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Comparison of Theories for Intensity Fluctuations in Strong Turbulence

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Theories for the intensity scintillations of a wave propagating through a thin, highly turbulent layer have recently been developed by both the optical and radio-astronomy communities. It is demonstrated here that these results, which are at first glance dissimilar, are in fact identical.

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Contents

Comparison of Theories for Intensity Fluctuations in Strong Turbulence

1. INTRODUCTION

For some years there appears to have existed a mutual distrust between optical and radio astronomy theorists on the subject of saturation of the intensity scintillations of an electromagnetic wave in turbulence. It seems the optical people have their doubts about the results obtained by the radio astronomers, and vice versa, as is clear from some remarks on page 111 of a recent paper by Rumsey.[1] In this paper we shall demonstrate that the two schools have actually obtained identical results for the spectrum

$$\tilde{\phi}(\underline{\kappa}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2r \, \langle \, I(\underline{\rho}) \, I(\underline{\rho} + \underline{r}) \, \rangle \, e^{-i\underline{\kappa}\cdot\underline{r}}$$

of the intensity fluctuations. In the above definition, $I(\underline{\rho})$ is the intensity and $\langle \; \rangle$ denotes an ensemble average.

In order to present both the optical and radio astronomy results from a common viewpoint we shall consider the case of propagation through a thin

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1. Rumsey, V. (1975) Scintillations due to a concentrated layer with a power law turbulence spectrum, Radio Science (New Series) 10:107-114.

turbulent layer, since this is the case generally considered by radio astronomers. As theories representative of the optical school we shall consider the two by Fante[2] and Prokhorov et al;[3] it will first be shown that these two are equivalent. It was shown earlier by Fante[4,2] that the physical theories of Clifford et al[5] and Yura[6] are qualitatively the same as that of Fante.[2] We will then compare the theories of Fante and Prokhorov et al with those developed by Rumsey[1] and Marians[7], which are representative of those developed by the radio-astronomy community.

2. ANALYSIS

As a starting point let us, therefore, consider Eq. (21) in Fante[2], which gives the spectral density $\tilde{\phi}(\underline{\kappa})$ of the intensity fluctuations for the case of plane wave propagation in an extended medium. We have

$$\tilde{\phi}(\underline{\kappa}) = \delta'(\underline{\kappa}) + 4\pi k_o^2 \int_c^z dz' \, \Phi_n(z', \underline{\kappa}) \left[1 - \cos\frac{\kappa^2 (z-z')}{k_o}\right]$$

$$\cdot \exp\left\{-\frac{\pi k_o^2}{2}\int_{z'}^{z} H\left[\xi, \frac{\underline{\kappa}(z-\xi)}{k_o}\right] d\xi - \frac{\pi k_o^2}{2}\int_0^{z'} d\xi \, H\left[\xi, \frac{\underline{\kappa}(z-z')}{k_o}\right]\right\}$$

$$+ \frac{k_o^2}{16\pi}\int_0^z dz' \exp\left\{-\frac{\pi k_o^2}{2}\int_{z'}^{z} d\xi \, H\left[\xi, \frac{\underline{\kappa}(z-\xi)}{k_o}\right]\right\} Y(z, z', \underline{\kappa}) , \qquad (1)$$

where $\Phi_n(z, \underline{\kappa})$ is the wavenumber spectrum of the index-of-refraction fluctuations, z is the path length, and k_o is the signal wavenumber. Also

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2. Fante, R. (1975) Electric field spectrum and intensity covariance of a wave in a random medium, Radio Science (New Series) 10:77-85.
3. Prokhorov, A., Bunkin, F., Gochelashvily, K., and Shishov, V. (1975) Laser irradiance propagation in turbulent media, Proc. IEEE 63:790-811.
4. Fante, R. (1974) Covariance of the Intensity Fluctuations of a Wave in a Random Medium, Air Force Cambridge Research Laboratories Technical Report, AFCRL-TR-74-0488.
5. Clifford, S., Ochs, G., and Lawrence, R. (1974) Saturation of optical scintillation by strong turbulence, J. Opt. Soc. Amer. 64:148-154.
6. Yura, H. (1974) Physical model for strong optical-amplitude fluctuations in a turbulent medium, J. Opt. Soc. Amer. 64:59-67.
7. Marians, M. (1975) Computed scintillation spectra for strong turbulence, Radio Science (New Series) 10:115-119.

$$H(\xi, \underline{\rho}) = 8 \iint_{-\infty}^{\infty} d^2\kappa \;\; \Phi_n(\xi, \underline{\kappa}) \; (1 - \cos \underline{\kappa} \cdot \underline{\rho}) \simeq \gamma(\xi) \left|\underline{\rho}\right|^{\nu} \tag{2}$$

where for a Kolmogorov spectrum $\gamma(\xi) = 1.88\, C_n^2(\xi)$ and $\nu = 5/3$, and for strong turbulence and $|\underline{\kappa}| > k_o z/r_c$, where r_c = coherence length

$$Y(z, z', \underline{\kappa}) \simeq 2\, H\left[z' \; \frac{\underline{\kappa}(z-z')}{k_o}\right] \iint_{-\infty}^{\infty} d^2 r_2 \left|\Gamma_2(z', \underline{r}_2)\right|^2 e^{i\underline{\kappa} \cdot \underline{r}_2} \tag{3}$$

and

$$\Gamma_2(z', \underline{r}_2) = \exp \left\{ -\frac{\pi k_o^2}{4} \int_0^{z'} H(\xi, \underline{r}_2)\, d\xi \right\} . \tag{4}$$

We now evaluate Eq. (1) in the limit when the medium is a thin, strong turbulent layer extending from $z' = 0$ to $z' = \Delta$. That is,

$$\gamma(\xi) = \begin{cases} \gamma_o & 0 \le \xi \le \Delta \\ 0 & \Delta \le \xi \le z \end{cases} \tag{5}$$

where $\Delta << z$. However, we also assume that Δ is such that $\sigma^2 = 1.23\, k_o^{7/6}\, C_n^2\, \Delta^{11/6} >> 1$ for the Kolmogorov spectrum, with an equivalent relation for other values of ν. In this case Eq. (1) simplifies considerably, and it is relatively straightforward to show that

$$\tilde{\Phi}(\underline{\kappa}) \simeq \delta(\underline{\kappa}) + 4\pi k_o^2 \Delta\, \Phi_n(\underline{\kappa}) \left[1 - \cos \frac{\kappa^2 z}{k_o}\right] \exp \left\{ -\frac{\pi}{2} k_o^2 \Delta\, H\left(\Delta, \frac{\underline{\kappa} z}{k_o}\right) \right\}$$

$$+ \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2 r_2 \left|\Gamma_2(\Delta, \underline{r}_2)\right|^2 e^{i\underline{\kappa} \cdot \underline{r}_2} . \tag{6}$$

For small values of $|\underline{\kappa}|$ the spectrum $\tilde{\Phi}(\underline{\kappa})$ is dominated by the first two terms in Eq. (6). That is, for $\kappa^2 z/k_o << 1$,

$$\tilde{\Phi}_{LOW}(\kappa) \simeq \delta(\underline{\kappa}) + 2\pi k_o^2 \Delta\, \Phi_n(\underline{\kappa}) \left(\frac{\kappa^2 z}{k_o}\right)^2 \exp \left\{ -\frac{\pi}{2} k_o^2 \Delta\, H\left(\Delta, \frac{\underline{\kappa} z}{k_o}\right) \right\}. \tag{7}$$

while for large values of $|\underline{\kappa}|$ the spectrum is dominated by the second term:

$$\tilde{\phi}_{HIGH}(\underline{\kappa}) \simeq \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2r_2 \left|\Gamma_2(\Delta,\underline{r}_2)\right|^2 e^{i\underline{\kappa}\cdot\underline{r}_2} \quad . \tag{8}$$

Now let us compare these equations with the recent Soviet (Prokhorov, et al[3]) results for the thin slab. If we define

$$\phi_s(\underline{\kappa}) = 2\pi k_o^2\Delta\, \Phi_n(\underline{\kappa})$$

$$D(\Delta,\underline{u}) = \frac{\pi}{2} k_o^2 \Delta\, H(\Delta,\underline{u}) \ ,$$

we can rewrite Eqs. (7) and (8) as

$$\tilde{\phi}_{LOW}(\underline{\kappa}) \simeq \delta(\underline{\kappa}) + \kappa^4\left(\frac{z}{k_o}\right)^2 \phi_s(\underline{\kappa}) \exp\left[-D\left(\Delta,\frac{\underline{\kappa} z}{k_o}\right)\right] \tag{9}$$

and

$$\tilde{\phi}_{HIGH}(\underline{\kappa}) \simeq \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2r_2\, e^{-D(\Delta,\underline{r}_2)}\, e^{i\underline{\kappa}\cdot\underline{r}_2} \quad , \tag{10}$$

which are identical with Eqs. (4.21) and (4.22) of Prokhorov et al.[3] Therefore, for a thin, strongly turbulent layer the results of Prokhorov and Fante[2] are identical. (A proof that these two theories give identical results for the extended medium is given in Appendix A.) As we pointed out earlier, the results of Clifford et al[5] and Yura[6] are qualitatively equivalent to those of Fante.[2] Therefore, what now remains is to compare the results of Rumsey[1] and Marians[7] with Eqs. (7) and (8) or (9) and (10).

If we combine Eqs. (12) and (18) of Rumsey and multiply by $(2\pi)^{-2}$ so that Rumsey's definition of $\Phi_I(\underline{\kappa})$ in his Eq. (4) is equivalent to our $\tilde{\phi}(\underline{\kappa})$, we get

$$\tilde{\phi}(\underline{\kappa}) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\beta\, e^{i\underline{\kappa}\cdot\underline{\beta}} \exp\left\{-Q\left[2|\underline{\beta}|^{\alpha-2} + 2\left|\frac{\underline{\kappa} z}{k_o}\right|^{\alpha-2} - \left|\underline{\beta}+\frac{\underline{\kappa} z}{k_o}\right|^{\alpha-2} - \left|\underline{\beta}-\frac{\underline{\kappa} z}{k_o}\right|^{\alpha-2}\right]\right\} \tag{11}$$

where $Q = T\,2^{2-\alpha}\,\Gamma(2-\alpha/2)/(\alpha-2)\,\Gamma(\alpha/2)$, T is Rumsey's turbulence strength parameter that can be related to the turbulence strength parameter σ^2, which is used in optical propagation studies, and α is defined by $\Phi_n(\underline{\kappa}) \sim |\underline{\kappa}|^{-\alpha}$. (An alternate derivation of Eq. (11) from the extended Huygens-Fresnel principle is given in Appendix B.) For a Kolmogorov spectrum $\alpha = 11/3$. Note that $\nu = \alpha-2$. Let us now consider Eq. (11) for the case of very large values of $|\underline{\kappa}|$. In this case the exponent in (11) can be expanded in a Taylor series, as

$$\exp\left\{-Q\,[\ldots]\right\} \simeq -2Q\,|\underline{\beta}|^{\alpha-2} + \ldots \quad . \tag{12}$$

If we use Eq. (12) in (11) we get

$$\tilde{\Phi}_{HIGH}(\underline{\kappa}) \simeq \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2\beta\, e^{i\underline{\kappa}\cdot\underline{\beta}}\; e^{-2Q|\underline{\beta}|^{\alpha-2}} \quad . \tag{13}$$

If we recall that the second order coherence function $\Gamma_2(\underline{\beta})$ is

$$\Gamma_2(\Delta, \underline{\beta}) = e^{-Q|\underline{\beta}|^{\alpha-2}} = e^{-\frac{1}{2}D(\Delta,\underline{\beta})} \;,$$

it is readily seen that Eq. (13) can be rewritten as

$$\tilde{\Phi}_{HIGH}(\underline{\kappa}) \simeq \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2\beta\, e^{i\underline{\kappa}\cdot\underline{\beta}} \left|\Gamma_2(\Delta,\underline{\beta})\right|^2$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2\beta\, e^{i\underline{\kappa}\cdot\underline{\beta}}\; e^{-D(\Delta,\underline{\beta})} \quad . \tag{14}$$

Upon comparing Eq. (14) and (8) or (10) we see that the two results are identical.

We next study Eq. (11) in the limit of small values of $\underline{\kappa}$. If we again expand the exponent in (11) in a Taylor series about $\underline{\kappa} = 0$, we get

$$\tilde{\Phi}_{LOW}(\underline{\kappa}) \simeq \left(\frac{1}{2\pi}\right)^2 \exp\left\{-2Q\left|\frac{\underline{\kappa}\,z}{k_o}\right|^{\alpha-2}\right\} \iint_{-\infty}^{\infty} d^2\beta \exp\left\{i\underline{\kappa}\cdot\underline{\beta}\right.$$

$$\left. + 2Q\,\frac{\kappa^2 z^2}{k_o^2}\,|\underline{\beta}|^{\alpha-4}\left(\frac{\alpha}{2}-1\right)\left[1+(\alpha-4)\cos^2\theta\right]\right\} \;, \tag{15}$$

where θ is the angle between $\underline{\kappa}$ and $\underline{\beta}$. We now expand $\exp\left\{2Q \ldots\right\}$ in a Taylor series and get

$$\tilde{\phi}_{LOW}(\underline{\kappa}) \simeq \delta(\underline{\kappa}) + \left|\underline{\kappa}\right|^{4-\alpha}\left(\frac{z}{k_o}\right)^2 A \exp\left\{-2Q\left|\frac{\underline{\kappa} z}{k_o}\right|^{\alpha-2}\right\}, \tag{16}$$

where

$$A = \left(\frac{2Q}{(2\pi)^2}\right)\left(\frac{\alpha}{2} - 1\right)\int_0^{2\pi} d\theta \int_0^{\infty} \frac{d\xi}{\xi^{3-\alpha}} e^{i\xi\cos\theta}\left[1 - (4-\alpha)\cos^2\theta\right]. \tag{17}$$

The double integral in Eq. (17) is readily evaluated and gives the result

$$2\pi\,(2)^{\alpha-2}\,\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(2 - \frac{\alpha}{2}\right)}$$

so that, upon substituting for Q, we get $A = T/2\pi$. Now

$$\phi_s(\underline{\kappa}) = \frac{T}{2\pi}\left|\underline{\kappa}\right|^{-\alpha} = \left(\frac{1}{2\pi}\right)^2 \Phi_\theta(\underline{\kappa}) \tag{18}$$

where $\Phi_\theta(\underline{\kappa})$ is the function defined in Eq. (15) of Rumsey.[1] (We note that Rumsey's definition of Φ_n differs by $(2\pi)^2$ from the definition of Φ_n used by optics researchers.) If we use $A = T/2\pi$ and Eq. (18) we can then rewrite (16) as

$$\tilde{\phi}_{LOW}(\underline{\kappa}) = \delta(\underline{\kappa}) + \kappa^4\left(\frac{z}{k_o}\right)^2 \phi_s(\underline{\kappa})\, e^{-D\left(\frac{\underline{\kappa} z}{k_o}\right)}, \tag{19}$$

where we have used the fact that $D(\Delta, \underline{u}) = 2Q|\underline{u}|^{\alpha-2}$. If we now compare Eq. (19) with (9) we see that the results are again identical. Therefore, the results of Rumsey[1] and Marians[7] for the thin, strongly turbulent layer are entirely equivalent to those of Fante[2] and Prokhorov et al.[3], as well as to the physical theories of Yura[6] and Clifford et al.[5] Of course, all these results will agree for specialized cases; for example, all predict (see, for example, Fante[2] Eq. (44) or Rumsey[1]) that the scintillation index m approaches unity as the strength of turbulence approaches infinity, although the approximate theories of Clifford et al[5] and Yura[6] do not give exactly unity, because they are not exact, but rather physically motivated approximations that explain the mechanisms underlying the saturation of the scintillations.

References

1. Rumsey, V. (1975) Scintillations due to a concentrated layer with a power law turbulence spectrum, Radio Science (New Series) 10:107-114.
2. Fante, R. (1975) Electric field spectrum and intensity covariance of a wave in a random medium, Radio Science (New Series) 10:77-85.
3. Prokhorov, A., Bunkin, F., Gochelashvily, K., and Shishov, V. (1975) Laser irradiance propagation in turbulent media, Proc. IEEE 63:790-811.
4. Fante, R. (1974) Covariance of the Intensity Fluctuations of a Wave in a Random Medium, Air Force Cambridge Research Laboratories Technical Report, AFCRL-TR-74-0488.
5. Clifford, S., Ochs, G., and Lawrence, R. (1974) Saturation of optical scintillation by strong turbulence, J. Opt. Soc. Amer. 64:148-154.
6. Yura, H. (1974) Physical model for strong optical-amplitude fluctuations in a turbulent medium, J. Opt. Soc. Amer. 64:59-67.
7. Marians, M. (1975) Computed scintillation spectra for strong turbulence, Radio Science (New Series) 10:115-119.
8. Feizulin, V., and Kravtsov, Y. (1967) Broadening of a laser beam in a turbulent medium, Radiophys. and Quantum Electronics 10:33-35.
9. Kon, A., and Feizulin, V. (1970) Fluctuations in the parameters of spherical waves propagating in a turbulent atmosphere, Radiophys. and Quantum Electronics 13:51-53.

Appendix A

Comparison of Theories for the Extended Medium

Here we show that the theories of Fante[2] and Prokhorov et al[3] are identical for the case of the extended medium. Let us assume that $H(\xi, \underline{\rho})$ is given by Eq. (2). Then in Eq. (1)

$$\exp\left\{-\frac{\pi k_o^2}{2}\int_{z'}^{z} H\left[\xi, \frac{\underline{\kappa}(z-\xi)}{k_o}\right]d\xi - \frac{\pi k_o^2}{2}\int_0^{z'} H\left[\xi, \frac{\underline{\kappa}(z-z')}{k_o}\right]d\xi\right\}$$

$$= \exp\left\{-\frac{\pi k_o^2}{2}\gamma_o z\left|\frac{\underline{\kappa} z}{k_o}\right|^{\nu}\left(1-\frac{z'}{z}\right)^{\nu}\left[\frac{z'}{z}\left(\frac{\nu}{\nu+1}\right)+\frac{1}{\nu+1}\right]\right\} \quad \text{(A 1)}$$

$$= \exp\left\{-D\left(z, \frac{\underline{\kappa} z}{k_o}\right)\left(1-\frac{z'}{z}\right)^{\nu}\left[\frac{z'}{z}\left(\frac{\nu}{\nu+1}\right)+\frac{1}{\nu+1}\right]\right\} \quad \text{(A2)}$$

where we have used the fact that $D(z, \underline{u}) = \frac{\pi}{2} k_o^2 z H(z, \underline{u})$. The last term in Eq. (1) can be simplified by using Eqs. (28) and (29) of Fante.[2] If we use Eq. (A2) in the second term in Eq. (1), and the simplification of Eq. (29) of Fante for the last term in Eq. (1) we get

$$\tilde{\phi}(\underline{\kappa}) = \delta(\underline{\kappa}) + 4\pi k_o^2 \int_0^z dz' \, \Phi_n(z', \underline{\kappa}) \left[1 - \cos \frac{\kappa^2 (z-z')}{k_o}\right]$$

$$\cdot \exp \left\{ -D\left(z, \frac{\underline{\kappa} z}{k_o}\right) \left(1 - \frac{z'}{z}\right)^{\nu} \left[\frac{z'}{z}\left(\frac{\nu}{\nu+1}\right) + \frac{1}{\nu+1}\right]\right\}$$

$$+ \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2 r_2 \left| \Gamma_2(z, \underline{r}_2)\right|^2 e^{+i\underline{\kappa} \cdot \underline{r}_2} \quad . \tag{A3}$$

The behavior of $\tilde{\phi}(\underline{\kappa})$ for small values of $\underline{\kappa}$ is determined principally by the first two terms in Eq. (A3). If we expand $[1 - \cos \kappa^2 (z-z')/k_o]$ in a Taylor series and recall our previous definition of ϕ_s we get

$$\tilde{\phi}_{LOW}(\underline{\kappa}) \simeq \delta(\underline{\kappa}) + \frac{\kappa^4}{k_o^2 z} \phi_s(\underline{\kappa}) \int_0^z dz' \, (z-z')^2$$

$$\cdot \exp \left\{ -D\left(z, \frac{\underline{\kappa} z}{k_o}\right) \left(1 - \frac{z'}{z}\right)^{\nu} \left[\frac{z'}{z}\left(\frac{\nu}{\nu+1}\right) + \frac{1}{\nu+1}\right]\right\} \tag{A4}$$

which is identical with Eq. (4.40) of Prokhorov et al.[3]

For large values of $|\underline{\kappa}|$ the last term in Eq. (A3) is dominant. Upon recalling that $\Gamma_2(z, \underline{u}) = \exp\ [-\frac{1}{2} D(z, \underline{u})]$, we get

$$\tilde{\phi}_{HIGH}(\underline{\kappa}) \simeq \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2 r_2 \, e^{-D(z, \underline{r}_2)} \, e^{i\underline{\kappa} \cdot \underline{r}_2} \tag{A5}$$

which is identical with Eq. (4.41) of Prokhorov et al.[3]

Appendix B

Derivation of Rumsey's Results by the Huygens-Fresnel Principle

Here we shall derive Rumsey's result as a limiting case of the extended Huygens-Fresnel principle, which is generally employed by optics researchers. This principle states that the field $u(z, \underline{r})$ at a point $\underline{r}$ in the plane z can be written (Feizulin and Kravtsov[8])

$$u(z, \underline{r}) = \frac{k_o e^{-ik_o z}}{2\pi i z} \iint_{-\infty}^{\infty} d^2 \rho_1 \, u_o(\underline{\rho}) \exp\left\{ i \frac{k_o}{2z} (\underline{r} - \underline{\rho}_1)^2 + \psi(\underline{r}, \underline{\rho}_1) \right\} \qquad \text{(B1)}$$

where $\psi(\underline{r}, \underline{\rho}_1) = \chi(\underline{r}, \underline{\rho}_1) + iS(\underline{r}, \underline{\rho}_1)$ is the additional complex phase, due to turbulence, of a spherical wave propagating from $(0, \underline{\rho}_1)$ to $(z, \underline{r})$, and $u_o(\underline{\rho}_1)$ is the field distribution in the z = 0 plane. We can use Eq. (B1) to form the quantity $\langle I(\underline{r}) I(\underline{r}') \rangle$ where $I(\underline{r}) = u(\underline{r}) u^*(\underline{r})$. If we assume that $u_o = \exp(-\rho_1^2/2a^2)$ we find after considerable manipulation (Feizulin and Kravtsov[8])

8. Feizulin, V., and Kravtsov, Y. (1967) Broadening of a laser beam in a turbulent medium, Radiophys. and Quantum Electronics 10:33-35.

$$\langle I(\underline{r})\, I(\underline{r}')\rangle = \left(\frac{k_o}{2\pi z}\right)^4 \iint_{-\infty}^{\infty} \cdots \iint d^2\rho_1\, d^2\rho_2\, d^2\rho_3\, d^2\rho_4$$

$$\cdot \exp\Bigg\{ -i\frac{k_o}{z}\, \underline{r}\cdot(\underline{\rho}_1-\underline{\rho}_2) - i\frac{k_o}{z}\underline{r}'\cdot(\underline{\rho}_3-\underline{\rho}_4)$$

$$-\left(\frac{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2}{2a^2}\right) + \frac{ik_o}{2z}\left(\rho_1^2-\rho_2^2+\rho_3^2-\rho_4^2\right)$$

$$-\frac{1}{2}D_1(0,\underline{\rho}_1-\underline{\rho}_2) - \frac{1}{2}D_1(\underline{r}-\underline{r}',\ \underline{\rho}_1-\underline{\rho}_4)$$

$$-\frac{1}{2}D_1(\underline{r}-\underline{r}',\ \underline{\rho}_2-\underline{\rho}_3) - \frac{1}{2}D_1(0,\ \underline{\rho}_3-\underline{\rho}_4)$$

$$+\frac{1}{2}D_1(\underline{r}-\underline{r}',\underline{\rho}_2-\underline{\rho}_4) + \frac{1}{2}D_1(\underline{r}-\underline{r}',\ \underline{\rho}_1-\underline{\rho}_3)$$

$$-D_\chi(\underline{r}-\underline{r}',\ \underline{\rho}_2-\underline{\rho}_4) - D_\chi(\underline{r}-\underline{r}',\ \underline{\rho}_1-\underline{\rho}_3)$$

$$+ i\left[D_{\chi s}(\underline{r}-\underline{r}',\ \underline{\rho}_2-\underline{\rho}_4) - D_{\chi s}(\underline{r}-\underline{r}',\ \underline{\rho}_1-\underline{\rho}_3)\right]\Bigg\} \qquad \text{(B2)}$$

where D_1 is the two-source spherical wave structure function, D_χ is the log-amplitude structure function, and $D_{\chi s}$ is the log-amplitude/phase structure function (Kon and Feizulin[9]). For a plane wave ($a \to \infty$) incident on a thin turbulent layer we can neglect D_χ and $D_{\chi s}$ in Eq. (B2). It is also convenient to define the new variables $\underline{R} = (\underline{\rho}_1 + \underline{\rho}_2 + \underline{\rho}_3 + \underline{\rho}_4)/4$, $\underline{r}_1 = (\underline{\rho}_1 - \underline{\rho}_3 + \underline{\rho}_2 - \underline{\rho}_4)/2$, $\underline{r}_2 = (\underline{\rho}_1 - \underline{\rho}_3 - \underline{\rho}_2 + \underline{\rho}_4)/2$, $\underline{\rho} = \underline{\rho}_1 + \underline{\rho}_3 - \underline{\rho}_2 - \underline{\rho}_4$. If these new variables are used in Eq. (B2) it is found that the $\underline{R}$ integration can readily be performed. The result then is

9. Kon, A., and Feizulin, V. (1970) Fluctuations in the parameters of spherical waves propagating in a turbulent atmosphere, Radiophys. and Quantum Electronics 13:51-53.

$$\langle I(\underline{r})\, I(\underline{r}')\rangle \simeq \frac{\pi a^2}{2}\left(\frac{k_o}{2\pi z}\right)^4 \int_{-\infty}^{\infty} \cdots \int d^2\rho\, d^2 r_1\, d^2 r_2 \exp\left\{ i\,\frac{k_o}{z}\,\underline{r}_1 \cdot \underline{r}_2 \right.$$

$$- \left(\frac{r_1^{\,2} + r_2^{\,2} + \frac{\mu}{4}\rho^2}{2a^2}\right) - i\frac{k_o}{z}\,(\underline{r} - \underline{r}')\cdot \underline{r}_2 - \frac{ik_o}{2z}\,(\underline{r} + \underline{r}')\cdot \underline{\rho}$$

$$- \frac{1}{2} D_1(0, \underline{r}_2 + \frac{\underline{\rho}}{2}) - \frac{1}{2} D_1\,(\underline{r} - \underline{r}', \underline{r}_1 + \frac{\underline{\rho}}{2}) - \frac{1}{2} D_1(\underline{r} - \underline{r}', \underline{r}_1 - \frac{\underline{\rho}}{2})$$

$$\left. - \frac{1}{2} D_1(0, \frac{\underline{\rho}}{2} - \underline{r}_2) + \frac{1}{2} D_1\,(\underline{r} - \underline{r}', \underline{r}_1 - \underline{r}_2) + \frac{1}{2} D_1\,(\underline{r} - \underline{r}', \underline{r}_1 + \underline{r}_2) \right\} \quad \text{(B3)}$$

where $\mu = 1 + \left(\frac{k_o a^2}{z}\right)^2$. Now as $a \to \infty$ (plane wave limit) it is clear that $\mu/a^2 \to \infty$ so that the integrand is sharply peaked about $\underline{\rho} = 0$. Therefore, we may do the integral in Eq. (B3), setting $\underline{\rho} = 0$ in all the terms of the exponent, except for $\mu\rho^2/8a^2$. The result is

$$\langle I(\underline{r})\, I(\underline{r}')\rangle = \left(\frac{k_o}{2\pi z}\right)^2 \iiiint_{-\infty}^{\infty} d^2 r_1\, d^2 r_2 \exp\left\{ i\,\frac{k_o}{z}\,\underline{r}_1 \cdot \underline{r}_2 - i\,\frac{k_o}{z}\,\underline{v}\cdot \underline{r}_2 \right.$$

$$\left. - D_1(0, \underline{r}_2) - D_1(\underline{v}, \underline{r}_1) + \frac{1}{2} D_1(\underline{v}, \underline{r}_1 - \underline{r}_2) + \frac{1}{2} D_1(\underline{v}, \underline{r}_1 + \underline{r}_2) \right\} \quad \text{(B4)}$$

where $\underline{v} = \underline{r} - \underline{r}'$. Because $\underline{v}$ is small compared with $\underline{r}_1$ and $\underline{r}_2$ over nearly all of the $\underline{r}_1$ and $\underline{r}_2$ integrations, we may replace $D_1(\underline{v}, \underline{r}_1)$ by $D_1(0, \underline{r}_1)$, etc. Furthermore,

$$\tilde{\Phi}\,(\underline{\kappa}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \langle I(\underline{r})\, I(\underline{r} + \underline{v})\rangle\, e^{-i\underline{\kappa}\cdot\underline{v}}\, d^2 v$$

so that if we take the Fourier transform of Eq. (B4) we get

$$\tilde{\phi}(\underline{\kappa}) = \left(\frac{k_o}{2\pi z}\right)^2 \iiiint_{-\infty}^{\infty} d^2r_1 d^2r_2 \exp\left\{ i\frac{k_o}{z}\underline{r}_1 \cdot \underline{r}_2 - D_1(0,\underline{r}_1)\right.$$

$$\left. - D_1(0,\underline{r}_2) + \frac{1}{2}D_1(0,\underline{r}_1 - \underline{r}_2) + \frac{1}{2}D_1(0,\underline{r}_1 + \underline{r}_2)\right\} \delta\left(\underline{\kappa} + \frac{k_o}{z}\underline{r}_2\right)$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2r_1\, e^{i\underline{\kappa}\cdot\underline{r}_1} \exp\left\{ - D_1(0,\underline{r}_1)\right.$$

$$\left. - D_1\left(0, \frac{\underline{\kappa} z}{k_o}\right) + \frac{1}{2}D_1\left(0, \underline{r}_1 - \frac{\underline{\kappa} z}{k_o}\right) + \frac{1}{2}D_1\left(0, \underline{r}_1 + \frac{\underline{\kappa} z}{k_o}\right)\right\}. \qquad \text{(B5)}$$

Upon recalling that (Kon and Feizulin[9])

$$D_1(\underline{x},\underline{y}) = \frac{\pi}{2} k_o^2 z \gamma_o \int_0^{\frac{\Delta}{z}} dt \left| \underline{x} t + \underline{y}(1-t)\right|^{\nu},$$

where ν is the same as in Eq. (2), it is easy to see that Eq. (B5) is identical with Rumsey's result in Eq. (11), because

$$D_1(0,\underline{y}) \simeq \frac{\pi}{2} k_o^2 \Delta \gamma_o \left|\underline{y}\right|^{\nu} = 2Q\left|\underline{y}\right|^{\nu}$$

where $\nu = \alpha - 2$.