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# CONTROLLING THE SEPARATION OF LAMINAR BOUNDARY LAYERS IN WATER: HEATING AND SUCTION

J. Aroesty, et al

RAND Corporation

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SANTA MONICA, CA. 90406

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## PREFACE

Under the sponsorship of the Tactical Technology Office of the Defense Advanced Research Projects Agency, Rand has been engaged in the development of hydrodynamic design criteria employing concepts of boundary-layer control.

In the absence of a comprehensive theory for the location and properties of the transition region between laminar and turbulent flow, much of the analysis and design is empirically based. However, a secondary, but still important issue can nevertheless be approached from an analytic viewpoint: the effect of boundary-layer control on the separation of a laminar flow. Using simple analytic methods and results from boundary-layer theory, this report compares the effectiveness of different types of boundary-layer control in delaying laminar separation.

The report should be useful to hydrodynamicists, designers of submersibles, and others engaged in applying fluid mechanical methodology to the improvement of underwater vehicle performance.

## SUMMARY

Boundary-layer control provides a potentially effective method for reducing the skin friction of submersible vehicles. Once the flow is laminarized, the focus often shifts to the problem of delaying the onset of laminar separation, where the surface shearing stress vanishes and the boundary layer is liberated from the stabilizing effect of the wall. Transition to turbulent boundary-layer flow often occurs as a consequence of this separation, because of the enormous rate of growth in disturbances in the separated boundary layer.

The effects of suction on the delay of separation are well known--analyses, computations, and measurements over the past seventy years have shown that laminar separation can be delayed almost indefinitely, if the additional complexities and costs of suction are tolerable. However, the effects of surface heating on delay of separation are less well known--there are no measurements available, and there are no directly applicable analyses or computations.

To fill this gap, we present an analysis of the minimum surface overheat, $T_w - T_\infty$, that will delay separation for a prescribed adverse pressure gradient in water. Laminar boundary layers usually separate when the pressure begins to rise aft of the maximum diameter on a body of revolution. The adverse pressure gradient ($dp/dx > 0$) that can be sustained without separation is small in the absence of heating or suction. However, slight alterations of the velocity profile near the wall can have strong effects on separation.

The analysis is for a classic type of boundary-layer flow, the Falkner-Skan wedge flow corresponding to negative values of $\beta$. We perform an asymptotic analysis of such flows, including the effects of viscosity variation with temperature. The energy and momentum equations are coupled through this variation, and we employ a high Prandtl number approximation to develop systematic approximations to the equations of motion and energy. We find, as might be expected intuitively, that the effect of heating and viscosity variation is localized to a

thin layer near the wall, well within the entire momentum boundary layer. The primary effect of this layer on separation is to provide a "slip" velocity for the outer main parts of the flow. This slip velocity enables the outer, shear-layer-like part of the flow to sustain a more adverse pressure gradient than it could in the absence of heating.

However, although heating is shown to delay separation, the magnitude of its effect is found to be small, particularly if practical values of wall overheat are considered. We show that the magnitude of the acceptable adverse pressure gradient parameter increases by about 25 percent for a 40°F overheat. For a wedge flow, this means that $U \sim x^{-0.11}$ rather than $U \sim x^{-0.090}$ as occurs when there is no heating.

Comparisons with available results for delaying separation using suction show that a suction velocity ratio of less than 0.0001 would have a comparable effect in maintaining an attached flow.

Our results suggest that the effects of surface heating on the delay of separation are probably real and significant, but heating does not have the almost unlimited capacity for maintaining an attached flow that suction appears to possess. In addition, it should be emphasized that neglecting the effects of heat transfer in prediction of the location of laminar separation produces design procedures that are essentially conservative.

## CONTENTS

# I. INTRODUCTION

We are interested in the feasibility of delaying the separation of a laminar boundary layer that is subject to an adverse pressure gradient in water. The significance of maintaining an attached boundary layer in regions of adverse pressure gradient is that transition to turbulent boundary-layer flow often occurs by laminar separation and turbulent reattachment slightly aft of the point of maximum diameter, and, as body size increases, the relative portion of body surface under turbulent flow increases. It is then desirable to consider additional means for controlling the location of laminar separation.

Two such methods for delaying separation in the presence of an adverse pressure gradient are suction and wall heating. The effects of such are well known. Schlichting[(1)] cites measurements made by Prandtl over seventy years ago that verified the powerful effect of suction on delaying separation. The effects of wall heating are not so well known and do not yet appear to have been measured. However, the calculations of Wazzan et al.[(2)] for a laminar water boundary layer of the Falkner-Skan type subject to an adverse pressure gradient indicate that surface heating in water could maintain an attached boundary layer, which would have separated in the absence of any temperature difference between surface and free stream. Wazzan's calculations for a Falkner-Skan $\beta = -0.198$, which is the maximum negative value for $\beta$ for unseparated flow, also indicated that moderate surface heating in water ($T_w - T_\infty < 40°F$, and $T_\infty = 60°F$) does not increase the minimum critical Reynolds number as much as it does for flat-plate flow. This result, although not bearing on the subject of separation, strongly suggests that even if surface heating could be used to maintain an unseparated flow, the stability characteristics of this flow would still make it susceptible to transition near the pressure minimum. The situation is probably different with suction, where experience suggests that the delay of separation also leads to the delay of transition.

In this report, we analyze the effect of wall heating on the delay of separation for the simplest important boundary-layer flows with adverse pressure gradient: the Falkner-Skan flows with negative $\beta$ and negative m, where

$$U \sim x^m$$

and

$$\beta = \frac{2m}{m + 1} .$$

Using the idea of local similarity, it is then possible to extend the results developed here to a more general boundary-layer flow with reasonable accur..cy.

We are primarily interested in sharper estimates of the effect of heating in order to compare the efficacy of suction versus heating for the prevention of separation. For that comparison, analyses based on the Falkner-Skan flows are useful because the effects of suction for such flows have already been examined carefully by Terrill[3] and Nickel.[4]

The question we consider is the following: How much adverse pressure gradient will a heated laminar boundary layer in water sustain without separation? In the language of Falkner-Skan flows, the question than becomes: What is the largest negative value of $\beta$ for an attached flow that a given wall overheat $\Delta T \equiv T_w - T_\infty$ will permit? The analysis is a systematic approximation to the boundary-layer equations, based on the properties of high Prandtl number solutions,[5] and the existence of a thermal boundary layer that is considerably thinner than the velocity boundary layer.

The Prandtl number of water at 60°F is about 8 and all the existing evidence is that this is large enough for the validity of methods that are based on high Prandtl number arguments.

For water flows in the range of 40°F to 100°F, the principal departure from constant property flow is due to viscosity variation. The density, thermal conductivity, and specific heat do not vary appreciably in this range, and the most important phenomena, because of the gradient of viscosity, are quite well represented by this model in which $\rho$, k, and $c_p$ are constant, and the variation of viscosity with temperature is preserved. There exists an analogy between wall heating and suction for a variable viscosity fluid. For example, the compatibility condition at a wall, for both suction and wall heating, is

$$\rho v_w \frac{\partial u}{\partial y} = - \frac{dp}{dx} + \mu_w \frac{\partial^2 u}{\partial y^2} + \frac{\partial \mu_w}{\partial y} \cdot \frac{\partial u}{\partial y}$$

or, since $\mu = \mu(T)$,

$$\left[\rho v_w - \left(\frac{d\mu}{dT} \frac{\partial T}{\partial y}\right)_w\right] \frac{\partial u}{\partial y} = - \frac{dp}{dx} + \mu_w \frac{\partial^2 u}{\partial y^2} .$$

For water, $d\mu/dT < 0$; for suction, $v_w < 0$. Therefore, since the temperature gradient dT/dy is negative if $T_w > T_\infty$, the effect of surface heating would be qualitatively similar to that of suction. The similarity is only qualitative--suction has a more powerful and global effect on a boundary layer because of its impact on boundary-layer profiles and the growth of the boundary layer. The effect of heating is localized to a smaller region (which nevertheless is an important one for stability), and there is little effect on boundary-layer growth.

# II. ANALYSIS

## LARGE PRANDTL NUMBER APPROXIMATIONS

The boundary-layer equations are written

Momentum: $$\rho_\infty u \frac{\partial u}{\partial x} + \rho_\infty v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu(T) \frac{\partial u}{\partial y}\right) \qquad (1)$$

Energy: $$\rho_\infty c_p u \frac{\partial T}{\partial x} + \rho_\infty c_p v \frac{\partial T}{\partial y} = k_\infty \frac{\partial^2 T}{\partial y^2} \qquad (2)$$

Continuity: $$\frac{\partial}{\partial x}(ur^j) + \frac{\partial}{\partial y}(vr^j) = 0 \qquad (3)$$

where j = 0 corresponds to plane flow,

j = 1 corresponds to axial-symmetry.

If the new variables

$$\xi = \rho_\infty \mu_\infty \int U_e r^{2j}\, dx$$

and

$$\eta = \frac{U_e r^j}{\sqrt{2\xi}}\, y$$

are used to replace x and y, and if

$$g = \frac{T - T_\infty}{T_w - T_\infty}, \quad \text{and} \quad \frac{\partial f}{\partial \eta} = \frac{u}{U_e}$$

are chosen as dependent variables, then the coupled equations of variable viscosity Falkner-Skan flow are obtained provided that

$$\beta \equiv \frac{2\xi}{U_e} \frac{dU_e}{dx} \frac{dx}{d\xi} \tag{4}$$

is constant. These equations are

$$\begin{cases} (Nf'')' + ff'' + \beta[1 - (f')^2] = 0 & (5) \\ g'' + Pr_\infty fg' = 0 & (6) \end{cases}$$

where $N = \mu/\mu_\infty$, $Pr_\infty = \mu_\infty c_{p\infty}/k_\infty$, and primes denote differentiation with respect to $\eta$. These equations must be supplemented by the appropriate boundary conditions.

$$\begin{cases} f(0) = f'(0) = 0 \ , & g(0) = 1 \\ f'(\infty) = 1 \ , & g(\infty) = 0 \ . \end{cases}$$

The coupling between Eqs. (5) and (6) is through the viscosity ratio, N, which is a strong function of temperature.

Since we are interested in the conditions for incipient separation, we seek solutions to the above set that also correspond to $f''(0) = 0$. These solutions would then show the vanishing of the shear stress at the wall, which is the usual definition of separation.

If the Prandtl number is large, then the thermal boundary layer is much thinner than the velocity boundary layer. This suggests there is a thin region near the wall where viscosity variation is important in the momentum balance. For such a layer, the dominant terms in the force balance are shear and pressure gradient; inertial effects are negligible.

We introduce a new variable, Z, such that $\eta = \epsilon Z$, so $Z = O(1)$ when $\eta = O(\epsilon)$; the dependence of this small parameter, $\epsilon$, on physical quantities is determined later. Introduce a new stream function variable, X(Z), such that

$$f_{\eta\eta\eta}(\eta) = \beta X_{ZZZ}(Z) \ , \tag{7}$$

$$f_{\eta\eta}(\eta) = \varepsilon\beta X_{ZZ}(Z) \ , \tag{8}$$

$$f_{\eta}(\eta) = \varepsilon^2\beta X_Z(Z) \ , \tag{9}$$

$$f(\eta) = \varepsilon^3\beta X(Z) \ , \tag{10}$$

where the conditions $f''(0) = f'(0) = f(0)$ have been used. This ordering of f in terms of $\varepsilon$ follows from the assumption that N is a quantity of O(1).

Then various terms in the momentum equation are of the order:

$$\begin{array}{ccc} (Nf_{\eta\eta})_\eta & + \ ff_{\eta\eta} & + \ \beta(1 - f_\eta^2) = 0 \\ O(1) & O(\varepsilon^4) & O(1 - \varepsilon^4) \end{array} \tag{11}$$

The determination of $\varepsilon$ follows from the consideration of the energy equation; where g is of O(1)

$$\begin{array}{cc} g_{\eta\eta} & + \ Pr_\infty f \ g_\eta = 0 \\ O\left(\frac{1}{\varepsilon^2}\right) & O\left(Pr_\infty \varepsilon^2 \beta\right) \end{array} \tag{12}$$

It is necessary that there be a balance between conduction and transport if the boundary conditions are to be satisfied at the wall and the edge of the thermal layer. From this balance, the order of magnitude of $\varepsilon$ is established: $\varepsilon = (-\beta \ Pr_\infty)^{-1/4}$. The minus sign before $\beta$ is chosen because $\beta$ is a negative quantity. While this quantity, $\varepsilon$, is not particularly small if $Pr_\infty \sim 10$ and $-\beta \sim 0.2$, there is empirical evidence from the work of Liepmann[6] and Narasimha and Vasantha[7] that even if $Pr_\infty$ is as small as 0.7, solutions obtained via the approximation $\varepsilon \to 0$ are very accurate. The reduced momentum equation becomes

$$(NX_{ZZ})_Z + 1 = 0 \tag{13}$$

and the reduced energy equation becomes

$$g_{ZZ} - Xg_Z = 0 \ . \tag{14}$$

In general, these two nonlinear equations must be solved simultaneously; they are coupled through the dependence of the viscosity ratio, N, on g, and the dependence of g on X. However, this solution would be easy to obtain numerically. The appropriate boundary conditions on X are that $X(0) = X_Z(0) = 0$, and the conditions on g are that $g(0) = 1$, and $g(\infty) = 0$. The condition that $X_{ZZ}(0) = 0$ follows from the additional requirement that the relationship between β and $T_w - T_\infty$ be such that the shear vanishes at the boundary.

While it would be interesting to solve the coupled nonlinear set (Eqs. (13) and (14)) numerically, we can proceed to obtain quantitative estimates by introducing an additional approximation in the energy equation. Prior to stating that approximation, however, it is necessary to integrate the momentum equation.

The first integral of the momentum equation yields

$$NX_{ZZ} = -Z \ , \tag{15}$$

the next integral yields

$$X_Z = - \int_0^Z \frac{Z \, dZ}{N(Z)} \tag{16}$$

and the last integral yields

$$X = \int_0^Z dZ' \int_0^{Z'} \frac{Z'' \, dZ''}{N(Z'')} \ . \tag{17}$$

We integrate Eq. (16) by parts

$$\frac{dX}{dZ} = -\frac{Z^2}{2N} + \frac{1}{2}\int Z^2 \frac{d}{dZ}\left(\frac{1}{N}\right) dZ \ . \qquad (18)$$

In the limit as $Z \to \infty$,

$$\frac{dX}{dZ} \sim \frac{-Z^2}{2} + \frac{1}{2}\int_0^\infty Z^2 \frac{d}{dZ}\left(\frac{1}{N}\right) dZ \ , \qquad (19)$$

where we anticipate that $N \to 1$ as $Z \to \infty$.

VISCOSITY VARIATION WITH TEMPERATURE

It is now necessary to specify N, the viscosity ratio, as a function of g. Gazley* has shown that a good approximation for the kinematic viscosity of water in the range of temperature between 40°F and 100°F is

$$\nu = \frac{10^{-5}}{0.0807 + 0.0126\ T} \qquad (20)$$

and between 100°F and 160°F it is

$$\nu = \frac{10^{-5}}{-0.1946 + 0.0153\ T} \qquad (21)$$

where T is in °F and $\nu$ is in $ft^2/sec$. This suggests a viscosity-temperature model of the form

$$\mu = \frac{1}{a + bT} \ . \qquad (22)$$

*Personal communication from Carl Gazley, Jr., The Rand Corporation, 1975.

For such a model,

$$\frac{1}{N} = \frac{a + bT}{a + bT_\infty} = \frac{a}{a + bT_\infty} + \frac{b\left[T_\infty + (T_w - T_\infty)g\right]}{a + bT_\infty}$$

$$= 1 + \alpha\Delta Tg \qquad (23)$$

where $\Delta T = T_w - T_\infty$, and $\alpha = b/(a + bT_\infty)$.
For water in the temperature range between 40°F to 100°F, $\alpha \approx 0.153\ (°F)^{-1}$.
Using the relationship

$$\frac{d\left(\frac{1}{N}\right)}{dZ} = \alpha\Delta T \frac{dg}{dZ}, \qquad (24)$$

the reduced velocity function dX/dZ is written

$$\frac{dX}{dZ} = \frac{-Z^2}{2N} + \frac{\alpha\Delta T}{2}\int_0^Z Z^2 \frac{dg}{dZ}\, dZ \qquad (25)$$

APPROXIMATE SOLUTION OF THE ENERGY EQUATION

It is traditional in heat transfer analyses to employ crude approximations to the convective terms in the energy equation and still obtain highly accurate results. Our approach continues that tradition, and we trust that further precise solution of Eqs. (13) and (14) will confirm our approximate analysis.

We assume that for purposes of solving the energy equation a suitable approximation to X is $X = -Z^3/6\bar{N}$, where $\bar{N}$ is an average reference value of N evaluated between $T_\infty$ and $T_w$. (See Appendix A for a more vigorous justification of this approximation.) This is equivalent to using the constant property version of X, where the viscosity is evaluated at some reference condition in the boundary layer. Notice that

$$X \to \frac{-Z^3}{6N_w} \quad \text{near } Z = 0 \ , \tag{26a}$$

and

$$X \to -\frac{Z^3}{6} \quad \text{at large values of } Z \ . \tag{26b}$$

This is equivalent to approximating $f(\eta)$ by

$$f(\eta) = -\frac{\beta\eta^3}{3\bar{N}} \ , \tag{27}$$

in the original variables.

The resulting energy equation is

$$\frac{d^2g}{dZ^2} + \frac{1}{6\bar{N}} Z^3 \frac{dg}{dZ} = 0 \ , \tag{28}$$

whose solution is

$$g = A \int_0^Z \exp(-Z^4/24\bar{N}) \, dZ + C \ . \tag{29}$$

The boundary conditions, $g(0) = 1$, $g(\infty) = 0$, fix the two constants A and C so that

$$g(Z) = 1 - \frac{\int_0^Z \exp(-Z^4/24\bar{N}) \, dZ}{\int_0^\infty \exp(-Z^4/24\bar{N}) \, dZ} \ . \tag{30}$$

Therefore,

$$\frac{d}{dZ}\left(\frac{1}{N}\right) = - \frac{\alpha\Delta T \cdot \exp(-Z^4/24\bar{N})}{\int_0^\infty \exp(-Z^4/24\bar{N})\, dZ} \tag{31}$$

and the asymptotic expression for dX/dZ (Eq. (19)) can now be evaluated. As $Z \to \infty$, the velocity function

$$\frac{dX}{dZ} \to \frac{Z^2}{2} - \frac{\alpha\Delta T}{2} \frac{\int_0^\infty Z^2 \exp(-Z^4/24\bar{N})\, dZ}{\int_0^\infty \exp(-Z^4/24\bar{N})\, dZ} . \tag{32}$$

(See Appendix B for evaluation of integrals.)

Thus

$$\frac{dX}{dZ} \to - \frac{Z^2}{2} - \frac{\alpha\Delta T}{2} \frac{\left\{\frac{(24\bar{N})^{3/4}}{4} \Gamma(3/4)\right\}}{\left\{\frac{(24\bar{N})^{1/4}}{4} \Gamma(1/4)\right\}} \tag{33}$$

or

$$\frac{dX}{dZ} \to - \frac{Z^2}{2} - \frac{\alpha\Delta T}{2} \frac{\Gamma(3/4)}{\Gamma(1/4)} (24\bar{N})^{1/2} \tag{34}$$

or

$$\frac{dX}{dZ} = - \frac{Z^2}{2} - \frac{\alpha\Delta T}{2} (24\bar{N})^{-1/2}\, c \tag{35}$$

where $c = \Gamma(3/4)/\Gamma(1/4)$. It is important to note that the constancy of the second term on the right-hand side of Eq. (35) is not a consequence of the approximation $X = -Z^3/6\bar{N}$, and that because of the inequality, Eq. (A.8) of Appendix A, the second term on the right-hand side of Eq. (35) must be a constant with lower and upper bounds given by

$$\frac{\alpha\Delta T}{2}(24N_w)^{-1/2} c \le \frac{\alpha\Delta T}{2}\int_0^\infty Z^2 \exp\left(B\int_0^Z X(Z')\,dZ'\right) dZ \le \frac{\alpha\Delta T}{2}(24)^{-1/2} c$$

(see Eq. (A.3)).

APPROXIMATE SOLUTION OF THE OUTER LAYER

We have thus determined the structure of the velocity profile within the thermal boundary layer. We recast the asymptotic formula for dX/dZ in terms of the original variables, $\eta$ and f:

$$\frac{df}{d\eta} = \beta\varepsilon^2 \frac{dX}{dZ}(Z) \tag{36}$$

$$= \beta(-\beta\,Pr_\infty)^{-1/2}\frac{dX}{dZ}(Z) \tag{37}$$

or

$$\frac{df}{d\eta} = \beta(-\beta\,Pr_\infty)^{-1/2}\left[-\frac{Z^2}{2} - \frac{\alpha\Delta Tc}{2}(24\bar{N})^{1/2}\right] \tag{38}$$

or

$$\frac{df}{d\eta} = -\frac{\beta}{2}\eta^2 - \frac{\beta}{2}(-\beta\,Pr_\infty)^{-1/2}\alpha\Delta Tc(24\bar{N})^{1/2} \tag{39}$$

Thus, as the variable $\eta \to 0$, the outer flow must be of the form

$$\frac{u}{U_e} \to -\frac{\beta}{2}\eta^2 + K_1 ,$$

where $K_1$ is a positive constant, in order to match to the velocity profile at the outer edge of the thermal boundary layer.

Stewartson[8] has examined solutions of the Falkner-Skan equations that have this property. They are two-dimensional wake-like flows that have a positive slip velocity at $\eta = 0$ as well as zero shear and possess values of $\beta$ between -0.198 and -0.5. Berger[9] discusses these flows and graphs the relationship between $df/d\eta\ (0)$ and $\beta$.* Thus it is only necessary to use that relationship between $f_\eta(0)$ and $\beta$, in addition to Eq. (39) evaluated as $\eta \to 0$,

$$\frac{df}{d\eta}(0) = -\beta(-\beta\ Pr_\infty)^{1/2} \frac{\alpha \Delta Tc}{2} (24\bar{N})^{1/2} \tag{40}$$

in order to determine the relationship between $\Delta T$ and $\beta$ that corresponds to incipient separation. (See Fig. 1 for a sketch of the outer and inner velocity profiles.)

Figure 26 of Ref. 9 is replotted in the convenient form

$$\frac{df}{d\eta}(0)/|\beta|^{1/2} \text{ versus } \beta$$

This is shown in Fig. 2.

Since

$$\frac{df}{d\eta}(0)/|\beta|^{1/2} = (Pr_\infty)^{-1/2} \frac{\alpha \Delta Tc}{2} (24\bar{N})^{1/2} , \tag{41}$$

the ordinate is proportional to $(T_w - T_\infty)$.

NUMERICAL VALUES

At $T_\infty = 60°F$,

$$\nu_\infty = 1.22 \times 10^{-5}\ ft^2/sec ,$$

$$b = 0.0126 \times 10^5 (ft^2/sec)^{-1} (°F)^{-1} ,$$

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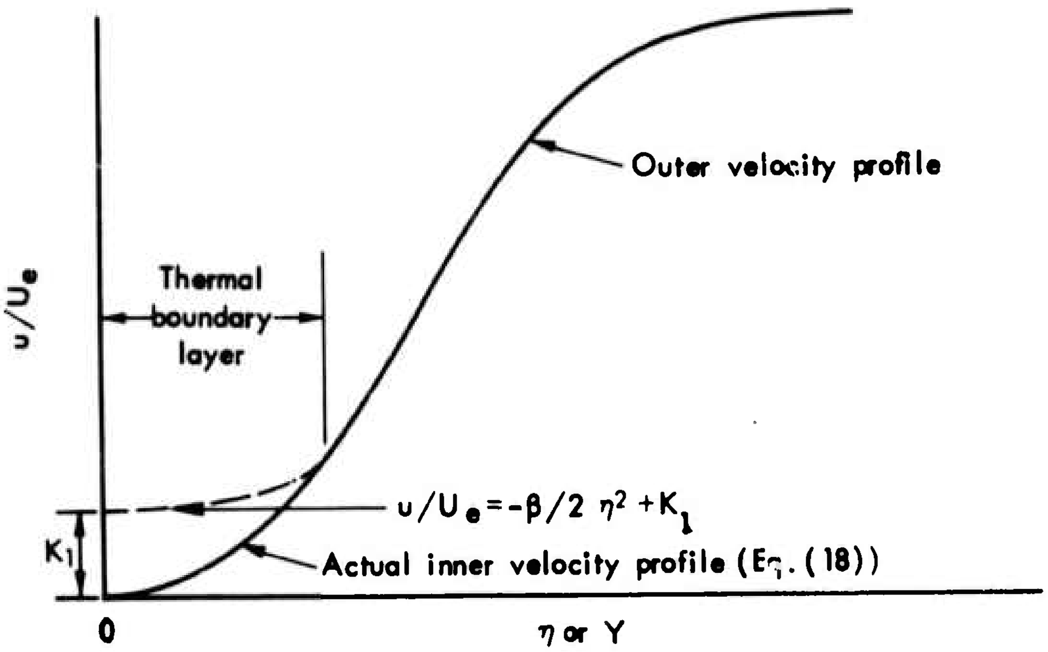
*See Fig. 26, p. 75, of Ref. 9.




NOTE: The outer velocity profile, if continued to the wall, would follow the dashed line and would correspond to a slip velocity, $K_1$, and zero shear. The inner velocity profile corrects this and brings the velocity at y = 0 to zero while still maintaining shear at y = 0. The dashed line is a parabola (see Eq. (39)) and corresponds to the effect of the surface heating on the outer velocity profile.

Fig. 1 — Sketch of the velocity profile

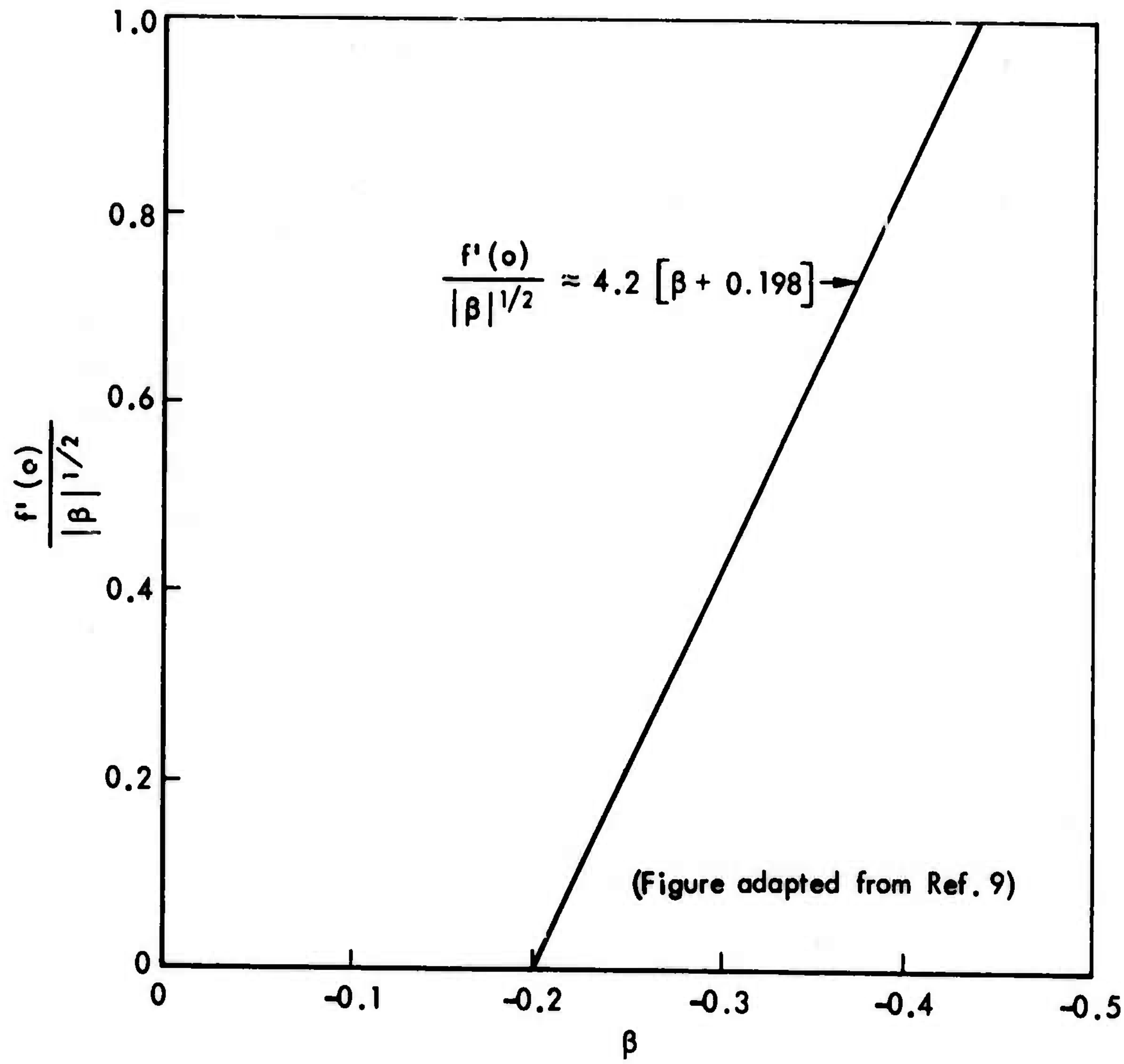
and

$$\alpha = 0.0153\ (°F)^{-1}\ ,$$

where the density variation is neglected.

If $Pr_\infty \approx 8$, then

$$(Pr_\infty)^{-1/2} \frac{\alpha c}{2} (24\bar{N})^{1/2} \Delta T = 0.00453\ \Delta T(\bar{N})^{1/2}\ . \qquad (42)$$




NOTE: $\frac{f'(o)}{|\beta|^{1/2}} = Pr_{\infty}^{-1/2} \frac{\alpha \Delta T c}{2} (24\bar{N})^{1/2}$

$= 0.0045\ \Delta T$ if $T_{\infty} = 60°F$

and $\bar{N} = 1$

Fig. 2 — Slip velocity versus Falkner-Skan β

## III. RESULTS AND CONCLUSIONS

The required wall overheat, $\Delta T_{min}$, is the minimum value of the temperature difference, $T_w - T_\infty$, that will prevent separation for a particular negative value of $\beta$. Its dependence on $\beta$ is shown in Fig. 3, where 60°F water is the ambient fluid. This figure is based on assuming a reference value, $\bar{N}$, of the viscosity ratio $N \equiv \mu/\mu_\infty$, which corresponds to unity. Since N decreases with increasing temperature, any other choice for the reference value $\bar{N}$ would be less than 1. From Eq. (42), it is easy to include any other choices for $\bar{N}$, such as

$$\bar{N} = \frac{\left(1 + \frac{\mu_w}{\mu_\infty}\right)}{2} ,$$

which corresponds to an average reference viscosity ratio. The straight-line relationship between $\Delta T_{min}$ and $\beta$ is

$$\Delta T_{min} \approx -1000\ [\beta + 0.198]\ . \qquad (43)$$

When an average reference condition is used, this straight line becomes one with positive curvature, and the resulting value of $\Delta T_{min}$ is always greater than the $\Delta T_{min}$ which is computed for $\bar{N}$ equal to one. For example, if $\Delta T_{min}$ is 40°F, Fig. 3 indicates that $\beta = -0.24$, and the choice of

$$\bar{N} = \frac{\left(1 + \frac{\mu_w}{\mu_\infty}\right)}{2}$$

leads to a value of $\beta$ which is about -0.235. This results because $\bar{N}$ is 0.9 instead of unity.

From Fig. 3, we see that a wall overheat of 40°F yields a pressure gradient parameter $\beta$ which is about -0.24 at separation. This corresponds

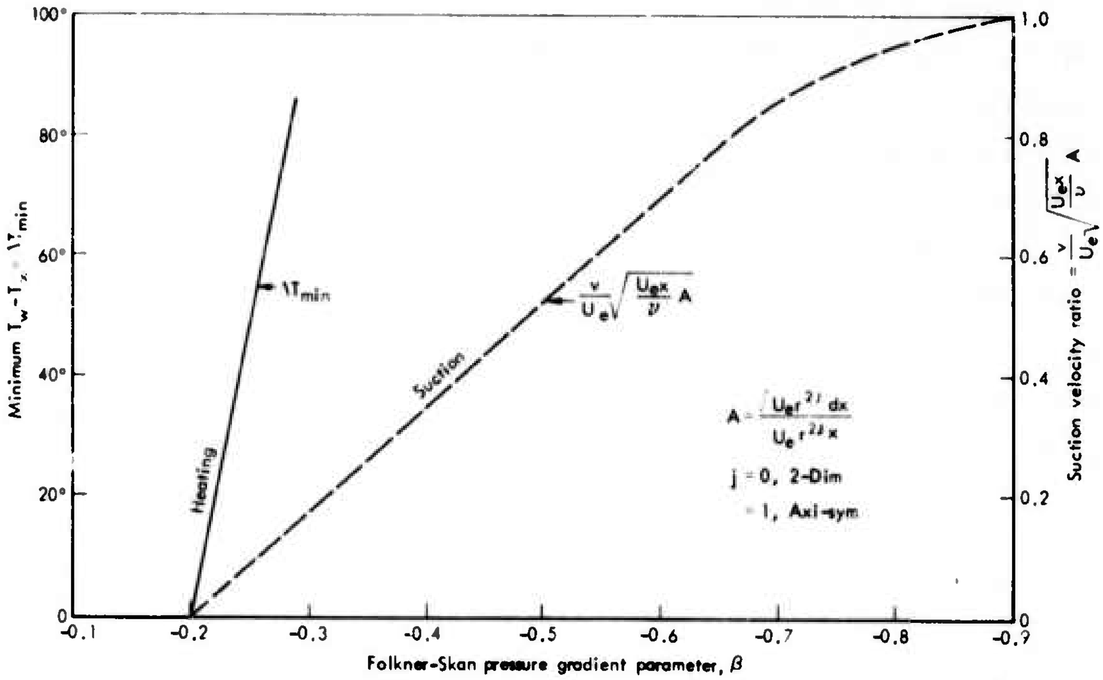



**Fig. 3— Required surface heating and suction to delay laminar separation beyond β = -0.198**

to a 25 percent increase in magnitude over the classic Falkner-Skan value of β = -0.198. It is easy to relate this to the pressure gradient or velocity gradient using the following definition of β:

$$\beta = 2 \frac{1}{U_e} \frac{dU_e}{dx} \frac{\int U_e r^{2j}\, dx}{U_e r^{2j}} .$$

If separation occurs aft of the maximum diameter, then $U_e \sim U_\infty$, r can be approximated as a constant, and the major variation in β comes from the velocity gradient $dU_e/dx$. The fractional increase in the magnitude of β then corresponds roughly to the fractional increase in the magnitude of $dU_e/dx$.

Thus, a 25 percent increase in the value of β corresponds to a similar increase in the permitted magnitude of the velocity gradient without separation. For the class of exact wedge flows, the free-stream velocity is now $U_e \sim x^{-0.11}$ rather than $U_e \sim x^{-0.091}$, as it is for the constant property case.

In order to illustrate the effectiveness of suction compared to wall heating in delaying separation, we have shown in Fig. 3 the required suction velocity ratio, $v/U_e$, which will prevent separation for the same class of Falkner-Skan flows. The quantity $U_e x/\nu$ is the Reynolds number and

$$A \equiv \frac{\int U_e r^{2j}\, dx}{U_e x r^{2j}} .$$

For a boundary layer near the maximum diameter point, the quantity A is close to unity.

The suction velocity ratio, $v_w/U_\infty$, which results in the same effect on separation as a 40°F wall overheat is about

$$\frac{0.1}{\sqrt{\frac{U_\infty x}{\nu}}} .$$

If the arc length Reynolds number at the maximum diameter is $10 \times 10^6$, then the suction velocity ratio is $0.03 \times 10^{-3}$; if the Reynolds number at the maximum diameter is $20 \times 10^6$, the value of $v_w/U_\infty$ is $0.02 \times 10^{-3}$.

From Fig. 3, it is clear that a small amount of suction has a very powerful effect on the delay of separation. Figure 4 is taken from Rosenhead[(10)] and shows that enormous unfavorable pressure gradients ($\beta < -1$) can be sustained without separation.* This, of course, is in accord with both intuition and experiment. Wall heating, as we have shown, has a much less dramatic effect, and it requires wall overheats,

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*See Fig. V.10, p. 249, of Ref. 10.

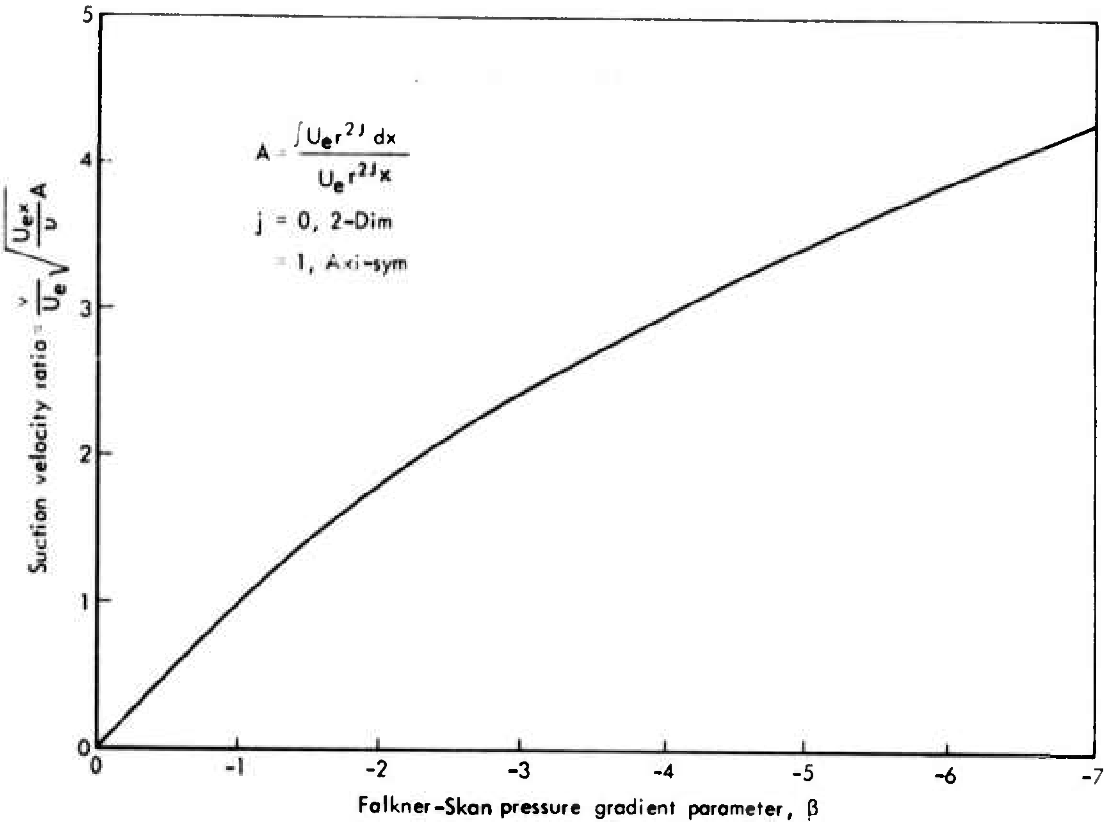


Fig. 4— Suction required to delay laminar separation beyond $\beta = -0.198$

$T_w - T_\infty$, which are unrealistically large, to have such significant effects.

The analysis we have shown here is an approximate one, but the approximations are well understood and are not expected to degrade the numerical results significantly. However, we have examined in detail only the Falkner-Skan separating flows. Both approximate and precise numerical results for other more realistic flows should be obtained, since the weakest part of our approach is the conceptual extension of our results, using a crude local similarity argument, to non-Falkner-Skan flows.

Our analysis predicts that laminar separation can be delayed significantly if the adverse pressure gradient is small in the region of laminar separation. Nevertheless, it appears that surface heating in water has much less impact on delaying separation than does even a small amount of suction.

## Appendix A

## THE APPROXIMATION $X \sim -Z^3$

A more rigorous justification of the use of the approximation $X \sim -Z^3$ in the energy equation can be given as follows.

The matching rule used in Eqs. (36) through (39) is formally

$$\lim_{\eta \to 0} \frac{df(\eta)}{d\eta} = \lim_{Z \to \infty} \frac{d}{d\eta} [\epsilon^3 \beta X(Z)]$$

$$= \epsilon^2 \beta \lim_{Z \to \infty} \frac{dX}{dZ}$$

or

$$\left(\frac{df(\eta)}{d\eta}\right)_0 = \epsilon^2 \beta \left(\frac{dX}{dZ}\right)_\infty \tag{A.1}$$

where $(dX/dZ)_\infty$ is determined from Eq. (25). Equation (25), which is exact, requires that $dg/dZ$ be known from the solution of the energy equation. In this connection, we note that the energy equation, Eq. (14), can formally be solved exactly for arbitrary $X(Z)$,

$$\frac{dg}{dZ} = -B^2 \exp\left(\int_0^Z X(Z)\, dZ\right) \tag{A.2}$$

where B is an arbitrary real constant determined by the boundary conditions on $g(Z)$, and the constant in Eq. (A.2) is written as $B^2$ to emphasize the fact that $dg/dZ < 0$ for all Z. Substituting this result into Eq. (25), we obtain

$$\frac{dX}{dZ} = -\frac{Z^2}{2N} - \frac{\alpha B^2 \Delta T}{2} \int_0^Z Z^2 \exp\left(\int_0^Z X(Z')dZ'\right) dZ \tag{A.3}$$

It is the behavior of this expression as $Z \to \infty$ that we are interested in.

Consider the integral appearing in Eq. (A.3). We must first begin by looking at the expression for $dg/dZ$ appearing therein. We note that it follows from the fact that $f(Z) > 0$ for all $Z$ and the definition of $X(Z)$ that $X(Z) < 0$ for all $Z$. It then follows for any $B$ that $dg/dZ$ approaches zero as $Z \to \infty$. We then have that $dg/dZ$ is an exponentially decaying function of $Z$ for large $Z$ and that $g(Z)$ decreases monotonically from 1 at the wall to zero at infinity.

For the purposes of this report it is sufficient that we have bounds on the second term on the right-hand side of Eq. (A.3). This can be done by analyzing Eq. (A.2) more carefully. First, recall that

$$X(Z) \sim -\frac{Z^3}{6N_w} \qquad \text{near } Z = 0 \,, \tag{26a}$$

$$X(Z) \sim -\frac{Z^3}{6} \qquad \text{for } Z \to \infty \,, \tag{26b}$$

and that with the assumption that $\Delta T = T_w - T_\infty > 0$, it follows from the definition of $N$ (see Eq. (23)) that $N_w < 1$. Thus $-Z^3/6N_w < -Z^3/6$. Further, from Eq. (16) we see that

$$X_Z < 0 \qquad \text{for all } Z \,. \tag{A.4}$$

Also, since from Eq. (23)

$$-\frac{1}{N^2}\frac{dN}{dZ} = \alpha\Delta T\,\frac{dg}{dZ}$$

it follows from $dg/dZ < 0$ that $dN/dZ > 0$. This in turn means that $N(Z)$ is monotonically increasing, and in particular it satisfies

$$N_w \le N(Z) \le 1 \tag{A.5}$$

If we use this last inequality, it follows immediately from Eq. (16) that

$$- \frac{1}{N_w} \int_0^Z ZdZ \leq - \int_0^Z \frac{ZdZ}{N(Z)} \leq - \int_0^Z ZdZ$$

or

$$- \frac{Z^2}{2N_w} \leq X_Z \leq - \frac{Z^2}{2} \qquad \text{(A.6)}$$

We also note that since $N = \mu/\mu_\infty > 0$, it follows from Eq. (15) that

$$X_{ZZ} < 0 \qquad \text{for all } Z \ . \qquad \text{(A.7)}$$

The consequence of Eqs. (A.4), (A.6), and (A.7) is that the curve X(Z) must lie between the two curves $-Z^3/6N_w$ and $-Z^3/6$, as shown in Fig. A.1, coinciding with the first of these near the origin and approaching the latter as $Z \to \infty$. For, since X(Z) starts at the origin with the same value, slope, and curvature as $-Z^3/6N_w$, it can "get outside" this curve only if it violates the left-hand part of inequality Eq. (A.6) or doubles back on itself. The latter of these alternatives is also ruled out since it would lead to X(Z) being a double-valued function of Z. On the other hand, X(Z) cannot "get outside" $-Z^3/6$, for to do so and still satisfy the right-hand part of inequality Eq. (A.6) would require that it double back on itself and again be double-valued. Thus X(Z) is bounded as shown in Fig. A.1.

From Fig. A.1 we note that

$$- \frac{1}{6N_w} \int_0^Z Z^3 dZ \leq \int_0^Z X(Z')dZ' \leq - \int_0^Z \frac{Z^3}{6} dZ \qquad \text{(A.8)}$$

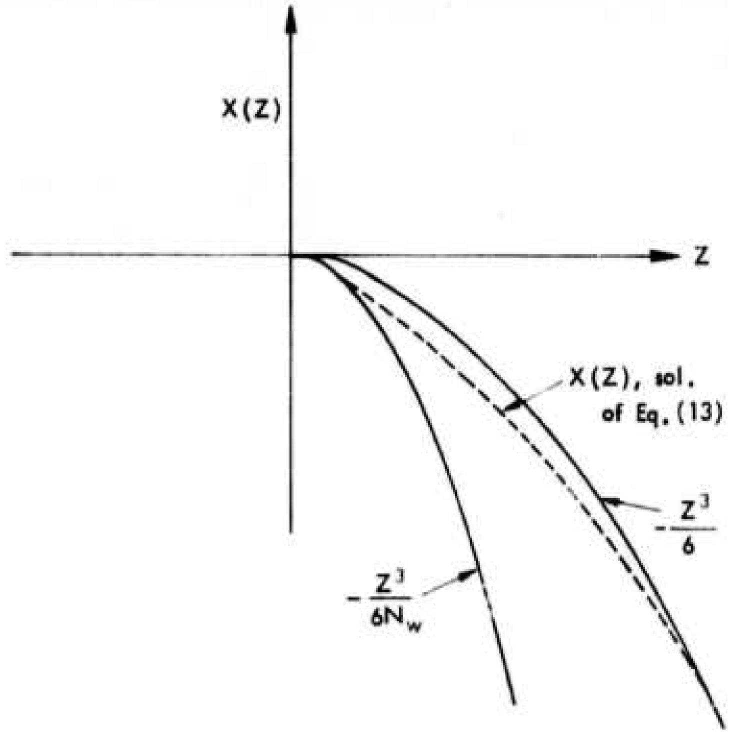


Fig. A.1—Sketch of approximate solution of energy equation

or

$$-\frac{Z^4}{24N_w} \le \int_0^Z X(Z')dZ' \le -\frac{Z^4}{24} \qquad \text{(A.9)}$$

Since for the cases of practical interest here $N_w$ is close to 1 (e.g., for water with $\Delta T = 40°F$, $N_w = 0.8$), we see from Eq. (A.5) that it is reasonable to make the approximation

$$\int_0^Z X(Z')dZ' = -\frac{Z^4}{24\bar{N}} \qquad \text{(A.10)}$$

where $\bar{N}$ is some average reference value of N evaluated for some T lying between $T_\infty$ and $T_w$. This is the approximation made in the main text.

However, it should be noted that for the purposes of this report what is required is the minimum value of the temperature difference, $T_w - T_\infty$, which will have a desired effect, the magnitude of which effect is determined by the value of the right-hand side of Eq. (A.3) for that $\Delta T$. This is obtained from Eq. (A.8) by assigning to

$$\int_0^Z X(Z')dZ'$$

its upper limit, namely $-Z^4/24$. In fact, the specific numerical estimates given in the report involve the use of this latter value, since they are based on $\bar{N} = 1$.

## Appendix B

## EVALUATION OF INTEGRALS

Two integrals, which must be evaluated (see Eq. (32)), are

$$I_1 = \int_0^\infty Z^2 \exp(-Z^4/24\bar{N})\, dZ$$

and

$$I_2 = \int_0^\infty \exp(-Z^4/24\bar{N})\, dZ \ .$$

We consider $I_1$ in detail. Let

$$\frac{Z^4}{24\bar{N}} = t \ , \qquad \text{(B.1)}$$

then

$$Z = (24\bar{N})^{1/4}\, t^{1/4} \ , \qquad \text{(B.2)}$$

and

$$dZ = \frac{(24\bar{N})^{1/4}}{4}\, t^{-3/4}\, dt \ . \qquad \text{(B.3)}$$

Thus, we can write $I_1$ as

$$I_1 = \frac{(24\bar{N})^{3/4}}{4} \int_0^\infty t^{-1/4}\, e^{-t}\, dt \ .$$

Since

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt ,$$

then

$$I_1 = \frac{(24\bar{N})^{3/4}}{4} \Gamma(3/4) . \tag{B.4}$$

Similarly,

$$I_2 = \frac{(24\bar{N})^{1/4}}{4} \int_0^\infty t^{-3/4} e^{-t} dt \tag{B.5}$$

so

$$I_2 = \frac{(24\bar{N})^{1/4}}{4} \Gamma(1/4) . \tag{B.6}$$

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Shows the magnitude of the heating effect on the laminar separation of a water boundary layer and compares its effectiveness in controlling separation to that of surface suction. An analysis is presented of the minimum surface overheat that will delay separation for a prescribed adverse pressure gradient in water. It is found that the effect of heating and viscosity variation is localized to a thin layer near the wall, well within the entire momentum boundary layer. However, although heating is shown to delay separation, the magnitude of its effect is found to be small, particularly if practical values of wall overheat are considered. Results suggest that the effects of surface heating on the delay of separation are probably real and significant, but heating does not have the almost unlimited capacity for maintaining an attached flow that suction appears to possess. Ref. (PB)