.79 of original Red strength, we observe that here again the victories are hardly overwhelming. Indeed, half the time when Blue "wins," it is with fewer survivors (numerically) than Red. When Red wins, it is essentially only by its incurring nearly all its allowable losses.

In Figure 8, force qualities are chosen so that f = 1.55--which (in view of Figure 4) makes the forces nearly equal, in the sense of being equally likely to win. Here Red's expected fraction of survivors, given a win, increases; and Blue's, decreases (relative to Figure 7), as one would expect.

A general conclusion to be drawn is that, even when one side is rather certain to win, the margin of victory is not (on the average) overwhelming.

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