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MANPOWER PLANNING MODELS - III. LOGITUDINAL MODELS

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Monterey, California

August 1975

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LONGITUDINAL MODELS

by

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and

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This is the third in a series of reports on Manpower Planning Models. The emphasis in this report is on the formulation, theory, and application of longitudinal models. The concepts of chain flows are introduced and exploited in a number of applications.

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1. Introduction.

The cross-sectional models discussed in Chapter II are useful because of their simplicity and their reliance on available cross-sectional data. More will be said about manpower data in a later Chapter. The cross-sectional model has a serious structural fault when used to describe manpower flow, especially when the flow fractions are interpreted probabilistically. Simply stated, this assumption says that flow from one class to another is independent of the time an individual has spent in a given class. Such an assumption is clearly not valid in many manpower systems, where time in a given class is a critical factor in determining availability for promotion or movement.

The models in this chapter do not require this restrictive assumption to hold. They are much more general than the cross-sectional models, and attempt to describe the flow of a group, or cohort, through the manpower system over time. The models are based on the entire history of the group, and hence are longitudinal models. As we shall see, the greater realism in the model is bought a price of a significant increase in data requirements.

Section 2 describes the basic longitudinal model and gives examples. Section 3 looks at a simple special case and Section 4 analyses the concept of equilibrium in a longitudinal model. Section 5 gives a probabilistic interpretation of the model. Sections 6, 7, 8, and 9 describe some examples in student forecasting, university planning, and military force structure planning. The Chapter ends with two advanced sections, 10 and 11, dealing with the concept of longitudinal conservation, followed by notes and comments.

2. The Assumption of Longitudinal Stability.

This section describes the general form of longitudinal models and gives several examples. As in previous chapters, we assume the organization contains N classes of manpower. The novel assumption in this chapter concerns the inflow into the system. The inflow is partitioned into K different categories. These categories are called either chains, cohorts, paths, or histories. For example, we could classify the students entering a university simply by their year of entry. In that case $K=1$. The students could also be classified according to eventual status. Of course when the manpower inflow is partitioned according to eventual status it is not possible to specify which individuals belong to which class when they enter. Fortunately this specific type of accounting is not necessary to answer a host of interesting questions as we shall see.

Example 1: Each year at matriculation ceremonies at TIM engineering school the dean speaks to the 600 new freshmen. To brace them for the hard work of the next four years he asks them to "look at the person to your left and to your right; only one of the three will graduate." The dean based his remark on the observation that the school has taken in 600 students per year for the past 20 years and has been awarding roughly 200 degrees per year over the same period. Thus the 600 new freshmen can be classified according to eventual status; 200 degree winners and 400 dropouts, even though it is not known which individuals fall into each class. ∎

Let $g(t)$ be a K-vector which gives the input of people in period t. Thus $g_k(t)$ is the number of people who enter chain k in period t. The fraction of people who enter chain k in period t who are counted in class i at time $t+u$ is $p_{ik}(u)$. The $N \times K$ matrix $P(u)$ describes the distribution of individuals in the K chains over the N classes the u^{th} time they are counted. If we assume M is the maximum number of times an individual is encountered, (i.e., M is the maximum number of period in the system) then the

$M+1$ matrices $P(0), P(1),\ldots,P(M)$ describe flow through the system. The fraction $p_{ik}(u)$, is independent both of the entering period t and the number of individuals, $g_k(t)$ that enter chain k (see the last paragraph in this section).

The contribution to stock in class i at time t is due to the inflow on chains $k = 1,2,\ldots,K$ in periods $t, t-1,\ldots,t-M$. Let $s_i(t;u)$ be the total stock in class i at time t that entered in period $t-u$. We say this group has *length of service* equal to u, since they have been counted at times $t-u, t-u+1,\ldots,t-1$. When $u = 0$, the length of service is zero since these individuals are being encountered for the first time.

The value of $s_i(t;u)$ is made up of contributions from each of the K chains

(1) $$s_i(t;u) = \sum_{k=1}^{K} p_{ik}(u)g_k(t-u) \ .$$

The total stock in class i at time t is given by

(2) $$s_i(t) = \sum_{u=0}^{M} s_i(t;u) = \sum_{u=0}^{M} \sum_{k=1}^{K} p_{ik}(u)g_k(t-u) \ .$$

We can also partition the individuals in class i by the chain on which they are flowing. Define $s_{ik}(t)$ to be the number of individuals in class i who are on chain k. Evidently

(3) $$s_{ik}(t) = \sum_{u=0}^{M} p_{ik}(u)g_k(t-u),$$

and by summing over k we again obtain equation (2) but with order of summation reversed.

Equation (2) describes the longitudinal flow model. It can be expressed in matrix notation as

(4) $$s(t) = P(0)g(t) + P(1)g(t-1) + \dots + P(M)g(t-M).$$

If we establish the convention that periods t for $t \leq 0$ are past periods, periods $t > 1$ are future periods, and that period 1 is the current period, then we can define the manpower *legacy* at time $t \geq 1$ as the contribution of past (prior to time 1) inputs $[g(0), g(-1), \dots, g(1-M)]$ to the stock at future time t. Let $\ell(t)$ (an N vector) be the legacy at time t. Then

$$\ell(t) = \begin{cases} P(t)g(0) + P(t+1)g(-1) + \dots + P(M)g(t-M) & \text{if } t \leq M, \\ 0 \quad \text{if } t > M \end{cases}$$

As usual the legacy is simply the sequence of stock levels that would be observed if no additional individuals entered the system: i.e., if $g(t) = 0$ for $t \geq 1$.

<u>Problem 1</u>: Determine an expression for the legacy in class i at time t that entered in period $t-u$, and for the legacy in class i at time t of individuals on chain k.

<u>Example 2</u>: Consider a two year junior college with two classes corresponding to freshmen (F) and sophomores (S). Let G stand for graduation, and D for dropout. We assume there are seven possible chains:

Chain	History
1	F S G
2	F F S G
3	F S S G
4	F D
5	F F D
6	F S D
7	F F S D

Note that $N = 2$ and $K = 7$.

Individuals on chains 1 through 3 eventually receive degrees; those on chains 4 through 7 eventually drop out. Individuals on chain 3, for example, repeat the sophomore year before graduating. The matrices $P(0)$, $P(1)$ and $P(2)$ are

		chain 1	2	3	4	5	6	7
P(0)	1	1	1	1	1	1	1	1
	2	0	0	0	0	0	0	0
P(1)	1	0	1	0	0	1	0	1
	2	1	0	1	0	0	1	0
P(2)	1	0	0	0	0	0	0	0
	0	0	1	1	0	0	0	1

The matrices $P(u)$, $u \geq 3$ are all zero matrices.

We reemphasize that when a student enters it is not known which of the seven chains he will follow. This is not determined until the student finally graduates or drops out. However, the model can still be useful as we show in later sections. It may be possible to estimate from past data the relative flows on the seven chains. One can then estimate, for example, the effects of instituting a policy of not allowing a freshman to repeat a year. Such a policy would eliminate chains 2, 5 and 7.

Problem 2: Given the flows below calculate $s(t)$ for $t = 1,2,3$. Use (1) and (3) to calculate $s_i(t;u)$ and $s_{ik}(t)$ for $t = 1$.

	Chain						
	1	2	3	4	5	6	7
g(-1)	7	1	4	1	2	0	1
g(0)	6	1	3	1	1	0	2
g(1)	7	3	1	0	2	1	0
g(2)	6	1	2	1	1	1	1
g(3)	8	1	1	3	0	1	0

Problem 3: (Continuation)

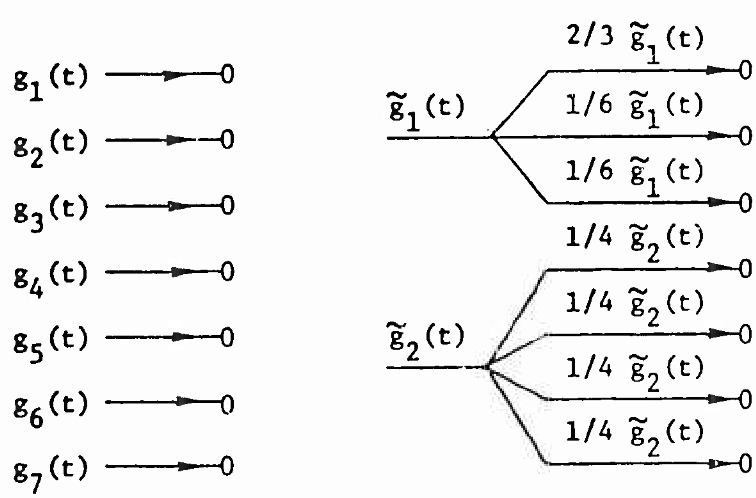
Calculate the legacy at time $t = 1,2,3$.

Problem 4: (Continuation)

Calculate $\sum_{k=4}^{7} s_{ik}(t)$ for $i = 1,2$, and $t = 1,2,3$. How would you interpret the fraction $\sum_{k=4}^{7} s_{ik}(t)/s_i(t)$?

Example 3: Suppose that flows in chains 1, 2, and 3 (the cohorts that eventually graduate) and flows in chains 4, 5, 6 and 7 (the cohorts that eventually drop-out) are aggregated. In the graduate group (aggregate chain 1) we assume that 2/3 of the flow follows the path of the old chain 1 and that 1/6 follows the flow of both chains 2 and 3. For the drop-out group (aggregate chain 2) we assume that 1/4 of the flow follows the same path as chains 4, 5, 6 and 7.

Instead of seven chains feeding the system as depicted at the left



we have the input scheme shown at the right.

Under this aggregation, the new values of P become,

$$P(0) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$P(1) = \begin{bmatrix} 1/6 & 1/2 \\ 5/6 & 1/4 \end{bmatrix}$$

$$P(2) = \begin{bmatrix} 0 & 0 \\ 1/3 & 1/4 \end{bmatrix}$$

Problem 5: (Based on Example 3)

Calculate $s_i(t)$, $\ell_i(t)$ and $s_{i2}(t)$ for $i = 1,2$ and $t = 1,2,3$ given the following flow data:

	Chain 1	Chain 2
$\tilde{g}(-1)$	12	4
$\tilde{g}(0)$	10	4
$\tilde{g}(1)$	11	3
$\tilde{g}(2)$	9	4
$\tilde{g}(3)$	10	4

Example 4: Consider the three class faculty example; nontenured (N), tenured (T), and retired (R). Suppose there are only seven possible career paths. We classify the paths according to the number of years individuals on that path spend in each manpower category.

History

Chain	N	T	R
1	4	0	0
2	5	30	15
3	5	20	20
4	5	10	0
5	0	25	15
6	0	20	20
7	0	10	0

Chain 1 leaves after four years of nontenured service. Chains 5 through 7 depict career paths of tenured appointments. We assume that individuals in chains 4 and 7 leave or die after ten years of tenured service. Those in chains 3 and 6 retire early, while those in chains 2 and 5 retire at age 65. Note that $M = 49$. Thus the 50 matrices $P(0),\ldots,P(49)$ which contain $50 \times 3 \times 7$ numbers are an inefficient way of storing the information summarized in the table above.

Example 5: Consider a four year undergraduate college with classes "eventual graduates" and "eventual dropouts" corresponding respectively to the indices 1 and 2. We assume there are four chains; eventual graduates and dropouts who enter as freshmen and eventual graduates and dropouts who enter as juniors. The four chains are listed below along with the average number of years a person in that chain attends the college.

chain	average number of years attended
1	4.5
2	2.3
3	2.2
4	0.8

Here we can take $M = 4$, and construct the P matrices

$$\begin{array}{c} \\ P(0) = \end{array}\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$

$$P(0) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.8 \end{bmatrix}$$

$$P(1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P(2) = \begin{bmatrix} 1 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0 \end{bmatrix}$$

$$P(3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P(4) = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the selection of the matrices is arbitrary to some extent. A more detailed specification of the model is necessary to fix correct values of the $P(u)$. For instance, if we assume people flow at a uniform rate (over a period) into the chains, and that a person in chain 2 stays exactly 2.3 periods then the model is correct.

If flow into the chains occurs on 15 September of each year and the inventory date is 1 November, then the model is correct if we say that every person on chain 2 stays more than 13.5 months, and that 30% stay more than 25.5 months and none stay 37.5 months. This gives an average of 2.3 years each. However, if we say that only 90% stay more than 1.5 months, 60% more than 13.5 months, 50% more than 25.5 months, 30% more than 37.5 months and 0% more than 49.5 months, then the expected number of years is still 2.3. However the fractions $p_{22}(u)$ change as indicated below.

$p_{22}(u)$

	u=0	u=1	u=2	u=3	u=4
old	1.0	1.0	0.3	0	0
new	0.9	0.6	0.5	0.3	0

Problem 6: (Based on Example 5)

Given the flows below, determine $s(t)$ and $\ell(t)$ for $t = 1,2,3,$ and 4.

Chain

	1	2	3	4
g(-3)	10	4	3	1
g(-2)	10	4	4	0
g(-1)	11	3	2	2
g(0)	9	5	6	0
g(1)	11	3	5	1
g(2)	12	2	5	2
g(3)	14	4	4	1
g(4)	13	4	5	1

This section has defined the longitudinal flow model, equation (4), and presented several examples of longitudinal flow processes. Example 4 indicated that longitudinal models may require a great deal of data, and in Example 5, that several sets of data can be consistent with the specifications of the model. The next section discusses a special application, after which the concept of equilibrium is investigated.

The reader might well question our basic assumption that $p_{ik}(u)$ is independent of the entering period t and the number of individuals $g_k(t)$

who enter chain k. We offer two sets of data, one supporting our assumption and one which to some extent violates it.

The first set of data is given in tables II.15, II.16 and II.17 in Chapter II. A study of these tables will show that for freshman entering the Berkeley Campus at the University of California at Berkeley in the fall of 1955 and the fall of 1960, the flow fractions of the two groups were essentially the same. Note that not only was there a five year time span between the groups, but that the numbers in each group were significantly different (2067 to 3228).

The second set of data is given in table III.1. Five groups are shown, each one a group of people who enlisted in the Marine Corps for an initial period of two years in July and August 1967, and January, February and June 1968. The table entries give the percentages of the groups remaining at the end of the given month after entry. For example, in January 1968, 4117 people entered the Marine Corps on a two year enlistment. After 12 months 89.6% of these were still in the Corps. After 24 months the percentage remaining was 11.2.

A close look at this data shows that for the first 17 months the percentages remaining are remarkedly similar between groups. Starting at 18 months however, the percentages start to vary significantly. The reader might also be wondering why, since all the people had enlisted for two years, less than 30% stayed in for the full enlisted period. The reason for both the significant attrition starting at about 18 months, and the instability between groups in the 18-30 month period, can be found by studying manpower policies used in the Marine Corps in the 1968-9 period. In that period the Marine Corps had problems manning overseas committments due to legal restrictions on personnel flows. To obtain feasible flows of people to overseas billets they had to institute an "early-out" policy, which meant that although some enlisted men had contracts covering 24 months, many were forced out earlier than this.

Month After Entry	Percent Remaining at End of Month After Entry Period (and Cohort Sizes)				
	Jul 1967 (1725)	Aug 1967 (1822)	Jan 1968 (4117)	Feb 1968 (3983)	Jun 1968 (4023)
0	100.0	100.0	100.0	100.0	100.0
1	97.9	98.1	98.2	97.9	97.2
2	96.8	97.0	96.8	96.7	95.4
3	96.0	96.5	96.3	95.8	94.4
4	95.6	96.0	95.9	95.4	94.0
5	95.1	95.7	95.6	95.1	93.6
6	94.4	95.2	94.8	94.3	93.2
7	92.9	94.5	93.4	93.4	92.7
8	92.0	94.0	92.5	92.5	92.0
9	91.2	91.8	91.6	91.6	91.2
10	89.2	91.2	90.9	91.0	90.6
11	88.4	90.2	90.2	90.1	89.9
12	87.5	89.6	89.6	89.1	89.3
13	86.9	88.8	88.7	88.3	87.3
14	86.0	88.5	87.5	86.8	85.6
15	85.4	87.7	86.5	85.3	84.2
16	84.5	87.3	82.8	82.4	82.0
17	83.1	85.8	80.7	80.7	79.5
18	76.1	80.3	72.1	73.9	65.9
19	59.1	65.1	55.7	51.1	59.3
20	52.6	51.5	44.2	45.5	47.3
21	38.4	46.3	40.8	40.9	40.6
22	30.6	34.9	37.7	37.1	32.8
23	25.3	30.6	30.1	30.7	29.2
24	9.4	8.4	11.2	10.3	7.4
25	7.4	5.9	8.9	7.7	6.0
26	5.4	4.9	7.8	6.4	5.1
27	4.2	4.2	7.0	5.7	4.4
28	3.7	3.6	6.2	4.9	4.1
29	3.3	3.3	5.7	4.5	3.6
30	3.0	3.2	5.2	4.1	3.4

Table III.1 Cohort Data for Selected Groups of 2-year enlisted Marines.

The reason for including the data in table III.1 is to show not only that the stability assumption can be violated, but also that the fractions $p_{ik}(u)$ are in certain situations <u>control variables</u>. In the example given, direct control of these fractions was used by the Marine Corps. Today, the problem is not to remove people early, but to retain them in an environment without a draft. To do this the fractions $p_{ik}(u)$ are being controlled indirectly through payment of selective bonuses to people with skills or attributes which the Marine Corps requires. In later sections in this report the longitudinal model is used in a number of ways. It is important for the reader to recognize the difference between using the model to <u>forecast</u> using $p_{ik}(u)$ estimated from historical data, and using the model for <u>planning</u>, where either the effects of certain $p_{ik}(u)$ values are analyzed, or the $p_{ik}(u)$ are determined to meet some objectives. When using historical estimates in forecasting it is important that the estimates can be expected to approximate actual future behavior. The model user must therefore be aware of significant policy changes which might affect the future values of $p_{ik}(u)$.

3. A Special Case: One Class, One Chain.

The special case of one manpower class and one chain allows us to examine the longitudinal flow model more easily and closely. This section presents several ways of visualizing the longitudinal stability of flow in a one class, one chain model.

We begin by simplifying notation, and write $g(t)$ for $g_1(t)$ and $p(u)$ for $p_{11}(u)$. The basic formulae are, for stocks,

$$s(t) = \sum_{u=0}^{M} p(u)g(t-u) \ ,$$

and for legacies,

$$\ell(t) = \begin{cases} \sum_{u=t}^{M} p(u)g(t-u) & \text{if } t \leq M \ , \\ 0 & \text{if } t > M \ , \end{cases}$$

where $s(t), p(u), g(t)$ and $\ell(t)$ are all scalars. As usual u measures the individual's length of service in the organization. The quantity $p(u)$ is called a survivor fraction. It is the fraction of those with length of service u that are still in the organization. For example, the entries in table III.1 are survivor fractions × 100.

Figure III.1 shows a graphical method of computing $p(u)g(t)$. The graph is for the case $M = 4$, and the particular values

u	0	1	2	3	4
p(u)	1.0	0.85	0.80	0.55	0.2

The input value $g(t)$ is plotted on the horizontal axis and the various values $p(u)g(t)$ can be read from the vertical axis. For example, when $g(t) = 70$,

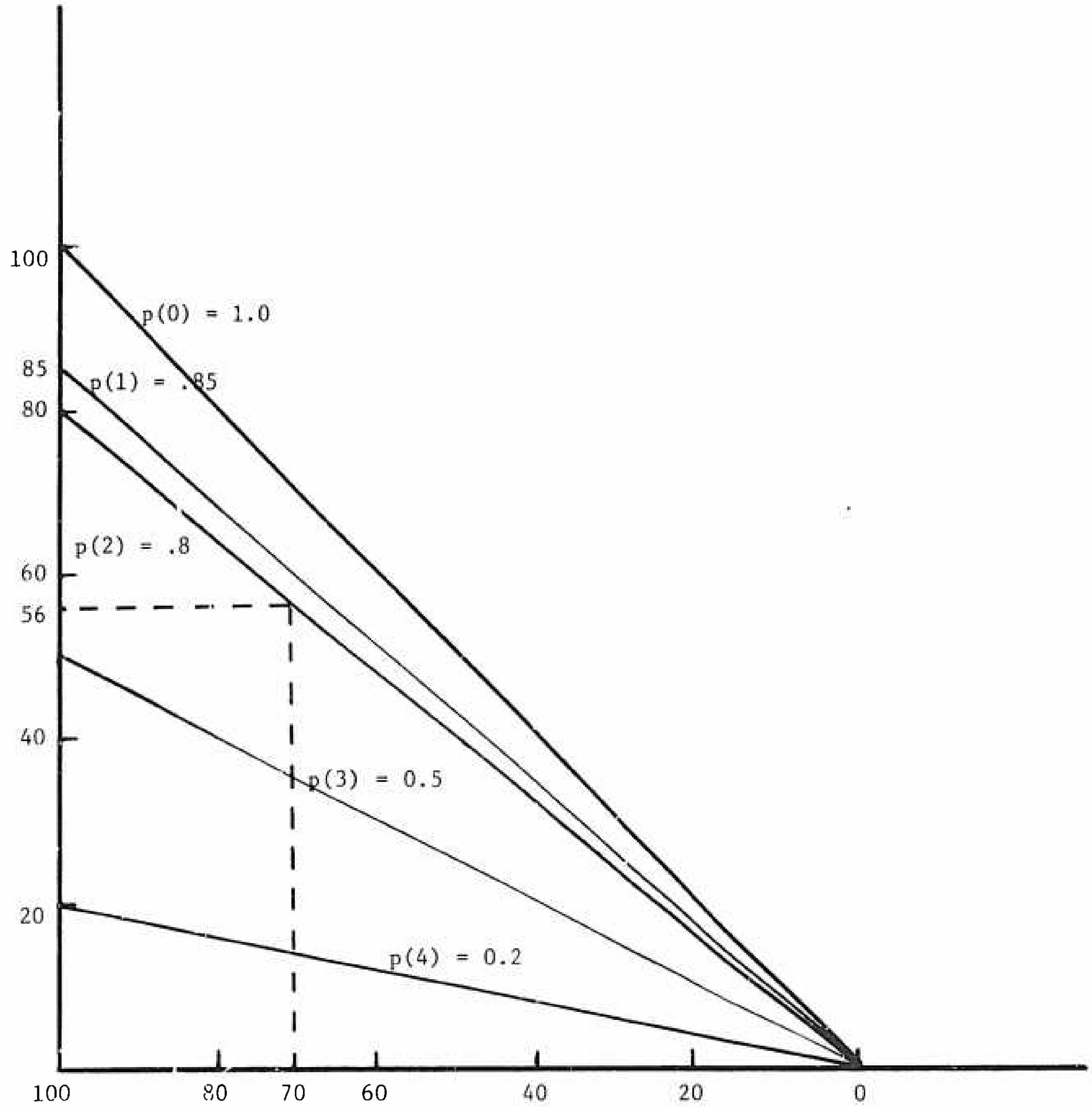


Figure III.1: Graphical Method of Calculating then we compute $p(2)g(t) = 0.8 \times 70 = 56$ by following the dotted line in Figure III.1.

Problem 7: Given the values of $g(t)$ below, use Figure III.1 to compute $p(u)g(t)$ for $u = 0,1,2,3,4$.

t	-7	-6	-5	-4	-3	-2	-1	0
g(t)	80	100	100	100	90	80	70	60

Figure III.2 shows how the stock at anytime is composed of groups according to the time period in which they joined the system. The number in each bar indicates the period in which they joined the system. In periods -3, -2, -1, and 0 we have five groups present since $M = 4$. The legacy of these past inputs at times 1, 2, 3, and 4 is also known. Notice the legacy at time t is made up from the inflow in period $0,-1,\ldots,t-M$.

Figure III.3 presents a third way in which the longitudinal flow process can be visualized. Reading across any row we have the size of the cohort as time proceeds. Reading down any column for $t \leq 0$ we have the contribution of each cohort to the system. If $t \geq 1$ we have the legacy of inputs in period $0,-1,\ldots$.

Example 6: The faculty of a university can be considered a one class system. The one chain assumption is valid if all appointments are made in the lowest ranks.

Example 7: The students at a two or four year college can be considered as a one class-one chain system particularly if almost all students enter as freshmen.

Example 8: The enlisted personnel in a skill category (rating) of the U.S. Navy can be treated as a one class-one chain system since all inflow into this system is from new Navy recruits.

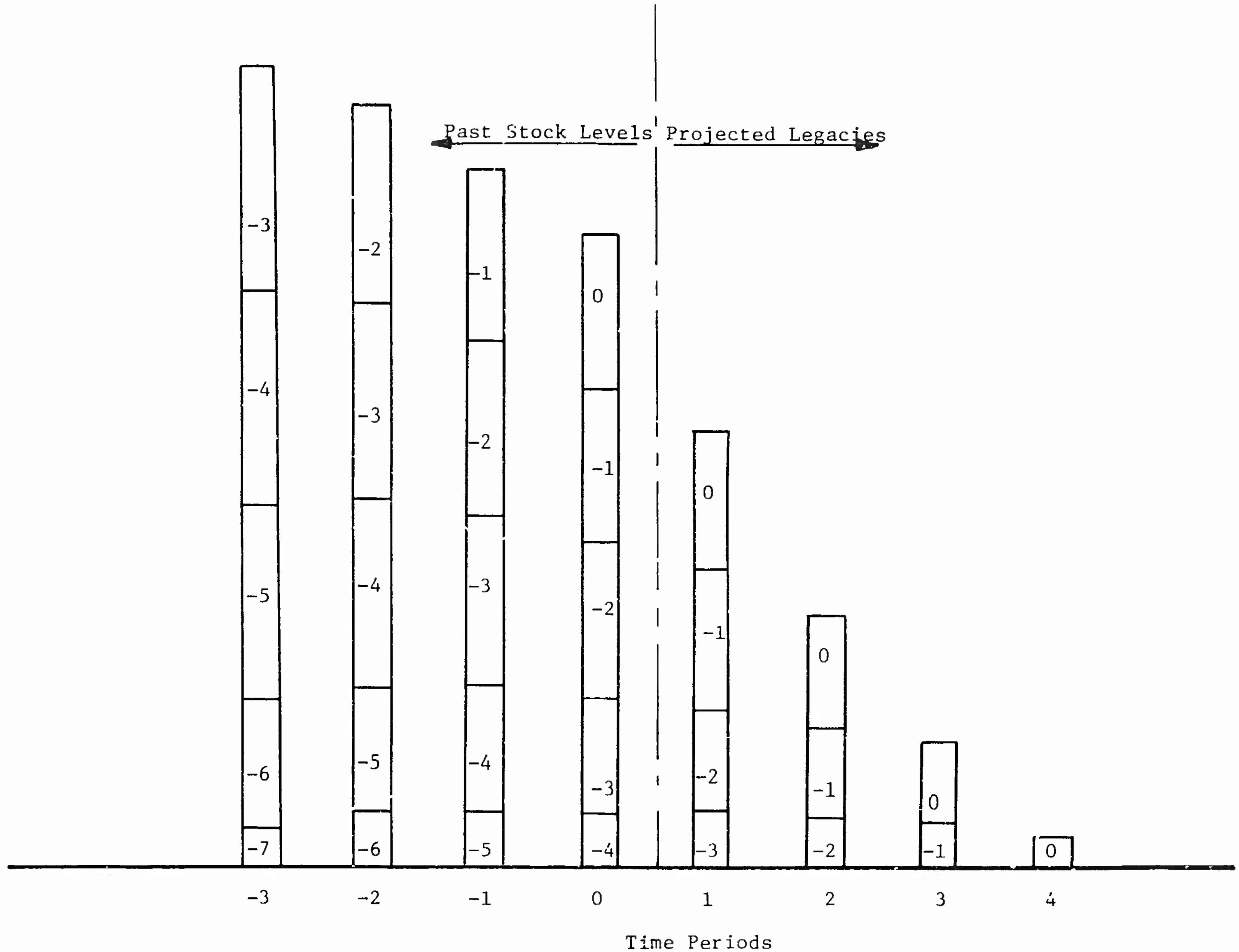


Figure III.2· Stocks at Each Period Viewed as a Superposition of Past Entering Groups.

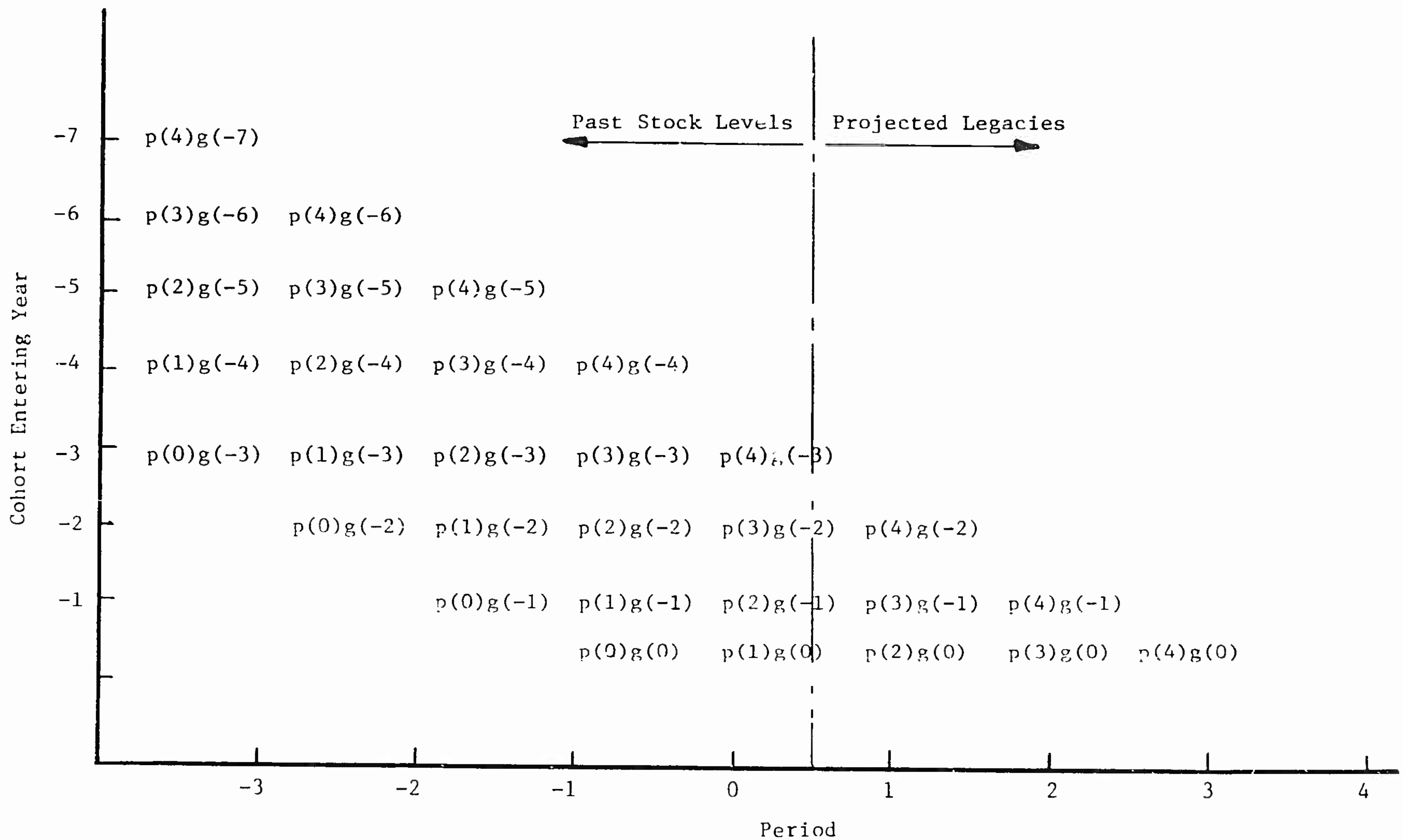


Figure III.3: Algebraic Representation of Stocks Seen in Figure III.2.

Example 9: The students at a four year college may either be admitted as freshmen or juniors. We can construct two 1 class (students) and 1 chain models that operate in parallel. The total number of students is thus the sum of the stocks in the two models. ∎

If we interpret the single chain and single class to be simply "still in the system," then $p(u)$ takes on a special meaning. If after having left a person cannot return to the system, then $p(u)$ must be non-increasing in u. If L represents the lifetime of an individual in the system then

$$p(u) = P[L > u] .$$

From this and a well known result in probability theory that $E[L] = \sum_{u=0}^{\infty} P[L > u]$, one can interpret the sum of the $p(u)$'s, i.e., $\sum_{u=0}^{M} p(u)$, as the average lifetime in the system of an individual. Also from (1), if the input in each period is equal to a constant g, then the stock at time t is given by $g\,E[L]$. This interpretation can be extended to the multiclass, multichain case, and this is done in the next section.

4. Equilibrium.

This section examines the longitudinal models at equilibrium. The most useful result is in the constant size system. In this case we find the data requirements for specifying a longitudinal model are greatly simplified and a more intuitive interpretation is given to the coefficients of the model. An analysis of geometric and arithmetic growth reaffirms the general principle that expansion allows for more flexibility in manpower systems while contraction restricts the range of possible decisions.

If $g(t)$ is a constant vector g then $s(t) = (\sum_{u=0}^{M} P(u))g$. Define $L = \sum_{u=0}^{M} P(u)$, an $N \times K$ matrix. The equilibrium cohort model is thus

$$s = Lg \quad (5)$$

In addition, we see that the coefficient ℓ_{ik} of L is the lifetime in class i of an individual in chain k. Thus an equilibrium chain can be specified by an N vector $\ell_k = [\ell_{1k}, \ell_{2k}, \ldots, \ell_{Nk}]$ where ℓ_{ik} is the number of times an individual on chain k will be counted in class i. Note that several nonstationary models, $(P(0), P(1), \ldots, P(M))$ lead to the same stationary model when these matrices add to the same matrix L.

Example 10: $(M = 3)$

	Length of Service u			
	0	1	2	3
Case 1, $p_{ik}(u)$	1	1/2	1/2	0
Case 2, $p_{ik}(u)$	1	0	0	1

In both cases $\ell_{ik} = 2$, however, in Case 2 the individual spends the first and last periods in class i. In Case 1 the individual spends the first and one half of the second and third periods in i.

With this interpretation of the ℓ_{ik} it is possible to write down equilibrium models directly without specifying the matrices P(u).

Example 11: Consider the three class, seven chain example presented in Example 4. It is obvious that

$$L = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 4 & 5 & 5 & 5 & 0 & 0 & 0 \\ 0 & 30 & 20 & 10 & 25 & 20 & 10 \\ 0 & 15 & 20 & 0 & 15 & 20 & 0 \end{bmatrix} \end{array} .$$

Problem 7: Calculate L for the systems described in Examples 2 and 3. Then, in the first case, calculate the equilibrium stock levels if the input is

$$g = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 8 & 1 & 2 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} .$$

If the input is changing geometrically in time $g(t) = \theta^t g$, then the stock levels will change at the same rate. From (4) we have $s(t) = \sum_{u=0}^{M} \theta^{t-u} P(u) g$. Let

(6) $$L(\theta) = \sum_{u=0}^{M} \theta^{-u} P(u) .$$

The model becomes

(7) $$\theta^t s = s(t) = \theta^t L(\theta) g .$$

Note that $L(1) = I$.

Example 12: (Continuation of 11)

If $\theta = 0.98$, then $L(\theta)$ is

$$\begin{bmatrix} 4.124 & 5.208 & 5.208 & 5.208 & 0 & 0 & 0 \\ 0 & 45.17 & 26.99 & 12.14 & 32.2 & 24.4 & 10.97 \\ 0 & 35.18 & 40.43 & 0 & 28.74 & 36.54 & 0 \end{bmatrix}$$

<u>Example 13:</u> For $\theta = 1.03$, $L(\theta)$ is

$$\begin{bmatrix} 3.829 & 4.717 & 4.717 & 4.717 & 0 & 0 & 0 \\ 0 & 17.41 & 13.22 & 7.579 & 17.94 & 15.32 & 8.786 \\ 0 & 4.37 & 7.319 & 0 & 5.873 & 8.484 & 0 \end{bmatrix}$$

■

An individual's view of the organization is determined by the input g and the matrices $P(0), P(1), \ldots, \ldots, P(M)$. However, the total organization is concerned with the matrix $L(\theta) = \sum_{u=0}^{M} \theta^{-u} P(u)$ and the input g. This discrepancy between the organization's view and the individual's view is extremely important. As we illustrate below it also seems to be sensitive to quite small changes in growth rates.

<u>Example 14:</u> (Continuation of 11)

Let the stationary input per period to each chain be

k	1	2	3	4	5	6	7
g_k	15	20	8	5	3	1	1

Then, using the same values of $L(\theta)$, the equilibrium $s = L(\theta)g$ is

	s_1	s_2	s_3
$\theta = 0.98$	234	1312	1150
$\theta = 1.00$	225	915	525
$\theta = 1.03$	213	570	172

It is a more meaningful comparison to contrast the number in each class with the number of active faculty, since the organization's budget and ability to generate retirement funds will most likely be closely tied to the number of active faculty.

Fraction in each class .

	Nontenured	Tenured	Retired
$\theta = 0.98$	.151	0.849	0.744
$\theta = 1.00$	.197	.803	0.461
$\theta = 1.03$	.272	.728	0.220

A small (3%) growth rate can make a significant difference over no growth and a very large difference over a 2% decay in input. Note first how larger values of θ, i.e. growth, shift the distribution of faculty toward the junior ranks and also keep the ratio of retired to working individuals low. There is a third advantage of growth. The 53 new appointments represented by g, are 3.4% of the size of the declining faculty, 4.7% of the size of the constant size faculty and 6.8% of the size of the growing faculty. The *percentage* of new faces in the growing faculty is twice as large as in the declining faculty. The reader should compare these results with those in Table II.5 of Chapter II to see that the longitudinal and cross-sectional models consistently lead to the same equilibrium behavior.

Example 15: Consider a university faculty with two chains. On chain one people spend 8 years in the nontenure ranks and 36 years in tenure ranks. The individuals on chain two spend 8 years in nontenure and then leave the system. If in each period we have 1 person enter chain one and 2 people enter chain two, then the equilibrium stock vector is $s = [24,36]$ which has 40% nontenured faculty. These data are summed up below.

	Chains 1	Chains 2
nontenure lifetime	8	8
tenure lifetime	36	0
flow	1	2

Now consider another university with 4% growth, i.e. $\theta = 1.04$. If the chain flows are organized as follows

	Chains 1	Chains 2
nontenure lifetime	6	6
tenure lifetime	38	0
flow	1.5	1.5

then the organization will retain 40% nontenure faculty. However, the prospect for an average appointee in the second university is much brighter: 50% of new appointments will eventually be promoted to tenure in 6 years. In the no growth case 33% attain tenure in 8 years.

The example above shows how growth gives the organization greater flexibility. The benefits of growth were passed on to the employees. Now suppose the organization is growing with $\theta = 1.04$, and the promotion rules implicit in the first university are followed; i.e., 8 years to a decision, and 1/3 are promoted. In this case the growing university will have 58.5% nontenure faculty. The benefits of growth have been assumed by the organization. The prospects for individuals in the growing university are the same as those in the constant size university.

As a third (intermediate) case, assume the university and employees share the benefits of growth. Let the chain flow be

	Chains 1	2
nontenure lifetime	7	7
tenure lifetime	37	0
flow	1.2	1.8

In this case 5/12 of the appointees are promoted in 7 years. The nontenure/ faculty ratio will be 0.488.

<u>Example 16</u>: Continuation of Example 15.

Four policies from example 15 are examined: 8 and 6 year nontenure periods with $\theta = 1$ and $\theta = 1.4$. In each case we wish to determine the equilibrium ratio of new appointments to total faculty size. If $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ is the appointment policy, then this ratio is (using equation (7))

$$\frac{eg}{eL(\theta)g} .$$

In Figure III.4 the new appointments, as a percentage of the total faculty size, are plotted against the percentage promoted to tenure for each of the four policies.

Points G and H on this graph represent a faculty with 40% nontenured. Point G corresponds to an 8 year nontenure period with a 33% promotion rate, while point H corresponds to a 6 year nontenure period with a 50% promotion rate. Clearly the employers are better off at point H. There is an additional

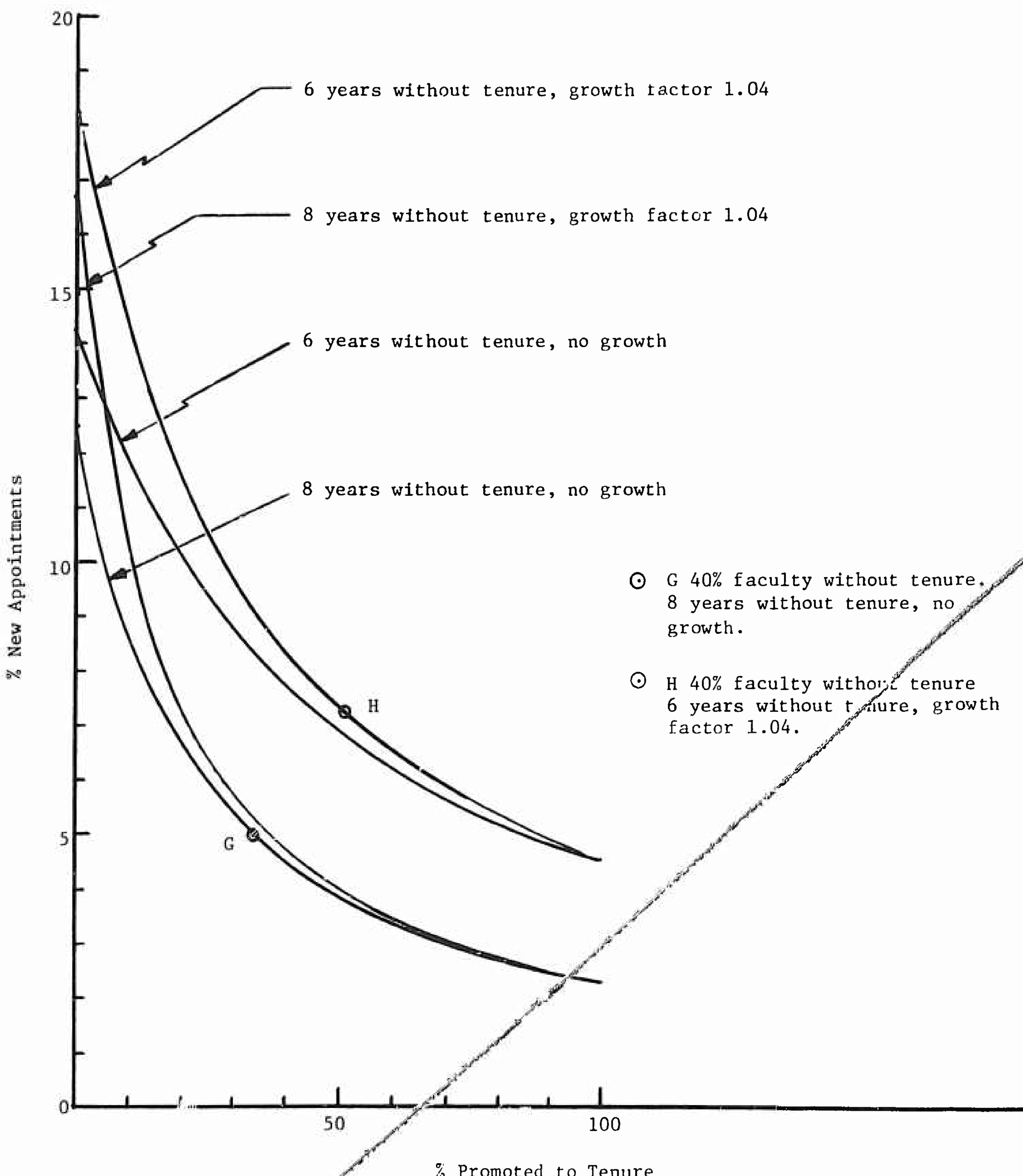


Figure III.4: New Appointments versus Promotion to Tenure benefit to the organization at H in the form of 7.2% new appointments per period. This should be contrasted with the 5% new appointments at point G.

Problem 8: If $g(t) = g + th$ for $t \geq 0$ ($g(t)$ grows arithmetically at rate h), then when $t \geq M$ show that

$$s(t) = tLh + Lg - \sum_{u=0}^{M} uP(u)h$$

Problem 9: (Based on Example 11)

Show that the second and fifth columns of $\sum_{u=0}^{M} uP(u)$ are given correctly in the table below

Column 2	Column 5
10	0
585	300
630	441

5. Probabilistic Interpretation of the Longitudinal Model.

To this point in this Chapter we have avoided discussing models which depend on the detailed movement of individuals. In this respect, sections 1-8 of Chapter II and sections 1-4 of this Chapter are similar. But if one wishes to describe or explain unpredictable variations in personnel flow one must somehow introduce randomness into the model. This can be done in a number of ways. The method described in this section follows that used in section 9 of Chapter II, and it allows us to use the longitudinal model already discussed.

Consider the path that an individual takes as he moves through the system. Assume he enters in period u on chain k. In what class will he be at time $(t+u)$? Let $p_{ik}(u)$ be the probability that this individual is in class i at $(t+u)$. Then $p_k(u) = [p_{1k}(u), p_{2k}(u), \ldots, p_{Nk}(u)]$ is a vector of probabilities which must be non-negative and sum to a number no bigger than 1. Note that $ep_k(u) = \sum_{i=1}^{N} p_{ik}(u)$ is the probability an individual is still in the system u periods after entrance. Since, by definition, once a person leaves the system he cannot return, $ep_k(u)$ must be nonincreasing in u, and $ep_k(0) = 1$.

Let $S_i(t;u)$ be the number of people in class i at t who entered the system in period $(t-u)$; this is now a random variable. Recall that $g_k(t)$ is the number of people who enter the system on chain k at any time t. Then

(8) $$E[S_i(t;u)] = \sum_{k=1}^{K} p_{ik}(u) g_k(t-u) .$$

Also, if $S_i(t)$ is the total in class i at time t, then

$$E[S_i(t) | S_i(t;u), u=0,1,\ldots,M] = \sum_{u=0}^{M} S_i(t;u) .$$

By unconditioning and using (8) we have

(9) $$E[S_i(t)] = \sum_{u=0}^{M} \sum_{k=1}^{K} p_{ik}(u) g_k(t-u) .$$

These are precisely the same equations as (1) and (2) in section 2, and if $s(t)$ represents the vector of expected values $s_i(t) = E[S_i(t)]$, equation (4) holds. Thus our probabilistic interpretation of the fractions $p_{ik}(u)$ is consistent with the earlier model.

This probabilistic model has a simple and logical interpretation in the one class, one chain case of section 3. Let Λ be the (random) lifetime of an individual in the system. Then $\Lambda > u$ if an only if an individual stays in the system at least u periods. Thus

$$p(u) = \text{Prob } [\Lambda > u] .$$

The expected lifetime (in the system) of an individual is

$$E[\Lambda] = \sum_{u=0}^{M} p(u) .$$

From equation (9) above, if $g(t-u)$ is the input flow in period $(t-u)$ the expected stock level at time t is

$$E[S(t)] = s(t) = \sum_{u=0}^{M} p(u)g(t-u) .$$

In the equilibrium case where $g(t) = g$ for all t, then

(10) $$s = E[\Lambda]g \quad \text{for all } t .$$

Equation (10) simply says that the expected stock levels are given by the input per period time the expected number of periods an individual stays in the system.

<u>Problem 10</u>: Show that element ℓ_{ik} of the matrix L in section 4 can be interpreted as the expected lifetime in class i of an individual on chain k. ∎

In a single class model, or in an aggregate model where $p_k(u) = \sum_{i=1}^{N} p_{ik}(u)$, it is possible to determine the variance of system lifetime for individuals on any chain. If the random variable Λ_k represents system life on chain k, then

$$\text{Prob}\,[\Lambda_k > u] = p_k(u)\,, \quad u = 0,1,2,\ldots,M\,.$$

It follows from this that

$$E[\Lambda_k] = \sum_{u=0}^{M} p_k(u)\,, \quad \text{and} \quad E[\Lambda_k^2] = \sum_{u=0}^{M} (2u+1)p_k(u)\,.$$

These imply that the variance in system lifetime on chain k is

$$2\sum_{u=0}^{M} u \sum_{i=1}^{N} p_{ik}(u) + \sum_{u=0}^{M}\sum_{i=1}^{N} p_{ik}(u) - \left(\sum_{u=0}^{M}\sum_{i=1}^{N} p_{ik}(u)\right)^2 .$$

<u>Example 17</u>: Suppose the matrices

$$P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad P(1) = \begin{bmatrix} .9 & 0 \\ 0 & .95 \end{bmatrix},$$

$$P(2) = \begin{bmatrix} .1 & 0 \\ .65 & .2 \end{bmatrix}, \qquad P(3) = \begin{bmatrix} 0 & 0 \\ .65 & .05 \end{bmatrix},$$

$$P(4) = \begin{bmatrix} 0 & 0 \\ .2 & 0 \end{bmatrix}, \qquad P(5) = \begin{bmatrix} 0 & 0 \\ .05 & 0 \end{bmatrix},$$

describe the flow in a 2 class (lower and upper division) undergraduate college, and the two chains are admission to lower and upper division. Using this data, the mean and variance of system life (years in college) in each chain is

	Mean	Variance	Standard Div
Chain 1	3.55	1.745	1.29
Chain 2	2.2	0.36	0.6

where the standard deviation is the square root of the variance.

6. A Student Enrollment Forecasting Model.

This section presents an actual example of real cohort flow data for undergraduate students entering the Berkeley campus of the University of California in the fall (beginning of the academic year). After we present and analyze the data we discuss several institutional and behavioral problems that made it difficult to implement these models in a straight forward manner. Throughout the section the notation FXX refers to the fall quarter of year XX. Thus F69 refers to the fall quarter in 1969.

We assume there are four classes of manpower: freshmen, sophomore, junior, and senior; and four chains: those entering as freshmen, sophomores, juniors or seniors. The time periods are taken to be 1 year and the entry data is given in Table III.2. Although students enter in other quarters in the academic year, by far the majority enter in the fall, and we concentrate on these cohorts. The matrices $P(u)$ for $u = 0,1,\ldots,6$ are given in Table III.3.

Given the data above we can calculate the stocks in F69 and the legacy of F63-F69 entrants in F70-F74. These results are shown in Table III.4.

If we wish to keep the stock level of fall entrants at a constant level, then it is possible to calculate the new admissions necessary in F70-F74 in order to maintain F69 stock levels. These are shown in Table III.5.

The steady state admission levels can be found by solving $s = Lg$, where

$$L = \begin{bmatrix} 1.283 & 0 & 0 & 0 \\ 0.835 & 1.137 & 0 & 0 \\ 0.790 & 0.842 & 1.413 & 0 \\ 0.525 & 0.554 & .741 & 1.501 \end{bmatrix}$$

The system is obviously quite close to equilibrium in F69. (See Table III.5).

As we remarked earlier, this data treats only those cohorts that entered in fall. Although this is the largest source of new students a sizeable number enter g(t)

	F63	F64	F65	F66	F67	F68	F69
Freshman	1883	2239	3303	3053	2579	3427	3620
Sophomores	258	542	843	733	390	602	728
Juniors	817	1366	1662	1418	1042	1442	1569
Seniors	48	124	175	205	125	202	199

Table III.2. Student Enrollment Input in Fall Quarters

	F69	F70	F71	F72	F73	F74	F75
Freshmen	4570	1010	96	55	32	21	11
Sophomores	3470	3040	906	147	50	22	12
Juniors	4780	3668	3120	1200	176	52	18
Seniors	3160	2980	2410	2020	753	187	64

Table III.4. Legacies of Entrant up to F69

	F70	F71	F72	F73	Steady State
Freshman	3560	3570	3569	3560	3560
Sophomores	431	437	439	439	440
Juniors	1110	1060	1140	1130	1130
Seniors	183	118	160	140	143

Table III.5. Future Fall Admissions Required to Maintain F69 Stock Levels

u	P(u)			
0	1	0	0	0
	0	1	0	0
	0	0	1	0
	0	0	0	1
1	.254			
	.584	.118		
	.009	.622	.265	
		.039	.493	.395
2	.012			
	.210	.013		
	.454	.189	.138	
	.009	.337	.192	.046
3	.007			
	.027	.003		
	.281	.022	.033	
	.318	.130	.042	.029
4	.004			
	.008	.003		
	.033	.005	.005	
	.152	.031	.008	.016
5	.003			
	.003			
	.009	.004	.001	
	.031	.010	.003	.015
6	.003			
	.003			
	.004	.001	.001	
	.015	.007	.003	0

Table III.3: The Matrices P(u) Up to Six Years in winter, spring and summer. If the yearly accounting point is the fall, then the total inventory in, say, F75 would be made of winter, spring and summer cohorts entering in calendar years 70 through 75. The matrices P(u) that apply to fall cohorts would not be applicable to cohorts that enter in other quarters, thus the data requirements are roughly four times as large as is shown in Tables III.2 and III.3.

We conclude this section with a discussion of the institutional difficulties involved in using the longitudinal model for the Berkeley campus. In F66 this campus switched from a semester system to a quarter system with year-round, 4 quarter operation. This caused problems in determining how to use data collected from a semester system, to predict enrollment in a quarter system.

The Berkeley campus operated on a semester system until the fall of 1966. It is reasonable to assume that students entering in the fall or spring would behave similarly under a semester or quarter system. However, the first winter and summer quarters ever to be offered were in 1967. The fractions of students who entered in these quarters and were enrolled in F69 are now applied to cohorts entering in the winter and summer of 1968 when forecasting for F70. It would certainly be expected that some students from the winter and summer quarters of 1967 would also be enrolled in F70, but how many? We have no fractions for winter or summer 1966 since there were no such quarters. These fractions have to be estimated in some reasonable way. An average was taken of the fractions from F65 and Sp66, (here W, Sp, Su, refer to Winter, Spring and Summer of the given year) for the winter quarter and from Sp66 and F66 from the summer quarter.

Another problem arose when, in 1970, the summer quarter was discontinued. This was in deciding what fractions to apply to the students who entered in Su69. These students had available only the winter and spring quarters of 1970 before

F70. The students who entered in Su68 could attend winter, spring and summer quarters before F69. It was felt that larger fractions of Su69 entrants would attend the fall of 1970 than the fractions of Su68 students attending F69. But how much larger? To estimate attendance of Su69 entrants it was assumed that the same fraction of these would attend F69 as did Su68 entrants in F68. Of these that enrolled in F69, they were then assumed to behave in the same way as new entrants in F69.

Besides these particular and rather confusing problems, caused by institutional operational changes, the stationarity of most of the fractions since the start of the summer quarter can be questioned. With such a major change in campus operations one might expect that it would take a number of years for the system to settle down, even if there were no changes between 3-quarter and 4-quarter operations. In light of this observation the results in Table III.5 are somewhat surprising.

7. A University Planning Model

This section describes a university planning model that is based on an equilibrium manpower flow model. The model relates the technology of the institution to the flows of students and faculty. The student faculty flow process is central to the model. We have a system containing nineteen classes of manpower shown in Table III.6. Notice that all abbreviations for stocks start with the letter S.

Class	Abbreviation	Description
1	SLA	Lower division Admission
2	SLS	Lower division Student
3	SLD	Lower division Dropout
4	SUA	Upper division Admission
5	SUS	Upper division Student
6	SUG	Upper division Graduate
7	SUD	Upper division Dropout
8	SMA	Masters Admission
9	SMS	Masters Student
10	SMT	Masters Teaching Assistant
11	SMG	Masters Graduate
12	SMD	Masters Dropout
13	SDA	Doctoral Admission
14	SDS	Doctoral Student
15	SDT	Doctoral Teaching Assistant
16	SDG	Doctoral Graduate
17	SDD	Doctoral Dropout
18	SFN	Faculty Nontenure
19	SFT	Faculty Tenure

Table III.6. Stock Classification Scheme

The model makes a distinction between students who are teaching assistants and students who are not. Thus the entire class of masters program students is actually the sum of classes 9 and 10.

This is an equilibrium model, thus we have the advantage of being able to rearrange the actual schedule of persons in a cohort in order to make a model that is easy to deal with. We illustrate this point with three examples.

Example 18: Harry enters the lower division in September 1975. After one year as a lower division student, Harry drops out.

Suppose our account period is one year, and the accounting date is April 1. Then Harry's history is summarized below:

Time	Class
1975	SLA
1976	SLS
1977	SLD

Example 19: Tom enters lower division in September 1976 and graduates from upper division in June 1980. Tom's history is:

Time	Class
1976	SLA
1977	SLS
1978	SLS
1979	SUS
1980	SUS
1981	SUG

Example 20: Dick is admitted to the Ph.D. program and enrolls in September 1976. Dick spends two years as a student. In one of those years he is a half-time teaching assistant. After two years Dick drops out of the Ph.D. program, takes a masters degree and leaves the University. Dick's history is:

Time	Class
1976	SDA
1977	SDS
1978	SDS (1/2) and SDT (1/2)
1979	SMG

We assume our equilibrium flow model has 10 chains. The chains and description are given in Table III.7.

Chain	Abbreviation	Description
1	FLD	Lower division Dropouts
2	FLG	Lower division Graduates
3	FUD	Upper division Dropouts
4	FUG	Upper division Graduates
5	FMD	Masters Dropouts
6	FMG	Masters Graduates
7	FDD	Doctors Dropouts
8	FDG	Doctors Graduates
9	FFN	Faculty Nontenure
10	FFT	Faculty Tenure

Table III.7. Chain Definitions and Descriptions

We assume that all lower division graduates enter the upper division, and that a certain fraction, see example 20, of the doctoral dropouts receive a masters degree. In addition, we assume that a certain fraction of the masters graduate chain enter the Ph.D. program.

The L matrix is given in Table III.8, where the entry in row i, column j, gives the number of time periods a person on chain j spends in class i. For example consider chain 3. The chain is FUD (upper division dropouts). The students on this chain spend one year in SUA (upper division admission), one year in SUS (upper division student) and one year in SUD (upper division dropout).

Stock Class \ Flow Chain	1	2	3	4	5	6	7	8	9	10
1	1	1								
2	1.2	2								
3	1									
4			1	1						
5		2	1	2						
6		0.8		1						
7		0.2	1							
8					1	1				
9					1.1	1.8				
10						0.05				
11						1	0.2			
12					1					
13							1	1		
14						0.16	1.5	3.2		
15						0.05	0.25	1		
16						0.05		1		
17							0.8			
18									5	
19									7	25

Table III.8. The L matrix for the University Planning Model.

Nineteen conservation relations between the nineteen variables s, and the 10 variables g, are given by

(11) $$s = Lg .$$

Example 21: Note that one simple use of the model is to choose the flows g, and calculate the stocks s. Three such calculations are presented below.

For the first calculation, let the chain flows be given by:

Chain	1	2	3	4	5	6	7	8	9	10
g	200	600	100	400	75	200	50	200	40	5

.

Using L in Table III.8, the resulting equilibrium stocks are:

Lower	Upper	Masters	Ph.D.	
800	500	275	250	admission
1440	2100	442.5	747	students
200	220	75	40	dropouts
-	880	210	210	graduates
-	-	10	222.5	teaching assistants

Nontenure	Tenure	
200	405	faculty

To see if these figures are reasonable we can check some meaningful ratios. First, the ratio of teaching assistants to undergraduates, $(s_{10} + s_{15})/(s_2 + s_5)$ is 0.066. The ratio of undergraduates to total students 0.71, the ratio of upper division to undergraduates is 0.59, and the ratio of students to faculty is 8.2.

These ratios are reasonable except the student/faculty ratio. Currently the input flow of faculty (g_9 and g_{10}) is 40 into non-tenure and 5 into tenure. For the second calculation we change g_9 and g_{10} to be 15 and 2. The same student results are obtained, but the faculty becomes:

Nontenure	Tenure
75	155

and the student faculty ratio is 21.6.

Finally, for the third calculation, we shorten the lifetime of lower division dropouts in student status from 1.2 to 0.6 years, we lower the stock of lower division students to 1320. Thus if we change g_1 and g_2 to new values 218 and 654, we would have the same student stocks.

Example 22: An alternate use of the equilibrium model is to specify the stocks s and then calculate the flows g necessary to maintain these stocks. In general, there does not exist a g such that $Lg = s$. However, we can calculate the g that gives stocks closest to s, in the sense of minimizing the inner product $(s - Lg)'(s - Lg)$. Here ' indicates the transpose operation. The g which minimizes this function is denoted g^*, and

$$L'Lg^* = L's , \tag{12}$$

where L' is the transpose of L. If a weighted measure is desired, then define W as a 19 x 19 diagonal matrix. Then (12) will still hold with s replaced by Ws and L replaced by WL, g^* minimizes $(s - Lg)'W'W(s - Lg)$. Two numerical calculations are shown below. Suppose the desired stocks are given by

Lower	Upper	Masters	Ph.D.	
1000	500	200	100	admissions
1700	3400	350	350	students
400	400	50	80	dropouts
600	2500	150	220	graduates
-	-	25	50	teacning assistant
		Nontenure	Tenure	
		85	240	faculty

The best flow approximations, g^*, (when W is an identity matrix) give stocks

Lower	Upper	Masters	Ph.D.	
1010	847	209	115	admissions
1850	3586	339	362	students
210	-133	52	15	dropouts
800	1780	160	104	graduates
-	-	8	109	teaching assistant
		Nontenure	Tenure	
		85	240	faculty

Suppose that on seeing the resulting input flows we decide to revise our desired stock plan to

Lower	Upper	Masters	Ph.D.	
1000	500	200	100	admissions
1750	2000	350	350	students
300	400	50	30	dropouts
700	1200	150	120	graduates
-	-	10	100	teaching assistant
		Nontenure	Tenure	
		85	240	faculty

Now the best flow approximations, g^*, give stocks

Lower	Upper	Masters	Ph.D.	
996	480	209	110	admissions
1726	2165	340	351	students
333	254	52	12	dropouts
663	889	160	102	graduates
-	-	8	106	teaching assistants
		Nontenure	Tenure	
		85	240	faculty

Note that this approximation is relatively close to the desired one. The largest error appears in the undergraduate degree category.

We can also use the basic flow model (10) in conjunction with other restrictions on the education process. We list several possibilities.

(i) Let λ_1 be the desired total student body size. Then

$$s_2 + s_5 + s_9 + s_{10} + s_{14} + s_{15} = \lambda_1 .$$

(ii) Let λ_2 be the desired total faculty size. Then

$$s_{18} + s_{19} = \lambda_2 .$$

(iii) Let α_1 be the desired ratio of undergraduate students to teaching assistants. Then

$$s_{10} + s_{15} = \alpha_1 (s_2 + s_5) .$$

(iv) Let α_2 and α_3 be the desired ratios of nontenure and tenure faculty to student. Then

$$s_{18} = \alpha_2 (s_2 + s_5 + s_9 + s_{10} + s_{14} + s_{15}) ,$$

$$s_{19} = \alpha_3 (s_2 + s_5 + s_9 + s_{10} + s_{14} + s_{15}) .$$

(v) Different categories of students present different workloads to faculty, and it is common to weigh the different categories to correct for this anomaly. Let w_1, w_2, w_3 and w_4 be the weights assigned Lower division, Upper division, Masters and Ph.D. students respectively, and let α_4 be the desired ratio of total faculty to weighted students. Then

$$(s_{18} + s_{19}) = \alpha_4 \, [w_1 s_2 + w_2 s_5 + w_3(s_9 + s_{10}) + w_4(s_{14} + s_{15})].$$

(vi) Let λ_3, λ_4 and λ_5 be the desired annual output of bachelor, masters and doctors degrees. Then

$$s_6 = \lambda_3, \; s_{11} = \lambda_4, \; s_{16} = \lambda_5 .$$

(vii) Let $\alpha_5, \alpha_6, \alpha_7, \alpha_8$ be the desired fractions of lower division, upper division, masters, and doctoral students respectively who dropout. Then

$$(g_1 + g_2)\alpha_s = g_1 ,$$

$$(g_3 + g_4)\alpha_6 = g_3 ,$$

$$(g_5 + g_6)\alpha_7 = g_5 ,$$

$$(g_7 + g_8)\alpha_8 = g_7 .$$

(viii) Let α_9 be the desired ratio of lower division to total undergraduate students. Then

$$s_2 = \alpha_9 \, (s_2 + s_5) .$$

Let α_{10} be the desired ratio of undergraduates to total students. Then

$$(s_2 + s_5) = \alpha_{10}(s_2 + s_5 + s_9 + s_{10} + s_{14} + s_{15}) .$$

Let α_{11} be the desired ratio of nontenured to total faculty. Then

$$s_{18} = \alpha_{11} \, (s_{18} + s_{19}) .$$

We see that there are a great many possibilities and that all the relations are linear in s and g. Suppose some restrictions are selected from the list. This leads to a system of equations

$$(13) \qquad Is - Lg = 0 ,$$

$$Hs + Fg = b ,$$

where H and F are coefficient matrices for the restrictions in question. These equations may have one, zero or an infinite number of solutions, and we are interested, in finding a single solution of (13). In this case we would try to build a 29 x 29 system of equations. If there is an inconsistancy in the requirements put on the system or if some of the parameters (α,λ) are unrealistic, then we will obtain unrealistic solutions of (13); for example, some values of stocks s and the flows g might be negative.

Example 23: We specify the following parameters λ, α, and w.

$\lambda_1 = 26500$	$\alpha_1 = 0.06$	$w_1 = 1$
$\lambda_3 = 4251$	$\alpha_4 = 1/29$	$w_2 = 1.5$
$\lambda_4 = 2370$	$\alpha_5 = 0.3$	$w_3 = 2.5$
$\lambda_5 = 634$	$\alpha_6 = 0.3$	$w_4 = 3.5$
	$\alpha_7 = 0.2$	
	$\alpha_{11} = 0.35$	

Under these conditions we obtain stocks

Lower	Upper	Masters	Ph.D.	
3840	3000	2672	1700	admissions
6761	10478	4436	3789	students
1172	1433	534	421	dropouts
-	4251	2373	634	graduates
-	-	106	927	teaching assistant
		Nontenure	Tenure	
		607	1128	faculty

The faculty input flows are 121 to nontenure, 11 to tenure.

If we change the weights used in the student faculty equation to

$$w_1 = 1,\ w_2 = 1,\ w_3 = 1.5,\ w_4 = 2$$

then the student stocks and flows remain unchanged; however the faculty stocks and flows become

Flows: Nontenure	Flows: Tenure	Stocks: Nontenure	Stocks: Tenure	
80	7	404	750	faculty

Example 24: Let λ_1, α_1, α_4, α_5, α_6, α_7, α_{11}, and w have the values originally presented in example 23. In addition let

$$\alpha_9 = 0.4\ ,\quad \alpha_{10} = 0.681\ ,\quad \text{and}\quad \alpha_8 = 0.3\ .$$

We will not specify degree output.

We obtain the stocks

Lower	Upper	Masters	Ph.D.	
4106	2995	2375	1153	admissions
7226	10839	3945	3406	students
1232	1473	475	277	dropouts
-	4395	1970	902	graduates
-	-	95	989	teaching assistant
		Nontenure	Tenure	
		397	738	faculty

If the lifetime of dropouts is shortened, we observe an increase in admissions.

Let

$$\ell_{21} = 0.6,\ \ell_{5,3} = 0.7,\ \ell_{9,5} = 0.7,\ \ell_{14,7} = 1.0\ .$$

Then we obtain the stocks

Lower	Upper	Masters	Ph.D.	
4573	2755	2628	1127	admissions
7226	10839	4152	3200	students
1372	1467	526	271	dropouts
-	4490	2170	894	graduates
-	-	105	979	teaching assistant
		Nontenure	Tenure	
		396	735	faculty

Example 25: Consider a university which is currently operating with stocks

Lower	Upper	Masters	Ph.D.	
4261	3108	2466	1197	admissions
7498	11248	4094	3535	students
1278	1529	493	287	dropouts
-	-	2045	936	graduates
-	-	99	1026	teaching assistants
		Nontenure	Tenure	
		412	766	faculty

and faculty flows of FFN 82, and FFT 8. The current constraints maintain 40% of all undergraduates in lower division.

We relax this constraint that 40% of undergraduates are in lower division and instead set $s_2 = 0$. The university then runs without a lower division, and the stocks become

Lower	Upper	Masters	Ph.D.	
-	11027	2466	1197	admissions
-	18746	4094	3535	students
-	3308	493	287	dropouts
-	-	2045	936	graduates
-	-	99	1026	teaching assistant
		Nontenure	Tenure	
		412	766	faculty

8. Applications of the One Class, One Chain Model.

This section describes applications of the single class single chain model presented in Section 3. A flexible package of interactive computer programs based on that model has been developed and used by manpower planners in the Navy and Marine Corps. This section describes a wide range of applications for these models. We assume that the organizations can be broken down into separate single class systems. For example, the enlisted force of the U.S. Navy can be classified by skill rating. There are approximately 90 of these skill ratings and, with the exception of recruits, each enlisted person is identified with a skill rating. In general, the models in this paper are used by treating each skill rating independently. However, we shall indicate how interactions between categories can be handled. These must frequently involve the transfer of either responsibility, (jobs, assignments) or people between the different categories. A second organization we shall examine is a particular subset of the Navy - the group of Navy captains. Within this group we can classify individuals according to year of entry in the Navy. Thus we partition the group of Navy captains into approximately 10 subgroups according to year of entry.

The single class, single chain model is extremely flexible and leads to simple calculations. In special cases when we are sure that the model's assumptions are not quite correct, the flexibility of the simple model can usually be used to modify the assumptions.

We first discuss the data requirements of our model, and then show several examples.

Recall that the index u measures periods of completed service or length of service (LOS), and that $p(u)$ is a survivor fraction, the fraction of those who entered u periods ago, and are still in the organization.

(i) Data

For each of the separate categories of manpower we need three blocks of data: the current stocks by length of service s(o;u), the future requirements z(t) at some time $t > 0$ and the survivor fractions p(u). In the motivating example of the 90 skill category Navy enlisted force with a five year planning period, we would require 6,030 items of data. For each skill category $M = 30$. Thus s(0;u) and p(u) together contain 62 elements. In addition we must know z(t) for $t = 1,2,\ldots,5$; this gives 67 elements for each category or 67 x 90 = 6,030 in all.

We shall, in general, only consider one skill category at any time so the variables s, p, and z will *not* be indexed to indicate to which category they apply.

The number s(o;u) and z(t) are reasonably easy to obtain with some accuracy. The difficult problem is determining the survivor fractions, p(u). The problem of estimating p(u) from past data will be treated in Chapter 7.

Example 26: In what follows we present several numerical examples. Many of these will be based on the illustrative data shown below. We indicate the current stocks by length of service (LOS), survivor fractions, and future requirements for 3 skill ratings; SM - signalman, QM - quartermaster, and BM - boatswain's mate, in the U.S. Navy

Current Stocks by LOS, s(o;u)

u	0	1	2	3	4	5	6
SM	2000	2200	1700	800	600	225	200
QM	1200	1600	1400	1200	600	150	300
BM	800	640	800	960	600	600	600

Survivor Fractions p(u)

u	0	1	2	3	4	5	6
SM	1.0	0.95	0.85	0.4	0.2	0.15	0.1
QM	1.0	0.90	0.90	0.6	0.3	0.1	0.1
BM	1.0	0.8	0.8	0.8	0.6	0.6	0.5

Requirements z(t).

t	1	2	3	4	5
SM	6000	5500	5000	4500	4500
QM	6500	7000	7500	8000	8000
BM	4700	4700	4700	4700	4700

Example 27: The manpower category Navy Captain can be considered as an aggregate of 10 manpower classifications, Navy Captains by year of entry in the Navy. Each separate year of entry group will have a known inventory in the rank captain--and it is possible to break down that inventory by length of service in rank captain. Some typical stocks are shown below.

Entry year	Length of Service as Captain 0	1	2	3	4	5	6	7	8	9	10
1942											2
1943									35	150	4
1944							14	30	170	12	
1945							18	225	10		
1946						14	200	20			
1947					11	185	10				
1948				17	120	19					
1949				165	20						
1950		22	190	44							
1951	32	209	35								
1952	215	18	5								

The survivor fractions are assumed to be the same for each entry year group.

Survivor Fractions

u	0	1	2	3	4	5	6	7	8	9	10
p(u)	1.0	0.985	0.97	0.956	0.941	0.927	0.881	0.749	0.635	0.225	0.051

The future requirements are for the aggregate of all entry year groups.

Requirements z(t)

t	1	2	3	4	5
Captains	2000	1800	1700	1600	1600

In general the gross requirements data is not actively stored. It is more convenient to calculate the legacy of the current manpower stock and to store net requirements data.

(ii) Future Legacies

Our first application of this model is to calculate the future legacy of our current stock of manpower. This is accomplished by solving

$$\ell(t) = \sum_{u=t}^{M} p(u)g(t-u).$$

The values of $g(t-u)$, the input flows in period $(t-u)$, are not explicitly known. However, $s(o;j) = p(j)g(-j)$ for $j \geqq 0$. Thus

$$\ell(t) = \begin{cases} \sum_{j=0}^{M-t} p(t+j)g(-j) = \sum_{j=0}^{M-t} \frac{p(t+j)}{p(j)} s(o;j), & \text{if } t \le M, \\ 0 & \text{if } t > M. \end{cases}$$

This calculation is in terms of the required data $s(o;u)$ and $p(u)$.

Example 28: The future legacies of the three enlisted ratings are given by:

t	1	2	3	4	5
SM	5668	3626	1763	947	531
QM	4563	3013	1608	693	297
BM	3940	3180	2360	1460	880

Example 29: The future legacies of the Captains are given by:

t	1	2	3	4	5
1942	0				
1943	46	3			
1944	100	32	6	1.0	
1945	210	81	20	1.0	
1946	200	161	62	14.0	1
1947	195	166	138	53	13
1948	153	144	124	99	41
1949	182	178	167	142	114
1950	252	248	243	227	195
1951	271	267	263	258	243
1952	234	230	227	223	219
Total	1845	1515	1252	1021	827

(iii) Net Requirements

It is only necessary to compute future legacies once and then store net requirements. Let $y(t) = z(t) - \ell(t)$ be the net requirements.

Example 30: Four our three enlisted ratings the net requirements are:

t	1	2	3	4	5
SM	331	1873	3236	3552	3968
QM	1936	3986	5891	7306	7702
BM	760	1520	2320	3240	3820

Example 31: For the Navy Captains the net requirements are:

t	1	2	3	4	5
Capt.	154	284	447	578	772

Problem 11: Assume there is a lower bound $\bar{g}$ on accessions. Show that net requirements are given by

$$y(t) = z(t) - \ell(t) - \left(\sum_{u=0}^{t} p(u)\right)\bar{g} .$$

Example 32: Assume that lower bounds of 700, 1000, 500 are imposed on the ratings SM, QM, and BM. The net requirements become:

t	1	2	3	4	5
SM	-368	508	1276	1312	1588
QM	936	2086	3091	3906	4002
BM	260	620	1040	1640	1920

Note that a negative entry implies that the legacy plus the future guaranteed accessions will more than satisfy requirements.

Example 33: If we assume a lower bound of 150 captains per year then the net requirements for captains become

t	1	2	3	4	5
Capt.	4	-13	4	-8	44

∎

(iv) Future Accessions

It is straightforward to calculate future accessions necessary to meet future requirements. If $y(t)$ represents net requirements and $g(t)$ accessions (with no lower bound on accessions), then

$$p(0)g(1) = y(1)$$

(14) $$p(1)g(1) + p(0)g(2) = y(2)$$

$$p(j)g(1) + \qquad + p(0)g(j) = y(j) .$$

In general,

$$\sum_{j=1}^{t} p(t-j)g(j) = y(t) , \quad t = 1,2,\ldots,$$

or

$$\sum_{u=0}^{t-1} p(u)g(t-u) = y(t) , \quad t = 1,2,\ldots, .$$

Example 34: The future accessions that exactly meet requirements for the three skill categories are:

Future Accessions g(t) .

t	1	2	3	4	5	∞
SM	331	1558	1474	694	1365	1233
QM	1936	2243	2128	2209	1870	2051
BM	760	912	1002	1252	1012	959

The final column gives the equilibrium accessions if requirements remain at the 5th period level.

Example 35: For Navy Captains future accessions are:

Future Accessions g(t) .

t	1	2	3	4	5	∞
Capt.	154	132	167	137	202	192

The accession level that meets requirements exactly can be negative. Typically this occurs when requirements are decreasing more rapidly than can be accounted for by natural attrition from the system. To find a simple accession policy that is nonnegative, we solve the recursive difference equation

$$g(t) = \text{Max}\left[0, \frac{y(t) - \sum_{j=1}^{t-1} p(t-j)g(j)}{p(0)}\right] . \tag{15}$$

This accession policy guarantees that future requirements will be met. They may be exceeded in certain time periods. If $s(t) = \sum_{u=0}^{M} s(t;u)$ is the total stock level, then $s(t) - z(t)$ measures the number of *redundant* personnel.

Example 36: Consider the net requirements for the three enlisted ratings when there are lower bounds on the accession levels. From our last calculation we see that accessions for QM and BM never drop below 1000 and 500 respectively. Thus the solution of (2) will agree with the equality solution of (1). However in periods 1 and 4 the accessions for SM drop below 700. The accession, stock, and redundancy levels for SM are shown below.

t	1	2	3	4	5
accession	700	1208	1493	826	1290
stock	6368	5500	5000	4500	4500
requirements	6000	5500	5000	4500	4500
surplus	368	0	0	0	0

We see there are 368 extra SM's in the first period, and also that the accessions in periods 2 through 5 are all above lower bound and are different from those calculated in example 34.

Example 37: A similar calculation can be made for Captains with a lower bound of 150 per year.

CAPTAINS

	1	2	3	4	5
accessions	154	150	150	150	190
stock	2000	1817	1700	1612	1600
requirements	2000	1800	1700	1600	1600
surplus	0	17	0	12	0

■

(v) Transfers of Jobs

A transfer of jobs simply is a change in the net requirements.

Example 38: The rating QM has increasing requirements while the rating SM has decreasing requirements. Suppose some of the functions traditionally performed by the rating QM could be transferred to SM. This transfer of responsibility might increase SM requirements by 500 per period and decrease QM requirements by 500. The future accessions needed to meet requirements after the change are:

Future Accessions g(t).

t	1	2	3	4	5
SM	832	1584	1526	925	1494
QM	1437	2194	2124	2059	1691
BM	760	912	1002	1253	1013

(vi) Transfers of Personnel.

Let $r(t;u)$ be the number of people with length of service u who are transferred *out* of the system at time t. We must have

$$r(t;u) \leq s(t;u) = \frac{p(u)s(u-t;o)}{p(u-t)} .$$

If $r(t;u) \leq 0$, then people are effectively transferred into the system. The increase in net requirements at time $t + k$ is given by

$$\frac{p(u+k)r(t;u)}{p(u)} .$$

Example 39: As pointed out above, the requirements for the QM rating are increasing, while those for SM are decreasing. It is possible to retrain individuals in the SM group and transfer them to QM. A typical retraining schedule is given below

Time t	1	1	1	2	2	2	3	3	3
LOS	0	1	2	1	2	3	2	3	4
Number	150	150	150	100	100	100	50	50	50

We will eventually transfer 300 people from each of the three youngest cohorts. The transfers are phased over time to provide some stability. The new legacies for SM and QM are:

Legacy $\ell(t)$.

t	1	2	3	4	5
SM	5218	3046	1286	644	403
QM	5013	3638	2083	960	383

The future accessions become:

Accessions $g(t)$

t	1	2	3	4	5
SM	781	1711	1433	735	1346
QM	1486	2023	2256	2295	1858

<u>Example 40</u>: The number of future accessions (promotions) to the rank of captain is limited by the decreasing future requirements and the large legacy that is a result of large requirements in the past. One way to deal with this problem is an "early retirement" program. This would allow for a smooth input into the rank of captain. A sample retirement schedule is shown below.

Time t	1	2	2	3	3	4	4	5
TIG	8	9	8	9	8	6	5	5
YRGR	45	45	46	46	47	50	51	52
Number	35	30	30	40	20	30	50	30

In this table TIG stand for time in grade, i.e., the number of years as a captain, and YRGR is for year group, i.e., the year the individual started his career.

With these changes the future legacies become:

		1	2	3	4	5
	42	0	0	0	0	0
	43	46	2	0	0	0
	44	100	32	5	0	0
	45	174	39	10	1	0
	46	200	131	11	3	0
YRGR	47	195	166	118	15	4
	48	153	144	124	99	41
	49	182	178	167	142	114
	50	252	248	243	197	171
	51	271	267	263	208	196
	52	234	230	227	223	189
	Total	1810	1443	1172	893	716

The circled numbers show the changes due to our early retirement policy. The new future accessions are

Accessions $q(t)$

t	1	2	3	4	5
Capt.	189	170	176	186	187

(vii) <u>Changes in Continuation Rates</u>

Let $q(0) = p(0)$ and for $u = 1,\ldots,M$, $q(u) = p(u)/p(u-1)$. The numbers $q(u)$ are the continuation rates, the fraction of people with LOS equal to $u - 1$ that continue in the system and appear one period later with LOS equal to u . Changes in continuation rates imply changes in the survivor fractions. For example, if we change $q(k)$ to $\tilde{q}(k)$, then

$$\tilde{p}(u) = \begin{cases} p(u) & \text{if } u < k , \\ \dfrac{\tilde{q}(k)\,p(u)}{q(k)} & \text{if } u \geq k . \end{cases}$$

The changed survivor fractions can apply to either the current stock of manpower or to the future inflows of manpower, or to both.

Example 41: To accelerate the release of SM we change q(3) from 8/17 to 4/17. The new survivor fractions are:

Survivor Fraction p(u).

u	0	1	2	3	4	5	6
SM	1	.95	.85	.2	.1	.075	.05

To increase the retention of our current stock of QM's we change q(u) from .5 to .8. The new Survivor Fractions for QM are

Survivor Fraction p(u).

u	0	1	2	3	4	5	6
QM	1	.9	.9	.6	.48	.16	.16

If these changes apply to the current stock of manpower we obtain a new legacy.

Legacy $\ell(t)$

t	1	2	3	4	5
SM	5268	2963	981	473	265
QM	4923	3413	2142	1109	476

Now assume the changed survivor fractions do *not* apply to future entrants. The future accessions become

Accessions g(t)

t	1	2	3	4	5
SM	731	1841	1646	603	1377
QM	1576	2167	1987	2204	1976

If the alternate survivor fractions apply to the future accessions *and* the current stock, then our required accession schedule is

Accessions g(t)

t	1	2	3	4	5
SM	731	1841	1646	749	1680
QM	1576	2167	1987	2204	1692

Example 42: Instead of an early retirement program for captains, we can change the survivor fractions by instituting a *mid-captain* review. The value of q(5) is currently 0.918. If this is changed to 0.5 the legacies become,

t	1	2	3	4	5
42	0	0	0	0	0
43	46	3	0	0	0
44	100	32	6	0	0
45	209	81	19	1	0
46	200	161	62	14	1
47	195	166	138	53	13
48	94	80	69	52	21
49	182	90	85	72	58
50	252	227	133	115	99
51	271	267	247	146	123
52	234	230	225	213	111
TOTAL	1772	1338	982	665	426

The legacies in the bordered section have changed (compared with the table in example 29). With these legacies and the new survivor fractions applied to future accessions we get the following accession schedule.

Accessions g(t)

t	1	2	3	4	5	∞
Capt.	227	237	262	227	254	241

9. A One Class, Many Chain Model.

In the one class, one chain model it is assumed that all individuals enter the system with zero periods of completed service. In the context of the Navy enlisted skill ratings discussed in section 8, this would mean that there are no significant flows between skill ratings, and that all accessions to the skill rating have zero length of completed service (LOS is 0). This assumption is not always valid. There is a pool of non-rated enlisted manpower that is not assigned to any particular skill rating. Individuals do move from the non-rated pool to the skill ratings with 1, 2 or more periods of completed service in the Navy. These movements are called "lateral accessions" to the skill rating.

In general it is difficult to handle lateral accessions because of the large number of degrees of freedom created by allowing such movements. However, we show how, under certain restrictions, lateral accessions can be treated as a one class, many chain model, and how this model can be reduced to a one class, one chain model similar to that in section 8.

We say that individuals who enter the "system" (say a Navy enlisted skill category) with k periods of completed service are on chain k. Thus we have $M+1$ possible chains $k = 0,1,2,\ldots,M$. Let $g_k(t)$ be the number of accessions in period t with LOS equal k. Then the total accessions in period t are

(16) $$f(t) = \sum_{k=0}^{M} g_k(t) = eg(t) ,$$

where $g(t) = [g_0(t), g_1(t), \ldots, g_M(t)]$.

Recall that in the one class, one chain model in section 8 that $p(u+1)/p(u)$ is the fraction of those with LOS u in the skill rating who remain and complete $(u+1)$ periods of service. We generalize this slightly and define

(17) $$p_k(u) = p(k+u)/p(k) .$$

In the one class, one chain model $p_k(u)$ is the fraction of those with LOS equal k who remain in service at least u more periods. In the $(M+1)$ chain model we assume that this fraction remains the same for individuals who enter with LOS less than k, and for those who enter with LOS equal k. Thus we assume that behavior affecting retention is the same for an individual with LOS k, independent of how he came to have LOS k.

Let $s(t)$ be the total stocks at time t (in the single class). From the basic equation (2),

$$s(t) = \sum_{u=0}^{M} \sum_{k=0}^{M} p_k(u) g_k(t-u). \tag{18}$$

This equation shows that there are $(M+1)^2$ input flow variables g. To reduce this number we introduce the concept of a proportional input policy. Let

$$r(k) = g_k(t)/f(t), \tag{19}$$

independent of t. Then $r(k)$ is the fraction of total input flow each period which enters on chain k, and this is assumed constant from period to period. Clearly from (16) and (19), $\sum_{k=0}^{M} r(k) = 1$.

Example 43: Let $M=2$ and $r(0) = 0.75$, $r(1) = 0.15$, $r(2) = 0.10$. The maximum LOS is 2 periods, 75% enter with LOS 0, 15% with LOS 1 and 10% with LOS 2. The following table shows the total input flows in periods 1,2,3, and the breakdown of these flows by the chains 0,1,2 .

Period t	Total Flow $f(t)$	Chain Flows $g_0(t)$	$g_1(t)$	$g_2(t)$
1	550	412	83	55
2	420	315	63	42
3	470	353	70	47

By using the proportional input policy (19) in (18), and using (17) the stocks at time t are given by

$$s(t) = \sum_{u=0}^{M} \sum_{k=0}^{M} \frac{p(u+k)}{p(k)} r(k) f(t-u) .$$

Since $p(u) = 0$ for $u > M$, then

$$s(t) = \sum_{u=0}^{M} \sum_{k=0}^{M-u} \frac{p(u+k)}{p(k)} r(k) f(t-u) .$$

Now let

(20) $$q(u) = \sum_{k=0}^{M-u} \frac{p(u+k)}{p(k)} r(k) .$$

Then $q(u)$ is a modified survivor fraction and the stocks can be written

$$s(t) = \sum_{u=0}^{M} q(u) f(t-u) ,$$

which is equivalent to the basic stock equation in section 3. Note that all that is required to calculate the q's are the p's and the r's. The modified survivor fraction $q(u)$ measures the fraction of accessions who entered the system in period $t-u$, regardless of length of service, who will be present in the system at time t.

Example 44: Let $M = 4$ and suppose p and r are given by

$$p = [1.00,\ 0.85,\ 0.80,\ 0.55,\ 0.20] ,$$

$$r = [0.75,\ 0.15,\ 0.10,\ 0,\ 0] .$$

Then the modified survivor fractions are

$$q = [1{,}00,\ 0.85,\ 0.72,\ 0.45,\ 0.15] .$$

Suppose that the proportional input policy is changed, and

$$r = [0.20,\ 0.40,\ 0.30,\ 0.10,\ 0] .$$

Then if p remains unchanged, the modified survivor fractions become

$$q = [1.00,\ 0.79,\ 0.49,\ 0.20,\ 0.04] .$$

■

In the set of equations (20) we calculate the modified survivor fractions q from the real survivor fractions p, and the given proportional inputs r. Suppose we are given desired survivor fractions q and real survivor fractions p. We can then ask whether or not there exist fractions r which satisfy (20).

Example 45: Let $M = 4$ and

$$p = [1.00, 0.85, 0.80, 0.55, 0.20] ,$$

$$q = [1.00, 0.80, 0.60, 0.40, 0.20] .$$

On solving (20) for r we find that

$$r = [1.00, -0.64, 0.85, -0.09, -0.12] .$$

Since r is uniquely defined by (20) there is no vector r with nonnegative coefficients which gives the above q. ∎

To find the set of q's that can be obtained by a feasible set of r's we simply calculate the q that would result from putting <u>all</u> accessions in with a certain length of service. If $r(k) = 1$ and $r(j) = 0$ for $j \neq k$, we obtain

$$q(u) = \begin{cases} p(u+k)/p(k) & \text{if } u < M-k , \\ 0 & \text{if } u > M-k . \end{cases}$$

Any feasible q must be a convex combination of the q's selected in this way.

<u>Example 45</u>: Let $M = 4$ and $p = [1.00, 0.85, 0.80, 0.55, 0.20]$. By setting $r(k) = 1$ in turn for $h = 0,1,2,3,4$, we get the q's as follows:

Case	0	1	2	3	4
r(0) = 1	1.0	0.85	0.80	0.55	0.2
r(1) = 1	1.0	0.94	0.64	0.24	0.0
r(2) = 1	1.0	0.68	0.25	0.0	0.0
r(3) = 1	1.0	0.36	0.0	0.0	0.0
r(4) = 1	1.0	0.0	0.0	0.0	0.0

Note if we wish to make $q(1) > p(1)$, then we must have $r(1) > 0$. This, in turn, will imply $q(2) < p(2)$, and $q(3) < p(3)$. To be specific, suppose $r(2) = r(3) = r(4) = 0$. Then

$$q(0) = r(0) + r(1) = 1$$

$$q(1) = 0.85\ r(0) + 0.94\ r(1)$$

$$q(2) = 0.80\ r(0) + 0.64\ r(1)$$

$$q(3) = 0.55\ r(0) + 0.24\ r(1)$$

$$q(4) = 0.2\ r(0)$$

The quantities $q(u)$ represent survivor fractions where u measures length of completed service in the skill rating. The quantities $p(u)$ represent the survivor fractions where u measures the length of completed service in the Navy.

The sum $\sum_{u=0}^{M} q(u)$ is the average lifetime of an accession in the skill rating. Note that $\sum_{u=0}^{M} q(u) = \sum_{k=0}^{M} \left[\sum_{j=0}^{M} \frac{p(j+k)}{p(k)}\right] r(k)$. The expression $\sum_{j=0}^{M} \frac{p(j+k)}{p(k)}$ is simply the average remaining lifetime of an individual with length of service k. Thus we see that the average length of service of an accession is a weighted average of the average remaining lifetimes.

Example 46: (Continuation of 45).

$$\sum_{u=0}^{4} q(u) = 4.4r(0) + 2.82r(1) + 1.93r(2) + 1.36r(3) + r(4)$$

Problem 12: Suppose instead of manpower we account for effective manpower. Let $a(j)$ for $j = 0,\dots,M$ measure the effectiveness of an individual with length of service j. Show that the contribution to effectiveness of an individual who has length of service u in the skill rating is

$$\sum_{j=u}^{M} a(j)\ \frac{p(j)r(j-u)}{p(j-u)}$$

Problem 13: Continuation of 12.

Interpret as an expected value the effectiveness measure above.

Problem 14: Continuation of 12.

If $s(0) = [s_0(0), s_1(0), \ldots, s_M(0)]$ is the length of service distribution at time 0, show that the effectiveness legacy at time t is

$$\sum_{j=t}^{M} \frac{a(j)p(j)s_{j-t}(0)}{p(j-t)}$$

Compare this result with the formula in problem 12.

10. Longitudinal Conservation

If the flow of manpower in the organization is conserved then we are not free to choose the parameters $p_{ik}(u)$ in an arbitrary way. This section examines possible restrictions on the choice of the $p_{ik}(u)$ and reveals the connection between the general flow model described in Chapter I and the longitudinal model presented in this chapter.

First it is obvious that the $p_{ik}(u)$ should be nonnegative and not greater than one. If we define

$$p_{ok}(u) = 1 - \sum_{i=1}^{N} p_{ik}(u) , \tag{21}$$

then $p_{ok}(u)$ is the fraction of the input on chain k in period $t - u$ that has left the system before time t. It is reasonable to assume that

(i) $$p_{ok}(0) = 0$$

(ii) $$0 \le p_{ok}(u) \le p_{ok}(u + 1) \le 1 .$$

These imply that no one can leave before they enter, and that the fraction who have left the system at some time t increases as the length of service increases.

Recall the basic flow conservation equations (I.3). We are going to modify these equations to obtain a sharper understanding of the longitudinal model and to make our concept of longitudinal accounting and conservation precise.

Let $\tilde{f}(t)$ be the vector obtained from $f(t)$ by omitting the terms $f_{oi}(t)$ for all $i = 1,2,\ldots,N$. Thus $\tilde{f}(t)$ is, in the general case, a $N(N + 1)$ vector with components $f_{ij}(t)$ for $i = 1,2,\ldots,N$ and $j = 0,1,2,\ldots,N$. Now define $\tilde{A}$ and $\tilde{B}$ as the matrices obtained from A and B by omitting the columns which correspond to flows $f_{oi}(t)$.

<u>Example 47</u>: For the faculty example in Chapter I (Example I.4) the modified system is

$$\tilde{f}(t) = [f_{10}(t), f_{11}(t), f_{12}(t), f_{20}(t), f_{22}(t), f_{23}(t), f_{30}(t), f_{23}(t)] ,$$

and

$$\tilde{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} .$$

Now we examine the evolution of the manpower system under three assumptions

$$\text{(22)} \quad \begin{array}{ll} \text{(i)} & es(0) = 1, \quad s(0) \geq 0 , \\ \text{(ii)} & f_{oi}(t) = 0 \quad \text{for} \quad i = 1,2,\ldots,N, \quad t \geq 1 , \\ \text{(iii)} & s(t) = 0 \quad \text{for} \quad t > M . \end{array}$$

The first assumption states that stock levels at time zero are nonnegative and the total size is 1. The second assumption insures there is no subsequent inflow into the system, and the third assumption says that all stock in the system at time zero leaves the system by time $M + 1$. The equations that describe this flow are

$$\text{(23)} \quad \begin{array}{ll} es(0) = 1 & \\ -Is(t-1) + \tilde{A}\tilde{f}(t) = 0 & \\ -\tilde{B}\tilde{f}(t) + Is(t) = 0 & \text{for } t = 1,2,\ldots,M , \\ s(0) \geq 0,\ s(t) \geq 0,\ \tilde{f}(t) \geq 0 & \text{for } t = 1,2,\ldots,M . \end{array}$$

These network flow equations describe all the possible ways an initial stock $s(0)$ can flow through the system for M periods.

It is easy to see that a solution of (23) is an extreme point solution if and only if each component of the solution, $s_i(t)$ (and therefore $f_{ij}(t)$), is either equal to zero or one. Thus an extreme point solution of (23) is the same as a personal history. The individual is in one and only one state at each accounting point.

We now refine and sharpen our definition of a chain. Let $p_k(u)$ be the N vector $p_k(u) = [p_{1k}(u), p_{2k}(u), \dots, p_{Nk}(u)]$, the fraction in each class of those who entered on chain k u periods ago. Chain k is defined by the sequence of vectors $\{p_k(u)\}_{u=0}^{M}$. Chain k is called <u>feasible</u> if

$$\text{(i)} \quad p_k(u) \geq 0 .$$

$$\text{(24)} \qquad \text{(ii)} \quad ep_k(0) = 1, \quad \text{and} \quad ep_k(u) \text{ is nonincreasing in } u \geq 0 ,$$

$$\text{(iii)} \quad \text{there exists a solution of (23) with } s(u) = p_k(u) \text{ for } u = 0,1,2,\dots,M .$$

<u>Problem 15</u>: Interpret each of the conditions in (24). ∎

Notice, that when $s(u) = p_k(u)$ and (i) and (ii) hold, then (23) reduces to

$$\tilde{A}\tilde{f}(t) = p_k(t-1) ,$$

$$\text{(25)} \qquad \tilde{B}\tilde{f}(t) = p_k(t),$$

$$\tilde{f}(t) > 0 .$$

These equations have a solution $\tilde{f}(t)$ for $t = 1,2,\dots,M$ if and only if the chain is feasible. Notice that the equations are separable and easy to solve. It is not difficult to construct infeasible chains as the following example indicates.

<u>Example 48</u>: Recall the three rank university model example (I.4). It is natural to assume that flows cannot occur from tenure to non-tenure. Consider a chain k where $p_k(0)$, $p_k(1)$ and $p_k(2)$ are given by

	$p_k(0)$	$p_k(1)$	$p_k(2)$
Nontenured	0	0	1
Tenured	1	1	0
Retired	0	0	0

A person on chain k moves from tenure to nontenure after two accounting periods. Figure III.5 below, shows the possible flows in this model. When $M = 2$, note that (i) and (ii) in (24) are satisfied, but (23) does not hold for $t = 2$.

<u>Example 49</u>: (Based on Figure III.5)

Consider the chain flows

	$p_k(0)$	$p_k(1)$	$p_k(2)$
Nontenured	1/2	1/4	1/8
Tenured	1/2	1/2	3/8
Retired	0	1/8	1/4

A feasible solution of (25) is given by

Arc ij	10	11	12	20	22	23	30	33
$\tilde{f}(1)$	0	1/4	1/4	1/8	1/4	1/8	0	0
$\tilde{f}(2)$	0	1/8	1/8	1/8	1/4	1/4	1/8	0

.

Another feasible solution of (25) is

Arc ij	10	11	12	20	22	23	30	33
$\tilde{f}(1)$	1/8	1/4	1/8	0	3/8	1/8	0	0
$\tilde{f}(2)$	1/8	1/8	0	0	3/8	1/8	0	1/8

We see that it is not necessary to have a unique solution of (25). ∎

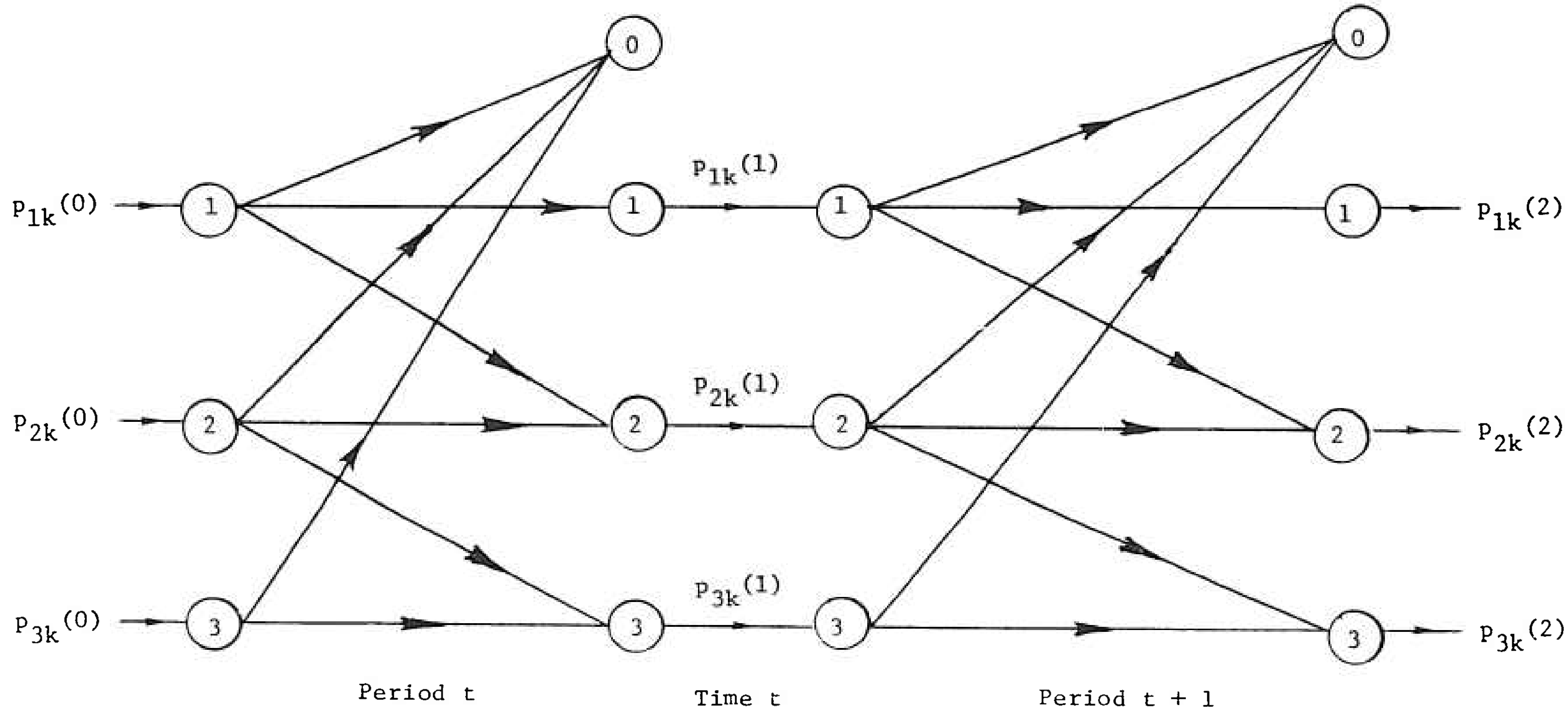


Figure III.5: Faculty Flow for Example 48.

We define a chain k to be *pure* if $p_{ik}(u) = 0$, or 1 for all i and u. If a chain is not pure it will be called *mixed*. A pure chain corresponds to a personal history in the organization, and thus to an extreme point solution of system of equations (23).

Let C be the set of all possible chains. Then

$$(26) \quad C = \left\{ s(t) \quad t = 0,1,\ldots,M \;\middle|\; \begin{array}{l} s(t),\ t = 0,1,\ldots,M \text{ and some } \tilde{f}(t) \\ t = 1,2,\ldots,M \text{ satisfy (23)} \end{array} \right\} .$$

A chain is feasible if and only if $s(t) = p_k(t)$ is in C. Pure chains correspond to extreme points of C, and as we saw above, for mixed chains it is possible there are several values of $\tilde{f}(t)$ that correspond to the chain.

Let us assume for a moment that each chain in the model is pure. Then for each chain k there is a *unique* solution $\tilde{f}^k(t)$ of (23). Now to make matters even simpler, assume that the only inflow is in period 0. Then the flows and stocks observed in periods 1 through M will be

$$(27) \quad \begin{aligned} \tilde{f}(t) &= \sum_{k=1}^{K} \tilde{f}^k(t) g_k(0) , \\ s(t) &= \sum_{k=1}^{K} p_k(t) g_k(0) . \end{aligned}$$

In this case it is possible to reconstruct the flows from the observed inputs g, and chain descriptions P by using (27). When the chains are mixed it is not possible to use (27), since there is no *unique* value of $\tilde{f}^k$ corresponding to chain k.

Example 50: Suppose there are only five extreme point solutions of (23). Then all solutions of (23) are a convex combination of the extreme point solutions. In Figure III.6 we have depicted the set C in an idealized situation. The flow in chain 1 can be expressed as a convex combination of pure chains α, β, and ε.

Let x_α, x_β, and x_ε be nonnegative numbers that sum to one. A unit inflow to chain k can be considered as being partitioned; x_α goes to pure chain α, x_β to pure chain β, and x_ε to pure chain ε. However, chain k can also be described in terms of pure chains α, δ, and κ. Let y_α, y_δ, and y_κ be the fraction of each unit inflow that goes into pure chains α, δ, and κ.

Example 51: (Continuation of Example 50)

Suppose we have four chains. The points of C that are obtained by taking convex combinations of the solutions are shown in Figure III.7.

In general, the normalization $s(0) = \sum_{k=1}^{K} g_k(0) = 1$, does not hold. Then we must consider C, (26), as the cross section of a *cone* of possible solutions, and the chains as generators of a subcone of allowable solutions. Thus Figure III.6 is the cone's cross section. Notice how the longitudinal model restricts flow to a subset of possibilities. The cones are demonstrated in Figure III.8.

The general flow solution is obtained by superimposing a sequence of systems exactly like (23), save the normalization $es(0) = 1$. One system starts at time $t = 0$, with $es(0) = \sum_{k=1}^{K} g_k(0)$, another at time 1 with $es(1) = \sum_{k=1}^{K} g_k(1)$. The stocks and flows in each of these systems are then added to obtain the total stocks and flows.

Example 52: (Continuation of Example 3, section 2)

Here we have $K = M = 2$. The chains are

$$\begin{array}{ll} \text{time} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ k = 1 & \begin{bmatrix} 1 & 1/6 & 0 \\ 0 & 5/6 & 2/6 \end{bmatrix} \\ k = 2 & \begin{bmatrix} 4/6 & 2/6 & 0 \\ 0 & 1/6 & 1/6 \end{bmatrix} \end{array} .$$

Period	0		1		2		3		
Chain	1	2	1	2	1	2	1	2	
Input	72	32	78	28	66	32	84	36	Totals
$s_1(0)$	72	32							104
$s_2(0)$	0	0							0
$s_1(1)$	12	16	78	28					134
$s_2(1)$	60	8	0	0					68
$s_1(2)$	0	0	13	14	66	32			125
$s_2(2)$	24	8	65	7	0	0			104
$s_1(3)$			0	0	11	16	84	36	147
$s_2(3)$			26	7	55	8	0	0	96
$s_1(4)$					0	0	14	18	32
$s_2(4)$					22	8	70	9	109
$s_1(5)$							0	0	0
$s_2(5)$							28	9	37

Table III.6. Stocks Obtained in Example 52.

Period		0		1		2		3		
Chain		1	2	1	2	1	2	1	2	
Input		72	32	78	28	66	32	84	36	Totals
	f_{10}	0	8							8
	f_{11}	12	16							28
t=1	f_{12}	60	8							68
	f_{20}	0	0							0
	f_{22}	0	0							0
	f_{10}	12	16	0	7					35
	f_{11}	0	0	13	14					27
t=2	f_{12}	0	0	65	7					72
	f_{20}	36	0	0	0					36
	f_{22}	24	8	0	0					32
	f_{10}			13	14	0	8			35
	f_{11}			0	0	11	16			27
t=3	f_{12}			0	0	55	8			63
	f_{20}			39	0	0	0			39
	f_{22}			26	7	0	0			33
	f_{10}					11	16	0	9	36
	f_{11}					0	0	14	18	32
t=4	f_{12}					0	0	70	9	79
	f_{20}					33	0	0	0	33
	f_{22}					22	8	0	0	30
	f_{10}							14	18	32
	f_{11}							0	0	0
t=5	f_{12}							0	0	0
	f_{20}							42	0	42
	f_{22}							28	9	37

Table III.7. Flows Obtained in Example 52.

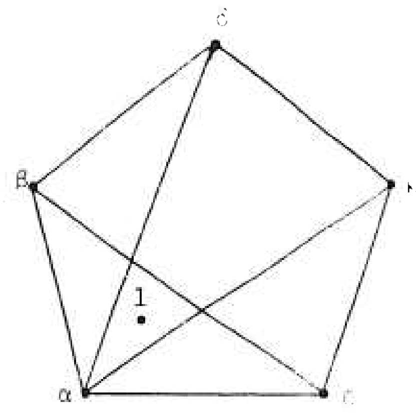


Figure III.6: The Set C With Five Pure Chains in Example 50.

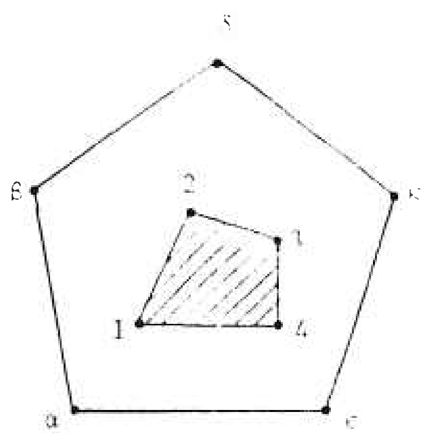


Figure III.7: The Set C for Example 51.

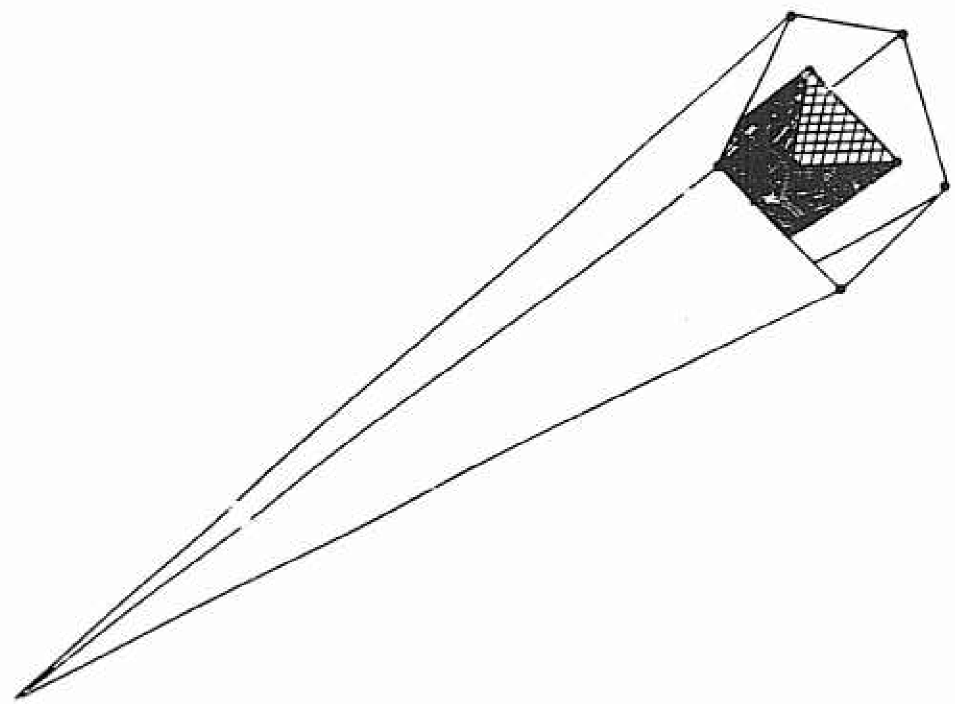


Figure III.8: The Cone of Possible Solutions and the Cone of Chain Flow Solutions.

Solutions of (25) corresponding to chains 1 and 2 are

		f_{10}	f_{11}	f_{12}	f_{20}	f_{22}
k=1	$\tilde{f}_1(1)$	0	.167	.833	0	0
	$\tilde{f}_1(2)$	.167	0	0	.50	.333
k=2	$\tilde{f}_2(1)$	.25	.5	.25	0	0
	$\tilde{f}_2(2)$	.5	0	0	0	.25

Now consider the following inflows into each chain each period.

	0	1	2	3
$g_1(t)$	72	78	66	84
$g_2(t)$	32	28	32	36

Using Equation (23), and overlapping the sequence of stocks we obtain Table III.6.

Periods 0 and 1 form a start up of the system. Times 2 and 3 are typical; the stock is composed of individuals who entered 0, 1, or 2 periods before. The stocks at times 4 and 5 can either be considered as a legacy, or the stocks that would result if no future inflows were allowed.

A similar calculation is carried out for the flows which are given in Table III.7.

Problem 16: The chains in Example 52 were obtained by a specific mixture of pure chains described in Example 3, section 2. The values of $\tilde{f}(t)$ presented in Example 52 are not consistent with the aggregation scheme described in Example 3. Find alternate values of $\tilde{f}(t)$ that are consistent with the aggregation scheme and recalculate, if necessary, the total stocks and flows.

Problem 17: (Based on section 6). Verify that the matrices P(u) given in section 6 satisfy conditions (i) and (ii) of (24).

Problem 18: (continuation) Construct $\tilde{A}$ and $\tilde{B}$ for the student flow model of section 6, and check condition (iii) of (24).

■

11. Systems Without Conservation

Frequently we encounter systems in which the normal conservation laws do not seem to apply. This section points out some of the ways in which models of this type arise and discusses their uses and possible pitfalls.

Nothing said about longitudinal conservation affects the model described in Equation (2.4). The discussion of longitudinal conservation was intended to introduce an additional degree of consistency into the model and to explain the relation of the longitudinal flow model to the earlier general flow model.

The examples below point out some anomalies which can occur if the model does not accurately represent real flows, or if the model is measuring the flow of some nonphysical commodity. As we shall point out these models can be redesigned to be more consistent with our sensibilities. However, such a redesign might make the model more complicated and no more useful.

Example 53: In the one class, one chain model that describes the separate ratings (skill categories) for the enlisted force in the U.S. Navy, in general, $p(0) = 1$, and $p(u) \geq p(u+1)$. Table III.8 lists $p(u)$, $u = 0,1,\ldots,24$ for the ratings "Boatswain's Mate" (BM) and "Electronics Technician" (ET). We find $p(0) = 1$, $p(1) = 3.43$, $p(2) = 6.28$, etc. for the BM rating. The source of the difficulty is that u measures length of service in the Navy. For most skill categories length of service in the Navy and skill category roughly coincide. However, for the BM skill category, length of service in the skill category is generally equal to length of service in the Navy minus 3. That is, most new "Boatswain's Mates" have completed 3 years of Navy duty in another skill category or as unrated personnel. Notice that for the ET rating the inequalities $p(u) \geq p(u+1)$ hold.

Example 54: In another one class, one chain example let the single manpower classification be Navy pilots. Inputs into the Navy pilot system are not capable

Length of Service, u . Years	P(u) Boatswains Mate (BM)	P(u) Electronic Technician (ET)
0	1	1
1	3.43	0.71
2	6.28	0.66
3	1.32	0.56
4	1.18	0.51
5	1.04	0.32
6	0.97	0.27
7	0.92	0.22
8	0.85	0.18
9	0.74	0.16
10	0.70	0.15
11	0.68	0.14
12	0.66	0.14
13	0.63	0.13
14	0.62	0.12
15	0.60	0.12
16	0.57	0.12
17	0.57	0.12
18	0.39	0.09
19	0.20	0.02
20	0.12	0.01
21	0.10	0.01
22	0.07	0.01
23	0.05	0.01
24	0.04	0.01

Table III.8: Fractions† p(u) for two U.S. Navy Ratings.

†Calculated from the enlisted force master files, Bureau of Naval Personnel, dated 6-30-71 and 6-30-72.

of flying planes; let us assume that it takes two years for each pilot trainee to qualify to fly. Thus $p(0) = p(1) = 0$, and, $p(2) = 1$. In later periods, the pilot may undergo retraining or be assigned to a nonflying job, thus $p(u)$ will depend on both the individual being still in the system and upon the individual being assigned to a flying job.

It is even possible to imagine $p(0)$ and $p(1)$ as negative. If it takes one qualified pilot to train two pilot trainees, then an increase in the number of trainees (those individuals in their first two years of service) will actually draw off qualified pilots from the stock available for assignment to flying units. With this interpretation we could have $p(0) = p(1) = -.5$.

Example 55: Many universities and other large organizations are governed by internal decision rules based on weighted measures of the student and faculty populations. Typical of these is that the ratio of full time equivalent faculty (FTE Faculty) to full time equivalent students (FTE Students) should be 29.

The concept of an FTE Faculty member is straightforward. It accounts for the convention that a great many faculty members have time off for research and leaves of absences, sabbaticals, etc. The concept of FTE Students is similar, accounting for light credit loads, quarters of vacation, etc. However, the concept of FTE Student is further complicated by a weighting scheme that counts masters and doctoral students as respectively 1.5 and 2.5 the weight of bachelor students. Attempts to model the manpower system using the classification FTE Student will run into difficulty.

These examples have only shown how nonconservation difficulties can arise. It is necessary that the model builder be aware of these problems. At the same time an imperfect model can sometimes be useful in answering some questions.

12. Notes and Comments

Longitudinal or cohort models are a traditional tool of actuarys, demographers, and health scientists. Much of Chapters 6, 7, and 8 of Bartholomew [1973] are devoted to models with a longitudinal aspect.

The treatment in this chapter was stimulated by Oliver [1969b]. Applications of these ideas can be found in Hopkins [1969], Oliver and Hopkins [1971], Grinold, Marshall and Oliver [1973], Grinold and Oliver [1973], and Marshall [1973]. The formal treatment of longitudinal models in secions 2, 4, 5, and 10 are the outgrowth of these papers. This treatment is novel and the framework should allow model builders to understand the power and limitations of longitudinal models.

The applications in sections 3, 8, and 9 are based on Grinold, Marshall, and Oliver [1973] and Grinold and Oliver [1973].

The data in section 6 is from Marshall, Oliver, and Suslow [1970]. The university application in section 7 is based on Oliver, Hopkins, and Armacost [1972], Oliver and Hopkins [1971], and Oliver [1973].

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